Noncommutative AdS spacetime: classical limits & boundary

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- I first met Antun in 1996. when he was on the fourth year of his undergraduate studies: it was a good generation, quite big, lively. Obviously, in November the lectures at the faculty stopped, due to the civil+student protests of 1996/97.

- Antun started to work at the Institute of Physics immediately after his graduation, at first as a member of the hep-th group. The group was very active, we organised and attended many meetings and workshops together, including those in Dubrovnik, Vienna, Kopaonik. Before he oriented his reseach towards the functional integrals, we wrote a paper on solitons in Born-Infeld electrodynamics,
A. Balaž, M. Burić, V. Radovanović, PRD **65** (2002) 065007.

- But strangely from the beginning, besides being a student and a collaborator, Antun was a friend. As he (to my annoyance) used to say that he learned to drink beer from me, I hope that meant that I was one of his why-so-serious? friends.

- Quantisation of gravity is a long-lasting effort of several generations. Although it produced many interesting mathematical ideas, important physical insights and new results, it has not yet given a satisfactory, consistent description of physics at the Planck scale.
- (Maybe because Einstein's idea about the relation between geometry and gravity is so perfect/true? Certainly, due to the lack of experimental data.)
- Quantum spacetime emerges in all descriptions which have been proposed/studied: either the gravitational field is quantised (implying the fluctuating lightcone, quantisation of volume, curvature, etc.); or as in string theory, spacetime coordinates are represented by quantum fields so they do not commute.
- In order to find properties of spacetime one measures fields on it, i.e., geometric characteristics are given via the field propagators.

- One physical motivation to study noncommutative geometry is a belief that the relation between gravity and geometry continues to exist beyond the classical regime, i.e. that in some interval of scales quantum gravity is described effectively as NC geometry.
- Noncommutative geometry is a generalisation of differential geometry to 'NC manifolds' generated by operators; recent models use even more advanced mathematical structures like nonassociative, deformed, twisted, L_∞ etc. algebras.
- On these generalised structures one defines differential geometry as well as physics, including classical fields, equations of motion, symmetries, quantisation, etc.
- Many models of NC geometry are being studied, based on different assumptions and with different properties. A common requirement is the existence of commutative or classical limit, often realised by defining the theory as some kind of deformation of the classical one.

- Geometry that we use is the noncommutative frame formalism, inspired by Cartan's description of differential geometry.
- Coordinates x^μ are elements of a C^*-algebra; derivations are given by commutators. There is a preferred set of vector fields, the moving frame e_α generated by p_α, that gives the quantisation rule

$$e_\alpha x^\mu \equiv [p_\alpha, x^\mu] = e^\mu_\alpha(x), \qquad e^\mu_\alpha \to e^\mu_\alpha|_{classical} \ .$$

Metric is in the frame basis locally flat; $g^{\mu\nu}(x) = e^\mu_\alpha(x)\, e^\nu_\beta(x)\, \eta_{\alpha\beta}$.

- Momenta p_α can but need not belong to coordinate algebra.
- NC differential-geometric structure is defined using momenta; algebraic structure is given by commutators of coordinates. Compatibility of the two structures and additional requirements give a number of constraints.
- Our strategy to define specific NC spaces: try to identify, within the known algebras, x^μ and p_α that give the moving frame, metric, Laplacian, symmetries, etc. with the desired classical limits.

• Solvable models in physics have symmetries (the more, the better). Our idea is to extend classical spaces of maximal symmetry to their NC or fuzzy versions. Leading example: the fuzzy sphere, based on the $so(3)$ algebra $[J^a, J^b] = i\varepsilon^{abc} J_c$,

$$x^a = (\bar{k}/r)\, J^a, \quad p_a = (1/ir)\, J_a, \quad r^2 = \bar{k} l/2$$

where l is the dimension of the UIR. Embedding in 3d is given by the Casimir relation; commutative limit by $\bar{k} \to 0,\ l \to \infty$.

• The fuzzy sphere proved to be extremely useful in discretisation of rotationally invariant problems and numerical simulations.

• We studied dS_2, dS_4 and AdS_2, AdS_3 spaces. Instead of the space of functions + a star/twisted product, we use UIRs of the symmetry group + their Hilbert space realisations.

• Momenta are chosen in the Lie algebra, coordinates – not necessarily. As # of momenta gives dimension of a NC space, we do not use all generators: this decreases symmetry.

• Commutative limit is given via Perelomov coherent states.

