Atomic Solitons in Optical Lattices

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In memory of Antun Balaž - insights into complex systems
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Collaboration with A. Cetoli, F. Lorenzi, B. Malomed, F. Toigo, and
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Summary

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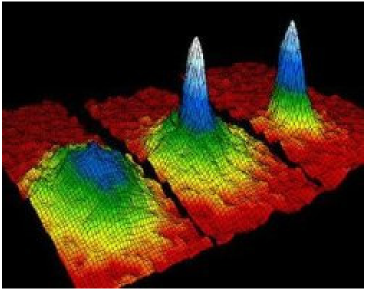
- Gross-Pitaevskii equation
- Dimensional reduction: from 3D to 1D
- 1D bright solitons
- Quasi-1D bright solitons
- Bright solitons in 2002 experiments
- Bright solitons in optical lattices
- Conclusions

Atomic BEC in optical lattices (I)

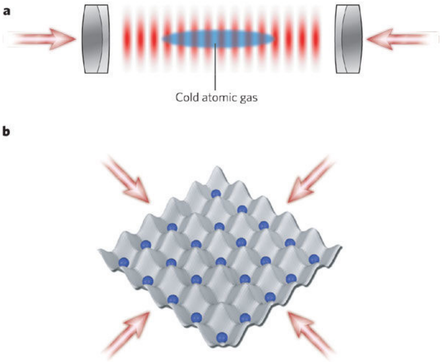
In 1995 Eric Cornell and Carl Wieman, Wolfgang Ketterle, and Randy Hulet achieved Bose-Einstein condensation (BEC) cooling very dilute gases of Rubidium (^{87}Rb), Sodium (^{23}Na), and Lithium (^{7}Li) atoms, respectively.



The BEC critical temperature is about $T_c \simeq 100$ nanoKelvin. The gas, made of dilute and ultracold neutral alkali-metal atoms, is in a meta-stable state which can survive for minutes.

Atomic BEC in optical lattices (II)

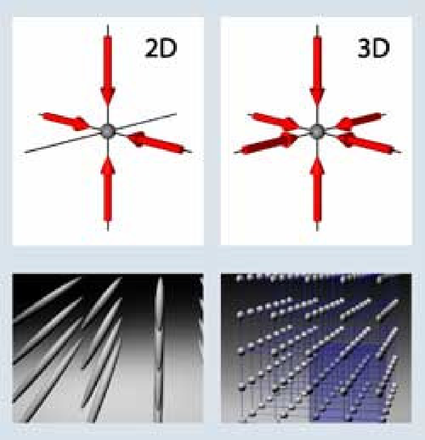
In 2002 the experimental group of Immanuel Bloch obtained with counter-propagating laser beams inside an optical cavity, stationary optical lattice which can trap ultracold atoms.



The resulting optical potential $U(\mathbf{r}) = -\langle \mathbf{d} \cdot \mathbf{E}(\mathbf{r}) \rangle$ can trap neutral atoms with electric dipole moment $\mathbf{d}$ in the minima of the optical lattice.

Atomic BEC in optical lattices (III)

Now the study of neutral atoms trapped with light is a very hot topic of research.



Changing the intensity and shape of the optical lattice, it is now possible to trap atoms in very different configurations. One can have many atoms per site bit also one atom per site.

Gross-Pitaevskii equation (I)

Static and dynamical properties of a pure Bose-Einstein condensate made of dilute and ultracold atoms are very well described by the Gross-Pitaevskii equation[1]

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + (N-1)\frac{4\pi\hbar^2 a_s}{m}|\psi(\mathbf{r},t)|^2\right]\psi(\mathbf{r},t) \;, \tag{1}$$

where $U(\mathbf{r})$ is the external trapping potential and a_s is the s-wave scattering length of the inter-atomic potential.

Here $\psi(\mathbf{r},t)$ is the wavefunction of the Bose-Einstein condensate normalized to one, i.e.

$$\int |\psi(\mathbf{r},t)|^2 \, d^3\mathbf{r} = 1 \;, \tag{2}$$

and such that $n(\mathbf{r}) = N|\psi(\mathbf{r},t)|^2$ is the local number density of the N condensed atoms.

[1] E.P. Gross, Nuovo Cimento **20**, 454 (1961); L.P. Pitaevskii, Sov. Phys. JETP. **13**, 451 (1961).

