From classical to fuzzy de Sitter spacetime and back

Dedicated to the memory of Antun Balaž (1973–2025)

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In memory of Antun Balaž (1973–2025)

- Friend, colleague, mentor — one of the central people at the **Institute of Physics Belgrade**.
- A passionate physicist, working on **path integrals**, **ultracold quantum gases**, Bose–Einstein condensation, and **complex systems**.
- He held important positions and received significant recognition, but he also gave a great deal of himself to **students, colleagues, and our community**.

Today's talk is dedicated to his memory.

> Antun was someone we could always rely on: always ready to help, to find a solution to every problem — or simply to point out that there is no problem.

Calm, generous, clever, and with a quiet sense of humour.

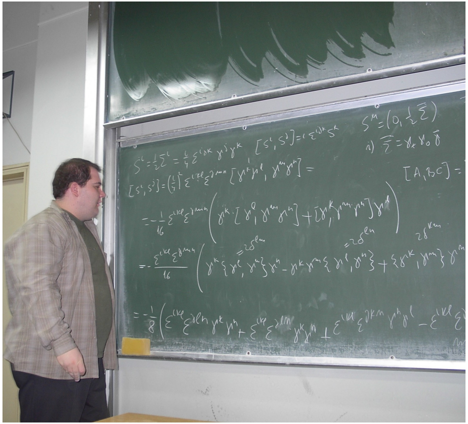
How we met — Petnica, 1994

- I first met Antun in 1994, at the **Petnica Science Center**.
- He was already a physics student — and one of the *young mentor* at the summer school. I was a high-school participant, still wondering what I would study.
- Those weeks in Petnica, and especially my conversations with Antun, are one of the reasons why I am standing here today as a physicist.
- Antun never lost his connection with Petnica. He kept coming back, as a lecturer and as a mentor. In recent years, he was also a member of the centre's Management Board.

Many of us in this room owe a great deal to Petnica — and to Antun.

While I was in the army — Antun took my place

- Almost ten years later, when I was doing my **mandatory military service**, Antun stepped in and took over my teaching duties at the Faculty of Physics in Belgrade.
- He taught my *problem-solving classes* — and did it brilliantly.
- In those classes, he gathered around himself a remarkable group of students — many of them later became the core of the **Scientific Computing Laboratory** (SCL) at the Institute of Physics Belgrade.



He knew how to teach, and he knew how to bring people together. He kept both gifts throughout his life.

A registered address, and a small daughter

- In the years that followed, Antun lived in different rented flats in Belgrade. He moved quite often. To save him from constantly updating his official documents, he was registered at our home address.
- From time to time, some letters for him would arrive in our mailbox. Then Antun would come to pick them up, usually on a Saturday evening.
- Those visits were **a small celebration** for my then very young daughter. Antun played with her and they both enjoyed it very much.
- One day she became suspiciously curious about the *mailbox*. When I asked why, she answered very seriously: *"So Antun will come again."*

Children always saw it. He loved them, and they loved him back.

Antun the textbook author

- There is one more side of Antun I want to mention. Together with N.V. and me, he wrote **physics textbooks** for the elementary school.
- Antun had a **remarkable instinct** for speaking to children. He did not speak to them from above, but with real respect for the child as someone who thinks, asks, and understands.
- I think he remembered very vividly what it felt like to learn physics for the first time, as a boy in elementary school.
- In that sense, he kept that boy alive in himself: curious, playful, and open to wonder.
- Perhaps this is why he wrote so naturally for children. In a way, he was writing for that boy.

A great mentor of researchers — and a great teacher of children.

Physics textbook, grade 7.

Now, to the physics —

the story Antun would have wanted me to tell.

Plan of the talk

A three-step story:

1. **Classical de Sitter spacetime**

   geometry, the Poincaré patch, scalar fields, vacua

2. **Fuzzy de Sitter spacetime**

   noncommutative algebra, frames, fuzzy Klein–Gordon

3. **Back to a classical description**

   semi-classical states, commutative limit, correlators

Based on works with M. Burić, B. Brkić, and I. Burić.

