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# Courant bracket found out to be T-dual to Roytenberg bracket

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**Abstract** Bosonic string moving in coordinate dependent background fields is considered. We calculate the generalized currents Poisson bracket algebra and find that it gives rise to the Courant bracket, twisted by a 2-form $2B_{\mu\nu}$. Furthermore, we consider the T-dual generalized currents and obtain their Poisson bracket algebra. It gives rise to the Roytenberg bracket, equivalent to the Courant bracket twisted by a bi-vector $\Pi^{\mu\nu}$, in case of $\Pi^{\mu\nu} = 2^\star B^{\mu\nu} = \kappa\theta^{\mu\nu}$. We conclude that the twisted Courant and Roytenberg brackets are T-dual, when the quantities used for their deformations are mutually T-dual.

## 1 Introduction

Non-geometric backgrounds [1–3] include various dualities. Duality symmetry is a way to show the equivalence between two apparently different theories. Specifically, T-duality [4,5] is a symmetry between two theories corresponding to different geometries and topologies. It was firstly noticed as the spectrum equivalence of the bosonic closed string with one dimension compactified to a radius $R$, with the bosonic closed string with one dimension compactified to a radius $\alpha'/R$.

The Courant bracket [6,7] is the generalization of the Lie bracket so that it includes both vectors and 1-forms. It is a fundamental structure of the generalized complex geometry. Vectors and 1-forms are treated on equal footing in the generalized complex structures. Many for string theory relevant geometries, such as complex, symplectic and Kähler geometry, are integrated into the framework of generalized complex structures. Moreover, the generalized complex geometry provides a framework for a unified description of diffeomorphisms and gauge transformations of the Kalb–Ramond field. Hitchin was the first one to introduce the generalized Calabi–Yau manifolds, that unified the concept of a Calabi–Yau manifold with the one of a symplectic manifold [8]. Gualtieri in his PhD thesis contributed further to the mathematical development of generalized complex geometry [9].

In generalized complex geometry, closure under the Courant bracket represents the integrability condition, in a same way that closure under the Lie bracket represents the integrability condition of almost complex structures. Moreover, the Courant bracket governs the gauge transformation in the double field string theory [10].

The Roytenberg bracket is the generalization of the Courant bracket, so that it includes a bi-vector. It was firstly introduced by Roytenberg [11]. In [12], the $\sigma$-model with both 2-form and a bi-vector was considered. The Poisson bracket algebra of the generalized currents was obtained. It has been observed that, while the current algebra is anomalous, the algebra of charges is closed and gives rise to the Roytenberg bracket. In [13], the Roytenberg bracket was obtained by lifting the topological sector of the first order action for the NS string $\sigma$-model to three dimensions. In [14], the higher order Roytenberg bracket is realized, by twisting by a p-vector.

In this paper, we consider the closed bosonic string moving in the coordinate dependent background fields. Generalized currents are defined as linear combinations of worldsheet basis vectors with arbitrary coordinate dependent coefficients, and their Poisson bracket algebra is calculated. We follow the work of [15], that analyzed the most general currents of the general $\sigma$ model, where it has been shown that the algebra of most general currents gives rise to the Courant bracket, twisted by the Kalb–Ramond field. Moreover, we consider the self T-duality, that is to say T-duality realized in the same phase space. The self T-duality interchanges momenta with coordinate derivatives, as well as the background fields with their T-dual background fields.

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Another set of generalized currents, T-dual to the aforementioned ones, are constructed and their algebra obtained. We find that their algebra gives rise to the Roytenberg bracket obtained by twisting the Courant bracket by the T-dual of the Kalb–Ramond field. Hence, we show that the twisted Courant bracket is T-dual to the corresponding Roytenberg one, obtaining the relation that connects the mathematically relevant structures with the T-duality.

## 2 Hamiltonian of the bosonic string

Consider the closed bosonic string in the nontrivial background defined by the symmetric metric tensor field $G_{\mu\nu}$ and the Kalb-Ramond antisymmetric tensor field $B_{\mu\nu}$, as well as the constant dilaton field $\Phi = const$. In the conformal gauge, the propagation is described by the action [16,17]

$$S = \int_\Sigma d^2\xi \mathcal{L} = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\eta^{\alpha\beta}G_{\mu\nu}(x) + \epsilon^{\alpha\beta}B_{\mu\nu}(x)\right]\partial_\alpha x^\mu \partial_\beta x^\nu, \tag{2.1}$$

where integration goes over a two-dimensional world-sheet $\Sigma$ parametrized by $\xi^\alpha (\xi^0 = \tau, \xi^1 = \sigma)$ with the worldsheet metric $\eta^{\alpha\beta}$. Coordinates of the D-dimensional space-time are $x^\mu(\xi),\ \mu = 0, 1, \ldots, D-1, \epsilon^{01} = -1$ and $\kappa = \frac{1}{2\pi\alpha'}$.

It is convenient to rewrite the action (2.1) using the light-cone coordinates $\xi^\pm = \xi^0 \pm \xi^1$ and derivatives $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$ as

$$S = \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu}(x)\partial_- x^\nu, \tag{2.2}$$

where

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2}G_{\mu\nu}(x). \tag{2.3}$$

The canonical momenta are given by

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = \kappa G_{\mu\nu}(x)\dot{x}^\nu - 2\kappa B_{\mu\nu}(x)x'^\nu. \tag{2.4}$$

The Hamiltonian is obtained in a usual way,

$$\mathcal{H}_C = \pi_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu - 2x'^\mu B_{\mu\nu}(G^{-1})^{\nu\rho}\pi_\rho + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu, \tag{2.5}$$

where

$$G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu} \tag{2.6}$$

is the effective metric.

Energy-momentum tensor components can be written as

$$T_\pm = \mp\frac{1}{4\kappa}(G^{-1})^{\mu\nu} j_{\pm\mu} j_{\pm\nu}, \tag{2.7}$$

where the currents $j_{\pm\mu}$ are given by

$$j_{\pm\mu}(x) = \pi_\mu + 2\kappa\Pi_{\pm\mu\nu}(x)x'^\nu. \tag{2.8}$$

In terms of the energy-momentum tensor components (2.7), the Hamiltonian is given by

$$\mathcal{H}_C = T_- - T_+ = \frac{1}{4\kappa}(G^{-1})^{\mu\nu}\left[j_{+\mu}j_{+\nu} + j_{-\mu}j_{-\nu}\right]. \tag{2.9}$$

In this paper, we are interested in these currents, currents T-dual to them, their generalizations, as well as their Poisson bracket algebra. Before that, let us present a short overview of T-duality.

### 2.1 Lagrangian approach to T-duality

In the Lagrangian approach to T-duality, the Buscher procedure of T-dualization has been developed [18–21]. It provides us with the procedure of transforming coordinates from one theory to the coordinates from its T-dual theory, when there is a global Abelian isometry of coordinates along which T-dualization is applied. The T-dualization rules for coordinates are given by [22,23]

$$\partial_\pm x^\mu \cong -\kappa\Theta^{\mu\nu}_\pm \partial_\pm y_\nu, \quad \partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu, \tag{2.10}$$

where we have introduced the T-dual coordinate $y_\mu$ and new fields $\Theta^{\mu\nu}_\pm$, defined by

$$\Theta^{\mu\nu}_\pm = -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}, \tag{2.11}$$

where $\theta^{\mu\nu}$ is the non-commutativity parameter, that first appeared in the context of open string coordinates non-commutativity in the presence of non-zero Kalb Ramond field [24], given by

$$\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}, \tag{2.12}$$

where $(G_E^{-1})^{\mu\nu}$ is the inverse of the effective metric defined in (2.6). It is straightforward to verify that $\Theta^{\mu\nu}_\pm$ fields are inverse to $\Pi_{\mp\mu\nu}$ fields

$$\Theta^{\mu\rho}_\pm \Pi_{\mp\rho\nu} = \frac{1}{2\kappa}\delta^\mu_\nu. \tag{2.13}$$

Successive application of the T-dualization (2.10) is involutive

$$\partial_\pm x^\mu \cong -\kappa \Theta_\pm^{\mu\nu} \partial_\pm y_\nu \cong 2\kappa \Theta_\pm^{\mu\nu} \Pi_{\mp\nu\rho} \partial_\pm x^\rho = \partial_\pm x^\mu, \tag{2.14}$$

where in the last step we have used (2.13).

Applying the T-dualization laws (2.10) to the action (2.2), we obtain the T-dual action

$$^\star S = \int d^2\xi \,^\star\mathcal{L} = \frac{\kappa^2}{2} \int d^2\xi \,\partial_+ y_\mu \Theta_-^{\mu\nu} \partial_- y_\nu. \tag{2.15}$$

Expressing the action (2.15) in the form of the initial action (2.2), we obtain

$$^\star\Pi_+^{\mu\nu} = \frac{\kappa}{2} \Theta_-^{\mu\nu}, \tag{2.16}$$

which allows us to read the T-dual background fields

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad ^\star B^{\mu\nu} = \frac{\kappa}{2} \theta^{\mu\nu}. \tag{2.17}$$

These relations correspond exactly to the expressions for T-dual background fields obtained by Buscher [18–21] in case of the existence of Abelian group of isometries along coordinates along which we perform T-duality.

## 2.2 Hamiltonian formulation of T-duality

Let us rewrite the T-dualization laws (2.10) in terms of phase space variables. Firstly, we need the expression for the T-dual canonical momentum. It is given by

$$^\star\pi^\mu = \frac{\partial \,^\star\mathcal{L}}{\partial \dot{y}_\mu} = \kappa (G_E^{-1})^{\mu\nu} \dot{y}_\nu - \kappa^2 \theta^{\mu\nu} y'_\nu. \tag{2.18}$$

Secondly, let us rewrite equations (2.10), separating the part that changes the sign from the part that does not. For the coordinates of the initial theory, we obtain

$$\dot{x}^\mu \cong -\kappa\theta^{\mu\nu} \dot{y}_\nu + (G_E^{-1})^{\mu\nu} y'_\nu, \quad x'^\mu \cong (G_E^{-1})^{\mu\nu} \dot{y}_\nu - \kappa\theta^{\mu\nu} y'_\nu, \tag{2.19}$$

and for the coordinates of the T-dual theory, we obtain

$$\dot{y}_\mu \cong -2B_{\mu\nu} \dot{x}^\nu + G_{\mu\nu} x'^\nu, \quad y'_\mu \cong G_{\mu\nu} \dot{x}^\nu - 2B_{\mu\nu} x'^\nu. \tag{2.20}$$

Comparing the second relation of (2.19) with (2.4), as well as the second relation of (2.20) with (2.18), we obtain the T-dualization laws (2.10) formulated in terms of the phase space variables

$$\kappa x'^\mu \cong \,^\star\pi^\mu, \quad \pi_\mu \cong \kappa y'_\mu. \tag{2.21}$$

When coordinate $\sigma$-derivatives and canonical momenta are integrated over the worldsheet space parameter $\sigma$, the winding numbers and momenta are respectively obtained [25]. Hence, we see that the T-dualization transforms the momenta of the initial theory into the winding numbers in its T-dual theory, and vice versa.

The T-duality can be considered as the canonical transformation generated by the type I functional [26,27]

$$F = \kappa \int d\sigma x^\mu y'_\mu, \tag{2.22}$$

which gives rise to momenta

$$\pi_\mu = \frac{\delta F}{\delta x^\mu} = \kappa y'_\mu, \quad ^\star\pi^\mu = \frac{-\delta F}{\delta y_\mu} = \kappa x'^\mu. \tag{2.23}$$

This is exactly the relation (2.21). The T-duality does not change the Hamiltonian, since the generating function (2.22) does not depend explicitly on time $\mathcal{H}_C \to \mathcal{H}_C + \frac{\partial F}{\partial t} = \mathcal{H}_C$.

In order to obtain the T-dual Hamiltonian, we apply relations (2.21)–(2.5), and obtain

$$^\star\mathcal{H}_C = \frac{1}{2\kappa} \,^\star\pi^\mu \, G^E_{\mu\nu} \,^\star\pi^\nu - 2 \,^\star\pi^\mu (BG^{-1})_\mu^{\ \nu} y'_\nu + \frac{\kappa}{2} y'_\mu (G^{-1})^{\mu\nu} y'_\nu. \tag{2.24}$$

Expressing the T-dual Hamiltonian in the form of the initial one (2.5), as

$$^\star\mathcal{H}_C = \frac{1}{2\kappa} \,^\star\pi^\mu \,^\star G^{-1}_{\mu\nu} \,^\star\pi^\nu - 2y'_\mu (^\star B \,^\star G^{-1})^\mu_{\ \nu} \,^\star\pi^\nu + \frac{\kappa}{2} y'_\mu \,^\star G_E^{\mu\nu} y'_\nu. \tag{2.25}$$

We are able to read once again the expressions for the T-dual background fields (2.17).

Given that we were able to write the Hamiltonian in terms of currents $j_{\pm\mu}$, we would like to write the T-dual Hamiltonian (2.25) in terms of T-dual currents. By analogy with the initial theory (2.7), we write the T-dual energy momentum tensor components as

$$^\star T_\pm = \mp \frac{1}{4\kappa} \,^\star G^{-1}_{\mu\nu} \,^\star j_\pm^\mu \,^\star j_\pm^\nu, \tag{2.26}$$

where $^\star j_\pm^\mu$ are T-dual currents, given by

$$^\star j_\pm^\mu = \,^\star\pi^\mu + 2\kappa \,^\star\Pi_\pm^{\mu\nu} y'_\nu. \tag{2.27}$$

The T-dual Hamiltonian is then given by

$$^\star\mathcal{H}_C = \,^\star T_- - \,^\star T_+ = \frac{1}{4\kappa} \,^\star G^{-1}_{\mu\nu} \left( ^\star j_+^\mu \,^\star j_+^\nu + \,^\star j_-^\mu \,^\star j_-^\nu \right), \tag{2.28}$$

We can check that substituting (2.27) into (2.28), the T-dual Hamiltonian in the form (2.25) is obtained. Therefore,

$$\mathcal{H}_C \cong {}^\star\mathcal{H}_C, \quad T_\pm \cong {}^\star T_\pm. \tag{2.29}$$

## 2.3 T-dual currents

Let us consider the transformation of the currents under T-duality. Applying (2.21)–(2.8), we obtain

$$j_{\pm\mu} \cong \kappa y'_\mu + 2\Pi_{\pm\mu\nu}{}^\star\pi^\nu = 2\Pi_{\pm\mu\nu}{}^\star j_\pm^\nu, \tag{2.30}$$

where we have used (2.13). Similarly, the T-dualization applied on the T-dual currents is as easily obtained

$${}^\star j_\pm^\mu \cong \kappa x'^\mu + \kappa\Theta_\mp^{\mu\nu}\pi_\nu = \kappa\Theta_\mp^{\mu\nu} j_{\pm\nu}. \tag{2.31}$$

The successive application of T-dualization on any current returns exactly that current.

Although the initial and T-dual theories are equivalent (2.29), the currents $j_{\pm\mu}$ and ${}^\star j_\pm^\mu$ do not transform exactly one into another by the T-dualization laws (2.21). There are couple of ways to see the nature of this fact. Firstly, the current $j_{\pm\mu}$ has the lower indices, while ${}^\star j_\pm^\mu$ has the upper indices.

Secondly, substituting (2.30) into (2.7), we obtain the T-dual transformation of the energy momentum tensor

$$\begin{aligned} T_\pm &\cong \pm\frac{1}{\kappa}{}^\star j_\pm^\mu(\Pi_\mp G^{-1}\Pi_\pm)_{\mu\nu}{}^\star j_\pm^\nu \\ &= \mp\frac{1}{4\kappa}{}^\star j_\pm^\mu G^E_{\mu\nu}{}^\star j_\pm^\nu = {}^\star T_\pm, \end{aligned} \tag{2.32}$$

where in the second step we used (2.3) and (2.6). The direct transformation of currents under T-duality $j_{\pm\mu} \cong {}^\star j_\pm^\mu$ would violate invariance of the energy momentum tensor. The effective metric $G^E_{\mu\nu}$ in the expression for T-dual energy momentum tensor is obtained from $-\Pi_\mp G^{-1}\Pi_\pm = \frac{1}{4}G_E$, which is only possible due to the non-trivial T-duality relation between currents (2.30).

Lastly, let us rewrite the expressions for currents in terms of coordinates, by substituting (2.4) into (2.8) and (2.18) into (2.27)

$$j_{\pm\mu} = \kappa G_{\mu\nu}\partial_\pm x^\nu, \quad {}^\star j_\pm^\mu = \kappa(G_E^{-1})^{\mu\nu}\partial_\pm y_\nu. \tag{2.33}$$

Hence, in the same way that coordinates $\partial_\pm x^\mu$ do not transform into T-dual coordinates $\partial_\pm y_\mu$ under (2.10), in the same way the currents $j_{\pm\mu}$ do not transform into T-dual currents ${}^\star j_\pm^\mu$. The transformation of variables under T-duality (2.21) is presented in the Table 1.

Lastly, let us define for future convenience the right hand side of (2.31), as a new current $l_\pm^\mu$

$$l_\pm^\mu = \kappa\Theta_\mp^{\mu\nu} j_{\pm\nu} = \kappa x'^\mu + \kappa\Theta_\mp^{\mu\nu}\pi_\nu. \tag{2.34}$$

**Table 1** Transformations under the T-dualization

| Initial theory | | T-dual theory |
|---|---|---|
| $\pi_\mu$ | $\cong$ | $\kappa y'_\mu$ |
| $\kappa x'^\mu$ | $\cong$ | ${}^\star\pi^\mu$ |
| $j_{\pm\mu}$ | $\cong$ | $2\Pi_{\pm\mu\nu}{}^\star j_\pm^\nu$ |
| $\kappa\Theta_\mp^{\mu\nu} j_{\pm\nu}$ | $\cong$ | ${}^\star j_\pm^\mu$ |

In the next chapter, we will see how we can avoid working in two phase spaces, and the currents $l_\pm^\mu$ will have an important role throughout the rest of the paper.

## 2.4 Self T-duality

So far we considered the case when two mutually T-dual theories are defined in two different phase spaces, that we have marked by $\{x^\mu, \pi_\mu\}$, and $\{y_\mu, {}^\star\pi^\mu\}$. It is in fact possible to realize T-duality in the same phase space, that we will call self T-duality.

To realize self T-duality, let us rewrite the second relation of (2.19), using (2.17)

$$\kappa x'^\mu \cong \kappa\, {}^\star G^{\mu\nu}\dot{y}_\nu - 2\kappa\, {}^\star B^{\mu\nu} y'_\nu. \tag{2.35}$$

Comparing it with the expression for momenta (2.4), we conclude that the exchange of coordinate with its T-dual $x^\mu \leftrightarrow y_\mu$ is equivalent to

$$\begin{aligned} &\pi_\mu \leftrightarrow \kappa x'^\mu, \quad B_{\mu\nu} \leftrightarrow {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}, \\ &G_{\mu\nu} \leftrightarrow {}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}. \end{aligned} \tag{2.36}$$

These are transformation rules for what we call self T-duality. Note that unlike in (2.21), here the background fields are transformed, as well.

The self T-duality gives the same expressions for T-dual background fields (2.17) as in case of Buscher procedure. It swaps the winding numbers with momenta as well, therefore preserving all features of T-duality, with the only difference being that it is realized in the same phase space.

The two currents $j_{\pm\mu}$ and $l_\pm^\mu$ transform into each other under the self T-duality (2.36)

$$j_{\pm\mu} = \pi_\mu + 2\kappa\Pi_{\pm\mu\nu}x'^\nu \leftrightarrow \kappa x'^\mu + \kappa\Theta_\mp^{\mu\nu}\pi_\nu = l_\pm^\mu. \tag{2.37}$$

On the other hand, under (2.36) the energy-momentum tensor is invariant

$$T_\pm = \mp\frac{1}{4\kappa}(G^{-1})^{\mu\nu} j_{\pm\mu} j_{\pm\nu} \leftrightarrow \mp\frac{1}{4\kappa}G^E_{\mu\nu} l_\pm^\mu l_\pm^\nu = T_\pm. \tag{2.38}$$

With the help of (2.9), we see that the Hamiltonian does not change under (2.36). Nevertheless, the Hamiltonian can be expressed in terms of new currents $l_\pm^\mu$

**Table 2** Transformations under the self T-duality

| Initial theory | | Self T-dual theory |
|---|---|---|
| $\pi_\mu$ | $\leftrightarrow$ | $\kappa x'_\mu$ |
| $\kappa x'^\mu$ | $\leftrightarrow$ | $\pi_\mu$ |
| $B_{\mu\nu}$ | $\leftrightarrow$ | $\frac{\kappa}{2}\theta^{\mu\nu}$ |
| $G_{\mu\nu}$ | $\leftrightarrow$ | $(G_E^{-1})^{\mu\nu}$ |
| $j_{\pm\mu}$ | $\leftrightarrow$ | $l^\mu_\pm$ |

$$\mathcal{H}_C = \frac{1}{4\kappa} G^E_{\mu\nu}\left(l^\mu_+ l^\nu_+ + l^\mu_- l^\nu_-\right), \tag{2.39}$$

but with the effective metric instead of the inverse metric. Substituting (2.6) and (2.34) into the previous equation, we obtain the initial form of the Hamiltonian (2.9).

It is important to point out that although the energy-momentum tensor components $T_\pm$ and the Hamiltonian $\mathcal{H}_C$ remain invariant under the self T-duality, the currents $j_{\pm\mu}$ and $l^\mu_\pm$ do not. Therefore although both currents $j_{\pm\mu}$ and $l^\mu_\pm$ are defined in terms of the initial theory variables, they have to change under self T-duality, due to the invariance of energy momentum tensor components (2.38). We summarize its transformation rules in the Table 2. Our next goal is to generalize these two currents and obtain the algebra of their generalizations.

## 3 Generalized currents in a new basis

In this chapter, we will construct two types of generalized currents. Generalized currents are arbitrary functionals of the fields, parametrized by a pair of vector field and covector field on the target space, treating both vectors and 1-forms on equal footing [9]. The convenient bases in which these generalized currents are defined are components of currents $j_{\pm\mu}$ and $l^\mu_\pm$.

Firstly, we will generalize the currents $j_{\pm\mu}$. From (2.8) we extract its $\tau$ and $\sigma$ components

$$j_{0\mu} = \frac{j_{+\mu} + j_{-\mu}}{2} = \pi_\mu + 2\kappa B_{\mu\nu}(x)x'^\nu,$$
$$j_{1\mu} = \frac{j_{+\mu} - j_{-\mu}}{2} = \kappa G_{\mu\nu}(x)x'^\nu. \tag{3.1}$$

We will mark

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}(x)x'^\nu, \tag{3.2}$$

as a new, auxiliary current. Therefore, $\{\kappa x'^\mu, i_\mu\}$ is a new convenient basis on the world-sheet. We can now write currents (2.8) in this basis as

$$j_{\pm\mu} = i_\mu \pm \kappa G_{\mu\nu}x'^\nu. \tag{3.3}$$

In the same way as in [15], we define the generalized currents in the new basis, as the linear combination of both coordinate $\sigma-$derivatives and auxiliary currents

$$J_{C(u,a)} = u^\mu(x)i_\mu + a_\mu(x)\kappa x'^\mu, \tag{3.4}$$

where $u^\mu(x)$ and $a_\mu(x)$ are the arbitrary coefficients. The charges of these currents are

$$Q_{C(u,a)} = \int d\sigma\, J_{C(u,a)}. \tag{3.5}$$

The charges exhibit additional symmetry. In order to see that, let us firstly rewrite the integral of the total derivative of an arbitrary function $\lambda$

$$\int_0^{2\pi} d\sigma\,(\lambda)' = \int_0^{2\pi} d\sigma\,\partial_\mu\lambda x'^\mu = 0, \tag{3.6}$$

which goes to zero for closed strings. From this fact, we obtain the reducibility relations for the charges

$$Q_{C(u,a+\partial\lambda)} = Q_{C(u,a)}. \tag{3.7}$$

The expression of the form (3.4) is particularly interesting, since it gives rise to many for string theory relevant structures. Firstly, for the special case of coefficients relation $a_\mu = \pm G_{\mu\nu}u^\nu$, we obtain

$$J_{C(u,\pm Gu)} = u^\mu j_{\pm\mu}. \tag{3.8}$$

Hence, the currents (2.8) indeed can be obtained from the generalized currents (3.4). On the other hand, for special case $a_\mu = -2B_{\mu\nu}u^\nu$, we obtain

$$J_{C(u,-2Bu)} = u^\mu\pi_\mu, \tag{3.9}$$

as well as for $u^\mu = 0$, we obtain

$$J_{C(0,a)} = a_\mu\kappa x'^\nu. \tag{3.10}$$

We see that the general current algebra for the appropriate coefficients reduces to non-commutativity relations of both coordinates and momenta.

We are also interested in another type of generalized current, that in analogous way generalizes $l^\mu_\pm$, in the basis related to its $\tau$ and $\sigma$ components

$$l^\mu_0 = \frac{l^\mu_+ + l^\mu_-}{2} = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu, \quad l^\mu_1 = \frac{l^\mu_+ - l^\mu_-}{2} = (G_E^{-1})^{\mu\nu}\pi_\nu. \tag{3.11}$$

The second set of generalized currents is defined by

$$J_{R(v,b)} = v^\mu(x)\pi_\mu + b_\mu(x)k^\mu, \tag{3.12}$$

where $v^\mu(x)$ and $b_\mu(x)$ are the arbitrary coefficients, and we have introduced another auxiliary current by

$$k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu. \tag{3.13}$$

Their charges are

$$Q_{R(v,b)} = \int d\sigma J_{R(v,b)}. \tag{3.14}$$

Similarly as in (3.7), these charges also exhibit additional symmetry. In order to see that, let us write the total derivative integral (3.6), using (3.13), in terms of new basis vectors

$$\int_0^{2\pi} d\sigma\kappa\partial_\mu\lambda x'^\mu = \int_0^{2\pi} d\sigma\partial_\mu\lambda(k^\mu - \kappa\theta^{\mu\nu}\pi_\nu). \tag{3.15}$$

As a result, we obtain the non-uniqueness of the charges

$$Q_{R(v+\kappa\theta\partial\lambda,b+\partial\lambda)} = Q_{R(v,b)}. \tag{3.16}$$

In a special case of $v^\mu = \pm(G_E^{-1})^{\mu\nu}b_\nu$, the generalized current (3.12) reduces to the current (2.30)

$$J_{R(\pm G_E^{-1}b,b)} = b_\mu l_\pm^\mu, \tag{3.17}$$

thus justifying calling it generalized current. Momenta $\pi_\mu$ and auxiliary currents $k^\mu$ can also be as easily obtained from it.

The two new bases transform into each other under (2.36):

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu \leftrightarrow \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu = k^\mu, \;\; \pi_\mu \leftrightarrow \kappa x'^\mu. \tag{3.18}$$

Therefore, the generalized currents are defined in the mutually T-dual bases, and their respective algebras are also going to be mutually T-dual.

At the end of this chapter, let us obtain the relations for coefficients when two generalized currents are equal. This will enable us to obtain the algebra of currents $J_{R(v,b)}$, provided that we have the algebra of $J_{C(u,a)}$, and vice versa. Let us start with rewriting the expressions for both generalized currents in the basis $\{\pi_\mu, x'^\mu\}$. Substituting the expression (3.2) into (3.4) we obtain

$$J_{C(u,a)} = u^\mu\pi_\mu + \kappa(a_\mu - 2B_{\mu\nu}u^\nu)x'^\mu, \tag{3.19}$$

while substituting the expression (3.13) into (3.12) we obtain

$$J_{R(v,b)} = (v^\mu - \kappa\theta^{\mu\nu}b_\nu)\pi_\mu + \kappa b_\mu x'^\mu. \tag{3.20}$$

Comparing (3.19)–(3.20), we see that generalized currents are equal when coefficients satisfy following relations

$$u^\mu = v^\mu - \kappa\theta^{\mu\nu}b_\nu, \\ a_\mu = 2B_{\mu\nu}v^\nu + (GG_E^{-1})_\mu^{\ \nu}b_\nu. \tag{3.21}$$

The above relations can be easily inversed. We obtain

$$v^\mu = (G_E^{-1}G)^\mu_{\ \nu}u^\nu + \kappa\theta^{\mu\nu}a_\nu, \\ b_\mu = a_\mu - 2B_{\mu\nu}u^\nu. \tag{3.22}$$

## 4 Courant bracket

We are interested in calculating the Poisson bracket algebra of the most general currents $J_{C(u,a)}$, defined in (3.4), as well as of their charges $Q_{C(u,a)}$, defined in (3.5). We will start with the generators $i_\mu$ and $x'^\mu$ algebra, that we calculate using the standard Poisson bracket relations

$$\{x^\mu(\sigma,\tau), \pi_\nu(\bar\sigma,\tau)\} = \delta^\mu_\nu\delta(\sigma-\bar\sigma), \\ \{x^\mu(\sigma,\tau), x^\nu(\bar\sigma,\tau)\} = 0, \\ \{\pi_\mu(\sigma,\tau), \pi_\nu(\bar\sigma,\tau)\} = 0. \tag{4.1}$$

In the accordance with [15], we will obtain that the algebra of generalized charges (3.5) gives rise to the twisted Courant bracket [6].

We obtain the algebra of generators (3.2)

$$\{i_\mu(\sigma), i_\nu(\bar\sigma)\} = -2\kappa B_{\mu\nu\rho}x'^\rho\delta(\sigma-\bar\sigma), \tag{4.2}$$

where the structural constants are the Kalb–Ramond field strength components, given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \tag{4.3}$$

The rest of the generators algebra is given by

$$\{i_\mu(\sigma), \kappa x'^\nu(\bar\sigma)\} = \kappa\delta_\mu^{\ \nu}\partial_\sigma\delta(\sigma-\bar\sigma), \quad \{\kappa x'^\mu(\sigma), \kappa x'^\nu(\bar\sigma)\} = 0. \tag{4.4}$$

The Poisson bracket of the most general currents (3.4) is obtained using (4.2) and (4.4). It reads

$$\begin{aligned}\{J_{C(u,a)}(\sigma), J_{C(v,b)}(\bar\sigma)\} &= (v^\nu\partial_\nu u^\mu - u^\nu\partial_\nu v^\mu)i_\mu\delta(\sigma-\bar\sigma) - 2\kappa B_{\mu\nu\rho}x'^\mu u^\nu v^\rho\delta(\sigma-\bar\sigma) \\ &- \kappa\left((\partial_\mu a_\nu - \partial_\nu a_\mu)v^\nu - (\partial_\mu b_\nu - \partial_\nu b_\mu)u^\nu\right)x'^\mu\delta(\sigma-\bar\sigma) \\ &+ \kappa\left(u^\mu(\sigma)b_\mu(\sigma) + v^\mu(\bar\sigma)a_\mu(\bar\sigma)\right)\partial_\sigma\delta(\sigma-\bar\sigma).\end{aligned} \tag{4.5}$$

We can modify the anomalous part in the following manner

$$\left(u^\mu(\sigma)b_\mu(\sigma) + v^\mu(\bar\sigma)a_\mu(\bar\sigma)\right)\partial_\sigma\delta(\sigma-\bar\sigma)$$

$$
\begin{aligned}
&= \frac{1}{2}\left((ub)(\sigma) + (va)(\bar\sigma)\right)\partial_\sigma\delta(\sigma-\bar\sigma) \\
&\quad - \frac{1}{2}(ub)(\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma) + \frac{1}{2}(va)(\bar\sigma)\partial_\sigma\delta(\sigma-\bar\sigma) \\
&= \frac{1}{2}\left((ub)(\sigma) + (ub)(\bar\sigma) + va(\sigma) + (va)(\bar\sigma)\right)\partial_\sigma\delta(\sigma-\bar\sigma) \\
&\quad + \frac{1}{2}\partial_\mu(va-ub)x'^\mu\delta(\sigma-\bar\sigma),
\end{aligned} \tag{4.6}
$$

where we have used the notation $(ub)(\sigma) = u^\mu(\sigma)b_\mu(\sigma)$, and the relation $f(\bar\sigma)\partial_\sigma\delta(\sigma-\bar\sigma) = f'(\sigma)\delta(\sigma-\bar\sigma) + f(\sigma)\partial_\sigma\delta(\sigma-\bar\sigma)$ in the last step. Substituting the previous equation in (4.5) we obtain

$$
\begin{aligned}
&\{J_{C(u,a)}(\sigma), J_{C(v,b)}(\bar\sigma)\} \\
&\quad = -J_{C(\bar w,\bar c)}(\sigma)\delta(\sigma-\bar\sigma) \\
&\quad\quad + \frac{\kappa}{2}\left((ub)(\sigma) + (ub)(\bar\sigma) + (va)(\sigma)\right. \\
&\quad\quad \left. + (va)(\bar\sigma)\right)\partial_\sigma\delta(\sigma-\bar\sigma),
\end{aligned} \tag{4.7}
$$

where the coefficients in the resulting current are

$$
\bar w^\mu = u^\nu\partial_\nu v^\mu - v^\nu\partial_\nu u^\mu, \tag{4.8}
$$

and

$$
\begin{aligned}
\bar c_\mu &= 2B_{\mu\nu\rho}u^\nu v^\rho + (\partial_\mu a_\nu - \partial_\nu a_\mu)v^\nu - (\partial_\mu b_\nu - \partial_\nu b_\mu)u^\nu \\
&\quad + \frac{1}{2}\partial_\mu(ub-va).
\end{aligned} \tag{4.9}
$$

The minus sign in front of the $J_{C(\bar w,\bar c)}$ is included for the future convenience. We see that $\bar w^\mu$ does not depend on background fields, while the coefficient $\bar c_\mu$ does, because of the $H$-flux term $B_{\mu\nu\rho}$.

The relation (4.7) defines the bracket, that acts on a pair of two ordered pairs consisting of a vector and a 1-form, that as a result has another ordered pair, that we can write like

$$
[(u,a),(v,b)]_C = (\bar w, \bar c). \tag{4.10}
$$

The bracket that we have obtained is the twisted Courant bracket [6]. The Courant bracket represents the generalization of the Lie bracket on spaces that contain both vectors and 1-forms. As a result, it gives an ordered pair of a vector $w = w^\mu\partial_\mu$ and a 1-form $c = c_\mu dx^\mu$.

Let us confirm the equivalence between the twisted Courant bracket and the bracket that we have obtained in (4.10). The coordinate free expression for the twisted Courant bracket is given by

$$
\begin{aligned}
[(u,a),(v,b)]_C &= \Big([u,v]_L, \mathcal{L}_u b - \mathcal{L}_v a - \frac{1}{2}d(i_u b - i_v a) \\
&\quad + H(u,v,.)\Big) \equiv (w,c),
\end{aligned} \tag{4.11}
$$

where $[u,v]_L$ is the Lie bracket and $H(u,v,.)$ is a 1-form obtained by contracting a three form. The Lie derivative $\mathcal{L}_u$ is defined in a usual way $\mathcal{L}_u = i_u d + d i_u$, where $d$ is the exterior derivative and $i_u$ the interior derivative. Their action on 1-forms is given by $da = \partial_\mu a_\nu dx^\mu dx^\nu$ and $i_u a = u^\mu a_\mu$.

The Lie bracket is given by

$$
[u,v]_L\,|^\mu = u^\nu\partial_\nu v^\mu - v^\nu\partial_\nu u^\mu. \tag{4.12}
$$

Using the definition of Lie derivative, we furthermore obtain

$$
\begin{aligned}
&\left(\mathcal{L}_u b - \mathcal{L}_v a - \frac{1}{2}d(i_u b - i_v a)\right)\Big|_\mu \\
&\quad = u^\nu(\partial_\nu b_\mu - \partial_\mu b_\nu) - v^\nu(\partial_\nu a_\mu - \partial_\mu a_\nu) \\
&\quad\quad + \frac{1}{2}\partial_\mu(ub-va).
\end{aligned} \tag{4.13}
$$

As for the last term in (4.11), it is given by

$$
H(u,v,.)|_\mu = 2B_{\mu\nu\rho}u^\nu v^\rho. \tag{4.14}
$$

The expression for the generalized current corresponding to the Courant bracket is obtained by substituting (4.12), (4.13) and (4.14) in (4.11)

$$
[(u,a),(v,b)]_C = (w,c), \tag{4.15}
$$

where $w^\mu$ and $c_\mu$ are exactly the same as $\bar w^\mu$ and $\bar c_\mu$ defined in (4.8) and (4.9), respectively. Therefore, we see that the bracket defined in (4.10) is indeed the twisted Courant bracket.

Besides the current algebra, we are interested in the algebra of charges (3.5). The anomalous term is canceled when integrated. For example, consider the first term in anomaly

$$
\begin{aligned}
&\int d\sigma d\bar\sigma\,(ub)(\sigma)\partial_\sigma\delta(\sigma-\bar\sigma) \\
&\quad = -\int d\bar\sigma\,\partial_{\bar\sigma}\int d\sigma\,(ub)(\sigma)\delta(\sigma-\bar\sigma) \\
&\quad = -\int d\bar\sigma\,\partial_{\bar\sigma}(ub(\bar\sigma)) = 0,
\end{aligned} \tag{4.16}
$$

since we are working with the closed strings. The other terms cancel in a similar manner. Integrating the generalized currents (4.7) over $\sigma$ and $\bar\sigma$ we obtain

$$
\{Q_{C(u,a)}, Q_{C(v,b)}\} = -Q_{C[(u,a),(v,b)]_C}. \tag{4.17}
$$

We see that the algebra of charges is anomaly free. The relation (4.17) was firstly obtained in [15] for the general case of the Hamiltonian formulation of string $\sigma$-model, in which momenta and coordinates satisfy the same Poisson bracket relations as auxiliary currents and coordinates in our theory.

Let us check whether the algebra (4.7) is consistent with the known results for the Poisson bracket algebra of the currents $j_{\pm\mu}$ [28]

$$
\{j_{\pm\mu}(\sigma), j_{\pm\nu}(\bar\sigma)\} = \pm 2\kappa\Gamma_{\mu,\nu\rho}x'^\rho\delta(\sigma-\bar\sigma)
$$

$$
\begin{aligned}
&\quad - 2\kappa B_{\mu\nu\rho} x'^{\rho} \delta(\sigma - \bar{\sigma}) \\
&\quad \pm 2\kappa G_{\mu\nu} \delta'(\sigma - \bar{\sigma}), \\
\{ j_{\pm\mu}(\sigma), j_{\mp\nu}(\bar{\sigma}) \} &= \pm 2\kappa \Gamma_{\mu,\nu\rho} x'^{\rho} \delta(\sigma - \bar{\sigma}) \\
&\quad - 2\kappa B_{\mu\nu\rho} x'^{\rho} \delta(\sigma - \bar{\sigma}), \qquad (4.18)
\end{aligned}
$$

where $\Gamma_{\mu,\nu\rho} = \frac{1}{2}(\partial_\nu G_{\rho\mu} + \partial_\rho G_{\nu\mu} - \partial_\mu G_{\nu\rho})$ are Christoffel symbols. If we substitute $a_\mu = \pm G_{\mu\nu} u^\nu$ and $b_\mu = \pm G_{\mu\nu} v^\nu$ for constants $u^\mu$ and $v^\mu$ in (4.7), with the help of (3.8) we obtain

$$
\begin{aligned}
&\{ u^\mu j_{\pm\mu}(\sigma), v^\nu j_{\pm\nu}(\bar{\sigma}) \} \\
&= u^\mu v^\nu \left( -2\kappa B_{\mu\nu\rho} \pm \kappa (\partial_\nu G_{\rho\mu} - \partial_\mu G_{\nu\rho}) \right) x'^{\rho} \delta(\sigma - \bar{\sigma}) \\
&\quad \pm \kappa u^\mu v^\nu (G_{\mu\nu}(\sigma) + G_{\mu\nu}(\bar{\sigma})) \partial_\sigma \delta(\sigma - \bar{\sigma}) \\
&= u^\mu v^\nu \left( -2\kappa B_{\mu\nu\rho} \pm \kappa (\partial_\nu G_{\rho\mu} + \partial_\rho G_{\mu\nu} - \partial_\mu G_{\nu\rho}) \right) x'^{\rho} \delta(\sigma - \bar{\sigma}) \\
&\quad \pm 2\kappa u^\mu v^\nu G_{\mu\nu}(\sigma) \partial_\sigma \delta(\sigma - \bar{\sigma}) \\
&= u^\mu v^\nu \{ j_{\pm\mu}, j_{\pm\nu} \}. \qquad (4.19)
\end{aligned}
$$

The consistency with the second relation in (4.18) can be as easily obtained.

## 5 Roytenberg bracket

The Roytenberg bracket appeared as a result of the current algebra firstly in [12], where the author twisted the Poisson structure by trading the 2-form $B_{\mu\nu}$ with the bi-vector $\Pi^{\mu\nu}$. In this paper, we firstly calculate the Poisson bracket algebra for the generalized currents $J_{R(v,b)}$ (3.12), in order to calculate the T-dual Poisson structure of the twisted Courant bracket.

While the currents (3.12) have the same form as the currents giving the Roytenberg bracket in [12], in [12] the momenta are redefined so that they are equal to the auxiliary currents $i_\mu$ (3.2) in our paper. As a result of this difference, the currents $J_{R(v,b)}$ and $J_{C(u,a)}$ are related by self T-duality, which is not the case for corresponding currents in [12]. Therefore, we will show that the Courant bracket twisted by a 2-form $2B_{\mu\nu}$ is T-dual to the Roytenberg bracket, obtained by twisting the Courant bracket by a bi-vector $\kappa\theta^{\mu\nu}$. When the fluxes are turned off, both of them reduce to the untwisted Courant bracket, that is T-dual to itself.

We will start with the algebra of auxiliary currents $k^\mu$ (3.13). Using (4.1), we obtain

$$
\begin{aligned}
\{ k^\mu(\sigma), k^\nu(\bar{\sigma}) \} &= -\kappa \partial_\rho \theta^{\mu\nu} x'^{\rho} \delta(\sigma - \bar{\sigma}) - \kappa^2 (\theta^{\mu\sigma} \partial_\sigma \theta^{\nu\rho} \\
&\quad - \theta^{\nu\sigma} \partial_\sigma \theta^{\mu\rho}) \pi_\rho \delta(\sigma - \bar{\sigma}), \qquad (5.1)
\end{aligned}
$$

where $\theta^{\mu\nu}$ is the non-commutativity parameter (2.12). From (3.13) we express the coordinate in terms of algebra generators and obtain

$$
\{ k^\mu(\sigma), k^\nu(\bar{\sigma}) \} = -\kappa Q_\rho{}^{\mu\nu} k^\rho \delta(\sigma - \bar{\sigma}) - \kappa^2 R^{\mu\nu\rho} \pi_\rho \delta(\sigma - \bar{\sigma}), \qquad (5.2)
$$

where we expressed the structure constants as fluxes

$$
Q_\rho{}^{\mu\nu} = \partial_\rho \theta^{\mu\nu}, \quad R^{\mu\nu\rho} = \theta^{\mu\sigma} \partial_\sigma \theta^{\nu\rho} + \theta^{\nu\sigma} \partial_\sigma \theta^{\rho\mu} + \theta^{\rho\sigma} \partial_\sigma \theta^{\mu\nu}. \qquad (5.3)
$$

These are the non-geometric fluxes [29]. They were firstly obtained by applying the Buscher rules [18–21] on the three-torus with non-trivial Kalb-Ramond field strength (4.3). After the T-duality transformations are applied along two isometry directions, one obtains the space that is locally geometric, but globally non-geometric. The flux for this background is $Q_\rho{}^{\mu\nu}$. After the T-duality transformation is applied along all directions, one obtains the space that is neither locally, nor globally geometric, characterized with the $R^{\mu\nu\rho}$ flux. When considering a generalized T-dualization, the $R$ flux is obtained when performing T-dualization over the arbitrary coordinate on which the background fields depend [30].

The rest of the generators algebra is calculated in a similar way

$$
\begin{aligned}
\{ k^\mu(\sigma), \pi_\nu(\bar{\sigma}) \} &= \kappa \delta^\mu_\nu \partial_\sigma \delta(\sigma - \bar{\sigma}) + \kappa Q_\nu{}^{\mu\rho} \pi_\rho \delta(\sigma - \bar{\sigma}), \\
\{ \pi_\mu(\sigma), \pi_\nu(\bar{\sigma}) \} &= 0. \qquad (5.4)
\end{aligned}
$$

We obtain the Poisson bracket of the most general currents $J_{R(u,a)}$, using (5.2) and (5.4). It reads

$$
\begin{aligned}
&\{ J_{R(u,a)}(\sigma), J_{R(v,b)}(\bar{\sigma}) \} \\
&= (v^\nu \partial_\nu u^\mu - u^\nu \partial_\nu v^\mu) \pi_\mu \delta(\sigma - \bar{\sigma}) - \kappa^2 R^{\mu\nu\rho} \pi_\mu a_\nu b_\rho \delta(\sigma - \bar{\sigma}) \\
&\quad - \kappa \left( \theta^{\nu\rho} \partial_\rho v^\mu a_\nu - v^\rho \partial_\nu a_\rho \theta^{\nu\mu} - \partial_\nu \theta^{\rho\mu} v^\nu a_\rho \right) \pi_\mu \delta(\sigma - \bar{\sigma}) \\
&\quad - \kappa \left( u^\rho \partial_\nu b_\rho \theta^{\nu\mu} + \kappa u^\rho \partial_\rho \theta^{\nu\mu} b_\nu - \kappa \theta^{\nu\rho} \partial_\rho u^\mu b_\nu \right) \pi_\mu \delta(\sigma - \bar{\sigma}) \\
&\quad + \left( u^\nu (\partial_\mu b_\nu - \partial_\nu b_\mu) - v^\nu (\partial_\mu a_\nu - \partial_\nu a_\mu) \right) k^\mu \delta(\sigma - \bar{\sigma}) \\
&\quad - \kappa \left( a_\rho b_\nu \partial_\mu \theta^{\rho\nu} - \theta^{\nu\rho} (\partial_\rho a_\mu b_\nu - \partial_\rho b_\mu a_\nu) \right) k^\mu \delta(\sigma - \bar{\sigma}) \\
&\quad + \kappa \left( u^\mu(\sigma) b_\mu(\sigma) + v^\mu(\bar{\sigma}) a_\mu(\bar{\sigma}) \right) \partial_\sigma \delta(\sigma - \bar{\sigma}). \qquad (5.5)
\end{aligned}
$$

Using (4.6) and (3.13) we can transform the anomaly in the following way

$$
\begin{aligned}
&\kappa \left( (ub)(\sigma) + (va)(\bar{\sigma}) \right) \partial_\sigma \delta(\sigma - \bar{\sigma}) \\
&= \frac{\kappa}{2} ((ub)(\sigma) + (ub)(\bar{\sigma}) + (va)(\sigma) + (va)(\bar{\sigma})) \partial_\sigma \delta(\sigma - \bar{\sigma}) \\
&\quad + \frac{1}{2} \partial_\mu (va - ub)(\sigma) (k^\mu - \theta^{\mu\rho} \pi_\rho) \delta(\sigma - \bar{\sigma}). \qquad (5.6)
\end{aligned}
$$

Substituting the last equation in (5.5), we obtain

$$
\begin{aligned}
&\{ J_{R(u,a)}(\sigma), J_{R(v,b)}(\bar{\sigma}) \} \\
&\quad = -J_{R(\bar{w},\bar{c})}(\sigma) \delta(\sigma - \bar{\sigma}) \\
&\quad\quad + \frac{\kappa}{2} \left( (ub)(\sigma) + (ub)(\bar{\sigma}) + (va)(\sigma) \right. \\
&\quad\quad \left. + (va)(\bar{\sigma}) \right) \partial_\sigma \delta(\sigma - \bar{\sigma}), \qquad (5.7)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{w}^\mu &= u^\nu \partial_\nu v^\mu - v^\nu \partial_\nu u^\mu + \kappa \theta^{\nu\rho} \partial_\rho v^\mu a_\nu \\
&\quad - \kappa v^\rho \partial_\nu a_\rho \theta^{\nu\mu} - \kappa Q_\nu{}^{\rho\mu} v^\nu a_\rho
\end{aligned}
$$

$$+\kappa u^\rho \partial_\nu b_\rho \theta^{\nu\mu} + \kappa u^\rho Q_\rho{}^{\nu\mu} b_\nu - \kappa\theta^{\nu\rho}\partial_\rho u^\mu b_\nu - \frac{\kappa}{2}\theta^{\mu\nu}\partial_\nu(va - ub) + \kappa^2 R^{\mu\nu\rho} a_\nu b_\rho, \tag{5.8}$$

and

$$\bar{c}_\mu = v^\nu(\partial_\mu a_\nu - \partial_\nu a_\mu) - u^\nu(\partial_\mu b_\nu - \partial_\nu b_\mu) - \frac{1}{2}\partial_\mu(va - ub) + \kappa a_\rho b_\nu Q_\mu{}^{\rho\nu} - \kappa\theta^{\nu\rho}(\partial_\rho a_\mu b_\nu - \partial_\rho b_\mu a_\nu), \tag{5.9}$$

where we have substituted $Q$ and $R$ fluxes (5.3). Unlike the coefficients in the previous case, here both coefficients depend on backgrounds, due to the presence of fluxes.

As expected, algebra is not closed due to the anomalous part. This Poisson bracket defines a new bracket

$$[(u, a), (v, b)]_R = (\bar{w}, \bar{c}), \tag{5.10}$$

which is equal to the Roytenberg bracket [11]. In case of only $R$ and $Q$ flux present in the generators algebra (5.3), the Roytenberg bracket is given by

$$[(u, a), (v, b)]_R = \Big([u, v]_L - [v, a\Pi]_L + [u, b\Pi]_L + \frac{1}{2}[\Pi, \Pi]_S(a, b, .) - \Big(\mathcal{L}_v a - \mathcal{L}_u b + \frac{1}{2}d(i_u b - i_v a)\Big)\Pi, + \mathcal{L}_u b - \mathcal{L}_v a - \frac{1}{2}d(i_u b - i_v a) - [a, b]_\Pi\Big), \tag{5.11}$$

where $\Pi = \Pi^{\mu\nu}\partial_\mu\partial_\nu$ is the bi-vector. The expression $[\Pi, \Pi]_S(a, b, .)$ represents the Schouten–Nijenhuis bracket [31] contracted with two 1-forms and $[a, b]_\Pi$ is the Koszul bracket [32] given by

$$[a, b]_\Pi = \mathcal{L}_{a\Pi} b - \mathcal{L}_{b\Pi} a + d(\Pi(a, b)). \tag{5.12}$$

The Koszul bracket is a generalization of the Lie bracket on the space of differential forms, while the Schouten-Nijenhuis bracket is a generalization of the Lie bracket on the space of multi-vectors.

The terms in (5.11) that we have not calculated yet can be written, using (4.13), as

$$\Big((\mathcal{L}_v a - \mathcal{L}_u b + \frac{1}{2}d(i_u b - i_v a))\Pi\Big)\Big|^\mu = \Big(u^\nu(\partial_\nu b_\rho - \partial_\rho b_\nu) - v^\nu(\partial_\nu a_\rho - \partial_\rho a_\nu) + \frac{1}{2}\partial_\rho(ub - va)\Big)\Pi^{\rho\mu}. \tag{5.13}$$

The Koszul bracket (5.12) can be further transformed in a following way

$$[a, b]_\Pi|_\mu = \Pi^{\rho\nu}(b_\rho\partial_\nu a_\mu - a_\rho\partial_\nu b_\mu) + \partial_\nu\Pi^{\nu\rho}a_\rho b_\mu, \tag{5.14}$$

while the remaining terms linear in $\Pi$ become

$$([-v, a\Pi]_L + [u, b\Pi]_L)|^\mu = v^\nu(\partial_\nu a_\rho \Pi^{\mu\rho} + a_\rho\partial_\nu\Pi^{\mu\rho}) + a_\rho\Pi^{\rho\nu}\partial_\nu v^\mu - u^\nu(\partial_\nu b_\rho\Pi^{\mu\rho} + b_\rho\partial_\nu\Pi^{\mu\rho}) - b_\rho\Pi^{\rho\nu}\partial_\nu u^\mu. \tag{5.15}$$

Lastly, we write the expression for the Schouten-Nijenhuis bracket for bi-vectors

$$[\Pi, \Pi]_S|^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}_{\alpha\beta\gamma}\Pi^{\sigma\alpha}\partial_\sigma\Pi^{\beta\gamma}, \tag{5.16}$$

where

$$\epsilon^{\mu\nu\rho}_{\alpha\beta\gamma} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\beta & \delta^\rho_\gamma \\ \delta^\nu_\alpha & \delta^\rho_\beta & \delta^\mu_\gamma \\ \delta^\rho_\alpha & \delta^\mu_\beta & \delta^\nu_\gamma \end{vmatrix} . \tag{5.17}$$

Thus, we get

$$([\Pi, \Pi]_S(a, b, .))|^\mu = 2R^{\mu\nu\rho}a_\nu b_\rho, \tag{5.18}$$

where $R^{\mu\nu\rho}$ is the flux defined in (5.3).

Combining the previously obtained terms, we obtain the expression for the generalized current corresponding to the Roytenberg bracket twisted by the non-commutativity parameter as a bi-vector

$$[(u, a), (v, b)]_R = (w, c), \tag{5.19}$$

where $w^\mu$ and $c_\mu$ are equal to $\bar{w}^\mu$ and $\bar{c}_\mu$, defined in (5.8) and (5.9), respectively, provided that $\Pi^{\mu\nu} = \kappa\theta^{\mu\nu}$.

Integrating the previous equation over $\sigma$ and $\bar{\sigma}$, we see that charges satisfy

$$\{Q_{R(u,a)}, Q_{R(v,b)}\} = -Q_{R[(u,a),(v,b)]_R}. \tag{5.20}$$

The bases in which these generalized currents have been defined are mutually T-dual (2.36). This means that the generalized currents also transform into each other

$$J_{C(u,a)} \leftrightarrow J_{R(v,b)}, \tag{5.21}$$

provided that we swap also coefficients $u^\mu \leftrightarrow b_\mu$, $a_\mu \leftrightarrow v^\mu$. We say that two types of brackets, one obtained by twisting the Courant bracket by a 2-form $2B_{\mu\nu}$, another obtained by twisting the Courant bracket by a bi-vector $\Pi^{\mu\nu}$, are mutually T-dual, as long as the aforementioned 2-form $B_{\mu\nu}$ is T-dual to the bi-vector $\Pi^{\mu\nu}$.

In [33] it has been proposed that T-duality can be understood as the isomorphism $\varphi$ between two Courant algebroids [7,9]. The relations connecting coefficients of two sets of generalized currents (3.21) can in fact be rewritten as

$$\varphi(u, a) = (u - \kappa\theta a, 2Bu + (G_E^{-1}G)a), \tag{5.22}$$

which can be interpreted as the isomorphism $\varphi(u,a) = (v,b)$ between two Courant algebroids with the trivial bundles over a point and with the twisted Courant and Roytenberg brackets as brackets that act on the Cartesian product of sections of these bundles, as well as the natural inner product $\langle .,. \rangle$ between generalized vectors, given by

$$\langle (u,a),(v,b)\rangle = \frac{1}{2}(ub+va). \tag{5.23}$$

In order for $\varphi$ to be the isomorphism that corresponds to T-duality, it has to satisfy the following conditions:

$$\begin{aligned} &\langle \varphi(u,a),\varphi(v,b)\rangle = \langle (u,a),(v,b)\rangle,\\ &[\varphi(u,a),\varphi(v,b)]_C = \varphi\left([(u,a),(v,b)]_R\right). \end{aligned} \tag{5.24}$$

To prove that the first condition is satisfied, using (5.22), we obtain

$$\begin{aligned} &\langle \varphi(u,a),\varphi(v,b)\rangle \\ &\quad = \langle (u-\kappa\theta a, 2Bu+(G_E^{-1}G)a),(v-\kappa\theta b, 2Bv \\ &\qquad +(G_E^{-1}G)b)\rangle \\ &\quad = \frac{1}{2}\Big(2B_{\mu\nu}u^\mu v^\nu + 2\kappa(B\theta)_\mu^{\ \nu} v^\mu a_\nu + (G_E^{-1}G)^\mu_{\ \nu} b_\mu u^\nu \\ &\qquad -\kappa(G_E^{-1}G)^\mu_{\ \nu}\theta^{\nu\rho}b_\mu a_\rho \\ &\qquad +2B_{\mu\nu}v^\mu u^\nu + 2\kappa(B\theta)_\mu^{\ \nu} u^\mu b_\nu + (G_E^{-1}G)^\mu_{\ \nu} a_\mu v^\nu \\ &\qquad -\kappa(G_E^{-1}G)^\mu_{\ \nu}\theta^{\nu\rho}a_\mu b_\rho\Big) \\ &\quad = \frac{1}{2}(u^\mu b_\nu + v^\mu a_\nu)\left((G_E^{-1}G)^\nu_{\ \mu} + 2\kappa(\theta B)^\nu_{\ \mu}\right) \\ &\quad = \langle (u,a),(v,b)\rangle, \end{aligned} \tag{5.25}$$

where we have used the fact that $B_{\mu\nu}$ and $(G_E^{-1}G\theta)^{\mu\nu}$ are both antisymmetric, as well as

$$(G_E^{-1}G)^\mu_{\ \nu} + 2\kappa(\theta B)^\mu_{\ \nu} = \delta^\mu_\nu, \tag{5.26}$$

which is the identity easily obtained from (2.6) and (2.12). As for the second relation of (5.24), it can be shown by writing the relation (4.17) for $\varphi$-transformed coefficients

$$\{Q_{C\varphi(u,a)}, Q_{C\varphi(v,b)}\} = -Q_{C[\varphi(u,a),\varphi(v,b)]_C}. \tag{5.27}$$

On the other hand, due to $Q_{C\varphi(u,a)} = Q_{R(u,a)}$, the terms on the right-hand sides of (5.27) and (5.20) are equal. By equating them, one obtains

$$Q_{R[(u,a),(v,b)]_R} = Q_{C[\varphi(u,a),\varphi(v,b)]_C}. \tag{5.28}$$

Lastly, using (5.22), we write the above relation in the form

$$Q_{C\varphi([(u,a),(v,b)]_R)} = Q_{C[\varphi(u,a),\varphi(v,b)]_C}, \tag{5.29}$$

from which the second condition of (5.24) is easily read.

Therefore, we have shown that the relations connecting two types of generalized currents (5.22) define the isomorphism between two Courant algebroids, characterized by twisted Courant and Roytenberg bracket, that according to [33] is interpreted as T-duality.

## 6 Conclusion

In this paper, we used the T-dualization rules (2.10) for coordinates in the Lagrangian approach, and (2.21) for the canonical variables in the Hamiltonian approach. The relation for T-dual background fields (2.17) stands in both approaches. These relations between the fields provide correct relations between the Courant and Roytenberg bracket.

The T-dualization rules we used, correspond exactly to Buscher's rules obtained in its original procedure [18–21] when there is an Abelian group of isometries of coordinates $x^a$ along which one T-dualizes: $B(x^a) = B(x^a + b^a)$, $G(x^a) = G(x^a + b^a)$. In the Buscher procedure the symmetry is gauged and the new action is obtained. Integrating out the gauge fields from that action, one obtains the T-dual Lagrangian. From that, the T-dual transformation law between the T-dual coordinate $\sigma$-derivatives and the canonical momenta of the initial theory can be obtained $\kappa y'_\mu \cong \pi_\mu$. This is exactly the relation (2.21) in our paper.