Gross-Pitaevskii equation (II)

The Gross-Pitaevskii equation, that is a nonlinear Schrödinger equation with cubic nonlinearity, can be deduced from the many-body quantum Hamiltonian of N identical spinless particles.
In the case of a pure Bose-Einstein condensate one assumes all bosons in the same time-dependent single-particle orbital (Hartree approximation)

$$\Psi(\mathbf{r}_1, ..., \mathbf{r}_N, t) = \prod_{i=1}^{N} \psi(\mathbf{r}_i, t) \, . \tag{3}$$

Moreover, in the case of dilute gases we assumes a very simple inter-atomic potential (Fermi pseudo-potential) that

$$V(\mathbf{r} - \mathbf{r}') \simeq g_{3D} \, \delta^{(3)}(\mathbf{r} - \mathbf{r}') \tag{4}$$

with $\delta^{(3)}(\mathbf{r})$ the Dirac delta function and

$$g_{3D} = \int V(\mathbf{r}) \, d^3\mathbf{r} = \frac{4\pi\hbar^2}{m} a_s \tag{5}$$

with a_S the s-wave scatering length a_s of the inter-atomic potential (Born approximation).

Dimensional reduction: from 3D to 1D (I)

We have seen that the Gross-Pitaevskii (GP) equation

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) + (N-1)g_{3D}|\psi(\mathbf{r},t)|^2 \right] \psi(\mathbf{r},t) \qquad (6)$$

is the Hartree equation for bosons, all in the same single-particle orbital $\psi(\mathbf{r},t)$. It is also the Euler-Lagrange equation of the following GP action functional

$$S = N \int dt\, d^3\mathbf{r}\, \psi^*(\mathbf{r},t) \Big(i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - U(\mathbf{r}) - \frac{N-1}{2} g_{3D}|\psi(\mathbf{r},t)|^2 \Big) \psi(\mathbf{r},t) \, . \qquad (7)$$

Let us now consider a very strong harmonic confinement of frequency $\omega_\perp$ along x and y and a generic confinement $\mathcal{U}(z)$ along z, namely

$$U(\mathbf{r}) = \frac{1}{2} m {\omega_\perp}^2 (x^2+y^2) + \mathcal{U}(z) \, . \qquad (8)$$

Dimensional reduction: from 3D to 1D (II)

On the basis of the chosen external confinement, we adopt the ansatz

$$\psi(\mathbf{r}, t) = f(z, t) \frac{1}{\pi^{1/2} a_\perp} \exp\left(\frac{x^2 + y^2}{2 {a_\perp}^2} \right) , \tag{9}$$

where $f(z, t)$ is the axial wave function and $a_\perp = \sqrt{\hbar/(m\omega_\perp)}$ is the characteristic length of the transverse harmonic confinement.

By inserting Eq. (9) into the GP action (7) and integrating along x and y, the resulting effective action functional depends only on the field $f(z, t)$.

One easily finds that the Euler-Lagrange equation of the axial wavefunction $f(z, t)$ reads

$$i\hbar \frac{\partial}{\partial t} f(z, t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \mathcal{U}(z) + \gamma |f(z, t)|^2 \right] f(z, t) , \tag{10}$$

where

$$\gamma = \frac{(N-1) g_{3D}}{2\pi {a_\perp}^2} \tag{11}$$

is the effective one-dimensional interaction strength and the additive constant $\hbar\omega_\perp$ has been omitted because it does not affect the dynamics.

1D bright solitons (I)

In the absence of axial confinement, i.e. $\mathcal{U}(z) = 0$, the 1D GPE becomes

$$i\hbar \frac{\partial}{\partial t} f(z,t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \gamma |f(z,t)|^2 \right] f(z,t) \; . \qquad (12)$$

This is a 1D nonlinear Schrödinger equation with cubic nonlinearity.
In 1972 Vladimir Zakharov and Aleksei Shabat[2] found that this equation admits solitonic solutions, such that

$$f(z,t) = \phi(z - vt)\, e^{i(mv^2/2-\mu)t/\hbar}, \qquad (13)$$

where v is the arbitrary velocity of propagation of the solution, which has a shape-invariant axial density profile:

$$n(z,t) = N|f(z,t)|^2 = N|\phi(z - vt)|^2 \; . \qquad (14)$$

[2] V.E. Zakharov and A.B. Shabat, Sov. Phys. JETP **34**, 62 (1972).