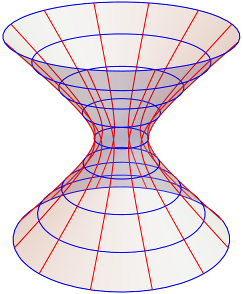
References: EPJC **75** (2015) 502; EPJC **78** (2018) 953; Class. Quantum Grav. **39** (2022); JHEP **10** (2024) 018.
Pedagogical: D. Anninos, "de Sitter Musings", Int. J. Mod. Phys. A **27** (2012) 1230013.

Why de Sitter space?

- De Sitter space is **the simplest curved spacetime** with a positive cosmological constant. It is like Minkowski space, but curved. This makes it a very natural place to study gravity and quantum fields.
- The observed Universe is approximately de Sitter in two important periods:
  - during **inflation**, when the energy scale was very high,
  - in the far future, when the expansion is dominated by **dark energy**.
- In inflationary cosmology, the quantities we want to compute are defined on **a background that is close to de Sitter space**. The information that cosmologists measure in the CMB is connected to physics on de Sitter space.

But de Sitter space also has many open problems:

- vacuum choice, horizons, holography, and possible quantum-gravity effects.



de Sitter hyperboloid

This is our motivation. Our NC construction is one way to look at **possible quantum-gravity effects**.

De Sitter space as a hyperboloid

A useful way to describe de Sitter space is to embed it in one higher-dimensional flat spacetime.

The d-dimensional de Sitter space is the hyperboloid in $(d+1)$-dimensional Minkowski space $\mathbb{R}^{1,d}$:

$$\mathrm{dS}_d = \left\{ X^A \in \mathbb{R}^{1,d} \;:\; \eta_{AB} X^A X^B = -X_0^2 + \sum_{i=1}^{d} X_i^2 = \ell^2 \right\}.$$

- ℓ is the **de Sitter radius**, fixed by Λ. From now on we set $\ell = 1$.
- The topology is $\mathbb{R} \times S^{d-1}$.
- The symmetry group is $\mathrm{SO}(1,d)$, the **de Sitter group**.
- It is an Einstein space with positive scalar curvature:

$$R_{ab} = \frac{d-1}{\ell^2} g_{ab}, \qquad R = \frac{d(d-1)}{\ell^2} > 0.$$

Main message

De Sitter geometry is controlled by the group $\mathrm{SO}(1,d)$.

Useful coordinates on dS_d

Different questions are best described in different coordinate systems.

Global coordinates (τ, Ω)

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \cosh^2\tau \, \mathrm{d}\Omega^2_{d-1}$$

- Cover the whole hyperboloid.
- Spatial slices are spheres S^{d-1}.
- The universe contracts and then expands.

Static patch (t, r, Ω)

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2_{d-2},$$

where $f(r) = 1 - r^2$.

- Describes the region seen by one observer.
- There is a cosmological horizon at $r = 1$.

Poincaré, or flat, patch (η, x^i)

$$\boxed{\mathrm{d}s^2 = \frac{1}{\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}\mathrm{x}^2\right)}$$

- $\eta \in (-\infty, 0)$ is conformal time.
- Spatial slices are flat $\mathbb{R}^{d-1}$.
- This patch covers only one half of dS_d.
- It is the natural patch for **inflationary cosmology**: perturbations are usually quantised here.

For this talk: we mostly use the Poincaré patch.

