Faraday and resonant waves in dipolar cigar-shaped Bose-Einstein condensates

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Contents

- Ultracold dipolar Bose gases
- Variational description of dipolar Bose gas
- Variational study of Faraday and resonant waves
- Faraday waves in ^{52}Cr, ^{168}Er and ^{164}Dy BECs
- Fourier spectrum in the time- and spatial-frequency domains
- Interaction effects on Faraday and resonant waves
- Conclusions

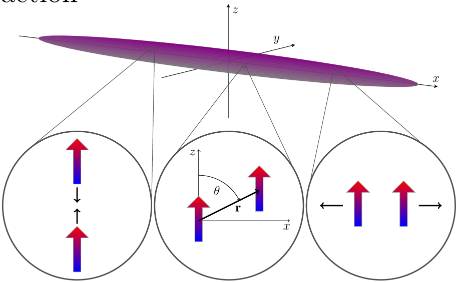
Ultracold dipolar Bose gases

- Experimental realization 2005 in ^{52}Cr, later in ^{164}Dy, ^{168}Er, ...
- Contact and dipol-dipol interaction

$$U_{\rm c}(\mathbf{r}-\mathbf{r}') = \frac{4\pi\hbar^2 a_s N}{m}\,\delta(\mathbf{r}-\mathbf{r}')$$

$$U_{\rm dd}(\mathbf{r}-\mathbf{r}') = \frac{\mu_0\bar{\mu}^2}{4\pi}\,\frac{1-3\cos^2\theta}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$U_{\rm int}(\mathbf{r}) = U_{\rm c}(\mathbf{r}) + U_{\rm dd}(\mathbf{r})$$



- Stability of the system and critical number of atoms
- Harmonically trapped system, with the frequencies $\omega_x = 7 \times 2\pi$ Hz, $\omega_y = 160.5 \times 2\pi$ Hz, and $\omega_z = 160.5 \times 2\pi$ Hz

P. Engels et al., Phys. Rev. Lett. **98**, 095301 (2007).

Mean-field theory for dipolar Bose gas

- Hamiltonian of the system

$$\hat{H} = \frac{\hbar^2}{2m}\int d\mathbf{r}\,\nabla\hat{\Psi}^\dagger(\mathbf{r},t)\,\nabla\hat{\Psi}(\mathbf{r},t) + \int d\mathbf{r}\,\hat{\Psi}^\dagger(\mathbf{r},t)\,U(\mathbf{r},t)\,\hat{\Psi}^\dagger(\mathbf{r}',t)$$
$$+\frac{1}{2}\int d\mathbf{r}\,d\mathbf{r}'\,\hat{\Psi}^\dagger(\mathbf{r},t)\hat{\Psi}^\dagger(\mathbf{r}',t)\,U_{\mathrm{int}}(\mathbf{r}-\mathbf{r}')\,\hat{\Psi}(\mathbf{r}',t)\hat{\Psi}(\mathbf{r},t)$$

- Dynamics of the system is governed by the Heisenberg equation

$$i\hbar\,\frac{\partial}{\partial t}\hat{\Psi}(\mathbf{r},t) = \left[\hat{\Psi}(\mathbf{r},t),\hat{H}\right]$$

- Dipolar Gross-Pitaevskii equation

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \Big[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r},t) + g|\psi(\mathbf{r},t)|^2$$
$$+\int d\mathbf{r}'\,\psi^*(\mathbf{r}',t)U_{\mathrm{dd}}(\mathbf{r}-\mathbf{r}',t)\psi(\mathbf{r}',t)\Big]\psi(\mathbf{r},t)$$

- Dimensionless dipolar GPE reads

$$i\,\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \Big[-\frac{1}{2}\nabla^2 + \frac{1}{2}\left(\gamma^2x^2+\nu^2y^2+\lambda^2z^2\right) + 4\pi N\frac{a_s}{l}\,|\psi(\mathbf{r},t)|^2$$
$$+3N\frac{a_{\mathrm{dd}}}{l}\int d\mathbf{r}'\,\frac{1-3\cos^2\theta}{|\mathbf{r}-\mathbf{r}'|^3}\left|\psi(\mathbf{r}',t)\right|^2\Big]\psi(\mathbf{r},t)$$

