Completing Feynman's Path Integral Program: Lagrangian Path Integral Measure

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Basics

- Stationarity of action $I = \int L(q, \dot{q})\, dt$ leads to N second order Euler-Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

- Defining momenta as $p_i = \frac{\partial L}{\partial \dot{q}_i}$ allows us to implement the Legendre transformation $H = p_i \dot{q}_i - L$. By construction $H = H(q, p)$. The Hamiltonian generates $2N$ first order equations of motion

$$\dot{q}_i - \frac{\partial H}{\partial p_i} = 0$$

$$\dot{p}_i + \frac{\partial H}{\partial q_i} = 0$$

Hessian matrix

- The Hessian is defined as

$$W_{ij} = \frac{\partial p_i}{\partial \dot{q}_j} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$$

  If it is not invertible (i.e. $\det W_{ij} = 0$) all the velocities can't be expressed in terms of momenta – system contains gauge symmetries and constraints (Hamiltonian follows from the Dirac-Bergmann algorithm).
- In the Lagrange formalism if the Hessian is not invertible we can't solve for all accelerations.
- From now on we focus on systems with invertible Hessians. Legendre transform gives H from L and vice versa.

External fields

- In the classical formalism introduction of external fields is trivial. Taking $L \to L + J_i q_i$ the equations of motion becomes
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} - J_i = 0$$
- Similarly for Hamiltonians we take $H \to H - J_i q_i - K_i p_i$ and the corresponding equations of motion become

$$\dot{q}_i - \frac{\partial H}{\partial p_i} + K_i = 0$$
$$\dot{p}_i + \frac{\partial H}{\partial q_i} - J_i = 0$$

Path integrals

- The phase space path integral for the generating functionl with external fields J_i, K_i is

$$Z[J,K] = \int [dqdp] \exp \frac{i}{\hbar} \int (p_i \dot{q}_i - H + J_i q_i + K_i p_i)\, dt$$

- Unitarity of time evolution implies a trivial path integral measure in phase space, i.e. $[dqdp] = \prod_{t,i} dq_i(t) dp_i(t)$
- The Lagrangian path integral is calculated by doing (if we can) the integrals over all the momenta. In general we find

$$Z[J] = Z[J,0] = \int [dq\,\mu] \exp \frac{i}{\hbar} \int (L + J_i q_i)\, dt$$

where μ is a non-trivial coordinate space measure.

Identities

- We will determine this measure for a general quantum theory by paralleling the relation between L and H formalisms using Schwinger-Dyson equations.
- The SD equations follow from the following two identities

$$0 = \int [dqdp] \, \frac{\delta}{\delta p_i(t)} \, e^{\frac{i}{\hbar} \int (p_i \dot{q}_i - H + J_i q_i + K_i p_i) dt}$$

$$0 = \int [dqdp] \, \frac{\delta}{\delta q_i(t)} \, e^{\frac{i}{\hbar} \int (p_i \dot{q}_i - H + J_i q_i + K_i p_i) dt}$$

(assuming integrand falls off fast enough on the boundary).

Identities (continued)

- The first identity gives

$$0 = \int [dqdp] \left(\dot{q}_i - \frac{\partial H}{\partial p_i} + K_i \right) e^{\frac{i}{\hbar} \int (p_i \dot{q}_i - H + J_i q_i + K_i p_i) dt}$$
$$= \left(\dot{Q}_i - \frac{\partial H}{\partial P_i} + K_i \right) Z[J, K]$$

  where we have introduced differential operators $Q_i \equiv \frac{\hbar}{i} \frac{\partial}{\partial J}$ and $P_i \equiv \frac{\hbar}{i} \frac{\partial}{\partial K}$. Hence $[Q_i, J_j] = [P_i, K_j] = \frac{\hbar}{i} \delta_{ij}$, while all other commutators vanish.

- We proceed similarly for the second identity

Schwinger-Dyson equations

- The two SD equations in the Hamiltonian formalism are

$$\left(\dot{Q}_i - \frac{\partial H}{\partial P_i} + K_i\right) Z[J,K] = 0$$

$$\left(\dot{P}_i + \frac{\partial H}{\partial Q_i} - J_i\right) Z[J,K] = 0\ .$$

- These equations have the same form as the classical Hamilton equations of motion in the presence of external fields. The key difference is in the non-trivial commutators. Because of this we can't absorb the external field terms into the Hamiltonian as we could classicaly.
- We next use the above equations to derive the SD equation in the Lagrange formalism. Note that in quantum theory we go from the Hamiltonian (which naturally encodes unitarity) to the Lagrangian not the other way around.