The most interesting case is when we try to perform the T-dualization along non-isometry directions $x^a$, such that background fields do depend on them. Then we should apply the generalized Buscher's procedure, developed in [25,34]. In this case, the expression for the T-dual background fields (2.17) remain the same but the argument of the T-dual background fields is not simply the T-dual variable $y_a$. It is the line integral $V$ that is a function of the world-sheet gauge fields $v_+^a$ and $v_-^a$, namely $V^a[v_+, v_-] \equiv \int_P d\xi^\alpha v_\alpha^a = \int_P (d\xi^+ v_+^a + d\xi^- v_-^a)$. The expressions for gauge fields can be obtained by varying the Lagrangian with respect to gauge fields and the expression for the argument of background fields has the form $V^a = -\kappa\,\theta^{ab} y_b + G_E^{-1ab}\,\tilde{y}_b$, where $\tilde{y}_a$ is a double of T-dual variable $y_a$, which satisfy relations $\dot{\tilde{y}}_a = y'_a$ and $\tilde{y}'_a = \dot{y}_a$. Let us point out that in such case the T-dual theory becomes locally non-geometric because the argument of the background fields is the line integral.

For example, in case of the weakly curved background [25] the initial theory is geometric and T-dual theory is non-geometric. In the initial theory the generalized current algebra gives rise to the twisted Courant bracket. However, in the T-dual theory, the presence of double variable $\tilde{y}_a$, makes the calculation of T-dual current algebra much more complicated. It is hard to believe that such a bracket or its corresponding self T-duality version will be equivalent to the

Roytenberg one. Therefore, in case of non-geometric theories, one might expect some new form of brackets.

Next, we introduced the T-duality in the same phase space, that we call self T-duality. It interchanges the momenta and coordinate $\sigma$-derivatives, as well as the background fields with the T-dual ones. The Hamiltonian was expressed in terms of currents $j_{\pm\mu}$ and metric tensor $G_{\mu\nu}$, as well as in terms of its T-dual currents $l_{\pm}^{\mu}$ and T-dual metric tensor ${}^{\star}G^{\mu\nu} = (G_E^{-1})^{\mu\nu}$. We considered two types of generalized currents, $J_{C(u,a)}$ and $J_{R(v,b)}$, that generalize currents $j_{\pm\mu}$ and $l_{\pm}^{\mu}$ respectively. The suitable basis for the current $J_{C(u,a)}$ consists of coordinate $\sigma$-derivatives $x'^{\mu}$ and the auxiliary currents $i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}x'^{\nu}$, and for the current $J_{R(v,b)}$, it consists of momenta $\pi_\mu$ and auxiliary currents $k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu$. These bases transform into each other under the self T-duality (2.36).

In this paper, we obtained two types of brackets, extracted from the generalized current Poisson bracket algebra. We have shown that one of them is equal to the twisted Courant bracket, while the other equals the Roytenberg bracket. The former can be obtained by twisting the Courant bracket by a 2-form, in our paper $2B_{\mu\nu}$, resulting in the appearance of $H-$flux in generators algebra. The latter bracket can be obtained by twisting the Courant bracket by a bi-vector $\Pi^{\mu\nu}$, resulting in the appearance of $Q-$ and $R-$fluxes, but not $H-$flux, in generators algebra. Since bases in which generalized currents are defined are mutually T-dual, we conclude that the brackets are mutually T-dual, when the bi-vector $\Pi^{\mu\nu}$ equals to the non-commutativity parameter $\kappa\theta^{\mu\nu}$.

We find these results important in itself. Both the Courant and the Roytenberg bracket are well understood mathematical structures. Relation between them and T-duality has a potential to help understand the T-duality better. Moreover, by analyzing characteristics of these brackets we can examine how certain aspects of the mutually T-dual theories relate to each other.

Suppose we turn off all the fluxes. That is equivalent to setting $B_{\mu\nu} = 0$ and $\Pi^{\mu\nu} = 0$, which reduce the auxiliary currents to canonical momentum and coordinate $\sigma$ derivative: $i_\mu \to \pi_\mu$ and $k^\mu \to \kappa x'^\mu$. The generalized currents now reduce to $J_{C(u,a)} = u^\mu\pi_\mu + a_\mu\kappa x'^\mu$ and $J_{R(v,b)} = v^\mu\pi_\mu + \kappa b_\mu x'^\mu$. It is easy to verify that these currents remain invariant under exchange of momenta and winding numbers, provided that we also change the coefficients in the particular way $J_{C(u,a)} \leftrightarrow \tilde{u}_\mu\kappa x'^\mu + \tilde{a}^\mu\pi_\mu = J_{C(\tilde{a},\tilde{u})}$. Therefore, we conclude that these currents are T-dual to themselves. They give rise to the Courant bracket, the untwisted one, which does not contain any fluxes.

It is interesting that both charges $Q_{C(u,a)}$ and $Q_{R(v,b)}$ can be expressed as the self T-dual symmetry generators in the form

$$\mathcal{G} = \int d\sigma \left[\xi^\mu\pi_\mu + \widetilde{\Lambda}_\mu\kappa x'^\mu\right]. \tag{6.1}$$

It is easy to show that if we define the new gauge parameter $\Lambda_\mu = \widetilde{\Lambda}_\mu + 2B_{\mu\nu}\xi^\nu$, the generators (6.1) are charges $Q_{C(\xi,\Lambda)}$; if we define $\tilde{\xi}^\mu = \xi^\mu + \kappa\theta^{\mu\nu}\widetilde{\Lambda}_\nu$ the generators are charges $Q_{R(\tilde{\xi},\widetilde{\Lambda})}$. Momenta $\pi_\mu$ are generators of general coordinate transformations and $x'^\mu$ generators of local gauge transformations $\delta_{\widetilde{\Lambda}}B_{\mu\nu} = \partial_\mu\widetilde{\Lambda}_\nu - \partial_\nu\widetilde{\Lambda}_\mu$, while $\xi^\mu$ and $\widetilde{\Lambda}_\mu$ are their corresponding parameters. These generators were studied in [28], where it was shown that general coordinate transformations are T-dual to gauge transformations.

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# Courant bracket as T-dual invariant extension of Lie bracket

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Abstract: We consider the symmetries of a closed bosonic string, starting with the general coordinate transformations. Their generator takes vector components $\xi^\mu$ as its parameter and its Poisson bracket algebra gives rise to the Lie bracket of its parameters. We are going to extend this generator in order for it to be invariant upon self T-duality, i.e. T-duality realized in the same phase space. The new generator is a function of a $2D$ double symmetry parameter $\Lambda$, that is a direct sum of vector components $\xi^\mu$, and 1-form components $\lambda_\mu$. The Poisson bracket algebra of a new generator produces the Courant bracket in a same way that the algebra of the general coordinate transformations produces Lie bracket. In that sense, the Courant bracket is T-dual invariant extension of the Lie bracket. When the Kalb-Ramond field is introduced to the model, the generator governing both general coordinate and local gauge symmetries is constructed. It is no longer self T-dual and its algebra gives rise to the $B$-twisted Courant bracket, while in its self T-dual description, the relevant bracket becomes the $\theta$-twisted Courant bracket. Next, we consider the T-duality and the symmetry parameters that depend on both the initial coordinates $x^\mu$ and T-dual coordinates $y_\mu$. The generator of these transformations is defined as an inner product in a double space and its algebra gives rise to the C-bracket.

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## Contents



## 1 Introduction

The Courant bracket [1, 2] and various generalizations obtained by its twisting had been relevant to the string theory since its appearance in the algebra of generalized currents [3–6]. It represents the generalization of the Lie bracket on spaces of generalized vectors, understood as the direct sum of the elements of the tangent bundle and the elements of the cotangent bundle. Although the Lie bracket satisfies the Jacobi identity, the Courant bracket does not. Its Jacobiator is equal to the exterior derivative of the Nijenhuis operator.

It is well known that the commutator of two general coordinate transformations along two vector fields produces another general coordinate transformation along the vector field equal to their Lie bracket. Since the Courant bracket represents its generalization, it is worth considering how it is related to symmetries of the bosonic string $\sigma$-model.

In [7], the field theory defined on the double torus, and its symmetries for restricted parameters were considered. The double space is seen as a direct sum of the initial and T-dual phase space, and the background fields depend on both of these coordinates. The symmetry algebra is closed only for restricted parameters, defined on the same isotropic space, in which case it gives rise to the C-bracket as the T-dual invariant bracket. The C-bracket [8, 9] is the bracket that generalizes the Lie bracket on double space.

In this paper, we analyze the general classical bosonic string $\sigma$-model and algebra of its symmetries generators, where both the background fields and symmetry parameters depend only on the coordinates $x^\mu$. We firstly consider the closed bosonic string moving



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in the background characterized solely by the metric tensor. We extend the generator of the general coordinate transformations so that it becomes invariant upon self T-duality, understood as T-duality realized in the same phase space [6]. We obtain the Courant bracket in the Poisson bracket algebra of this extended generator. The Courant bracket is therefore a self T-dual invariant extension of the Lie bracket.

Furthermore, we consider the bosonic string $\sigma$-model that includes the antisymmetric Kalb-Ramond field too. The antisymmetric field is introduced by the action of B-transformation on the generalized metric. We construct the symmetry generator and recognize that it generates both the general coordinate and the local gauge transformations [10]. In this case, the symmetry generator is not invariant upon self T-duality and it gives rise to the twisted Courant bracket. The matrix that governs this twist is exactly the matrix of B-shifts.

Next, we consider the self T-dual description of the theory, that we construct in the analogous manner, this time with the action of $\theta$-transformation, T-dual to the B-transformation. We obtain the bracket governing the generator algebra that turns out to be the $\theta$-twisted Courant bracket, also known as the Roytenberg bracket [4, 11]. The twisted Courant and Roytenberg brackets had been shown to be related by self T-duality [6].

Lastly, we consider the more conventional T-duality, connecting different phase spaces. We generalize our results, by demanding that the symmetry parameters depend on both the initial and T-dual coordinates. We consider the symmetry generator that is a sum of the generator of general coordinate transformations and its analogous generator in the T-dual phase space. In this case, additional constraints, similar to the ones in [7–9], have to be imposed on symmetry parameters, in order for the generator algebra to be closed. We extend the Poisson bracket relations for both initial and T-dual phase spaces and obtain the generator algebra, which produces the $C$-bracket. The $C$ bracket is the generalization of the Courant bracket when parameters depend on both initial and T-dual coordinates. The invariance upon T-duality is guaranteed from the way how the bracket is obtained. If parameters do not depend on T-dual coordinates, $C$-bracket reduces to the Courant bracket.

## 2 Bosonic string moving in the background characterized by the metric field

Consider the closed bosonic string, moving in the background defined by the coordinate dependent metric field $G_{\mu\nu}(x)$, with the Kalb-Ramond field set to zero $B_{\mu\nu} = 0$ and the constant dilaton field $\Phi = const$. In the conformal gauge, the Lagrangian density is given by [12, 13]

$$\mathcal{L} = \frac{\kappa}{2}\eta^{\alpha\beta}G_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{2.1}$$

where $x^\mu(\xi)$, $\mu = 0, 1, \ldots, D-1$ are coordinates on the $D$-dimensional space-time, and $\eta^{\alpha\beta}$, $\alpha, \beta = 0, 1$ is the worldsheet metric, $\epsilon^{01} = -1$ is the Levi-Civita symbol, and $\kappa = \frac{1}{2\pi\alpha'}$ with $\alpha'$ being the Regge slope parameter. The Legendre transformation of the Lagrangian gives the canonical Hamiltonian

$$\mathcal{H}_C = \pi_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu + \frac{\kappa}{2} x'^\mu G_{\mu\nu} x'^\nu \,, \tag{2.2}$$

where $\pi_\mu$ are canonical momenta conjugate to coordinates $x^\mu$, given by

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu}(x)\dot{x}^\nu \,. \tag{2.3}$$

The Hamiltonian can be rewritten in the matrix notation

$$\mathcal{H}_C = \frac{1}{2\kappa}(X^T)^M G_{MN} X^N \,, \tag{2.4}$$

where $X^M$ is a double canonical variable, given by

$$X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix} \,, \tag{2.5}$$

and $G_{MN}$ is the so called generalized metric, that in the absence of the Kalb-Ramond field takes the diagonal form

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & (G^{-1})^{\mu\nu} \end{pmatrix} \,. \tag{2.6}$$

In this paper, we firstly consider the T-duality realized without changing the phase space, which is called the self T-duality [6]. Two quantities are said to be self T-dual if they are invariant upon

$$\kappa x'^\mu \leftrightarrow \pi_\mu \,, \quad G_{\mu\nu} \leftrightarrow {}^\star G^{\mu\nu} = (G^{-1})^{\mu\nu} \,. \tag{2.7}$$

The first part of (2.7) corresponds to the T-duality interchanging the winding and momentum numbers, which are respectively obtained by integrating $\kappa x'^\mu$ and $\pi_\mu$ over the worldsheet space parameter $\sigma$ [14]. The second part of (2.7) corresponds to swapping the background fields for the T-dual background fields. Our approach gives the same expression for the T-dual metric as the usual T-dualization procedure obtained by Buscher in the special case of zero Kalb-Ramond field [15–17].

### 2.1 Symmetry generator

Let us consider symmetries of the closed bosonic string. The canonical momenta $\pi_\mu$ generate the general coordinate transformations. The generator is given by [10]

$$\mathcal{G}_{\rm GCT}(\xi) = \int_0^{2\pi} d\sigma \xi^\mu(x)\pi_\mu \,, \tag{2.8}$$

with $\xi^\mu$ being a symmetry parameter. The general coordinate transformations of the metric tensor are given by [7, 10]

$$\delta_\xi G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu} \,, \tag{2.9}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field $\xi$. Its action on the metric field is

$$\mathcal{L}_\xi G_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu \,, \tag{2.10}$$

where $D_\mu$ are covariant derivatives defined in a usual way

$$D_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho_{\mu\nu}\xi_\rho\,, \tag{2.11}$$

and $\Gamma^\mu_{\nu\rho} = \frac{1}{2}(G^{-1})^{\mu\sigma}(\partial_\nu G_{\rho\sigma} + \partial_\rho G_{\sigma\nu} - \partial_\sigma G_{\nu\rho})$ are Christoffel symbols. It is easy to verify, using the standard Poisson bracket relations

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu{}_\nu \delta(\sigma - \bar\sigma)\,, \tag{2.12}$$

that the Poisson bracket of these generators can be written as

$$\Big\{\mathcal{G}_{\rm GCT}(\xi_1),\, \mathcal{G}_{\rm GCT}(\xi_2)\Big\} = -\mathcal{G}_{\rm GCT}\Big([\xi_1, \xi_2]_L\Big)\,, \tag{2.13}$$

where $[\xi_1, \xi_2]_L$ is the Lie bracket. The Lie bracket is the commutator of two Lie derivatives

$$[\xi_1, \xi_2]_L = \mathcal{L}_{\xi_1}\xi_2 - \mathcal{L}_{\xi_2}\xi_1 \equiv \mathcal{L}_{\xi_3}\,, \tag{2.14}$$

which results in another Lie derivative along the vector $\xi_3^\mu$, given by

$$\xi_3^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu\,. \tag{2.15}$$

Let us now construct the symmetry generator that is related to the generator of general coordinate transformations by self T-duality (2.7)

$$\mathcal{G}_{LG}(\lambda) = \int_0^{2\pi} d\sigma \lambda_\mu(x)\kappa x'^\mu\,, \tag{2.16}$$

where $\lambda_\mu$ is a gauge parameter.

The symmetry parameters $\xi^\mu$ and $\lambda_\mu$ are vector and 1-form components, respectively. They can be combined in a double gauge parameter, given by

$$\Lambda^M = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix}\,. \tag{2.17}$$

The double gauge parameter is a generalized vector, defined on the direct sum of elements of tangent and cotangent bundle. Combining (2.8) and (2.16), we obtain the symmetry generator that is self T-dual (2.7)

$$\mathcal{G}(\xi, \lambda) = \mathcal{G}_{\rm GCT}(\xi) + \mathcal{G}_{LG}(\lambda) = \int_0^{2\pi} d\sigma \Big[\xi^\mu \pi_\mu + \lambda_\mu \kappa x'^\mu\Big] = \int_0^{2\pi} d\sigma (\Lambda^T)^M \eta_{MN} X^N\,, \tag{2.18}$$

where $\eta_{MN}$ is the $O(D, D)$ invariant metric [18], given by

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{2.19}$$

The expression $(\Lambda^T)^M \eta_{MN} X^N$ can be recognized as the natural inner product on the space of generalized vectors

$$\langle \Lambda^M, X^N \rangle = (\Lambda^T)^M \eta_{MN} X^N\,. \tag{2.20}$$



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We are interested in obtaining the algebra of this extended symmetry generator (2.18), analogous to (2.13). Using the Poisson bracket relations (2.12), we obtain

$$\Big\{\mathcal{G}(\xi_1,\lambda_1),\,\mathcal{G}(\xi_2,\lambda_2)\Big\} = \int d\sigma\Big[\pi_\mu(\xi_2^\nu\partial_\nu\xi_1^\mu - \xi_1^\nu\partial_\nu\xi_2^\mu) + \kappa x'^\mu(\xi_2^\nu\partial_\nu\lambda_{1\mu} - \xi_1^\nu\partial_\nu\lambda_{2\mu})\Big] + \int d\sigma d\bar\sigma\kappa\Big[\lambda_{1\mu}(\sigma)\xi_2^\mu(\bar\sigma) + \lambda_{2\mu}(\bar\sigma)\xi_1^\mu(\sigma)\Big]\partial_\sigma\delta(\sigma-\bar\sigma)\,. \tag{2.21}$$

In order to transform the anomalous part, we note that

$$\partial_\sigma\delta(\sigma-\bar\sigma) = \frac{1}{2}\partial_\sigma\delta(\sigma-\bar\sigma) - \frac{1}{2}\partial_{\bar\sigma}\delta(\sigma-\bar\sigma)\,, \tag{2.22}$$

and

$$f(\bar\sigma)\partial_\sigma\delta(\sigma-\bar\sigma) = f(\sigma)\partial_\sigma\delta(\sigma-\bar\sigma) + f'(\sigma)\delta(\sigma-\bar\sigma)\,. \tag{2.23}$$

Applying the previous two relations to the right hand side of (2.21), one obtains

$$\Big\{\mathcal{G}(\xi_1,\lambda_1),\,\mathcal{G}(\xi_2,\lambda_2)\Big\} = -\mathcal{G}(\xi,\lambda)\,, \tag{2.24}$$

where the resulting gauge parameters are given by

$$\xi^\mu = \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu\,, \tag{2.25}$$

$$\lambda_\mu = \xi_1^\nu(\partial_\nu\lambda_{2\mu} - \partial_\mu\lambda_{2\nu}) - \xi_2^\nu(\partial_\nu\lambda_{1\mu} - \partial_\mu\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\lambda_2 - \xi_2\lambda_1)\,.$$

These relations define the Courant bracket $[(\xi_1,\lambda_1),(\xi_2,\lambda_2)]_C = (\xi,\lambda)$ [1, 2], allowing us to rewrite the generator algebra (2.24)

$$\Big\{\mathcal{G}(\xi_1,\lambda_1),\,\mathcal{G}(\xi_2,\lambda_2)\Big\} = -\mathcal{G}\left(\Big[(\xi_1,\lambda_1),\;(\xi_2,\lambda_2)\Big]_{\mathcal{C}}\right)\,. \tag{2.26}$$

The Courant bracket represents the self T-dual invariant extension of the Lie bracket.

In the coordinate-free notation, the Courant bracket can be written as

$$\Big[(\xi_1,\lambda_1),(\xi_2,\lambda_2)\Big]_{\mathcal{C}} = \Big([\xi_1,\xi_2]_L,\; \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d\,(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big)\,, \tag{2.27}$$

with $i_\xi$ being the interior product along the vector field $\xi$, and $d$ being the exterior derivative. The Lie derivative $\mathcal{L}_\xi$ can be written as their anticommutator

$$\mathcal{L}_\xi = i_\xi d + d i_\xi\,. \tag{2.28}$$

The Courant bracket does not satisfy the Jacobi identity. Nevertheless, the Jacobiator of the Courant bracket is an exact 1-form [20]

$$\Big[(\xi_1,\lambda_1),\Big[(\xi_2,\lambda_2),(\xi_3,\lambda_3)\Big]_{\mathcal{C}}\Big]_{\mathcal{C}} + cycl. = d\varphi, \quad (d\varphi)_\mu = \partial_\mu\varphi\,. \tag{2.29}$$

However, if one makes the following change of parameters $\lambda_\mu \to \lambda_\mu + \partial_\mu\varphi$, the generator (2.18) does not change

$$\mathcal{G}(\xi,\lambda+\partial\varphi) = \mathcal{G}(\xi,\lambda) + \kappa\int_0^{2\pi}\varphi' d\sigma = \mathcal{G}(\xi,\lambda)\,, \tag{2.30}$$



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since the total derivative integral vanishes for the closed string. Therefore, the deviation from Jacobi identity contributes to the trivial symmetry, and we say that the symmetry is reducible.

The theory with the metric tensor was already discussed in [19], where it was proven that the invariance under both diffeomorphisms and dual diffeomorphisms requires the introduction of the Kalb-Ramond field. In our approach, if we want to include the T-dual of the general coordinate transformation in the same theory, we obtain the local gauge transformation that constitutes a trivial symmetry, since $\delta_\lambda G_{\mu\nu} = 0$ [7, 10]. Therefore, it is necessary to include the Kalb-Ramond field, in order to have non-trivial local gauge transformations, which we do in the next section.

## 3 Bosonic string moving in the background characterized by the metric field and the Kalb-Ramond field

In this section, we extend the Hamiltonian so that it includes the antisymmetric Kalb-Ramond field. It is possible to obtain this Hamiltonian from the transformation of the generalized metric $G_{MN}$ (2.6) under the so called B-transformations. The B-transformations (or B-shifts) [20] are realized by $e^{\hat{B}}$, where

$$\hat{B}^M{}_N = \begin{pmatrix} 0 & 0 \\ 2B_{\mu\nu} & 0 \end{pmatrix} . \tag{3.1}$$

As a result of $\hat{B}^2 = 0$, the full transformation is easily obtained

$$(e^{\hat{B}})^M{}_N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta_\mu^\nu \end{pmatrix} . \tag{3.2}$$

Its transpose is given by

$$((e^{\hat{B}})^T)_M{}^N = \begin{pmatrix} \delta_\mu^\nu & -2B_{\mu\nu} \\ 0 & \delta^\mu_\nu \end{pmatrix} , \tag{3.3}$$

from which it is easy to verify that

$$((e^{\hat{B}})^T)_M{}^K \, \eta_{KL} \, (e^{\hat{B}})^L{}_N = \eta_{MN} , \tag{3.4}$$

meaning they are the elements of the $O(D, D)$ group.

The transformation of generalized metric $G_{MN}$ (2.6) under the B-shifts is given by

$$G_{MN} \to ((e^{\hat{B}})^T)_M{}^K \, G_{KQ} \, (e^{\hat{B}})^Q{}_N \equiv H_{MN} , \tag{3.5}$$

where $H_{MN}$ is the generalized metric

$$H_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix} , \tag{3.6}$$

and $G^E_{\mu\nu}$ is the effective metric perceived by the open strings, given by

$$G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu} . \tag{3.7}$$

It is straightforward to write the canonical Hamiltonian

$$\begin{aligned}\hat{\mathcal{H}}_C &= \frac{1}{2\kappa}(X^T)^M H_{MN} X^N \\ &= \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu - 2x'^\mu B_{\mu\rho}(G^{-1})^{\rho\nu}\pi_\nu \,,\end{aligned} \tag{3.8}$$

as well as the Lagrangian in the canonical form

$$\begin{aligned}\hat{\mathcal{L}}(\dot{x}, x', \pi) &= \pi_\mu \dot{x}^\mu - \hat{\mathcal{H}}_C(x', \pi) \\ &= \pi_\mu \dot{x}^\mu - \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu - \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu + 2x'^\mu B_{\mu\rho}(G^{-1})^{\rho\nu}\pi_\nu .\end{aligned} \tag{3.9}$$

On the equations of motion for $\pi_\mu$, we obtain

$$\pi_\mu = \kappa G_{\mu\nu}\dot{x}^\nu - 2\kappa B_{\mu\nu}x'^\nu \,. \tag{3.10}$$

Substituting (3.10) into (3.9) we find the well known expression for bosonic string Lagrangian [12, 13]

$$\begin{aligned}&\hat{\mathcal{L}}(\dot{x}, x') = \frac{\kappa}{2}\dot{x}^\mu G_{\mu\nu}\dot{x}^\nu - \frac{\kappa}{2}x'^\mu G_{\mu\nu}x'^\nu - 2\kappa\dot{x}^\mu B_{\mu\nu}x'^\nu = \kappa\partial_+ x^\mu \Pi_{+\mu\nu}\partial_- x^\nu \,, \\ &\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu} \,, \ \partial_\pm x^\mu = \dot{x}^\mu \pm x'^\mu \,.\end{aligned} \tag{3.11}$$

It is possible to rewrite the canonical Hamiltonian (3.8) in terms of the generalized metric $G_{MN}$, that characterizes background with the metric only tensor. Substituting (3.5) into (3.8), we obtain

$$\hat{\mathcal{H}}_C = \frac{1}{2\kappa}(X^T)^M ((e^{\hat{B}})^T)_M^{\ K}\, G_{KL}\, (e^{\hat{B}})^L_{\ N} X^N = \frac{1}{2\kappa}(\hat{X}^T)^M\, G_{MN}\, \hat{X}^N \,, \tag{3.12}$$

where

$$\hat{X}^M = (e^{\hat{B}})^M_{\ N}\, X^N = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu \end{pmatrix} \equiv \begin{pmatrix} \kappa x'^\mu \\ i_\mu \end{pmatrix} , \tag{3.13}$$

with $i_\mu$ being the auxiliary current, given by

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu \,. \tag{3.14}$$

The algebra of auxiliary currents $i_\mu$ gives rise to the $H$-flux [6]

$$\{i_\mu(\sigma), i_\nu(\bar{\sigma})\} = -2\kappa B_{\mu\nu\rho}x'^\rho \delta(\sigma - \bar{\sigma}) \,, \tag{3.15}$$

where the structural constants are the Kalb-Ramond field strength components, given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \,. \tag{3.16}$$

### 3.1 Symmetry generator

Let us extend the symmetry transformations of the background fields for the theory with the non-trivial Kalb-Ramond field. The infinitesimal general coordinate transformations of the background fields are given by

$$\delta_\xi G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu}\,, \;\; \delta_\xi B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu}\,, \tag{3.17}$$

where the action of the Lie derivative $\mathcal{L}_\xi$ (2.28) on the Kalb-Ramond field is given by [10]

$$\mathcal{L}_\xi B_{\mu\nu} = \xi^\rho \partial_\rho B_{\mu\nu} + \partial_\mu \xi^\rho B_{\rho\nu} - \partial_\nu \xi^\rho B_{\rho\mu}\,, \tag{3.18}$$

while its action on the metric field is the same as in (2.10). The local gauge transformations of the background fields are [10]

$$\delta_\lambda G_{\mu\nu} = 0\,, \;\; \delta_\lambda B_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu\,. \tag{3.19}$$

Rewriting the symmetry generator $\mathcal{G}(\xi,\lambda)$ (2.18) in terms of the basis defined by components of $\hat{X}^M$ (3.13), one obtains

$$\begin{aligned}\mathcal{G}(\xi,\lambda) &= \int d\sigma (\Lambda^T)^M \eta_{MN} X^N = \int d\sigma (\hat{\Lambda}^T)^M ((e^{-\hat{B}})^T)_M^{\ K} \eta_{KL} (e^{-\hat{B}})^L_{\ N} \hat{X}^N \\ &= \int d\sigma (\hat{\Lambda}^T)^M \eta_{MN} \hat{X}^N\,,\end{aligned} \tag{3.20}$$

where (3.4) was used in the last step, and $\hat{\Lambda}^M$ is a new double gauge parameter, given by

$$\hat{\Lambda}^M = (e^{\hat{B}})^M_{\ N} \Lambda^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu \\ \lambda_\mu + 2B_{\mu\nu}\xi^\nu \end{pmatrix} \equiv \begin{pmatrix} \xi^\mu \\ \hat{\lambda}_\mu \end{pmatrix}. \tag{3.21}$$

We are going to mark the right hand side of (3.20) as a new generator

$$\mathcal{G}^{\hat{B}}(\xi,\hat{\lambda}) = \int d\sigma \Big[\xi^\mu i_\mu + \hat{\lambda}_\mu \kappa x'^\mu\Big]\,, \tag{3.22}$$

which equals the generator (2.18), when the relations between the gauge parameters (3.21) are satisfied $\mathcal{G}(\xi, \hat{\lambda} - 2B_{\mu\nu}\xi^\nu) = \mathcal{G}^{\hat{B}}(\xi,\hat{\lambda})$. The expression (3.22) exactly corresponds to the symmetry generator obtained in [10], where $\xi^\mu$ are parameters of general coordinate transformations and $\hat{\lambda}_\mu$ are parameters of local gauge transformations, that respectively correspond to transformations of the background fields (3.17) and (3.19).

Our goal is to obtain the algebra in the form

$$\Big\{\mathcal{G}^{\hat{B}}(\xi_1,\hat{\lambda}_1), \mathcal{G}^{\hat{B}}(\xi_2,\hat{\lambda}_2)\Big\} = -\mathcal{G}^{\hat{B}}(\xi,\hat{\lambda})\,, \tag{3.23}$$

where

$$\lambda_{i\mu} = \hat{\lambda}_{i\mu} - 2B_{\mu\nu}\xi_i^\nu\,, \;\; i = 1,2\,; \;\;\; \lambda_\mu = \hat{\lambda}_\mu - 2B_{\mu\nu}\xi^\nu\,, \tag{3.24}$$

due to (3.21). The Poisson bracket between canonical variables (2.12) remains the same after the introduction of the Kalb-Ramond field. Therefore the results from previous section, as well as mutual relations between coefficients in different bases can be used to obtain the algebra (3.23). Firstly, substituting (3.24) into the second equation in (2.25), one obtains

$$\lambda_\mu = \xi_1^\nu(\partial_\nu\hat\lambda_{2\mu} - \partial_\mu\hat\lambda_{2\nu}) - \xi_2^\nu(\partial_\nu\hat\lambda_{1\mu} - \partial_\mu\hat\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\hat\lambda_2 - \xi_2\hat\lambda_1) \\ +2B_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho - 2B_{\mu\nu}(\xi_1^\rho\partial_\rho\xi_2^\nu - \xi_2^\rho\partial_\rho\xi_1^\nu)\,. \tag{3.25}$$

Secondly, substituting the previous equation in (3.21), one obtains

$$\xi^\mu = \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu\,, \\ \hat\lambda_\mu = \xi_1^\nu(\partial_\nu\hat\lambda_{2\mu} - \partial_\mu\hat\lambda_{2\nu}) - \xi_2^\nu(\partial_\nu\hat\lambda_{1\mu} - \partial_\mu\hat\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\hat\lambda_2 - \xi_2\hat\lambda_1) + 2B_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho\,. \tag{3.26}$$

The above relations define the twisted Courant bracket $[(\xi_1,\hat\lambda_1),(\xi_2,\hat\lambda_2)]_{\mathcal{C}_B} = (\xi,\hat\lambda)$ [21]. This is the bracket of the symmetry transformations

$$\left\{\mathcal{G}^{\hat B}(\xi_1,\hat\lambda_1),\,\mathcal{G}^{\hat B}(\xi_2,\hat\lambda_2)\right\} = -\mathcal{G}^{\hat B}\left(\Big[(\xi_1,\hat\lambda_1),(\xi_2,\hat\lambda_2)\Big]_{\mathcal{C}_B}\right)\,, \tag{3.27}$$

in the theory defined by both metric and Kalb-Ramond field.

In the coordinate free notation, the twisted Courant bracket is given by

$$\Big[(\xi_1,\hat\lambda_1),(\xi_2,\hat\lambda_2)\Big]_{\mathcal{C}_B} = \Big([\xi_1,\xi_2]_L, \mathcal{L}_{\xi_1}\hat\lambda_2 - \mathcal{L}_{\xi_2}\hat\lambda_1 - \frac{1}{2}d(i_{\xi_1}\hat\lambda_2 - i_{\xi_2}\hat\lambda_1) + H(\xi_1,\xi_2,.)\Big)\,, \tag{3.28}$$

where $H(\xi_1,\xi_2,.)$ represents the contraction of the $H$-flux $H = dB$ (3.16) with two gauge parameters $\xi_1$ and $\xi_2$. This term is the corollary of the non-commutativity of the auxiliary currents $i_\mu$ (3.15), due to twisting of the Courant bracket with the Kalb-Ramond field. In special case when the Kalb-Ramond field $B$ is a closed form $dB = 0$, the twisted Courant bracket (3.28) reduces to the Courant bracket (2.27). This can also be seen from the well known fact that $B$-shifts (3.2) are symmetries of the Courant bracket when $B$ is a closed form [20].

## 4 Courant bracket twisted by $\theta^{\mu\nu}$

When both the metric and the Kalb-Ramond field are present in the theory, the expressions for T-dual fields are given by [15]

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad ^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}\,, \tag{4.1}$$

where $\theta^{\mu\nu}$ is the non-commutativity parameter for the string endpoints on a D-brane [22], given by

$$\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1})^{\mu\rho}B_{\rho\sigma}(G^{-1})^{\sigma\nu}\,. \tag{4.2}$$

We say that two quantities are self T-dual, if they are invariant under the interchange [6]

$$\pi_\mu \leftrightarrow \kappa x'^\mu\,,\; G_{\mu\nu} \leftrightarrow (G_E^{-1})^{\mu\nu}\,,\; B_{\mu\nu} \leftrightarrow \frac{\kappa}{2}\theta^{\mu\nu}\,. \tag{4.3}$$

When the Kalb-Ramond field is set to zero $B_{\mu\nu} = 0$, (4.3) reduces to the self T-duality transformation laws in the background without the $B$ field (2.7).

From the relations (4.3), it is apparent that the introduction of Kalb-Ramond field breaks down the self T-duality invariance of the symmetry generator (3.22). To find a new self T-dual invariant generator, we will analogously to the prior construction start with the background containing only T-dual metric. The Hamiltonian in the metric only background, similar to (2.2), reads

$$
{}^\star\mathcal{H}_C = \frac{1}{2\kappa}\pi_\mu (G_E^{-1})^{\mu\nu}\pi_\nu + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu = (X^T)^M\, {}^\star G_{MN} X^N\,, \tag{4.4}
$$

where ${}^\star G_{MN}$ is the T-dual generalized metric for the above Hamiltonian, given by

$$
{}^\star G_{MN} = \begin{pmatrix} G^E_{\mu\nu} & 0 \\ 0 & (G_E^{-1})^{\mu\nu} \end{pmatrix}. \tag{4.5}
$$

Note that the self T-duality is realized as the joint action of the permutation of the coordinate $\sigma$-derivatives with the canonical momenta and the swapping all the fields in (2.6) for their T-duals. This is equivalent to the Buscher's procedure [15–17], when it is done in the same phase space.

In order to construct the Hamiltonian in the self T-dual description, we consider how the T-dual generalized metric (4.5) is transformed with respect to the so called $\theta$-transformations $e^{\hat\theta}$, where

$$
\hat\theta^M{}_N = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2\,{}^\star B^{\mu\nu} \\ 0 & 0 \end{pmatrix}. \tag{4.6}
$$

The full exponential $e^{\hat\theta}$ is given by

$$
(e^{\hat\theta})^M{}_N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta_\mu^\nu \end{pmatrix}, \tag{4.7}
$$

and its transpose by

$$
((e^{\hat\theta})^T)_M{}^N = \begin{pmatrix} \delta_\mu^\nu & 0 \\ -\kappa\theta^{\mu\nu} & \delta^\mu_\nu \end{pmatrix}. \tag{4.8}
$$

They are elements of the $O(D,D)$ group as well, i.e.

$$
((e^{\hat\theta})^T)_M{}^L\, \eta_{LK}\, (e^{\hat\theta})^K{}_N = \eta_{MN}\,. \tag{4.9}
$$

Under (4.7), the T-dual generalized metric (4.5) transforms in the following way

$$
{}^\star G_{MN} \to\ ((e^{\hat\theta})^T)_M{}^L\, {}^\star G_{LK}(e^{\hat\theta})^K{}_N \equiv\ {}^\star H_{MN}\,, \tag{4.10}
$$

where

$$
{}^\star H_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}, \tag{4.11}
$$

which is exactly equal to the generalized metric (3.6). From it we can write the T-dual Hamiltonian

$$
\begin{aligned}
{}^\star\mathcal{H}_C &= \frac{1}{2\kappa}(X^T)^M\, {}^\star H_{MN} X^N \qquad\qquad (4.12)\\
&= \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu - 2x'^\mu B_{\mu\rho}(G^{-1})^{\rho\nu}\pi_\nu \equiv \hat{\mathcal{H}}_C\,.
\end{aligned}
$$

The canonical Lagrangian is given by

$$\begin{aligned}{}^\star\mathcal{L}(\pi,\dot{x},x) &= \pi_\mu \dot{x}^\mu - {}^\star\mathcal{H}_C(x',\pi) \\ &= \pi_\mu \dot{x}^\mu - \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu - \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu + 2x'^\mu B_{\mu\rho}(G^{-1})^{\rho\nu}\pi_\nu\,,\end{aligned} \tag{4.13}$$

from which one easily obtains

$$\pi_\mu = \kappa G_{\mu\nu}\dot{x}^\nu - 2\kappa B_{\mu\nu}x'^\nu\,. \tag{4.14}$$

We see that the canonical momentum remains the same, which is expected, since the self T-duality is realized in the same phase space. Substituting (4.14) into (4.13), one obtains

$${}^\star\mathcal{L}(\dot{x},x) = \frac{\kappa}{2}\dot{x}^\mu G_{\mu\nu}\dot{x}^\nu - \frac{\kappa}{2}x'^\mu G_{\mu\nu}x'^\nu - 2\kappa\dot{x}^\mu B_{\mu\nu}x'^\nu = \kappa\partial_+ x^\mu \Pi_{+\mu\nu}\partial_- x^\nu\,. \tag{4.15}$$

It is obvious that both the Hamiltonian and the Lagrangian are invariant under the self T-duality.

In the same manner as in the previous section, substituting (4.10) into (4.12), we rewrite the Hamiltonian

$${}^\star\hat{\mathcal{H}}_C = \frac{1}{2\kappa}(X^T)_M^{\ L}\,((e^{\hat{\theta}})^T)_L^{\ K}\,{}^\star G_{KJ}\,(e^{\hat{\theta}})^J_{\ N}\,X^N = \frac{1}{2\kappa}\tilde{X}^M\,{}^\star G_{MN}\tilde{X}^N\,, \tag{4.16}$$

where

$$\tilde{X}^M = (e^{\hat{\theta}})^M_{\ N}X^N = \begin{pmatrix}\delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu\end{pmatrix}\begin{pmatrix}\kappa x'^\nu \\ \pi_\nu\end{pmatrix} = \begin{pmatrix}\kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu \\ \pi_\mu\end{pmatrix} \equiv \begin{pmatrix}k^\mu \\ \pi_\mu\end{pmatrix}\,, \tag{4.17}$$

and $k^\mu$ is an auxiliary current, given by

$$k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu\,. \tag{4.18}$$

The Poisson bracket algebra of these currents is obtained in [6]

$$\{k^\mu(\sigma), k^\nu(\bar{\sigma})\} = -\kappa Q_\rho^{\ \mu\nu}k^\rho\delta(\sigma-\bar{\sigma}) - \kappa^2 R^{\mu\nu\rho}\pi_\rho\delta(\sigma-\bar{\sigma})\,, \tag{4.19}$$

where $Q$ and $R$ are non-geometric fluxes [23], given by

$$Q_\rho^{\ \mu\nu} = \partial_\rho\theta^{\mu\nu}, \qquad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}\,. \tag{4.20}$$

We now define a new double gauge parameter

$$\tilde{\Lambda}^M = (e^{\hat{\theta}})^M_{\ N}\Lambda^N = \begin{pmatrix}\delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu\end{pmatrix}\begin{pmatrix}\xi^\nu \\ \lambda_\nu\end{pmatrix} = \begin{pmatrix}\xi^\mu + \kappa\theta^{\mu\nu}\lambda_\nu \\ \lambda_\mu\end{pmatrix} \equiv \begin{pmatrix}\hat{\xi}^\mu \\ \lambda_\mu\end{pmatrix}\,. \tag{4.21}$$

The generator (2.18) written in terms of new gauge parameters $\mathcal{G}(\hat{\xi}-\kappa\theta\lambda,\lambda) \equiv \mathcal{G}^{\hat{\theta}}(\hat{\xi},\lambda)$ is given by

$$\mathcal{G}^{\hat{\theta}}(\hat{\xi},\lambda) = \int d\sigma\Big[\hat{\xi}^\mu\pi_\mu + \lambda_\mu k^\mu\Big]\,. \tag{4.22}$$

The auxiliary currents $i_\mu$ (3.14) and $k^\mu$ (4.18) are related by the self T-duality relations (4.3). Moreover, one easily demonstrates that the self T-dual image of the generator $\mathcal{G}^{\hat{B}}$ (3.22) is the generator $\mathcal{G}^{\hat{\theta}}$ (4.22).

Like in a previous case, we want to obtain the algebra in the form

$$\left\{\mathcal{G}^{\hat{\theta}}(\hat{\xi}_1,\lambda_1),\, \mathcal{G}^{\hat{\theta}}(\hat{\xi}_2,\lambda_2)\right\} = -\mathcal{G}^{\hat{\theta}}(\hat{\xi},\lambda)\,, \tag{4.23}$$

where from (4.21) we read the relations between the old and new gauge parameters

$$\xi_i^\mu = \hat{\xi}_i^\mu - \kappa\theta^{\mu\nu}\lambda_{i\nu}\,,\ i=1,2;\quad \xi^\mu = \hat{\xi}^\mu - \kappa\theta^{\mu\nu}\lambda_\nu\,. \tag{4.24}$$

Combining (4.24), (2.25) and (4.21), one obtains

$$\begin{aligned}
\hat{\xi}^\mu &= \hat{\xi}_1^\nu\partial_\nu\hat{\xi}_2^\mu - \hat{\xi}_2^\nu\partial_\nu\hat{\xi}_1^\mu + \\
&+\kappa\theta^{\mu\nu}\left(\hat{\xi}_1^\rho(\partial_\nu\lambda_{2\rho}-\partial_\rho\lambda_{2\nu}) - \hat{\xi}_2^\rho(\partial_\nu\lambda_{1\rho}-\partial_\rho\lambda_{1\nu}) - \frac{1}{2}\partial_\nu(\hat{\xi}_1\lambda_2-\hat{\xi}_2\lambda_1)\right)\\
&+\kappa\hat{\xi}_1^\nu\partial_\nu(\lambda_{2\rho}\theta^{\rho\mu}) - \kappa\hat{\xi}_2^\nu\partial_\nu(\lambda_{1\rho}\theta^{\rho\mu}) + \kappa(\lambda_{1\nu}\theta^{\nu\rho})\partial_\rho\hat{\xi}_2^\mu - \kappa(\lambda_{2\nu}\theta^{\nu\rho})\partial_\rho\hat{\xi}_1^\mu\\
&+\kappa^2 R^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\,,\\
\lambda_\mu &= \hat{\xi}_1^\nu(\partial_\nu\lambda_{2\mu}-\partial_\mu\lambda_{2\nu}) - \hat{\xi}_2^\nu(\partial_\nu\lambda_{1\mu}-\partial_\mu\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\hat{\xi}_1\lambda_2-\hat{\xi}_2\lambda_1)\\
&+\kappa\theta^{\nu\rho}(\lambda_{1\nu}\partial_\rho\lambda_{2\mu}-\lambda_{2\nu}\partial_\rho\lambda_{1\mu}) + \kappa\lambda_{1\rho}\lambda_{2\nu}Q_\mu{}^{\rho\nu}\,.
\end{aligned} \tag{4.25}$$

The relations (4.25) define a bracket $[(\hat{\xi}_1,\lambda_1),(\hat{\xi}_2,\lambda_2)]_{\mathcal{C}_\theta} = (\hat{\xi},\lambda)$ that is known as the $\theta$-twisted Courant bracket, or Roytenberg bracket. It is related by self T-duality with the twisted Courant bracket, when the relations between the fields (4.1) hold [6].

In the coordinate free notation, the $\theta$-twisted Courant bracket is given by

$$\begin{aligned}
\Big[(\hat{\xi}_1,\lambda_1),(\hat{\xi}_2,\lambda_2)\Big]_{\mathcal{C}_\theta} =\Big(&[\hat{\xi}_1,\hat{\xi}_2]_L - \kappa[\hat{\xi}_2,\lambda_1\theta]_L + \kappa[\hat{\xi}_1,\lambda_2\theta]_L + \frac{\kappa^2}{2}[\theta,\theta]_S(\lambda_1,\lambda_2,.)\\
&+\kappa\left(\mathcal{L}_{\hat{\xi}_2}\lambda_1 - \mathcal{L}_{\hat{\xi}_1}\lambda_2 + \frac{1}{2}d(i_{\hat{\xi}_1}\lambda_2 - i_{\hat{\xi}_2}\lambda_1)\right)\theta\\
&\mathcal{L}_{\hat{\xi}_1}\lambda_2 - \mathcal{L}_{\hat{\xi}_2}\lambda_1 - \frac{1}{2}d(i_{\hat{\xi}_1}\lambda_2 - i_{\hat{\xi}_2}\lambda_1) - \kappa[\lambda_1,\lambda_2]_\theta\Big)\,,
\end{aligned} \tag{4.26}$$

where $[\theta,\theta]_S(\lambda_1,\lambda_2,.)$ represents the Schouten-Nijenhuis bracket [24] contracted with two 1-forms, that when having bi-vectors as domain is given by

$$[\theta,\theta]_S|^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}_{\alpha\beta\gamma}\theta^{\sigma\alpha}\partial_\sigma\theta^{\beta\gamma} = 3R^{\mu\nu\rho}\,, \tag{4.27}$$

where

$$\epsilon^{\mu\nu\rho}_{\alpha\beta\gamma} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\beta & \delta^\rho_\gamma \\ \delta^\nu_\alpha & \delta^\rho_\beta & \delta^\mu_\gamma \\ \delta^\rho_\alpha & \delta^\mu_\beta & \delta^\nu_\gamma \end{vmatrix}\,, \tag{4.28}$$

and $[\lambda_1,\lambda_2]_\theta$ is the Koszul bracket [25] given by

$$[\lambda_1,\lambda_2]_\theta = \mathcal{L}_{\theta\lambda_1}\lambda_2 - \mathcal{L}_{\theta\lambda_2}\lambda_1 + d(\theta(\lambda_1,\lambda_2))\,. \tag{4.29}$$

The Koszul bracket is a generalization of the Lie bracket on the space of differential forms, while the Schouten-Nijenhuis bracket is a generalization of the Lie bracket on the space of multi-vectors.



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## 5 C-bracket

In this section, we will show how our results can be generalized, so that they give rise to the $C$-bracket [8, 9] as the T-dual invariant bracket, in the accordance with [7]. Consider that T-dual theory is defined in the T-dual phase space, characterized by T-dual coordinates $y_\mu$ and the T-dual momenta $^\star\pi^\mu$. They are related with the initial phase space by T-duality relations [15]

$$\pi_\mu \simeq \kappa y'_\mu \,, \qquad {}^\star\pi^\mu \simeq \kappa x'^\mu \,. \tag{5.1}$$

We can define a double phase space obtained as a sum of two canonical phase spaces. Let us introduce the double coordinate

$$X^M = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix} \,, \tag{5.2}$$

as well as the double canonical momentum

$$\Pi_M = \begin{pmatrix} \pi_\mu \\ {}^\star\pi^\mu \end{pmatrix} \,. \tag{5.3}$$

In this notation, the T-duality laws (5.1) take a form

$$\Pi_M \simeq \kappa\, \eta_{MN} X'^M \,, \tag{5.4}$$

where $\eta_{MN}$ is the $O(D,D)$ metric (2.19).

### 5.1 Poisson brackets of canonical variables

The standard Poisson bracket algebra is assumed for both initial and T-dual phase space

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu \delta(\sigma-\bar\sigma) \,, \qquad \{y_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = \delta^\nu_\mu \delta(\sigma-\bar\sigma) \,, \tag{5.5}$$

with other bracket of canonical variables within the same phase space being zero.

For the remaining Poisson bracket relations, one must use the consistency with T-duality relations. Firstly, applying the T-dualization along all initial coordinates $x^\mu$, i.e. the second relation of (5.1) on the Poisson bracket algebra between coordinates derivatives, one obtains

$$\{\kappa x'^\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} \simeq \{{}^\star\pi^\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} = \kappa \delta^\mu_\nu \delta'(\sigma-\bar\sigma) \,. \tag{5.6}$$

Similarly, applying the T-dualization along all T-dual coordinates $y_\mu$, i.e. the first relation of (5.1), one obtains

$$\{\kappa x'^\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} \simeq \{\kappa x'^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \kappa \delta^\mu_\nu \delta'(\sigma-\bar\sigma) \,. \tag{5.7}$$

Hence, we conclude

$$\{\kappa x'^\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} = \kappa \delta^\mu_\nu \delta'(\sigma-\bar\sigma) \,. \tag{5.8}$$

The successive integration along both $\sigma$ and $\bar\sigma$ for the appropriate choice of the integration constant produces the relation [26]

$$\{\kappa x^\mu(\sigma), \kappa y_\nu(\bar\sigma)\} = -\kappa \delta^\mu_\nu \theta(\sigma-\bar\sigma) \,, \tag{5.9}$$



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where

$$\theta(\sigma)=\begin{cases}-\frac{1}{2} & \sigma=-\pi\\ 0 & -\pi<\sigma<\pi\\ \frac{1}{2} & \sigma=\pi\end{cases}. \tag{5.10}$$

Secondly, taking into the account T-duality (5.1), the Poisson bracket algebra of momenta is easily transformed into

$$\{\pi_\mu(\sigma),{}^\star\pi^\nu(\bar\sigma)\}\simeq\kappa\{\pi_\mu(\sigma),x'^\nu(\bar\sigma)\}=\kappa\delta_\mu^\nu\delta'(\sigma-\bar\sigma)\,, \tag{5.11}$$

when T-dualization is applied along the coordinates $y_\mu$, and

$$\{\pi_\mu(\sigma),{}^\star\pi^\nu(\bar\sigma)\}\simeq\kappa\{y'_\mu(\sigma),{}^\star\pi^\nu(\bar\sigma)\}=\kappa\delta_\mu^\nu\delta'(\sigma-\bar\sigma)\,, \tag{5.12}$$

when it is applied along the coordinates $x^\mu$. As in the previous case, we obtain

$$\{\pi_\mu(\sigma),{}^\star\pi^\nu(\bar\sigma)\}=\kappa\delta_\mu^\nu\delta'(\sigma-\bar\sigma)\,. \tag{5.13}$$

In a same manner, it is easy to demonstrate that

$$\{x^\mu(\sigma),{}^\star\pi^\nu(\bar\sigma)\}=0\,,\;\;\{y_\mu(\sigma),\pi_\nu(\bar\sigma)\}=0\,. \tag{5.14}$$

In a double space, the above relations can be simply written as

$$\Big\{\kappa X^M(\sigma),\kappa X^N(\bar\sigma)\Big\}=-\kappa\eta^{MN}\theta(\sigma-\bar\sigma)\,,\quad\{\Pi_M(\sigma),\Pi_N(\bar\sigma)\}=\kappa\,\eta_{MN}\delta'(\sigma-\bar\sigma)\,. \tag{5.15}$$

### 5.2 Generator in double space

Now let us extend the generator of general coordinate transformations, so that it includes the T-dual version of that generator

$$G(\xi,\lambda)=\int d\sigma\mathcal{G}(\xi,\lambda)=\int d\sigma\Big[\xi^\mu(x,y)\pi_\mu+\lambda_\mu(x,y){}^\star\pi^\mu\Big]\,, \tag{5.16}$$

where the symmetry parameters $\xi$ and $\lambda$ depend on both initial coordinates $x^\mu$ and T-dual coordinates $y_\mu$. The generator $\mathcal{G}(\xi,\lambda)$ can be rewritten in terms of double canonical variables as

$$\mathcal{G}(\Lambda)=\Lambda^M(x,y)\eta_{MN}\Pi^N\qquad\Longleftrightarrow\qquad\mathcal{G}_\Lambda=\langle\Lambda,\Pi\rangle\,, \tag{5.17}$$

where

$$\Lambda^M(X)=\begin{pmatrix}\xi^\mu(x^\mu,y_\mu)\\ \lambda_\mu(x^\mu,y_\mu)\end{pmatrix}. \tag{5.18}$$

This generator is manifestly $O(D,D)$ invariant.

We are interested in the algebra of the form

$$\{G(\Lambda_1),G(\Lambda_2)\}=-G(\Lambda)\,. \tag{5.19}$$

To obtain it, it is convenient to introduces double derivative

$$\partial_M=\begin{pmatrix}\partial_\mu\\ \tilde\partial^\mu\end{pmatrix}\qquad\left(\partial_\mu\equiv\frac{\partial}{\partial x^\mu}\,,\quad\tilde\partial^\mu\equiv\frac{\partial}{\partial y_\mu}\right), \tag{5.20}$$

so that the following Poisson bracket relations can be written

$$\Big\{\Lambda^M(\sigma),\Pi_N(\bar\sigma)\Big\}=\partial_N\Lambda^M\delta(\sigma-\bar\sigma),\Big\{\Lambda^M(\sigma),\Lambda^N(\bar\sigma)\Big\}=-\frac{1}{\kappa}\,\partial^P\Lambda_1^M\partial_P\Lambda_2^N\theta(\sigma-\bar\sigma)\,. \quad (5.21)$$

The second relation makes the situation more complicated, since it would result in the symmetry algebra not closing on another generator. However, in the accordance with [7, 9], we can consider restricted parameters on isotropic spaces, for which $\Delta=\eta^{PQ}\partial_P\partial_Q=\partial^Q\partial_Q$ anihilates all gauge parameters, as well as their products. Therefore, we write

$$\Delta\left(\Lambda_1^M\Lambda_2^N\right)=\Delta\Lambda_1^M\Lambda_2^N+2\partial_Q\Lambda_1^M\partial^Q\Lambda_2^N+\Lambda_1^M\Delta\Lambda_2^N=0\,, \quad (5.22)$$

from which one obtains

$$\partial_Q\Lambda_1^M\partial^Q\Lambda_2^N=0\,. \quad (5.23)$$

Substituting (5.23) into (5.21), we obtain

$$\Big\{\Lambda^M(\sigma),\Lambda^N(\bar\sigma)\Big\}=0\,. \quad (5.24)$$

We see that the restriction of gauge parameters to isotropic spaces is necessary for the algebra of generator (5.16) to be closed.

Now we are ready to calculate the algebra. Using the second relation of (5.15), the first relation of (5.21), and (5.24), we have

$$\{\mathcal{G}_{\Lambda_1}(\sigma),\mathcal{G}_{\Lambda_2}(\bar\sigma)\}=-\Big(\Lambda_1^N\partial_N\Lambda_2^M-\Lambda_2^N\partial_N\Lambda_1^M\Big)\Pi_M\delta(\sigma-\bar\sigma)+\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma)\,. \quad (5.25)$$

Using (2.23), the anomalous term can be rewritten as

$$\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma)=\kappa\langle\Lambda_1(\sigma),\Lambda_2(\sigma)\rangle\delta'(\sigma-\bar\sigma)+\kappa\langle\Lambda_1(\sigma),\Lambda_2'(\sigma)\rangle\delta(\sigma-\bar\sigma)\,, \quad (5.26)$$

which with the help of (2.22) can be further transformed into

$$\begin{aligned}\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma)=&\,\frac{\kappa}{2}\Big(\langle\Lambda_1,\Lambda_2'\rangle-\langle\Lambda_1',\Lambda_2\rangle\Big)\delta(\sigma-\bar\sigma)\\&+\frac{\kappa}{2}\Big(\langle\Lambda_1,\Lambda_2\rangle(\sigma)+\langle\Lambda_1,\Lambda_2\rangle(\bar\sigma)\Big)\delta'(\sigma-\bar\sigma)\,,\end{aligned} \quad (5.27)$$

where the dependence of $\sigma$ has been omitted, where all terms depend solely on it.

Next, we write

$$\kappa\Lambda'^M=\kappa\,X'^N\partial_N\Lambda^M\,, \quad (5.28)$$

and with the help of (5.4)

$$\kappa\Lambda'^M\simeq\eta^{NR}\Pi_R\partial_N\Lambda^M\,. \quad (5.29)$$

The full anomalous term can now be written as

$$\begin{aligned}\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma)=&\,\frac{1}{2}\eta_{PQ}\,\eta^{MN}\Big(\Lambda_1^P\partial_N\Lambda_2^Q-\Lambda_2^P\partial_N\Lambda_1^Q\Big)\Pi_M\delta(\sigma-\bar\sigma)\\&+\frac{\kappa}{2}\Big(\langle\Lambda_1,\Lambda_2\rangle(\sigma)+\langle\Lambda_1,\Lambda_2\rangle(\bar\sigma)\Big)\delta'(\sigma-\bar\sigma)\,.\end{aligned} \quad (5.30)$$

The second line of the previous equation disappears after the integration with respect to $\sigma$ and $\bar{\sigma}$.

Consequently,

$$\Big\{G_{\Lambda_1}(\sigma), G_{\Lambda_2}(\bar{\sigma})\Big\} = -\Big[\Lambda_1^N \partial_N \Lambda_2^M - \Lambda_2^N \partial_N \Lambda_1^M \\ -\frac{1}{2}\eta_{PQ}\,\eta^{MN}\Big(\Lambda_1^P \partial_N \Lambda_2^Q - \Lambda_2^P \partial_N \Lambda_1^Q\Big)\Big]\Pi_M \delta(\sigma-\bar{\sigma})\,. \tag{5.31}$$

We recognize that we can write the relation (5.19) as

$$\{G(\Lambda_1), G(\Lambda_2)\} = -G\Big([\Lambda_1, \Lambda_2]_C\Big)\,, \tag{5.32}$$

where $[\Lambda_1, \Lambda_2]_C$ is the $C$-bracket, given by

$$[\Lambda_1, \Lambda_2]_C{}^M = \Lambda_1^N \partial_N \Lambda_2^M - \Lambda_2^N \partial_N \Lambda_1^M - \frac{1}{2}\Big(\Lambda_1^N \partial^M \Lambda_{2N} - \Lambda_2^N \partial^M \Lambda_{1N}\Big)\,. \tag{5.33}$$

The $C$-bracket was firstly obtained in [8, 9] as the generalization of the Lie derivative in the double space. For ${}^\star\pi^\mu = 0$, and $y = 0$ the double phase space reduces to the initial one, while the generator (5.17) reduces to the generator of general coordinate transformations (2.8), which gives rise to the Lie bracket.

We could have obtained $C$-bracket within the framework of self T-duality as well, by demanding that the parameters depend on both $x$ and $y$, substituting ${}^\star\pi^\mu = \kappa x'^\mu$ in (5.17)

$$\mathcal{G}(\xi, \lambda) = \xi^\mu(x,y)\pi_\mu + \kappa\lambda_\mu(x,y)x'^\mu\,. \tag{5.34}$$

If we additionally demand that the symmetry parameters do not depend on the T-dual coordinates $y_\mu$, this generator turns out to be exactly the Courant bracket generator (3.20). It is in the accordance with [7] that the C-bracket reduces to the Courant bracket, in case when there is no dependence on $y$.