A scalar field on the Poincaré patch

Consider a free massive scalar field:

$$(\Box - m^2)\phi = 0, \qquad \Box = \frac{1}{\sqrt{-g}}\partial_a\big(\sqrt{-g}\, g^{ab}\partial_b\big).$$

On the Poincaré patch, the equation separates:

$$\phi(\eta, \mathrm{x}) = \int \mathrm{d}^{d-1}k \left[a_{\mathrm{k}}\, u_{\mathrm{k}}(\eta, \mathrm{x}) + a_{\mathrm{k}}^{\dagger}\, u_{\mathrm{k}}^{*}(\eta, \mathrm{x})\right],$$

where the spatial part satisfies

$$\partial_i \partial^i h_{\mathrm{k}}(\mathrm{x}) = -k^2 h_{\mathrm{k}}(\mathrm{x}).$$

For each momentum k, the time-dependent part has two independent solutions:

$$u_{\mathrm{k}}(\eta, \mathrm{x}) \propto (-\eta)^{(d-1)/2}\left[c_1 H^{(1)}_{i\kappa}(-k\eta) + c_2 H^{(2)}_{i\kappa}(-k\eta)\right] h_{\mathrm{k}}(\mathrm{x}), \qquad \kappa^2 = m^2 - \left(\frac{d-1}{2}\right)^2.$$

- $H^{(1,2)}_{i\kappa}$ are Hankel functions.
- Choosing the coefficients (c_1, c_2) means choosing which modes are called **positive-frequency modes**.
- That choice defines the **vacuum**.

Invariant vacua

In flat spacetime, there is a natural notion of positive frequency. In de Sitter space, this choice is not unique.

De Sitter symmetry does not choose a unique vacuum:

- **Bunch–Davies vacuum**: selected by the usual positive-frequency behaviour in the far past. This is the standard choice in inflationary cosmology.
- α-**vacua**: other de Sitter-invariant vacua. They are obtained by mixing Bunch–Davies modes with their complex conjugates:

  $$u_{\mathrm{k}}^{(\alpha)} = A_\alpha u_{\mathrm{k}}^{BD} + B_\alpha (u_{\mathrm{k}}^{BD})^*.$$

  This is a Bogoliubov transformation.
- Different choices of vacuum give different two-point functions:

  $$\langle 0|\phi(x)\phi(x')|0\rangle.$$

Observation

In Minkowski space, ∂_t is timelike everywhere and fixes a preferred positive-frequency split. In dS_d, no Killing vector of $SO(1,d)$ is timelike everywhere. So symmetry alone does not select a unique vacuum.

Why noncommutativity?

- In quantum gravity, we expect smooth spacetime to stop being a good picture at the **Planck scale**.
- One simple way to describe this is to say that the **coordinates become operators**, with $[x^\mu, x^\nu] \neq 0$. This is similar to ordinary quantum mechanics (position and momentum do not commute). Here, the new idea is that two spacetime coordinates do not commute.
- The simplest choice is to take

  $$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$

  where $\theta^{\mu\nu}$ is constant. This is the usual **Moyal-type noncommutativity**. But this is not the only possibility. Noncommutativity can also be introduced in a different, more geometric way.
- Inflation happened at very high energies, possibly close to the scale where quantum-gravity effects matter. If spacetime was noncommutative at that time, it could leave a small trace in **primordial fluctuations** — and in the CMB.
  - This is why the construction is not only a formal idea. It may also have a **physical meaning**.

Why *fuzzy* (rather than Moyal)?

- **History.** The fuzzy approach started with the fuzzy sphere — one of the first and most important examples of a noncommutative fuzzy space. There, the coordinates are $SU(2)$ generators, and the algebra of functions on the sphere is replaced by a finite-dimensional matrix algebra.
  - Fuzzy de Sitter is the natural Lorentzian version of the same idea.
- **Symmetry.** We build the algebra directly from **unitary irreducible representations of $SO(1,d)$**. So de Sitter symmetry is the input, not something added later.
- **Curvature.** It is built into the group structure. In Moyal-type approaches, one usually starts with flat space and a constant noncommutativity tensor.
- **No preferred constant tensor.** In Moyal space one writes $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ with constant $\theta^{\mu\nu}$. This is simple, but it introduces extra fixed data.
  - In the fuzzy approach, noncommutativity comes from the geometry itself.