5. Fuzzy AdS_2

- we start with the classical AdS_2 symmetry $SO(1,2) \simeq SL(2,\mathbb{R})$
- algebra: H, E_+, E_-: $[H, E_+] = E_+$, $[H, E_-] = -E_-$, $[E_+, E_-] = 2H$
- representation: discrete series T_l^- $(l \leq -1)$, $\mathcal{H} = \{f(\xi), \xi > 0\}$

$$H = \xi\partial_\xi + l + 1\,, \quad E_+ = -i\xi\,, \quad E_- = -i\left(\xi\partial_\xi^2 + 2(l+1)\partial_\xi\right)$$

$$(f,g) = 2^{2l+1}\pi \int_0^\infty d\xi\, \xi^{2l+1}\, f(\xi)^* g(\xi)$$

lowest weight state: $\Psi_0(\xi) = N\xi^{-2l-1}e^{-\xi}$

- fuzzy Poincaré coordinates: $z = i\hbar E_+,\ p_z = H,\ t = -i\hbar H,\ p_t = E_+$
- Laplacian: $\Box\phi = [p_z,[p_z,\phi]] - [p_z,\phi] - [p_t,[p_t,\phi]]$
- semiclassical (coherent) states: $|\lambda, c\rangle \equiv \lambda^{-p_z}\, e^{-cp_t}\, |\Psi_0\rangle$

$\langle\lambda, c\,|z|\lambda, c\rangle = -l\hbar\lambda, \quad \langle\lambda, c\,|t|\lambda, c\rangle = -l\hbar c$

- symmetry group: $SO(2,2) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$
- algebra: $H,\, E_+,\, E_-,\, \bar{H},\, \bar{E}_+,\, \bar{E}_-$
- representation: $T_l^- \otimes \bar{T}_l^-$, $\mathcal{H} = \{f(\xi,\bar{\xi}),\, \xi,\bar{\xi} > 0\}$
  lowest weight state: $|\Psi_0 \otimes \bar{\Psi}_0\rangle$
- fuzzy Poincaré coordinates:

$$z = -2i\kappa\sqrt{E_+\bar{E}_+}, \qquad p_z = H + \bar{H}$$

$$x = i\kappa\left(\sqrt{\frac{\bar{E}_+}{E_+}}\left(H - \frac{1}{4}\right) + \sqrt{\frac{E_+}{\bar{E}_+}}\left(\bar{H} - \frac{1}{4}\right)\right), \qquad p_x = E_+ + \bar{E}_+$$

$$t = i\kappa\left(\sqrt{\frac{\bar{E}_+}{E_+}}\left(H - \frac{1}{4}\right) - \sqrt{\frac{E_+}{\bar{E}_+}}\left(\bar{H} - \frac{1}{4}\right)\right), \qquad p_t = E_+ - \bar{E}_+$$

- Laplacian: $\Box\phi = [p_z,[p_z,\phi]] - 2[p_z,\phi] - [p_t,[p_t,\phi]] + [p_x,[p_x,\phi]]$
- semiclassical states: $|\lambda, b, c\rangle \equiv \lambda^{-p_z}\, e^{-cp_t} e^{-bp_x}\, |\Psi_0 \otimes \bar{\Psi}_0\rangle$

• Classical scalar field $\Phi(x^\mu)$ is a function of NC coordinates that satisfies the Klein-Gordon equation, in the AdS_2 case,

$$\Box\phi = [p_z,[p_z,\phi]] - [p_z,\phi] - [p_t,[p_t,\phi]] = m^2\phi \ .$$

How to solve this equation?

• Usually, if fields are represented as functions of commutative coordinates with a $\star$-product, equation is solved perturbatively.

• Or, coordinates can be treated as operators and equation is solved for a specific ordering. It is possible to do this for fuzzy dS and AdS by introducing new coordinates, such that the frame derivations do not change the ordering. Completeness?

• Our strategy: represent the scalar field in the concrete representation, in some basis of $\mathcal{H}$, as an operator kernel; momenta, likewise. The KG equation becomes a linear partial differential equation that can be solved by standard methods.

• The algebra of functions on fuzzy AdS$_2$ and the generators of momenta are $\mathcal{A} \cong T_l^- \otimes (T_l^-)^* = \{\phi(\xi_L, \xi_R),\ \xi_L, \xi_R > 0\}$, and $\mathrm{ad}_{p_z} = \xi_L \partial_{\xi_L} + \xi_R \partial_{\xi_R} + 2l + 2\,, \quad \mathrm{ad}_{p_t} = -i(\xi_L - \xi_R)\,.$

• We change the variables, $\xi_L = \rho \sin\varphi\,, \xi_R = \rho\cos\varphi\,,\ \varphi \in (0, \frac{\pi}{2})$ and solve the KG equation: we obtain the scalar field modes $\phi_\beta(\rho,\varphi) = c_\beta\,\delta(\varphi - \varphi_0)\,\rho^{-2l-\frac{3}{2}}\,J_\nu(\beta\rho)\,,\ \beta = \sqrt{2}\cos(\varphi_0 + \frac{\pi}{4}) \in (-1, 1)\,.$