## 6 Conclusion

In this paper, we firstly considered the bosonic string moving in the background defined solely by the metric tensor, in which the generalized metric $G_{MN}$ has a simple diagonal form (2.6). The general coordinate transformations are generated by canonical momenta $\pi_\mu$, parametrised with vector components $\xi^\mu$. We have extended this generator, so that it is self T-dual, adding the symmetry generated by coordinate $\sigma$-derivative $x'^\mu$, that are T-dual to the canonical momenta $\pi_\mu$ (2.7). The extended generator of both of these symmetries is a function of a double gauge parameter $\Lambda^M$ (2.17). The latter is a generalized vector, i.e. an element of a space obtained from a direct sum of vectors and 1-forms. The symmetry generator $\mathcal{G}(\Lambda) = \mathcal{G}(\xi, \lambda)$ of both of aforementioned symmetries was expressed as the standard $O(D,D)$ inner product of two generalized vectors (2.18). The Poisson bracket between the extended generators $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$ resulted up to a sign in the generator $\mathcal{G}(\Lambda)$, with its argument being equal to the Courant bracket of the double gauge parameters

$\Lambda = [\Lambda_1, \Lambda_2]_{\mathcal{C}}$. As this is analogous to an appearance of the Lie bracket in the algebra of general coordinate transformations generators, we concluded that the Courant bracket is the self T-dual extension of the Lie bracket.

Afterwards, we added the Kalb-Ramond field $B_{\mu\nu}$ to the background, transforming the diagonal generalized metric $G_{MN}$ acting by the $B$-transformation $e^{\hat{B}}$ (3.2). The standard generalized metric for bosonic string $H_{MN}$ was obtained (3.6), as well as the well known expressions for the Hamiltonian (3.8) and the Lagrangian (3.11). We noted that it is possible to express the Hamiltonian in terms of the diagonal generalized metric $G_{MN}$, on the expense of transforming the double canonical variable $X^M$ by the $B$-shift. This newly obtained canonical variable $\hat{X}$ was suitable for rewriting the symmetry generator $\mathcal{G}$ as $\mathcal{G}^{\hat{B}}$, which is no longer self T-dual. This is the generator of both general coordinate, and local gauge transformations. The Poisson bracket algebra of this new generator was calculated and as an argument of the resulting generator the Courant bracket twisted by the Kalb-Ramond field was obtained. It deviates from the Courant bracket by the term related to the $H$-flux, which is the term that breaks down the self T-duality invariance.

We considered the self T-dual description of the bosonic string $\sigma$-model. Analogously as in the first description, the complete Hamiltonian was constructed starting from the background characterized only by the T-dual metric ${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}$. We applied the $\theta$-transformations $e^{\hat{\theta}}$ (4.7), T-dual to B-shifts, and obtained the same canonical Hamiltonian. Similarly to the previous case, the action of $\theta$-transformation on the double canonical variable was chosen for an appropriate basis. In this basis, the symmetry generator dependent upon some new gauge parameters was constructed and its algebra gave rise to the $\theta$-twisted Courant bracket. This bracket is characterized by the presence of terms related to non-geometric $Q$ and $R$ fluxes.

It would be interesting to obtain the bracket that includes all of the fluxes, while remaining invariant upon the self T-duality. The natural candidate for this is the Courant bracket twisted by both the Kalb-Ramond field and the non-commutativity parameter. This could be done by the matrix $e^{\breve{B}}$, where

$$\breve{B} = \hat{B} + \hat{\theta} = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 2B_{\mu\nu} & 0 \end{pmatrix} . \tag{6.1}$$

This transformation is not trivial, as the square of the matrix $\breve{B}$ is not zero. Nevertheless, the transformation is also an element of the $O(D,D)$ group, and it remains an interesting idea for future research [27].

Lastly, we considered the symmetry generator in the double phase space that is a sum of the initial and T-dual phase space. The generator of general coordinate transformations is extended so that it includes the analogous generator in the T-dual phase space, generated by T-dual momenta ${}^\star\pi^\mu$. Both symmetry parameters were taken to depend on both the initial and T-dual coordinates, in which case the $C$-bracket is obtained as the bracket of the algebra of those generators. The $C$ bracket has already been established as the T-dual invariant bracket [7–9], from the gauge algebra in the double space. We obtain its Poisson bracket representation, using the T-duality relations between canonical variables of different, mutually T-dual, phase spaces. These T-duality relations gave rise to the non-trivial Poisson bracket between the initial and T-dual momenta, which makes a crucial step in obtaining $C$-bracket.

We conclude that both Courant and $C$-bracket are T-dual invariant extension of the Lie bracket. The former is their extension in the initial phase space, that governs both the local gauge and general coordinate transformations. The latter is the extension of Lie bracket in the double phase space, that is a direct sum of the initial and T-dual phase space. Though the algebra of the generators that gives rise to the Courant bracket always closes, the algebra of generators in a double phase space that produces $C$-bracket only closes on a restricted parameters on an isotropic space. If all variables are independent of T-dual coordinates $y_\mu$, the $C$-bracket reduces to the Courant bracket, which confirms results from our paper.

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# Courant bracket twisted both by a 2-form $B$ and by a bi-vector $\theta$

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**Abstract** We obtain the Courant bracket twisted simultaneously by a 2-form $B$ and a bi-vector $\theta$ by calculating the Poisson bracket algebra of the symmetry generator in the basis obtained acting with the relevant twisting matrix. It is the extension of the Courant bracket that contains well known Schouten–Nijenhuis and Koszul bracket, as well as some new star brackets. We give interpretation to the star brackets as projections on isotropic subspaces.

## 1 Introduction

The Courant bracket [1,2] represents the generalization of the Lie bracket on spaces of generalized vectors, understood as the direct sum of the elements of the tangent bundle and the elements of the cotangent bundle. It was obtained in the algebra of generalized currents firstly in [3]. Generalized currents are arbitrary functionals of the fields, parametrized by a pair of vector field and covector field on the target space. Although the Lie bracket satisfies the Jacobi identity, the Courant bracket does not.

In bosonic string theory, the Courant bracket is governing both local gauge and general coordinate transformations, invariant upon T-duality [4,5]. It is a special case of the more general $C$-bracket [6,7]. The $C$-bracket is obtained as the T-dual invariant bracket of the symmetry generator algebra, when the symmetry parameters depend both on the initial and T-dual coordinates. It reduces to the Courant bracket once when parameters depend solely on the coordinates from the initial theory.

It is possible to obtain the twisted Courant bracket, when the self T-dual generator algebra is considered in the basis obtained from the action of the appropriate $O(D, D)$ transformation [8]. The Courant bracket is usually twisted by a 2-form $B$, giving rise to what is known as the twisted Courant bracket [9], and by a bi-vector $\theta$, giving rise to the $\theta$-twisted Courant bracket [10]. In [3,8,11,12], the former bracket was obtained in the generalized currents algebra, and it was shown to be related to the latter by self T-duality [13], when the T-dual of the $B$ field is the bi-vector $\theta$.

The $B$-twisted Courant bracket contains $H$ flux, while the $\theta$-twisted Courant bracket contains non-geometric $Q$ and $R$ fluxes. The fluxes are known to play a crucial role in the compactification of additional dimensions in string theory [14]. Non-geometric fluxes can be used to stabilize moduli. In this paper, we are interested in obtaining the Poisson bracket representation of the twisted Courant brackets that contain all fluxes from the generators algebra. Though it is possible to obtain various twists of the $C$-bracket as well [15], we do not deal with them in this paper.

The realization of all fluxes using the generalized geometry was already considered, see [16] for a comprehensive review. In [17], one considers the generalized tetrads originating from the generalized metric of the string Hamiltonian. As the Lie algebra of tetrads originating from the initial metric defines the geometric flux, it is suggested that all the other fluxes can be extracted from the Courant bracket of the generalized tetrads. Different examples of $O(D, D)$ and $O(D) \times O(D)$ transformations of generalized tetrads lead to the Courant bracket algebras with different fluxes as its structure constants.

In [18], one considers the standard Lie algebroid defined with the Lie bracket and the identity map as an anchor on the tangent bundle, as well as the Lie algebroid with the Koszul bracket and the bi-vector $\theta$ as an anchor on the cotangent bundle. The tetrad basis in these Lie algebroids is suitable for defining the geometric $f$ and non-geometric $Q$ fluxes. It was shown that by twisting both of these Lie algebroids by $H$-flux one can construct the Courant algebroid, which gives rise to all of the fluxes in the Courant bracket algebra.

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Unlike previous approaches where generalized fluxes were defined using the Courant bracket algebra, in a current paper we obtain them in the Poisson bracket algebra of the symmetry generator.

Firstly, we consider the symmetry generator of local gauge and global coordinate transformations, defined as a standard inner product in the generalized tangent bundle of a double gauge parameter and a double canonical variable. The $O(D, D)$ group transforms the double canonical variable into some other basis, in terms of which the symmetry generator can be expressed. We demonstrate how the Poisson bracket algebra of this generator can be used to obtain twist of the Courant bracket by any such transformation. We give a brief summary of how $e^{\hat{B}}$ and $e^{\hat{\theta}}$ produce respectively the $B$-twisted and $\theta$-twisted Courant bracket in the Poisson bracket algebra of generators [8].

Secondly, we consider the matrix $e^{\breve{B}}$ used for twisting the Courant bracket simultaneously by a 2-form and a bi-vector. The argument $\breve{B}$ is defined simply as a sum of the arguments $\hat{B}$ and $\hat{\theta}$. Unlike $\hat{B}$ or $\hat{\theta}$, the square of $\breve{B}$ is not zero. The full Taylor series gives rise to the hyperbolic functions of the parameter depending on the contraction of the 2-form with the bi-vector $\alpha^{\mu}_{\nu} = 2\kappa\theta^{\mu\rho}B_{\rho\nu}$. We represent the symmetry generator in the basis obtained acting with the twisting matrix $e^{\breve{B}}$ on the double canonical variable. This generator is manifestly self T-dual and its algebra closes on the Courant bracket twisted by both $B$ and $\theta$.

Instead of computing the $B-\theta$ twisted Courant bracket directly, we introduce the change of basis in which we define some auxiliary generators, in order to simplify the calculations. This change of basis is also realized by the action of an element of the $O(D, D)$ group. The structure constants appearing in the Poisson bracket algebra have exactly the same form as the generalized fluxes obtained in other papers [16–18]. The expressions for fluxes is given in terms of new auxiliary fields $\mathring{B}$ and $\mathring{\theta}$, both being the function of $\alpha^{\mu}$.

The algebra of these new auxiliary generators closes on another bracket, that we call $\mathring{C}$-twisted Courant bracket. We obtain its full Poisson bracket representation, and express it in terms of generalized fluxes. We proceed with rewriting it in the coordinate free notation, where many terms are recognized as the well known brackets, such as the Koszul or Schouten–Nijenhuis bracket, but some new brackets, that we call star brackets, also appear. These star brackets as a domain take the direct sum of tangent and cotangent bundle, and as a result give the graph of the bi-vector $\mathring{\theta}$ in the cotangent bundle, i.e. the sub-bundle for which the vector and 1-form components are related as $\xi^{\mu} = \kappa\mathring{\theta}^{\mu\nu}\lambda_{\nu}$. We show that they can be defined in terms of the projections on isotropic subspaces acting on different twists of the Courant bracket.

Lastly, we return to the previous basis and obtain the full expression for the Courant bracket twisted by both $B$ and $\theta$. It has a similar form as $\mathring{C}$-twisted Courant bracket, but in this case the other brackets contained within it are also twisted. The Courant bracket twisted by both $B$ and $\theta$ and the one twisted by $\mathring{C}$ are directly related by a $O(D, D)$ transformation represented with the block diagonal matrix.

## 2 The bosonic string essentials

The canonical Hamiltonian for closed bosonic string, moving in the $D$-dimensional space-time with background characterized by the metric field $G_{\mu\nu}$ and the antisymmetric Kalb–Ramond field $B_{\mu\nu}$ is given by [19,20]

$$\mathcal{H}_C = \frac{1}{2\kappa}\pi_\mu(G^{-1})^{\mu\nu}\pi_\nu + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu \\ -2x'^\mu B_{\mu\rho}(G^{-1})^{\rho\nu}\pi_\nu, \tag{2.1}$$

where $\pi_\mu$ are canonical momenta conjugate to coordinates $x^\mu$, and

$$G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu} \tag{2.2}$$

is the effective metric. The Hamiltonian can be rewritten in the matrix notation

$$\mathcal{H}_C = \frac{1}{2\kappa}(X^T)^M H_{MN} X^N, \tag{2.3}$$

where $X^M$ is a double canonical variable given by

$$X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix}, \tag{2.4}$$

and $H_{MN}$ is the so called generalized metric, given by

$$H_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}, \tag{2.5}$$

with $M, N \in \{0, 1\}$. In the context of generalized geometry [21], the double canonical variable $X^M$ represents the generalized vector. The generalized vectors are $2D$ structures that combine both vector and 1-form components in a single entity.

The standard T-duality [22,23] laws for background fields have been obtained by Buscher [24]

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad ^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}, \tag{2.6}$$

where $(G_E^{-1})^{\mu\nu}$ is the inverse of the effective metric (2.2), and $\theta^{\mu\nu}$ is the non-commutativity parameter, given by

$$\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1})^{\mu\rho} B_{\rho\sigma}(G^{-1})^{\sigma\nu}. \tag{2.7}$$

The T-duality can be realized without changing the phase space, which is called the self T-duality [13]. It has the same transformation rules for the background fields like T-duality (2.6), with additionally interchanging the coordinate $\sigma$-derivatives $\kappa x'^{\mu}$ with canonical momenta $\pi_\mu$

$$\kappa x'^{\mu} \cong \pi_\mu. \tag{2.8}$$

Since momenta and winding numbers correspond to $\sigma$ integral of respectively $\pi_\mu$ and $\kappa x'^{\mu}$, we see that the self T-duality, just like the standard T-duality, swaps momenta and winding numbers.

### 2.1 Symmetry generator

We consider the symmetry generator that at the same time governs the general coordinate transformations, parametrized by $\xi^\mu$, and the local gauge transformations, parametrized by $\lambda_\mu$. The generator is given by [25]

$$G(\xi,\lambda) = \int_0^{2\pi} d\sigma \mathcal{G}(\xi,\lambda) = \int_0^{2\pi} d\sigma \Big[\xi^\mu \pi_\mu + \lambda_\mu \kappa x'^{\mu}\Big]. \tag{2.9}$$

It has been shown that the general coordinate transformations and the local gauge transformations are related by self T-duality [25], meaning that this generator is self T-dual. If one makes the following change of parameters $\lambda_\mu \to \lambda_\mu + \partial_\mu \varphi$, the generator (2.9) does not change

$$G(\xi,\lambda+\partial\varphi) = G(\xi,\lambda) + \kappa \int_0^{2\pi} \varphi' d\sigma = G(\xi,\lambda), \tag{2.10}$$

since the total derivative integral vanishes for the closed string. Therefore, the symmetry is reducible.

Let us introduce the double gauge parameter $\Lambda^M$, as the generalized vector, given by

$$\Lambda^M = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix}, \tag{2.11}$$

where $\xi^\mu$ represent the vector components, and $\lambda_\mu$ represent the 1-form components. The space of generalized vectors is endowed with the natural inner product

$$\langle \Lambda_1, \Lambda_2\rangle = (\Lambda_1^T)^M \eta_{MN} \Lambda_2^N \Leftrightarrow \langle(\xi_1,\lambda_1),(\xi_2,\lambda_2)\rangle = i_{\xi_1}\lambda_2 + i_{\xi_2}\lambda_1 = \xi_1^\mu \lambda_{2\mu} + \xi_2^\mu \lambda_{1\mu}, \tag{2.12}$$

where $i_\xi$ is the interior product along the vector field $\xi$, and $\eta_{MN}$ is $O(D,D)$ metric, given by

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.13}$$

Now it is possible to rewrite the generator (2.9) as

$$G(\Lambda) = \int d\sigma \langle \Lambda, X\rangle. \tag{2.14}$$

In [8], the Poisson bracket algebra of generator (2.9) was obtained in the form

$$\Big\{G(\Lambda_1),\ G(\Lambda_2)\Big\} = -G\Big([\Lambda_1,\Lambda_2]_{\mathcal{C}}\Big), \tag{2.15}$$

where the standard Poisson bracket relations between coordinates and canonical momenta were assumed

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu \delta(\sigma-\bar\sigma). \tag{2.16}$$

The bracket $[\Lambda_1,\Lambda_2]_{\mathcal{C}}$ is the Courant bracket [1], defined by

$$[\Lambda_1,\Lambda_2]_{\mathcal{C}} = \Lambda \Leftrightarrow [(\xi_1,\lambda_1),(\xi_2,\lambda_2)]_{\mathcal{C}} = (\xi,\lambda), \tag{2.17}$$

where

$$\xi^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu,$$

and

$$\lambda_\mu = \xi_1^\nu(\partial_\nu \lambda_{2\mu} - \partial_\mu \lambda_{2\nu}) - \xi_2^\nu(\partial_\nu \lambda_{1\mu} - \partial_\mu \lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\lambda_2 - \xi_2\lambda_1). \tag{2.18}$$

It is the generalization of the Lie bracket on spaces of generalized vectors.

## 3 $O(D,D)$ group

Consider the orthogonal transformation $\mathcal{O}$, i.e. the transformation that preserves the inner product (2.12)

$$\langle \mathcal{O}\Lambda_1, \mathcal{O}\Lambda_2\rangle = \langle\Lambda_1,\Lambda_2\rangle \Leftrightarrow (\mathcal{O}\Lambda_1)^T \eta (\mathcal{O}\Lambda_2) = \Lambda_1^T \eta \Lambda_2, \tag{3.19}$$

which is satisfied for the condition

$$\mathcal{O}^T\ \eta\ \mathcal{O} = \eta. \tag{3.20}$$

There is a solution for the above equation in the form $\mathcal{O} = e^T$, see Sec. 2.1 of [21], where

$$T = \begin{pmatrix} A & \theta \\ B & -A^T \end{pmatrix}, \tag{3.21}$$

with $\theta : T^\star M \to TM$ and $B : TM \to T^\star M$ being anti-symmetric, and $A : TM \to TM$ being the endomorphism. In general case, $B$ and $\theta$ can be independent for $\mathcal{O}$ to satisfy condition (3.20).

Consider now the action of some element of $O(D, D)$ on the double coordinate $X$ (2.4) and the double gauge parameter $\Lambda$ (2.11)

$$\hat{X}^M = \mathcal{O}^M_{\ N}\, X^N, \;\; \hat{\Lambda}^M = \mathcal{O}^M_{\ N}\, \Lambda^N, \tag{3.22}$$

and note that the relation (2.15) can be written as

$$\int d\sigma \Big\{ \langle \Lambda_1, X\rangle, \langle \Lambda_2, X\rangle \Big\} = -\int d\sigma \langle [\Lambda_1, \Lambda_2]_{\mathcal{C}}, X\rangle, \tag{3.23}$$

and using (3.19) and (3.22) as

$$\int d\sigma \Big\{ \langle \hat{\Lambda}_1, \hat{X}\rangle, \langle \hat{\Lambda}_2, \hat{X}\rangle \Big\} = -\int d\sigma \langle [\Lambda_1, \Lambda_2]_{\mathcal{C}}, X\rangle = -\int d\sigma \langle [\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T}, \hat{X}\rangle, \tag{3.24}$$

where we expressed the right hand side in terms of some new bracket $[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T}$. Moreover, using (3.19) and (3.22), the right hand side of (3.23) can be written as

$$\langle [\Lambda_1, \Lambda_2]_{\mathcal{C}}, X\rangle = \langle [\mathcal{O}^{-1}\hat{\Lambda}_1, \mathcal{O}^{-1}\hat{\Lambda}_2]_{\mathcal{C}}, \mathcal{O}^{-1}\hat{X}\rangle = \langle \mathcal{O}[\mathcal{O}^{-1}\hat{\Lambda}_1, \mathcal{O}^{-1}\hat{\Lambda}_2]_{\mathcal{C}}, \hat{X}\rangle. \tag{3.25}$$

Using (3.24) and (3.25), one obtains

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T} = \mathcal{O}[\mathcal{O}^{-1}\hat{\Lambda}_1, \mathcal{O}^{-1}\hat{\Lambda}_2]_{\mathcal{C}} = e^T[e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_{\mathcal{C}}. \tag{3.26}$$

This is a definition of a $T$-twisted Courant bracket. Throughout this paper, we use the notation where $[,]_{\mathcal{C}}$ is the Courant bracket, while when $\mathcal{C}$ has an additional index, it represents the twist of the Courant bracket by the indexed field, e.g. $[,]_{\mathcal{C}_B}$ is the Courant bracket twisted by $B$.

In a special case, when $A = 0$, $\theta = 0$, the bracket (3.26) becomes the Courant bracket twisted by a 2-form $B$ [9]

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = e^{\hat{B}}[e^{-\hat{B}}\Lambda_1, e^{-\hat{B}}\Lambda_2]_{\mathcal{C}}, \tag{3.27}$$

where $e^{\hat{B}}$ is the twisting matrix, given by

$$e^{\hat{B}} = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix}, \;\; \hat{B}^M_{\ N} = \begin{pmatrix} 0 & 0 \\ 2B_{\mu\nu} & 0 \end{pmatrix}. \tag{3.28}$$

This bracket has been obtained in the algebra of generalized currents [11,13].

In case of $A = 0$, $B = 0$, the bracket (3.26) becomes the Courant bracket twisted by a bi-vector $\theta$

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_\theta} = e^{\hat{\theta}}[e^{-\hat{\theta}}\Lambda_1, e^{-\hat{\theta}}\Lambda_2]_{\mathcal{C}}, \tag{3.29}$$

where $e^{\hat{\theta}}$ is the twisting matrix, given by

$$e^{\hat{\theta}} = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix}, \;\; \hat{\theta}^M_{\ N} = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 0 & 0 \end{pmatrix}. \tag{3.30}$$

The $B$-twisted Courant bracket (3.27) and $\theta$-twisted Courant bracket (3.29) are related by self T-duality [13]. It is easy to demonstrate that both $e^{\hat{B}}$ and $e^{\hat{\theta}}$ satisfy the condition (3.20).

We can now deduce a simple algorithm for finding the Courant bracket twisted by an arbitrary $O(D, D)$ transformation. One rewrites the double symmetry generator $G(\xi, \lambda)$ in the basis obtained by the action of the matrix $e^T$ on the double coordinate (2.4). Then, the Poisson bracket algebra between these generators gives rise to the appropriate twist of the Courant bracket. In this paper, we apply this algorithm to obtain the Courant bracket twisted by both $B$ and $\theta$.

## 4 Twisting matrix

The transformations $e^{\hat{B}}$ and $e^{\hat{\theta}}$ do not commute. That is why we define the transformations that simultaneously twists the Courant bracket by $B$ and $\theta$ as $e^{\breve{B}}$, where

$$\breve{B} = \hat{B} + \hat{\theta} = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 2B_{\mu\nu} & 0 \end{pmatrix}. \tag{4.1}$$

The Courant bracket twisted at the same time both by a 2-form $B$ and by a bi-vector $\theta$ is given by

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_{B\theta}} = e^{\breve{B}}[e^{-\breve{B}}\Lambda_1, e^{-\breve{B}}\Lambda_2]_{\mathcal{C}}. \tag{4.2}$$

The full expression for $e^{\breve{B}}$ can be obtained from the well known Taylor series expansion of exponential function

$$e^{\breve{B}} = \sum_{n=0}^{\infty} \frac{\breve{B}^n}{n!}. \tag{4.3}$$

The square of the matrix $\breve{B}$ is easily obtained

$$\breve{B}^2 = 2\begin{pmatrix} \kappa(\theta B)^\mu{}_\nu & 0 \\ 0 & \kappa(B\theta)_\mu{}^\nu \end{pmatrix}, \tag{4.4}$$

as well as its cube

$$\breve{B}^3 = 2\begin{pmatrix} 0 & \kappa^2(\theta B\theta)^{\mu\nu} \\ 2\kappa(B\theta B)_{\mu\nu} & 0 \end{pmatrix}. \tag{4.5}$$

The higher degree of $\breve{B}$ are given by

$$\breve{B}^{2n} = \begin{pmatrix} (\alpha^n)^\mu{}_\nu & 0 \\ 0 & ((\alpha^T)^n)_\mu{}^\nu \end{pmatrix}, \tag{4.6}$$

for even degrees, and for odd degrees by

$$\breve{B}^{2n+1} = \begin{pmatrix} 0 & \kappa(\alpha^n\theta)^{\mu\nu} \\ 2(B\alpha^n)_{\mu\nu} & 0 \end{pmatrix}, \tag{4.7}$$

where we have marked

$$\alpha^\mu{}_\nu = 2\kappa\theta^{\mu\rho}B_{\rho\nu}. \tag{4.8}$$

Substituting (4.6) and (4.7) into (4.3), we obtain the twisting matrix

$$e^{\breve{B}} = \begin{pmatrix} \left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n)!}\right)^\mu{}_\nu & \kappa\left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\right)^\mu{}_\rho \theta^{\rho\nu} \\ 2B_{\mu\rho}\left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\right)^\rho{}_\nu & \left(\sum_{n=0}^\infty \frac{(\alpha^T)^n}{(2n)!}\right)_\mu{}^\nu \end{pmatrix}. \tag{4.9}$$

Taking into the account the Taylor's expansion of hyperbolic functions

$$cosh(x) = \sum_{n=0}^\infty \frac{x^{2k}}{(2k)!}, \quad sinh(x) = \sum_{n=0}^\infty \frac{x^{2k+1}}{(2k+1)!}, \tag{4.10}$$

the twisting matrix (4.9) can be rewritten as

$$e^{\breve{B}} = \begin{pmatrix} \mathcal{C}^\mu{}_\nu & \kappa\mathcal{S}^\mu{}_\rho\theta^{\rho\nu} \\ 2B_{\mu\rho}\mathcal{S}^\rho{}_\nu & (\mathcal{C}^T)_\mu{}^\nu \end{pmatrix}, \tag{4.11}$$

with $\mathcal{S}^\mu{}_\nu = \left(\frac{\sinh\sqrt{\alpha}}{\sqrt{\alpha}}\right)^\mu{}_\nu$ and $\mathcal{C}^\mu{}_\nu = \left(\cosh\sqrt{\alpha}\right)^\mu{}_\nu$. Its determinant is given by

$$\det(e^{\breve{B}}) = e^{Tr(\breve{B})} = 1, \tag{4.12}$$

and the straightforward calculations show that its inverse is given by

$$e^{-\breve{B}} = \begin{pmatrix} \mathcal{C}^\mu{}_\nu & -\kappa\mathcal{S}^\mu{}_\rho\theta^{\rho\nu} \\ -2B_{\mu\rho}\mathcal{S}^\rho{}_\nu & (\mathcal{C}^T)_\mu{}^\nu \end{pmatrix}. \tag{4.13}$$

One easily obtains the relation

$$(e^{\breve{B}})^T\ \eta\ e^{\breve{B}} = \eta, \tag{4.14}$$

therefore the transformation (4.11) is indeed an element of $O(D, D)$.

It is worth pointing out characteristics of the matrix $\alpha^\mu{}_\nu$. It is easy to show that $\alpha^\mu{}_\rho\theta^{\rho\nu} = \theta^{\mu\rho}(\alpha^T)_\rho{}^\nu$ and $B_{\mu\rho}\alpha^\rho{}_\nu = (\alpha^T)_\mu{}^\rho B_{\rho\nu}$, which is further generalized to

$$(f(\alpha))^\mu{}_\rho\theta^{\rho\nu} = \theta^{\mu\rho}(f(\alpha^T))_\rho{}^\nu, \quad B_{\mu\rho}(f(\alpha))^\rho{}_\nu = (f(\alpha^T))_\mu{}^\rho B_{\rho\nu}, \tag{4.15}$$

for any analytical function $f(\alpha)$. Moreover, the well known hyperbolic identity $\cosh(x)^2 - \sinh(x)^2 = 1$ can also be expressed in terms of newly defined tensors

$$(\mathcal{C}^2)^\mu{}_\nu - \alpha^\mu{}_\rho(\mathcal{S}^2)^\rho{}_\nu = \delta^\mu_\nu. \tag{4.16}$$

Lastly, the self T-duality relates the matrix $\alpha$ to its transpose $\alpha \cong \alpha^T$, due to (2.6). Consequently, we write the following self T-duality relations

$$\mathcal{C} \cong \mathcal{C}^T, \quad \mathcal{S} \cong \mathcal{S}^T. \tag{4.17}$$

## 5 Symmetry generator in an appropriate basis

The direct computation of the bracket (4.2) would be difficult, given the form of the matrix $e^{\breve{B}}$ . Therefore, we use the indirect computation of the bracket, by computing the Poisson bracket algebra of the symmetry generator (2.9), rewritten in the appropriate basis. As elaborated at the end of the Chapter 3, this basis is obtained by the action of the matrix (4.11) on the double coordinate (2.4)

$$\breve{X}^M = (e^{\breve{B}})^M{}_N\ X^N = \begin{pmatrix} \breve{k}^\mu \\ \breve{\imath}_\mu \end{pmatrix}, \tag{5.18}$$

where

$$\breve{k}^\mu = \kappa\mathcal{C}^\mu{}_\nu x'^\nu + \kappa(\mathcal{S}\theta)^{\mu\nu}\pi_\nu, \quad \breve{\imath}_\mu = 2(B\mathcal{S})_{\mu\nu}x'^\nu + (\mathcal{C}^T)_\mu{}^\nu\pi_\nu, \tag{5.19}$$

are new currents. Applying (2.6), (2.8) and (4.17) to currents $\breve{k}^\mu$ and $\breve{\imath}_\mu$ we obtain $\breve{\imath}_\mu$ and $\breve{k}^\mu$ respectively, meaning that these currents are directly related by self T-duality. Multiplying the Eq. (5.18) with the matrix (4.13), we obtain the relations inverse to (5.19)

$$\kappa x'^\mu = \mathcal{C}^\mu{}_\nu\breve{k}^\nu - \kappa(\mathcal{S}\theta)^{\mu\nu}\breve{\imath}_\nu, \quad \pi_\mu = -2(B\mathcal{S})_{\mu\nu}\breve{k}^\nu + (\mathcal{C}^T)_\mu{}^\nu\breve{\imath}_\nu. \tag{5.20}$$

Applying the transformation (4.11) to a double gauge parameter (2.11), we obtain new gauge parameters

$$\breve{\Lambda}^M = \begin{pmatrix} \breve{\xi}^\mu \\ \breve{\lambda}_\mu \end{pmatrix} = (e^{\breve{B}})^M{}_N \, \Lambda^N = \begin{pmatrix} \mathcal{C}^\mu{}_\nu \xi^\nu + \kappa(\mathcal{S}\theta)^{\mu\nu}\lambda_\nu \\ 2(B\mathcal{S})_{\mu\nu}\xi^\nu + (\mathcal{C}^T)_\mu{}^\nu \lambda_\nu \end{pmatrix}. \tag{5.21}$$

The symmetry generator (2.9) rewritten in a new basis $\mathcal{G}(\mathcal{C}\xi + \kappa\mathcal{S}\theta\lambda, 2(B\mathcal{S})\xi + \mathcal{C}^T\lambda) \equiv \breve{\mathcal{G}}(\breve{\xi}, \breve{\lambda})$ is given by

$$\breve{G}(\breve{\Lambda}) = \int d\sigma \langle \breve{\Lambda}, \breve{X} \rangle \Leftrightarrow \breve{G}(\breve{\xi}, \breve{\lambda}) = \int d\sigma \Big[ \breve{\xi}^\mu \breve{\imath}_\mu + \breve{\lambda}_\mu \breve{k}^\mu \Big]. \tag{5.22}$$

Substituting (5.18) and (5.21) into (5.22), the symmetry generator in the initial canonical basis (2.9) is obtained. Due to mutual self T-duality between basis currents (5.19), this generator is invariant upon self T-duality.

Rewriting the Eq. (2.15) in terms of new gauge parameters (5.21) in the basis of auxiliary currents (5.19), the Courant bracket twisted by both a 2-form $B_{\mu\nu}$ and by a bi-vector $\theta^{\mu\nu}$ is obtained in the new generator (5.22) algebra

$$\Big\{ \breve{G}(\breve{\Lambda}_1),\ \breve{G}(\breve{\Lambda}_2) \Big\} = -\breve{G}\Big( [\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathcal{C}_{B\theta}} \Big). \tag{5.23}$$

### 5.1 Auxiliary generator

Let us define a new auxiliary basis, so that both the matrices $\mathcal{C}$ and $\mathcal{S}$ are absorbed in some new fields, giving rise to the generator algebra that is much more readable. When the algebra in this basis is obtained, simple change of variables back to the initial ones will provide us with the bracket in need.

Multiplying the second equation of (5.19) with the matrix $\mathcal{C}^{-1}$, we obtain

$$\breve{\imath}_\nu (\mathcal{C}^{-1})^\nu{}_\mu = \pi_\mu + 2\kappa (B\mathcal{S}\mathcal{C}^{-1})_{\mu\nu} x'^\nu, \tag{5.24}$$

where we have used $(B\mathcal{S})_{\nu\rho}(\mathcal{C}^{-1})^\nu{}_\mu = -(B\mathcal{S}\mathcal{C}^{-1})_{\rho\mu} = (B\mathcal{S}\mathcal{C}^{-1})_{\mu\rho}$, due to tensor $B\mathcal{S}$ being antisymmetric, and properties (4.15). We will mark the result as a new auxiliary current, given by

$$\mathring{\imath}_\mu = \pi_\mu + 2\kappa \mathring{B}_{\mu\nu} x'^\nu, \tag{5.25}$$

where $\mathring{B}$ is an auxiliary B-field, given by

$$\mathring{B}_{\mu\nu} = B_{\mu\rho} \mathcal{S}^\rho{}_\sigma (\mathcal{C}^{-1})^\sigma{}_\nu. \tag{5.26}$$

On the other hand, multiplying the first equation of (5.19) with the matrix $\mathcal{C}$, we obtain

$$\mathcal{C}^\mu{}_\nu \breve{k}^\nu = (\mathcal{C}^2)^\mu{}_\nu \kappa x'^\nu + \kappa (\mathcal{C}\mathcal{S}\theta)^{\mu\nu} \pi_\nu. \tag{5.27}$$

Substituting (4.16) in the previous equation, and keeping in mind that $\mathcal{C}$, $\mathcal{S}$ and $\theta$ commute (4.15), we obtain

$$\mathcal{C}^\mu{}_\nu \breve{k}^\nu = \kappa x'^\mu + \kappa (\mathcal{C}\mathcal{S}\theta)^{\rho\nu} (\pi_\nu + 2\kappa (B\mathcal{S}\mathcal{C}^{-1})_{\nu\sigma} x'^\sigma). \tag{5.28}$$

Using (5.25), the results are marked as a new auxiliary current

$$\mathring{k}^\mu = \kappa x'^\mu + \kappa \mathring{\theta}^{\mu\nu} \mathring{\imath}_\nu, \tag{5.29}$$

where $\mathring{\theta}$ is given by

$$\mathring{\theta}^{\mu\nu} = \mathcal{C}^\mu{}_\rho \mathcal{S}^\rho{}_\sigma \theta^{\sigma\nu}. \tag{5.30}$$

There is no explicit dependence on either $\mathcal{C}$ nor $\mathcal{S}$ in redefined auxiliary currents, rather only on canonical variables and new background fields. From (5.29), it is easy to express the coordinate $\sigma$-derivative in the basis of new auxiliary currents

$$\kappa x'^\mu = \mathring{k}^\mu - \kappa \mathring{\theta}^{\mu\nu} \mathring{\imath}_\nu. \tag{5.31}$$

The first equation of (5.19) could have been multiplied with $\mathcal{C}$, instead of $\mathcal{C}^{-1}$, given that the latter would also produce a current that would not explicitly depend on $\mathcal{C}$. However, the expression for coordinate $\sigma$-derivative $\kappa x'^\mu$ would explicitly depend on $\mathcal{C}^2$ in that case, while with our choice of basis it does not (5.31).

Substituting (5.24) and (5.28) in the expression for the generator (5.22), we obtain

$$\breve{G}(\breve{\xi}, \breve{\lambda}) = \int d\sigma \Big[ \breve{\lambda}_\mu (\mathcal{C}^{-1})^\mu{}_\nu \mathring{k}^\nu + \breve{\xi}^\mu (\mathcal{C}^T)_\mu{}^\nu \mathring{\imath}_\nu \Big], \tag{5.32}$$

from which it is easily seen that the generator (5.22) is equal to an auxiliary generator

$$\mathring{G}(\mathring{\Lambda}) = \int d\sigma \langle \mathring{X}, \mathring{\Lambda} \rangle \Leftrightarrow \mathring{G}(\mathring{\xi}, \mathring{\lambda}) = \int d\sigma \Big[ \mathring{\lambda}_\mu \mathring{k}^\mu + \mathring{\xi}^\mu \mathring{\imath}_\mu \Big], \tag{5.33}$$

provided that

$$\mathring{\Lambda}^M = \begin{pmatrix} \mathring{\xi}^\mu \\ \mathring{\lambda}_\mu \end{pmatrix}, \quad \mathring{\lambda}_\mu = \breve{\lambda}_\nu (\mathcal{C}^{-1})^\nu{}_\mu, \quad \mathring{\xi}^\mu = \mathcal{C}^\mu{}_\nu \breve{\xi}^\nu, \tag{5.34}$$

and

$$\mathring{X}^M = \begin{pmatrix} \mathring{k}^\mu \\ \mathring{\imath}_\mu \end{pmatrix}. \tag{5.35}$$

Once that the algebra of (5.33) is known, the algebra of generator (5.22) can be easily obtained using (5.34).

The change of basis to the one suitable for the auxiliary generator (5.33) corresponds to the transformation

$$A^M_{\ N} = \begin{pmatrix} (\mathcal{C})^\mu_{\ \nu} & 0 \\ 0 & ((\mathcal{C}^{-1})^T)_\mu^{\ \nu} \end{pmatrix}, \ \mathring{\Lambda}^M = A^M_{\ N}\,\check{\Lambda}^N, \ \mathring{X}^M = A^M_{\ N}\,\check{X}^N, \tag{5.36}$$

that can be rewritten as

$$\mathring{X}^M = (Ae^{\check{B}})^M_{\ N}\, X^N, \quad \mathring{\Lambda}^M = (Ae^{\check{B}})^M_{\ N}\,\Lambda^N, \tag{5.37}$$

where (5.18) and (5.21) were used. It is easy to show that the transformation $A^M_{\ N}$, and consequentially $(Ae^{\check{B}})^M_{\ N}$, is the element of $O(D,D)$ group

$$A^T\eta\, A = \eta, \ (Ae^{\check{B}})^T\, \eta\, (Ae^{\check{B}}) = \eta, \tag{5.38}$$

which means that there is $\mathring{C}$, for which [21]

$$e^{\mathring{C}} = Ae^{\check{B}}. \tag{5.39}$$

The generator (5.33) gives rise to algebra that closes on $\mathring{C}$-twisted Courant bracket

$$\Big\{\mathring{G}(\mathring{\Lambda}_1), \mathring{G}(\mathring{\Lambda}_1)\Big\} = -\mathring{G}\Big([\mathring{\Lambda}_1, \mathring{\Lambda}_2]_{\mathcal{C}_{\mathring{C}}}\Big), \tag{5.40}$$

where the $\mathring{C}$-twisted Courant bracket is defined by

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]_{\mathcal{C}_{\mathring{C}}} = e^{\mathring{C}}[e^{-\mathring{C}}\mathring{\Lambda}_1, e^{-\mathring{C}}\mathring{\Lambda}_2]_{\mathcal{C}}. \tag{5.41}$$

In the next chapter, we will obtain this bracket by direct computation of the generators Poisson bracket algebra.

Lastly, let us briefly comment on reducibility conditions for the $\mathring{C}$-twisted Courant bracket. Since we are working with the closed strings, the total derivatives vanishes when integrated out over the worldsheet. Using (5.31), we obtain

$$\int d\sigma\kappa\varphi' = \int d\sigma\kappa x'^\mu\partial_\mu\varphi = \int d\sigma\Big(\mathring{k}^\mu\partial_\mu\varphi + \kappa\mathring{\imath}_\mu\mathring{\theta}^{\mu\nu}\partial_\nu\varphi\Big) = 0, \tag{5.42}$$

for any parameter $\lambda$. Hence, the generator (5.33) remains invariant under the following change of parameters

$$\mathring{\xi}^\mu \to \mathring{\xi}^\mu + \kappa\mathring{\theta}^{\mu\nu}\partial_\nu\varphi, \quad \mathring{\lambda}_\mu \to \mathring{\lambda}_\mu + \partial_\mu\varphi. \tag{5.43}$$

These are reducibility conditions (2.10) in the basis spanned by $\mathring{k}^\mu$ and $\mathring{\imath}_\mu$.

## 6 Courant bracket twisted by $\mathring{C}$ from the generator algebra

In order to obtain the Poisson bracket algebra for the generator (5.33), let us firstly calculate the algebra of basis vectors, using the standard Poisson bracket relations (2.16). The auxiliary currents $\mathring{\imath}_\mu$ algebra is

$$\{\mathring{\imath}_\mu(\sigma), \mathring{\imath}_\nu(\bar\sigma)\} = -2\mathring{B}_{\mu\nu\rho}\mathring{k}^\rho\delta(\sigma-\bar\sigma) - \mathring{\mathcal{F}}_{\mu\nu}^{\ \ \rho}\,\mathring{\imath}_\rho\delta(\sigma-\bar\sigma), \tag{6.1}$$

where $\mathring{B}_{\mu\nu\rho}$ is the generalized H-flux, given by

$$\mathring{B}_{\mu\nu\rho} = \partial_\mu\mathring{B}_{\nu\rho} + \partial_\nu\mathring{B}_{\rho\mu} + \partial_\rho\mathring{B}_{\mu\nu}, \tag{6.2}$$

and $\mathring{\mathcal{F}}^\rho_{\mu\nu}$ is the generalized f-flux, given by

$$\mathring{\mathcal{F}}_{\mu\nu}^{\ \ \rho} = -2\kappa\mathring{B}_{\mu\nu\sigma}\mathring{\theta}^{\sigma\rho}. \tag{6.3}$$

The algebra of currents $\mathring{k}^\mu$ is given by

$$\{\mathring{k}^\mu(\sigma), \mathring{k}^\nu(\bar\sigma)\} = -\kappa\mathring{\mathcal{Q}}_\rho^{\ \mu\nu}\mathring{k}^\rho\delta(\sigma-\bar\sigma) - \kappa^2\mathring{\mathcal{R}}^{\mu\nu\rho}\mathring{\imath}_\rho\delta(\sigma-\bar\sigma), \tag{6.4}$$

where

$$\mathring{\mathcal{Q}}_\mu^{\ \nu\rho} = \mathring{Q}_\mu^{\ \nu\rho} + 2\kappa\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\rho\tau}\mathring{B}_{\mu\sigma\tau}, \quad \mathring{Q}_\mu^{\ \nu\rho} = \partial_\mu\mathring{\theta}^{\nu\rho} \tag{6.5}$$

and

$$\mathring{\mathcal{R}}^{\mu\nu\rho} = \mathring{R}^{\mu\nu\rho} + 2\kappa\mathring{\theta}^{\mu\lambda}\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\rho\tau}\mathring{B}_{\lambda\sigma\tau},$$
$$\mathring{R}^{\mu\nu\rho} = \mathring{\theta}^{\mu\sigma}\partial_\sigma\mathring{\theta}^{\nu\rho} + \mathring{\theta}^{\nu\sigma}\partial_\sigma\mathring{\theta}^{\rho\mu} + \mathring{\theta}^{\rho\sigma}\partial_\sigma\mathring{\theta}^{\mu\nu}. \tag{6.6}$$

The terms in (6.4) containing both $\mathring{\theta}$ and $\mathring{B}$ are the consequence of non-commutativity of auxiliary currents $\mathring{\imath}_\mu$. The remaining algebra of currents $\mathring{k}^\mu$ and $\mathring{\imath}_\mu$ can be as easily obtained

$$\{\mathring{\imath}_\mu(\sigma), \mathring{k}^\nu(\bar\sigma)\} = \kappa\delta_\mu^\nu\delta'(\sigma-\bar\sigma) + \mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu}\,\mathring{k}^\rho\delta(\sigma-\bar\sigma) - \kappa\mathring{\mathcal{Q}}_\mu^{\ \nu\rho}\mathring{\imath}_\rho\delta(\sigma-\bar\sigma). \tag{6.7}$$

The basic algebra relations can be summarized in a single algebra relation where the structure constants contain all generalized fluxes

$$\{\mathring{X}^M, \mathring{X}^N\} = -\mathring{F}^{MN}_{\ \ \ \ P}\,\mathring{X}^P\delta(\sigma-\bar\sigma) + \kappa\eta^{MN}\delta'(\sigma-\bar\sigma), \tag{6.8}$$

with

$$F^{MN\rho} = \begin{pmatrix} \kappa^2 \mathring{\mathcal{R}}^{\mu\nu\rho} & -\kappa \mathring{\mathcal{Q}}_\nu^{\ \mu\rho} \\ \kappa \mathring{\mathcal{Q}}_\mu^{\ \nu\rho} & \mathring{\mathcal{F}}_{\mu\nu}^{\ \ \rho} \end{pmatrix},$$
$$F^{MN}{}_\rho = \begin{pmatrix} \kappa \mathring{\mathcal{Q}}_\rho^{\ \mu\nu} & \mathring{\mathcal{F}}_{\nu\rho}^{\ \ \mu} \\ -\mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu} & 2\mathring{B}_{\mu\nu\rho} \end{pmatrix}. \tag{6.9}$$

The form of the generalized fluxes is the same as the ones already obtained using the tetrad formalism [16–18]. In our approach, the generalized fluxes are obtained in the Poisson bracket algebra, only from the fact that the generalized canonical variable $X^M$ is transformed with an element of the $O(D, D)$ group that twists the Courant bracket both by $B$ and $\theta$ at the same time. Consequentially, the fluxes obtained in this paper are functions of some new effective fields, $\mathring{B}_{\mu\nu}$ (5.26) and $\mathring{\theta}^{\mu\nu}$ (5.30).

We now proceed to obtain the full bracket. Let us rewrite the generator (5.33) algebra

$$\begin{aligned}
&\Big\{\mathring{G}(\mathring{\xi}_1, \mathring{\lambda}_1)(\sigma),\ \mathring{G}(\mathring{\xi}_2, \mathring{\lambda}_2)(\bar{\sigma})\Big\} \\
&= \int d\sigma d\bar{\sigma}\Big[\Big\{\mathring{\xi}_1^\mu(\sigma)\mathring{\imath}_\mu(\sigma), \mathring{\xi}_2^\nu(\bar{\sigma})\mathring{\imath}_\nu(\bar{\sigma})\Big\} \\
&\quad+\Big\{\mathring{\lambda}_{1\mu}(\sigma)\mathring{k}^\mu(\sigma), \mathring{\lambda}_{2\nu}(\bar{\sigma})\mathring{k}^\nu(\bar{\sigma})\Big\} \\
&\quad+\Big\{\mathring{\xi}_1^\mu(\sigma)\mathring{\imath}_\mu(\sigma), \mathring{\lambda}_{2\nu}(\bar{\sigma})\mathring{k}^\nu(\bar{\sigma})\Big\} \\
&\quad+\Big\{\mathring{\lambda}_{1\mu}(\sigma)\mathring{k}^\mu(\sigma), \mathring{\xi}_2^\nu(\bar{\sigma})\mathring{\imath}_\nu(\bar{\sigma})\Big\}\Big].
\end{aligned} \tag{6.10}$$

The first term of (6.10) is obtained, using (6.1)

$$\begin{aligned}
&\int d\sigma d\bar{\sigma}\Big\{\mathring{\xi}_1^\mu(\sigma)\mathring{\imath}_\mu(\sigma), \mathring{\xi}_2^\nu(\bar{\sigma})\mathring{\imath}_\nu(\bar{\sigma})\Big\} \\
&= \int d\sigma\Big[\mathring{\imath}_\mu\Big(\mathring{\xi}_2^\nu\partial_\nu\mathring{\xi}_1^\mu - \mathring{\xi}_1^\nu\partial_\nu\mathring{\xi}_2^\mu - \mathring{\mathcal{F}}_{\nu\rho}^{\ \ \mu}\,\mathring{\xi}_1^\nu\mathring{\xi}_2^\rho\Big) \\
&\quad-2\mathring{B}_{\mu\nu\rho}\mathring{k}^\mu\mathring{\xi}_1^\nu\mathring{\xi}_2^\rho\Big].
\end{aligned} \tag{6.11}$$

The second term is obtained, using (6.4)

$$\begin{aligned}
&\int d\sigma d\bar{\sigma}\Big\{\mathring{\lambda}_{1\mu}(\sigma)\mathring{k}^\mu(\sigma), \mathring{\lambda}_{2\nu}(\bar{\sigma})\mathring{k}^\nu(\bar{\sigma})\Big\} \\
&= \int d\sigma\Big[\mathring{k}^\mu\Big(\kappa\mathring{\theta}^{\nu\rho}(\mathring{\lambda}_{2\nu}\partial_\rho\mathring{\lambda}_{1\mu} - \mathring{\lambda}_{1\nu}\partial_\rho\mathring{\lambda}_{2\mu}) - \kappa\mathring{\mathcal{Q}}_\mu^{\ \nu\rho} \\
&\quad\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}\Big) - \mathring{\imath}_\mu\kappa^2\mathring{\mathcal{R}}^{\mu\nu\rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}\Big].
\end{aligned} \tag{6.12}$$

The remaining terms are antisymmetric with respect to $1 \leftrightarrow 2,\ \sigma \leftrightarrow \bar{\sigma}$ interchange. Therefore, it is sufficient to calculate only the first term in the last line of (6.10)

$$\begin{aligned}
&\int d\sigma d\bar{\sigma}\Big\{\mathring{\xi}_1^\mu(\sigma)\mathring{\imath}_\mu(\sigma), \mathring{\lambda}_{2\nu}(\bar{\sigma})\mathring{k}^\nu(\bar{\sigma})\Big\} \\
&= \int d\sigma\Big[\mathring{k}^\mu\Big(-\mathring{\xi}_1^\nu\partial_\nu\mathring{\lambda}_{2\mu} - \mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu}\,\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu}\Big) \\
&\quad+\mathring{\imath}_\mu\Big(\kappa(\mathring{\lambda}_{2\nu}\mathring{\theta}^{\nu\rho})\partial_\rho\mathring{\xi}_1^\mu - \kappa\mathring{\mathcal{Q}}_\rho^{\ \nu\mu}\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu}\Big)\Big] \\
&\quad+\int d\sigma d\bar{\sigma}\kappa\mathring{\xi}_1^\nu(\sigma)\mathring{\lambda}_{2\nu}(\bar{\sigma})\partial_\sigma\delta(\sigma-\bar{\sigma}).
\end{aligned} \tag{6.13}$$

In order to transform the anomalous part, we note that

$$\partial_\sigma\delta(\sigma-\bar{\sigma}) = \frac{1}{2}\partial_\sigma\delta(\sigma-\bar{\sigma}) - \frac{1}{2}\partial_{\bar{\sigma}}\delta(\sigma-\bar{\sigma}), \tag{6.14}$$

and

$$f(\bar{\sigma})\partial_\sigma\delta(\sigma-\bar{\sigma}) = f(\sigma)\partial_\sigma\delta(\sigma-\bar{\sigma}) + f'(\sigma)\delta(\sigma-\bar{\sigma}). \tag{6.15}$$

Applying (6.14) and (6.15) to the last row of (6.13), we obtain

$$\begin{aligned}
&\int d\sigma d\bar{\sigma}\kappa\mathring{\xi}_1^\nu(\sigma)\mathring{\lambda}_{2\nu}(\bar{\sigma})\partial_\sigma\delta(\sigma-\bar{\sigma}) \\
&= \frac{1}{2}\int d\sigma\kappa x'^\mu\Big(\mathring{\xi}_1^\nu\partial_\mu\mathring{\lambda}_{2\nu} - \partial_\mu\mathring{\xi}_1^\nu\mathring{\lambda}_{2\nu}\Big) \\
&\quad+\frac{\kappa}{2}\int d\sigma d\bar{\sigma}\Big(\mathring{\xi}_1^\nu(\sigma)\mathring{\lambda}_{2\nu}(\sigma)\partial_\sigma\delta(\sigma-\bar{\sigma}) \\
&\quad-\mathring{\xi}_1^\nu(\bar{\sigma})\mathring{\lambda}_{2\nu}(\bar{\sigma})\partial_{\bar{\sigma}}\delta(\sigma-\bar{\sigma})\Big) \\
&= \frac{1}{2}\int d\sigma\Big[\mathring{k}^\mu\Big(\mathring{\xi}_1^\nu\partial_\mu\mathring{\lambda}_{2\nu} - \partial_\mu\mathring{\xi}_1^\nu\mathring{\lambda}_{2\nu}\Big) \\
&\quad+\mathring{\imath}_\mu\kappa\mathring{\theta}^{\mu\rho}\Big(\mathring{\xi}_1^\nu\partial_\rho\mathring{\lambda}_{2\nu} - \partial_\rho\mathring{\xi}_1^\nu\mathring{\lambda}_{2\nu}\Big)\Big],
\end{aligned} \tag{6.16}$$

where (5.31) was used, as well as antisymmetry of $\mathring{\theta}$. Substituting (6.16) to (6.13), we obtain

$$\begin{aligned}
&\int d\sigma d\bar{\sigma}\Big\{\mathring{\xi}_1^\mu(\sigma)\mathring{\imath}_\mu(\sigma), \mathring{\lambda}_{2\nu}(\bar{\sigma})\mathring{k}^\nu(\bar{\sigma})\Big\} \\
&= \int d\sigma\Big[\mathring{k}^\mu\Big(\mathring{\xi}_1^\nu(\partial_\mu\mathring{\lambda}_{2\nu} - \partial_\nu\mathring{\lambda}_{2\mu}) \\
&\quad-\frac{1}{2}\partial_\mu(\mathring{\xi}_1\mathring{\lambda}_2) - \mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu}\,\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu}\Big) \\
&\quad+\mathring{\imath}_\mu\Big(\kappa(\mathring{\lambda}_{2\nu}\mathring{\theta}^{\nu\rho})\partial_\rho\mathring{\xi}_1^\mu + \kappa\mathring{\theta}^{\mu\rho}\Big(\mathring{\xi}_1^\nu\partial_\rho\mathring{\lambda}_{2\nu} - \frac{1}{2}\partial_\rho(\mathring{\xi}_1\mathring{\lambda}_2)\Big) \\
&\quad-\kappa\mathring{\mathcal{Q}}_\rho^{\ \nu\mu}\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu}\Big)\Big].
\end{aligned} \tag{6.17}$$

Substituting (6.11), (6.12) and (6.17) into (6.10), we write the full algebra of generator in the form

$$\begin{aligned}
&\Big\{\mathring{G}(\mathring{\Lambda}_1),\ \mathring{G}(\mathring{\Lambda}_2)\Big\} \\
&= -\mathring{G}(\mathring{\Lambda}) \Leftrightarrow \Big\{\mathring{G}(\mathring{\xi}_1, \mathring{\lambda}_1),\ \mathring{G}(\mathring{\xi}_2, \mathring{\lambda}_2)\Big\} = -\mathring{G}(\mathring{\xi}, \mathring{\lambda}),
\end{aligned} \tag{6.18}$$

where

$$\begin{aligned}\mathring{\xi}^\mu &= \mathring{\xi}_1^\nu\partial_\nu\mathring{\xi}_2^\mu - \mathring{\xi}_2^\nu\partial_\nu\mathring{\xi}_1^\mu - \kappa\mathring{\theta}^{\mu\rho}\\ &\times\Big(\mathring{\xi}_1^\nu\partial_\rho\mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\nu\partial_\rho\mathring{\lambda}_{1\nu} - \frac{1}{2}\partial_\rho(\mathring{\xi}_1\mathring{\lambda}_2 - \mathring{\xi}_2\mathring{\lambda}_1)\Big)\\ &+\kappa\mathring{\theta}^{\nu\rho}(\mathring{\lambda}_{1\nu}\partial_\rho\mathring{\xi}_2^\mu - \mathring{\lambda}_{2\nu}\partial_\rho\mathring{\xi}_1^\mu)\\ &+\kappa^2\mathring{\mathcal{R}}^{\mu\nu\rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho} + \mathring{\mathcal{F}}_{\rho\sigma}^{\ \ \mu}\,\mathring{\xi}_1^\rho\mathring{\xi}_2^\sigma\\ &+\kappa\mathring{\mathcal{Q}}_\rho^{\ \nu\mu}(\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\rho\mathring{\lambda}_{1\nu}),\end{aligned} \tag{6.19}$$

and

$$\begin{aligned}\mathring{\lambda}_\mu &= \mathring{\xi}_1^\nu(\partial_\nu\mathring{\lambda}_{2\mu} - \partial_\mu\mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu\mathring{\lambda}_{1\mu} - \partial_\mu\mathring{\lambda}_{1\nu})\\ &+\frac{1}{2}\partial_\mu(\mathring{\xi}_1\mathring{\lambda}_2 - \mathring{\xi}_2\mathring{\lambda}_1)\\ &+\kappa\mathring{\theta}^{\nu\rho}(\mathring{\lambda}_{1\nu}\partial_\rho\mathring{\lambda}_{2\mu} - \mathring{\lambda}_{2\nu}\partial_\rho\mathring{\lambda}_{1\mu})\\ &+2\mathring{B}_{\mu\nu\rho}\mathring{\xi}_1^\nu\mathring{\xi}_2^\rho + \kappa\mathring{\mathcal{Q}}_\mu^{\ \nu\rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho} + \mathring{\mathcal{F}}_{\mu\sigma}^{\ \ \nu}\\ &\times(\mathring{\xi}_1^\sigma\mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\sigma\mathring{\lambda}_{1\nu}).\end{aligned} \tag{6.20}$$

It is possible to rewrite the previous two equations, if we note the relations between the generalized fluxes

$$\mathring{\mathcal{R}}^{\mu\nu\rho} = \mathring{R}^{\mu\nu\rho} + \mathring{\theta}^{\mu\sigma}\mathring{\theta}^{\nu\tau}\mathring{\mathcal{F}}_{\sigma\tau}^{\ \ \rho}, \quad \mathring{\mathcal{Q}}_\mu^{\ \nu\rho} = \mathring{Q}_\mu^{\ \nu\rho} + \mathring{\theta}^{\nu\sigma}\mathring{\mathcal{F}}_{\mu\sigma}^{\ \ \rho}. \tag{6.21}$$

Now we have

$$\begin{aligned}\mathring{\xi}^\mu &= \mathring{\xi}_1^\nu\partial_\nu\mathring{\xi}_2^\mu - \mathring{\xi}_2^\nu\partial_\nu\mathring{\xi}_1^\mu\\ &+\kappa\mathring{\theta}^{\mu\rho}\Big(\mathring{\xi}_1^\nu(\partial_\nu\mathring{\lambda}_{2\rho} - \partial_\rho\mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu\mathring{\lambda}_{1\rho} - \partial_\rho\mathring{\lambda}_{1\nu})\\ &+\frac{1}{2}\partial_\rho(\mathring{\xi}_1\mathring{\lambda}_2 - \mathring{\xi}_2\mathring{\lambda}_1)\Big)\\ &+\kappa\mathring{\xi}_1^\rho\partial_\rho(\mathring{\lambda}_{2\nu}\mathring{\theta}^{\nu\mu}) - \kappa(\mathring{\lambda}_{2\nu}\mathring{\theta}^{\nu\rho})\partial_\rho\mathring{\xi}_1^\mu - \kappa\mathring{\xi}_2^\rho\partial_\rho(\mathring{\lambda}_{1\nu}\mathring{\theta}^{\nu\mu})\\ &+\kappa(\mathring{\lambda}_{1\nu}\mathring{\theta}^{\nu\rho})\partial_\rho\mathring{\xi}_2^\mu + \kappa^2\mathring{R}^{\mu\nu\rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}\\ &+\mathring{\mathcal{F}}_{\rho\sigma}^{\ \ \mu}\,\mathring{\xi}_1^\rho\mathring{\xi}_2^\sigma + \kappa\mathring{\theta}^{\mu\sigma}\mathring{\mathcal{F}}_{\sigma\rho}^{\ \ \nu}\,(\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\rho\mathring{\lambda}_{1\nu})\\ &+\kappa^2\mathring{\theta}^{\mu\sigma}\mathring{\theta}^{\nu\tau}\mathring{\mathcal{F}}_{\sigma\tau}^{\ \ \rho}\,\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho},\end{aligned} \tag{6.22}$$

and

$$\begin{aligned}\mathring{\lambda}_\mu &= \mathring{\xi}_1^\nu(\partial_\nu\mathring{\lambda}_{2\mu} - \partial_\mu\mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu\mathring{\lambda}_{1\mu} - \partial_\mu\mathring{\lambda}_{1\nu})\\ &+\frac{1}{2}\partial_\mu(\mathring{\xi}_1\mathring{\lambda}_2 - \mathring{\xi}_2\mathring{\lambda}_1)\\ &+\kappa\mathring{\theta}^{\nu\rho}(\mathring{\lambda}_{1\nu}\partial_\rho\mathring{\lambda}_{2\mu} - \mathring{\lambda}_{2\nu}\partial_\rho\mathring{\lambda}_{1\mu}) + \kappa\mathring{Q}_\mu^{\ \nu\rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}\\ &+2\mathring{B}_{\mu\nu\rho}\mathring{\xi}_1^\nu\mathring{\xi}_2^\rho + \mathring{\mathcal{F}}_{\mu\sigma}^{\ \ \nu}\\ &\times(\mathring{\xi}_1^\sigma\mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\sigma\mathring{\lambda}_{1\nu}) + \kappa\mathring{\theta}^{\nu\sigma}\mathring{\mathcal{F}}_{\mu\sigma}^{\ \ \rho}\,\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho},\end{aligned} \tag{6.23}$$

where the partial integration was used in the equation (6.22).