The fuzzy de Sitter algebra

We build everything from one structure: the Lie algebra $\mathfrak{so}(1,d)$ of the de Sitter group. The antisymmetric generators $M_{AB} = -M_{BA}$ satisfy

$$[M_{AB}, M_{CD}] = i\big(\eta_{AC} M_{BD} - \eta_{BC} M_{AD} - \eta_{AD} M_{BC} + \eta_{BD} M_{AC}\big).$$

This is the symmetry algebra of classical de Sitter space. Here we use the same algebra to build the coordinates themselves.

For the spacetime coordinates we use the **Pauli–Lubanski vector** W^A, which is *quadratic* in the generators. In $d = 4$:

$$\boxed{X^A \;=\; \ell\, W^A, \qquad W^A = \tfrac{1}{8}\,\epsilon^{ABCDE} M_{BC} M_{DE}}$$

- Coordinates are not single generators — they are particular *quadratic* combinations.
- Acting on a Hilbert space, the X^A are **operators**, not numbers — spacetime points become quantum observables.

Fuzzy hyperboloid

Why this choice of coordinates? Classically, dS_d is the surface

$$\eta_{AB}\, x^A x^B \;=\; \ell^2.$$

In $d = 4$, the operator counterpart $\eta_{AB}\, W^A W^B$ is exactly the **quartic Casimir** of $\mathfrak{so}(1,d)$, which **takes a single fixed value in any irreducible representation**. So

$$\eta_{AB}\, X^A X^B \;=\; \text{const}$$

holds *exactly* at the operator level: a **fuzzy hyperboloid**.

The value of this constant depends on the chosen irreducible representation. Non-zero values are obtained, for example, for the **principal continuous series**. Thus the *fuzzy geometry is selected not only by the algebra, but also by the representation*.

From algebra to geometry

Having an algebra is not enough. To do geometry, we also need:

metric → **derivatives** → **Laplacian**

For this we use the **noncommutative frame formalism** of John Madore.

The classical idea is Cartan's moving frame. Instead of working directly with the components of the metric, we choose an orthonormal **local frame** $\{e_a\}$, or vielbein, such that

$$g(e_a, e_b) = \eta_{ab}.$$

- In this local frame, the metric looks flat.
- But the space is still curved.
- The curvature is encoded in how the local frame changes from point to point.

This is the idea that we now translate to the noncommutative case.

Noncommutative frame formalism

In Madore's formalism, the frame is built algebraically, using commutators. The basic ingredients are operators p_a, called **momenta**. They are dual to the one-forms and define the derivatives:

$$e_a(f) = [p_a, f].$$

- The operators p_a play the role of the frame vectors.
- The metric in this frame is fixed to be constant:

$$g(e_a, e_b) = \eta_{ab}.$$

- The non-trivial geometric information is **encoded in the algebra of the momenta**, in particular in the commutators $[p_a, p_b]$.

The main choice is which $\mathfrak{so}(1,d)$ generators we use as the momenta p_a. Different choices give different commutative limits. The choice we use, introduced by Burić and Madore in 2015, gives the **Poincaré patch of de Sitter space** in the commutative limit.

Fuzzy coordinates in the Poincaré patch

In $d = 2$ this becomes especially simple. The embedding-coordinate operators can be written as generators of $\mathfrak{so}(1,2)$. For the Poincaré patch we use the combinations

$$\boxed{\hat{\eta} = -i\bar{k}P, \qquad \hat{x} = -i\bar{k}D}$$

Here P is a translation generator and D is a dilatation generator.

The corresponding **momenta** are also generators:

$$\hat{p}_0 = D, \qquad \hat{p}_1 = -P.$$

The coordinates do **not commute:**

$$[\hat{\eta}, \hat{x}] = i\bar{k}\,\hat{\eta}.$$

- This is not Moyal-type noncommutativity.
- The right-hand side is not constant; it depends on $\hat{\eta}$.
- The noncommutativity is therefore adapted to the curved de Sitter geometry.