Legendre Transform (kind of)

- Consider the quantity $\mathbb{L} \equiv P\dot{Q} - H(Q,P)$. $P's$ and $Q's$ all commute so just as in the classical case we find

$$\frac{\partial \mathbb{L}}{\partial Q_i} = -\frac{\partial H}{\partial Q_i}$$
$$\frac{\partial \mathbb{L}}{\partial \dot{Q}_i} = P_i$$
$$\frac{\partial \mathbb{L}}{\partial P_i} = \dot{Q}_i - \frac{\partial H}{\partial P_i}$$

- Unlike in the classical case the last equation is not zero, so we have $\mathbb{L} = \mathbb{L}(Q, \dot{Q}, P)$, i.e we have not gotten rid of momentums. Though useful, $\mathbb{L}$ is not the Lagrangian.

Towards the Lagrangian

- The Hamiltonian SD equations can be written in terms of $\mathbb{L}$ as

$$\left(\frac{\partial \mathbb{L}}{\partial P_i} + K_i\right) Z[J,K] = 0$$

$$\left(\frac{d}{dt}\frac{\partial \mathbb{L}}{\partial \dot{Q}_i} - \frac{\partial \mathbb{L}}{\partial Q_i} - J_i\right) Z[J,K] = 0$$

- The second equation above seems to be exactly what we need for the Lagrangian formalism, however, we still hove to get rid of the P dependence in $\mathbb{L}$. To do this we use the first equation to determine how P acts on $Z[J,K]$ and then to use that to get rid of the P dependence in the second equation.
- Before we tackle the general case we first look at a simple and illustrative example.

Illustrative Example

- $L = \frac{1}{2}G(q)\dot{q}^2 - V(q)$ implies $H = \frac{1}{2}G^{-1}(q)p^2 + V(q)$. The Hamiltonian SD equations for this model read

$$\left(\dot{Q} - G^{-1}(Q)P + K\right) Z[J,K] = 0$$

$$\left(\dot{P} - \frac{1}{2}G^{-2}(Q)G'(Q)P^2 + V'(Q) - J\right) Z[J,K] = 0$$

- The first equation gives $P\,Z[J,K] = G\,(\dot{Q}+K)Z[J,K]$. Differentiating this with respect to time gives

$$\dot{P}Z[J,K] = \left(G'\dot{Q}^2 + G'\dot{Q}K + G\ddot{Q} + G\dot{K}\right) Z[J,K]$$

On the other hand, applying P once more (and using the non-trivial $[P,K]$ commutator) we find

$$P^2 Z[J,K] = \left(G^2(\dot{Q}+K)^2 + \frac{\hbar}{i}G\right) Z[J,K]$$

Illustrative Example (Continued)

- We can now eliminate all P dependance from the second SD equation. Having done that we can set $K = 0$ to obtain the SD equation for $Z[J] = Z[J, 0]$ as

$$\left(G\ddot{Q} + \frac{1}{2} G'\dot{Q}^2 - \frac{1}{2}\frac{\hbar}{i} G^{-1}G' + V' - J \right) Z[J] = 0$$

- The classical action $I = \int L dt$ satisfies

$$\frac{\delta I}{\delta Q} = \frac{\partial L}{\partial Q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{Q}} = -\frac{1}{2} G'\dot{Q}^2 - G\ddot{Q} - V'$$

- In addition, we also have $G^{-1}G' = \frac{\delta}{\delta Q} \int dt \ln G$

Illustrative Example (Concluded)

- Putting everything together we can write the Lagrangian SD equation in its standard form as

$$\left(\frac{\delta S}{\delta Q} + J\right) Z[J] = 0$$

where $S = I + \frac{1}{2}\,\frac{\hbar}{i}\int \ln G\, dt$.

- The path integral expression for $Z[J]$ is is just the functional Fourier transform of the above SD equation

$$Z[J] = \int [dq] \exp\frac{i}{\hbar}\left(S + \int Jq\, dt\right) =$$

$$= \int [dq\,\mu] \exp\frac{i}{\hbar}\left(I + \int Jq\, dt\right)$$

where the measure is simply $\mu = \sqrt{G}$. In the case of many components q_i the measure is simply $(\det G_{ij})^{1/2}$.