• We find, next, the expectation values of the field modes in semiclassical states, $\langle\phi_\beta\rangle = \langle\lambda, c|\phi_\beta|\lambda, c\rangle$:

$$\langle\phi_\beta\rangle = \frac{\Gamma(\nu - 2l + \frac{1}{2})}{\Gamma(\nu+1)\Gamma(-2l)}\,\frac{2\pi c_\beta\,\lambda^{2l}(\frac{\beta}{2})^\nu}{\alpha^{\nu-2l+1/2}}\ {}_2F_1\Big(\frac{\nu-2l+\frac{1}{2}}{2}\,, \frac{\nu-2l+\frac{3}{2}}{2}\,; \nu+1; -\frac{\beta^2}{\alpha^2}\Big)$$

with $\mu = -2l + \frac{1}{2}\,,\ \alpha = \frac{\sqrt{2-\beta^2} + ic\beta}{\lambda}\,.$

- In the previous formula, parameter λ is proportional to the expectation value of coordinate $\langle z\rangle$, and c is proportional to time $\langle t\rangle$. Using the limiting formula for the hypergeometric function,

$$J_\nu(x) = \lim_{a,b\to\infty} \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_2F_1(a,b;\nu+1;-\tfrac{x^2}{4ab})\ ,$$

and identifying $2a = \nu - 2l + \frac{1}{2}$, $2b = \nu - 2l + \frac{3}{2}$, we find that the classical/commutative limit of the modes is given by $l \to -\infty$:

$$\langle\phi_\beta\rangle \sim \frac{1}{\sqrt{2}}\, e^{-i\omega\langle t\rangle}\sqrt{\langle z\rangle}\, J_\nu\left(\omega\langle z\rangle\right)\ .$$

- Here the normalisation is fixed as $c_\beta = \frac{\sqrt{\bar{k}}}{4\pi}(\frac{8}{4+\bar{k}^2\omega^2})^{-l+\frac{1}{4}}$ and we introduced frequency $\omega = \frac{2\beta}{\bar{k}\sqrt{2-\beta^2}}$, $\omega \in \left(-\frac{2}{\bar{k}}, \frac{2}{\bar{k}}\right)$.
- In terms of the dependence on $\langle z\rangle$ and $\langle t\rangle$, these functions are precisely the modes of the scalar field on commutative AdS_2.

• Quantisation is introduced as usual: the free scalar field is quantised via expansion in modes,

$$\Phi = \int d\omega\,(a_\omega \phi_\omega + a^\dagger_\omega \phi^*_\omega)\,, \quad [a_\omega, a^\dagger_{\omega'}] = \delta(\omega - \omega')\,, \quad a_\omega|0\rangle = 0\,.$$

• the n-point function in the semiclassical states is given by

$$G_n(x_1, \dots, x_n) = \langle x_1, \dots, x_n|\,\langle 0|\Phi \otimes \dots \otimes \Phi|0\rangle\,|x_1, \dots, x_n\rangle\,.$$

• the 2-point function is the integral over positive frequencies,

$$G_2(x_1, x_2) = \int_0^{2/\not{k}} d\omega\,\langle x_1|\phi_\omega|x_1\rangle\,\langle x_2|\phi^*_\omega|x_2\rangle\,.$$

• This integral is, in principle, difficult to evaluate as it contains the product of two hypergeometric functions. It is however possible to calculate it on the AdS boundary defined as $\langle z_1\rangle \to 0$, $\langle z_2\rangle \to 0$, after the usual rescaling by $\langle z_1\rangle^{-\Delta}\langle z_2\rangle^{-\Delta}$ $(\Delta = \nu + \frac{1}{2})$.

• The boundary-to-boundary 2-point function $G_{\partial\partial}$ can be obtained exactly, in terms of the Appell function F_1,

$$G_{\partial\partial} \sim F_1(2\Delta, \Delta - 2l, \Delta - 2l, 2\Delta + 1; \frac{i\langle t_1\rangle}{l\bar{k}}, \frac{-i\langle t_2\rangle}{l\bar{k}})$$

where $F_1(a, b, b'; c; x, y) = \sum_{m,n} \frac{(a)_{m+n}(b)_m(b')_n}{m!\, n!\, (c)_{m+n}}\, x^m y^n$.