Variational description of dipolar Bose gas

- Dipolar GPE can be written as the Euler-Lagrange equation for the following Lagrangian density

$$\mathcal{L}(\psi,\psi^*) = \frac{i}{2}\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right) + \frac{1}{2}\psi^*\nabla^2\psi - U\,|\psi|^2 - 2\pi N a_s\,|\psi|^4 - \frac{3Na_{\mathrm{dd}}}{2}\,|\psi|^2\int d\mathbf{r}'\,\frac{1-3\cos^2\theta}{|\mathbf{r}-\mathbf{r}'|^3}\,|\psi(\mathbf{r}')|^2$$

- For the variational study, we use the Gaussian ansatz

$$\psi(x,y,z,t) = A\,e^{-\frac{x^2}{2u_x^2}-\frac{y^2}{2u_y^2}-\frac{z^2}{2u_z^2}+ix^2\phi_x+iy^2\phi_y+iz^2\phi_z}\,[1+(\alpha+i\beta)\cos kx]$$

- By integration we calculate Lagrangian of the system

$$L(t) = L_1(t) + L_2(t) + L_3(t) + L_4(t) + L_5(t)$$

D. Vudragović et al., Symmetry **11**, 1090 (2019).
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Variational description of dipolar Bose gas

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- By integration we calculate Lagrangian of the system

$$L(t) = L_1(t) + L_2(t) + L_3(t) + L_4(t) + L_5(t)$$

$$L_1(t) = \frac{i}{2}\int d\mathbf{r}\,\left(\psi^*\dot{\psi} - \psi\dot{\psi}^*\right) = -\frac{1}{2}\left(u_x^2\dot{\phi}_x + u_y^2\dot{\phi}_y + u_z^2\dot{\phi}_z\right) - \frac{\alpha\dot{\beta}-\beta\dot{\alpha}}{2+\alpha^2+\beta^2}$$

$$L_2(t) = -\frac{1}{4}\left(\frac{1}{u_x^2}+\frac{1}{u_y^2}+\frac{1}{u_z^2}+4u_x^2\phi_x^2+4u_y^2\phi_y^2+4u_z^2\phi_z^2\right) - \frac{(\alpha^2+\beta^2)\,k^2}{2(2+\alpha^2+\beta^2)}$$

Variational description of dipolar Bose gas

- Dipolar GPE can be written as the Euler-Lagrange equation for the following Lagrangian density

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- For the variational study, we use the Gaussian ansatz

$$\psi(x,y,z,t) = A\,e^{-\frac{x^2}{2u_x^2}-\frac{y^2}{2u_y^2}-\frac{z^2}{2u_z^2}+ix^2\phi_x+iy^2\phi_y+iz^2\phi_z}\,[1+(\alpha+i\beta)\cos kx]$$

- By integration we calculate Lagrangian of the system

$$L(t) = L_1(t) + L_2(t) + L_3(t) + L_4(t) + L_5(t)$$

$$L_3(t) = -\int d\mathbf{r}\,U\,|\psi|^2 = -\frac{1}{4}\left(\gamma^2 u_x^2 + \nu^2 u_y^2 + \lambda^2 u_z^2\right)$$

$$L_4(t) = -2\pi N a_s \int d\mathbf{r}\,|\psi|^4 = -\frac{Na_s}{\sqrt{2\pi}\,u_x u_y u_z}\left(1+\frac{\alpha^4+16\alpha^2+2\alpha^2\beta^2+\beta^4)}{2(2+\alpha^2+\beta^2)^2}\right)$$