1D bright solitons (II)

Setting $\zeta = z - vt$, the 1D stationary GP equation

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{d\zeta^2} + \gamma |\phi(\zeta)|^2 \right] \phi(\zeta) = \mu \, \phi(\zeta) \; , \tag{15}$$

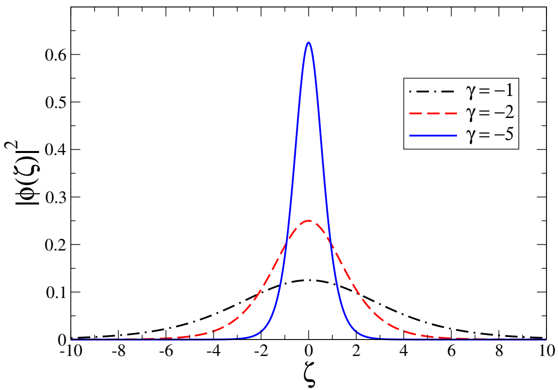
with $\gamma < 0$ (self-focusing), admits the bright-soliton solution

$$\phi(\zeta) = \sqrt{\frac{m|\gamma|}{8\hbar^2}} \; Sech \left[\frac{m|\gamma|}{4\hbar^2} \zeta \right] \tag{16}$$

with $Sech[x] = \frac{2}{e^x + e^{-x}}$ and

$$\mu = -\frac{m \, \gamma^2}{16 \, \hbar^2} \; . \tag{17}$$

1D bright solitons (III)



Probability density $|\phi(\zeta)|^2$ of the bright soliton for three values of the nonlinear strength γ. We set $\hbar = m = 1$.

Quasi-1D bright solitons (I)

The 1D GPE has been derived from the 3D GPE assuming a transverse Gaussian with a constant transverse width $a_\perp$.
A more general assumption[3], is based on a space-time dependent transverse width

$$\psi(\mathbf{r},t) = f(x,t)\frac{1}{\pi^{1/2}\sigma(x,t)}\exp\left(-\frac{y^2+z^2}{2\sigma(x,t)^2}\right), \qquad (18)$$

where $f(x,t)$ is the axial wave function and $\sigma(x,t)$ is the space-time dependent transverse width.
Inserting this ansatz in the 3D GP action functional, after neglecting the spatial derivatives of $\sigma(x,t)$, the Euler-Lagrange equations of $f(x,t)$ and $\sigma(x,t)$ give the **1D nonpolynomial Schrödinger equation** (1D NPSE)

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}f &= \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) + \frac{\hbar^2}{2m\sigma^2} + \frac{m\omega_\perp^2\sigma^2}{2} + \frac{2\hbar^2 a_s(N-1)|f|^2}{m\sigma^2}\right] f\,, \\ \sigma &= a_\perp\left(1+2a_s(N-1)|f|^2\right)^{1/4}\,. \end{aligned} \qquad (19)$$

[3] LS, A. Parola, L. Reatto, Phys. Rev. A **65**, 043614 (2002).

Quasi-1D bright solitons (II)

In the weak-coupling regime $|a_s||f|^2 \ll 1$ one finds $\sigma \simeq a_\perp$ and the 1D NPSE becomes the familiar 1D GPE.

However, contrary to the 1D GPE bright soliton, the 1D NPSE bright soliton does not exist anymore, **collapsing** at[4]

$$\gamma_c = \left(\frac{a_s(N-1)}{a_\perp} \right)_c = -\frac{2}{3} \, . \tag{20}$$

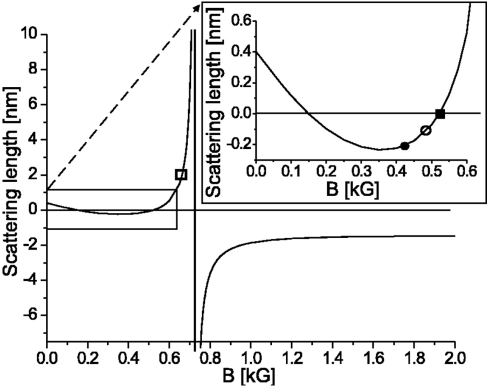
This analytical result is in extremely good agreement with the numerical solution of the 3D GPE: $\gamma_c = -0.67$.

Main message: a real bright soliton will collapse when the absolute value of the interaction strength is sufficiently large!