The relation (6.18) defines the $\mathring{C}$-twisted Courant bracket

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]_{\mathcal{C}_{\mathring{C}}} = \mathring{\Lambda} \Leftrightarrow [(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]_{\mathcal{C}_{\mathring{C}}} = (\mathring{\xi}, \mathring{\lambda}), \tag{6.24}$$

that gives the same bracket as (5.41). Both (6.19)–(6.20) and (6.22)–(6.23) are the products of $\mathring{C}$-twisted Courant bracket. The former shows explicitly how the gauge parameters depend on the generalized fluxes. In the latter, similarities between the expressions for two parameters is easier to see.

## 6.1 Special cases and relations to other brackets

Even though the non-commutativity parameter $\theta$ and the Kalb Ramond field $B$ are not mutually independent, while obtaining the bracket (6.24) the relation between these fields (2.7) was not used. Therefore, the results stand even if a bi-vector and a 2-form used for twisting are mutually independent. This will turn out to be convenient to analyze the origin of terms appearing in the Courant bracket twisted by $\mathring{C}$.

Primarily, consider the case of zero bi-vector $\theta^{\mu\nu} = 0$ with the 2-form $B_{\mu\nu}$ arbitrary. Consequently, the parameter $\alpha$ (4.8) is zero, while the hyperbolic functions $\mathcal{C}$ and $\mathcal{S}$ are identity matrices. Therefore, the auxiliary fields (5.26) and (5.30) simplify in a following way

$$\mathring{B}_{\mu\nu} \to B_{\mu\nu} \quad \mathring{\theta}^{\mu\nu} \to 0, \tag{6.25}$$

and the twisting matrix $e^{\breve{B}}$ (4.11) becomes the matrix $e^{\hat{B}}$ (3.28). The expressions (6.19) and (6.20) respectively reduce to

$$\mathring{\xi}^\mu = \mathring{\xi}_1^\nu\partial_\nu\mathring{\xi}_2^\mu - \mathring{\xi}_2^\nu\partial_\nu\mathring{\xi}_1^\mu, \tag{6.26}$$

and

$$\begin{aligned}\mathring{\lambda}_\mu &= \mathring{\xi}_1^\nu(\partial_\nu\mathring{\lambda}_{2\mu} - \partial_\mu\mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu\mathring{\lambda}_{1\mu} - \partial_\mu\mathring{\lambda}_{1\nu})\\ &+\frac{1}{2}\partial_\mu(\mathring{\xi}_1\mathring{\lambda}_2 - \mathring{\xi}_2\mathring{\lambda}_1) + 2B_{\mu\nu\rho}\mathring{\xi}_1^\nu\mathring{\xi}_2^\rho,\end{aligned} \tag{6.27}$$

where $B_{\mu\nu\rho}$ is the Kalb–Ramond field strength, given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \tag{6.28}$$

The equations (6.26) and (6.27) define exactly the $B$-twisted Courant bracket (3.27) [9].

Secondarily, consider the case of zero 2-form $B_{\mu\nu} = 0$ and the bi-vector $\theta^{\mu\nu}$ arbitrary. Similarly, $\alpha = 0$ and $\mathcal{C}$ and $\mathcal{S}$ are identity matrices. The auxiliary fields $\mathring{B}_{\mu\nu}$ and $\mathring{\theta}^{\mu\nu}$ are given by

$$\mathring{B}_{\mu\nu} \to 0 \quad \mathring{\theta}^{\mu\nu} \to \theta^{\mu\nu}. \tag{6.29}$$

The twisting matrix $e^{\breve{B}}$ becomes the matrix of $\theta$-transformations $e^{\hat{\theta}}$ (3.30). The gauge parameters (6.19) and (6.20) are respectively given by

$$\begin{aligned}\mathring{\xi}^\mu &= \mathring{\xi}_1^\nu \partial_\nu \mathring{\xi}_2^\mu - \mathring{\xi}_2^\nu \partial_\nu \mathring{\xi}_1^\mu \\ &+ \kappa\theta^{\mu\rho}\Big(\mathring{\xi}_1^\nu(\partial_\nu \mathring{\lambda}_{2\rho} - \partial_\rho \mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu \mathring{\lambda}_{1\rho} - \partial_\rho \mathring{\lambda}_{1\nu}) \\ &+ \frac{1}{2}\partial_\rho(\mathring{\xi}_1 \mathring{\lambda}_2 - \mathring{\xi}_2 \mathring{\lambda}_1)\Big) \\ &+ \kappa \mathring{\xi}_1^\nu \partial_\nu(\mathring{\lambda}_{2\rho}\theta^{\rho\mu}) - \kappa \mathring{\xi}_2^\nu \partial_\nu(\mathring{\lambda}_{1\rho}\theta^{\rho\mu}) \\ &+ \kappa(\mathring{\lambda}_{1\nu}\theta^{\nu\rho})\partial_\rho \mathring{\xi}_2^\mu - \kappa(\mathring{\lambda}_{2\nu}\theta^{\nu\rho})\partial_\rho \mathring{\xi}_1^\mu \\ &+ \kappa^2 R^{\mu\nu\rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}, \end{aligned} \tag{6.30}$$

and

$$\begin{aligned}\mathring{\lambda}_\mu &= \mathring{\xi}_1^\nu(\partial_\nu \mathring{\lambda}_{2\mu} - \partial_\mu \mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu \mathring{\lambda}_{1\mu} - \partial_\mu \mathring{\lambda}_{1\nu}) \\ &+ \frac{1}{2}\partial_\mu(\mathring{\xi}_1 \mathring{\lambda}_2 - \mathring{\xi}_2 \mathring{\lambda}_1) \\ &+ \kappa\theta^{\nu\rho}(\mathring{\lambda}_{1\nu}\partial_\rho \mathring{\lambda}_{2\mu} - \mathring{\lambda}_{2\nu}\partial_\rho \mathring{\lambda}_{1\mu}) + \kappa \mathring{\lambda}_{1\rho}\mathring{\lambda}_{2\nu} Q_\mu^{\ \rho\nu}, \end{aligned} \tag{6.31}$$

where by $Q_\mu^{\ \nu\rho}$ and $R^{\mu\nu\rho}$ we have marked the non-geometric fluxes, given by

$$Q_\mu^{\ \nu\rho} = \partial_\mu \theta^{\nu\rho}, \ \ R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}. \tag{6.32}$$

The bracket defined by these relations is $\theta$-twisted Courant bracket (3.29) [8] and it features the non-geometric fluxes only.

Let us comment on terms in the obtained expressions for gauge parameters (6.22) and (6.23). The first line of (6.22) appears in the Courant bracket and in all brackets that can be obtained from its twisting by either a 2-form or a bi-vector. The next two lines correspond to the terms appearing in the $\theta$-twisted Courant bracket (6.30). The other terms do not appear in either $B$- or $\theta$-twisted Courant bracket.

Similarly, the first line of (6.20) appears in the Courant bracket (2.18) and in all other brackets obtained from its twisting, while the terms in the second line appear exclusively in the $\theta$ twisted Courant bracket (6.27). The first term in the last line appear in the $B$-twisted Courant bracket (6.31), while the rest are some new terms. We see that all the terms that do not appear in neither of two brackets are the terms containing $\mathring{\mathcal{F}}$ flux.

### 6.2 Coordinate free notation

In order to obtain the formulation of the $\mathring{C}$-twisted Courant bracket in the coordinate free notation, independent of the local coordinate system that is used on the manifold, let us firstly provide definitions for a couple of well know brackets and derivatives.

The Lie derivative along the vector field $\xi$ is given by

$$\mathcal{L}_{\mathring{\xi}} = i_{\mathring{\xi}} d + d i_{\mathring{\xi}}, \tag{6.33}$$

with $i_{\mathring{\xi}}$ being the interior product along the vector field $\mathring{\xi}$ and $d$ being the exterior derivative. Using the Lie derivative one easily defines the Lie bracket

$$[\mathring{\xi}_1, \mathring{\xi}_2]_L = \mathcal{L}_{\mathring{\xi}_1}\mathring{\xi}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\xi}_1. \tag{6.34}$$

The generalization of the Lie bracket on a space of 1-forms is a well known Koszul bracket [26]

$$[\mathring{\lambda}_1, \mathring{\lambda}_2]_\theta = \mathcal{L}_{\theta\mathring{\lambda}_1}\mathring{\lambda}_2 - \mathcal{L}_{\theta\mathring{\lambda}_2}\mathring{\lambda}_1 + d(\mathring{\theta}(\mathring{\lambda}_1, \mathring{\lambda}_2)). \tag{6.35}$$

The expressions (6.19) and (6.20) in the coordinate free notation are given by

$$\begin{aligned}\mathring{\xi} &= [\mathring{\xi}_1, \mathring{\xi}_2]_L - [\mathring{\xi}_2, \mathring{\lambda}_1\kappa\mathring{\theta}]_L + [\mathring{\xi}_1, \mathring{\lambda}_2\kappa\mathring{\theta}]_L \\ &-\Big(\mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1 - \frac{1}{2}d(i_{\mathring{\xi}_1}\mathring{\lambda}_2 - i_{\mathring{\xi}_2}\mathring{\lambda}_1)\Big)\kappa\mathring{\theta} \\ &+\mathring{\mathcal{F}}(\mathring{\xi}_1, \mathring{\xi}_2, .) - \kappa\mathring{\theta}\mathring{\mathcal{F}}(\mathring{\lambda}_1, ., \mathring{\xi}_2) + \kappa\mathring{\theta}\mathring{\mathcal{F}}(\mathring{\lambda}_2, ., \mathring{\xi}_1) \\ &+\mathring{\mathcal{R}}(\mathring{\lambda}_1, \mathring{\lambda}_2, .), \end{aligned} \tag{6.36}$$

and

$$\begin{aligned}\mathring{\lambda} &= \mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1 - \frac{1}{2}d(i_{\mathring{\xi}_1}\mathring{\lambda}_2 - i_{\mathring{\xi}_2}\mathring{\lambda}_1) - [\mathring{\lambda}_1, \mathring{\lambda}_2]_{\kappa\mathring{\theta}} \\ &+ \mathring{H}(\mathring{\xi}_1, \mathring{\xi}_2, .) - \mathring{\mathcal{F}}(\mathring{\lambda}_1, ., \mathring{\xi}_2) + \mathring{\mathcal{F}}(\mathring{\lambda}_2, ., \mathring{\xi}_1) \\ &+ \kappa\mathring{\theta}\mathring{\mathcal{F}}(\mathring{\lambda}_1, \mathring{\lambda}_2, .), \end{aligned} \tag{6.37}$$

where

$$\mathring{H} = 2d\mathring{B}. \tag{6.38}$$

We have marked the geometric $H$ flux as $\mathring{H}$, so that it is distinguished from the 2-form $\mathring{B}$. In the local basis, the full term containing $H$-flux is given by

$$\mathring{H}(\mathring{\xi}_1, \mathring{\xi}_2, .)\Big|_\mu = 2\mathring{B}_{\mu\nu\rho}\,\mathring{\xi}_1^\nu \mathring{\xi}_2^\rho. \tag{6.39}$$

Similarly are defined the terms containing $\mathring{\mathcal{F}}$ flux

$$\mathring{\mathcal{F}}(\mathring{\xi}_1, \mathring{\xi}_2, .)\Big|^\mu = \mathring{\mathcal{F}}_{\nu\rho}^{\ \mu}\,\mathring{\xi}_1^\nu \mathring{\xi}_2^\rho, \tag{6.40}$$

and the non-geometric $\mathring{\mathcal{R}}$ flux

$$\mathring{\mathcal{R}}(\mathring{\lambda}_1, \mathring{\lambda}_2, .)\Big|^\mu = \mathring{\mathcal{R}}^{\mu\nu\rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}, \tag{6.41}$$

as well as

$$\mathring{\theta}\mathring{\mathcal{F}}(\mathring{\lambda}_1, ., \mathring{\xi}_2)\Big|^\mu = \mathring{\theta}^{\nu\sigma}\mathring{\mathcal{F}}_{\sigma\rho}^{\ \mu}\,\mathring{\lambda}_{1\nu}\mathring{\xi}_2^\rho. \tag{6.42}$$

It is possible to rewrite the coordinate free notation in terms of the $\mathring{H}$-flux and $\mathring{\theta}$ bi-vector only. The geometric $\mathring{\mathcal{F}}$ flux is just the contraction of the $\mathring{H}$-flux with a bi-vector

$$\mathring{\mathcal{F}} = \kappa\mathring{\theta}\,\mathring{H}. \tag{6.43}$$

The non-geometric $\mathring{\mathcal{R}}$ flux can be rewritten as

$$\mathring{\mathcal{R}} = \frac{1}{2}[\mathring{\theta},\mathring{\theta}]_S + \wedge^3(\kappa\mathring{\theta})\mathring{H}, \tag{6.44}$$

where $\wedge$ is the wedge product, and by $[\mathring{\theta},\mathring{\theta}]_S$ we have marked the Schouten–Nijenhuis bracket [27], given by

$$[\mathring{\theta},\mathring{\theta}]_S\Big|^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}_{\alpha\beta\gamma}\mathring{\theta}^{\sigma\alpha}\partial_\sigma\mathring{\theta}^{\beta\gamma} = 2\mathring{R}^{\mu\nu\rho}, \tag{6.45}$$

where

$$\epsilon^{\mu\nu\rho}_{\alpha\beta\gamma} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\beta & \delta^\rho_\gamma \\ \delta^\nu_\alpha & \delta^\rho_\beta & \delta^\mu_\gamma \\ \delta^\rho_\alpha & \delta^\mu_\beta & \delta^\nu_\gamma \end{vmatrix}. \tag{6.46}$$

Expressing both $\mathring{\mathcal{F}}$ and $\mathring{\mathcal{R}}$ fluxes in terms of the bi-vector $\mathring{\theta}$ and 3-form $\mathring{H}$, we obtain

$$\begin{aligned}\mathring{\xi} &= [\mathring{\xi}_1,\mathring{\xi}_2]_L - [\mathring{\xi}_2,\mathring{\lambda}_1\kappa\mathring{\theta}]_L + [\mathring{\xi}_1,\mathring{\lambda}_2\kappa\mathring{\theta}]_L \\ &\quad -\Big(\mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1 - \frac{1}{2}d(i_{\mathring{\xi}_1}\mathring{\lambda}_2 - i_{\mathring{\xi}_2}\mathring{\lambda}_1)\Big)\kappa\mathring{\theta} \\ &\quad +\frac{\kappa^2}{2}[\mathring{\theta},\mathring{\theta}]_S(\mathring{\lambda}_1,\mathring{\lambda}_2,.) \\ &\quad +\kappa\mathring{\theta}\mathring{H}(.,\mathring{\xi}_1,\mathring{\xi}_2) - \wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2) \\ &\quad + \wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_2,.,\mathring{\xi}_1) + \wedge^3\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,\mathring{\lambda}_2,.),\end{aligned} \tag{6.47}$$

and

$$\begin{aligned}\mathring{\lambda} &= \mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1 - \frac{1}{2}d(i_{\mathring{\xi}_1}\mathring{\lambda}_2 - i_{\mathring{\xi}_2}\mathring{\lambda}_1) - [\mathring{\lambda}_1,\mathring{\lambda}_2]_{\kappa\mathring{\theta}} \\ &\quad +\mathring{H}(\mathring{\xi}_1,\mathring{\xi}_2,.) - \kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2) \\ &\quad +\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2) + \wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,\mathring{\lambda}_2,.).\end{aligned} \tag{6.48}$$

The term $\kappa\mathring{\theta}\mathring{H}(.,\mathring{\xi}_1,\mathring{\xi}_2)$ is the wedge product of a bi-vector with a 3-form, contracted with two vectors, given by

$$\Big(\kappa\mathring{\theta}\mathring{H}(.,\mathring{\xi}_1,\mathring{\xi}_2)\Big)^\mu = 2\kappa\mathring{\theta}^{\mu\nu}\mathring{B}_{\nu\rho\sigma}\mathring{\xi}_1^\rho\mathring{\xi}_2^\sigma, \tag{6.49}$$

and $\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2)$ is similarly defined, with the 1-form contracted instead of one vector field

$$\Big(\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2)\Big)_\mu = 2\kappa\mathring{\theta}^{\nu\rho}\mathring{B}_{\rho\mu\sigma}\mathring{\lambda}_{1\nu}\mathring{\xi}_2^\sigma. \tag{6.50}$$

The terms like $\wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2)$ are the wedge product of two bi-vectors with a 3-form, contracted with the 1-form $\mathring{\lambda}_1$ and the vector $\mathring{\xi}_2$

$$\Big(\wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2)\Big)^\mu = 2\kappa^2\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\mu\rho}\,\mathring{B}_{\sigma\rho\tau}\mathring{\lambda}_{1\nu}\mathring{\xi}_2^\tau, \tag{6.51}$$

and similarly when contraction is done with two forms

$$\Big(\wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,\mathring{\lambda}_2,.)\Big)_\mu = 2\kappa^2\mathring{\theta}^{\tau\rho}\mathring{\theta}^{\nu\sigma}\,\mathring{B}_{\rho\sigma\mu}\mathring{\lambda}_{1\tau}\mathring{\lambda}_{2\nu}. \tag{6.52}$$

Lastly, the term $\wedge^3\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,\mathring{\lambda}_2,.)$ is obtained by taking a wedge product of three bi-vectors with a 3-form and than contracting it with two 1-forms. It is given by

$$\Big(\wedge^3\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,\mathring{\lambda}_2,.)\Big)^\mu = 2\kappa^3\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\rho\tau}\mathring{\theta}^{\mu\lambda}\mathring{B}_{\sigma\tau\lambda}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}, \tag{6.53}$$

## 7 Star brackets

The expressions for gauge parameters (6.36) and (6.37) produce some well known bracket, such as Lie bracket and Koszul bracket. The remaining terms can be combined so that they are expressed by some new brackets, acting on pairs of generalized vectors. It turns out that these brackets produce a generalized vector, where the vector part $\mathring{\xi}^\mu$ and the 1-form part $\mathring{\lambda}_\mu$ are related by $\mathring{\xi}^\mu = \kappa\mathring{\theta}^{\mu\nu}\mathring{\lambda}_\nu$, effectively resulting in the graphs in the generalized cotangent bundle $T^\star M$ of the bi-vector $\mathring{\theta}$, i.e. $\xi = \kappa\theta(.,\lambda)$. The star brackets can be interpreted in terms of projections on isotropic subspaces.

### 7.1 $\theta$-star bracket

Let us firstly consider the second line of (6.22) and the first line of (6.23). When combined, they define a bracket acting on a pair of generalized vectors

$$[\mathring{\Lambda}_1,\mathring{\Lambda}_2]^\star_{\mathring{\theta}} = \mathring{\Lambda}^\star \Leftrightarrow [(\mathring{\xi}_1,\mathring{\lambda}_1),(\mathring{\xi}_2,\mathring{\lambda}_2)]^\star_{\mathring{\theta}} = (\mathring{\xi}_\star,\mathring{\lambda}^\star), \tag{7.1}$$

where

$$\begin{aligned}\mathring{\xi}^\mu_\star &= \kappa\mathring{\theta}^{\mu\rho}\Big(\mathring{\xi}_1^\nu(\partial_\nu\mathring{\lambda}_{2\rho} - \partial_\rho\mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu\mathring{\lambda}_{1\rho} - \partial_\rho\mathring{\lambda}_{1\nu}) \\ &\quad + \frac{1}{2}\partial_\rho(\mathring{\xi}_1\mathring{\lambda}_2 - \mathring{\xi}_2\mathring{\lambda}_1)\Big),\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}\mathring{\lambda}^\star_\mu &= \mathring{\xi}_1^\nu(\partial_\nu\mathring{\lambda}_{2\mu} - \partial_\mu\mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu\mathring{\lambda}_{1\mu} - \partial_\mu\mathring{\lambda}_{1\nu}) \\ &\quad +\frac{1}{2}\partial_\mu(\mathring{\xi}_1\mathring{\lambda}_2 - \mathring{\xi}_2\mathring{\lambda}_1),\end{aligned} \tag{7.3}$$

from which one easily reads the relation

$$\mathring{\xi}^\mu_\star = \kappa\mathring{\theta}^{\mu\rho}\mathring{\lambda}^\star_\rho. \tag{7.4}$$

In a coordinate free notation, this bracket can be written as

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]^{\star}_{\mathring{\theta}} = [(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]^{\star}_{\mathring{\theta}} = \Big(\kappa\mathring{\theta}\Big(., \mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1\Big), \mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1\Big). \tag{7.5}$$

### 7.2 $B\theta$-star bracket

The remaining terms contain geometric $\mathring{H}$ and $\mathring{\mathcal{F}}$ fluxes. Note that they are the only terms that depend on the new effective Kalb–Ramond field $\mathring{B}$. Firstly, we mark the last line of (6.23) as

$$\mathring{\lambda}^*_\mu = 2\mathring{B}_{\mu\nu\rho}\mathring{\xi}_1^\nu\mathring{\xi}_2^\rho + \mathring{\mathcal{F}}_{\mu\sigma}^{\ \ \nu}\left(\mathring{\xi}_1^\sigma\mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\sigma\mathring{\lambda}_{1\nu}\right) + \kappa\mathring{\theta}^{\nu\sigma}\mathring{\mathcal{F}}_{\mu\sigma}^{\ \ \rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}. \tag{7.6}$$

Secondly, using the definition of $\mathring{\mathcal{F}}$ (6.3) and the fact that $\mathring{\theta}$ is antisymmetric, the last line of (6.22) can be rewritten as

$$\begin{aligned}\mathring{\xi}^\mu_* &= 2\kappa\mathring{\theta}^{\mu\nu}\mathring{B}_{\nu\rho\sigma}\mathring{\xi}_1^\rho\mathring{\xi}_2^\sigma + \kappa\mathring{\theta}^{\mu\sigma}\mathring{\mathcal{F}}_{\sigma\rho}^{\ \ \nu}\left(\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\rho\mathring{\lambda}_{1\nu}\right)\\ &\quad + \kappa^2\mathring{\theta}^{\mu\nu}\mathring{\theta}^{\tau\sigma}\mathring{\mathcal{F}}_{\nu\sigma}^{\ \ \rho}\mathring{\lambda}_{1\tau}\mathring{\lambda}_{2\rho}\\ &= \kappa\mathring{\theta}^{\mu\nu}\mathring{\lambda}^*_\nu.\end{aligned} \tag{7.7}$$

Now relations (7.6) and (7.7) define the $B\theta$-star bracket by

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]^*_{\mathring{B}\mathring{\theta}} = \mathring{\Lambda}^* \Leftrightarrow [(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]^*_{\mathring{B}\mathring{\theta}} = (\mathring{\xi}_*, \mathring{\lambda}^*), \tag{7.8}$$

We can write the full bracket (6.24) as

$$\begin{aligned}&[(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]_{\mathcal{C}_{\mathring{C}}}\\ &= \Big([\mathring{\xi}_1, \mathring{\xi}_2]_L - [\mathring{\xi}_2, \mathring{\lambda}_1\kappa\mathring{\theta}]_L + [\mathring{\xi}_1, \mathring{\lambda}_2\kappa\mathring{\theta}]_L\\ &\quad + \frac{\kappa^2}{2}[\mathring{\theta}, \mathring{\theta}]_S(\mathring{\lambda}_1, \mathring{\lambda}_2, .), -[\mathring{\lambda}_1, \mathring{\lambda}_2]_{\kappa\mathring{\theta}}\Big)\\ &\quad + [(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]^*_{\mathring{B},\mathring{\theta}} + [(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]^\star_{\mathring{\theta}}.\end{aligned} \tag{7.9}$$

### 7.3 Isotropic subspaces

In order to give an interpretation to newly obtained starred brackets, it is convenient to consider isotropic subspaces. A subspace $L$ is isotropic if the inner product (2.12) of any two generalized vectors from that sub-bundle is zero

$$\langle\Lambda_1, \Lambda_2\rangle = 0, \quad \Lambda_1, \Lambda_2 \in L. \tag{7.10}$$

From (2.12), one easily finds that

$$\xi_i^\mu = \kappa\,\theta^{\mu\nu}\lambda_{i\nu}. \quad (i = 1, 2) \quad \theta^{\mu\nu} = -\theta^{\nu\mu}, \tag{7.11}$$

for any bi-vector $\theta$, and

$$\lambda_{i\mu} = 2B_{\mu\nu}\xi_i^\mu. \quad (i = 1, 2) \quad B_{\mu\nu} = -B_{\nu\mu}, \tag{7.12}$$

for any 2-form $B$ satisfy the condition (7.10).

Furthermore, it is straightforward to introduce projections on these isotropic subspaces by

$$\mathcal{I}^\theta(\Lambda^M) = \mathcal{I}^\theta(\xi^\mu, \lambda_\mu) = (\kappa\,\theta^{\mu\nu}\lambda_\nu, \lambda_\mu), \tag{7.13}$$

and

$$\mathcal{I}_B(\Lambda^M) = \mathcal{I}_B(\xi^\mu, \lambda_\mu) = (\xi^\mu, 2B_{\mu\nu}\xi^\nu). \tag{7.14}$$

Now it is easy to give an interpretation to star brackets. The $\theta$-star bracket (7.1) can be defined as the projection of the Courant bracket (3.29) on the isotropic subspace (7.13)

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]^\star_{\mathring{\theta}} = \mathcal{I}^{\mathring{\theta}}\Big([\mathring{\Lambda}_1, \mathring{\Lambda}_2]_{\mathcal{C}}\Big). \tag{7.15}$$

Similarly, note that all the terms in (6.37) that do not appear in the $\theta$-twisted Courant bracket, contribute exactly to the $B\theta$-star bracket. From that, it is easy to obtain the definition of the $B\theta$-star bracket (7.8)

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]^*_{\mathring{B}\mathring{\theta}} = \mathcal{I}^{\mathring{\theta}}\Big([\mathring{\Lambda}_1, \mathring{\Lambda}_2]_{\mathcal{C}_{\mathring{C}}}\Big) - \mathcal{I}^{\mathring{\theta}}\Big([\mathring{\Lambda}_1, \mathring{\Lambda}_2]_{\mathcal{C}_{\mathring{\theta}}}\Big). \tag{7.16}$$

## 8 Courant bracket twisted by $B$ and $\theta$

Now it is possible to write down the expression for the Courant bracket twisted by $B$ and $\theta$ (4.2), using the expression for $\mathring{C}$-twisted Courant bracket

$$[\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathcal{C}_{B\theta}} = A^{-1}[A\breve{\Lambda}_1, A\breve{\Lambda}_2]_{\mathcal{C}_{\mathring{C}}}, \tag{8.1}$$

where $A$ is defined in (5.36). Substituting (8.1) into (6.36), we obtain

$$\begin{aligned}\breve{\xi} &= \mathcal{C}^{-1}[\mathcal{C}\breve{\xi}_1, \mathcal{C}\breve{\xi}_2]_L\\ &\quad - \mathcal{C}^{-1}[\mathcal{C}\breve{\xi}_2, \breve{\lambda}_1\kappa\mathcal{C}^{-1}\mathring{\theta}]_L + \mathcal{C}^{-1}[\mathcal{C}\breve{\xi}_1, \breve{\lambda}_2\kappa\mathcal{C}^{-1}\mathring{\theta}]_L\\ &\quad - \Big(\mathcal{L}_{\mathcal{C}\breve{\xi}_1}(\breve{\lambda}_2\mathcal{C}^{-1}) - \mathcal{L}_{\mathcal{C}\breve{\xi}_2}(\breve{\lambda}_1\mathcal{C}^{-1})\\ &\quad - \frac{1}{2}d(i_{\breve{\xi}_1}\breve{\lambda}_2 - i_{\breve{\xi}_2}\breve{\lambda}_1)\Big)\kappa\mathring{\theta}\mathcal{C}^{-1}\\ &\quad + \frac{\kappa^2}{2}\mathcal{C}^{-1}[\mathring{\theta}, \mathring{\theta}]_S(\breve{\lambda}_1\mathcal{C}^{-1}, \breve{\lambda}_2\mathcal{C}^{-1}, .)\\ &\quad + \kappa\mathcal{C}^{-1}\mathring{\theta}\mathring{H}(., \mathcal{C}\breve{\xi}_1, \mathcal{C}\breve{\xi}_2)\\ &\quad - \mathcal{C}^{-1}\wedge^2\kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_1\mathcal{C}^{-1}, ., \mathcal{C}\breve{\xi}_2)\\ &\quad + \mathcal{C}^{-1}\wedge^2\kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_2\mathcal{C}^{-1}, ., \mathcal{C}\breve{\xi}_1)\\ &\quad + \mathcal{C}^{-1}\wedge^3\kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_1\mathcal{C}^{-1}, \breve{\lambda}_2\mathcal{C}^{-1}, .),\end{aligned} \tag{8.2}$$

and similarly, substituting (8.1) into (6.37), we obtain

$$\begin{aligned}\breve{\lambda} = &\Big(\mathcal{L}_{\mathcal{C}\breve{\xi}_1}(\breve{\lambda}_2\mathcal{C}^{-1}) - \mathcal{L}_{\mathcal{C}\breve{\xi}_2}(\breve{\lambda}_1\mathcal{C}^{-1}) - \frac{1}{2}d(i_{\breve{\xi}_1}\breve{\lambda}_2 - i_{\breve{\xi}_2}\breve{\lambda}_1)\Big)\mathcal{C}\\ &+\mathring{H}(\mathcal{C}\breve{\xi}_1, \mathcal{C}\breve{\xi}_2, .)\mathcal{C}\\ &-[\breve{\lambda}_1\mathcal{C}^{-1}, \breve{\lambda}_2\mathcal{C}^{-1}]_{\kappa\mathring{\theta}}\mathcal{C} - \kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_1\mathcal{C}^{-1}, ., \mathcal{C}\breve{\xi}_2)\mathcal{C}\\ &+\kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_2\mathcal{C}^{-1}, ., \mathcal{C}\breve{\xi}_1)\mathcal{C}\\ &+\wedge^2\kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_1\mathcal{C}^{-1}, \breve{\lambda}_2\mathcal{C}^{-1}, .)\mathcal{C}, \end{aligned} \tag{8.3}$$

where $\mathcal{C}^\mu_{\ \nu} = \Big(\cosh\sqrt{\alpha}\Big)^\mu_{\ \nu}$ and $\breve{\Lambda} = (\breve{\xi}, \breve{\lambda})$ (5.21). This is somewhat a cumbersome expression, making it difficult to work with. To simplify it, with the accordance of our convention, we define the twisted Lie bracket by

$$[\breve{\xi}_1, \breve{\xi}_2]_{L_\mathcal{C}} = \mathcal{C}^{-1}[\mathcal{C}\breve{\xi}_1, \mathcal{C}\breve{\xi}_2]_L, \tag{8.4}$$

as well as the twisted Schouten–Nijenhuis bracket

$$\Big([\breve{\theta}, \breve{\theta}]_{S_\mathcal{C}}\Big)^{\mu\nu\rho} = (\mathcal{C}^{-1})^\mu_{\ \sigma}(\mathcal{C}^{-1})^\nu_{\ \lambda}(\mathcal{C}^{-1})^\rho_{\ \tau}\Big([\mathcal{C}\breve{\theta}, \mathcal{C}\breve{\theta}]_S\Big)^{\sigma\lambda\tau}, \tag{8.5}$$

and twisted Koszul bracket

$$[\breve{\lambda}_1, \breve{\lambda}_2]_{\theta_\mathcal{C}} = (\mathcal{C}^T)^{-1}[\mathcal{C}^T\breve{\lambda}_1, \mathcal{C}^T\breve{\lambda}_2]_{\theta\mathcal{C}}, \tag{8.6}$$

where the transpose of the matrix is necessary because the Koszul bracket acts on 1-forms. Now, the first three terms of (8.2) can be written as

$$[\breve{\xi}_1, \breve{\xi}_2]_{L_\mathcal{C}} - [\breve{\xi}_2, \breve{\lambda}_1\kappa\mathcal{C}^{-1}\breve{\theta}]_{L_\mathcal{C}} + [\breve{\xi}_1, \breve{\lambda}_2\kappa\mathcal{C}^{-1}\breve{\theta}]_{L_\mathcal{C}}, \tag{8.7}$$

where

$$\breve{\theta}^{\mu\nu} = (\mathcal{C}^{-1})^\mu_{\ \rho}\mathring{\theta}^{\rho\nu} = \mathcal{S}^\mu_{\ \rho}\theta^{\rho\nu}. \tag{8.8}$$

The second line of (8.2) and the first line of (8.3) originating from $\mathring{\theta}$ star bracket (7.1) can be easily combined into

$$[(\mathcal{C}\breve{\xi}_1, \breve{\lambda}_1\mathcal{C}^{-1}), (\mathcal{C}\breve{\xi}_2, \breve{\lambda}_2\mathcal{C}^{-1})]^\star_{\mathcal{C}^{-1}\breve{\theta}}\,\mathcal{C}. \tag{8.9}$$

The terms originating from $\mathring{B}\mathring{\theta}$ star bracket (7.8) are combined into

$$[(\breve{\xi}_1, \breve{\lambda}_1), (\breve{\xi}_2, \breve{\lambda}_2]^*_{\breve{B}, \mathcal{C}^{-1}\breve{\theta}}, \tag{8.10}$$

where

$$\begin{aligned}\breve{B}_{\mu\nu\rho} = \mathring{B}_{\alpha\beta\gamma}\mathcal{C}^\alpha_{\ \mu}\mathcal{C}^\beta_{\ \nu}\mathcal{C}^\gamma_{\ \rho} = &\Big(\partial_\alpha(B\mathcal{S}\mathcal{C}^{-1})_{\beta\gamma}\\ &+\partial_\beta(B\mathcal{S}\mathcal{C}^{-1})_{\gamma\alpha} + \partial_\gamma(B\mathcal{S}\mathcal{C}^{-1})_{\alpha\beta}\Big)\mathcal{C}^\alpha_{\ \mu}\mathcal{C}^\beta_{\ \nu}\mathcal{C}^\gamma_{\ \rho}. \end{aligned} \tag{8.11}$$

The expressions for the Courant bracket twisted by both $B$ and $\theta$ can be written in a form

$$\begin{aligned}&[(\breve{\xi}_1, \breve{\lambda}_1), (\breve{\xi}_2, \breve{\lambda}_2)]_{\mathcal{C}_{B\theta}}\\ &= \Big([\breve{\xi}_1, \breve{\xi}_2]_{L_\mathcal{C}} - [\breve{\xi}_2, \breve{\lambda}_1\kappa\mathcal{C}^{-1}\breve{\theta}]_{L_\mathcal{C}} + [\breve{\xi}_1, \breve{\lambda}_2\kappa\mathcal{C}^{-1}\breve{\theta}]_{L_\mathcal{C}}\\ &+\frac{\kappa^2}{2}[\breve{\theta}, \breve{\theta}]_{S_\mathcal{C}}(\breve{\lambda}_1, \breve{\lambda}_2, .), -[\breve{\lambda}_1, \breve{\lambda}_2]_{\theta_\mathcal{C}}\Big)\\ &+[(\mathcal{C}\breve{\xi}_1, \breve{\lambda}_1\mathcal{C}^{-1}), (\mathcal{C}\breve{\xi}_2, \breve{\lambda}_2\mathcal{C}^{-1})]^\star_{\mathcal{C}^{-1}\breve{\theta}}\,\mathcal{C}\\ &+[(\breve{\xi}_1, \breve{\lambda}_1), (\breve{\xi}_2, \breve{\lambda}_2]^*_{\breve{B}, \mathcal{C}^{-1}\breve{\theta}}. \end{aligned} \tag{8.12}$$

When the Courant bracket is twisted by both $B$ and $\theta$, it results in a bracket similar to $\mathring{C}$-twisted Courant bracket, where Lie brackets, Schouten Nijenhuis bracket and Koszul bracket are all twisted as well.

## 9 Conclusion

We examined various twists of the Courant bracket, that appear in the Poisson bracket algebra of symmetry generators written in a suitable basis, obtained acting on the double canonical variable (2.4) by the appropriate elements of $O(D, D)$ group. In this paper, we considered the transformations that twists the Courant bracket simultaneously by a 2-form $B$ and a bi-vector $\theta$. When these fields are mutually T-dual, the generator obtained by this transformation is invariant upon self T-duality.

We obtained the matrix elements of this transformation, that we denoted $e^{\breve{B}}$ (4.11), expressed in terms of the hyperbolic functions of a parameter $\alpha$ (4.8). In order to avoid working with such a complicated expression, we considered another $O(D, D)$ transformation $A$ (5.36) and introduced a new generator, written in a basis of auxiliary currents $\mathring{\imath}_\mu$ and $\mathring{k}^\mu$. The Poisson bracket algebra of a new generator was obtained and it gave rise to the $\mathring{C}$-twisted Courant bracket, which contains all of the fluxes.

The generalized fluxes were obtained using different methods [10–12,16–18]. In our approach, we started by an $O(D, D)$ transformation that twists the Courant bracket simultaneously by a 2-form $B$ and bi-vector $\theta$, making it manifestly self T-dual. We obtained the expressions for all fluxes, written in terms of the effective fields

$$\begin{aligned}\mathring{B}_{\mu\nu} &= B_{\mu\rho}\Big(\frac{\tanh\sqrt{2\kappa\theta B}}{\sqrt{2\kappa\theta B}}\Big)^\rho_{\ \nu},\\ \mathring{\theta}^{\mu\nu} &= \Big(\frac{\sinh 2\sqrt{2\kappa\theta B}}{2\sqrt{2\kappa\theta B}}\Big)^\mu_{\ \sigma}\theta^{\sigma\nu}. \end{aligned} \tag{9.1}$$

The fluxes, as a function of these effective fields, appear naturally in the Poisson bracket algebra of such generators.

Similar bracket was obtained in the algebra of generalized currents in [11,12] and is sometimes referred to as the Roytenberg bracket [10]. In that approach, phase space has been changed, so that the momentum algebra gives rise to the $H$-flux, after which the generalized currents were defined in terms of the open string fields. The bracket obtained this way corresponds to the Courant bracket that was firstly twisted by $B$ field, and then by a bi-vector $\theta$. The matrix of that twist is given by

$$e^R = e^{\hat{\theta}} e^{\hat{B}} = \begin{pmatrix} \delta^\mu_\nu + \alpha^\mu_\nu & \kappa\theta^{\mu\nu} \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix}. \tag{9.2}$$

In our approach, we obtained the transformations that twists the Courant bracket at the same time by $B$ and $\theta$, resulting in a $\mathring{C}$-twisted Courant bracket. As a consequence, the $\mathring{C}$-twisted Courant bracket is defined in terms of auxiliary fields $\mathring{B}$ (5.26) and $\mathring{\theta}$ (5.30), that are themselves function of $\alpha$. This is not the case in [11,12]. The Roytenberg bracket calculated therein can be also obtained following our approach by twisting with the matrix

$$e^C = A e^{\breve{B}} = \begin{pmatrix} \mathcal{C}^2 & \kappa(\mathcal{C}\mathcal{S}\theta) \\ 2B\mathcal{C}\mathcal{S} & 1 \end{pmatrix}, \tag{9.3}$$

demanding that the background fields are infinitesimal $B \sim \epsilon$, $\theta \sim \epsilon$ and keeping the terms up to $\epsilon^2$. With these conditions, $e^C$ (9.3) becomes exactly $e^R$ (9.2), and the bracket becomes the Roytenberg bracket.

Analyzing the $\mathring{C}$-twisted Courant bracket, we recognized that certain terms can be seen as new brackets on the space of generalized vectors, that we named star brackets. We demonstrated that they are closely related to projections on isotropic spaces. It is well established that the Courant bracket does not satisfy the Jacobi identity in general case. The sub-bundles on which the Jacobi identity is satisfied are known as Dirac structures, which as a necessary condition need to be subsets of isotropic spaces. Therefore, the star brackets might provide future insights into integrability conditions for the $\mathring{C}$-twisted Courant bracket [28].

In the end, we obtained the Courant bracket twisted at the same time by $B$ and $\theta$ by considering the generator in the basis spanned by $\breve{\iota}$ and $\breve{k}$, equivalent to undoing $A$ transformation, used to simplify calculations. With the introduction of new fields $\breve{B}_{\mu\nu}$ and $\breve{\theta}^{\mu\nu}$, this bracket has a similar form as $\mathring{C}$-twisted Courant bracket, whereby the Lie, Schouten–Nijenhuis and Koszul brackets became their twisted counterparts.

It has already been established that $B$-twisted and $\theta$-twisted Courant brackets appear in the generator algebra defined in bases related by self T-duality [13]. When the Courant bracket is twisted by both $B$ and $\theta$, it is self T-dual, and as such, represent the self T-dual extension of the Lie bracket that includes all fluxes. It has been already shown [8] how the Hamiltonian can be obtained acting with $B$-transformations on diagonal generalized metric. The same method could be replicated with the twisting matrix $e^{\breve{B}}$, that would give rise to a different Hamiltonian, whose further analysis can provide interesting insights in the role that the Courant bracket twisted by both $B$ and $\theta$ plays in understanding T-duality.

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# Twisted C-Brackets

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**We consider the double field formulation of the closed bosonic string theory, and calculate the Poisson bracket algebra of the symmetry generators governing both general coordinate and local gauge transformations. Parameters of both of these symmetries depend on a double coordinate, defined as a direct sum of the initial and T-dual coordinate. When no antisymmetric field is present, the *C*-bracket appears as the Lie bracket generalization in a double theory. With the introduction of the Kalb-Ramond field, the *B*-twisted *C*-bracket appears, while with the introduction of the non-commutativity parameter, the $\theta$-twisted *C*-bracket appears. We present the derivation of these brackets and comment on their relations to analogous twisted Courant brackets and T-duality.**

## 1. Introduction

The development of string theory led to the discovery of T-duality,[1–3] a transformation needed to establish a connection between the existing string theories. In case of closed strings, the emergence of T-duality is closely related to the fact that they can wind around compact dimension. Two theories defined in different geometries can create the same string spectrum, and are said to be T-dual. The T-duality was generalized into procedures of finding mutually physically equivalent descriptions of a string.[4,5]

The T-duality opened a search for the connections between the relevant mathematical structures within T-dual theories. Some of them appear upon consideration of the generators of symmetries, which in the classical theory act on the energy-momentum tensor via Poisson bracket. The changes of energy-momentum tensor under the symmetry transformations can be interpreted as the changes in the space-time fields under diffeomorphisms and local gauge transformations. Under these transformations, the Virasoro algebra is not broken, and the corresponding conformal field theories of the string are isomorphic.[6,7] In the Poisson bracket algebra of symmetry generator governing both diffeomorphisms and local gauge transformations the Courant bracket[8,9] is obtained. It is known that under T-duality the bosonic string symmetries transform into one another,[10] and as such, the Courant bracket represents the self T-dual extension of the Lie bracket.

The Courant bracket and its various deformations have found their appearances in string theory in multiple occasions. For instance, the relevant string theory fluxes[11] appeared in the twisted Courant brackets,[12] that were obtained from the world-sheet[13–15] and in the algebroid relations of the appropriate vielbeins.[16–18] They can be obtained in the algebra of symmetry generators as well, provided the generator is expressed in the appropriate non-canonical basis. These non-canonical variables, also known as currents, give rise to the string fluxes as its structure constants in their Poisson bracket algebra. In the presence of Kalb-Ramond field, the *B*-twisted Courant bracket was obtained in the generator algebra.[19] In the self T-dual picture, obtained by swapping the canonical momenta with coordinate $\sigma$-derivatives, and the background fields with their T-duals, the algebra bracket becomes the $\theta$-twisted Courant bracket.[19] The former twisted Courant bracket contains geometric *H*-flux, while the latter contains non-geometric *Q* and *R* fluxes. It was showed that they are mutually related by T-duality.[20] The Courant bracket twisted simultaneously by both *B* and $\theta$ has also been constructed from the symmetry generator algebra.[21] The bracket contains all string fluxes and is invariant under T-duality.

The observations regarding T-duality interchanging different twisted Courant brackets were done in the single theory. However, there is a growing interest in the double theory approach, in which all background fields are function both of the initial coordinate and the T-dual coordinate. The T-duality is a symmetry of a double action, and can be simply realized by permutation of the coordinates.[22–25] We would like to see how the interchange of different twisted Courant brackets and their fluxes under T-duality is manifested in the double field theory.

The first steps towards the double field theory development were done in the early 1990s[26–29] (for nice reviews on the subject, see [30, 31]). In [32, 33], generalized Lie derivative was constructed in the double space, which acting on another vector defined in the double space gives rise to what is now known as the *C*-bracket. It is an extension of the Lie bracket to the double space, and found important role in governing symmetries algebra in double field theory. The action on the double torus invariant under both initial and T-dual diffeomorphisms was formulated in [34], where background fields were expanded around constant values up to the cubic order. It was later generalized to a generalized metric formulation of the double theory.[35] In [36],

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it was demonstrated that the $C$-bracket reduces to the Courant bracket, when dependence on T-dual coordinates is neglected. From this correspondence between $C$- and Courant bracket we expect that the twisted Courant brackets are related to some twisted $C$-bracket, which we consider in this paper by analyzing the Poisson bracket algebra of appropriate generators.

We begin by outlining the properties of a closed bosonic string in the double space, when only metric tensor is present. We consider the standard Poisson bracket relations within the initial and T-dual phase space, while the algebra of the variables from different phase spaces is obtained from the requirement that the T-duality transformations commute with the Poisson bracket relations. The algebra of both generalized coordinate transformations and their T-duals in this background was already considered in [19], where it was showed that it produces the $C$-bracket. We include this case in this paper for completeness and as a prerequisite for future chapters.

Afterwards, we outline a simple procedure of obtaining the twisted Courant bracket from the generator written in a suitable basis, introduced in [21]. We demonstrate that this procedure is applicable to the double theory as well, and that it can be used to define new brackets in double theory. By analogy with how twisted Courant brackets are defined, we define these brackets as the twisted $C$-brackets. From a differential graded algebra perspective the twisted $C$-brackets were briefly discussed in [37].

Subsequently, we introduce the Kalb-Ramond field dependent on both coordinates $x^\mu$ and $y_\mu$ to the metric only background. We show that the usual expressions for the Lagrangian and the Hamiltonian are obtained, when $B$-transformations act on a diagonal generalized metric. The same transformation can be seen as keeping the generalized metric diagonal, while transforming the basis. The transformed basis includes non-canonical momenta, the Poisson bracket of which includes the double theory flux. We calculate the Poisson bracket algebra of this new generator, and obtain what in our notation corresponds to the $B$-twisted $C$-bracket.

We consider the background characterized with the T-dual metric tensor only, and introduce the non-commutative parameter with the action of $\theta$-transformations. The Hamiltonian once again can be expressed in a diagonal form, in the basis spanned by non-canonical momenta. In their Poisson bracket algebra, we show that non-canonical momenta feature $\hat{\Theta}$ double flux. We express the symmetry generator in this basis, and obtain the $\theta$-twisted $C$-bracket in their algebra governed by Poisson bracket. By construction, this bracket is T-dual to the $B$-twisted $C$-bracket.

We note that both the initial theory, in which all variables depend solely on $x^\mu$, and the T-dual theory, in which all variables depend solely on $y_\mu$ can be obtained from the double theory, by demanding that there is no dependence on $y_\mu$ in the former, and no dependence on $x^\mu$ in the latter case. In the initial theory, the $B$-twisted $C$-bracket reduces to the $B$-twisted Courant bracket, while the $\theta$-twisted $C$-bracket becomes the $\theta$-twisted Courant bracket. On the other hand, in the T-dual theory the $B$-twisted $C$-bracket reduces to the $\theta$-twisted Courant bracket, while the $\theta$-twisted $C$-bracket becomes the $B$-twisted Courant bracket. This way, we show that projections of the twisted $C$-brackets to mutually T-dual phase spaces establish the twisted Courant brackets that are mutually T-dual.[20]

## 2. Bosonic String in a Double Metric Space

Consider the closed bosonic string in the background defined solely by the coordinate dependent metric field $G_{\mu\nu}(X)$. The metric tensor is a function of a double coordinate $X^M$, defined in a direct sum of the initial coordinate space, characterized by $x^\mu$, and T-dual coordinate space, characterized by $y_\mu$

$$X^M = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix}, \tag{2.1}$$

where $\mu = 0, 1 \dots D-1,\ D = 26$. In the conformal gauge, the Lagrangian density is given by [38, 39]

$$\mathcal{L} = \frac{\kappa}{2}\partial_+ X^M G_{MN} \partial_- X^N\,,\ \partial_\pm X^M = \dot{X}^M \pm X'^N\,, \tag{2.2}$$

where

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & (G^{-1})^{\mu\nu} \end{pmatrix}. \tag{2.3}$$

The canonical momenta are obtained from the variation of the Lagrangian

$$\Pi_M = \kappa G_{MN}\dot{X}^N \equiv \begin{pmatrix} \pi_\mu \\ {}^\star\pi^\mu \end{pmatrix},$$

$$\pi_\mu = \kappa G_{\mu\nu}\dot{x}^\nu\,,\ {}^\star\pi^\mu = \kappa (G^{-1})^{\mu\nu}\dot{y}_\nu\,, \tag{2.4}$$

while the Legendre transformation of the Lagrangian gives the canonical Hamiltonian

$$\mathcal{H}_C = \frac{1}{2\kappa}\Pi_M G^{MN}\Pi_N + \frac{\kappa}{2}X'^M G_{MN} X'^N\,, \tag{2.5}$$

where

$$G^{MN} = \begin{pmatrix} (G^{-1})^{\mu\nu} & 0 \\ 0 & G_{\mu\nu} \end{pmatrix}. \tag{2.6}$$

The indices on $G$ are lowered by the $O(D, D)$ invariant metric, given by

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.7}$$

i.e. $G^{MN} = \eta^{MP}\eta^{NQ}G_{PQ}$.

In the conventional field theory, the transformation of the metric tensor under diffeomorphisms parametrized with $\xi$ is governed by the Lie derivative $\delta_\xi G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu}$, and generated by $\xi^\mu\pi_\mu$. The Poisson bracket algebra of such a generator gives rise to the Lie bracket.

Analogously, one defines the diffeomorphisms in the double space by

$$G(\Lambda) = \int d\sigma \mathcal{G}_\Lambda\,, \tag{2.8}$$

with

$$\mathcal{G}_\Lambda = (\Lambda^T)^M(X)\eta_{MN}\Pi^N \qquad \Longleftrightarrow \qquad \mathcal{G}_\Lambda = \langle \Lambda, \Pi \rangle\,, \tag{2.9}$$

where $\Lambda^M$ are the parameters, given by

$$\Lambda^M(X) = \begin{pmatrix} \xi^\mu(x,y) \\ \lambda_\mu(x,y) \end{pmatrix}, \tag{2.10}$$

and $\langle , \rangle$ is the natural inner product in a double space

$$\langle \Lambda, X\rangle = (\Lambda^T)^M \eta_{MN} X^N . \tag{2.11}$$

To give an interpretation to the generator (2.9), we expand the inner product and obtain

$$\mathcal{G}_\Lambda = \xi^\mu(x,y)\pi_\mu + \lambda_\mu(x,y)^\star\pi^\mu . \tag{2.12}$$

The first term represents the diffeomorphisms, while the second term represents the local gauge transformations, equivalent to the T-dual diffeomorphisms.[10] The algebra of this generator leads to the double theory generalization of the Lie bracket.

### 2.1. *C*-Bracket

To obtain the Poisson bracket algebra of the generator $\mathcal{G}_\Lambda$, one uses the Poisson bracket relations between the double variables (for details, see [19])

$$\{\Pi_M(\sigma), \Pi_N(\bar\sigma)\} \simeq \kappa\, \eta_{MN}\delta'(\sigma-\bar\sigma), \tag{2.13}$$
$$\{X'^M(\sigma), \Pi_N(\bar\sigma)\} = \delta^M_N\, \delta'(\sigma-\bar\sigma),$$

the same relations as used in [32, 33]. The terms stemming from bracket between only coordinates $\{X^M(\sigma), X^N(\bar\sigma)\}$ are canceled by the appropriate choice of Heaviside theta function, which sets all parameters and fields annihilated by the operator

$$\eta^{MN}\partial_M\partial_N = \partial^M\partial_M = 0, \tag{2.14}$$

where $\partial^M$ are the derivatives in a double theory, given by

$$\partial_M = \begin{pmatrix} \partial_\mu \\ \tilde\partial^\mu \end{pmatrix}, \qquad \left(\partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \quad \tilde\partial^\mu \equiv \frac{\partial}{\partial y_\mu}\right). \tag{2.15}$$

Furthermore, we also require that the product of any two fields $\phi$ and $\psi$ is also annihilated by (2.14), i.e.

$$\begin{aligned}\partial^M\partial_M(\phi\psi) &= (\partial^M\partial_M\phi)\,\psi + 2\partial^M\phi\,\partial_M\psi + \phi\,\partial^M\partial_M\psi \\ &= 2\partial^M\phi\,\partial_M\psi = 0 .\end{aligned} \tag{2.16}$$

The conditions (2.14) and (2.16) were also used in [30, 32, 33], and are referred to as strong condition.

It follows that

$$\begin{aligned}\{\mathcal{G}_{\Lambda_1}(\sigma), \mathcal{G}_{\Lambda_2}(\bar\sigma)\} &= -\mathcal{G}_{[\Lambda_1,\Lambda_2]_C}(\sigma)\delta(\sigma-\bar\sigma) \\ &\quad + \frac{\kappa}{2}\big(\langle\Lambda_1,\Lambda_2\rangle(\sigma) + \langle\Lambda_1,\Lambda_2\rangle(\bar\sigma)\big)\delta'(\sigma-\bar\sigma) .\end{aligned} \tag{2.17}$$

where $[\Lambda_1,\Lambda_2]_C$ stands for the $C$-bracket, firstly defined by Siegel.[32,33] It was defined by

$$\begin{aligned}{[\Lambda_1,\Lambda_2]_C}^Q &= \Lambda_1^N\partial_N\Lambda_2^Q - \Lambda_2^N\partial_N\Lambda_1^Q \\ &\quad -\frac{1}{2}\eta_{NP}\left(\Lambda_1^N\partial^Q\Lambda_2^P - \Lambda_2^N\partial^Q\Lambda_1^P\right).\end{aligned} \tag{2.18}$$

It is a well known fact that if the parameters $\Lambda$ do not depend on the T-dual coordinates $y_\mu$, the $C$-bracket reduces to the Courant bracket[8]

$$[\Lambda_1,\Lambda_2]_C \to [\Lambda_1,\Lambda_2]_{\mathcal{C}} = \Lambda \equiv \begin{pmatrix} \xi \\ \lambda \end{pmatrix}, \tag{2.19}$$

where

$$\xi^\mu = \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu, \tag{2.20}$$

$$\lambda_\mu = \xi_1^\nu(\partial_\nu\lambda_{2\mu} - \partial_\mu\lambda_{2\nu}) - \xi_2^\nu(\partial_\nu\lambda_{1\mu} - \partial_\mu\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\lambda_2 - \xi_2\lambda_1).$$

The Courant bracket is defined on the generalized tangent bundle, that is a sum of the tangent and the cotangent bundle over a manifold. The elements of the generalized tangent bundle are the generalized vectors that combine vectors and 1-forms into a single entity.[40]

We see that the symmetry generator governing general coordinate and local gauge transformations produces the $C$-bracket, in a same way that the generator of general coordinate transformations produces the Lie bracket. The $C$-bracket is therefore the T-dual invariant generalization of the Lie bracket in a double space, when there is no antisymmetric field.

### 2.2. *O*(*D*, *D*) Group

At the end of this chapter, we outline the mathematical requirements that allows us to introduce the antisymmetric field to the double theory, and obtain the twisted $C$-bracket for any orthogonal transformation. Assume a transformation $\mathcal{O}$ that transforms momenta and gauge parameters as $\hat\Pi = \mathcal{O}\Pi$, and $\hat\Lambda = \mathcal{O}\Lambda$. The generator (2.9) in terms of these new variables is given by

$$\hat{\mathcal{G}}_{\hat\Lambda} = \langle\hat\Lambda, \hat\Pi\rangle = \langle\mathcal{O}\Lambda, \mathcal{O}\Pi\rangle . \tag{2.21}$$

If furthermore, we demand that the transformation is orthogonal,[1] so that

$$\mathcal{O}^T\eta\mathcal{O} = \eta, \tag{2.22}$$

the relation (2.17) becomes

$$\begin{aligned}&\left\{\langle\hat\Lambda_1,\hat\Pi\rangle(\sigma), \langle\hat\Lambda_2,\hat\Pi\rangle(\bar\sigma)\right\} \\ &\quad = -\langle\mathcal{O}[\mathcal{O}^{-1}\hat\Lambda_1, \mathcal{O}^{-1}\hat\Lambda_2]_C, \hat\Pi\rangle(\sigma)\delta(\sigma-\bar\sigma) \\ &\quad + \frac{\kappa}{2}\big(\langle\hat\Lambda_1,\hat\Lambda_2\rangle(\sigma) + \langle\hat\Lambda_1,\hat\Lambda_2\rangle(\bar\sigma)\big)\delta'(\sigma-\bar\sigma),\end{aligned} \tag{2.23}$$

the right hand side of which gives rise to the twisted $C$-bracket

$$[\hat\Lambda_1,\hat\Lambda_2]_{C_\mathcal{O}} = \mathcal{O}[\mathcal{O}^{-1}\hat\Lambda_1, \mathcal{O}^{-1}\hat\Lambda_2]_C . \tag{2.24}$$

 

This is the twisted $C$-bracket, in a similar way that twisted Poisson, or twisted Courant brackets are defined.

We see that the every $O(D,D)$ transformation has a corresponding twisted $C$-bracket (2.24). It can be obtained in the Poisson bracket algebra of generator in the basis of new momenta $\hat{\Pi} = \mathcal{O}\Pi$. This procedure, that was applied for different twisted Courant brackets in[19,21], will be used to obtain the $C$-bracket twisted by a 2-form, and by a bi-vector.