Lagrangian DDI term calculation

- Lagrangian DDI term
$$L_5(t) = -\frac{3Na_{\rm dd}}{2}\int d\mathbf{r}\,|\psi(\mathbf{r})|^2\int d\mathbf{r}'\,U_{\rm dd}(\mathbf{r}-\mathbf{r}')\,|\psi(\mathbf{r}')|^2$$
- The $\mathbf{r}'$ integral can be calculated using the convolution theorem
$$L_5(t) = -\frac{3Na_{\rm dd}}{2}\int d\mathbf{r}\,|\psi(\mathbf{r})|^2\,\mathcal{F}^{-1}\left\{\mathcal{F}\left[U_{\rm dd}\right](\mathbf{k})\,\mathcal{F}\left[|\psi|^2\right](\mathbf{k})\right\}(\mathbf{r})$$
- If we explicitly write the inverse Fourier transform
$$L_5(t) = -\frac{3Na_{\rm dd}}{2\,(2\pi)^3}\int d\mathbf{k}\,\mathcal{F}\left[U_{\rm dd}\right](\mathbf{k})\,\mathcal{F}\left[|\psi|^2\right](\mathbf{k})\int d\mathbf{r}\,|\psi(\mathbf{r})|^2\,e^{i\,\mathbf{k}\cdot\mathbf{r}}$$
$$= -\frac{3Na_{\rm dd}}{(2\pi)^2}\int d\mathbf{k}\left(3\,\frac{k_z^2}{k_z^2+k_y^2+k_z^2}-1\right)B^2\,e^{-\frac{1}{2}(k_x^2u_x^2+k_y^2u_y^2+k_z^2u_z^2)}$$
- Can be expressed in terms of the anisotropy function
$$f(x,y) = -\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\sin\theta\left(\frac{3\,x^2y^2\cos^2\theta}{(x^2\sin^2\varphi+y^2\cos^2\varphi)\sin^2\theta+x^2y^2\cos^2\theta}-1\right)$$
$$L_5(t) = \frac{Na_{\rm dd}}{\sqrt{2\pi}\,u_xu_yu_z}\,f\left(\frac{u_x}{u_z},\frac{u_y}{u_z}\right)\left(1-\frac{8\alpha^2}{(2+\alpha^2+\beta^2)^2}\right)$$

Equations of motion

- Euler-Lagrange equations for the variational parameters

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0\,, \quad q_i \in \{u_x, u_y, u_z, \phi_x, \phi_y, \phi_z, \alpha, \beta\}$$

$$\ddot{u}_x + \gamma^2 u_x - \frac{1}{u_x^3} - \sqrt{\frac{2}{\pi}}\frac{N}{u_x^2 u_y u_z}\left[a_s - a_{\mathrm{dd}} f\left(\frac{u_x}{u_z}, \frac{u_y}{u_z}\right) + a_{\mathrm{dd}}\frac{u_x}{u_z} f_1\left(\frac{u_x}{u_z}, \frac{u_y}{u_z}\right)\right] = 0$$

$$\ddot{u}_y + \nu^2 u_y - \frac{1}{u_y^3} - \sqrt{\frac{2}{\pi}}\frac{N}{u_x u_y^2 u_z}\left[a_s - a_{\mathrm{dd}} f\left(\frac{u_x}{u_z}, \frac{u_y}{u_z}\right) + a_{\mathrm{dd}}\frac{u_y}{u_z} f_2\left(\frac{u_x}{u_z}, \frac{u_y}{u_z}\right)\right] = 0$$

$$\begin{aligned}\ddot{u}_z + \lambda^2 u_z - \frac{1}{u_z^3} - \sqrt{\frac{2}{\pi}}\frac{N}{u_x u_y u_z^2}\Bigg[a_s - a_{\mathrm{dd}} f\left(\frac{u_x}{u_z}, \frac{u_y}{u_z}\right) &- a_{\mathrm{dd}}\frac{u_x}{u_z} f_1\left(\frac{u_x}{u_z}, \frac{u_y}{u_z}\right)\\ &- a_{\mathrm{dd}}\frac{u_y}{u_z} f_2\left(\frac{u_x}{u_z}, \frac{u_y}{u_z}\right)\Bigg] = 0\end{aligned}$$

$$\ddot{\alpha} + \left\{\frac{k^4}{4} + \sqrt{\frac{2}{\pi}}\frac{N}{u_x u_y u_z}\left[a_s + a_{\mathrm{dd}}\, f\left(\frac{u_x}{u_z}, \frac{u_y}{u_z}\right)\right] k^2\right\}\alpha = 0$$