• The limit $l \to -\infty$ corresponds to the confluent limit of F_1 ,

$$\lim_{l\to-\infty} G_{\partial\partial} = \frac{\bar{k}^{-2\Delta}}{2\Gamma(\Delta + \frac{1}{2})^2\,\Delta}\, {}_1F_1(2\Delta; 2\Delta + 1; \frac{-2i\langle\, t_{12}\rangle}{\bar{k}})$$

• A further limit $\bar{k} \to 0$ gives

$$\lim_{\bar{k}\to 0} \lim_{l\to-\infty} G_{\partial\partial} = \frac{\bar{k}^{-2\Delta}\Gamma(2\Delta + 1)}{2\Gamma(\Delta + \frac{1}{2})^2\,\Delta} \left(\frac{2i\langle t_{12}\rangle}{\bar{k}}\right)^{-2\Delta} = C_\Delta\, \langle i\hat{t}_{12}\rangle^{-2\Delta} ,$$

precisely the commutative 2-point function on AdS_2 .

- numerically:

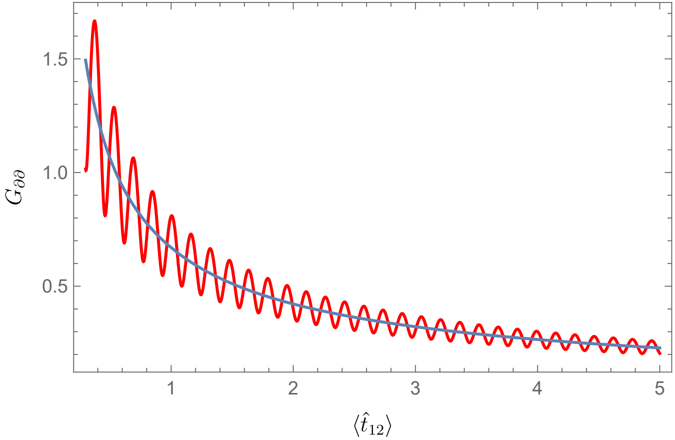

Figure: Commutative and fuzzy $G_{\partial\partial}$: $l = -\infty$, $\hbar = 1/20$, $\Delta = 1/3$

- Since $SO(2,2) \sim SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, UIRs that describe the fuzzy AdS_3 are tensor products, $T_l^- \otimes T_l^-$. The Hilbert space consists of functions of two variables, $f(\xi,\bar{\xi})$, the operator kernels depend on four arguments, etc: on the calculational level, everything is doubled.
- However, the rules of how to assign operators to coordinates and momenta, such that geometry of fuzzy AdS_3 has the correct commutative limit, differ. In particular, we have 3 coordinates and 3 momenta (and not 2×2=4) .
- The semiclassical states are defined by group translations of the highest weight state, but they do not not form a set of coherent states, in the sense that they do not span the whole representation space: there is an internal (i.e., non-geometric) subspace of states.
- The KG equation can be solved by change of variables and has a correct semiclassical limit for $l \to -\infty$. The analysis of the boundary 2-point function is more difficult (no closed expression found), but again, the classical limit gives exactly the correct 2-point function.

- numerically:

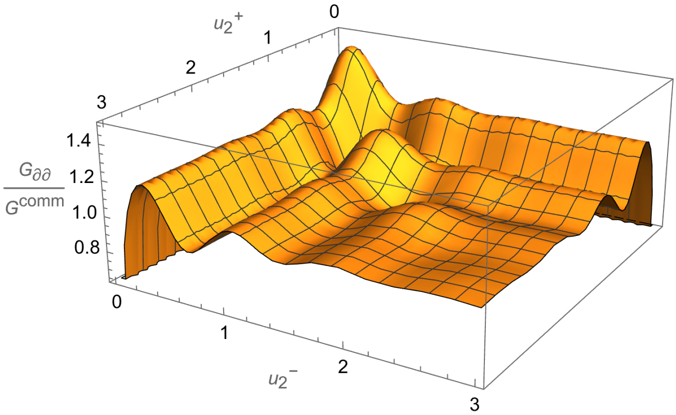

Figure: Commutative and fuzzy $G_{\partial\partial}$: $l = -20$, $\hbar = 1/2$, $\Delta = 1/3$

- We found an interesting fuzzy extensions of AdS spaces, which have good classical limits both on the level of geometry and on the level of propagating classical and quantum fields.
- These geometries have less symmetries than the original ones; further, in dimensions $d > 2$, the construction imposes the existence of internal, nongeometric (genuinely quantum?) degrees of freedom.
- Also, we found a new set of models with modified AdS/CFT, with the exact 2-point functions in the bulk and on the boundary.

Things to do:

- to understand the obtained fuzzy geometries and field theories beyond the semiclassical approximation: 'deep quantum' states?
- to study symmetries in the bulk and on the boundary: perhaps, deformed symmetries, quantum groups?
- to include interactions: e.g. calculate the 3-point functions via Witten diagrams and check the applicability of AdS/CFT prescription
- to check if the correlation functions can be obtained by introducing additional conformal field, e.g. defect on the boundary?