Frame, metric and dynamics

The coordinate commutator is important, $[\hat{\eta}, \hat{x}] = i\bar{k}\,\hat{\eta}$, but the differential geometry is determined by the frame.

The frame relations are

$$[\hat{p}_0, \hat{\eta}] = \hat{\eta}, \qquad [\hat{p}_1, \hat{x}] = \hat{\eta},$$

so that the only nonzero components of the orthonormal frame are:

$$e^0{}_0 = e^1{}_1 = \hat{\eta}.$$

With the flat frame metric η^{ab}, this gives

$$g^{00} = -\hat{\eta}^2, \qquad g^{11} = \hat{\eta}^2,$$

the inverse metric of the Poincaré patch of dS_2.

The momentum algebra

$$[\hat{p}_0, \hat{p}_1] = \hat{p}_1$$

determines the Laplacian. Since the momenta are Lie-algebra generators, the fuzzy Klein–Gordon operator has a natural algebraic form, with no extra ordering prescription.

The commutative limit is: $\bar{k} \to 0$, with physical coordinates kept finite.

The fuzzy Laplacian

We now turn to equations of motion.

In the noncommutative frame formalism, derivatives are commutators with the momenta $\hat{p}_a$. Therefore **the scalar Laplacian is also built from commutators.**

In the two-dimensional Poincaré patch, this takes the form

$$\Delta\Phi = -[\hat{p}_0, [\hat{p}_0, \Phi]] + [\hat{p}_0, \Phi] + [\hat{p}_1, [\hat{p}_1, \Phi]].$$

- This is **the fuzzy analogue** of the de Sitter box operator.
- In the commutative limit, with physical coordinates kept finite, it reduces to $\Box_{\mathrm{dS}_d}$.
- The fuzzy Klein–Gordon equation is $(\Delta - m^2)\hat{\phi} = 0$.
- The spectrum of the symmetric version $p_a p^a$ was studied in Brkić, Burić, Latas (Class. Quantum Grav. **39**, 2022).

What is the fuzzy field?

The field is a function of the noncommutative coordinates:

$$\hat{\phi} = \phi(\hat{\eta}, \hat{x}).$$

Since $\hat{\eta}$ and $\hat{x}$ are operators, $\hat{\phi}$ is also an operator: it is **an element of the noncommutative algebra.**

- It is not an ordinary function on spacetime.
- It is not yet a second-quantised field.

Thus, solving the fuzzy Klein–Gordon equation means finding elements of the algebra that satisfy

$$(\Delta - m^2)\hat{\phi} = 0.$$

New coordinates: from $\hat{x}$ to $\hat{y}$

The problem. The coordinates $\hat{x}^\mu = (\hat{\eta}, \hat{x}^i)$ are natural geometrically, but they are not the best coordinates for solving the fuzzy Klein–Gordon equation. The action of the momenta on products of $\hat{\eta}$ and $\hat{x}^i$ leads to ordering difficulties.

The fix. Introduce a new coordinate, here in dS_2:

$$\hat{y} = \tfrac{1}{2}\left(\hat{\eta}^{-1}\hat{x} + \hat{x}\,\hat{\eta}^{-1}\right), \qquad [\hat{\eta}, \hat{y}] = i\bar{k}.$$

Its brackets with the momenta are remarkably simple:

$$[\hat{p}_0, \hat{\eta}] = \hat{\eta}, \quad [\hat{p}_1, \hat{\eta}] = 0, \qquad [\hat{p}_0, \hat{y}] = -\hat{y}, \quad [\hat{p}_1, \hat{y}] = 1.$$

- These relations mean that the frame derivations **preserve the chosen ordering** of $\hat{\eta}$ and $\hat{y}$.
- So we can fix an ordering and **separate variables**, as in the commutative theory.
- In $d = 4$:

$$\hat{y}^i = \tfrac{1}{2}(\hat{\eta}^{-1}\hat{x}^i + \hat{x}^i\hat{\eta}^{-1}),$$

  with $[\hat{p}_0, \hat{y}^j] = -\hat{y}^j$ and $[\hat{p}_i, \hat{y}^j] = \delta_i^j$.