[4] LS, A. Parola, L. Reatto, Phys. Rev. A **66**, 043603 (2002); Phys. Rev. Lett. **91**, 080405 (2003).

Bright solitons in 2002 experiments (I)

In 2002 there were two relevant experiments about bright solitons with BECs made of ^{7}Li atoms.



Both experiments used the technique of Fano-Feschbach resonance to tune the s-wave scattering length a_s of the inter-atomic potential by means of an **external constant magnetic field**. In the figure: scattering length a_s for ^{7}Li in state $|F = 1, m_F = 1\rangle$.

Bright solitons in 2002 experiments (II)

At ENS of Paris, Khaykovich et al. [Science **296**, 1290 (2002)] reported the production of bright solitons in an ultracold ^{7}Li gas. The interaction was tuned with a Feshbach resonance from repulsive to attractive before release in a one-dimensional optical waveguide. Propagation of the soliton without dispersion over a macroscopic distance of 1.1 millimeter was observed.

At Rice University, Strecker et al. [Nature **417**, 150 (2002)] reported the formation of a train of bright solitons of ^{7}Li atoms in a quasi-one-dimensional optical trap. The solitons were set in motion by offsetting the optical potential, and were observed to propagate in the potential for many oscillatory cycles without spreading.[5]

[5] For collective modes of bright-soliton trains: J.H.V. Nguyen, D. Luo, R.G. Hulet, Science **356**, 422 (2017).

Bright solitons in optical lattices (I)

In 2007 we analyzed quasi-1D solitons in the cigar-shaped trap equipped with a periodic potential, which can be created as an optical lattice (OL)[6]. The energy-per-atom E of the self-attractive BEC are given by

$$\begin{aligned} E &= \int d^3\mathbf{r}\, \psi^*(\mathbf{r}) \Big[-\frac{1}{2}\nabla^2 + \frac{1}{2}(x^2+y^2) + \mathcal{U}(z) \\ &+ \pi\, g\, |\psi(\mathbf{r})|^2 \Big] \psi(\mathbf{r}) \; , \end{aligned} \tag{21}$$

in adimensional form, with $g = 2(N-1)a_s/a_\perp < 0$. Here, the OL potential acting along axis z is

$$\mathcal{U}(z) = -V_0 \cos\left(2k_L z\right) \tag{22}$$

with lattice spacing $d_L = \pi/k_L$ and recoil energy $E_r = \pi^2\hbar^2/(2md_L^2)$. In 2025 this specific problem has been experimentally investigated at Strathclyde University.[7]

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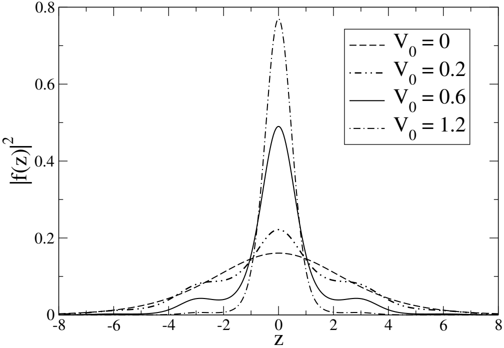
[6] LS, A. Cetoli, B. A. Malomed, and F. Toigo, Phys. Rev. A **75**, 033622 (2007).

[7] R. Cruickshank, F. Lorenzi, A. La Rooij, E. Kerr, T. Hilker, S. Kuhr, LS, E. Haller, Phys. Rev. Lett. **135**, 263404 (2025).

Bright solitons in optical lattices (II)



The axial density profile, $|f(z)|^2$, of the bright soliton in OL periodic potential $\mathcal{U}(z)$, Eq. (22), with $k_L = 1$ and four different values of V_0. The self-attraction strength is fixed at $g = -0.5$. Numerical solution of the 1D NPSE.[8]

[8]LS, A. Cetoli, B. A. Malomed, and F. Toigo, Phys. Rev. A **75**, 033622 (2007).