## 3. Bosonic String in a Complete Double Space

To obtain the action describing the string moving in doubled space described by both metric $G(X)$ and the antisymmetric Kalb-Ramond field $B(X)$, one considers the $B$-transformations

$$(e^{\hat{B}})^M{}_N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta_\mu^\nu \end{pmatrix}, \tag{3.1}$$

where

$$\hat{B}^M{}_N = \begin{pmatrix} 0 & 0 \\ 2B_{\mu\nu} & 0 \end{pmatrix}. \tag{3.2}$$

One easily finds

$$((e^{\hat{B}})^T)_M{}^N = \begin{pmatrix} \delta^\nu_\mu & -2B_{\mu\nu} \\ 0 & \delta^\mu_\nu \end{pmatrix}, \tag{3.3}$$

and that the $B$-transformations are $O(D,D)$ (2.22), i.e.

$$((e^{\hat{B}})^T)_M{}^K\, \eta_{KL}\, (e^{\hat{B}})^L{}_N = \eta_{MN}. \tag{3.4}$$

The doubled action is governed by the generalized metric $H_{MN}$, the $B$-transformation of the doubled metric $G_{MN}$, defined in (2.3)

$$H_{MN} = ((e^{\hat{B}})^T)_M{}^K\, G_{KL}\, (e^{\hat{B}})^L{}_N = \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}, \tag{3.5}$$

where $G^E$ is the effective metric, given by

$$G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}. \tag{3.6}$$

The Lagrangian density for closed bosonic string is now given by

$$\mathcal{L} = \frac{\kappa}{2}\partial_+ X^M H_{MN} \partial_- X^N. \tag{3.7}$$

This theory was considered in a constant and weakly curved background in [24, 25], where it was shown that this representation of doubled theory makes T-duality just a coordinate permutation, and represents all T-dual theories in an unified manner.

The equation of motion for (3.7) is

$$\partial_+(H_{MN}\partial_- X^N) + \partial_-(H_{MN}\partial_+ X^N) = 0. \tag{3.8}$$

The conjugate momenta in a double space

$$\Pi_M = \begin{pmatrix} \pi_\mu \\ {}^\star\pi^\mu \end{pmatrix}, \tag{3.9}$$

are as usual obtained by varying the Lagrangian density over $\dot{X}^M$

$$\Pi_M = \kappa H_{MN}\dot{X}^N, \tag{3.10}$$

from which we have

$$\pi_\mu = G^E_{\mu\nu}\dot{x}^\nu - 2(BG^{-1})_\mu{}^\nu \dot{y}_\nu, \tag{3.11}$$

and

$${}^\star\pi^\mu = (G^{-1})^{\mu\nu}\dot{y}_\nu + 2(G^{-1}B)^\mu{}_\nu \dot{x}^\nu. \tag{3.12}$$

The canonical Hamiltonian is given by

$$\mathcal{H}_C = \frac{1}{2\kappa}\Pi_M H^{MN}\Pi_N + \frac{\kappa}{2}X'^M H_{MN} X'^N. \tag{3.13}$$

### 3.1. T-Duality

The T-duality relations were firstly obtained by Buscher[4] in case of existence of a global isometry. As the initial theory is described with the metric tensor $G_{\mu\nu}$ and the anti-symmetric Kalb-Ramond field $B_{\mu\nu}$, the T-dual theory is described with the T-dual metric tensor ${}^\star G^{\mu\nu}$ and T-dual ${}^\star B^{\mu\nu}$ field, where

$${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}, \tag{3.14}$$

and $\theta^{\mu\nu}$ is the non-commutativity parameter, given by

$$\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1})^{\mu\rho}B_{\rho\sigma}(G^{-1})^{\sigma\nu}. \tag{3.15}$$

In a Lagrangian approach, relations between the coordinates of mutually T-dual phase spaces are given by

$$\partial_\pm x^\mu \simeq -\kappa\theta^{\mu\nu}_\pm \partial_\pm y_\nu, \quad \partial_\pm y_\mu \simeq -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu, \tag{3.16}$$

where

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}, \quad \theta^{\mu\nu}_\pm = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}. \tag{3.17}$$

In a double space approach, the above T-duality relations are combined so that they appear simply as a permutation of doubled coordinates[22–25]

$$\partial_\pm X^M \simeq \pm\eta^{MN}H_{NK}\,\partial_\pm X^K, \tag{3.18}$$

or equivalently as

$$\Pi_M \simeq \kappa\,\eta_{MN}X'^M. \tag{3.19}$$

The consistency condition[24] for the T-dual transformation (3.18) is actually the equation of motion of the doubled theory (3.8).

When the T-duality rules (3.16) are applied to relations (3.11) and (3.12), one easily obtains the usual expressions for the initial theory momenta

$$\pi_\mu = \kappa G_{\mu\nu}\dot{x}^\nu - 2\kappa B_{\mu\nu}x'^\nu, \tag{3.20}$$

and similarly for the T-dual momenta

$$ {}^\star\pi^\mu = \kappa (G_E^{-1})^{\mu\nu} \dot{y}_\nu - \kappa^2 \theta^{\mu\nu} y'_\nu \,. \tag{3.21}$$

Lastly, let us comment on the fact that T-duality transforms the equations of motions into the Bianchi identities of the T-dual theory.[5] In the double space formulation, the equations of motion and Bianchi identities are united[24] in a single relation (3.8).

### 3.2. *C*-Bracket Twisted by *B*

To obtain the $C$-bracket twisted by $B$, we rewrite the Hamiltonian (3.13) in terms of diagonal $G_{MN}$

$$\mathcal{H}_C = \frac{1}{2\kappa} \hat{\Pi}_M G^{MN} \hat{\Pi}_N + \frac{\kappa}{2} \hat{X}'^M G_{MN} \hat{X}'^N \,, \tag{3.22}$$

where the new momenta are

$$\hat{\Pi}^M = (e^{\hat{B}})^M{}_N \Pi^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} {}^\star\pi^\nu \\ \pi_\nu \end{pmatrix} = \begin{pmatrix} {}^\star\pi^\mu \\ \pi_\mu + 2B_{\mu\nu}{}^\star\pi^\nu \end{pmatrix} \equiv \begin{pmatrix} {}^\star\pi^\mu \\ \hat{\pi}_\mu \end{pmatrix} , \tag{3.23}$$

and new coordinates

$$\hat{X}'^M = (e^{\hat{B}})^M{}_N X'^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} x'^\nu \\ y'_\nu \end{pmatrix} = \begin{pmatrix} x'^\mu \\ y'_\mu + 2B_{\mu\nu} x'^\nu \end{pmatrix} \equiv \begin{pmatrix} x'^\mu \\ \hat{y}'_\mu \end{pmatrix} . \tag{3.24}$$

We see that the $\sigma$-derivative $X'^M$ transforms as a vector under the $B$-shifts. The T-duality relations are preserved, i.e.

$$\hat{\Pi}_M \simeq \kappa \eta_{MN} \hat{X}'^N \,. \tag{3.25}$$

Let us now introduce new symmetry parameters

$$\hat{\Lambda}^M = (e^{\hat{B}})^M{}_N \Lambda^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu \\ \lambda_\mu + 2B_{\mu\nu}\xi^\nu \end{pmatrix} \equiv \begin{pmatrix} \xi^\mu \\ \hat{\lambda}_\mu \end{pmatrix} . \tag{3.26}$$

The symmetry generator can be written as

$$\hat{\mathcal{G}}_{\hat{\Lambda}} = \hat{\Lambda}^M \eta_{MN} \hat{\Pi}^N \,, \tag{3.27}$$

and its algebra as

$$\{\hat{\mathcal{G}}_{\hat{\Lambda}_1}(\sigma), \hat{\mathcal{G}}_{\hat{\Lambda}_2}(\bar{\sigma})\} = -\hat{\mathcal{G}}_{[\hat{\Lambda}_1,\hat{\Lambda}_2]_{C_B}}(\sigma)\delta(\sigma-\bar{\sigma}) + \frac{\kappa}{2}\big(\langle\hat{\Lambda}_1,\hat{\Lambda}_2\rangle(\sigma) + \langle\hat{\Lambda}_1,\hat{\Lambda}_2\rangle(\bar{\sigma})\big)\delta'(\sigma-\bar{\sigma}) \,, \tag{3.28}$$

where

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{C_B} = e^{\hat{B}}[e^{-\hat{B}}\hat{\Lambda}_1, e^{-\hat{B}}\hat{\Lambda}_2]_C \,. \tag{3.29}$$

This is a definition of the $B$-twisted C-bracket. To obtain its expression, let us firstly obtain the expressions for Poisson brackets between the new momenta and symmetry parameters. Using (3.23), one obtains

$$\begin{aligned}\{\hat{\Pi}^M(\sigma), \hat{\Pi}^N(\bar{\sigma})\} &= \{(e^{\hat{B}}\Pi)^M(\sigma), (e^{\hat{B}}\Pi)^N(\bar{\sigma})\} \\ &= (e^{\hat{B}})^M_J(\sigma)(e^{\hat{B}})^N_K(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} \\ &\quad -(e^{\hat{B}})^M_J \partial^J (e^{\hat{B}})^N_K \Pi^K \delta(\sigma-\bar{\sigma}) \\ &\quad +(e^{\hat{B}})^N_J \partial^J (e^{\hat{B}})^M_K \Pi^K \delta(\sigma-\bar{\sigma}) \,,\end{aligned} \tag{3.30}$$

where unless written otherwise, dependence on $\sigma$ is assumed. To obtain the first term, we apply the T-dual relations (2.13)

$$(e^{\hat{B}})^M_J(\sigma)(e^{\hat{B}})^N_K(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} \simeq \kappa\left[e^B(\sigma)\eta e^{B^T}(\bar{\sigma})\right]^{MN}\delta'(\sigma-\bar{\sigma}) \,. \tag{3.31}$$

Using the properties of the $\delta$ function

$$f(\bar{\sigma})\partial_\sigma\delta(\sigma-\bar{\sigma}) = f(\sigma)\partial_\sigma\delta(\sigma-\bar{\sigma}) + f'(\sigma)\delta(\sigma-\bar{\sigma}) \,, \tag{3.32}$$

the term becomes

$$\kappa\eta^{MN}\delta'(\sigma-\bar{\sigma}) + \kappa(e^{\hat{B}})^M_P \eta^{PR}\partial_Q((e^{\hat{B}})^T)_R{}^N X'^Q \delta(\sigma-\bar{\sigma}) \,, \tag{3.33}$$

where we have used the orthogonal property of $B$-transformation (3.4)

$$\kappa(e^{\hat{B}})^M_P \eta^{PR}((e^{\hat{B}})^T)_R{}^N \delta'(\sigma-\bar{\sigma}) = \kappa\eta^{MN}\delta'(\sigma-\bar{\sigma}) \,. \tag{3.34}$$

After another application of T-duality, the other term becomes

$$\kappa(e^{\hat{B}})^M_P \eta^{PR}\partial_Q((e^{\hat{B}})^T)_R{}^N X'^Q \delta(\sigma-\bar{\sigma}) \simeq (e^{\hat{B}})^M_P \partial_Q \hat{B}^{PN}\Pi^Q \delta(\sigma-\bar{\sigma}) \,. \tag{3.35}$$

Using the properties of $\hat{B}^M_N$ matrix (3.2)

$$\hat{B}^M{}_K \hat{B}^K{}_N = 0, \quad \hat{B}^M{}_K \partial^Q \hat{B}^K{}_N = 0, \quad (e^{\hat{B}})^M{}_N = \delta^M{}_N + \hat{B}^M{}_N, \tag{3.36}$$

and substituting (3.23) and (3.33) into (3.30), one obtains

$$\{\hat{\Pi}^M(\sigma), \hat{\Pi}^N(\bar{\sigma})\} = -\hat{B}^{MNQ}\hat{\Pi}_Q\delta(\sigma-\bar{\sigma}) + A^{MN}(\sigma-\bar{\sigma}) \,, \tag{3.37}$$

where $A^{MN}$ is the anomalous term, where

$$A^{MN}(\sigma-\bar{\sigma}) \simeq \kappa\eta^{MN}\delta'(\sigma-\bar{\sigma}) \,, \tag{3.38}$$

and $\hat{B}^{MNQ}$ is the flux, where

$$\begin{aligned}\hat{B}^{MNQ} &= B^{MNQ} + S^{MNQ} \\ B^{MNQ} &= \partial^M \hat{B}^{NQ} + \partial^N \hat{B}^{QM} + \partial^Q \hat{B}^{MN} \\ S^{MNQ} &= \hat{B}^M{}_K \partial^K \hat{B}^{NQ} + \hat{B}^N{}_K \partial^K \hat{B}^{QM} + \hat{B}^Q{}_K \partial^K \hat{B}^{MN} \,.\end{aligned} \tag{3.39}$$

The flux can be also written in a more compact way as

$$\hat{B}^{MNQ} = \hat{\partial}^M \hat{B}^{NQ} + \hat{\partial}^N \hat{B}^{QM} + \hat{\partial}^Q \hat{B}^{MN}\,, \tag{3.40}$$

where we have introduced new derivatives

$$\hat{\partial}^M = (e^{\hat{B}})^M{}_K \partial^K = \partial^M + \hat{B}^M_K\, \partial^K\,. \tag{3.41}$$

The other necessary algebra is straightforwardly obtained

$$\{\hat{\Lambda}^M(\sigma), \hat{\Pi}^N(\bar{\sigma})\} = \hat{\partial}^N \hat{\Lambda}^M \delta(\sigma - \bar{\sigma})\,. \tag{3.42}$$

The careful reader might notice that we wrote the equality instead of T-duality sign in the relations (3.37), even though the relations between the momenta $\Pi^M$ were given in terms of their T-duals (2.13). This is possible since the two successive applications of T-duality act as the identity operation. However, on the anomalous part we actually applied T-duality relations only once, the second application will follow soon.

Now we are ready to calculate the full bracket

$$\begin{aligned}\{\hat{\mathcal{G}}_{\hat{\Lambda}_1}(\sigma), \hat{\mathcal{G}}_{\hat{\Lambda}_2}(\bar{\sigma})\} = {} & \hat{\Lambda}^M_1(\sigma)\hat{\Lambda}^N_2(\bar{\sigma}) A_{MN} \\ & - \hat{\Lambda}_{1M}\hat{\Lambda}_{2N}\hat{B}^{MNQ}\hat{\Pi}_Q \delta(\sigma - \bar{\sigma}) \\ & + \hat{\Pi}_Q\left[\hat{\Lambda}^N_2 \hat{\partial}_N \hat{\Lambda}^Q_1 - \hat{\Lambda}^N_1 \hat{\partial}_N \hat{\Lambda}^Q_2\right]\delta(\sigma - \bar{\sigma})\,.\end{aligned} \tag{3.43}$$

Again, using (3.32) and (3.38), we obtain

$$\begin{aligned} & \hat{\Lambda}^M_1(\sigma)\hat{\Lambda}^N_2(\bar{\sigma}) A_{MN}(\sigma - \bar{\sigma}) \\ & \simeq \kappa\langle \hat{\Lambda}_1(\sigma), \hat{\Lambda}_2(\sigma)\rangle \delta'(\sigma - \bar{\sigma}) + \kappa \langle \hat{\Lambda}_1(\sigma), \hat{\Lambda}'_2(\sigma)\rangle \delta(\sigma - \bar{\sigma}) \\ & = \frac{\kappa}{2}\left(2\langle \hat{\Lambda}_1, \hat{\Lambda}_2\rangle \delta'(\sigma - \bar{\sigma}) + \langle \hat{\Lambda}_1, \hat{\Lambda}_2\rangle' \delta(\sigma - \bar{\sigma})\right) \\ & \quad + \frac{\kappa}{2}\left(\langle \hat{\Lambda}_1, \hat{\Lambda}'_2\rangle - \langle \hat{\Lambda}'_1, \hat{\Lambda}_2\rangle\right)\delta(\sigma - \bar{\sigma}) \\ & = \frac{\kappa}{2}\left(\langle \hat{\Lambda}_1, \hat{\Lambda}_2\rangle(\sigma) + \langle \hat{\Lambda}_1, \hat{\Lambda}_2\rangle(\bar{\sigma})\right)\delta'(\sigma - \bar{\sigma}) \\ & + \frac{\kappa}{2}\left(\langle \hat{\Lambda}_1, \hat{\Lambda}'_2\rangle - \langle \hat{\Lambda}'_1, \hat{\Lambda}_2\rangle\right)\delta(\sigma - \bar{\sigma})\,. \end{aligned} \tag{3.44}$$

The anomalous part correspond exactly to the anomalous part of the relation (3.28). It does not contribute to the twisted $C$-bracket. Using the T-duality relations (3.19) and definitions (2.11) and (3.41), we write the remaining part as

$$\begin{aligned}\frac{\kappa}{2}\left(\langle \hat{\Lambda}_1, \hat{\Lambda}'_2\rangle - \langle \hat{\Lambda}'_1, \hat{\Lambda}_2\rangle\right) & = \frac{\kappa}{2}\eta_{MN}\left(\hat{\Lambda}^M_1 \partial_Q \hat{\Lambda}^N_2 - \hat{\Lambda}^M_2 \partial_Q \hat{\Lambda}^N_1\right) X'^Q \\ & \simeq \frac{1}{2}\eta_{MN}\left(\hat{\Lambda}^M_1 \partial^Q \hat{\Lambda}^N_2 - \hat{\Lambda}^M_2 \partial^Q \hat{\Lambda}^N_1\right)\Pi_Q \\ & = \frac{1}{2}\eta_{MN}\left(\hat{\Lambda}^M_1 \hat{\partial}^Q \hat{\Lambda}^N_2 - \hat{\Lambda}^M_2 \hat{\partial}^Q \hat{\Lambda}^N_1\right)\hat{\Pi}_Q\,.\end{aligned} \tag{3.45}$$

Therefore, we applied the T-duality the second time on the last term contributing to the $B$-twisted $C$-bracket. The full bracket is given by

$$\begin{aligned} & [\hat{\Lambda}_1, \hat{\Lambda}_2]_{C_B}{}^Q = \hat{\Lambda}^N_1 \hat{\partial}_N \hat{\Lambda}^Q_2 - \hat{\Lambda}^N_2 \hat{\partial}_N \hat{\Lambda}^Q_1 \\ & -\frac{1}{2}\eta_{MN}\left(\hat{\Lambda}^M_1 \hat{\partial}^Q \hat{\Lambda}^N_2 - \hat{\Lambda}^M_2 \hat{\partial}^Q \hat{\Lambda}^N_1\right) + \hat{\Lambda}_{1M}\hat{\Lambda}_{2N}\hat{B}^{MNQ}\,. \end{aligned} \tag{3.46}$$

This is the $C$-bracket twisted by $B$. Substituting the expansion of derivative (3.41), we can separate terms that contain $B$ from those that do not

$$\begin{aligned}[\hat{\Lambda}_1, \hat{\Lambda}_2]_{C_B}{}^Q = {} & \hat{\Lambda}^N_1 \partial_N \hat{\Lambda}^Q_2 - \hat{\Lambda}^N_2 \partial_N \hat{\Lambda}^Q_1 \\ & -\frac{1}{2}\eta_{MN}\left(\hat{\Lambda}^M_1 \partial^Q \hat{\Lambda}^N_2 - \hat{\Lambda}^M_2 \partial^Q \hat{\Lambda}^N_1\right) \\ & + \hat{B}^N{}_R\left(\hat{\Lambda}_{1N}\partial^R \hat{\Lambda}^Q_2 - \hat{\Lambda}_{2N}\partial^R \hat{\Lambda}^Q_1\right) \\ & -\frac{1}{2}\hat{B}^Q{}_R\left(\hat{\Lambda}_{1N}\partial^R \hat{\Lambda}^N_2 - \hat{\Lambda}_{2N}\partial^R \hat{\Lambda}^N_1\right) \\ & + \hat{\Lambda}_{1M}\hat{\Lambda}_{2N}\hat{B}^{MNQ} \end{aligned} \tag{3.47}$$

The first line is the $C$-bracket, while the other two lines are corrections due to twisting. If the Kalb-Ramond field is zero, the bracket reduces to the $C$-bracket.

In [41], authors discussed the conditions under which $B$-shifts are automorphisms of the $C$-bracket, i.e. $e^{\hat{B}}[\Lambda_1, \Lambda_2]_C = [e^{\hat{B}}\Lambda_1, e^{\hat{B}}\Lambda_2]_C$, which they found to be correct for $\hat{B}^M_N \partial^N = 0$ and $\partial_M \hat{B}_{NR} + cyclic = 0$. We considered a more general case of arbitrary $B$-shifts and obtained $B$-twisted $C$-bracket, from which above conditions for automorphism can be read directly.

## 4. C-Bracket Twisted by $\theta$

In this chapter, we will show how the $\theta$-twisted $C$-bracket can be obtained from the generator's algebra. By analogy, consider the string moving in the background characterized only with the T-dual metric tensor. The generalized metric is given by

$${}^\star G_{MN} = \begin{pmatrix} {}^\star G^{-1}_{\mu\nu} & 0 \\ 0 & {}^\star G^{\mu\nu}\end{pmatrix} = \begin{pmatrix} G^E_{\mu\nu} & 0 \\ 0 & (G^{-1}_E)^{\mu\nu}\end{pmatrix}\,, \tag{4.1}$$

where $G_E$ is defined in (3.6). Now let us consider another important $O(D, D)$ transformation realized by

$$(e^{\hat{\theta}})^M{}_N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix}\,, \tag{4.2}$$

where

$$\hat{\theta}^M{}_N = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 0 & 0\end{pmatrix} = \begin{pmatrix} 0 & 2\,{}^\star B^{\mu\nu} \\ 0 & 0\end{pmatrix}\,. \tag{4.3}$$

This is known as a $\theta$-transformation. We have

$$((e^{\hat{\theta}})^T)_M{}^N = \begin{pmatrix} \delta^\nu_\mu & 0 \\ -\kappa\theta^{\mu\nu} & \delta^\mu_\nu\end{pmatrix}\,, \tag{4.4}$$

from which we verify that it is indeed $O(D, D)$

$$((e^{\hat{\theta}})^T)_M^{\ L}\,\eta_{LK}\,(e^{\hat{\theta}})^K_{\ N} = \eta_{MN}\,. \tag{4.5}$$

Under this transformation, the diagonal generalized metric (4.1) goes to

$$\begin{aligned} {}^\star H_{MN} &= ((e^{\hat{\theta}})^T)_M^{\ L}\,{}^\star G_{LK}(e^{\hat{\theta}})^K_{\ N} \\ &= \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}, \end{aligned} \tag{4.6}$$

which is exactly equal to the generalized metric (3.5).

The Hamiltonian (3.13) can be rewritten in terms of ${}^\star G_{MN}$

$$\mathcal{H}_C = \frac{1}{2\kappa}\breve{\Pi}_M{}^\star G^{MN}\breve{\Pi}_N + \frac{\kappa}{2}\breve{X}'^M{}^\star G_{MN}\breve{X}'^N\,, \tag{4.7}$$

where

$$\begin{aligned} \breve{X}'^M &= (e^{\hat{\theta}})^M_{\ N}X'^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix}\begin{pmatrix} x'^\nu \\ y'_\nu \end{pmatrix} \\ &= \begin{pmatrix} x'^\mu + \kappa\theta^{\mu\nu}y'_\nu \\ y'_\mu \end{pmatrix} \equiv \begin{pmatrix} \breve{x}'^\mu \\ y'_\mu \end{pmatrix}. \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \breve{\Pi}^M &= (e^{\hat{\theta}})^M_{\ N}\Pi^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix}\begin{pmatrix} {}^\star\pi^\nu \\ \pi_\nu \end{pmatrix} \\ &= \begin{pmatrix} {}^\star\pi^\mu + \kappa\theta^{\mu\nu}\pi_\nu \\ \pi_\mu \end{pmatrix} \equiv \begin{pmatrix} {}^\star\breve{\pi}^\mu \\ \pi_\mu \end{pmatrix}. \end{aligned} \tag{4.9}$$

The new symmetry parameters are given by

$$\begin{aligned} \breve{\Lambda}^M &= (e^{\hat{\theta}})^M_{\ N}\Lambda^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix}\begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu + \kappa\theta^{\mu\nu}\lambda_\nu \\ \lambda_\mu \end{pmatrix} \\ &\equiv \begin{pmatrix} \breve{\xi}^\mu \\ \lambda_\mu \end{pmatrix}, \end{aligned} \tag{4.10}$$

in terms of which we write the symmetry generator as

$$\breve{\mathcal{G}}_{\breve{\Lambda}} = \breve{\Lambda}^M\eta_{MN}\breve{\Pi}^N\,. \tag{4.11}$$

Its algebra results in the $\theta$-twisted $C$-bracket

$$\begin{aligned} \{\breve{\mathcal{G}}_{\breve{\Lambda}_1}(\sigma), \breve{\mathcal{G}}_{\breve{\Lambda}_2}(\bar{\sigma})\} &= -\breve{\mathcal{G}}_{[\breve{\Lambda}_1,\breve{\Lambda}_2]_{C_\theta}}(\sigma)\delta(\sigma-\bar{\sigma}) \\ &\quad + \frac{\kappa}{2}\big(\langle\breve{\Lambda}_1,\breve{\Lambda}_2\rangle(\sigma) + \langle\breve{\Lambda}_1,\breve{\Lambda}_2\rangle(\bar{\sigma})\big)\delta'(\sigma-\bar{\sigma})\,, \end{aligned} \tag{4.12}$$

where

$$[\breve{\Lambda}_1, \breve{\Lambda}_2]_{C_\theta} = e^{\hat{\theta}}[e^{-\hat{\theta}}\breve{\Lambda}_1, e^{-\hat{\theta}}\breve{\Lambda}_2]_C\,. \tag{4.13}$$

Let us outline the algebra relations necessary for obtaining this bracket. Firstly, we have

$$\begin{aligned} \{\breve{\Pi}^M(\sigma), \breve{\Pi}^N(\bar{\sigma})\} &= \{(e^{\hat{\theta}}\Pi)^M(\sigma), (e^{\hat{\theta}}\Pi)^N(\bar{\sigma})\} \\ &= (e^{\hat{\theta}})^M_{\ J}(\sigma)(e^{\hat{\theta}})^N_{\ K}(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} \\ &\quad -(e^{\hat{\theta}})^M_{\ J}\partial^J(e^{\hat{\theta}})^N_{\ Q}\Pi^Q\delta(\sigma-\bar{\sigma}) \\ &\quad +(e^{\hat{\theta}})^N_{\ J}\partial^J(e^{\hat{\theta}})^M_{\ Q}\Pi^Q\delta(\sigma-\bar{\sigma})\,. \end{aligned} \tag{4.14}$$

Next, using (2.13) and (3.32), we obtain

$$\begin{aligned} &(e^{\hat{\theta}})^M_{\ J}(\sigma)(e^{\hat{\theta}})^N_{\ K}(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} \\ &\quad = A^{MN}(\sigma-\bar{\sigma}) + (e^{\hat{\theta}})^M_{\ P}\partial_Q\hat{\theta}^{PN}\Pi^Q\delta(\sigma-\bar{\sigma})\,, \end{aligned} \tag{4.15}$$

where $A^{MN}$ is the same anomaly as in the previous chapter (3.38). Substituting (4.15) and (4.9) into (4.14), we obtain

$$\{\breve{\Pi}^M(\sigma), \breve{\Pi}^N(\bar{\sigma})\} = -\breve{\Theta}^{MNQ}\breve{\Pi}_Q\delta(\sigma-\bar{\sigma}) + A^{MN}(\sigma-\bar{\sigma})\,, \tag{4.16}$$

where

$$\begin{aligned} \breve{\Theta}^{MNQ} &= \Theta^{MNQ} + R^{MNQ} \\ \Theta^{MNQ} &= \partial^M\hat{\theta}^{NQ} + \partial^N\hat{\theta}^{QM} + \partial^Q\hat{\theta}^{MN} \\ R^{MNQ} &= \hat{\theta}^M{}_K\partial^K\hat{\theta}^{NQ} + \hat{\theta}^N{}_K\partial^K\hat{\theta}^{QM} + \hat{\theta}^Q{}_K\partial^K\hat{\theta}^{MN}\,. \end{aligned} \tag{4.17}$$

This flux can be also written as

$$\breve{\Theta}^{MNR} = \breve{\partial}^M\hat{\theta}^{NR} + \breve{\partial}^N\hat{\theta}^{RM} + \breve{\partial}^R\hat{\theta}^{MN}\,. \tag{4.18}$$

where $\breve{\partial}^M$ are new derivatives, given by

$$\breve{\partial}^M = (e^{\hat{\theta}})^M_{\ N}\,\partial^N = \partial^M + \hat{\theta}^M_{\ N}\,\partial^N\,. \tag{4.19}$$

Next, we have

$$\{\breve{\Lambda}^M(\sigma), \breve{\Pi}^N(\bar{\sigma})\} = \breve{\partial}^N\breve{\Lambda}^M\delta(\sigma-\bar{\sigma})\,. \tag{4.20}$$

Comparing relations (4.16) to (3.37), and (4.20) to (3.42), we see that the basic algebra relations have the same form. Therefore, we write

$$\begin{aligned} \{\breve{\mathcal{G}}_{\breve{\Lambda}_1}(\sigma), \breve{\mathcal{G}}_{\breve{\Lambda}_2}(\bar{\sigma})\} &= \breve{\Lambda}^M_1(\sigma)\breve{\Lambda}^N_2(\bar{\sigma})A_{MN}(\sigma-\bar{\sigma}) \\ &\quad -\breve{\Lambda}_{1M}\breve{\Lambda}_{2N}\breve{\Theta}^{MNQ}\breve{\Pi}_Q\delta(\sigma-\bar{\sigma}) \\ &\quad +\breve{\Pi}_Q\Big[\breve{\Lambda}^N_2\breve{\partial}_N\breve{\Lambda}^Q_1 - \breve{\Lambda}^N_1\breve{\partial}_N\breve{\Lambda}^Q_2\Big]\delta(\sigma-\bar{\sigma})\,, \end{aligned} \tag{4.21}$$

and after using (3.44) and (3.45), we obtain the full $\theta$-twisted $C$-bracket

$$\begin{aligned} [\breve{\Lambda}_1, \breve{\Lambda}_2]_{C_\theta}{}^Q &= \breve{\Lambda}^N_1\breve{\partial}_N\breve{\Lambda}^Q_2 - \breve{\Lambda}^N_2\breve{\partial}_N\breve{\Lambda}^Q_1 \\ &\quad - \frac{1}{2}\eta_{MN}\left(\breve{\Lambda}^M_1\breve{\partial}^Q\breve{\Lambda}^N_2 - \breve{\Lambda}^M_2\breve{\partial}^Q\breve{\Lambda}^N_1\right) + \breve{\Lambda}_{1M}\breve{\Lambda}_{2N}\breve{\Theta}^{MNQ}\,. \end{aligned} \tag{4.22}$$

The only difference between the $B$- and $\theta$-twisted $C$-brackets is that the derivatives are defined with $B$-shifts in the former and $\theta$-transformations in the latter case, and the flux term is given in terms of different, mutually T-dual fields. Due to the T-duality relations between these fields (3.14), the two twisted $C$-brackets are mutually T-dual as well.

Using (4.19), we can rewrite the above bracket as

$$\begin{aligned}[\check{\Lambda}_1, \check{\Lambda}_2]_{C_\theta}{}^Q = &\check{\Lambda}_1^N \partial_N \check{\Lambda}_2^Q - \check{\Lambda}_2^N \partial_N \check{\Lambda}_1^Q \\ &-\frac{1}{2}\eta_{MN}\left(\check{\Lambda}_1^M \partial^Q \check{\Lambda}_2^N - \check{\Lambda}_2^M \partial^Q \check{\Lambda}_1^N\right) \\ &+\hat{\theta}^N{}_R\left(\check{\Lambda}_{1N}\partial^R\check{\Lambda}_2^Q - \check{\Lambda}_{2N}\partial^R\check{\Lambda}_1^Q\right) \\ &-\frac{1}{2}\hat{\theta}^Q{}_R\left(\check{\Lambda}_{1N}\partial^R\check{\Lambda}_2^N - \check{\Lambda}_{2N}\partial^R\check{\Lambda}_1^N\right) \\ &+\check{\Lambda}_{1M}\check{\Lambda}_{2N}\check{\Theta}^{MNQ} \end{aligned} \tag{4.23}$$

The first line is the $C$-bracket, while the remaining terms are corrections due to its twisting. If the bi-vector $\theta$ is zero, this bracket reduces to the standard $C$-bracket.

From the double theory it is possible to obtain both the initial $\sigma$ model, as well its T-dual description. The former is obtained when all fields and parameters depend on the coordinate $x^\mu$ only, and the latter when they depend only on $y_\mu$. In the following chapters, we analyze the twisted $C$-brackets realization in these theories.

## 5. Initial Theory

Firstly consider the $B$-twisted $C$-bracket (3.46) and its projection to the initial theory, characterized solely by coordinates $x^\mu$. If the parameters $\hat{\Lambda}$ and Kalb-Ramond field $B$ do not depend on the T-dual coordinate $y$, the derivatives $\hat{\partial}^Q$ become just derivatives along the initial coordinates $x^\mu$

$$\hat{\partial}^Q \to \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix}\begin{pmatrix} 0 \\ \partial_\nu \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_\mu \end{pmatrix}. \tag{5.1}$$

It is easy to obtain that

$$\hat{\Lambda}_1^N \partial_N \hat{\Lambda}_2^Q \to \begin{pmatrix} \xi_1^\nu \partial_\nu \xi_2^\mu \\ \xi_1^\nu \partial_\nu \hat{\lambda}_{2\mu} \end{pmatrix}, \tag{5.2}$$

and

$$\eta_{MN}\hat{\Lambda}_1^M \hat{\partial}^Q \hat{\Lambda}_2^N \to \begin{pmatrix} 0 \\ \hat{\lambda}_{1\nu}\partial_\mu \xi_2^\nu + \xi_1^\nu \partial_\mu \hat{\lambda}_{2\nu} \end{pmatrix}. \tag{5.3}$$

We see that the derivative $\hat{\partial}^Q$ no longer depends on the Kalb-Ramond field. The flux $\hat{B}^{MNQ}$ therefore reduces to $B^{MNQ}$, and

$$\hat{B}^{MNQ}\hat{\Lambda}_{1M}\hat{\Lambda}_{2N} \to \begin{pmatrix} 0 \\ 2B_{\mu\nu\rho}\xi_1^\nu \xi_2^\rho \end{pmatrix}, \tag{5.4}$$

where $B_{\mu\nu\rho}$ is the Kalb-Ramond field strength, given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \tag{5.5}$$

This term represents the geometric $H$ flux in the initial theory.[16] Combining previous relations and using the chain rule, the $B$-twisted $C$-bracket becomes

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{C_B} \to [\hat{\Lambda}_1, \hat{\Lambda}_2]_{C_B} = \hat{\Lambda} \equiv \begin{pmatrix} \xi \\ \hat{\lambda} \end{pmatrix}, \tag{5.6}$$

where

$$\begin{aligned} \xi^\mu &= \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu, \\ \hat{\lambda}_\mu &= \xi_1^\nu(\partial_\nu \hat{\lambda}_{2\mu} - \partial_\mu \hat{\lambda}_{2\nu}) - \xi_2^\nu(\partial_\nu \hat{\lambda}_{1\mu} - \partial_\mu \hat{\lambda}_{1\nu}) \\ &\quad + \frac{1}{2}\partial_\mu(\xi_1 \hat{\lambda}_2 - \xi_2 \hat{\lambda}_1) + 2B_{\mu\nu\rho}\xi_1^\nu \xi_2^\rho. \end{aligned} \tag{5.7}$$

This bracket is a well known Courant bracket twisted by a 2-form $B$.[12]

Now let us consider the realization of the $\theta$-twisted $C$-bracket (4.22) in the initial phase space. If we omit the dependence form $y_\mu$, the derivative $\check{\partial}$ becomes

$$\check{\partial}^Q \to \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix}\begin{pmatrix} 0 \\ \partial_\nu \end{pmatrix} = \begin{pmatrix} \kappa\theta^{\mu\nu}\partial_\nu \\ \partial_\mu \end{pmatrix}. \tag{5.8}$$

Now, we have

$$\check{\Lambda}_1^N \check{\partial}_N \check{\Lambda}_2^Q \to \begin{pmatrix} \lambda_{1\nu}\kappa\theta^{\nu\rho}\partial_\rho \check{\xi}_2^\mu + \check{\xi}_1^\nu \partial_\nu \check{\xi}_2^\mu \\ \lambda_{1\nu}\kappa\theta^{\nu\rho}\partial_\rho \lambda_{2\mu} + \check{\xi}_1^\nu \partial_\nu \lambda_{2\mu} \end{pmatrix}, \tag{5.9}$$

and

$$\eta_{MN}\check{\Lambda}_1^M \check{\partial}^Q \check{\Lambda}_2^N \to \begin{pmatrix} \kappa\theta^{\mu\nu}(\check{\xi}_1^\rho \partial_\nu \lambda_{2\rho} + \lambda_{1\rho}\partial_\nu \check{\xi}_2^\rho) \\ \check{\xi}_1^\rho \partial_\mu \lambda_{2\rho} + \lambda_{1\rho}\partial_\mu \check{\xi}_2^\rho \end{pmatrix}. \tag{5.10}$$

The flux term is given by

$$\check{\Lambda}_{1M}\check{\Lambda}_{2N}\check{\Theta}^{MNQ} \to \begin{pmatrix} \kappa^2 R^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho} + \kappa Q_\nu{}^{\rho\mu}(\check{\xi}_1^\nu \lambda_{2\rho} - \check{\xi}_2^\nu \lambda_{1\rho}) \\ \kappa Q_\mu{}^{\rho\nu}\lambda_{1\rho}\lambda_{2\nu} \end{pmatrix}, \tag{5.11}$$

where $Q$ and $R$ are non-geometric fluxes,[16] given by

$$Q_\mu{}^{\nu\rho} = \partial_\mu \theta^{\nu\rho}, \quad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma \theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma \theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma \theta^{\mu\nu}. \tag{5.12}$$

We see that $\Theta^{MNQ}$ gave rise to non-geometric $Q$-flux, while $R^{MNQ}$ gave rise to the non-geometric $R$-flux.

Substituting (5.9), (5.10) and (5.11) into (4.22), with the help of chain rule, we obtain

$$[\check{\Lambda}_1, \check{\Lambda}_2]_{C_\theta} \to [\check{\Lambda}_1, \check{\Lambda}_2]_{C_\theta} = \check{\Lambda} \equiv \begin{pmatrix} \check{\xi} \\ \lambda \end{pmatrix}, \tag{5.13}$$

where

$$\begin{aligned} \check{\xi}^\mu = &\ \check{\xi}_1^\nu \partial_\nu \check{\xi}_2^\mu - \check{\xi}_2^\nu \partial_\nu \check{\xi}_1^\mu + \\ &-\kappa\theta^{\mu\nu}\Big(\check{\xi}_1^\rho(\partial_\nu \lambda_{2\rho} - \partial_\rho \lambda_{2\nu}) - \check{\xi}_2^\rho(\partial_\nu \lambda_{1\rho} - \partial_\rho \lambda_{1\nu}) \\ &\quad -\frac{1}{2}\partial_\nu(\check{\xi}_1 \lambda_2 - \check{\xi}_2 \lambda_1)\Big) \end{aligned} \tag{5.14}$$

$$
\begin{aligned}
&+\kappa\breve{\xi}_1^\nu\partial_\nu(\lambda_{2\rho}\theta^{\rho\mu})-\kappa\breve{\xi}_2^\nu\partial_\nu(\lambda_{1\rho}\theta^{\rho\mu})\\
&+\kappa(\lambda_{1\nu}\theta^{\nu\rho})\partial_\rho\breve{\xi}_2^\mu-\kappa(\lambda_{2\nu}\theta^{\nu\rho})\partial_\rho\breve{\xi}_1^\mu\\
&+\kappa^2(\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho}+\theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu}+\theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu})\lambda_{1\nu}\lambda_{2\rho}\,,
\end{aligned}
$$

$$
\begin{aligned}
\lambda_\mu=\;&\breve{\xi}_1^\nu(\partial_\nu\lambda_{2\mu}-\partial_\mu\lambda_{2\nu})-\breve{\xi}_2^\nu(\partial_\nu\lambda_{1\mu}-\partial_\mu\lambda_{1\nu})+\frac{1}{2}\partial_\mu(\breve{\xi}_1\lambda_2-\breve{\xi}_2\lambda_1)\\
&+\kappa\theta^{\nu\rho}(\lambda_{1\nu}\partial_\rho\lambda_{2\mu}-\lambda_{2\nu}\partial_\rho\lambda_{1\mu})+\kappa\lambda_{1\rho}\lambda_{2\nu}\partial_\mu\theta^{\rho\nu}\,.
\end{aligned}
$$

This is the Courant bracket twisted by a bi-vector $\theta$. As expected, the twisted $C$-brackets reduce to their twisted Courant counterparts in the initial theory, in the same way that the $C$-bracket reduces to the Courant one, once there is no dependence on $y_\mu$.

## 6. T-Dual Theory

Now let us consider the twisted $C$-brackets in the T-dual theory, that is to say the theory obtained after applying the T-dualization along all of the initial coordinates $x^\mu$. Effectively, this is obtained when we demand all quantities to depend solely on $y_\mu$.

For the $B$-twisted $C$-bracket, we have

$$
\hat{\partial}^Q\to\begin{pmatrix}\delta^\mu_\nu & 0\\ 2B_{\mu\nu} & \delta_\mu^\nu\end{pmatrix}\begin{pmatrix}\tilde{\partial}^\nu\\ 0\end{pmatrix}=\begin{pmatrix}\tilde{\partial}^\mu\\ 2B_{\mu\nu}\tilde{\partial}^\nu\end{pmatrix}\,. \tag{6.1}
$$

Moreover, we obtain

$$
\hat{\Lambda}_1^N\partial_N\hat{\Lambda}_2^Q\to\begin{pmatrix}\hat{\lambda}_{1\nu}\tilde{\partial}^\nu\xi_2^\mu+2B_{\nu\rho}\xi_1^\rho\tilde{\partial}^\nu\xi_2^\mu\\ \hat{\lambda}_{1\nu}\tilde{\partial}^\nu\hat{\lambda}_{2\mu}+2B_{\nu\rho}\xi_1^\rho\tilde{\partial}^\nu\hat{\lambda}_{2\mu}\end{pmatrix}\,, \tag{6.2}
$$

and

$$
\eta_{MN}\hat{\Lambda}_1^M\hat{\partial}^Q\hat{\Lambda}_2^N\to\begin{pmatrix}\hat{\lambda}_{1\nu}\tilde{\partial}^\mu\xi_2^\nu+\xi_1^\nu\tilde{\partial}^\mu\hat{\lambda}_{2\nu}\\ 2\hat{\lambda}_{1\nu}B_{\mu\rho}\tilde{\partial}^\rho\xi_2^\nu+2\xi_1^\nu B_{\mu\rho}\tilde{\partial}^\rho\hat{\lambda}_{2\nu}\end{pmatrix}\,. \tag{6.3}
$$

The term containing flux $\hat{B}^{MNQ}$ becomes

$$
\hat{B}^{MNQ}\hat{\Lambda}_{1M}\hat{\Lambda}_{2N}\to\begin{pmatrix}\kappa\,{}^\star Q^\mu{}_{\nu\rho}\xi_1^\nu\xi_2^\rho\\ \kappa\,{}^\star Q^\nu{}_{\rho\mu}(\xi_1^\rho\hat{\lambda}_{2\nu}-\xi_2^\rho\hat{\lambda}_{1\nu})+\kappa^{2\star}R_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho\end{pmatrix}, \tag{6.4}
$$

where we have marked the non-geometric fluxes in T-dual theory by

$$
\kappa\,{}^\star Q^\mu{}_{\nu\rho}=2\tilde{\partial}^\mu B_{\nu\rho}=\kappa\,\tilde{\partial}^\mu\,{}^\star\theta_{\nu\rho}\,, \tag{6.5}
$$

and

$$
\begin{aligned}
\kappa^{2\star}R_{\mu\nu\rho}&=4B_{\mu\sigma}\tilde{\partial}^\sigma B_{\nu\rho}+4B_{\nu\sigma}\tilde{\partial}^\sigma B_{\rho\mu}+4B_{\rho\sigma}\tilde{\partial}^\sigma B_{\mu\nu}\,, \qquad (6.6)\\
&=\kappa^2\,{}^\star\theta_{\mu\sigma}\tilde{\partial}^\sigma\,{}^\star\theta_{\nu\rho}+\kappa^2\,{}^\star\theta_{\nu\sigma}\tilde{\partial}^\sigma\,{}^\star\theta_{\rho\mu}+\kappa^2\,{}^\star\theta_{\rho\sigma}\tilde{\partial}^\sigma\,{}^\star\theta_{\mu\nu}\,.
\end{aligned}
$$

Combining previous relations and using the chain rule, the $B$-twisted $C$-bracket becomes

$$
[\hat{\Lambda}_1,\hat{\Lambda}_2]_{\mathrm{C}_B}\to[\hat{\Lambda}_1,\hat{\Lambda}_2]_{\mathcal{C}_B}=\hat{\Lambda}\equiv\begin{pmatrix}\xi\\ \hat{\lambda}\end{pmatrix}\,, \tag{6.7}
$$

where

$$
\begin{aligned}
\xi^\mu=\;&\hat{\lambda}_{1\nu}(\tilde{\partial}^\nu\xi_2^\mu-\tilde{\partial}^\mu\xi_2^\nu)-\hat{\lambda}_{2\nu}(\tilde{\partial}^\nu\xi_1^\mu-\tilde{\partial}^\mu\xi_1^\nu)+\tilde{\partial}^\mu(\xi_1\hat{\lambda}_2-\xi_2\hat{\lambda}_1)\\
&2B_{\nu\rho}(\xi_1^\nu\tilde{\partial}^\rho\xi_2^\mu-\xi_2^\nu\tilde{\partial}^\rho\xi_1^\mu)+2\tilde{\partial}^\mu B_{\nu\rho}\xi_1^\nu\xi_2^\rho\,,\\
\hat{\lambda}_\mu=\;&\hat{\lambda}_{1\nu}\tilde{\partial}^\nu\hat{\lambda}_{2\mu}-\hat{\lambda}_{2\nu}\tilde{\partial}^\nu\hat{\lambda}_{1\mu}\\
&-2B_{\mu\nu}\Big(\hat{\lambda}_{1\rho}(\tilde{\partial}^\nu\xi_2^\rho-\tilde{\partial}^\rho\xi_2^\nu)-\hat{\lambda}_{2\rho}(\tilde{\partial}^\nu\xi_1^\rho-\tilde{\partial}^\rho\xi_1^\nu)\\
&\quad-\frac{1}{2}\tilde{\partial}^\nu(\hat{\lambda}_1\xi_2-\hat{\lambda}_2\xi_1)\Big)\\
&+2\hat{\lambda}_{1\nu}\tilde{\partial}^\nu(\xi_2^\rho B_{\rho\mu})-2\hat{\lambda}_{2\nu}\tilde{\partial}^\nu(\xi_1^\rho B_{\rho\mu})\\
&+2(\xi_1^\nu B_{\nu\rho})\tilde{\partial}^\rho\hat{\lambda}_{2\mu}-2(\xi_2^\nu B_{\nu\rho})\tilde{\partial}^\rho\hat{\lambda}_{1\mu}\\
&+4\big(B_{\mu\sigma}\tilde{\partial}^\sigma B_{\nu\rho}+B_{\nu\sigma}\tilde{\partial}^\sigma B_{\rho\mu}+B_{\rho\sigma}\tilde{\partial}^\sigma B_{\mu\nu}\big)\xi_1^\nu\xi_2^\rho\,.
\end{aligned} \tag{6.8}
$$

In the T-dual description, $B_{\mu\nu}(y)$ plays the role of the T-dual bi-vector $B_{\mu\nu}=\frac{\kappa}{2}{}^\star\theta_{\mu\nu}$, while the parameters of general coordinate and local gauge transformations correspond to the parameters of local gauge and general coordinate transformations of the initial theory, respectively[10,34]

$$
{}^\star\hat{\lambda}^\mu=\xi^\mu\,,\quad {}^\star\xi_\mu=\hat{\lambda}_\mu\,. \tag{6.9}
$$

As such, the bracket defined by (6.8) is ${}^\star\theta$-twisted Courant bracket.

Similarly, for the derivatives appearing in the $\theta$-twisted $C$-bracket we have

$$
\breve{\partial}^Q\to\begin{pmatrix}\delta^\mu_\nu & \kappa\theta^{\mu\nu}\\ 0 & \delta_\mu^\nu\end{pmatrix}\begin{pmatrix}\tilde{\partial}^\nu\\ 0\end{pmatrix}=\begin{pmatrix}\tilde{\partial}^\mu\\ 0\end{pmatrix}\,. \tag{6.10}
$$

Next, we write

$$
\breve{\Lambda}_1^N\breve{\partial}_N\breve{\Lambda}_2^Q\to\begin{pmatrix}\lambda_{1\nu}\tilde{\partial}^\nu\breve{\xi}_2^\mu\\ \lambda_{1\nu}\tilde{\partial}^\nu\lambda_{2\mu}\end{pmatrix} \tag{6.11}
$$

and

$$
\eta_{MN}\breve{\Lambda}_1^M\breve{\partial}^Q\breve{\Lambda}_2^N\to\begin{pmatrix}\lambda_{1\nu}\tilde{\partial}^\mu\breve{\xi}_2^\nu+\breve{\xi}_1^\nu\tilde{\partial}^\mu\lambda_{2\nu}\\ 0\end{pmatrix}\,, \tag{6.12}
$$

while the flux term is simply given by

$$
\breve{\Lambda}_{1M}\breve{\Lambda}_{2N}\breve{\Theta}^{MNQ}\to\begin{pmatrix}\kappa\,{}^\star B^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\\ 0\end{pmatrix}\,, \tag{6.13}
$$

where ${}^\star B^{\mu\nu\rho}$ is the $H$ flux in T-dual theory

$$
\begin{aligned}
\kappa{}^\star B^{\mu\nu\rho}&=\kappa\,\tilde{\partial}^\mu\theta^{\nu\rho}+\kappa\,\tilde{\partial}^\nu\theta^{\rho\mu}+\kappa\,\tilde{\partial}^\rho\theta^{\mu\nu}\\
&=2\tilde{\partial}^\mu\,{}^\star B^{\nu\rho}+2\tilde{\partial}^\nu\,{}^\star B^{\rho\mu}+2\tilde{\partial}^\rho\,{}^\star B^{\mu\nu}\,.
\end{aligned} \tag{6.14}
$$

The full bracket

$$
[\breve{\Lambda}_1,\breve{\Lambda}_2]_{\mathrm{C}_\theta}\to[\breve{\Lambda}_1,\breve{\Lambda}_2]_{\mathcal{C}_\theta}=\breve{\Lambda}\equiv\begin{pmatrix}\breve{\xi}\\ \lambda\end{pmatrix}\,, \tag{6.15}
$$

where

$$\begin{aligned}\check{\xi}^{\mu} &= \lambda_{1\nu}(\tilde{\partial}^{\nu}\check{\xi}_2^{\mu} - \tilde{\partial}^{\mu}\check{\xi}_2^{\nu}) - \lambda_{2\nu}(\tilde{\partial}^{\nu}\check{\xi}_1^{\mu} - \tilde{\partial}^{\mu}\check{\xi}_1^{\nu}) \\ &\quad + \frac{1}{2}\tilde{\partial}^{\mu}(\check{\xi}_1\lambda_2 - \check{\xi}_2\lambda_1) \\ &\quad + \kappa\ {}^{\star}B^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\,, \\ \lambda_{\mu} &= \lambda_{1\nu}\tilde{\partial}^{\nu}\lambda_{2\mu} - \lambda_{2\nu}\tilde{\partial}^{\nu}\lambda_{1\mu}\,.\end{aligned} \tag{6.16}$$

We recognize the bracket as the Courant bracket twisted by a 2-form ${}^{\star}B$, written in terms of T-dual variables.

In the double theory we naturally obtain both the Courant bracket twisted by a 2-form $B$ and bi-vector $\theta$ from a single twisted $C$-bracket. These twisted Courant brackets define Courant algebroids that are mutually isomorphic, where the isomorphism connecting them represents the T-duality.[20,42]

## 7. Conclusions

We started with the theory characterized solely by the metric tensor and considered firstly generator of diffeomorphisms and local gauge transformations, equivalent to the T-dual diffeomorphisms. This generator is parametrized by a double parameter, whose components depend on both the initial and T-dual coordinates. It has already been shown to give rise to the $C$-bracket, which is the double theory generalization of the well known Courant bracket. We followed the method of [19] to obtain the twisted $C$-brackets.

Primarily, we considered the twist by a 2-form $B_{\mu\nu}(x,y)$ and obtained the $C$ bracket twisted by $B$. The resulting bracket can be separated into two parts. The first part of the bracket has identical form as the $C$-bracket, though the derivatives $\hat{\partial}^M = \partial^M + \hat{B}^M_{\ N}\,\partial^N$ are found in place of the usual $\partial^M$ derivatives. The second part of the twisted $C$-bracket contains a generalized flux $\hat{B}^{MNQ}$ (3.40).

Secondarily, we obtained the $C$-bracket twisted by a bi-vector $\theta$, which also contains a part of the same form as the $C$-bracket, and the flux contracting two gauge parameters. Similarly to the previous case, the bracket is written in terms of different derivatives $\breve{\partial}^M = \partial^M + \hat{\theta}^M_{\ N}\,\partial^N$, while the generalized flux is $\breve{\Theta}^{MNQ}$ (4.18). There are a couple of ways how the twisted $C$-brackets differ from their twisted Courant counterparts.

Firstly, we noted that the $B$-twisted (3.46) and $\theta$-twisted (4.22) $C$-brackets have the same form, with only difference being that the derivatives and fluxes (3.39) of the former are given in terms of 2-form $B$, while of the latter in terms of bi-vector $\theta$ (4.17). As it can be easily seen from comparing (5.7) to (5.14), this is not the case for their analogous twisted Courant brackets. If we take into the account that the Kalb-Ramond field and the non-commutativity parameter are mutually T-dual, the T-duality between the $B$- and $\theta$-twisted $C$-brackets is obvious.

Secondly, the twisting of the Courant bracket can be realized by adding the terms with fluxes to the standard Courant bracket expression. When the $C$-bracket is twisted, apart from adding the fluxes, the derivatives also change and transform by the same twisting matrix. The derivatives transform in the same way as $O(D,D)$ vectors, and do not result in the derivatives in the new basis.

Thirdly, both of the twisted $C$-brackets separately encapsulate the isomoprhism between two Courant algebroids, which represents the T-duality. This was shown by considering the initial and T-dual theories separately, by neglecting the T-dual coordinate dependence in the former and the initial coordinate dependence in the latter case. We have shown that in the same way that $C$-bracket becomes the Courant bracket, the twisted $C$-brackets become their twisted Courant counterparts. Moreover, we showed that in the T-dual description, the $B$-twisted $C$-bracket becomes in fact $\theta$-twisted Courant bracket, and vice versa.

## Conflict of Interest

The authors declare no conflict of interest.

## Keywords

c-bracket, double theory, string theory, t-duality

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# Courant and Roytenberg bracket and their relation via T-duality *

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Abstract

We consider the $\sigma$-model for closed bosonic string propagating in the coordinate dependent metric and Kalb-Ramond field. Firstly, we consider the generator of both general coordinate and local gauge transformations. The Poisson bracket algebra of these generators is obtained and we see that it gives rise to the Courant bracket. Secondly, we consider generators in a new basis, consisting of the $\sigma$ derivative of coordinates, as well as the auxiliary currents. Their Poisson bracket algebra gives rise to the Courant bracket, twisted by the Kalb-Ramond field. Finally, we calculate the algebra of the T-dual generator. The Poisson bracket algebra of T-dual generator gives rise to the Roytenberg bracket, equivalent to the bracket obtained by twisting the Courant bracket by the non-commutativity parameter, which is T-dual to the Kalb-Ramond field. We show that the twisted Courant bracket and the Roytenberg bracket are mutually related via T-duality.

## 1. Bosonic string action

The closed bosonic string is moving in a curved background, associated with the symmetric metric tensor field $G_{\mu\nu} = G_{\nu\mu}$, the antisymmetric Kalb-Ramond tensor field $B_{\mu\nu} = -B_{\nu\mu}$, as well as the scalar dilaton field $\Phi$. If we consider the case of constant dilation field, in a conformal gauge

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$g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$, the action is given by [1]

$$S = \int_\Sigma d^2\xi \mathcal{L} = \kappa \int_\Sigma d^2\xi \Big[\frac{1}{2}\eta^{\alpha\beta}G_{\mu\nu}(x) + \epsilon^{\alpha\beta}B_{\mu\nu}(x)\Big]\partial_\alpha x^\mu \partial_\beta x^\nu \,. \tag{1}$$

The integration goes over two-dimensional world-sheet $\Sigma$ parametrized by $\xi^\alpha(\xi^0 = \tau, \xi^1 = \sigma)$. Coordinates of the D-dimensional space-time are $x^\mu(\xi)$, $\mu = 0, 1, ..., D-1$, $\epsilon^{01} = -1$ and $\kappa = \frac{1}{2\pi\alpha'}$. In a usual way, we derive the expression for the canonical momenta

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu}(x)\dot{x}^\nu - 2\kappa B_{\mu\nu}(x)x'^\nu, \tag{2}$$

and obtain the Hamiltonian

$$\mathcal{H}_C = \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu - 2x'^\mu B_{\mu\nu}(G^{-1})^{\nu\rho}\pi_\rho + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu \,, \tag{3}$$

where $G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$ is the effective metric. It is possible to rewrite the Hamiltonian in a more convenient form

$$\mathcal{H}_C = \frac{1}{4\kappa}(G^{-1})^{\mu\nu}\Big[j_{+\mu}j_{+\nu} + j_{-\mu}j_{-\nu}\Big] \,. \tag{4}$$

where the currents $j_{\pm\mu}$ are defined by

$$j_{\pm\mu}(x) = j_{0\mu} \pm j_{1\mu} = \pi_\mu + 2\kappa \Pi_{\pm\mu\nu}x'^\nu \,, \tag{5}$$

and $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}$ are composite fields. The $\tau-$component of current $j_\mu$ will be marked as an auxiliary current $j_{0\mu} = \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu \equiv i_\mu$. Now, we rewrite the equation (5)

$$j_{\pm\mu} = i_\mu \pm \kappa G_{\mu\nu}x'^\nu \,. \tag{6}$$

The generalized current is defined by

$$J_C(\xi, \Lambda^C) = \xi^\mu(x)i_\mu + \Lambda^C_\mu(x)\kappa x'^\mu \,, \tag{7}$$

where $\xi^\mu$ and $\Lambda^C_\mu$ are some coordinate dependent parameters. Due to their index structure, the former parameters can be regarded as vector field components and the latter as 1-form components. The suitable language for description of generalized currents is the one of generalized geometry [2]. The generalized vectors are direct sum of elements of tangent and cotangent bundle over some manifold, meaning that the generalized geometry treats vectors and 1-forms on equal footing. Therefore, it is possible to consider the generalized current as the function on generalized vector $\xi \oplus \Lambda^C$.