Solving fuzzy KG: ordering matters

Strategy Start from commutative modes written in the coordinates (η, y^i). In these coordinates, the modes separate:

$$v(\eta, y^i) = (-\eta)^{-i\omega} F(y^i).$$

Promote the variables to operators:

$$\eta \to \hat{\eta}, \qquad y^i \to \hat{y}^i,$$

and **choose a fixed ordering:**

$$v(\eta, y^i) \longrightarrow \hat{v}(\hat{\eta}, \hat{y}^i).$$

Why this works

- The frame derivations preserve the chosen ordering.
- The fuzzy Laplacian acts on $\hat{v}$ in the same algebraic way as the classical Laplacian acts on v.
- Therefore every commutative KG mode gives a fuzzy KG solution.

Result

$$(\Delta_{dS} - m^2)v(\eta, y^i) = 0 \Rightarrow (\hat{\Delta} - m^2)\hat{v}(\hat{\eta}, \hat{y}^i) = 0.$$

Extra modes in fuzzy de Sitter

To find all solutions, we represent operators as **integral kernels**

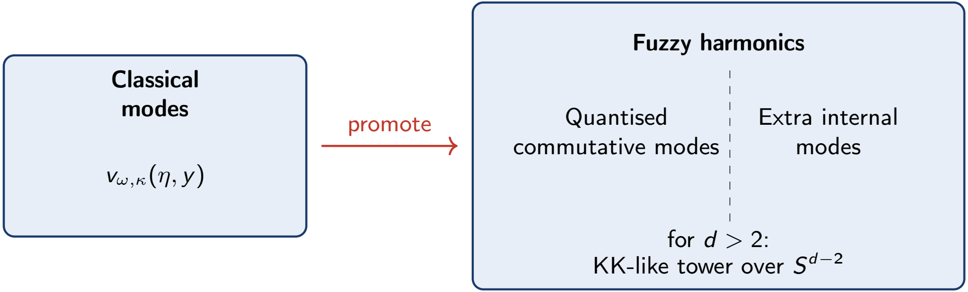
$$f^a{}_b(z_L, z_R),$$

that is, as functions of two sets of variables:

$$\mathcal{A} = \mathrm{End}(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^*.$$

- In $d = 2$, the quantised modes already form a complete set: there are **no extra modes**.
- In $d = 4$, the fuzzy KG equation has **more solutions than the commutative one**.
- To each commutative solution there corresponds a family of noncommutative solutions, with additional quantum numbers
$$l, \qquad \vec{m}, \qquad e_a \otimes e_b.$$
- The labels $l, \vec{m}$ come from the internal S^2, while $e_a \otimes e_b$ labels the discrete matrix part.
- These additional labels are the **internal data** of the fuzzy mode.
- Their role, and their precise commutative limit, remain an important open problem.

Two kinds of fuzzy modes



Classical modes

$v_{\omega,\kappa}(\eta, y)$

promote

Fuzzy harmonics

Quantised commutative modes

Extra internal modes

for $d > 2$:
KK-like tower over S^{d-2}

For $d > 2$, fuzzy de Sitter has **more modes** than classical de Sitter.

The way back: semi-classical states

How does a smooth classical geometry re-emerge from fuzzy de Sitter?