Variational study of Faraday and resonant waves

- The radial component of the trap is harmonically modulated

$$\omega_y(t) = \omega_z(t) = \Omega_0(1 + \epsilon \sin \omega_m t)\,, \quad \lambda(t) = \nu(t) = \lambda_0(1 + \epsilon \sin \eta_m t)$$

- Usually cast into the form of the Mathieu-like equation

$$\ddot{\alpha} + \left[a(k) + \epsilon b(k) \sin 2\tau\right] \alpha = 0$$

- Equation of motion for the wave amplitude

$$\ddot{\alpha} + \left\{ \frac{k^4}{4} + \sqrt{\frac{2}{\pi}} \frac{N}{u_x u_y u_z} \left[a_s + a_{\rm dd}\, f\left(\frac{u_x}{u_z}, \frac{u_y}{u_z} \right) \right] k^2 \right\} \alpha = 0$$

- Mathematical form of the instability condition appears for $a(k) = n^2 \Leftrightarrow \omega = n \cdot \omega_m/2,\, n \in \mathbb{N}$
- We assume that the condensate width u_x, u_y and u_z slowly varies with respect to time
- For simplicity, we assume $u_y \approx u_z \equiv u_\rho$

Variational study of Faraday and resonant waves

- Equation for the variational parameter α can be written as

$$\ddot{\alpha} + \left[\frac{k^4}{4} + \frac{\Lambda k^2}{4}\lambda_0(1+\epsilon\sin\omega_m t)\right]\alpha = 0$$

$$\Lambda = \frac{4\sqrt{\frac{2}{\pi}}\,N\left[a_s - \frac{a_{\rm dd}}{2}\,f_s\left(\frac{u_\rho}{u_x}\right)\right]}{u_x\left\{1+\sqrt{\frac{2}{\pi}}\frac{N}{u_x}\left[a_s + \frac{a_{\rm dd}}{2}f_s\left(\frac{u_\rho}{u_x}\right) - a_{\rm dd}f_s'\left(\frac{u_\rho}{u_x}\right)\right]\right\}^{1/2}}$$

- After variable change $\omega_m t \to 2\tau$

$$a(k) = \frac{k^4}{\omega_m^2} + \frac{\lambda_0\Lambda k^2}{\omega_m^2}\,, \qquad b(k) = \frac{\lambda_0\Lambda k^2}{\omega_m^2}$$

- Instability condition for the Faraday waves $a(k) = 1^2$, $\omega = \omega_m/2$

$$k_F = \sqrt{-\frac{\lambda_0\Lambda}{2} + \sqrt{\frac{\lambda_0^2\Lambda^2}{4} + \omega_m^2}}$$

- Emergence of resonant waves corresponds to $a(k) = 2^2$, $\omega = \omega_m$

$$k_R = \sqrt{-\frac{\lambda_0\Lambda}{2} + \sqrt{\frac{\lambda_0^2\Lambda^2}{4} + 4\omega_m^2}}$$

Faraday waves in ^{52}Cr BEC

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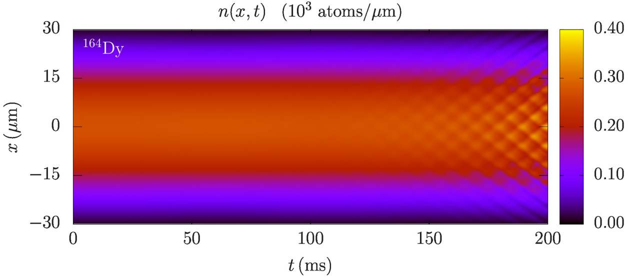
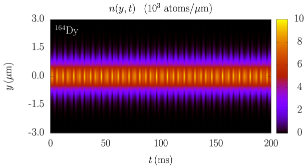
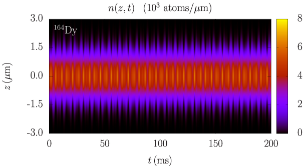
Time evolution of the integrated density