Besides the algebra of these generalized currents, we are also interested in algebra of T-dual generalized currents. T-duality [1, 3] is inherent to string theory and represents the equivalence of two seemingly different physical theories. The equivalence manifests itself in a way that all quantities in one theory are identified with quantities in its T-dual theory. It was firstly noticed in case of the bosonic closed string with one dimension compactified to a radius $R$. In that case, mass spectrum is given by [1]

$$M^2 = \frac{K^2}{R^2} + W^2 \frac{R^2}{\alpha'^2}\,, \tag{8}$$

where $K$ is momentum and $W$ winding number. It is obvious that the mass spectrum remains invariant upon simultaneously swapping $K \leftrightarrow W$ and $R \leftrightarrow \frac{\alpha'}{R}$. What can be concluded is that spectrums of two theories that both have one dimension compactified to a circle, where in one case the circle is of small and in the other of large radius, are indistinguishable. The momenta in one theory are winding numbers in its T-dual theory, and vice versa.

We consider the T-duality realized without changing the phase space [4]. Its transformation rules between the canonical variables are given by

$$\pi_\mu \cong \kappa x'^\mu\,. \tag{9}$$

When the above relation is integrated over space parameter $\sigma$, we obtain that the T-duality interchanges momenta with the winding numbers. The background fields have their T-dual counterparts as well [5], namely

$${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}\,, \tag{10}$$

where $\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1} B G^{-1})^{\mu\nu}$ is the non-commutativity parameter. Two variables are said to be mutually T-dual if they transform one into another when simultaneously both (9) and (10) are applied [4].

Now we follow the same procedure as above for constructing the T-dual generalized currents. Applying (9) and (10) to (5), we obtain the current $l_\pm^\mu$, T-dual to current $j_{\pm\mu}$

$$l_\pm^\mu = k^\mu \pm (G_E^{-1})^{\mu\nu}\pi_\nu, \quad k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu\,. \tag{11}$$

The Hamiltonian can be expressed in terms of these currents by

$$\mathcal{H}_C = \frac{1}{4\kappa} G^E_{\mu\nu}\Big(l_+^\mu l_+^\nu + l_-^\mu l_-^\nu\Big)\,. \tag{12}$$

Substituting the expression (11) in (12), we obtain the Hamiltonian in the form (4). The Hamiltonian remains invariant under T-duality.

The generalization of current $l_\pm^\mu$ is given by

$$J_R(\xi_R, \Lambda) = \xi_R^\mu(x)\pi_\mu + \Lambda_\mu(x)k^\mu\,. \tag{13}$$

Instead of auxiliary current $i_\mu$ and coordinate $\sigma-$derivative $x'^\mu$, the basis for T-dual generalized currents consist of T-dual auxiliary current $k^\mu$ and canonical momenta $\pi_\mu$. The coefficients $\xi^\mu_R$ are arbitrary vector field components, while $\Lambda_\mu$ are arbitrary 1-form components. Next, we will see how these currents are related to the symmetry generators.

## 2. Symmetry generators algebra

Let us start with the general coordinate transformations. They are generated by

$$\mathcal{G}_{GCT}(\xi) = \int_0^{2\pi} d\sigma \xi^\mu(x)\pi_\mu \,. \tag{14}$$

Action of the general coordinate transformation on background fields is governed by the action of Lie derivative [6]

$$\delta_\xi G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu},\ \delta_\xi B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu} \,. \tag{15}$$

The Lie derivative is defined by $\mathcal{L}_\xi = i_\xi d + d i_\xi$, where the interior product $i_\xi$ acts as a contraction with a vector field $\xi$, while the exterior derivative $d$ extends the concept of the differential of a function to differential forms. The commutator of two Lie derivatives results in another Lie derivative

$$\mathcal{L}_{\xi_1}\mathcal{L}_{\xi_2} - \mathcal{L}_{\xi_2}\mathcal{L}_{\xi_1} = \mathcal{L}_{[\xi_1,\xi_2]_L} \,, \tag{16}$$

where by $[\xi_1, \xi_2]_L$, we marked the Lie bracket between two vector fields $\xi_1$ and $\xi_2$. Explicitly, it is given by

$$([\xi_1,\xi_2]_L)^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu \,. \tag{17}$$

Calculating the Poisson bracket algebra of general coordinate transformations generators $\mathcal{G}_{GCT}$ (14), we obtain the relation:

$$\{\mathcal{G}_{GCT}(\xi_1),\, \mathcal{G}_{GCT}(\xi_2)\} = -\mathcal{G}_{GCT}([\xi_1,\xi_2]_L) \,. \tag{18}$$

We note that this algebra gives rise to Lie bracket.

We would like to extend the generator $G_{GCT}$ so that it includes the local gauge transformations as well. They are generated by [6]

$$\mathcal{G}_{LGT}(\Lambda) = \int d\sigma \Lambda_\mu \kappa x'^\mu \,, \tag{19}$$

where $\Lambda_\mu$ are gauge parameters that are 1-form components. The action of local gauge transformations on background fields is given by

$$\delta_\Lambda G_{\mu\nu} = 0,\ \delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \,. \tag{20}$$

When gauge parameter is changed so that total derivative of arbitrary function is added $\Lambda_\mu \to \Lambda_\mu + \partial_\mu \lambda$, the action of generator does not change.

Hence, different gauge parameters generate same symmetry and therefore the generator is reducible.

Local gauge transformations are T-dual to general coordinate transformations, due to relation (9). The sum of both generators (14) and (19) results in symmetry generator that is T-dual to itself

$$\mathcal{G}(\xi \oplus \Lambda) = \int d\sigma \Big[\xi^\mu(x)\pi_\mu + \Lambda_\mu \kappa x'^\mu\Big] \,. \tag{21}$$

Using the language of generalized geometry, we treat the generator (21) as a function of generalized gauge parameter, defined as the direct sum of vector and 1-form parameter $\xi \oplus \Lambda$.

## 2.1. Courant bracket

The Poisson bracket algebra of the generators (21) is given by

$$\{\mathcal{G}(\xi_1 \oplus \Lambda_1),\, \mathcal{G}(\xi_2 \oplus \Lambda_2)\} = -\mathcal{G}([\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C)\,, \tag{22}$$

where $[\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C$ is the Courant bracket [7]. It is considered to be the generalized geometry extension of the Lie bracket. Just like the Lie bracket acts on vectors, Courant bracket acts on generalized vectors. Its full expression is given by

$$[\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C = [\xi_1, \xi_2]_L \oplus \Big(\mathcal{L}_{\xi_1}\Lambda_2 - \mathcal{L}_{\xi_2}\Lambda_1 - \frac{1}{2}d(i_{\xi_1}\Lambda_2 - i_{\xi_2}\Lambda_1)\Big)\,. \tag{23}$$

The first term $[\xi_1, \xi_2]_L$ on the right hand side of previous equation gives the vector, while the other terms give 1-form. It has been shown before [8] that the algebra of symmetry transformations gives rise to the Courant bracket.

Courant bracket cannot be a bracket of a Lie algebra, as it does not satisfy the Jacobi identity. However, this does not pose a problem, as the deviation from Jacobi identity is closed form that disappears after the integration and hence correspond to a trivial symmetry. The Jacobiator of Courant bracket is given by [2]

$$[[\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C, \xi_3 \oplus \Lambda_3]_C + cyclic = dNij(\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2, \xi_3 \oplus \Lambda_3)_C\,, \tag{24}$$

where the Nijenhuis operator is defined by

$$Nij(\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2, \xi_3 \oplus \Lambda_3)_C = \frac{1}{3} < (\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2)_C, \xi_3 \oplus \Lambda_3 > + cycl. \tag{25}$$

and $< \xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2 >= \frac{1}{2}(\xi_1(\Lambda_2) - \xi_2(\Lambda_1))$ is the natural inner product between generalized vectors.

### 2.2. Twisted Courant bracket

We claimed that it is possible to obtain the generalized currents from the symmetry generators. To demonstrate that, let us now define the new gauge parameter

$$\Lambda^C_\mu = \Lambda_\mu + 2B_{\mu\nu}\xi^\nu \,. \tag{26}$$

This change of parameter can be interpreted as the B-transformation acting on the generalized gauge parameter

$$\mathcal{O}_B \Lambda = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \cdot \begin{pmatrix} \xi^\nu \\ \Lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu \\ \Lambda^C_\mu \end{pmatrix} \,. \tag{27}$$

It is suitable to rewrite the generator (21) in the new basis

$$\mathcal{G}_\mathcal{C}(\xi \oplus \Lambda^C) = \int d\sigma \Big[ \xi^\mu i_\mu + \kappa \Lambda^C_\mu x'^\mu \Big] \,. \tag{28}$$

Comparing the expression for the symmetry generator $\mathcal{G}_\mathcal{C}$ (28) with the expression for generalized currents (7), we see that this generator is the charge corresponding to the current $J_C$.

The change of basis results in the appearance of H-flux in the basis algebra

$$\{ i_\mu(\sigma), i_\nu(\bar\sigma) \} = -2\kappa B_{\mu\nu\rho} x'^\rho \delta(\sigma - \bar\sigma) \,, \tag{29}$$

where

$$B_{\mu\nu\rho} = (dB)_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{30}$$

is the H-flux. The Poisson bracket algebra of generators was obtained in [4, 9] and written in the form

$$\{ \mathcal{G}_\mathcal{C}(\xi_1 \oplus \Lambda^C_1),\, \mathcal{G}_\mathcal{C}(\xi_2 \oplus \Lambda^C_2) \} = -\mathcal{G}_\mathcal{C}([\xi_1 \oplus \Lambda^C_1, \xi_2 \oplus \Lambda^C_2]_B) \,. \tag{31}$$

The bracket $[\xi_1 \oplus \Lambda^C_1, \xi_2 \oplus \Lambda^C_2]_B$ is called the Courant bracket, twisted by a 2-form $2B_{\mu\nu}$. This twist is realized by

$$[\mathcal{O}_B(\xi_1 \oplus \Lambda_1), \mathcal{O}_B(\xi_2 \oplus \Lambda_2)]_C - \mathcal{O}_B[\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C = H(\xi_1, \xi_2, .) \,. \tag{32}$$

We see that it differs from the Courant bracket (23) by the contraction of H-flux $H = dB$ (30) with two gauge parameters $\xi_1$ and $\xi_2$. The full expression for this bracket is given by

$$\begin{aligned} [\xi_1 \oplus \Lambda^C_1, \xi_2 \oplus \Lambda^C_2]_B = [\xi_1, \xi_2]_L \oplus \Big( & \mathcal{L}_{\xi_1} \Lambda^C_2 - \mathcal{L}_{\xi_2} \Lambda^C_1 \\ & - \frac{1}{2} d(i_{\xi_1} \Lambda^C_2 - i_{\xi_2} \Lambda^C_1) + H(\xi_1, \xi_2, .) \Big) \,. \end{aligned} \tag{33}$$

### 2.3. Roytenberg bracket

Finally, we are going to consider one more transformation of the gauge parameter

$$\xi_R^\mu = \xi^\mu + \kappa\theta^{\mu\nu}\Lambda_\nu \,. \tag{34}$$

This can be written as a generalized gauge parameter transformation, characterized by the action of antisymmetric $\theta^{\mu\nu}$ bi-vector

$$\mathcal{O}_\theta\Lambda = \begin{pmatrix} \delta_\nu^\mu & \kappa\theta^{\mu\nu} \\ 0 & \delta_\mu^\nu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \Lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi_\mu^R \\ \Lambda_\mu \end{pmatrix} \,. \tag{35}$$

The symmetry generator can be rewritten with new parameters in a new basis

$$\mathcal{G}_{\mathcal{R}}(\xi_R \oplus \Lambda) = \int d\sigma \Big[\xi_R^\mu \pi_\mu + \Lambda_\mu k^\mu\Big] \,. \tag{36}$$

It is obvious that the generator is the charge corresponding to current $J_R$ (13) and that the T-duality relation $\mathcal{G}_{\mathcal{R}}(\xi_R \oplus \Lambda) \cong \mathcal{G}_{\mathcal{C}}(\xi \oplus \Lambda^C)$ holds.

There is a presence of non-geometric fluxes in basis algebra

$$\{k^\mu(\sigma), k^\nu(\bar\sigma)\} = -\kappa Q_\rho{}^{\mu\nu} k^\rho \delta(\sigma - \bar\sigma) - \kappa^2 R^{\mu\nu\rho}\pi_\rho \delta(\sigma - \bar\sigma) \,, \tag{37}$$

where $Q_\rho{}^{\mu\nu} = \partial_\rho\theta^{\mu\nu}$ is Q-flux and $R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}$ is R-flux. The new generator algebra [4, 10] is given by

$$\{\mathcal{G}_{\mathcal{R}}(\xi_1^R \oplus \Lambda_1),\, \mathcal{G}_{\mathcal{R}}(\xi_2^R \oplus \Lambda_2)\} = -\mathcal{G}_{\mathcal{R}}([\xi_1^R \oplus \Lambda_1, \xi_2^R \oplus \Lambda_2]_R) \,, \tag{38}$$

where the resulting bracket is the Roytenberg bracket [11]. It is a generalization of Courant bracket, obtained by twisting the Courant bracket by some bi-vector. The corresponding bi-vector in our case is the non-commutativity parameter $\kappa\theta^{\mu\nu}$. The Roytenberg bracket differs from the Courant bracket by

$$[\mathcal{O}_\theta(\xi_1 \oplus \Lambda_1), \mathcal{O}_\theta(\xi_2 \oplus \Lambda_2)]_C - \mathcal{O}_\theta[\xi_1 \oplus \Lambda_1, \xi_2 \oplus \Lambda_2]_C \,, \tag{39}$$

In its most general form, Roytenberg bracket encompasses all fluxes. In this case, canonical momenta are commutative, meaning that there is no H-flux present in basis algebra. The full expression is

$$\begin{aligned} [\xi_1 \oplus \Lambda_1^R, \xi_2 \oplus \Lambda_2^R]_R = &\Big([\xi_1, \xi_2]_L - \kappa[\xi_2, \Lambda_1^R\theta]_L + \kappa[\xi_1, \Lambda_2^R\theta]_L \\ &+ \frac{\kappa^2}{2}[\theta,\theta]_S(\Lambda_1^R, \Lambda_2^R, .) - (\mathcal{L}_{\xi_2}\Lambda_1^R - \mathcal{L}_{\xi_1}\Lambda_2^R + \frac{1}{2}d(i_{\xi_1}\Lambda_2^R - i_{\xi_2}\Lambda_1^R))\kappa\theta\Big) \\ &\oplus \Big(\mathcal{L}_{\xi_1}\Lambda_2^R - \mathcal{L}_{\xi_2}\Lambda_1^R - \frac{1}{2}d(i_{\xi_1}b - i_{\xi_2}\Lambda_1^R) - [\Lambda_1^R, \Lambda_2^R]_\theta\Big) \,. \end{aligned} \tag{40}$$

The expression $[\theta,\theta]_S(\Lambda_1^R, \Lambda_2^R, .)$ is the Schouten-Nijenhuis bracket [12] contracted with two 1-forms. It is a generalization of the Lie bracket on the space of multi-vectors. The expression $[\Lambda_1^R, \Lambda_2^R]_\theta$ is the Koszul bracket [13], a generalization of the Lie bracket on the space of differential forms.

We note that 2-form $2B_{\mu\nu}$ and bi-vector $\kappa\theta^{\mu\nu}$ used for twisting the Courant bracket in two cases are mutually T-dual (10). Moreover, both of these brackets appeared in considering charges corresponding to the generalized currents, defined in mutually T-dual bases. Therefore, we can conclude that the T-duality connects twisted Courant and Roytenberg bracket, as long as they are twisted by the mutually T-dual fields.

## 3. Conclusions

We considered a standard $\sigma$-model for closed bosonic string. We wrote Hamiltonian in terms of two types of currents. The components of these currents have been used as bases in which generalized currents were defined. These generalized currents are defined in mutually T-dual bases.

Afterwards, we considered the self T-dual symmetry generator $\mathcal{G}(\xi \oplus \Lambda)$. It takes the generalized gauge parameter $\xi \oplus \Lambda$ as a parameter and generates both general coordinate transformations and local gauge transformations. The Poisson bracket algebra of these generators was calculated and the Courant bracket has been obtained. The Courant bracket appeared in a same way in which Lie bracket was obtained in the case of general coordinate transformations algebra.

Next, we considered the action of B-transformation to the generalized gauge parameter. The symmetry generator written in terms of the resulting gauge parameter is the charge corresponding to the generalized current $J_C$. The Poisson bracket algebra of these generators gives rise to the Courant bracket twisted by a 2-form $2B$.

In the end, we consider the action of $\theta$-transformation on the generalized gauge parameter. The newly obtained generator is the charge for the generalized current $J_R$ and it gives rise to the Roytenberg bracket. The Roytenberg bracket in general represent the Courant bracket twisted by a bi-vector. In this specific case, the bi-vector is $\kappa\theta^{\mu\nu}$, which is T-dual to the Kalb-Ramond field $B_{\mu\nu}$. Consequently, we conclude that both the twisted Courant bracket and the Roytenberg bracket appear when generalized currents defined in two mutually T-dual bases are considered.

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# COURANT ALGEBROIDS IN BOSONIC STRING THEORY

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# KURANTOVI ALGEBROIDI U BOZONSKOJ TEORIJI STRUNA

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# Abstract

Generalized geometry is a new mathematical paradigm in which vectors and 1-forms are united and investigated as single objects - generalized vectors. In this dissertation, we explore symmetries of bosonic string theory and their relations with T-duality in the formalism of generalized geometry. The generator of both diffeomorphisms and local gauge transformations is constructed and expressed as an $O(D, D)$ invariant inner product of two generalized vectors. In the same way that the Poisson bracket algebra of generators of diffeomorphism gives rise to the Lie bracket, the algebra of the extended generators gives rise to the Courant bracket. Taking into account the T-duality relation between two string symmetries, we interpret the Courant bracket as the T-dual extension of the Lie bracket [1].

We then develop a simple procedure for twisting the Courant bracket with any $O(D, D)$ transformation, allowing us to obtain Courant brackets deformed with different fluxes. The crux of this method consists of expressing the generator in the basis of non-canonical currents, which are connected with canonical variables via the $O(D, D)$ transformation. We show that the Poisson bracket algebra of generators in the basis of currents closes on the appropriate twisted Courant bracket. We prove that there is a natural way to define a Courant algebroid using these twisted Courant brackets. We provide many examples of $O(D, D)$ transformations and their corresponding twisted Courant brackets, including the $B$-twisted Courant bracket and the $\theta$-twisted Courant bracket. The $B$-twisted Courant bracket is characterized by $H$ flux appearing in the algebra of currents, while the $\theta$-twisted Courant bracket is characterized by the so-called non-geometric $Q$ and $R$ fluxes. It has been shown that these brackets are mutually T-dual [2].

In addition, we construct the generator that produces the Courant bracket twisted simultaneously by $B$ and $\theta$ in its Poisson bracket algebra. This generator is expressed in terms of currents that contain all string fluxes in their Poisson bracket relations. Moreover, we show that the Courant bracket twisted simultaneously by $B$ and $\theta$ is invariant under the T-duality [3]. We also demonstrate that all fluxes can exist on the Dirac structures associated with the Courant algebroid for this bracket, without any restrictions imposed on fluxes.

In the end, results are generalized to a double theory, in which variables depend on both initial and T-dual coordinates. The algebra of generators that include both initial and T-dual diffeomorphisms closes on the double field extension of the Courant bracket called $C$-bracket. Following the same procedure as in the single theory, we obtained the $B$-twisted and $\theta$-twisted $C$-brackets [4]. We demonstrate that by projecting the twisted $C$-brackets to the initial and T-dual phase spaces, the mutually T-dual twisted Courant brackets are obtained.

**Key words**: Bosonic string, T-duality, Generalized geometry
**Scientific field**: Physics
**Research area**: String theory

# Sažetak

Generalisana geometrija podrazumeva novu matematičku paradigmu u kojoj se vektori i 1-forme objedinjuju i razmatraju kao jedinstveni objekti - generalisani vektori. U ovoj disertaciji istražujemo simetrije bozonske teorije struna i njihove veze sa T-dualnošću korišćenjem formalizma generalisane geometrije. Konstruisan je jedinstven generator difeomorfizama i lokalnih gradijentnih transformacija i predstavljen kao $O(D, D)$ invarijantan skalarni proizvod između dva generalisana vektora. Na isti način kao što u algebri Poasonovih zagrada generatora difeomorfizama nastaje Lijeva zagradi, algebra proširenog generatora daje Kurantovu zagradu. Uzimajući u obzir T-dualne veze između ove dve simetrije, Kurantova zagrada je interpretirana kao ekstenzija Lijeve zagrade invarijantna na T-dualnost [1].

Zatim razvijamo jednostavnu proceduru za pronalaženje Kurantovih zagrada zavrnutih proizvoljnim $O(D, D)$ transformacijama, što nam omogućava da dobijemo Kurantove zagrade deformisane različitim fluksevima. Osnova metode je predstavljanje generatora u bazisu nekanonskih struja, koje su $O(D, D)$ transformacijom povezane sa kanonskim promenljivama. Pokazano je da se algebra Poasonovih zagrada između generatora izraženih preko struja zatvara na odgovarajućoj zavrnutoj Kurantovoj zagradi. Dokazano je i da takva zavrnuta Kurantova zagrada definiše na prirodan način Kurantov algebroid. Dali smo više primera $O(D, D)$ transformacija i odredili njima odgovarajuće zavrnute Kurantove zagrade, uključujući i $B$-zavrnutu i $\theta$-zavrnutu Kurantovu zagradu. Kurantovu zagradu zavrnutu poljem $B$ karakteriše pojavljivanje $H$ fluksa u algebri struja, dok Kurantovu zagradu zavrnutu poljem $\theta$ karakteriše pojavljivanje takozvanih negeometrijskih $Q$ i $R$ flukseva. Pokazano je da su ove dve zagrade međusobno T-dualne [2].

Dodatno, konstruisan je i generator koji daje Kurantovu zagradu istovremeno zavrnutu poljima $B$ i $\theta$. Ovaj generator izražen je preko pomoćnih struja u čijim algebarskim relacijama izraženim preko Poasonovih zagrada se dobijaju svi fluksevi teorije struna. Dodatno, pokazali smo da je na ovakav način zavrnuta Kurantova zagrada i invarijantna na T-dualnost [3]. Takođe smo pokazali da svi fluksevi mogu postojati na Dirakovim strukturama Kurantovog algebroida definisanog ovom zagradom, bez ikakvih ograničenja na tim fluksevima.

Na kraju, uopštili smo rezultate na duplu teoriju, u kojoj sve promenljive zavise i od početnih i od T-dualnih koordinata. Algebra generatora koji obuhvata difeomorfizme i T-dualne difeomorfizme zatvara se na $C$-zagradi, što je generalizacija Kurantove zagrade na dupli fazni prostor. Koristeći se istom procedurom kao i u neudupliranoj teoriji, dobili smo $C$-zagrade zavrnute poljima $B$ i $\theta$ [4]. Projektujući ove zagrade na međusobno T-dualne fazne prostore, dobili smo međusobno T-dualne zavrnute Kurantove zagrade.

**Ključne reči**: Bozonska struna, T-dualnost, Generalisana geometrija
**Naučna oblast**: Fizika
**Uža naučna oblast**: Teorija struna

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# Part I

# String theory essentials

# Chapter 1

# Introduction

Relation between physics and mathematics is a long and close one. Since Newton's publication of Philosophiæ Naturalis Principia Mathematica, the accepted paradigm was that laws of physics should have a precise mathematical formulation. Newton's three laws of motion formulated in Principia require only a simple calculus. Physical properties of the system, such as momentum, energy or angular momentum, are modeled by smooth functions. It was later demonstrated that the introduction of Lagrangian in configuration space is suitable for deriving equations of motions from the principle of minimal action. Equivalently, everything could be derived from the Hamiltonian in the phase space, obtained as a Legendre transformation of the Lagrangian. The need for more complex mathematical formalism grew with the expansion of knowledge of physical phenomena. In the formulation of quantum mechanics, physical variables are expressed as operators on the Hilbert space, and their values are the eigenvalues of these operators. For the formulation of general relativity, physicists incorporated Riemannian geometry.

Classical mechanics is able to predict the motion of many different objects, from billiard balls to rocket ships. Given the initial conditions, i.e. initial coordinates and velocities, the motion of any object is fully determined by the action of its forces. However, when it comes to the description of objects on very short or very large scales, classical mechanics is unable to make any meaningful predictions. It is important to point out that these extremes in scales brought fundamental changes in our understanding of space-time and challenged intuitive but wrong premises about the world around us.

Quantum theory is fundamentally predicated upon different premises than classical mechanics. For instance, Heisenberg's uncertainty principle disproves the assumption that coordinate and velocity can be simultaneously well-defined and expressed by single values. Waves and particles are not seen as fundamentally different phenomena, but each particle has a wave property. The energy of light is not continuous but is divided into a finite number of chunks of energy. These ideas were necessary to describe real physics phenomena, like the emission of electrons when metal is hit by light. What followed was the development of the quantum field theory, where particles arise as excitations of the fields, and their interactions are described by the coupling of their fields. This is all well combined into the Standard Model, which adequately describes electromagnetic, weak, and strong nuclear interaction, as was confirmed with a myriad of experiments on different energy scales with very high precision [5]. Though impressive, the Standard Model has a major limitation: it does not include gravity.

Gravity is best described by Einstein's general theory of relativity. It is formulated in a curved background, and gravity arises as a consequence of the space-time geometry. Many experiments confirmed Einstein's theory [6], including the light bending and time dilatation near a massive body, as well as recent observations of gravitational waves by LIGO [7, 8]. Gravity is a weak force, 25 orders of magnitude weaker than weak nuclear interactions, and even more times weaker than strong nuclear and electromagnet interactions. As such, on smaller scales, its effects are usually negligible. However, on large scales, when we are considering massive objects such as galaxies and black holes, it is the dominant force, due to its long range, and in these situations, the general theory of relativity applies. However, some situations do not quite fit in either of the descriptions. On very large energies, in the interior of black holes, or in the moments right after the Big Bang, we do not have a proper argument to rule out either relativistic or quantum effects. Therefore, it is believed that quantum mechanics and general relativity are effective theories of a more general quantum theory of gravity.

There is almost a universal belief that the quantum theory of gravity exists. The electromagnetic and weak interactions are unified at larger energies, and by symmetry breaking mechanism they separate. So the argument goes that on even higher energies all interactions should be unified. Moreover, the quantum field theory is formulated in a fixed background, while the nature of general relativity is that space-time itself is dynamic, so a more universal theory has to exist.

However, the formulation of this unified theory turned out to be challenging. If one proceeds with the quantization of gravity in the framework of perturbative quantum field theory, the theory is not renormalizable by the standard renormalization techniques. Moreover, the quantum effects of gravity are expected to appear at energies of $10^{19} GeV$, which are inaccessible with current accelerators, making experimental testing of quantum gravity very difficult. It can be argued that understanding the quantum theory of gravity will bring a better understanding of space-time and the fundamental properties of the world around us, in the same way as quantum field theory and general relativity did. As it was historically true, we can expect that description of the theory of quantum gravity will also require new areas of mathematics.

## 1.1 String theory

String theory [9, 10, 11] is a theoretical framework that postulates that the basic elements of Nature are one-dimensional strings, rather than point particles. Unlike Standard Model, which requires nineteen parameters to be obtained experimentally and put into theory by hand, the string theory requires only one dimensionless parameter, which is the fundamental string length. The quantized string has a discrete spectrum of vibrating modes, with different modes appearing as different particles at distances larger than the fundamental string length. The string propagating in space-time spans the two-dimensional world-sheet parametrized with a time-like parameter $\tau$ and space-like parameter $\sigma$, analogous to a relativistic point particle moving along the world-line. The world-sheet can either be a torus, in which case we have a closed string, or a strip, in which case we have an open string. In the former case, the periodic boundary conditions are imposed on the string target space. In the latter case, we can impose either Neumann conditions, where end-points of an open string are fixed in space, or Dirichlet boundary conditions, where they are on a dynamical object, called D-brane. While it might appear as having odd assumptions, there are good reasons that make string theory the candidate for the universal description of all interactions.

First of all, one of the string vibration modes produces a massless spin 2 particle, which has never been observed. Though initially, this led to the abandonment of the theory, in 1974 it has been shown that such a particle obeys the Ward identities and can be interpreted as graviton [12]. Formulation of the quantum theory of gravity is arguably the biggest challenge in contemporary physics, and the fact that string theory predicts a consistent quantum theory of gravity makes it appealing for research. Moreover, for one-loop corrections, the gravity emerging from string theory is renormalizable [13]. In particle physics, Feynman diagrams are webs of world lines that juxtapose in points, which are sources of singularities. In string theory, Feynman diagrams are two-dimensional surfaces, that intersect on smooth areas, and as such there are no local singularities. The divergences in string gravity cancel each other out. The fundamental string length provides a natural ultraviolet cut-off for graviton scattering amplitudes.

The first string theory that was developed is the bosonic string theory. Besides graviton, bosonic string theory also includes gauge bosons. It is an incomplete theory since it does not include fermions. Moreover, it predicts the existence of tachyons, particles of negative energy. These issues are resolved with the introduction of supersymmetry, leading to the development of superstring theory. Given that all particles appear in the spectrum of supersymmetric strings, the superstring theory is a promising candidate for a unified theory of all fundamental interactions. There are five different superstring theories that are anomaly free. These are type I, IIA, IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$.

As was the case with previous breakthroughs in theoretical physics, string theory is also enriched with more complicated mathematical apparatus. The world-sheet poses conformal symmetry. The conformal invariance on the quantum level imposes the number of dimensions of space-time. In bosonic string theory, the critical number of dimensions of space-time is twenty-six, while for the superstring theories, it is ten. From the optimistic perspective, one can claim that this is an advantage of string theory because the number of dimensions is predicted by the theory. From the pessimistic perspective, we do not observe ten dimensions and this is a challenge that the theory has to resolve.

Interestingly enough, the idea that all interactions can be unified in space-time with extra dimensions is older than the string theory itself. Einstein's general relativity and Maxwell equations were derived from gravity action in five dimensions in the works of Kaluza and Klein [14, 15]. They proposed that an additional dimension is compactified on a circle. This motivated the string theorists to consider the approach where supplementary dimensions are compactified.

## 1.2 T-duality

The propagation of closed strings in space-time with one dimension being compactified to a circle led to the discovery of T-duality, a striking feature unique to string theory. It was observed that two string theories, one with a dimension compactified on a circle with radius $R$ and the other one with a dimension compactified on a circle with radius proportional to $\frac{1}{R}$ have the same mass spectrum, and therefore are physically indistinguishable. The winding number, that is to say, the number of times a closed string winds around a compact dimension, in one theory represents the momentum number in its T-dual theory, and vice versa. This is a simple example of a more general string phenomenon, that two theories can be defined on backgrounds with different geometries, or even topologies, but still, predict the same physics.

Dualities are relations between different actions that lead to the same observable quantities that are an integral part of the string theory. For the closed string moving in the background characterized by constant fields, the procedure of obtaining T-dual theory was developed by Buscher [16]. The open string with Neumann boundary conditions is T-dual to open strings with Dirichlet boundary conditions. Moreover, the T-duality connects IIA and IIB superstring theories, and two types of heterotic string theories. Together with S-duality, which connects theories with a strong and weak coupling constant, T-duality connects different supersymmetric string theories with a single, eleven-dimensional M-theory. This was observed by Witten in 1995, marking the so-called second superstring revolution. Before that, it was unclear which superstring theory is to be preferred.

As backgrounds become more complicated, obtaining the T-dualization procedure becomes more challenging. After T-dualization one can obtain so-called non-geometric backgrounds, where background fields are non-local. Though many advances in understanding this intriguing feature, we still do not have the universal procedure of obtaining T-duality for string moving in the arbitrary background. Given its importance in relating different superstring theories, a better understanding of T-duality is one of the string theory priorities.

The T-duality gives rise to a plethora of geometries and topologies. Therefore, its universal formulation requires a general mathematical framework, which can include these different spaces. The promising candidate for such a framework is generalized geometry. It is the geometry of the generalized tangent bundle, which is just a direct sum of the tangent and cotangent bundle over a manifold. Vectors and 1-forms are combined into single objects, called generalized vectors. On the space of generalized vectors, there is a natural way to define both a symmetric and antisymmetric inner product. The former is invariant under the $O(D, D)$ transformations, which are transformations that govern T-duality.

Furthermore, the generalized tangent bundle is equipped with the Courant bracket. It can be understood as a generalization of the Lie bracket on the generalized tangent bundle. Unlike the Lie bracket, the Courant bracket does not satisfy the Leibniz rule and Jacobi identity, though there are sub-bundles on which it does satisfy both of them and can be seen as the bracket of Lie algebra. Initially, the Courant bracket was constructed as a double of Lie bialgebroid, which is just the ordered pair of two Lie algebroids on mutually dual vector bundles. Soon after its construction, it was observed that one of the Lie algebroids can be twisted by an exact 3-form, and a twisted Courant bracket was obtained. Subsequently, Roytenberg showed that both Lie algebroids can be twisted, and constructed what is known as the Roytenberg bracket. The additional terms that appeared due to twisting can be interpreted as string theory fluxes.

Fluxes in string theory appear among others in the context of background compactification [17, 18, 19], and generalized geometry [20, 21, 22]. The vacuum in the compactification background is degenerated. Different possible configurations are parametrized by moduli, that appear as massless scalar fields in the lower dimensional theory. These fields are problematic from a phenomenological point of view since they would carry long-range interactions that are unphysical. The problem can be resolved by introducing fluxes to the background [23, 24, 25]. The fluxes generate the potential that stabilizes the vacuum expectation value and gives mass to moduli.

## 1.3 Overview of the thesis

In this dissertation, we explore symmetries of bosonic string theory and their relations with T-duality in the formalism of generalized geometry. The generator of both diffeomorphisms and local gauge transformations is constructed and expressed as an $O(D, D)$ invariant inner product of two generalized vectors. In the same way that the Poisson bracket algebra of generators of diffeomorphism gives rise to the Lie bracket, the algebra of the extended generators gives rise to the Courant bracket. Taking into account the T-duality relation between two string symmetries, we interpret the Courant bracket as the T-dual extension of the Lie bracket [1].

We then develop a simple procedure for twisting the Courant bracket with any $O(D, D)$ transformation, allowing us to obtain Courant brackets deformed with different fluxes. The crux of this method consists of expressing the generator in the basis of non-canonical currents, which are connected with canonical variables via the $O(D, D)$ transformation. We show that the Poisson bracket algebra of generators in the basis of currents closes on the appropriate twisted Courant bracket. We prove that there is a natural way to define a Courant algebroid using these twisted Courant brackets. We provide many examples of $O(D, D)$ transformations and their corresponding twisted Courant brackets, including the $B$-twisted Courant bracket and the $\theta$-twisted Courant bracket. The $B$-twisted Courant bracket is characterized by $H$ flux appearing in the algebra of currents, while the $\theta$-twisted Courant bracket is characterized by the so-called non-geometric $Q$ and $R$ fluxes. It has been shown that these brackets are mutually T-dual [2].

In addition, we construct the generator that produces the Courant bracket twisted simultaneously by $B$ and $\theta$ in its Poisson bracket algebra. This generator is expressed in terms of currents that contain all string fluxes in their Poisson bracket relations. Moreover, we show that the Courant bracket twisted simultaneously by $B$ and $\theta$ is invariant under the T-duality [3]. We also demonstrate that all fluxes can exist on the Dirac structures associated with the Courant algebroid for this bracket, without any restrictions imposed on fluxes.

In the end, results are generalized to a double theory, in which variables depend on both initial and T-dual coordinates. The algebra of generators that include both initial and T-dual diffeomorphisms closes on the double field extension of the Courant bracket called $C$-bracket. Following the same procedure as in the single theory, we obtained the $B$-twisted and $\theta$-twisted $C$-brackets [4]. We demonstrate that by projecting the twisted $C$-brackets to the initial and T-dual phase spaces, the mutually T-dual twisted Courant brackets are obtained.

# Chapter 2

# Action for bosonic string

In this chapter, we will give a brief overview of the action for a relativistic particle, after which we will by analogy, construct the first action for a bosonic string. We will then provide a non-linear $\sigma$-model, that describes propagation of the closed bosonic string in coordinately dependent background fields.

## 2.1 Relativistic particle

Let us first consider a relativistic free particle, moving in a curved $D$-dimensional background. The movement of the particle sweeps the one-dimensional world-line parametrized with a time-like parameter $\tau$. The action of a relativistic particle is proportional to the invariant length of its trajectory

$$S_0 = m \int \sqrt{G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} d\tau \, , \tag{2.1}$$

where $G_{\mu\nu}$ is the metric of the background, $x^\mu$ represent the coordinates of the particle, and $\dot{x}^\mu$ their derivatives with respect to $\tau$. The proportionality constant $m$ is obtained from the dimensional analysis and represents the mass of the particle. Under the reparametrization of the world-line $\tau' = f(\tau)$ we have

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau'}\frac{\partial \tau'}{\partial \tau} = \dot{f}\frac{\partial x^\mu}{\partial \tau'} \, , \quad d\tau' = \dot{f} d\tau \, , \tag{2.2}$$

and therefore the action (2.1) remains invariant under the reparametrization of the world-line.

We are not quite satisfied with the action that features the square root, because it is impossible to quantize it by Fenyman's path integral formalism. Recall that in this formalism, the quantum mechanical propagator is obtained by integrating over different contributions of all paths in configuration space, with weights being expressed as $e^{-i\frac{S}{\hbar}}$. We can avoid integrating over square root by using another linear action that is equivalent to the action (2.1) on the classical level. With the introduction of an independent auxiliary field $e(\tau)$ on the world-line, we define the action

$$S = \frac{1}{2}\int d\tau\Big(\frac{1}{e}G_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + m^2 e\Big)\,. \tag{2.3}$$

The variation of the action along $e$ produces the equations of motions

$$e = \frac{1}{m}\sqrt{G_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}\,, \tag{2.4}$$

on which the action (2.3) becomes (2.1). The variation of action along the coordinates $x^{\mu}$ provides the well-known equation of motion for a free relativistic particle along the geodesic, given by

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho}\dot{x}^{\nu}\dot{x}^{\rho} = 0\,, \tag{2.5}$$

where $\Gamma^{\rho}_{\mu\nu}$ are Christoffel symbols, given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}(G^{-1})^{\rho\sigma}\Big(\partial_{\mu}G_{\nu\sigma} + \partial_{\nu}G_{\mu\sigma} - \partial_{\sigma}G_{\mu\nu}\Big)\,. \tag{2.6}$$

In case of a relativistic particle moving in the electromagnetic field, one should add the interacting term to the action

$$S_{int} = \int d\tau q A_{\mu}\dot{x}^{\mu}\,, \tag{2.7}$$

where $q$ is the electric charge of the particle, while $A_{\mu}$ is the vector potential. The action (2.7) is invariant under gauge transformations

$$A_{\mu} \to A_{\mu} + \partial_{\mu}\lambda\,, \tag{2.8}$$

due to

$$\int d\tau q\partial_{\mu}\lambda\dot{x}^{\mu} = \int d\tau q\dot{\lambda} = 0\,. \tag{2.9}$$

## 2.2 Action for non-interacting string

Now consider a one-dimensional string and suppose we want to introduce its action by analogy with the relativistic particle. In the same way that the particle sweeps the world-line, a string sweeps the world-sheet. The Nambu-Goto action [26] that describes the string is proportional to the area of the worldsheet. It is given by

$$S_{NG} = \kappa\int_{\Sigma} d^2\xi\sqrt{-\det\Big(\partial_{\alpha}x^{\mu}\partial_{\beta}x^{\nu}G_{\mu\nu}\Big)}\,, \quad d^2\xi = d\sigma d\tau\,, \tag{2.10}$$

where $\kappa = \frac{1}{2\pi l_s^2}$, with $l_s$ being the string length scale. It is the only parameter in string theory. The string is moving in a $D$-dimensional space-time characterized with a constant metric $G_{\mu\nu}$, where $\mu,\ \nu$ are the coordinates of the space-time $\mu,\ \nu \in 0,\ 1, ..., D-1$. The indices $\alpha,\ \beta = 0,\ 1$ are coordinates on the world-sheet $\Sigma$, parametrized with one time-like parameter $\xi^0 = \tau$ and one space-like parameter $\xi^1 = \sigma$. For closed strings, the topology of world-sheet is a torus $\mathbb{R} \times S^1$, where $-\infty \leq \tau \leq \infty$ and $0 \leq \sigma < 2\pi$.

Obviously, the same problem as with the action for relativistic particle (2.1) persists - the Nambu-Goto action cannot be quantized with the Feynman path integral procedure due to its nonlinearity. In an analogous way, it is possible to introduce the world-sheet metric $g_{\alpha\beta}$ and construct the Polyakov action

$$S_P = \frac{\kappa}{2} \int d^2\sigma \sqrt{-g}\; g^{\alpha\beta} G_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{2.11}$$

where $g^{\alpha\beta}$ is the inverse of the world-sheet metric $g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$, and $g$ is determinant of the world-sheet metric $g = \det g_{\alpha\beta}$. The Polyakov action is invariant under the global Poincaré transformations

$$x^{\mu\prime}(\xi) = \Lambda^\mu{}_\nu x^\nu(\xi) + a^\mu \,, \quad g'_{\alpha\beta} = g_{\alpha\beta} \,, \tag{2.12}$$

where $a^\mu$ are translation parameters, and $\Lambda^\mu{}_\nu$ are Lorentz transformations that satisfy $\Lambda^T \eta \Lambda = \eta$, for the Minkowski metric $\eta$. Additionally, the action is invariant under the reparametrization of the world-sheet

$$\xi^\alpha \to \xi^{\alpha\prime}(\xi) \,, \quad x^{\mu\prime}(\xi') = x^\mu(\xi) \,, \quad g'_{\alpha\beta}(\xi') = \frac{\partial \xi^\gamma}{\partial \xi'^\alpha} \frac{\partial \xi^\delta}{\partial \xi'^\beta} g_{\gamma\delta} \,, \tag{2.13}$$

as well as under the Weyl transformations

$$g'_{\alpha\beta}(\xi) = e^{\phi(\xi)} g_{\alpha\beta}(\xi) \,, \quad x^{\mu\prime}(\xi) = x^\mu(\xi) \,. \tag{2.14}$$

Reparametrization and Weyl transformations are local transformations and can be used to choose the gauge. The theory invariant under the Weyl transformations is said to be conformally invariant. The string theory is therefore the conformal field theory.

The variation of the Polyakov action (2.11) with respect to the worldsheet metric gives rise to the equations of motions, on which the Nambu-Goto action (2.10) is obtained. To demonstrate this, we will find useful the following relations

$$\delta g = -g g_{\alpha\beta} \delta g^{\alpha\beta} \,, \quad \delta\sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\alpha\beta} \delta g^{\alpha\beta} \,, \tag{2.15}$$

so that we can obtain the equations of motions by varying the Polyakov action with respect to the worldsheet metric $\frac{\delta S_P}{\delta g^{\alpha\beta}}$

$$\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu} - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma x^\mu \partial_\delta x^\nu G_{\mu\nu} = 0 \,. \tag{2.16}$$

Its solution is in the form

$$g_{\alpha\beta} = \lambda\, \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}\,, \tag{2.17}$$

where $\lambda$ is an arbitrary scalar, due to Weyl symmetry (2.14). Substituting (2.17) into the Polyakov action (2.11), one obtains the Nambu-Goto action (2.10). The second set of equations of motion is obtained from the action variation with respect to $x^\mu$, in which case the wave equations are obtained

$$\partial^\alpha \partial_\alpha x^\mu = \ddot{x}^\mu - x''^\mu = 0\,. \tag{2.18}$$

If we introduce the light-cone coordinates by

$$\xi^\pm = \tau \pm \sigma\,, \quad \partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)\,, \tag{2.19}$$

the equation of motion (2.18) can be rewritten as

$$\partial_+ \partial_- x^\mu = 0\,. \tag{2.20}$$

Now its general solution will have a decomposition to the left-movers $x_L^\mu$ and right-movers $x_R^\mu$

$$x^\mu = x_L^\mu(\xi_+) + x_R^\mu(\xi_-)\,, \tag{2.21}$$

which can be expanded in modes by

$$\begin{aligned} x_L^\mu &= \frac{1}{2}x_0^\mu + \frac{1}{2}l_s^2 p^\mu(\tau+\sigma) + i\frac{l_s}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in(\tau+\sigma)} \\ x_R^\mu &= \frac{1}{2}x_0^\mu + \frac{1}{2}l_s^2 p^\mu(\tau-\sigma) + i\frac{l_s}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau-\sigma)}\,, \end{aligned} \tag{2.22}$$

where $x_0^\mu$ is the center of mass position of a string, and $p^\mu$ is total string momentum. The exponential terms represent the string excitation modes, and $\alpha_n^\mu$ and $\tilde{\alpha}_n^\mu$ its coefficients.

## 2.3 Bosonic string $\sigma$-model

The Polyakov action can be quantized, in which case oscillatory modes $\alpha_n^\mu$ and $\tilde{\alpha}_n^\mu$ are promoted to operators that satisfy the harmonic oscillator algebra relations, so that they can be interpreted as creation and annihilation operators. The ground state is defined as the state annihilated by annihilation operators. Bosonic string theory also predicts the existence of negative norm states, which are removed in case of strings moving in 26-dimensional space-time. This is what we call a critical bosonic string theory [1]. In this case, the first excited state gives a set of $24^2 = 576$ of the states that correspond to the tensor products of two $SO(24)$ representations. It includes the symmetric traceless part, the antisymmetric part, and the trace.

The symmetric traceless part transforms as a massless particle of spin 2, and for that reason the background field corresponding to it is space-time metric $G_{\mu\nu}$. The antisymmetric part is represented by the Kalb-Ramond field $B_{\mu\nu}$, and the trace is represented with the scalar dilaton field $\Phi$. The string $\sigma$-model is described by the following action:

$$S = \kappa \int_\Sigma d^2\xi \Big[\frac{1}{2}\sqrt{-g}g^{\alpha\beta}G_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu + \epsilon^{\alpha\beta}B_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu + \frac{\pi}{\kappa}\sqrt{-g}\Phi(x)R^{(2)}\Big] , \quad (2.23)$$

where $\epsilon^{\alpha\beta}$ is the antisymmetric tensor density, with $\epsilon^{01} = 1$, and $R^{(2)}$ is the scalar curvature of the world-sheet metric $g$, given by

$$R^{(2)} = g_{\mu\nu}R^{\mu\nu} , \quad R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} , \quad R^\rho{}_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\tau_{\mu\nu}\Gamma^\rho_{\tau\sigma} - \Gamma^\tau_{\mu\sigma}\Gamma^\rho_{\tau\nu} , \quad (2.24)$$

where $\Gamma^\rho_{\mu\nu}$ are Christoffel symbols (2.6) of the world-sheet metric $g$. The coupling of the string with the metric tensor is the same as in the case of Polyakov action. The Kalb-Ramond field is analogous to the potential $A_\mu$, and hence this term can be seen as the interacting term. The dilaton is a quantum effect, that is added to preserve the conformal invariance on the quantum level. The conformal invariance on the quantum level results in the space-time equations of motion for the background fields, that to the lowest order in slope parameter $\alpha' = \frac{2\pi}{\kappa}$ are [29]

$$R_{\mu\nu} - \frac{1}{4}B_{\mu\rho\sigma}B_\nu{}^{\rho\sigma} + 2D_\mu D_\nu \Phi = 0 , \quad (2.25)$$

$$D_\rho B^\rho{}_{\mu\nu} - 2D_\rho \Phi B^\rho{}_{\mu\nu} = 0 , \quad (2.26)$$

$$4(D\Phi)^2 - 4D_\mu D^\mu \Phi + \frac{1}{12}B_{\mu\nu\rho}B^{\mu\nu\rho} - R = 0 , \quad (2.27)$$

where $B_{\mu\nu\rho}$ is the Kalb-Ramond field strength, given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} , \quad (2.28)$$

$R$ and $R_{\mu\nu}$ are respectively the Ricci scalar and Ricci tensor (2.24) of the space-time metric $G_{\mu\nu}$, and by $D_\mu$ we have marked the covariant derivative, which acts on a vector field $V_\mu$ by

$$D_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\sigma_{\mu\nu}V_\sigma , \quad (2.29)$$

[1] It is possible to define non-critical bosonic string theory in a space-time with dimensions $D < 26$, provided that an appropriate Liouville term is added to the action. This results in a Liouville field theory that is not linear and does not possess the Weyl invariance but is classically integrable. For reviews of the Liouville field theory, see [27, 28].

which is easily generalized to the terms appearing in the equations of motions

$$\begin{aligned} D_\mu D_\nu \Phi &= \partial_\mu D_\nu \Phi - \Gamma^\sigma_{\mu\nu} D_\sigma \Phi \,, \\ D_\mu B^\nu{}_{\rho\sigma} &= \partial_\mu B^\nu{}_{\rho\sigma} + \Gamma^\nu_{\mu\tau} B^\tau{}_{\rho\sigma} - \Gamma^\tau_{\mu\rho} B^\nu{}_{\tau\sigma} - \Gamma^\tau_{\mu\sigma} B^\nu{}_{\rho\tau} \,. \end{aligned} \tag{2.30}$$

The conformal symmetry allows us to chose the conformal gauge $g_{\alpha\beta} = e^{2\varphi}\eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is a flat Minkowski metric. Furthermore, if we do not take into account quantum effects, the dilaton field can be taken to be zero, in which case the action simplifies to

$$S = \int_\Sigma d^2\xi \mathcal{L} = \kappa \int_\Sigma d^2\xi \Big( \frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu} \Big) \partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{2.31}$$

which in the light-cone coordinates becomes

$$S = \kappa \int d\xi^2 \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \,, \tag{2.32}$$

where $\Pi_{\pm\mu\nu}$ fields are given by

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu} \,. \tag{2.33}$$

The action in the form (2.31) will mostly be used in this thesis. The symmetries of this action will be analyzed in detail.

## 2.4 Canonical Hamiltonian

We finish this chapter with the derivation of the canonical Hamiltonian for string $\sigma$-model. The canonical momenta corresponding to the coordinate $x^\mu$ are obtained from variation of the Lagrangian (2.31) with respect to coordinate time derivative

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu} \dot{x}^\nu - 2\kappa B_{\mu\nu} x'^\nu \,. \tag{2.34}$$

The canonical Hamiltonian is the Legendre transformation of the Lagrangian

$$\mathcal{H}_C = \pi_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2\kappa} \pi_\mu (G^{-1})^{\mu\nu} \pi_\nu + \frac{\kappa}{2} x'^\mu G^E_{\mu\nu} x'^\nu - 2x'^\mu B_{\mu\rho} (G^{-1})^{\rho\nu} \pi_\nu \,, \tag{2.35}$$

where $G_E$ is the effective metric, given by

$$G_E^{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu} \,. \tag{2.36}$$

The effective metric is the open string metric. From the symmetric properties of the background fields $B$ and $G$, one easily shows that the effective metric is symmetric.

Often we will find convenient to express the Hamiltonian in matrix notation

$$\mathcal{H}_C = \frac{1}{2\kappa}(X^T)^M H_{MN} X^N \,, \tag{2.37}$$

where $H_{MN}$ is the generalized metric, given by

$$H_{MN} = \begin{pmatrix} G^E_{\mu\nu} & 2(BG^{-1})_\mu{}^\nu \\ -2(G^{-1}B)^\mu{}_\nu & (G^{-1})^{\mu\nu} \end{pmatrix} \,, \tag{2.38}$$

and

$$X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix} \,. \tag{2.39}$$

The index $M$ can take values from 0 to $2D-1$, and includes both covariant and contravariant indices.

# Chapter 3

# T-duality

Dualities are well known to appear in physics. There is a unique form of duality that appears in string theory and relates theories formulated in different geometries or topologies, called T-duality. In this chapter, we will start with the presentation of the first emergence of T-duality, in the case of a string moving in a background with one dimension compactified to a circle. Next, we will show how to obtain the T-dual theory from the initial one by means of the Buscher procedure. Lastly, we will comment on how the procedure can be generalized and what are the other important features of T-duality.

## 3.1 First appearance of T-duality

The most well-known example where T-duality emerges is a closed bosonic string in the space-time with one dimension compactified to a circle of radius $R$. In that case, the space-time is the tensor product of Minkowski space-time and a circle $\mathbb{R}^{1,24} \times S^1$. The compactification on a circle has a couple of peculiar consequences. Firstly, the generator of translation by $a$ along the dimension $x_{25}$ is proportional to the factor $e^{ip_{25}a}$. The translation by $2\pi R$ should by design be the identity operator $e^{ip_{25}\cdot 2\pi R} = 1$, from which we obtain

$$p^{25} = \frac{n}{R}, \quad n \in \mathbb{Z}, \tag{3.1}$$

where $n$ are integers known as momentum numbers. Secondly, the string can wind around the compact dimension, which we can express by

$$x^{25}(\sigma + 2\pi) = x^{25}(\sigma) + 2\pi m R, \quad m \in \mathbb{Z}, \tag{3.2}$$

where $m$ is the winding number, equals the number of times the string winds around the compact dimension. The mass spectrum of particles can be obtained in the form

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{l_s^4}. \tag{3.3}$$

The spectrum remains invariant under the exchange of winding numbers $m$ and momentum numbers $n$, provided that one makes the transformation $R \leftrightarrow \frac{l_s^2}{R}$.

The closed string moving on a dimension compactified on a radius $R$ has the indistinguishable physics from the string moving on a dimension compactified on a radius $\frac{l_s^2}{R}$. Though two strings would be described by different actions, the observable quantities would be the same. Moreover, the self T-dual radius $R = l_s$ can be seen as the critical, or minimal radius. All theories with a dimension compactified on radii lower than the critical one are in fact T-dual to theories with a dimension compactified on a larger one.

In the case of a large radius $R \to \infty$, winding modes become very heavy and hence require a lot of energy to be created. As such, they become irrelevant for the dynamics of a string. The momentum modes become very light, and the differences between two modes $\frac{l_s^2}{R}$ becomes very small, effectively meaning that momenta are continuous. The large radius limit $R \to \infty$ describes a string with no winding and continuous momenta along that dimension, which is equivalent to that dimension being effectively non-compactified. In its corresponding T-dual radius $R \to 0$, we have the opposite case. The momentum modes are very heavy, but the winding modes are very light and basically make the continuum.

## 3.2 Buscher procedure

The Buscher procedure is the first formal method of obtaining T-dual theory from the closed string $\sigma$-model. The procedure requires a shift of coordinate

$$\delta x^\mu = \lambda^\mu = const \ , \tag{3.4}$$

to be a global symmetry of the action (2.31), which corresponds to the case of constant background fields

$$B_{\mu\nu} = const\,, \quad G_{\mu\nu} = const\,. \tag{3.5}$$

The first step in the procedure is a localization of the global symmetry. The partial derivative is replaced with the covariant derivative

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v_\alpha^\mu\,, \tag{3.6}$$

where $v_\alpha^\mu$ are gauge fields. We will require the covariant derivatives to be gauge invariant

$$\delta D_\alpha x^\mu = 0\,, \tag{3.7}$$

from which we can easily read the transformation laws for the gauge fields

$$\delta v_\alpha^\mu = -\partial_\alpha \lambda^\mu\,. \tag{3.8}$$

Since the gauge fields are not physical, we will demand that their field strength is zero

$$F^{\mu}_{\alpha\beta} = \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha = 0\,. \tag{3.9}$$

This condition can be assured by adding the appropriate term with the Lagrangian multiplier $y_\mu$ to the action (2.31). The gauge action invariant under the localized symmetries is then given by

$$S_{loc}(x,y,v) = \kappa \int d^2\xi \Big[ (\frac{\eta^{\alpha\beta}}{2} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}) D_\alpha x^\mu D_\beta x^\nu + \frac{1}{2} y_\mu \epsilon^{\alpha\beta} F^\mu_{\alpha\beta} \Big]\,. \tag{3.10}$$

In the second step, we will fix the gauge by demanding $x^\mu(\xi) = x^\mu(\xi_0)$, so the gauge fixed action is

$$S_{fix}(y,v) = \kappa \int d^2\xi \Big[ (\frac{\eta^{\alpha\beta}}{2} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu}) v^\mu_\alpha v^\nu_\beta + \frac{1}{2} y_\mu \epsilon^{\alpha\beta} F^\mu_{\alpha\beta} \Big]\,, \tag{3.11}$$

or in the light-cone coordinates (2.19)

$$S_{fix}(y,v) = \kappa \int d^2\xi \Big[ v^\mu_+ \Pi_{+\mu\nu} v^\nu_- + \frac{1}{2} y_\mu (\partial_+ v^\mu_- - \partial_- v^\mu_+) \Big]\,. \tag{3.12}$$

Equations of motions can be obtained from the variation principle. By varying with respect to the Lagrange multiplier $y_\mu$, one obtains the condition (3.9), as required. Its solution is

$$v^\mu_\pm = \partial_\pm x^\mu\,, \tag{3.13}$$

which when substituted in the gauge fixed action (3.12) gives rise to the initial action (2.32). On the other hand, the variation with respect to the gauge field produces another set of equations of motions

$$\partial_\pm y_\mu + 2\Pi_{\mp\mu\nu} v^\nu_\pm = 0\,. \tag{3.14}$$

To find their solution, we introduce another set of fields by

$$\Theta^{\mu\nu}_\pm = \theta^{\mu\nu} \mp \frac{1}{\kappa} (G_E^{-1})^{\mu\nu}\,, \tag{3.15}$$

where $\theta^{\mu\nu}$ is the non-commutativity parameter, given by

$$\theta^{\mu\nu} = -\frac{2}{\kappa} (G_E^{-1} B G^{-1})^{\mu\nu}\,, \tag{3.16}$$

and $G_E^{-1}$ is the inverse of the effective metric (2.36). The non-commutativity parameter appears in non-commutative relations on the open string endpoints, in the presence of non-zero Kalb-Ramond field [30]. It is an antisymmetric tensor, while the inverse of the effective metric is symmetric, so one easily proves that

$$\Theta^{\mu\nu}_\pm = -\Theta^{\nu\mu}_\mp\,, \tag{3.17}$$

and moreover that

$$\Pi_{\pm\mu\rho}\Theta^{\rho\nu}_{\mp} = \frac{1}{2\kappa}\delta^{\nu}_{\mu}\,. \tag{3.18}$$

We can multiply the equation of motion (3.14) with $\Theta_\pm$ and obtain its solution

$$v^{\mu}_{\pm} = -\kappa\Theta^{\mu\nu}_{\pm}\partial_\pm y_\nu\,. \tag{3.19}$$

When substituted (3.19) in the gauge fixed action, the T-dual action is obtained

$$^\star S(y) = \int d\xi^{2\star}\mathcal{L} = \kappa\int d\xi^2 \partial_+ y_\mu {}^\star\Pi^{\mu\nu}_+\partial_- y_\nu\,, \quad {}^\star\Pi^{\mu\nu}_+ = \frac{\kappa}{2}\Theta^{\mu\nu}_-\,, \tag{3.20}$$

where $y_\mu$ is the T-dual coordinate, and the T-dual fields are given by

$$^\star G^{\mu\nu} = (G^{-1}_E)^{\mu\nu}\,, \quad {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}\,. \tag{3.21}$$

By comparing (3.13) and (3.18), one obtains the T-duality relations between the coordinates

$$\partial_\pm x^\mu \simeq -\kappa\Theta^{\mu\nu}_{\pm}\partial_\pm y_\nu\,, \tag{3.22}$$

or

$$\begin{aligned} \dot{x}^\mu &\simeq -\kappa\theta^{\mu\nu}\dot{y}_\nu + (G^{-1}_E)^{\mu\nu}y'_\nu\,, \\ x'^\mu &\simeq (G^{-1}_E)^{\mu\nu}\dot{y}_\nu - \kappa\theta^{\mu\nu}y'_\nu\,. \end{aligned} \tag{3.23}$$

When these transformations are applied to the coordinates in the initial Lagrangian, the T-dual Lagrangian is obtained, i.e.

$$\begin{aligned} \kappa\partial_+ x^\mu\Pi_{+\mu\nu}\partial_- x^\nu &\simeq \kappa^3\Theta^{\mu\rho}_+\partial_+ y_\rho\Pi_{+\mu\nu}\Theta^{\nu\sigma}_-\partial_- y_\sigma = -\kappa^3\partial_+ y_\rho\Theta^{\rho\mu}_-\Pi_{+\mu\nu}\Theta^{\nu\sigma}_-\partial_- y_\sigma \\ &= -\frac{\kappa^2}{2}\partial_+ y_\rho\Theta^{\rho\sigma}_-\partial_- y_\sigma\,. \end{aligned} \tag{3.24}$$

The Buscher procedure can be applied to the T-dual action (3.20) as well. We will not go into details, as the procedure is exactly the same as when it is applied to the initial action. The T-duality transformation laws on the T-dual coordinates are given by [16]

$$\partial_\pm y_\mu \simeq -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu\,. \tag{3.25}$$

When these relations are applied to the T-dual action (3.20), the initial action is obtained (2.32).