How P acts on Z

- For simplicity in this section we present results for a general single component model.
- The first SD equation (used to get rid of P dependence) can be written as
$$\frac{\partial H}{\partial P}\, Z = x\, Z$$
Note that Z is short for $Z[J,K]$. We have also introduced $x = \dot{Q} + K$.
- Relevant commutators are $[P,x] = \frac{\hbar}{i}$, while for for general function $f(x)$ we have $[P, f(x)] = \frac{\hbar}{i} f'(x)$.
- The action of P on Z is $PZ = p(Q,x)Z$ and p is chosen so that this is consistent with the above SD equation. To do this we have to first look at how all powers of P act on Z.

How P^n acts on Z

- It. obviously follows that

$$P^2\, Z = Pp\, Z = \Big(pP + [P, p\,]\Big) Z = \left(p^2 + \frac{\hbar}{i}\, p'\right) Z$$

- In the same way we find

$$P^3\, Z = \left(p^3 + 3\,\frac{\hbar}{i}\, pp' + \left(\frac{\hbar}{i}\right)^2 p''\right) Z$$

- It is relatively easy to determine the action of P^n on Z. For example to $O(\hbar^2)$ we find

$$P^n\, Z = \left[p^n + \frac{\hbar}{i}\binom{n}{2} p^{n-2}\, p' + \left(\frac{\hbar}{i}\right)^2 \binom{n}{3}\, p^{n-3}\, p'' + \right.$$
$$\left. + 3\left(\frac{\hbar}{i}\right)^2 \binom{n}{4} p^{n-4}\, (p')^2\right] Z$$

How $f(P)$ acts on Z

- For a general function $f(P) = \sum_n f_n P^n$ (to order $O(\hbar)$) we find
$$f(P)\,Z = \left(f(p) + \frac{1}{2}\,\frac{\hbar}{i}\,p'\,\frac{\partial^2 f}{\partial p^2} \right) Z$$
In what follows we will only present results to this order.
- Set $f = \frac{\partial H}{\partial P}$. Consistency with the first SD equation now gives the equation that determines p to be

$$\frac{\partial H}{\partial p} + \frac{1}{2}\,\frac{\hbar}{i}\,p'\,\frac{\partial^3 H}{\partial p^3} = x$$

Solving for $p(Q, x)$

- To solve the above consistency equation we expand $p(Q, x) = p_0 + \frac{\hbar}{i}\, p_1$, substitute this in the above equation and equate like powers of $\hbar$.
- Doing this we find $\frac{\partial H}{\partial p_0} = x$, so p_0 is just the classical momentum.
- The first correction p_1 equals

$$p_1 = -\,\frac{1}{2}\, p_0'\, \frac{\frac{\partial^3 H}{\partial p_0^3}}{\frac{\partial^2 H}{\partial p_0^2}} = -\,\frac{1}{2}\, p_0'\, \frac{\partial}{\partial p_0} \ln\left(\frac{\partial^2 H}{\partial p_0^2}\right)$$

Towards Lagrangian SD equation

- Having solved for p we can now input this into the remaining SD equation

$$\left(\dot{P} + \frac{\partial H}{\partial Q} - J\right) Z = 0$$

- Differentiating the P equation we find

$$\dot{P}\, Z = \left[\frac{\partial p}{\partial Q}\dot{Q} + p'(\ddot{Q} + \dot{K})\right] Z$$

- Setting $f = \frac{\partial H}{\partial Q}$ we determine how $\frac{\partial H}{\partial Q}$ acts on Z. Having gotten rid all P's we can now also set $K = 0$ to obtain

$$\left[p'\ddot{Q} + \frac{\partial p}{\partial Q}\dot{Q} + \frac{\partial H(Q,p)}{\partial Q} + \frac{1}{2}\left(\frac{\hbar}{i}\right) p' \frac{\partial^3 H(Q,p)}{\partial Q \partial p^2} - J\right] Z[J] = 0$$

Towards Lagrangian SD equation (Continued)

- Finally the second SD equation becomes

$$\Big[p_0'\ddot{Q} + \frac{\partial p_0}{\partial Q}\hat{Q} + \frac{\partial H\,(Q_1 p_0)}{\partial Q} + \\ + \left(\frac{\hbar}{i}\right)\left(\frac{\partial p_1}{\partial Q}\dot{Q} + \frac{\partial^2 H}{\partial Q \partial p_0} p_1 + p_1'\ddot{Q} + \frac{1}{2}\,p_0'\,\frac{\partial^3 H}{\partial Q \partial p_0^2}\right) - J\Big] Z[J] = 0$$

- Note that this is the Lagrangian SD equation we seek. We just have to write all the expressions above in terms of L and not H.