Bright solitons in optical lattices (III)

To study lattice solitons in an approximate analytical form, one can use the Gaussian ansatz

$$\psi(\mathbf{r}) = \frac{1}{\pi^{3/4}\sigma\eta^{1/2}} \exp\left\{-\frac{(x^2+y^2)}{2\sigma^2} - \frac{z^2}{2\eta^2}\right\}, \tag{23}$$

where σ and η are, respectively, the transverse width and axial length of the localized pattern. Inserting this ansatz into Eq. (21) one obtains

$$E = \frac{1}{2}\left(\frac{1}{2\eta^2} + \frac{1}{\sigma^2} + \sigma^2\right) + \frac{g}{2\sqrt{2\pi}}\frac{1}{\sigma^2\eta} - V_0 \exp\left(-k_L^2\eta^2\right) . \tag{24}$$

Bright solitons in optical lattices (IV)

Aiming to predict the ground state in the framework of the above approximation, we look for values of σ and η that minimize energy E [as given by Eq. (24)], using equations $\partial E/\partial\sigma = \partial E/\partial\eta = 0$. This way, we derive coupled equations,

$$-\frac{1}{\eta^3} + \frac{g}{(2\pi)^{1/2}}\frac{1}{\sigma^2\eta^2} + 4V_0k_L^2\eta\exp\left(-k_L^2\eta^2\right) = 0, \qquad (25)$$

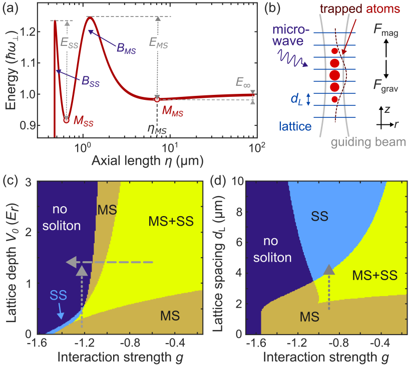
$$-\frac{1}{\sigma^3} + \sigma + \frac{g}{(2\pi)^{1/2}}\frac{1}{\sigma^3\eta} = 0. \qquad (26)$$

Quite remarkably, we can get σ as a function of η, i.e.

$$\sigma(\eta) = \left(1 + \frac{g}{(2\pi)^{1/2}}\frac{1}{\eta}\right)^{1/4} \qquad (27)$$

and also $E(\eta) = E(\eta, \sigma(\eta))$.

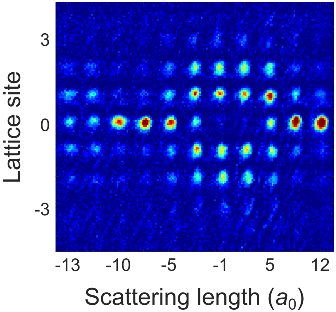
Bright solitons in optical lattices (V)



(a) Energy $E(\eta)$ for a Gaussian wave packet with $V_0 = 1.1\,E_r$, $a_s = -6.2\,a_0$, $d_L = 2\,\mu$m. Single-site (SS) and multi-site (MS) solitons are stable at minima M_{SS} and M_{MS} with barriers B_{SS} and B_{MS}. (b) Sketch of experimental setup. (c) Stable regions of SS and MS solitons for varying parameters g and V_0, with $N = 1800$, $\omega_\perp = 2\pi \times 30\,$Hz, $d_L = 3.2\,\mu$m. No solitons exist in dark blue regions. (d) Stable regions for varying d_L, same parameters as (c) with constant $V_0 = 1.3\,E_r$. [9]

[9] R. Cruickshank *et al.*, Phys. Rev. Lett. **135**, 263404 (2025).

Bright solitons in optical lattices (VI)



Measured density distribution changing scattering length a_s. There are 4 regimes (from left to right): (i) delocalization due to the collapse for strong attraction (ii) a bright soliton for moderate attraction, (iii) free expansion near zero interaction, and (iv) (self-trapping) localization for strong repulsion. These regimes arise from the interplay between the effective potential barriers B_{SS} and B_{MS}. Here $d_L = 3.2(2)\,\mu$m, $V_0 = 1.3(1)\,E_r$, $\omega_\perp = 2\pi \times 40(1)\,$Hz, $N \approx 1800$. [10]

[10] R. Cruickshank *et al.*, Phys. Rev. Lett. **135**, 263404 (2025).

Conclusions

- Periodic potentials are now easily created with laser beams (optical lattices).
- Ultracold atoms are an ideal platform to study many-body problems on a lattice.
- Atomic BEC are very useful to study coherent nonlinear phenomena, such as solitons (bright and dark) and quantized vortices.
- In these atomic systems there is an excellent synergy between theory and experiments.

Acknowledgements

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