The T-duality relations are as easily expressed in Hamiltonian formalism. The variation of the T-dual Lagrangian with respect to the T-dual coordinate $\tau$-derivative is equal to the T-dual canonical momentum

$$^\star\pi^\mu = \frac{\partial\,{}^\star\mathcal{L}}{\partial\dot{y}_\mu} = \kappa(G^{-1}_E)^{\mu\nu}\dot{y}_\nu - \kappa^2\theta^{\mu\nu}y'_\nu\,. \tag{3.26}$$

Therefore, in terms of the canonical variables, T-dual relations (3.23) are

$$\pi_\mu \simeq \kappa y'_\mu\,, \quad \kappa x'^\mu \simeq {}^\star\pi^\mu\,. \tag{3.27}$$

The T-duality transforms momenta into $\sigma$-derivatives of the T-dual coordinates, and vice versa. To give a further interpretation of these relations, we notice that the integrals of the canonical momenta produce the momentum numbers $P^\mu$, and the integrals of the $\sigma$-derivatives of the coordinates along compact dimension produce the winding numbers $W^\mu$

$$P^\mu = \int d\sigma \pi_\mu\,, \quad W^\mu = \int d\sigma \kappa x'^\mu\,. \tag{3.28}$$

The Buscher procedure demonstrates that the winding numbers of the initial theory are the momenta in its T-dual theory, and vice versa.

## 3.3 Beyond Buscher procedure

Due to its requirement for global shift symmetry, the Buscher procedure cannot be applied in the majority of cases. Let us consider a so-called weakly curved background, characterized by the constant metric field $G_{\mu\nu} = const$ and the Kalb-Ramond field only linearly dependent on coordinate $B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho\,, \quad b_{\mu\nu} = const$. If the field strength $B_{\mu\nu\rho}$ is taken to be infinitesimal and its higher orders are neglected, it is straightforward to demonstrate the fields in weakly curved background satisfy equations of motion (2.25) - (2.27). In this case, it is possible to construct the adapted Buscher procedure [31]. One introduces the gauge fields $v^\mu_\alpha$ and replaces the partial derivatives with the covariant ones. This is insufficient, because the Kalb-Ramond field $B$ depends on $x^\mu$, and the coordinate $x^\mu$ itself is not locally gauge invariant. This obstacle can be overcome by replacing the coordinate with the line integral

$$V^\mu = \int_P d\xi^\alpha v^\mu_\alpha = \int_P (d\xi^+ v^\mu_+ + d\xi^- v^\mu_-)\,, \tag{3.29}$$

where the integration is done along the path from the initial point $\xi^\alpha_0(\tau_0, \sigma_0)$ to the final one $\xi^\alpha(\tau, \sigma)$. With this change in mind, one can follow Buscher's ideas. On the equations of motions related to the Lagrangian multiplier, the initial theory is obtained. On the equations of motions for the gauge fields $v^\mu_\alpha$, the T-dual theory is obtained. The background fields in the T-dual theory depend on the line integral of T-dual coordinate $y_\mu$. The line integral is a non-local object, and we say that this theory is non-geometric. The situation becomes even more complicated when other backgrounds are included, where the metric also depends on coordinates [32].

## 3.4 T-duality in superstring theories

Even though this thesis is primarily focused on bosonic strings, we are going to briefly touch on the importance of T-duality in superstring theories. The bosonic string theory is incomplete because it does not include fermions. They can be added to the theory with the introduction of supersymmetry. We can require that the theory is either supersymmetric on the world-sheet, in which case we obtain the Ramond-Neveu-Schwarz (RNS) strings, or that the theory is supersymmetric in the ten-dimensional Minkowski space-time, in which case we obtain the Green-Schwarz (GS) strings.

The RNS formalism [33, 34] relies on adding the standard Dirac action for $D$ massless fermions. The action is invariant under the $N = 2$ supersymmetry group. The left and right moving ground fermionic states can be chosen to have the same or the opposite chirality. The former case corresponds to the type IIA superstring theory, while the latter case corresponds to the type IIB superstring theory. These two superstring theories contain only closed strings.

The supersymmetric theory that contains both open and closed strings can be constructed with the help of GS mechanism [35, 36, 37], where the action is constructed by requiring the manifest Poincaré supersymmetry. This action can be only invariant under $N = 1$ supersymmetry, in which case we obtain type $I$ superstring theory.

Heterotic string theories [38] can be constructed by combining the left movers of the 26-dimensional bosonic string theory, with the right movers of the ten-dimensional superstring theory. The spectrum contains Yang-Mills multiplets that are either based on $SO(32)$ or $E_8 \times E_8$ gauge symmetry.

The fact that we are able to construct five different but self-consistent superstring theories was initially seen as a weakness of string theory since there was no obvious choice of which one should be preferable. However, these theories are mutually related by dualities. Specifically, T-duality connects IIA and IIB superstring theories [39, 40], and also two heterotic string theories [41, 42, 43].

Apart from T-duality, different superstring theories can also be related via S-duality. It is a duality between theories that have different coupling constants. Type I superstring theory with a coupling constant $g$ is S-dual to the $SO(32)$ heterotic string theory with coupling constant $\frac{1}{g}$ [44]. Moreover, type IIB superstring theory is invariant under $S$-duality.

In 1995, Edward Witten suggested that all superstring theories could be related with dualities to an eleven-dimensional $M$-theory. The $M$-theory has not been formulated on all orders of perturbations, but its effective low-energy action is an eleven-dimensional theory of supergravity. Type IIA superstring theory in strong coupling is equivalent to an eleven-dimensional supergravity theory with one dimension compactified on $T^1$ [44]. Similarly, $E_8 \times E_8$ heterotic string theory with strong coupling constant $g$ becomes M-theory compactified to a $Z_2$ orbifold of the circle [45, 46]. These observations make strong arguments for the existence of the $M$-theory which is connected with all superstring theories by a web of dualities.

So far, there is no universal description of T-duality that can be applied to an arbitrary string field configuration. The intricate geometric and non-geometric spaces that can be related via T-duality necessitate a novel mathematical framework to be able to accommodate them. In the next part, we will introduce the reader to the basic elements of generalized geometry, which appears to come with a suitable apparatus for describing T-duality and other relevant string phenomena.

# Part II

# Generalized geometry

# Chapter 4

# Differential geometry

This chapter begins with definitions of fundamental geometric terms that will be used throughout the thesis. We introduce vectors, the Lie derivative of a vector field, and define the Lie bracket. Next, we define differential forms, together with the exterior derivative and the interior product. Finally, we extend the definition of Lie bracket to multi-vectors by introducing the Schouten-Nijenhuis bracket.

## 4.1 Tangent and cotangent bundle

We start with the definition of a manifold:

**Definition 1 (Manifold)**
*A $n$-dimensional smooth manifold is a Hausdorff* [1] *topological space $\mathcal{M}$ such that:*

- *For each point $p \in \mathcal{M}$ there is an open neighborhood $U_\alpha$ that is homeomorphic to $\mathbb{R}^n$, i.e. there is a smooth map $\phi_\alpha : U_\alpha \to \mathbb{R}^n$. The ordered pairs $U_\alpha, \phi_\alpha$ are called charts, and the collection of charts covering all topological space is called atlas.*
- *For each two non-disjoint neighborhoods $U_\alpha$ and $U_\beta$, transition maps on their intersections $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ are smooth maps.*

This means that locally $n$-dimensional manifold resembles the $n$-dimensional Euclidean space, and that in each chart a local coordinate system is defined. The resemblance with $\mathbb{R}^n$ manifests in the ability to use the calculus techniques on manifolds.

[1] Hausdorff topological space is a topological space where for any two distinct points, there exists neighborhoods of each which are disjoint from each other.

**Definition 2 (Tangent vector)**

*Let $\mathcal{M}$ be a smooth n-dimensional manifold. Given a point $p$ on it, let $\mathcal{F}_p\mathcal{M}$ be the family of real valued smooth functions on $\mathcal{M}$. A function $\xi : \mathcal{F}_p\mathcal{M} \to \mathbb{R}$ is called the tangent vector $\xi$ at $p \in \mathcal{M}$ if it satisfies:*

- *$\xi$ is linear - $\xi(af + bg) = a\xi(f) + b\xi(g)$ , for $f,\ g \in \mathcal{F}_p\mathcal{M}$ , and for $a,\ b \in \mathbb{R}$ ,*
- *$\xi$ satisfies Leibniz property - $\xi(fg) = \xi(f)g + f\xi(g)$ , for $f,\ g \in \mathcal{F}_p\mathcal{M}$.*

If we chose a chart at $p$ with coordinates $x^\mu$, the vector field in point $p$ can be represented in the basis of partial derivatives $\xi = \xi^\mu \partial_\mu$. If there is no global covering of the manifold, i.e. if it cannot be covered by a single chart, then it is not possible to define partial derivatives globally. The set of all tangent vectors at a point $p$ forms a vector space.

**Definition 3 (Tangent space)**

*The set $T_p\mathcal{M}$ of all tangent vectors through a point $p$ is called the tangent space of $\mathcal{M}$ at $p$.*

Intuitively, we can associate a $n$-dimensional Euclidean vector space to each point of a manifold. This is by no means sufficient to describe all physical phenomena. For example, there are internal degrees of freedom, that are associated to each point on the space-time, and as such we would like to have generalizations of manifolds, such that to each point on a manifold we can attribute another manifold. To achieve this, let us define the fiber bundle.

**Definition 4 (Fiber bundle)**

*A bundle is a triple $\Big(V,\ \pi,\ \mathcal{M}\Big)$ consisting of a base manifold $\mathcal{M}$, a total manifold $V$ and a surjective map $\pi : V \to \mathcal{M}$ called projection. The inverse image $\pi^{-1}(p)$ is the fiber over $p$, for $p \in \mathcal{M}$. If fibers over all points on the base manifold are homeomorphic to a space $F$, the triple $\Big(V,\ \pi,\ \mathcal{M}\Big)$ is said to be a fiber bundle, with $F$ being a fiber.*

The physical fields are usually represented as functions that depend on specific points on manifolds, and appear as sections.

**Definition 5 (Section)**

*A section of a bundle $\Big(V,\ \pi,\ \mathcal{M}\Big)$ is a map $\sigma : \mathcal{M} \to V$ such that the image of each point $p \in \mathcal{M}$ lies in the fiber $\pi^{-1}(p)$ over $p$, i.e. $\pi \circ \sigma =$ Id, where Id is an identity operator on the base manifold.*

The first example of a fiber bundle can simply be obtained as the disjoint union of the tangent spaces of a manifold $\mathcal{M}$. This way, we obtain the tangent bundle, which we define below.

**Definition 6 (Tangent bundle)**

*The tangent bundle is a triple* $\left(T\mathcal{M},\ \pi,\ \mathcal{M}\right)$*, where* $T\mathcal{M}$ *is the disjoint union of the tangent spaces of a base manifold* $\mathcal{M}$*, and the projection* $\pi$ *is trivial projection* $\pi : T_p\mathcal{M} \to p$*.*

The fibers of the tangent bundle are the tangent spaces $T_p\mathcal{M}$. Its section is some function that will take some point $p$ on the manifold as a domain and map it to the fiber $T_p\mathcal{M}$. Therefore, we conclude that vector fields are elements of the smooth section of the tangent bundle.

The tangent bundle is also a smooth manifold, so we can define the higher order tangent bundles by $T^n\mathcal{M} = T(T^{n-1}\mathcal{M})$. Their sections' elements are antisymmetric multi-vectors.

We also define the cotangent space and cotangent bundle:

**Definition 7 (Cotangent bundle)**

*The cotangent space* $T_p^\star\mathcal{M}$ *to a smooth manifold* $\mathcal{M}$ *at the point* $p$ *is the dual space of the tangent space* $T_p\mathcal{M}$*. The disjoint union of cotangent spaces to the manifold is the cotangent bundle* $T^\star\mathcal{M}$*.*

The elements of the smooth section of the cotangent bundle are differential 1-forms. In some local chart, 1-forms can be written in the basis of coordinate differentials, i.e. $\lambda = \lambda_\mu dx^\mu$.

The cotangent bundle is also a manifold, so higher order of cotangent bundles are defined as the cotangent bundles of the cotangent bundles, i.e. $\bigwedge^p T^\star\mathcal{M} = T^\star \bigwedge^{p-1} T^\star\mathcal{M}$. The elements of their sections are differential $p$-forms $\omega$ - totally antisymmetric tensors of type $(0, p)$, which in some coordinate basis can be expressed by

$$\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\ , \tag{4.1}$$

where $\wedge$ denotes the wedge product $\wedge : \bigwedge^p T^\star\mathcal{M} \times \bigwedge^q T^\star\mathcal{M} \to \bigwedge^{p+q} T^\star\mathcal{M}$, which is totally antisymmetric, e.g. $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$.

## 4.2 Lie derivative

Lie derivative represents the coordinate invariant evaluation of the change of a tensor field along the flow defined by a vector field. There are multiple equivalent ways to define a Lie derivative. We present the so-called classical definition:

**Definition 8 (Lie derivative)**

*Let* $T$ *be a tensor field of type (p, q) (i.e. contravariant of order p and covariant of order q) defined over a manifold* $\mathcal{M}$*. The Lie derivative of the tensor field T with respect to the vector field* $\xi$ *is another tensor field* $\mathcal{L}_\xi T$ *of type (p, q) with components*

$$\mathcal{L}_\xi T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q} = \xi^\rho \partial_\rho T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q} - \sum_{i=0}^{p} T^{\mu_1\ldots\rho\hat{\mu}_i\ldots\mu_p}_{\nu_1\nu_2\ldots\nu_q}\partial_\rho\xi^{\mu_i} + \sum_{i=1}^{q} T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\rho\hat{\nu}_i\ldots\nu_q}\partial_{\nu_i}\xi^\rho\ , \tag{4.2}$$

*where $\hat{\nu}_i$ denotes omission of such index, e.g. $T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\ldots\rho\hat{\nu}_q}\partial_{\nu_q}\xi^\rho = T^{\mu_1\mu_2\ldots\mu_p}_{\nu_1\nu_2\ldots\rho}\partial_{\nu_q}\xi^\rho$*

We will now provide explicit expressions for the action of the Lie derivative on the tensors that we will encounter most frequently. Lie derivative of a function is defined as the directional derivative of a function

$$\mathcal{L}_\xi f = \xi^\mu \partial_\mu f\,. \tag{4.3}$$

The Lie derivative of a vector field produces another vector field, that is known as the Lie bracket, defined by

$$[\xi_1,\xi_2]_L f = (\mathcal{L}_{\xi_1}\xi_2)f = \mathcal{L}_{\xi_1}(\mathcal{L}_{\xi_2}f) - \mathcal{L}_{\xi_2}(\mathcal{L}_{\xi_1}f)\,, \tag{4.4}$$

or in local coordinate basis by

$$\Big([\xi_1,\xi_2]_L\Big)^\mu = \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu\,. \tag{4.5}$$

The Lie bracket satisfies the Leibniz rule

$$[\xi_1, f\xi_2]_L = f[\xi_1,\xi_2]_L + (\mathcal{L}_{\xi_1}f)\,\xi_2\,, \tag{4.6}$$

and the Jacobi identity

$$[\xi_1,[\xi_2,\xi_3]_L]_L + [\xi_2,[\xi_3,\xi_1]_L]_L + [\xi_3,[\xi_1,\xi_2]_L]_L = 0\,. \tag{4.7}$$

We can use the Lie derivative properties in order to obtain the action of Lie derivative on the 1-forms. There is a way to obtain a scalar from a vector $\xi$ with a 1-form $\lambda$ using the contraction defined by $\lambda(\xi) = \lambda_\mu\xi^\mu$. From the fact that Lie derivative satisfies the Leibniz rule, we write

$$\mathcal{L}_{\xi_1}(\lambda(\xi_2)) \;=\; (\mathcal{L}_{\xi_1}\lambda)\xi_2 + \lambda(\mathcal{L}_{\xi_1}\xi_2) = (\mathcal{L}_{\xi_1}\lambda)_\mu\xi_2^\mu + \lambda_\mu\Big(\xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu\Big)\,. \tag{4.8}$$

On the other hand, $\lambda(\xi_2)$ is a scalar, and using (4.3) we obtain

$$\mathcal{L}_{\xi_1}(\lambda(\xi_2)) = \xi_1^\mu\partial_\mu(\lambda_\nu\xi_2^\nu) = \xi_1^\mu\partial_\mu\lambda_\nu\xi_2^\nu + \xi_1^\mu\lambda_\nu\partial_\mu\xi_2^\nu\,. \tag{4.9}$$

By comparing the relations (4.8) and (4.9), we obtain the action of the Lie derivative on 1-form expressed in some coordinate basis by

$$\Big(\mathcal{L}_{\xi_1}\lambda\Big)_\mu = \xi_1^\nu(\partial_\nu\lambda_\mu - \partial_\mu\lambda_\nu)\,. \tag{4.10}$$

## 4.3 Exterior algebra of differential forms

Differential $p$-forms (4.1) are part of the smooth section of $\bigwedge^p T\mathcal{M}$. The antisymmetric wedge product $\wedge$ defines exterior algebra between differential forms, equipped with natural grading related to the degree of differential forms. It is often convenient working with the exterior derivative and interior product, which we define below.

**Definition 9 (Exterior derivative)**
*The exterior derivative of a $p$-form $\lambda$ is a $p+1$ -form $d\lambda$, such that*

$$\begin{aligned} d\lambda(\xi_0, ..., \xi_p) &= \sum_{i=0}^{p}(-1)^i \mathcal{L}_{\xi_i}\Big(\lambda(\xi_0, ..., \hat{\xi}_i, ..., \xi_p)\Big) \\ &\quad + \sum_{i<j}(-1)^{i+j}\lambda([\xi_i, \xi_j]_L, \xi_0, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_p)\,, \end{aligned} \tag{4.11}$$

*where $d\lambda(\xi_0, ..., \xi_p)$ stands for the contraction of a $p+1$ -form $d\lambda$ with $p+1$ vectors $\xi_0 ... \xi_p$, and $\hat{\xi}_i$ denotes the omission of $\xi_i$ in such contractions.*

The exterior derivative extends the notion of the differential of the function to differential forms of degree $p$. It is the antiderivation of degree 1 on the graded algebra of differential forms $\bigwedge^p T^\star\mathcal{M} \to \bigwedge^{p+1} T^\star\mathcal{M}$ that is also nilpotent, i.e.

$$d(d\lambda) = 0\,, \tag{4.12}$$

for any $p$-form $\lambda$, and it satisfies the graded Leibniz identity

$$d(\lambda_1 \wedge \lambda_2) = d\lambda_1 \wedge \lambda_2 + (-1)^p \lambda_1 \wedge d\lambda_2\,, \tag{4.13}$$

where $\lambda_1,\ \lambda_2$ are a $p$-form and a $q$-form, respectively.

**Definition 10 (Interior product)**
*The interior product $i_\xi : \bigwedge^p T^\star\mathcal{M} \to \bigwedge^{p-1} T^\star\mathcal{M}$ is defined to be the contraction of a differential form with a vector field $\xi$ by*

$$(i_\xi\lambda)(\xi_1, \xi_2, ..., \xi_{p-1}) = \lambda(\xi, \xi_1, \xi_2, ..., \xi_{p-1})\,. \tag{4.14}$$

The interior product reduces the degree of a differential form by one, and also satisfies the graded Leibniz identity

$$i_\xi(\lambda_1 \wedge \lambda_2) = i_\xi\lambda_1 \wedge \lambda_2 + (-1)^p \lambda_1 \wedge i_\xi\lambda_2\,, \tag{4.15}$$

for any $p$-form $\lambda_1$, and $q$-form $\lambda_2$, so it is an antiderivation of degree $-1$ on the graded algebra of differential forms.

The Lie derivative on a function (4.3) can be written in terms of interior product by

$$\mathcal{L}_\xi f = i_\xi df \,, \tag{4.16}$$

while the Lie derivative of a 1-form (4.10) can be rewritten in coordinate independent form, also known as Cartan formula, by

$$\mathcal{L}_\xi \lambda = i_\xi d\lambda + di_\xi \lambda \,. \tag{4.17}$$

The Cartan formula stands for any $p$-form $\lambda$. Combining Cartan formula with Leibniz rule, we can obtain the useful identity for Lie derivative

$$\mathcal{L}_{(f\xi)}\lambda = f\mathcal{L}_\xi \lambda + df i_\xi \lambda \,, \tag{4.18}$$

where $f$ is a smooth function.

For two vector fields $\xi_1$ and $\xi_2$, the interior product satisfies

$$i_{\xi_1} i_{\xi_2} \lambda = -i_{\xi_2} i_{\xi_1} \lambda \,, \tag{4.19}$$

and

$$i_{[\xi_1,\xi_2]_L} = \mathcal{L}_{\xi_1} i_{\xi_2} - i_{\xi_2} \mathcal{L}_{\xi_1} \,. \tag{4.20}$$

## 4.4 Schouten-Nijenhuis bracket

The Schouten-Nijenhuis bracket [47, 48, 49] is a bracket that extends the notion of the Lie bracket to the space of multi-vectors. Formally, multi-vectors and Schouten-Nijenhuis bracket constitute a Gerstenhaber algebra, which is a graded-commutative algebra with a Lie bracket of degree -1 satisfying the Poisson identity.

Let $\theta_1 \in \Gamma(T \bigwedge^p \mathcal{M})$ and $\theta_2 \in \Gamma(T \bigwedge^q \mathcal{M})$ be multi-vectors of order $p$ and $q$ respectively, and $0 \leq p, q \leq dim(\mathcal{M})$. Suppose that in some local coordinate basis they are given by

$$\theta_1 = \frac{1}{p!}\theta_1^{\mu_1,\ldots\mu_p}\partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_p} \,, \quad \theta_2 = \frac{1}{q!}\theta_2^{\nu_1,\ldots\nu_q}\partial_{\nu_1} \wedge \ldots \wedge \partial_{\nu_q} \,. \tag{4.21}$$

The Schouten-Nijenhuis bracket of $\theta_1$ and $\theta_2$ is the function $[,]_S : T \bigwedge^p \mathcal{M} \times T \bigwedge^q \mathcal{M} \to T \bigwedge^{p+q-1} \mathcal{M}$, given by

$$\begin{aligned} [\theta_1, \theta_2]_S &= \frac{1}{(p+q-1)!}[\theta_1, \theta_2]_S^{\mu_1\ldots\mu_{p+q-1}}\partial_{\mu_1} \wedge \ldots \wedge \partial_{\mu_{p+q-1}} \,, \\ [\theta_1, \theta_2]_S^{\mu_1\ldots\mu_{p+q-1}} &= \frac{1}{(p-1)!q!}\epsilon^{\mu_1\ldots\mu_{p+q-1}}_{\nu_1\ldots\nu_{p-1}\rho_1\ldots\rho_q}\theta_1^{\sigma\nu_1\ldots\nu_{p-1}}\partial_\sigma \theta_2^{\rho_1\ldots\rho_q} \\ &\quad + \frac{(-1)^p}{p!(q-1)!}\epsilon^{\mu_1\ldots\mu_{p+q-1}}_{\nu_1\ldots\nu_p\rho_1\ldots\rho_{q-1}}\theta_2^{\sigma\rho_1\ldots\rho_{q-1}}\partial_\sigma \theta_1^{\nu_1\ldots\nu_p} \,, \end{aligned} \tag{4.22}$$

where the antisymmetric Levi Civita symbol is defined by

$$\epsilon^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_p} = \begin{vmatrix} \delta^{\mu_1}_{\nu_1} & \dots & \delta^{\mu_1}_{\nu_p} \\ . & & . \\ . & & . \\ . & & . \\ \delta^{\mu_p}_{\nu_1} & \dots & \delta^{\mu_p}_{\nu_p} \end{vmatrix} . \tag{4.23}$$

To get a more grasp into definition of the Schouten-Nijenhuis bracket, let us consider some simple examples. Firstly, for two vector fields $\xi_1 = \xi_1^\mu \partial_\mu$ and $\xi_2^\nu \partial_\nu$, we have $p = q = 1$, and Levi Civita symbol (4.23) becomes just the trivial Kroneker delta $\epsilon^\mu_\nu = \delta^\mu_\nu$. The second expression in (4.22) becomes

$$[\xi_1, \xi_2]_S^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu \,, \tag{4.24}$$

which is just the expression for Lie bracket. Another case that we will consider in the thesis is of a Schouten-Nijenhuis bracket of the bi-vector $\theta = \frac{1}{2}\theta^{\mu\nu}\partial_\mu \wedge \partial_\nu$ with itself. Its expression is given by

$$[\theta, \theta]_S|^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}_{\alpha\beta\gamma} \theta^{\alpha\sigma}\partial_\sigma \theta^{\beta\gamma} \,, \tag{4.25}$$

and

$$\epsilon^{\mu\nu\rho}_{\alpha\beta\gamma} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\beta & \delta^\rho_\gamma \\ \delta^\nu_\alpha & \delta^\rho_\beta & \delta^\mu_\gamma \\ \delta^\rho_\alpha & \delta^\mu_\beta & \delta^\nu_\gamma \end{vmatrix} , \tag{4.26}$$

resulting in

$$[\theta, \theta]_S|^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma \theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma \theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma \theta^{\mu\nu} \,. \tag{4.27}$$

It turns out that this coincides with the expression for the string $R$-flux. The bi-vector $\theta$ such that its Schouten-Nijenhuis bracket gives zero is a Poisson bi-vector, which can be used to define Poisson manifolds (see Appendix [A]).

The Schouten-Nijenhuis bracket is graded-commutative,

$$[\theta_1, \theta_2]_S = -(-1)^{(p-1)(q-1)} [\theta_2, \theta_1]_S \,, \tag{4.28}$$

and it satisfies the graded Jacobi identity

$$(-1)^{(p-1)(r-1)}[\theta_1, [\theta_2, \theta_3]_S]_S + (-1)^{(q-1)(p-1)}[\theta_2, [\theta_3, \theta_1]_S]_S + (-1)^{(r-1)(q-1)}[\theta_3, [\theta_1, \theta_2]_S]_S = 0 \,, \tag{4.29}$$

where $p, \ q, \ r$ are oders of multi-vectors $\theta_1, \ \theta_2, \ \theta_3$, respectively. Moreover, it satisfies the graded Leibniz identity

$$[\theta_1, [\theta_2 \wedge \theta_3]_S]_S = [\theta_1, \theta_2]_S \wedge \theta_3 + (-1)^{(p-1)q} \theta_2 \wedge [\theta_1, \theta_3]_S \,. \tag{4.30}$$

In fact, the Schouten-Nijenhuis bracket can be alternatively defined by relations [22]

$$[f, g]_S = 0\,, \quad [\xi, f]_S = \mathcal{L}_\xi(f)\,, \quad [\xi_1, \xi_2]_S = [\xi_1, \xi_2]_L\,, \tag{4.31}$$

where other relations for multi-vectors are obtained by demanding the graded commutativity (4.28) and graded Leibniz identity (4.30).

# Chapter 5

# Lie algebroid

In general, vector fields can be defined on a smooth section of a vector bundle, that is not necessarily a tangent bundle. This motivates the question of the change of tensors along these vector fields, which Lie algebroids can explain. In this chapter, we provide a definition of Lie algebroid and demonstrate how one can extend the notions of Lie and exterior derivative to some other vector bundles. We also provide a definition of the Koszul bracket, which is the generalization of the Lie bracket to the space of 1-forms. Lastly, we define Lie bialgebroids, which will be useful in introducing the Courant bracket on the generalized tangent bundle.

## 5.1 Lie algebroid and its corresponding Lie derivative

**Definition 11 (Lie algebroid)**

*Lie algebroid is a triple* $\Big(V, [,], \rho\Big)$ *consisting of a vector bundle* $V$*, the anchor* $\rho : V \to T\mathcal{M}$*, and the skew-symmetric bracket* $[,]$ *on the space of smooth section of* $V$*, so that the following compatibility conditions are satisfied:*

$$\rho[\xi_1, \xi_2] = [\rho(\xi_1), \rho(\xi_2)]_L \,, \tag{5.1}$$

$$[\xi_1, f\xi_2] = f[\xi_1, \xi_2] + (\mathcal{L}_{\rho(\xi_1)} f)\xi_2 \,, \tag{5.2}$$

$$[\xi_1, [\xi_2, \xi_3]] + [\xi_2, [\xi_3, \xi_1]] + [\xi_3, [\xi_1, \xi_2]] = 0 \,. \tag{5.3}$$

The Lie algebroid [50, 51] generalizes the notion of the tangent bundle. The first compatibility condition tells us that the anchor $\rho$ is the morphism between vector bundle $V$ and the tangent bundle over a manifold $T\mathcal{M}$ that is compatible with the Lie bracket. This way, we relate the Lie algebroid bracket, defined on some vector bundle to the well-known Lie bracket on the tangent bundle. The remaining two conditions require that the new bracket satisfies the Leibniz rule and Jacobi identity, both satisfied by the Lie bracket.

From the compatibility conditions in the Lie algebroid definition, we see that on this structure we can define a Lie derivative. Its action on functions is defined by

$$\hat{\mathcal{L}}_\xi f = \mathcal{L}_{\rho(\xi)} f \,, \tag{5.4}$$

and on vectors by the Lie algebroid bracket

$$\hat{\mathcal{L}}_{\xi_1} \xi_2 = [\xi_1, \xi_2] \,. \tag{5.5}$$

The second compatibility condition from the definition above ensures that this derivative satisfies the Leibniz property, i.e.

$$\hat{\mathcal{L}}_{\xi_1}(f\xi_2) = f\hat{\mathcal{L}}_{\xi_1}\xi_2 + (\hat{\mathcal{L}}_{\xi_1} f)\xi_2 \,, \tag{5.6}$$

which can be used to obtain its action on dual vectors.

We can also define the exterior derivation by [52]

$$\begin{aligned} \hat{d}\lambda(\xi_0, ..., \xi_p) \;\; &= \;\; \sum_{i=0}^{p} (-1)^i \mathcal{L}_{\rho(\xi_i)} \Big( \lambda(\xi_0, ..., \hat{\xi}_i, ..., \xi_p) \Big) \\ &\quad + \sum_{i<j} (-1)^{i+j} \lambda([\xi_i, \xi_j], \xi_0, ..., \hat{\xi}_i, ..., \hat{\xi}_j, ..., \xi_p) \,, \end{aligned} \tag{5.7}$$

where we effectively in relation (4.11) substituted the Lie derivative with its algebroid counterpart $\hat{\mathcal{L}}_\xi$. This exterior derivative is also nilpotent. The Lie derivative on the dual vectors can be expressed by Cartan formula, i.e.

$$\hat{\mathcal{L}}_\xi \lambda = i_\xi \hat{d} \, \lambda + \hat{d} i_\xi \, \lambda \,. \tag{5.8}$$

There is a myriad of examples of Lie algebroids. The easiest and the most straightforward is the Lie algebroid with the tangent bundle over a manifold $T\mathcal{M}$ as a vector bundle, the Lie bracket $[,]_L$ and the identity operator Id as its anchor. The compatibility conditions from Definition 11 are the well-known characteristics of the Lie bracket (4.6) and (4.7).

## 5.2 Koszul bracket

We now want to introduce the Lie algebroid on a cotangent bundle over a manifold $T^\star\mathcal{M}$. We will define a morphism from cotangent to tangent bundle by

$$\theta(\lambda_1)\lambda_2 = \theta(\lambda_1, \lambda_2) \,, \quad \Big(\theta(\lambda_1)\Big)^\mu = \lambda_{1\nu}\theta^{\nu\mu} \,, \tag{5.9}$$

where $\theta$ is a bi-vector ($\theta^{\mu\nu} = -\theta^{\nu\mu}$) that satisfies

$$[\theta, \theta]_S = 0\,, \tag{5.10}$$

where $[,]_S$ is the Schouten-Nijenhuis bracket (4.25). The bi-vector that satisfies this condition is called Poisson bi-vector (see Appendix [A]).

We define the Koszul bracket [53] between two 1-forms by

$$[\lambda_1, \lambda_2]_\theta = \mathcal{L}_{\theta(\lambda_1)}\lambda_2 - \mathcal{L}_{\theta(\lambda_2)}\lambda_1 - d(\theta(\lambda_1, \lambda_2))\,, \tag{5.11}$$

which in some local basis $dx^\mu$ has the components

$$\Big([\lambda_1, \lambda_2]_\theta\Big)_\mu = \theta^{\nu\rho}(\lambda_{1\nu}\partial_\rho\lambda_{2\mu} - \lambda_{2\nu}\partial_\rho\lambda_{1\mu}) + \lambda_{1\rho}\lambda_{2\nu}\partial_\mu\theta^{\rho\nu}\,. \tag{5.12}$$

To show that the structure $\Big(T^\star\mathcal{M}, [,]_\theta, \theta\Big)$ is a Lie algebroid, we need to prove that the three Lie algebroid conditions (5.1)-(5.3) are satisfied. Firstly, in order to show that $\theta$ is really a correct anchor, we express the left-hand side of the (5.1) by

$$\Big(\theta([\lambda_1, \lambda_2]_\theta)\Big)_\mu = \theta^{\nu\rho}\theta^{\sigma\mu}(\lambda_{1\nu}\partial_\rho\lambda_{2\sigma} - \lambda_{2\nu}\partial_\rho\lambda_{1\sigma}) + \lambda_{1\rho}\lambda_{2\sigma}\theta^{\nu\mu}\partial_\nu\theta^{\rho\sigma}\,. \tag{5.13}$$

On the other hand, we express the right-hand side of (5.1) by

$$\begin{aligned}([\theta(\lambda_1), \theta(\lambda_2)]_L)_\mu &= \lambda_{1\nu}\theta^{\nu\rho}\partial_\rho(\lambda_{2\sigma}\theta^{\sigma\mu}) - \lambda_{2\nu}\theta^{\nu\rho}\partial_\rho(\lambda_{1\sigma}\theta^{\sigma\mu}) \\ &= \theta^{\nu\rho}\theta^{\sigma\mu}\Big(\lambda_{1\nu}\partial_\rho\lambda_{2\sigma} - \lambda_{2\nu}\partial_\rho\lambda_{1\sigma}\Big) + \lambda_{1\rho}\lambda_{2\sigma}\Big(\theta^{\rho\nu}\partial_\nu\theta^{\sigma\mu} + \theta^{\sigma\nu}\partial_\nu\theta^{\mu\rho}\Big)\,.\end{aligned} \tag{5.14}$$

Now combining (5.13) and (5.14), we obtain

$$\begin{aligned}\Big(\theta([\lambda_1, \lambda_2]_\theta)\Big)_\mu &= ([\theta(\lambda_1), \theta(\lambda_2)]_L)_\mu - \lambda_{1\rho}\lambda_{2\sigma}\Big(\theta^{\rho\nu}\partial_\nu\theta^{\sigma\mu} + \theta^{\sigma\nu}\partial_\nu\theta^{\mu\rho}\Big) + \lambda_{1\rho}\lambda_{2\sigma}\theta^{\nu\mu}\partial_\nu\theta^{\rho\sigma} \\ &= ([\theta(\lambda_1), \theta(\lambda_2)]_L)_\mu - \lambda_{1\rho}\lambda_{2\sigma}(\theta^{\mu\nu}\partial_\nu\theta^{\rho\sigma} + \theta^{\rho\nu}\partial_\nu\theta^{\sigma\mu} + \theta^{\sigma\nu}\partial_\nu\theta^{\mu\rho}) \\ &= ([\theta(\lambda_1), \theta(\lambda_2)]_L)_\mu\,,\end{aligned} \tag{5.15}$$

where we used (4.22) and the condition (5.10). If the bi-vector is not Poisson, this condition would not be satisfied.

Next, let us show that the Koszul bracket satisfies the Leibniz rule (5.2)

$$\begin{aligned}[\lambda_1, f\lambda_2]_\theta &= \mathcal{L}_{\theta(\lambda_1)}(f\lambda_2) - \mathcal{L}_{f\theta(\lambda_2)}\lambda_1 - d\theta(\lambda_1, f\lambda_2) \\ &= f\mathcal{L}_{\theta(\lambda_1)}\lambda_2 + (\mathcal{L}_{\theta(\lambda_1)}f)\lambda_2 - f\mathcal{L}_{\theta(\lambda_2)}\lambda_1 \\ &\quad -df\theta(\lambda_2, \lambda_1) - df\theta(\lambda_1, \lambda_2) - fd\theta(\lambda_1, \lambda_2) \\ &= f[\lambda_1, \lambda_2]_\theta + \mathcal{L}_{\theta(\lambda_1)}f\lambda_2\,.\end{aligned} \tag{5.16}$$

We used the fact that the Lie derivative satisfies the Leibniz rule (4.6), its property (4.18), and that the bi-vector is antisymmetric. We did not use the fact that $\theta$ is a Poisson bi-vector, so the Koszul bracket satisfies the Leibniz rule regardless of that.

Lastly, the associativity (5.3) is easily proven from the first Lie algebroid condition (5.15) and associativity of Lie derivative (4.7). We write

$$\begin{aligned} [\theta(\lambda_1), [\theta(\lambda_2), \theta(\lambda_3)]_L]_L + cyclic &= [\theta(\lambda_1), \theta([\lambda_2, \lambda_3]_\theta)]_L + cyclic \\ &= \theta([\lambda_1, [\lambda_2, \lambda_3]_\theta]_\theta + cyclic) = 0\,. \end{aligned} \tag{5.17}$$

We showed that the triple $\Big(T^\star\mathcal{M}, [,]_\theta, \theta\Big)$ is a Lie algebroid, for a Poisson bi-vector $\theta$. In the case of the bi-vector not being Poisson, the Koszul bracket can still be defined and it will satisfy the Leibniz rule. However, the anchor $\theta$ is not algebra homomorphism, and the Jacobi identity does not stand. This structure is then referred to as quasi-Lie algebroid.

For the Lie algebroid associated with the Koszul bracket, we can define its corresponding Lie derivative. On functions, it can be defined from (5.4)

$$\hat{\mathcal{L}}_{\lambda_1} f = \lambda_{1\nu}\theta^{\nu\mu}\partial_\mu f\,, \tag{5.18}$$

while on 1-forms, it acts as the Koszul bracket

$$\hat{\mathcal{L}}_{\lambda_1}\lambda_2 = [\lambda_1, \lambda_2]_\theta\,. \tag{5.19}$$

Therefore, the Koszul bracket is interpreted as the generalization of the Lie bracket on 1-forms.

Since this Lie algebroid is defined on the cotangent bundle, and therefore acts on 1-forms, it defines the exterior derivation on the smooth section of the tangent bundle and higher orders of the tangent bundle. From (5.7), we obtain the action of exterior derivative on functions and vectors

$$(d_\theta f)^\mu = \theta^{\mu\nu}\partial_\nu f\,, \quad (d_\theta \xi)^{\mu\nu} = \theta^{\mu\rho}\partial_\rho\xi^\nu - \theta^{\nu\rho}\partial_\rho\xi^\mu - \xi^\rho\partial_\rho\theta^{\mu\nu}\,. \tag{5.20}$$

The generalized formula for the exterior derivative can also be written in terms of the Schouten-Nijenhuis bracket [21]

$$d_\theta = [\theta, .]_S\,. \tag{5.21}$$

The exterior derivative is nilpotent for the Poisson bi-vector $\theta$. We can prove that easily for functions

$$\begin{aligned} (d_\theta d_\theta f)^{\mu\nu} &= \theta^{\mu\rho}\partial_\rho(\theta^{\nu\sigma}\partial_\sigma f) - \theta^{\nu\rho}\partial_\rho(\theta^{\mu\sigma}\partial_\sigma f) - \theta^{\rho\sigma}\partial_\sigma f\partial_\rho\theta^{\mu\nu} \\ &= \partial_\sigma f(\theta^{\mu\rho}\partial_\rho\theta^{\nu\sigma} + \theta^{\nu\rho}\partial_\rho\theta^{\sigma\mu} + \theta^{\sigma\rho}\partial_\rho\theta^{\mu\nu})\,, \end{aligned} \tag{5.22}$$

which goes to zero if and only if the condition (5.10) is satisfied. For a general multi-vector $\beta$ of rank $r$, the nilpotence of the exterior derivative $d_\theta$ is a consequence of the graded Jacobi identity (4.29)

$$(-1)^{r-1}[\theta, [\theta, \beta]_S]_S - [\theta, [\beta, \theta]_S]_S + (-1)^{r-1}[\beta, [\theta, \theta]_S]_S = 0\,. \tag{5.23}$$

The third term is zero due to condition (5.10). After applying the graded-commutative relation (4.28) to the second term, with the help of definition (5.21), we obtain

$$d_\theta d_\theta \beta = 0\,, \tag{5.24}$$

for any multi-vector $\beta$. If the bi-vector $\theta$ is not Poisson, and the relation (5.10) does not hold, one can still define the exterior derivative $d_\theta$. The relation (5.23) has the form

$$2 d_\theta d_\theta \beta + [\beta, [\theta, \theta]_S]_S = 0\,, \tag{5.25}$$

so the exterior derivative is no longer nilpotent, but it does satisfy the Leibniz rule.

## 5.3 Lie bialgebroid

We saw that Lie algebroid can be defined on a tangent bundle, for example, with the Lie bracket as its bracket, and on the cotangent bundle, for example with the Koszul bracket as its bracket. In general, Lie algebroids can be defined on mutually dual bundles. Of particular interest is the case when the exterior derivative corresponding to one algebroid commute with the bracket of the other algebroid, in which case we obtain the Lie bialgebroid [54, 55], for which we provide a definition below:

**Definition 12 (Lie bialgebroid)**
*Let $\Big(V, [,]_L, \rho\Big)$ be a Lie algebroid and supposee that $\Big(V^\star, [,]_{L^\star}, \rho^\star\Big)$ is also a Lie algebroid, where bundles $V$ and $V^\star$ are dual to each other. The structure $(V, V^\star)$ is said to define a Lie bialgebroid if*

$$d^\star [\xi_1, \xi_2]_L = [d^\star \xi_1, \xi_2]_S + [\xi_1, d^\star \xi_2]_S\,, \tag{5.26}$$

*where $d^\star$ is a Lie algebroid differential of $V^\star$, and $\xi_1, \xi_2$ are from smooth section of $V$, and $[,]_S$ is a Schouten-Nijenhuis bracket on a smooth section of multi-vectors $T \bigwedge^p \mathcal{M}$ defined graded symmetric via Leibniz rule.*

A simple corollary can be proven - if $(V, V^\star)$ is a Lie bialgebroid, then $(V^\star, V)$ is also a Lie bialgebroid. The condition (5.26) suggests that the exterior derivative related to the one algebroid bracket acts as the graded bracket on the dual bundle. To illustrate this, let us recall two structures - $\Big(T\mathcal{M}, [,]_L, \mathrm{Id}\Big)$, and $\Big(T^\star\mathcal{M}, [,]_\theta, \theta\Big)$ for Poisson bi-vector $\theta$, that we demonstrated are both examples of Lie algebroids. They are defined on mutually dual bundles and in fact, constitute a Lie bialgebroid. The condition (5.26) becomes

$$\begin{aligned}
& d_\theta [\xi_1, \xi_2]_L = [d_\theta \xi_1, \xi_2]_S + [\xi_1, d_\theta \xi_2]_S \\
& [\theta, [\xi_1, \xi_2]_S]_S = [[\theta, \xi_1]_S, \xi_2]_S + [\xi_1, [\theta, \xi_2]_S]_S \\
& [\theta, [\xi_1, \xi_2]_S]_S + [\xi_1, [\xi_2, \theta]_S]_S + [\xi_2, [\theta, \xi_1]_S]_S = 0\,,
\end{aligned} \tag{5.27}$$

where we firstly used (5.20), and then the graded identity (4.28). The final expression is just the graded Jacobi identity (4.29), so we proved the condition (5.26). The exterior derivative related to the Koszul bracket $d_\theta$ acts on multi-vectors as the Schouten-Nijenhuis bracket.

The attempts to construct structures similar to Lie algebroids on the double of Lie bialgebroid $V \oplus V^\star$ led to the development of generalized geometry and construction of Courant algebroids, which we will introduce in the next chapter.

# Chapter 6

# Generalized tangent bundle

In this chapter, we will consider the so-called generalized geometry, that is to say, the geometry of the generalized tangent bundle. Primarily, we will define two natural inner products on the space of generalized vectors. Secondarily, we will obtain the Courant bracket as the extension of the Lie bracket to the section of the generalized tangent bundle. Lastly, we will define the Courant algebroid together with the conditions for it to be the Lie algebroid.

## 6.1 Inner product and $O(D, D)$ group

The research of generalized geometry was pioneered in the early 2000s, in the works of Hitchin and his introduction of generalized Calabi-Yau manifolds [56], and later in the works of his student Gualtieri [57]. It is a geometry of the generalized tangent bundle, defined as a direct sum of tangent and cotangent bundle over a manifold $T\mathcal{M} \oplus T^\star\mathcal{M}$. The elements of its section are generalized vectors, that have both the vector and 1-form components

$$\Lambda^M = \xi^\mu \oplus \lambda_\mu = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix} , \tag{6.1}$$

where $\xi$ represents the vector components, and $\lambda$ represents the 1-form components of generalized vectors.

The interior product (4.14) defines a natural way to combine vectors and 1-forms into a scalar $i_\xi \lambda = \xi^\mu \lambda_\mu$. We can use this to define an inner product on the smooth section of the generalized tangent bundle. We can define the inner product between generalized vectors in two ways - so that it is symmetric and antisymmetric. In the former case, it is defined by

$$\langle \Lambda_1, \Lambda_2 \rangle = \langle \xi_1 \oplus \lambda_1, \xi_2 \oplus \lambda_2 \rangle = i_{\xi_1} \lambda_2 + i_{\xi_2} \lambda_1 \, . \tag{6.2}$$

The signature of the symmetric inner product is $(D, D)$, where $D$ is the dimension of the manifold $\mathcal{M}$. It is well known that the Lie group of linear transformations that leaves the inner product of such a signature invariant is the indefinite orthogonal $O(D, D)$ group. This group plays a very important role in string theory, as it governs the T-duality transformations. The fact that a transformation $\mathcal{O}$ keeps the inner product (6.2) invariant can be expressed in matrix notation by

$$(\mathcal{O}^T)_M^{\ \ P}\, \eta_{PQ}\, \mathcal{O}^Q_{\ \ N} = \eta_{MN}\,, \tag{6.3}$$

where $\eta$ is $O(D, D)$ invariant metric, given by

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{6.4}$$

The metric can be used for lowering or raising indices $M,\ N$.

Some of more notable examples of $O(D, D)$ transformations include $B$-transformations (or $B$-shifts), which are given by

$$e^{\hat{B}} = \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix} \qquad \hat{B}^M_{\ \ N} = \begin{pmatrix} 0 & 0 \\ 2B_{\mu\nu} & 0 \end{pmatrix}\,. \tag{6.5}$$

Their inverse is easily obtained from (B.4)

$$e^{-\hat{B}} = \begin{pmatrix} 1 & 0 \\ -2B & 1 \end{pmatrix}\,. \tag{6.6}$$

The $B$-transformations act on the generalized metric (2.38) by shifting the Kalb-Ramond field. The second example we will outline here is the $\theta$-transformations, which are given by

$$e^{\hat{\theta}} = \begin{pmatrix} 1 & \kappa\theta \\ 0 & 1 \end{pmatrix}\,, \quad \hat{\theta}^M_{\ \ N} = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 0 & 0 \end{pmatrix}\,, \tag{6.7}$$

and when inverted by

$$e^{-\hat{\theta}} = \begin{pmatrix} 1 & -\kappa\theta \\ 0 & 1 \end{pmatrix}\,. \tag{6.8}$$

It is easy to show that indeed

$$\langle e^{\hat{B}}\Lambda_1, e^{\hat{B}}\Lambda_2\rangle = \langle \Lambda_1, \Lambda_2\rangle\,, \quad \langle e^{\hat{\theta}}\Lambda_1, e^{\hat{\theta}}\Lambda_2\rangle = \langle \Lambda_1, \Lambda_2\rangle\,. \tag{6.9}$$

We will encounter these and other $O(D, D)$ transformations throughout the thesis. For more mathematically rigorous details of $O(D, D)$ group, see Appendix [B].

For completeness, let us also define the antisymmetric inner product by

$$\langle \Lambda_1, \Lambda_2 \rangle_- = \langle \xi_1 \oplus \lambda_1, \xi_2 \oplus \lambda_2 \rangle_- = i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1 \,. \tag{6.10}$$

As we will see in the next section, this inner product is used in a definition of the Courant bracket. We will primarily be interested in the symmetric inner product, which we will simply refer to as the inner product from now onward.

## 6.2 Courant bracket

We would like to generalize the notion of the Lie bracket, defined on generalized vectors. The definition of the generalized tangent bundle as a direct sum of two dual bundles $T\mathcal{M}$ and $T^\star\mathcal{M}$ is suitable for considering a Lie bialgebroid structure of the form $(T\mathcal{M}, T^\star\mathcal{M})$, with their respective brackets $[,]_L$ and $[,]_{L^\star}$. We can define a skew-symmetric bracket on the smooth section of the generalized tangent bundle by

$$\begin{aligned}[\Lambda_1, \Lambda_2] = \quad & \Big( [\xi_1, \xi_2]_L + {\mathcal{L}^\star}_{\lambda_1}\xi_2 - {\mathcal{L}^\star}_{\lambda_2}\xi_1 - \frac{1}{2} d^\star \langle \Lambda_1, \Lambda_2 \rangle_- \Big) \\ & \oplus \Big( [\lambda_1, \lambda_2]_{L^\star} + \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2} d \langle \Lambda_1, \Lambda_2 \rangle_- \Big) \,.\end{aligned} \tag{6.11}$$

Here, $\mathcal{L}_\xi$ and $d$ represent the Lie derivative and exterior derivative defined on the Lie algebroid on $T\mathcal{M}$, while ${\mathcal{L}^\star}_\lambda$ and $d^\star$ are analogous operations corresponding to the Lie algebroid of $T^\star\mathcal{M}$.

Let us construct a simple example of a skew-symmetric bracket using (6.11). For a bracket on the tangent bundle, we will use the usual Lie bracket. On the cotangent bundle, we can use the trivial bracket that is zero between any two forms. This corresponds to the Lie algebroid with the anchor $\rho^\star = 0$, and the Lie bialgebroid compatibility condition (5.26) is satisfied. The above relation for the bracket gives rise to the well-known Courant bracket given by

$$\begin{aligned}[\Lambda_1, \Lambda_2]_\mathcal{C} &= \xi \oplus \lambda \\ \xi &= [\xi_1, \xi_2]_L \,, \\ \lambda &= \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2} d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1) \,.\end{aligned} \tag{6.12}$$

The Courant bracket is the generalization of the Lie bracket to the generalized tangent bundle. The right-hand side vector and 1-form components of (6.12) can be expressed in some local coordinate basis by

$$\begin{aligned}\xi^\mu &= \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu \,, \\ \lambda_\mu &= \xi_1^\nu (\partial_\nu \lambda_{2\mu} - \partial_\mu \lambda_{2\nu}) - \xi_2^\nu (\partial_\nu \lambda_{1\mu} - \partial_\mu \lambda_{1\nu}) + \frac{1}{2} \partial_\mu (\xi_1 \lambda_2 - \xi_2 \lambda_1) \,.\end{aligned} \tag{6.13}$$

There is a natural way to introduce the projections to the tangent and cotangent bundle $\pi$ and $\tilde{\pi}$ respectively by

$$\pi(\Lambda) = \pi(\xi \oplus \lambda) = \xi\,, \quad \tilde{\pi}(\Lambda) = \tilde{\pi}(\xi \oplus \lambda) = \lambda\,. \tag{6.14}$$

From the expression (6.12), we see that the Courant bracket on vectors reduces to Lie bracket, while on 1-form it becomes zero

$$[\pi(\Lambda_1), \pi(\Lambda_2)]_{\mathcal{C}} = [\xi_1, \xi_2]_L\,, \quad [\tilde{\pi}(\Lambda_1), \tilde{\pi}(\Lambda_2)]_{\mathcal{C}} = 0\,. \tag{6.15}$$

Effectively, the Courant bracket on smooth sections of tangent and cotangent bundles reduces to the respective Lie algebroid brackets from which it was constructed. Therefore, though the generalized tangent bundle treats vectors and 1-forms in a symmetrical manner, the Courant bracket defined on it does not. Moreover, we have

$$\pi\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}}\Big) = [\pi(\Lambda_1), \pi(\Lambda_2)]_{\mathcal{C}}\,, \quad \tilde{\pi}\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}}\Big) \neq [\tilde{\pi}(\Lambda_1), \tilde{\pi}(\Lambda_2)]_{\mathcal{C}}\,, \tag{6.16}$$

and hence the projection on the tangent bundle is involutive with respect to the Courant bracket, while the projection on the cotangent bundle is not.

In general, the Courant bracket satisfies neither the Leibniz rule nor Jacobi identity (see Appendix [C] for proof). In fact, the deviation from these identities can be expressed in terms of the exterior derivative of the inner product

$$[\Lambda_1, f\Lambda_2]_{\mathcal{C}} = f[\Lambda_1, \Lambda_2]_{\mathcal{C}} + (\mathcal{L}_{\pi(\Lambda_1)} f)\Lambda_2 - \frac{1}{2}\langle \Lambda_1, \Lambda_2 \rangle df\,, \tag{6.17}$$

$$\text{Jac}(\Lambda_1, \Lambda_2, \Lambda_3) = d\text{Nij}(\Lambda_1, \Lambda_2, \Lambda_3)\,, \tag{6.18}$$

where Jac is the Jacobiator, given by

$$\text{Jac}(\Lambda_1, \Lambda_2, \Lambda_3) = [[\Lambda_1, \Lambda_2]\,, \Lambda_3] + [[\Lambda_2, \Lambda_3]\,, \Lambda_1] + [[\Lambda_3, \Lambda_1]\,, \Lambda_2]\,, \tag{6.19}$$

and Nij is the Nijenhuis operator, given by

$$\text{Nij}(\Lambda_1, \Lambda_2, \Lambda_3) = \frac{1}{6}\Big(\langle [\Lambda_1, \Lambda_2], \Lambda_3 \rangle + \langle [\Lambda_2, \Lambda_3], \Lambda_1 \rangle + \langle [\Lambda_3, \Lambda_1], \Lambda_2 \rangle\Big)\,. \tag{6.20}$$

## 6.3 Courant algebroid

We saw that the Courant bracket does not satisfy the Lie algebroid conditions, and as such, we cannot in general define the Lie algebroid with the Courant bracket as its bracket. However, one can define the Lie algebroid generalization, called Courant algebroid [58]. We provide its definition below:

**Definition 13 (Courant algebroid)**

*Let $V$ be a vector bundle, $\langle , \rangle$ the non-degenerate inner product and $[,]$ a skew-symmetric bracket on a smooth section of a vector bundle $V$, and let $\rho : V \to TM$ be a smooth bundle map called anchor. Let $\mathcal{D}$ be a differential operator on smooth functions defined by*

$$\langle \mathcal{D}f, \Lambda \rangle = \mathcal{L}_{\rho(\Lambda)} f \, . \tag{6.21}$$

*The structure $\Big(V, \langle , \rangle, [,], \rho\Big)$ is called the* **Courant algebroid** *if it satisfies the following compatibility relations*

$$\rho[\Lambda_1, \Lambda_2] = [\rho(\Lambda_1), \rho(\Lambda_2)]_L \, , \tag{6.22}$$

$$[\Lambda_1, f\Lambda_2] = f[\Lambda_1, \Lambda_2] + (\mathcal{L}_{\rho(\Lambda_1)} f)\Lambda_2 - \frac{1}{2}\langle \Lambda_1, \Lambda_2 \rangle \mathcal{D}f \, , \tag{6.23}$$

$$\mathcal{L}_{\rho(\Lambda_1)}\langle \Lambda_2, \Lambda_3 \rangle = \langle [\Lambda_1, \Lambda_2] + \frac{1}{2}\mathcal{D}\langle \Lambda_1, \Lambda_2 \rangle, \Lambda_3 \rangle + \langle \Lambda_2, [\Lambda_1, \Lambda_3] + \frac{1}{2}\mathcal{D}\langle \Lambda_1, \Lambda_3 \rangle \rangle \, , \tag{6.24}$$

$$\langle \mathcal{D}f, \mathcal{D}g \rangle = 0 \, , \tag{6.25}$$

$$Jac(\Lambda_1, \Lambda_2, \Lambda_3) = \mathcal{D}Nij(\Lambda_1, \Lambda_2, \Lambda_3) \, , \tag{6.26}$$

*for all $\Lambda_1$, $\Lambda_2$, $\Lambda_3$ from smooth section of a vector bundle $V$, and for all smooth functions $f$ and $g$ on the manifold.*

Any double of Lie bialgebroid defines the Courant algebroid. The reverse however is not the case, and the Courant algebroid brackets encompass a larger set of brackets [59].

A straightforward example of Courant algebroids, that we will refer to as the standard Courant algebroid, consists of the generalized tangent bundle, the $O(D, D)$ invariant inner product (6.2), the projection $\pi$ from the generalized tangent bundle to the tangent bundle as its anchor (6.14), and the Courant bracket (6.12). It is not difficult to show that the five compatibility conditions are satisfied. We present the proof in the Appendix [C].

There is an alternative definition of the Courant algebroid, in which the Courant algebroid bracket is defined so that it is not skew-symmetric, but it does satisfy the Leibniz rule and Jacobi identity [60]. This definition is proven to be equivalent to the one provided above. The skew-symmetric brackets are more suitable to describe the algebra of symmetries, and we will use the definition (Def. 13) exclusively in this thesis.

Perhaps the most striking application of Courant algebroids is in the attempt to explain the T-duality. For instance, in [61, 62], authors investigated two manifolds $\mathcal{M}$ and $\tilde{\mathcal{M}}$, both principle torus bundles over a common manifold $B$. The manifolds had 3-form fluxes $H$ and $\tilde{H}$ on them. The conditions for the theories defined on these two manifolds with fluxes to be mutually T-dual were obtained, which were shown in [63] to be equivalent to the isomorphism $\phi$ between two Courant algebroids, that satisfies

$$\langle \phi(\Lambda_1), \phi(\Lambda_2)\rangle = \langle \Lambda_1, \Lambda_2\rangle\,, \quad \phi\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}_H}\Big) = [\phi(\Lambda_1), \phi(\Lambda_2)]_{\mathcal{C}_{\tilde{H}}}\,, \tag{6.27}$$

where $[,]_{\mathcal{C}_H}$ denotes the Courant bracket deformed with the flux $H$. As such, the Courant algebroids appear convenient to describe T-duality. We will further investigate it in the analysis of bosonic string $\sigma$-model symmetries.