Expressions in terms of L

- Note that $p_0 = \frac{\partial L}{\partial \dot{Q}}$ implies $\frac{\partial H}{\partial p_0} = \dot{Q}$. It follows that $p_0' = \frac{\partial p_0}{\partial \dot{Q}} = \frac{\partial^2 L}{\partial \dot{Q}^2}$
- To express higher derivatives of H in terms of L we use

$$\frac{\partial}{\partial p_0} = \left(\frac{\partial \dot{Q}}{\partial p_0}\right) \frac{\partial}{\partial \dot{Q}} = \left(\frac{\partial p_0}{\partial \dot{Q}}\right)^{-1} \frac{\partial}{\partial \dot{Q}} = \left(\frac{\partial^2 L}{\partial \dot{Q}^2}\right)^{-1} \frac{\partial}{\partial \dot{Q}}$$

- We can now easily turn all remaining quantities from the H to the L world. For example $\frac{\partial^2 H}{\partial p_0^2} = \left(\frac{\partial^2 L}{\partial \dot{Q}^2}\right)^{-1}$ and so

$$p_1 = -\frac{1}{2}\, p_0' \frac{\partial}{\partial p_0} \ln\left(\frac{\partial^2 H}{\partial p_0^2}\right) = \frac{1}{2} \frac{\partial}{\partial \dot{Q}} \ln\left(\frac{\partial^2 L}{\partial \dot{Q}^2}\right)$$

Canonical Form of Lagrangian SD equation

- We could now cast the remaining SD equation into the canonical form

$$\left(\frac{d}{dt}\frac{\partial L_{\text{eff}}}{\partial \dot{Q}} - \frac{\partial L_{\text{eff}}}{\partial Q} - J\right) Z[J] = 0$$

  determining the effective Lagrangian $L_{\text{eff}} = L + \left(\frac{\hbar}{i}\right) M$. The measure is just $\mu = \exp M$. M depends on $\hbar$ and is regular as $\hbar$ goes to zero, so $M = M_0 + \left(\frac{\hbar}{i}\right) M_1 + \ldots$

- Due to the overall factor of $\frac{\hbar}{i}$ in the expression relating L_{eff} and M, to obtain results to order $O(\hbar^n)$ we need to expanding M to $O(\hbar^{n-1})$.

Focus on the Measure

- Rather than continuing with casting the Lagrangian SD equation into its canonical form we will now focus directly on the M term
- Note that $p_0 = \frac{\partial L}{\partial \dot{Q}}$ and $p = \frac{\partial L_{\text{eff}}}{\partial \dot{Q}}$. Their difference is $p - p_0 = \frac{\partial}{\partial \dot{Q}}\,(L_{\text{eff}} - L)$, so that $M_0 = \ln \mu_0 = \frac{1}{2}\,\ln\left(\frac{\partial^2 L}{\partial \dot{Q}^2}\right)$. Therefore, to leading order, the measure is simply $\mu_0 = \left(\frac{\partial^2 L}{\partial \dot{Q}^2}\right)^{\frac{1}{2}}$
- For a general multi component non-singular theory (to leading order) we find

$$\mu_0 = (\det W_{ij})^{\frac{1}{2}}$$

- One can easily extend this procedure to higher orders.

Summary

- We consider general theories with an invertible Hessian matrix (second derivatives of the Lagrangian with respect to the velocities). For these theories it is possible to calculate velocities in terms of the momenta $p = \frac{\partial L}{\partial \dot{q}}$, to define the corresponding Hamiltonian as the Legendre transformation $H(q,p) = p\dot{q} - L(q,\dot{q})$ and to obtain the phase space equations of motion.
- We extend this to general non-singular quantum theories by looking at SD equations for the generating functional in L and H formalisms. In so doing we establish the relation between L and H and uniquely determine the path integral measure μ of the Lagrangian path integral. The role of the measure is to maintain unitarity of time evolution in the coordinate space formalism.