Use **semi-classical, coherent-like states**. In fuzzy dS_2, choose a reference vector $|\xi_0\rangle$:

$$\eta_0 = \langle \xi_0 | \hat{\eta} | \xi_0 \rangle, \qquad x_0 = \langle \xi_0 | \hat{x} | \xi_0 \rangle.$$

This associates $|\xi_0\rangle$ with the classical point

$$\xi_0 = (\eta_0, x_0).$$

Other points are reached by the momentum/frame flow:

$$|\xi_1\rangle = \lambda^{\hat{p}_0} e^{b\hat{p}_1} |\xi_0\rangle.$$

Key property

For the Poincaré coordinate operators, this construction reproduces the classical coordinates exactly:

$$\langle \xi_1 | \hat{\eta} | \xi_1 \rangle = \eta_1, \qquad \langle \xi_1 | \hat{x} | \xi_1 \rangle = x_1.$$

Expectation values and correlators

Once semi-classical states are chosen, any fuzzy observable can be compared with a classical quantity by taking expectation values:

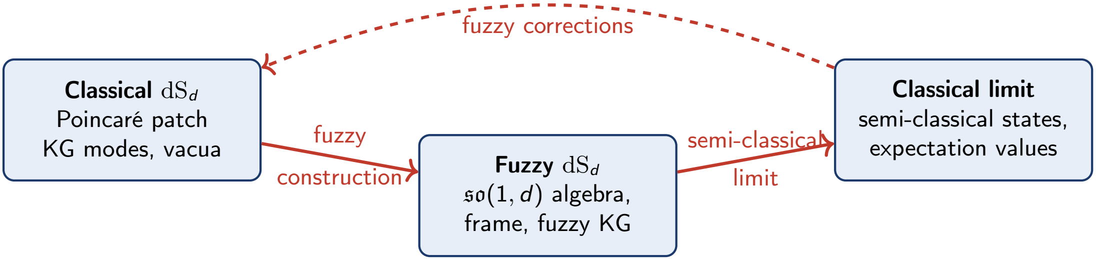
$$A_\xi = \langle \xi | \hat{A} | \xi \rangle .$$

- For coordinate operators in fuzzy dS_2, this gives the classical Poincaré coordinates exactly.
- For general observables, expectation values give the classical result plus **noncommutative corrections**.
- The same idea should be applied to the fuzzy scalar field: fuzzy modes can be used to build the two-point function.
- Its commutative limit, especially in $d = 4$, remains to be understood.

Outlook

A symmetry-preserving way to study **noncommutative corrections** to cosmological correlators.

The three-step story, in one slide



fuzzy corrections

Classical dS_d
Poincaré patch
KG modes, vacua

fuzzy
construction

Fuzzy dS_d
$\mathfrak{so}(1,d)$ algebra,
frame, fuzzy KG

semi-classical
limit

Classical limit
semi-classical states,
expectation values

The dashed arrow is the next target: **symmetry-preserving fuzzy corrections** to cosmological observables.

Open directions

- Classify and interpret the **internal $S^2 \times W$ modes** in fuzzy dS_4.
- Understand how the internal modes enter physical observables, especially two-point functions.
- Compute **late-time correlators** relevant to inflation, including possible noncommutative corrections.
- Complete the **"back to classical" construction**: semi-classical states are understood in $d = 2$, while $d = 4$ is still a major task.
- Study interactions and higher-point functions on fuzzy dS_d.
- Compare with matrix-model approaches and IKKT-type emergent cosmologies.

Summary

- Fuzzy de Sitter space is built from $\mathfrak{so}(1,d)$ in a UIR, with geometry described by the **noncommutative frame formalism**.
- The fuzzy construction selects useful coordinates (η, y^i), adapted to the Poincaré patch.
- With a compatible ordering, the fuzzy Klein–Gordon equation can be solved at the operator level in $d = 2$ and $d = 4$.
- Every commutative mode can be **promoted** to a fuzzy mode; for $d > 2$ there are also genuinely noncommutative **internal modes**.
- Semi-classical states give a way back to classical observables: worked out in $d = 2$, with $d = 4$ still in progress.
- This opens a route toward studying noncommutative effects in inflationary physics.

Thank you.

In gratitude to Antun — for all the conversations we will not have, and for the example he set.