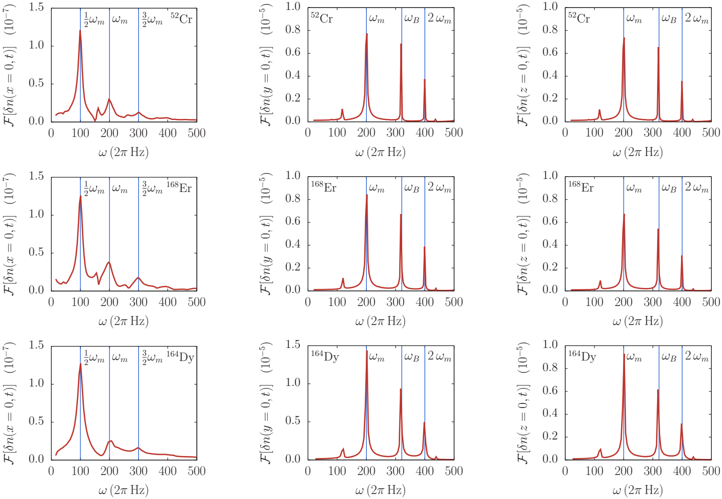






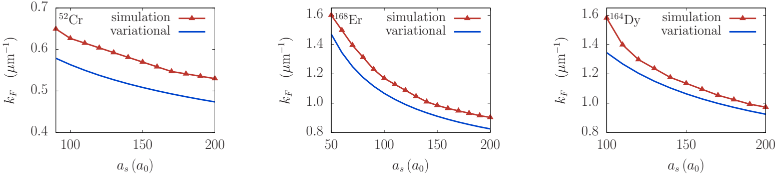
I. Vasić et al., arXiv:2603.11870 (2026).
D. Vudragović et al., Symmetry **11**, 1090 (2019).
D. Vudragović et al., Comput. Phys. Commun. **183**, 2021 (2012).

Fourier spectrum in the time-frequency domain

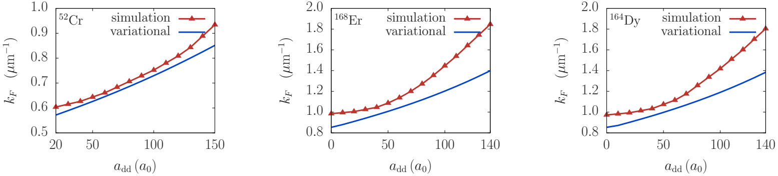


Interaction effects

- Wave vector of the Faraday waves as a function of a_s



- Wave vector of the Faraday waves as a function of a_{dd}



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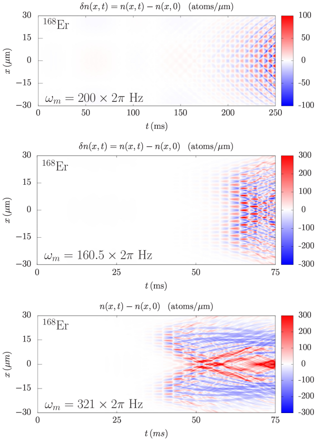
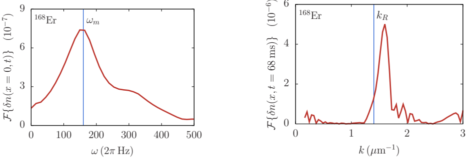
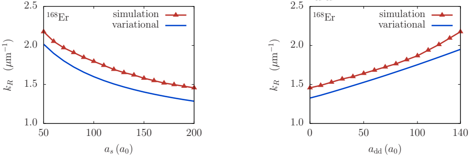
Resonant waves



- FFT in in the time-frequency and spatial-frequency domain



- Wave vector of the resonant waves as a function of a_s and $a_{\rm dd}$



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Conclusions

- We have developed a variational approach and used it for study of Faraday and resonant waves in dipolar BECs
- We have developed split-step semi-implicit Crank-Nicolson method used to solve numerically the dipolar GPE
- We have derived analytical expressions for the periods of Faraday/resonant waves
- Periods of Faraday/resonant waves were calculated numerically
- Comparison of variational and numerical results shows very good agreement

In memory of Antun Balaž

I would like to express my sincere gratitude to Antun Balaž for his support, inspiration, and friendship throughout the years. His guidance and influence will always remain an important part of both my scientific and personal life.