## 6.4 Dirac structures

**Definition 14 (Dirac structures)**

*For a sub-bundle to be isotropic with respect to the inner product means that the inner product of any two generalized vectors from its section is zero*

$$\langle \Lambda_1, \Lambda_2\rangle = 0\,. \tag{6.28}$$

*Dirac structures are defined as the isotropic sub-bundles with the maximal dimension that are closed under the skew-symmetric Courant algebroid bracket.*

For any 2-form $B$, the sub-bundle

$$\mathcal{V}_B(\Lambda) = \xi^\mu \oplus 2B_{\mu\nu}\xi^\nu \tag{6.29}$$

is going to be isotropic with respect to the $O(D,D)$ invariant inner product (6.2) due to the antisymmetric properties of a 2-form

$$\langle \xi_1^\mu \oplus 2B_{\mu\rho}\xi_1^\rho, \xi_2^\nu \oplus 2B_{\nu\sigma}\xi_2^\sigma\rangle = 2B_{\mu\nu}(\xi_1^\mu\xi_2^\nu + \xi_1^\nu\xi_2^\mu) = 0\,. \tag{6.30}$$

Similarly, for any bi-vector $\theta$, we can construct another isotropic sub-bundle by

$$\mathcal{V}_\theta(\Lambda) = \kappa\theta^{\mu\nu}\lambda_\nu \oplus \lambda_\mu\,. \tag{6.31}$$

The sub-bundle of the form $\mathcal{V}_B$ and $\mathcal{V}_\theta$ from the mathematical perspective represent a graph of 2-form over tangent bundle, and a graph of bi-vector over a cotangent bundle, respectively.

The importance of Dirac structures lies in the fact that the Courant algebroid on them reduces to the Lie algebroid. To see this, let us take a look at the second Courant algebroid compatibility condition (6.23), and note that the algebroid bracket satisfies Leibniz rule up to the term $\frac{1}{2}\langle \Lambda_1, \Lambda_2\rangle \mathcal{D}f$. This term is zero on isotropic sub-spaces (6.28), and therefore the Leibniz identity will be satisfied on them. Likewise, the fifth compatibility condition (6.26) becomes the usual Jacobi identity on Dirac structures. This can be deduced by evaluating the Nijenhuis operator (6.20) of some vectors from the Dirac structure $\mathcal{V}$. Without loss of generality, we have

$$\begin{aligned} &\Lambda_1, \Lambda_2 \in \mathcal{V} \Rightarrow [\Lambda_1, \Lambda_2] \in \mathcal{V}\,, \\ &[\Lambda_1, \Lambda_2], \Lambda_3 \in \mathcal{V} \Rightarrow \langle [\Lambda_1, \Lambda_2], \Lambda_3 \rangle = 0\,, \end{aligned} \tag{6.32}$$

where the first line is the consequence of Dirac structures being closed under the bracket, and the second line stands from the definition of isotropic spaces (6.28).

### 6.4.1 Dirac structures of the standard Courant algebroid

Let us calculate the Dirac structures associated with the standard Courant bracket. Substituting $\lambda_{1\mu} = 2B_{\mu\nu}\xi_1^\nu$ and $\lambda_{2\mu} = 2B_{\mu\nu}\xi_2^\nu$ into the second relation of (6.13), we obtain

$$\begin{aligned} \lambda_\mu &= 2\xi_1^\nu \Big( \partial_\nu (B_{\mu\rho}\xi_2^\rho) - \partial_\mu (B_{\nu\rho}\xi_2^\rho) \Big) - 2\xi_2^\nu \Big( \partial_\nu (B_{\mu\rho}\xi_1^\rho) - \partial_\mu (B_{\nu\rho}\xi_1^\rho) \Big) + 2\partial_\mu (B_{\nu\rho}\xi_1^\nu \xi_2^\rho) \\ &= 2B_{\mu\rho}(\xi_1^\nu \partial_\nu \xi_2^\rho - \xi_2^\nu \partial_\nu \xi_1^\rho) + 2B_{\nu\rho}(\xi_2^\nu \partial_\mu \xi_1^\rho + \xi_2^\rho \partial_\mu \xi_1^\nu) \\ &\quad + 2\xi_1^\nu \xi_2^\rho (\partial_\nu B_{\mu\rho} - \partial_\mu B_{\nu\rho} - \partial_\rho B_{\mu\nu} + \partial_\mu B_{\rho\nu} + \partial_\mu B_{\nu\rho}) \\ &= 2B_{\mu\nu}\xi^\nu - 2(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu})\xi_1^\nu \xi_2^\rho\,, \end{aligned} \tag{6.33}$$

where we first applied the chain rule, and then the skew-symmetric properties of $B$, together with expressing the first relation of (6.13) to express $\xi^\mu$. Therefore, the sub-bundle $\mathcal{V}_B$ (6.29) is a Dirac structure for a closed 2-form $B$

$$dB = 0 \Leftrightarrow \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} = 0\,. \tag{6.34}$$

Mathematically, the manifold $\mathcal{M}$ with a closed non-degenerate 2-form $B$ is a symplectic structure. If we interpret a 2-form $B$ as the Kalb-Ramond field, we obtained that its field strength is zero, or equivalently that H-flux is zero.

On the other hand, for sub-bundle $\mathcal{V}_\theta$ we firstly note that it can be written as

$$\mathcal{V}_\theta(\Lambda) = -\theta(\lambda) \oplus \lambda\,, \tag{6.35}$$

where $\theta(\lambda)$ is defined as in (5.9). For $\mathcal{V}_\theta$ to be closed under the Courant bracket, we can use the coordinate-free expression for the Courant bracket (6.12), and obtain

$$\begin{aligned} \xi &= [\theta(\lambda_1), \theta(\lambda_2)]_L \\ \lambda &= -\Big( \mathcal{L}_{\theta(\lambda_1)}\lambda_2 - \mathcal{L}_{\theta(\lambda_2)}\lambda_1 - d(\theta(\lambda_1, \lambda_2)) \Big) = -[\lambda_1, \lambda_2]_\theta\,, \end{aligned} \tag{6.36}$$

where we recognized in the second line the expression for the Koszul bracket (5.11). We can use the relation (5.15) which is correct only for $[\theta, \theta]_S = 0$ (5.10). On this Dirac structure, we obtained the Poisson manifold [A]. If the bi-vector $\theta$ is interpreted as the non-commutative parameter (3.16), this translates into the $R$-flux being zero.

In general, Dirac structures define integrability conditions for Courant algebroids. We saw how symplectic and Poisson manifolds are obtained from the Dirac structures for the standard Courant algebroid. Moreover, we observed that Dirac structures associated with the standard Courant algebroid put severe restrictions on string fluxes. In the following chapters of the thesis, we will investigate twisted Courant brackets and their corresponding Courant algebroids. We will demonstrate that these restrictions on string fluxes are relaxed on Dirac structures related to the twisted Courant algebroids [64].

# Part III

# Single theory

# Chapter 7

# Symmetries of bosonic string

In this chapter, we will obtain generators of both diffeomorphisms and local gauge transformations for the bosonic string $\sigma$-model. We will show that these symmetries are not independent, rather they are related by T-duality. In the end, we will consider the double generator governing both of these symmetries and show that its Poisson bracket algebra produces the Courant bracket.

## 7.1 Symmetry generators

Symmetry is generally understood as a change in space-time fields that does not change the physically observable quantities. In the standard approach of quantum field theory, symmetries can be seen as transformations of the background fields that keep classical action invariant. In string theory, background fields are defined as functions on the world-sheet, which possess conformal invariance. The physical observable quantities, like scattering amplitudes, are obtained from conformal field theory. Therefore, symmetries imply the existence of physically equivalent solutions to the string equations of motions, which correspond to the mutually isomorphic conformal field theories [65, 66].

Symmetries in string theory $\sigma$-model are governed by generators. They are scalars $\mathcal{G}$, that in classical theory act on Hamiltonian via Poisson bracket. If the Poisson bracket between the generator and Hamiltonian can be interpreted as the change in fields, we say that $\mathcal{G}$ generates a symmetry, i.e. if

$$\mathcal{H}_{(G,B)} + \{\mathcal{G}, \mathcal{H}_{(G,B)}\} = \mathcal{H}_{(G+\delta G, B+\delta B)} \,, \tag{7.1}$$

we say that $B \to B + \delta B$, and $G \to G + \delta G$ are symmetry transformations of the background fields generated by $\mathcal{G}$. We will seek symmetry generators $\mathcal{G}$ such that its action on Hamiltonian seeks a change that can be interpreted as the difference in background fields.

### 7.1.1 Diffeomorphisms

The first generator to be considered will be in the form [67]

$$\mathcal{G}_\xi = \int_0^{2\pi} d\sigma \xi^\mu(x(\sigma))\pi_\mu(\sigma)\,, \tag{7.2}$$

with $\xi^\mu$ being a symmetry parameter. The usual equal-time Poisson bracket relations are assumed

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu \delta(\sigma - \bar\sigma)\,, \quad \{\pi_\mu(\sigma), \pi_\nu(\bar\sigma)\} = \{x^\mu(\sigma), x^\nu(\bar\sigma)\} = 0\ . \tag{7.3}$$

The change in Hamiltonian generated by $\mathcal{G}_\xi$ (7.2) can be expressed as

$$\{\mathcal{G}_\xi, \mathcal{H}\} = \delta_\xi \mathcal{H}\,. \tag{7.4}$$

The transformation of each term in the Hamiltonian (2.35) will be considered separately. For the change in the first term, we have

$$\begin{aligned}
\delta_\xi\Big(\frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu\Big) &= \frac{1}{2\kappa}\int d\bar\sigma \Big\{\xi^\rho(\bar\sigma)\pi_\rho(\bar\sigma), \pi_\mu(\sigma)(G^{-1})^{\mu\nu}(\sigma)\pi_\nu(\sigma)\Big\} \\
&= \frac{1}{2\kappa}\int d\bar\sigma \Big(\pi_\rho \partial_\mu \xi^\rho (G^{-1})^{\mu\nu}\pi_\nu + \pi_\mu (G^{-1})^{\mu\nu}\partial_\nu \xi^\rho \pi_\rho \\
&\qquad -\pi_\mu \xi^\rho \partial_\rho (G^{-1})^{\mu\nu}\pi_\nu\Big)\delta(\sigma - \bar\sigma) \\
&= \frac{1}{2\kappa}\pi_\mu\Big( - \xi^\rho \partial_\rho (G^{-1})^{\mu\nu} + \partial_\rho \xi^\mu (G^{-1})^{\nu\rho} + (G^{-1})^{\mu\rho}\partial_\rho \xi^\nu\Big)\pi_\nu\,,
\end{aligned} \tag{7.5}$$

where in the second step we omitted dependence on $\sigma$, and relabeled some dummy indices, to make the expression more readable. For the second term of Hamiltonian (2.35), we write

$$\begin{aligned}
\delta_\xi\Big(\frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu\Big) &= \frac{\kappa}{2}\int d\bar\sigma \Big\{\xi^\rho(\bar\sigma)\pi_\rho(\bar\sigma), x'^\mu(\sigma)G^E_{\mu\nu}(\sigma)x'^\nu(\sigma)\Big\} \\
&= \frac{\kappa}{2}\int d\bar\sigma \Big[\Big(\xi^\mu(\bar\sigma)G^E_{\mu\nu}x'^\nu + x'^\mu G^E_{\mu\nu}\xi^\nu(\bar\sigma)\Big)\delta'(\sigma - \bar\sigma) - x'^\mu \xi^\rho \partial_\rho G^E_{\mu\nu}x'^\nu \delta(\sigma - \bar\sigma)\Big] \\
&= \frac{\kappa}{2}\int d\bar\sigma \Big[\Big(x'^\rho \partial_\rho \xi^\mu G^E_{\mu\nu}x'^\nu + x'^\mu G^E_{\mu\nu}x'^\rho \partial_\rho \xi^\nu - x'^\mu \xi^\rho \partial_\rho G^E_{\mu\nu}x'^\nu\Big)\delta(\sigma - \bar\sigma) \\
&\qquad +\Big(\xi^\mu G^E_{\mu\nu}x'^\nu + x'^\mu G^E_{\mu\nu}\xi^\nu\Big)\delta'(\sigma - \bar\sigma)\Big] \\
&= \frac{\kappa}{2}x'^\mu \Big(\partial_\mu \xi^\rho G^E_{\rho\nu} + G^E_{\mu\rho}\partial_\nu \xi^\rho - \xi^\rho \partial_\rho G^E_{\mu\nu}\Big)x'^\nu\,,
\end{aligned} \tag{7.6}$$

where in the third line we used the property of the delta function

$$f(\bar\sigma)\delta'(\sigma - \bar\sigma) = f'(\sigma)\delta(\sigma - \bar\sigma) + f(\sigma)\delta'(\sigma - \bar\sigma)\,, \tag{7.7}$$

and in the last line, we used

$$\int d\bar{\sigma} f(\sigma)\delta'(\sigma-\bar{\sigma}) = f(\sigma)\frac{\partial}{\partial_\sigma}\int d\bar{\sigma}\delta(\sigma-\bar{\sigma}) = 0\,, \tag{7.8}$$

which makes the anomalous part become zero in (7.6).

The last term in the Hamiltonian (2.35) transforms as

$$\begin{aligned}
\delta_\xi\Big(-2x'^\mu (BG^{-1})_\mu^{\ \nu}\,\pi_\nu\Big) &= -2\int d\bar{\sigma}\Big\{\xi^\rho(\bar{\sigma})\pi_\rho(\bar{\sigma}), x'^\mu(\sigma)(BG^{-1})_\mu^{\ \nu}(\sigma)\,\pi_\nu(\sigma)\Big\} \qquad (7.9)\\
&= -2\int d\bar{\sigma}\Big[\Big(x'^\mu(BG^{-1})_\mu^{\ \nu}\,\partial_\nu\xi^\rho\pi_\rho - x'^\mu\xi^\rho\partial_\rho(BG^{-1})_\mu^{\ \nu}\,\pi_\nu\Big)\delta(\sigma-\bar{\sigma})\\
&\qquad +\xi^\mu(\bar{\sigma})(BG^{-1})_\mu^{\ \nu}\,\pi_\nu\delta'(\sigma-\bar{\sigma})\Big]\\
&= -2\int d\bar{\sigma}\Big[\Big(x'^\mu(BG^{-1})_\mu^{\ \nu}\,\partial_\nu\xi^\rho\pi_\rho - x'^\mu\xi^\rho\partial_\rho(BG^{-1})_\mu^{\ \nu}\,\pi_\nu\\
&\qquad +x'^\rho\partial_\rho\xi^\mu(BG^{-1})_\mu^{\ \nu}\,\pi_\nu\Big)\delta(\sigma-\bar{\sigma}) + \xi^\mu(BG^{-1})_\mu^{\ \nu}\,\pi_\nu\delta'(\sigma-\bar{\sigma})\Big]\\
&= -2x'^\mu\Big((BG^{-1})_\mu^{\ \rho}\,\partial_\rho\xi^\nu + \partial_\mu\xi^\rho(BG^{-1})_\rho^{\ \nu} - \xi^\rho\partial_\rho(BG^{-1})_\mu^{\ \nu}\Big)\pi_\nu\,,
\end{aligned}$$

where we once again used (7.7) and (7.8). Substituting (7.5), (7.6) and (7.9) into (7.4), we can read the following transformation laws

$$\delta_\xi(G^{-1})^{\mu\nu} = -\xi^\rho\partial_\rho(G^{-1})^{\mu\nu} + \partial_\rho\xi^\mu(G^{-1})^{\nu\rho} + (G^{-1})^{\mu\rho}\partial_\rho\xi^\nu\,, \tag{7.10}$$

$$\delta_\xi G^E_{\mu\nu} = \partial_\mu\xi^\rho G^E_{\rho\nu} + G^E_{\mu\rho}\partial_\nu\xi^\rho - \xi^\rho\partial_\rho G^E_{\mu\nu}\,, \tag{7.11}$$

$$\delta_\xi(BG^{-1})_\mu^{\ \nu} = (BG^{-1})_\mu^{\ \rho}\,\partial_\rho\xi^\nu + \partial_\mu\xi^\rho(BG^{-1})_\rho^{\ \nu} - \xi^\rho\partial_\rho(BG^{-1})_\mu^{\ \nu}\,. \tag{7.12}$$

From these relations, we can easily obtain the transformation laws for metric and Kalb-Ramond tensor. For instance, using

$$\delta_\xi(G_{\mu\rho}(G^{-1})^{\rho\nu}) = \delta_\xi G_{\mu\rho}(G^{-1})^{\rho\nu} + G_{\mu\rho}\delta_\xi(G^{-1})^{\rho\nu} = 0\,, \tag{7.13}$$

we obtain

$$\delta_\xi G_{\mu\nu} = -\xi^\rho\partial_\rho G_{\mu\nu} - \partial_\mu\xi^\rho G_{\rho\nu} - \partial_\nu\xi^\rho G_{\rho\mu}\,. \tag{7.14}$$

Similarly, substituting $\delta_\xi(B_{\mu\rho}(G^{-1})^{\rho\nu}) = \delta_\xi B_{\mu\rho}(G^{-1})^{\rho\nu} + B_{\mu\rho}\delta_\xi(G^{-1})^{\rho\nu}$ into (7.12), we obtain

$$\delta_\xi B_{\mu\nu} = -\xi^\rho\partial_\rho B_{\mu\nu} + \partial_\mu\xi^\rho B_{\rho\nu} - B_{\mu\rho}\partial_\nu\xi^\rho\,. \tag{7.15}$$

Without loss of generality, we can change the sign of the parameter $\xi\to-\xi$, and write

$$\delta_\xi G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu} = \xi^\rho\partial_\rho G_{\mu\nu} + \partial_\mu\xi^\rho G_{\rho\nu} + \partial_\nu\xi^\rho G_{\rho\mu}\,, \tag{7.16}$$

$$\delta_\xi B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu} = \xi^\rho\partial_\rho B_{\mu\nu} - \partial_\mu\xi^\rho B_{\rho\nu} + B_{\mu\rho}\partial_\nu\xi^\rho\,. \tag{7.17}$$

These are the general coordinate transformations or diffeomorphisms. The Poisson bracket satisfies the Jacobi identity, so we can write

$$\begin{aligned} 0 &= \{\mathcal{G}_{\xi_1}, \{\mathcal{G}_{\xi_2}, \mathcal{H}\}\} + \{\mathcal{G}_{\xi_2}, \{\mathcal{H}, \mathcal{G}_{\xi_1}, \}\} + \{\mathcal{H}, \{\mathcal{G}_{\xi_1}, \mathcal{G}_{\xi_2}\}\} \\ &= \{\mathcal{G}_{\xi_1}, \{\mathcal{G}_{\xi_2}, \mathcal{H}\}\} - \{\mathcal{G}_{\xi_2}, \{\mathcal{G}_{\xi_1}, \mathcal{H}\}\} - \{\{\mathcal{G}_{\xi_1}, \mathcal{G}_{\xi_2}\}, \mathcal{H}\}\,, \end{aligned} \tag{7.18}$$

from which we obtain that the algebra of generators governing diffeomorphisms closes on the Lie bracket (4.5)

$$\Big\{\mathcal{G}_{\xi_1}\,, \mathcal{G}_{\xi_2}\Big\} = -\mathcal{G}_{[\xi_1,\xi_2]_L}\,. \tag{7.19}$$

### 7.1.2 Local gauge transformations

We now seek the generator in the form

$$\mathcal{G}_\lambda = \int_0^{2\pi} d\sigma \lambda_\mu(x(\sigma))\kappa x'^\mu(\sigma)\,, \tag{7.20}$$

so that its action on the Hamiltonian via Poisson bracket

$$\{\mathcal{G}_\lambda, \mathcal{H}\} = \delta_\lambda \mathcal{H}\,. \tag{7.21}$$

can be interpreted as the change of background fields. With the help of delta function identity (7.7), we obtain

$$\begin{aligned} \delta_\lambda\Big(\frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu\Big) &= \frac{1}{2\kappa}\int d\bar\sigma \Big\{\kappa\lambda_\rho(\bar\sigma)\partial_{\bar\sigma}x^\rho(\bar\sigma), \pi_\mu(\sigma)(G^{-1})^{\mu\nu}(\sigma)\pi_\nu(\sigma)\Big\} \\ &= \int d\bar\sigma\Big( - \lambda_\mu(\bar\sigma)(G^{-1})^{\mu\nu}\pi_\nu\delta'(\sigma-\bar\sigma) + x'^\rho\partial_\mu\lambda_\rho (G^{-1})^{\mu\nu}\pi_\nu\delta(\sigma-\bar\sigma)\Big) \\ &= x'^\mu(\partial_\rho\lambda_\mu - \partial_\mu\lambda_\rho)(G^{-1})^{\rho\nu}\pi_\nu\,, \end{aligned} \tag{7.22}$$

where the anomalous part goes to zero due to (7.8). The second Hamiltonian term (2.35) does not depend on momenta, so we have

$$\delta_\lambda\Big(\frac{1}{2}x'^\mu G^E_{\mu\nu}x'^\nu\Big) = \frac{1}{2}\int d\bar\sigma\Big\{\lambda_\rho(\bar\sigma)\partial_{\bar\sigma}x^\rho(\bar\sigma), x'^\mu(\sigma)G^E_{\mu\nu}(\sigma)x'^\nu(\sigma)\Big\} = 0\,, \tag{7.23}$$

and similarly using (7.7), we have

$$\begin{aligned} \delta_\lambda\Big( - 2x'^\mu (BG^{-1})_\mu{}^\nu\pi_\nu\Big) &= -2\kappa\int d\bar\sigma\Big\{\lambda_\rho(\bar\sigma)\partial_{\bar\sigma}x^\rho(\bar\sigma), x'^\mu(\sigma)(BG^{-1})_\mu{}^\nu(\sigma)\pi_\nu(\sigma)\Big\} \\ &= -2\kappa x'^\mu (BG^{-1})_\mu{}^\nu \int d\bar\sigma\Big( - \lambda_\nu(\bar\sigma)\delta'(\sigma-\bar\sigma) + \partial_\nu\lambda_\rho x'^\rho\delta(\sigma-\bar\sigma)\Big) \\ &= 2\kappa x'^\mu (BG^{-1})_\mu{}^\rho(\partial_\nu\lambda_\rho - \partial_\rho\lambda_\nu)x'^\nu\,. \end{aligned} \tag{7.24}$$

Substituting contributions (7.22), (7.23) and (7.24) into (7.21), we obtain

$$\begin{aligned}\delta_\lambda (G^{-1})^{\mu\nu} &= 0\,, &(7.25)\\ \delta_\lambda G^E_{\mu\nu} &= 4(BG^{-1})_\mu{}^\rho (\partial_\rho \lambda_\nu - \partial_\nu \lambda_\rho)\,, &(7.26)\\ \delta_\lambda (BG^{-1})_\mu{}^\nu &= \frac{1}{2}(\partial_\mu \lambda_\rho - \partial_\rho \lambda_\mu)(G^{-1})^{\rho\nu}\,, &(7.27)\end{aligned}$$

from which we read the transformation of background fields

$$\begin{aligned}\delta_\lambda G_{\mu\nu} &= 0\,, \\ \delta_\lambda B_{\mu\nu} &= (d\lambda)_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu\,,\end{aligned} \tag{7.28}$$

where without loss of generality we redefined $\lambda \to \frac{1}{2}\lambda$. These transformations of the background fields are known as local gauge transformations. There are analogous to the gauge transformations of the vector potential in electromagnetism (2.8).

Local gauge transformations are reducible transformations, due to the nilpotency of the exterior derivative. To demonstrate this, we consider the transformation of the Kalb-Ramond field governed by the generator of local gauge transformations with the parameter that is a sum of parameter $\lambda$ and the exterior derivative of some smooth function $f$

$$\delta_{\lambda+df} B = d\lambda + d^2 f = \delta_\lambda B\,, \tag{7.29}$$

or in some coordinate basis

$$\delta_{\lambda+df} B_{\mu\nu} = \partial_\mu(\lambda_\nu + \partial_\nu f) - \partial_\nu(\lambda_\mu + \partial_\mu f) = \delta_\lambda B_{\mu\nu}\,. \tag{7.30}$$

The T-duality exchanges momenta with the winding numbers (3.28). Since canonical momenta and the coordinate $\sigma$-derivatives are also generators of the diffeomorphisms and local gauge transformations, respectively, we conclude that the general coordinate transformations and local gauge transformations are not independent, rather they are related by T-duality.

## 7.2 Double generator and Courant bracket

The mutual relation of generators by T-duality motivates us to consider a single generator that will govern both of these symmetry transformations. The parameters of diffeomorphism $\xi^\mu$ are vector components, while the parameters of the local gauge transformations $\lambda_\mu$ are components of the 1-forms. Therefore, we can combine two parameters into a generalized vector $\Lambda^M$, where

$$\Lambda^M = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix}\,. \tag{7.31}$$

The generator governing both symmetry transformations is just the sum of generators $\mathcal{G}_\xi$ (7.2) and $\mathcal{G}_\lambda$ (7.20)

$$\mathcal{G}_\Lambda = \mathcal{G}_\xi + \mathcal{G}_\lambda = \int_0^{2\pi} d\sigma \Big( \xi^\mu \pi_\mu + \lambda_\mu \kappa x'^\mu \Big) \,, \tag{7.32}$$

which can be recognized as the inner product (6.2) on the generalized tangent bundle

$$\mathcal{G}_\Lambda = \int d\sigma \langle \Lambda, X \rangle \,. \tag{7.33}$$

Let us now proceed with the algebra of double generator $\mathcal{G}_\Lambda$. Using the Poisson bracket relations (7.3), we obtain

$$\begin{aligned}\Big\{ \mathcal{G}(\xi_1 \oplus \lambda_1),\, \mathcal{G}(\xi_2 \oplus \lambda_2) \Big\} = & \int d\sigma \Big( \pi_\mu (\xi_2^\nu \partial_\nu \xi_1^\mu - \xi_1^\nu \partial_\nu \xi_2^\mu) + \kappa x'^\mu (\xi_2^\nu \partial_\nu \lambda_{1\mu} - \xi_1^\nu \partial_\nu \lambda_{2\mu}) \Big) \\ & + \int d\sigma d\bar\sigma \kappa \Big( \lambda_{1\mu}(\sigma) \xi_2^\mu(\bar\sigma) + \lambda_{2\mu}(\bar\sigma) \xi_1^\mu(\sigma) \Big) \delta'(\sigma - \bar\sigma) \,,\end{aligned} \tag{7.34}$$

where we adopt the notation in which $\xi_1$ and $\lambda_1$ are vector and 1-form components of a generalized vector $\Lambda_1$, etc. To transform the anomalous terms, we note the identity

$$\delta'(\sigma - \bar\sigma) = -\partial_{\bar\sigma} \delta(\sigma - \bar\sigma) \,. \tag{7.35}$$

Now, the first term in the last line on the right-hand side of the equation (7.34) can be rewritten as

$$\begin{aligned}\kappa \int d\sigma d\bar\sigma \lambda_{1\mu}(\sigma) \xi_2^\mu(\bar\sigma) \delta'(\sigma - \bar\sigma) &= \frac{\kappa}{2} \int d\sigma d\bar\sigma \Big( \lambda_{1\mu}(\sigma) \xi_2^\mu(\bar\sigma) \delta'(\sigma - \bar\sigma) - \lambda_{1\mu}(\sigma) \xi_2^\mu(\bar\sigma) \partial_{\bar\sigma} \delta(\sigma - \bar\sigma) \Big) \\ &= \frac{\kappa}{2} \int d\sigma d\bar\sigma \Big( \lambda_{1\mu} \xi_2^\mu \delta'(\sigma - \bar\sigma) - \lambda_{1\mu}(\bar\sigma) \xi_2^\mu(\bar\sigma) \partial_{\bar\sigma} \delta(\sigma - \bar\sigma) \Big) \\ &\quad + \frac{\kappa}{2} \int d\sigma \Big( \lambda_{1\mu} \partial_\nu \xi_2^\mu x'^\nu - \partial_\nu \lambda_{1\mu} \xi_2^\mu x'^\nu \Big) \\ &= \kappa \int d\sigma \Big( \frac{1}{2} \partial_\nu (\lambda_{1\mu} \xi_2^\mu) - \xi_2^\mu \partial_\nu \lambda_{1\mu} \Big) x'^\nu \,.\end{aligned} \tag{7.36}$$

We first made the relation symmetric with the help of (7.35). Afterward, we used the relation (7.7), and in the end, we used the chain rule, and (7.8).

Similarly, the second anomalous term in (7.34) is as easily transformed

$$\begin{aligned}\kappa \int d\sigma d\bar\sigma \lambda_{2\mu}(\bar\sigma) \xi_1^\mu(\sigma) \delta'(\sigma - \bar\sigma) &= \frac{\kappa}{2} \int d\sigma d\bar\sigma \Big( \lambda_{2\mu}(\bar\sigma) \xi_1^\mu(\sigma) \delta'(\sigma - \bar\sigma) - \lambda_{2\mu}(\bar\sigma) \xi_1^\mu(\sigma) \partial_{\bar\sigma} \delta(\sigma - \bar\sigma) \Big) \\ &= \frac{\kappa}{2} \int d\sigma d\bar\sigma \Big( \lambda_{2\mu} \xi_1^\mu \delta'(\sigma - \bar\sigma) - \lambda_{2\mu}(\bar\sigma) \xi_1^\mu(\bar\sigma) \partial_{\bar\sigma} \delta(\sigma - \bar\sigma) \Big) \\ &\quad + \frac{\kappa}{2} \int d\sigma \Big( \partial_\nu \lambda_{2\mu} \xi_1^\mu x'^\nu - \lambda_{2\mu} \partial_\nu \xi_1^\mu x'^\nu \Big) \\ &= \kappa \int d\sigma \Big( \xi_1^\mu \partial_\nu \lambda_{2\mu} - \frac{1}{2} \partial_\nu (\lambda_{2\mu} \xi_1^\mu) \Big) x'^\nu \,.\end{aligned} \tag{7.37}$$

Substituting (7.36) and (7.37) into (7.34), we obtain

$$\Big\{\mathcal{G}_{\Lambda_1},\, \mathcal{G}_{\Lambda_2}\Big\} = -\mathcal{G}_{\Lambda}\,, \tag{7.38}$$

where the resulting gauge parameters are given by

$$\begin{aligned} \xi^\mu &= \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu\,, \\ \lambda_\mu &= \xi_1^\nu(\partial_\nu \lambda_{2\mu} - \partial_\mu \lambda_{2\nu}) - \xi_2^\nu(\partial_\nu \lambda_{1\mu} - \partial_\mu \lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\lambda_2 - \xi_2\lambda_1)\,. \end{aligned} \tag{7.39}$$

These relations can be recognized from the previous chapter, as relations (6.13) defining the Courant bracket

$$\Big\{\mathcal{G}_{\Lambda_1},\, \mathcal{G}_{\Lambda_2}\Big\} = -\mathcal{G}_{[\Lambda_1,\Lambda_2]_\mathcal{C}}\,. \tag{7.40}$$

We see that by extending the generator of diffeomorphisms with the local gauge transformations, the Courant bracket appears [68]. Due to the T-dual relation between two symmetry transformations, we conclude that the Courant bracket is the T-dual extension of the Lie bracket.

The Courant bracket, together with the generalized tangent bundle and its symmetric bilinear form (6.2), and the natural projection to the tangent bundle as an anchor, defines the standard Courant algebroid (see Appendix [C]). In the previous chapter, we obtained its Dirac structures and saw that it puts a severe restriction on $H$- and $R$-fluxes. In the next chapter, we will introduce the twisted Courant algebroid, that is to say, the Courant algebroid defined with the twisted version of the Courant bracket.

# Chapter 8

# Twisted Courant algebroid

We will define the Courant bracket twisted by any element of the $O(D, D)$ group and then show that it defines a Courant algebroid, where all the compatibility conditions are a priori satisfied.

## 8.1 Twisted Courant bracket

Let $e^T$ be an $O(D, D)$ transformation, keeping the inner product (6.2) invariant. Its action on the basis $X^M$ (2.39) produces another basis

$$\hat{X}^M = (e^T)^M_{\ N}\, X^N\,. \tag{8.1}$$

We can express the generator $\mathcal{G}_\Lambda$ (7.33) in this basis, using the invariance of the inner product with respect to $e^T$

$$\mathcal{G}_\Lambda = \int d\sigma \langle \Lambda, X\rangle = \int d\sigma \langle e^T\Lambda, e^T X\rangle = \int d\sigma \langle \hat{\Lambda}, \hat{X}\rangle = \mathcal{G}^{(T)}_{\hat{\Lambda}}\,, \tag{8.2}$$

where we marked the resulting generator as $\mathcal{G}^{(T)}_{\hat{\Lambda}}$, and where

$$\hat{\Lambda}^M = (e^T)^M_{\ N}\, \Lambda^N\,. \tag{8.3}$$

Using the $O(D, D)$ invariance of the inner product, the algebra relations of the generator written in a new basis becomes

$$\begin{aligned} \left\{\mathcal{G}^{(T)}_{\hat{\Lambda}_1}, \mathcal{G}^{(T)}_{\hat{\Lambda}_2}\right\} &= -\int d\sigma \langle [\Lambda_1, \Lambda_2]_C, X\rangle = -\int d\sigma \langle [e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_C, e^{-T}\hat{X}\rangle \\ &= -\int d\sigma \langle e^T[e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_{\mathcal{C}}, \hat{X}\rangle = -\mathcal{G}^{(T)}_{[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T}}\,, \end{aligned} \tag{8.4}$$

where we defined the $T$-twisted Courant bracket by

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T} = e^T[e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_{\mathcal{C}}\,. \tag{8.5}$$

For each $O(D,D)$ invariant transformation $e^T$, there is a corresponding twisted Courant bracket. A straightforward method of obtaining the twisted Courant bracket involves using the transformation $e^T$ to change the basis in which the generator is represented, followed by calculating the Poisson bracket algebra of said generator.

## 8.2 Courant algebroid related to the twisted Courant bracket

Let us demonstrate that the twisted Courant bracket defines a Courant algebroid. We are looking for an anchor that satisfies

$$\rho\Big([\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_\mathcal{T}}\Big) = [\rho(\hat{\Lambda}_1), \rho(\hat{\Lambda}_2)]_L \,. \tag{8.6}$$

Using (8.3) and (8.5), we rewrite the previous relation as

$$\rho\Big(e^T [\Lambda_1, \Lambda_2]_\mathcal{C}\Big) = [\rho(e^T \Lambda_1), \rho(e^T \Lambda_2)]_L \,. \tag{8.7}$$

Now from the fact that the natural projection $\pi$ (6.14) is the anchor for the standard Courant algebroid (6.16), we obtain

$$\rho(\hat{\Lambda}) = \pi(e^{-T} \hat{\Lambda}) \,. \tag{8.8}$$

The corresponding differential operator is obtained from substituting (8.3) and (8.8) into the definition of the Courant algebroid differential operator (6.21)

$$\langle \mathcal{D} f, \hat{\Lambda} \rangle = \mathcal{L}_{\rho(\hat{\Lambda})}\, f = \mathcal{L}_{\pi(\Lambda)} f = \langle \mathcal{D}^{(0)} f, \Lambda \rangle = \langle \mathcal{D}^{(0)} f, e^{-T} \hat{\Lambda} \rangle = \langle e^T \mathcal{D}^{(0)} f, \hat{\Lambda} \rangle \,, \tag{8.9}$$

from which we obtain

$$\mathcal{D} f = e^T \mathcal{D}^{(0)} f \,, \tag{8.10}$$

where $\mathcal{D}^{(0)}$ is differential operator of standard Courant algebroid (C.4).

We still need to verify that the compatibility conditions in the Courant algebroid definition (6.23) - (6.26) are satisfied for the above choice of anchor, bracket, and differential operator. For the second property (6.23), we have

$$\begin{aligned}
[\hat{\Lambda}_1, f\hat{\Lambda}_2]_{\mathcal{C}_\mathcal{T}} &= e^T [e^{-T} \hat{\Lambda}_1, f e^{-T} \hat{\Lambda}_2]_\mathcal{C} \\
&= e^T \Big( f [e^{-T} \hat{\Lambda}_1, e^{-T} \hat{\Lambda}_2]_\mathcal{C} + (\mathcal{L}_{\pi(e^{-T}\hat{\Lambda}_1)} f)(e^{-T} \hat{\Lambda}_2) - \frac{1}{2} \langle e^{-T} \hat{\Lambda}_1, e^{-T} \hat{\Lambda}_2 \rangle \mathcal{D}^{(0)} f \Big) \\
&= f [\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_\mathcal{T}} + (\mathcal{L}_{\rho(\hat{\Lambda}_1)} f) \hat{\Lambda}_2 - \frac{1}{2} \langle \hat{\Lambda}_1, \hat{\Lambda}_2 \rangle \mathcal{D} f \,,
\end{aligned} \tag{8.11}$$

where we first used the definition of the twisted Courant bracket (8.5), afterward we applied (6.17), and in the end used the expressions for the anchor $\rho$ (8.8) and the differential operator $\mathcal{D}$ (8.10), as well as the fact that $O(D,D)$ transformations keep the inner product invariant.

For the third condition (6.24), we firstly write

$$\begin{aligned}\langle[\hat\Lambda_1,\hat\Lambda_2]_{\mathcal{C}_\mathcal{T}}+\frac{1}{2}\mathcal{D}\langle\hat\Lambda_1,\hat\Lambda_2\rangle,\hat\Lambda_3\rangle &= \langle e^T[e^{-T}\hat\Lambda_1,e^{-T}\hat\Lambda_2]_\mathcal{C}+\frac{1}{2}e^T\mathcal{D}^{(0)}\langle\hat\Lambda_1,\hat\Lambda_2\rangle,\hat\Lambda_3\rangle \\ &= \langle[\Lambda_1,\Lambda_2]_\mathcal{C}+\frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1,\Lambda_2\rangle,\Lambda_3\rangle\,,\end{aligned} \tag{8.12}$$

where we used (8.5), (8.3) and (8.10). Similarly, we obtain

$$\langle\hat\Lambda_2,[\hat\Lambda_1,\hat\Lambda_3]_{\mathcal{C}_\mathcal{T}}+\frac{1}{2}\mathcal{D}\langle\hat\Lambda_1,\hat\Lambda_3\rangle\rangle=\langle\Lambda_2,[\Lambda_1,\Lambda_3]_\mathcal{C}+\frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1,\Lambda_3\rangle\rangle\,, \tag{8.13}$$

and

$$\mathcal{L}_{\rho(\hat\Lambda_1)}\langle\hat\Lambda_2,\hat\Lambda_3\rangle=\mathcal{L}_{\pi(\Lambda_1)}\langle\Lambda_2,\Lambda_3\rangle\,. \tag{8.14}$$

Adding (8.12) and (8.13), we obtain

$$\begin{aligned}&\langle[\hat\Lambda_1,\hat\Lambda_2]_{\mathcal{C}_\mathcal{T}}+\frac{1}{2}\mathcal{D}\langle\hat\Lambda_1,\hat\Lambda_2\rangle,\hat\Lambda_3\rangle+\langle\hat\Lambda_2,[\hat\Lambda_1,\hat\Lambda_3]_{\mathcal{C}_\mathcal{T}}+\frac{1}{2}\mathcal{D}\langle\hat\Lambda_1,\hat\Lambda_3\rangle\rangle=\\ &\langle[\Lambda_1,\Lambda_2]_\mathcal{C}+\frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1,\Lambda_2\rangle,\Lambda_3\rangle+\langle\Lambda_2,[\Lambda_1,\Lambda_3]_\mathcal{C}+\frac{1}{2}\mathcal{D}^{(0)}\langle\Lambda_1,\Lambda_3\rangle\rangle=\\ &\mathcal{L}_{\pi(\Lambda_1)}\langle\Lambda_2,\Lambda_3\rangle=\mathcal{L}_{\rho(\hat\Lambda_1)}\langle\hat\Lambda_2,\hat\Lambda_3\rangle\,,\end{aligned} \tag{8.15}$$

where in the end we used (C.11) and (8.14).

The fourth condition (6.25) is as easily obtained from the orthogonality of $e^T$ with respect to the inner product

$$\langle\mathcal{D}f,\mathcal{D}g\rangle=\langle e^T\mathcal{D}^{(0)}f,e^T\mathcal{D}^{(0)}g\rangle=\langle\mathcal{D}^{(0)}f,\mathcal{D}^{(0)}g\rangle=0\,. \tag{8.16}$$

Lastly, we note that

$$[[\hat\Lambda_1,\hat\Lambda_2]_{\mathcal{C}_\mathcal{T}},\hat\Lambda_3]_{\mathcal{C}_\mathcal{T}}=e^T[[e^{-T}\hat\Lambda_1,e^{-T}\hat\Lambda_2]_\mathcal{C},e^{-T}\hat\Lambda_3]_\mathcal{C}=e^T[[\Lambda_1,\Lambda_2]_\mathcal{C},\Lambda_3]_\mathcal{C}\,, \tag{8.17}$$

from which we express the Jacobiator (6.19) for the twisted Courant bracket in terms of the Jacobiator for the Courant bracket by

$$\mathrm{Jac}_{\mathcal{C}_\mathcal{T}}(\hat\Lambda_1,\hat\Lambda_2,\hat\Lambda_3)=e^T\mathrm{Jac}_\mathcal{C}(\Lambda_1,\Lambda_2,\Lambda_3)\,. \tag{8.18}$$

Similarly, we note that

$$\langle[\hat\Lambda_1,\hat\Lambda_2]_{\mathcal{C}_\mathcal{T}},\hat\Lambda_3\rangle=\langle e^T[e^{-T}\hat\Lambda_1,e^{-T}\hat\Lambda_2]_\mathcal{C},\hat\Lambda_3\rangle=\langle[\Lambda_1,\Lambda_2]_\mathcal{C},\Lambda_3\rangle\,, \tag{8.19}$$

from which one easily obtains the relation between the Nijenhuis operator (6.20) of the twisted and standard Courant bracket

$$\mathrm{Nij}_{\mathcal{C}_\mathcal{T}}(\hat\Lambda_1,\hat\Lambda_2,\hat\Lambda_3)=\mathrm{Nij}_\mathcal{C}(\Lambda_1,\Lambda_2,\Lambda_3)\,. \tag{8.20}$$

Substituting (8.10), (8.18) and (8.20) into (6.18), we obtain the last compatibility condition of (6.26).

Thus, it has been demonstrated that $\left(T\mathcal{M} \oplus T^\star\mathcal{M}, \langle,\rangle, [,]_{\mathcal{C}_T}, \rho\right)$ is a Courant algebroid with the appropriate twisted Courant bracket (8.5) as its bracket. This is a simple consequence of the fact that the Courant bracket defines the standard Courant algebroid and that the inner product (6.2) remains invariant under $O(D, D)$ transformations.

It is worth reiterating that the crucial step in obtaining the twisted Courant brackets is a change of basis by the action of $O(D, D)$ transformation. As we will see in the following chapters, it is possible to choose different bases in which the generalized metric is diagonal, and Hamiltonian has the form of a non-interacting Hamiltonian, expressed in terms of non-canonical currents. The Poisson bracket relations of these currents will contain fluxes.

# Chapter 9

# B-twisted Courant bracket

The procedure for obtaining the twisted Courant algebroid from the Poisson bracket algebra will be applied in this chapter for case of $B$-transformations. We will obtain the $B$-twisted Courant bracket, its related Courant algebroid and its Dirac structures.

## 9.1 Free form Hamiltonian

Consider the background field characterized only with the metric tensor, so that the generalized metric has a simple diagonal form

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & (G^{-1})^{\mu\nu} \end{pmatrix} . \tag{9.1}$$

Acting with the $B$-transformations $e^{\hat{B}}$ (6.5), we can obtain the usual expression for the generalized metric $H_{MN}$ (2.38)

$$H_{MN} = ((e^{\hat{B}})^T)_M^{\ K} \, G_{KL} \, (e^{\hat{B}})^L_{\ N} . \tag{9.2}$$

Therefore, we can rewrite the canonical Hamiltonian (2.35) in the form of a free Hamiltonian

$$\mathcal{H}_{\mathcal{C}} = \frac{1}{2\kappa}(X^T)^M H_{MN} X^N = \frac{1}{2\kappa}(e^{\hat{B}}X)^M G_{MN} (e^{\hat{B}}X)^N = \frac{1}{2\kappa}\hat{X}^M G_{MN} \hat{X}^N , \tag{9.3}$$

where

$$\hat{X}^M = (e^{\hat{B}})^M_{\ N} \, X^N = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu \end{pmatrix} \equiv \begin{pmatrix} \kappa x'^\mu \\ i_\mu \end{pmatrix} , \tag{9.4}$$

where $i_\mu$ is an auxiliary current given by

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu . \tag{9.5}$$

The Poisson bracket algebra of auxiliary currents $i_\mu$ is easily obtained with the help of the standard Poisson bracket relations between canonical variables (7.3)

$$\{i_\mu(\sigma), i_\nu(\bar{\sigma})\} = -2\kappa B_{\mu\nu\rho} x'^\rho \delta(\sigma - \bar{\sigma}) \,, \tag{9.6}$$

where $B_{\mu\nu\rho}$ is the Kalb-Ramond field strength (2.28). In the context of flux compactification, the Kalb-Ramond field strength is known as the $H$-flux. Mathematically, this is the exterior derivative of a 2-form $B$.

Let us now express the generator (7.32) in the non-canonical basis $\hat{X}^M$. It is given by

$$\mathcal{G}^{\hat{B}}_{\hat{\Lambda}} = \int d\sigma \langle \hat{\Lambda}, \hat{X} \rangle = \int d\sigma \Big( \xi^\mu i_\mu + \hat{\lambda}_\mu \kappa x'^\mu \Big) \,, \tag{9.7}$$

which is exactly equal to the generator written in canonical basis when the following relation between gauge parameters is satisfied

$$\hat{\Lambda}^M = (e^{\hat{B}})^M{}_N \, \Lambda^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu \\ \lambda_\mu + 2B_{\mu\nu}\xi^\nu \end{pmatrix} \equiv \begin{pmatrix} \xi^\mu \\ \hat{\lambda}_\mu \end{pmatrix} . \tag{9.8}$$

In the previous chapter, we saw that the $O(D,D)$ transformation on the basis in which the generator is expressed gives rise to the new basis, in which generator closes on the twisted Courant bracket. In this case, we have $e^{\hat{B}}$ as $O(D,D)$ transformation, which when substituted in (8.4) becomes

$$\Big\{ \mathcal{G}^{\hat{B}}_{\Lambda_1}, \mathcal{G}^{\hat{B}}_{\Lambda_2} \Big\} = -\mathcal{G}^{\hat{B}}_{[\Lambda_1, \Lambda_2]_{\mathcal{C}_B}} \,, \tag{9.9}$$

where we have marked the $B$-twisted Courant bracket (8.5) by

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = e^{\hat{B}} [e^{-\hat{B}} \Lambda_1, e^{-\hat{B}} \Lambda_2]_{\mathcal{C}} \,. \tag{9.10}$$

## 9.2 B-twisted Courant bracket

We see how the $B$-twisted Courant bracket can be obtained from the newly defined generator $\mathcal{G}^{\hat{B}}_{\Lambda}$ (9.7). Before that, we require the Poisson bracket relations between the auxiliary currents (9.5) and parameters (9.8), which are easily obtained using the standard Poisson bracket relations

$$\{\xi^\mu(\sigma), i_\nu(\bar{\sigma})\} = \partial_\nu \xi^\mu \delta(\sigma - \bar{\sigma}) \,, \quad \{\lambda_\mu(\sigma), i_\nu(\bar{\sigma})\} = \partial_\nu \lambda_\mu \delta(\sigma - \bar{\sigma}) \,, \tag{9.11}$$

where we assume the $\sigma$ dependence unless stated otherwise. We note that the part containing only vector parameters $\xi$ in (9.9) produces additional term containing $H$-flux, compared to the standard Courant bracket

$$\Big\{ \xi^\mu_1(\sigma) i_\mu(\sigma), \xi^\nu_2(\bar{\sigma}) i_\nu(\bar{\sigma}) \Big\} = -\Big[ \Big( \xi^\nu_1 \partial_\nu \xi^\mu_2 - \xi^\nu_2 \partial_\nu \xi^\mu_1 \Big) i_\mu + 2B_{\mu\nu\rho} \xi^\nu_1 \xi^\rho_2 \kappa x'^\mu \Big] \delta(\sigma - \bar{\sigma}) \,. \tag{9.12}$$

The other contributions remain the same as in the case of the Courant bracket, since

$$\begin{aligned}\Big\{\lambda_{1\mu}(\sigma)\kappa x'^{\mu}(\sigma),\xi_2^{\nu}(\bar\sigma)i_{\nu}(\bar\sigma)\Big\} &= \Big\{\lambda_{1\mu}(\sigma)\kappa x'^{\mu}(\sigma),\xi_2^{\nu}(\bar\sigma)\pi_{\nu}(\bar\sigma)\Big\} \\ \Big\{\lambda_{1\mu}(\sigma)\kappa x'^{\mu}(\sigma),\lambda_{2\nu}(\bar\sigma)\kappa x'^{\nu}(\bar\sigma)\Big\} &= 0\,.\end{aligned} \tag{9.13}$$

Substituting (9.12) and (9.13) into (9.9), we obtain the expression for resulting symmetry parameter $\hat\Lambda = \xi \oplus \hat\lambda$ for the B-twisted Courant bracket

$$\begin{aligned}\xi^{\mu} &= \xi_1^{\nu}\partial_{\nu}\xi_2^{\mu} - \xi_2^{\nu}\partial_{\nu}\xi_1^{\mu}\,, \\ \hat\lambda_{\mu} &= \xi_1^{\nu}(\partial_{\nu}\hat\lambda_{2\mu} - \partial_{\mu}\hat\lambda_{2\nu}) - \xi_2^{\nu}(\partial_{\nu}\hat\lambda_{1\mu} - \partial_{\mu}\hat\lambda_{1\nu}) + \frac{1}{2}\partial_{\mu}(\xi_1\hat\lambda_2 - \xi_2\hat\lambda_1) + 2B_{\mu\nu\rho}\xi_1^{\nu}\xi_2^{\rho}\,,\end{aligned} \tag{9.14}$$

or in the coordinate invariant notation

$$\begin{aligned}\xi &= [\xi_1,\xi_2]_L\,, \\ \lambda &= \mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1) + dB(\xi_1,\xi_2,.)\,.\end{aligned} \tag{9.15}$$

The $B$-twisted Courant bracket was firstly obtained in [69], where the authors considered a double of a trivial Lie algebroid on cotangent bundle with the bracket $[\lambda_1,\lambda_2] = 0$, and a quasi-Lie algebroid whose bracket is defined as

$$[\xi_1,\xi_2]_{\tilde L} = [\xi_1,\xi_2]_L + dB(\xi_1,\xi_2,.)\,. \tag{9.16}$$

Then, the relations (6.11) gives rise to the $B$-twisted Courant bracket. The bracket (9.16) does not satisfy the Jacobi identity, and for that reason defines a quasi-Lie algebroid. The flux term can be seen as the deformation from the Lie algebroid.

## 9.3 Courant algebroid

We saw how the $B$-twisted Courant bracket is directly obtained from the symmetry generator in basis obtained from the appropriate $O(D,D)$ transformation. Substituting $e^T = e^{\hat B}$ into (8.8) and (8.10), we see that the anchor and the derivative operator are defined the same as in the case of non-twisted, standard Courant algebroid, i.e.

$$\rho^{(\hat B)} = \pi\,, \quad \mathcal{D}^{(\hat B)}f = \mathcal{D}^{(0)}f = 0 \oplus df\,. \tag{9.17}$$

Let us seek the Dirac structures in the form of $\mathcal{V}_B$ (6.29). On this sub-bundle, the symmetry generator becomes

$$\begin{aligned}\mathcal{G}^{\hat B}_{\mathcal{V}_B(\Lambda)} &= \int d\sigma\Big(\xi^{\mu}i_{\mu} + 2B_{\mu\nu}\xi^{\nu}\kappa x'^{\mu}\Big) = \int d\sigma\Big(\xi^{\mu}\pi_{\mu} + 2B_{\mu\nu}(\xi^{\mu}\kappa x'^{\nu} + \xi^{\nu}\kappa x'^{\mu})\Big) \\ &= \int d\sigma\xi^{\mu}\pi_{\mu}\,,\end{aligned} \tag{9.18}$$

due to $B$ being antisymmetric. This generator is known to be a generator of diffeomorphisms (7.2), and gives rise to the Lie bracket (7.19) in its Poisson bracket algebra. Hence, $\mathcal{V}_B$ is going to be a Dirac structure no matter what value of the Kalb-Ramond field strength $dB$, i.e.

$$\Big[\mathcal{V}_B(\Lambda_1), \mathcal{V}_B(\Lambda_2)\Big]_{\mathcal{C}_B} = \mathcal{V}_B\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}_B}\Big)\,, \quad \forall\, dB\,. \tag{9.19}$$

For the standard Courant algebroid, $\mathcal{V}_B$ is a Dirac structure only for a closed 2-form $dB = 0$ (6.34). The twisting of the Courant bracket by $B$ lifted the restriction it imposed on its Dirac structures in the form of $\mathcal{V}_B$.

As for the Dirac structures in the form of $\mathcal{V}_\theta$ (6.31), the restrictions on fluxes remain. The easiest way to see that is to substitute $\xi^\mu = \kappa\theta^{\mu\nu}\lambda_\nu$ into relation (9.14). We obtain

$$\begin{aligned}
\xi^\mu &= \kappa^2\theta^{\mu\sigma}\theta^{\nu\rho}(\lambda_{1\rho}\partial_\nu\lambda_{2\sigma} - \lambda_{2\rho}\partial_\nu\lambda_{1\sigma}) + \kappa^2(\theta^{\nu\rho}\partial_\nu\theta^{\mu\sigma} - \theta^{\nu\sigma}\partial_\nu\theta^{\mu\rho})\lambda_{1\rho}\lambda_{2\sigma} \\
\lambda_\mu &= \kappa\theta^{\nu\rho}(\lambda_{1\rho}\partial_\nu\lambda_{2\mu} - \lambda_{2\rho}\partial_\nu\lambda_{1\mu}) + \kappa\partial_\mu\theta^{\nu\rho}\lambda_{1\rho}\lambda_{2\nu} + 2\kappa^2 B_{\mu\nu\rho}\theta^{\nu\alpha}\theta^{\rho\beta}\lambda_{1\alpha}\lambda_{2\beta}\,.
\end{aligned} \tag{9.20}$$

For this to define a Dirac structure, the condition $\xi^\mu = \kappa\theta^{\mu\nu}\lambda_\nu$ has to be true on resulting parameters. However, we have instead the relation

$$\xi^\mu = \kappa\theta^{\mu\nu}\lambda_\nu - \kappa^2(\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu} + 2\kappa\theta^{\mu\alpha}\theta^{\nu\beta}\theta^{\rho\gamma}B_{\alpha\beta\gamma})\lambda_{1\nu}\lambda_{2\rho}\,, \tag{9.21}$$

and therefore $\mathcal{V}_\theta$ will be a Dirac structure for

$$[\mathcal{V}_\theta(\Lambda_1), \mathcal{V}_\theta(\Lambda_2)]_{\mathcal{C}_B} = \mathcal{V}_\theta\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}_B}\Big)\,, \quad \mathcal{R} = 0\,, \tag{9.22}$$

where $\mathcal{R}$ is generalized $R$-flux, given by

$$\mathcal{R}^{\mu\nu\rho} = R^{\mu\nu\rho} + 2\kappa\theta^{\mu\alpha}\theta^{\nu\beta}\theta^{\rho\gamma}B_{\alpha\beta\gamma}\,, \quad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}\,. \tag{9.23}$$

In the coordinate free notation, the generalized $\mathcal{R}$-flux has the expression

$$\mathcal{R} = \frac{1}{2}[\theta, \theta]_S + 2\kappa \wedge^3 \theta\, dB\,, \tag{9.24}$$

where $\wedge^3\theta\, dB$ represents the multiplication of a bi-vector $\theta$ three times with the 3-form $dB$.

The condition $\mathcal{R} = 0$ defines twisted Poisson structures [69]. In that case, one can define the twisted Poisson bracket, using the relation (A.1), which will not satisfy the Jacobi identity. The twisted Poisson structures appeared in many instances in the context of string theory. For instance, they are a suitable mathematical language for describing the non-commutative and non-associative string backgrounds [70]. As we see, in generalized geometry we obtain twisted Poisson structures as Dirac structures of the $B$-twisted Courant algebroids.

# Chapter 10

# $\theta$-twisted Courant bracket in symmetry algebra

In this chapter, we will consider the background obtained by the action of $\theta$-transformation, acting on the background characterized solely by the T-dual metric tensor. We will show that Hamiltonian can be written in the diagonal form in a non-canonical basis and that the symmetry generator algebra in that basis closes on $\theta$-twisted Courant bracket. In the end, we will show that this bracket is in fact T-dual to $B$-twisted Courant bracket.

## 10.1 Free form Hamiltonian

We will begin with the background characterized solely by the T-dual metric tensor. The T-dual of the diagonal generalized metric $G_{MN}$ (9.1) is given by

$$ {}^\star G_{MN} = \begin{pmatrix} {}^\star(G^{-1})_{\mu\nu} & 0 \\ 0 & {}^\star G^{\mu\nu} \end{pmatrix} = \begin{pmatrix} G^E_{\mu\nu} & 0 \\ 0 & (G_E^{-1})^{\mu\nu} \end{pmatrix} , \tag{10.1} $$

where the relation (3.21) was used. We will introduce the antisymmetric field with the T-dual of $B$-transformations, which are $\theta$-transformations $e^{\hat{\theta}}$ (6.7). The T-dual generalized metric becomes

$$ \begin{aligned} {}^\star H_{MN} &= ((e^{\hat{\theta}})^T)_M^{\ L}\, {}^\star G_{LK} (e^{\hat{\theta}})^K_{\ N} = \begin{pmatrix} 1 & 0 \\ -2{}^\star B & 1 \end{pmatrix} \begin{pmatrix} {}^\star G^{-1} & 0 \\ 0 & {}^\star G \end{pmatrix} \begin{pmatrix} 1 & 2{}^\star B \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} {}^\star G^{-1} & 2{}^\star G^{-1}{}^\star B \\ -2{}^\star B {}^\star G^{-1} & {}^\star G - 4{}^\star B {}^\star G^{-1}{}^\star B \end{pmatrix} = \begin{pmatrix} G_E & 2BG^{-1} \\ -2G^{-1}B & G^{-1} \end{pmatrix} = H_{MN} , \end{aligned} \tag{10.2} $$

where in the second line we used (3.21) and (2.38).

We rewrite the Hamiltonian in a non-canonical basis, so that the T-dual generalized metric is diagonal

$$ {}^\star\hat{\mathcal{H}}_C = \frac{1}{2\kappa}(X^T)_M^{\ L}\,((e^{\hat\theta})^T)_L^{\ K}\,{}^\star G_{KJ}\,(e^{\hat\theta})^J_{\ N}\,X^N = \frac{1}{2\kappa}\hat{X}^M\,{}^\star G_{MN}\hat{X}^N\,, \tag{10.3} $$

where

$$ \hat{X}^M = (e^{\hat\theta})^M_{\ N}\,X^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix}\begin{pmatrix} \kappa x'^\nu \\ \pi_\nu \end{pmatrix} = \begin{pmatrix} \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu \\ \pi_\mu \end{pmatrix} \equiv \begin{pmatrix} k^\mu \\ \pi_\mu \end{pmatrix}\,, \tag{10.4} $$

and where $k^\mu$ is an auxiliary current, given by

$$ k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu\,. \tag{10.5} $$

The Poisson bracket algebra of these currents is easily obtained from the standard Poisson bracket relations between canonical variables (7.3)

$$ \begin{aligned} \{k^\mu(\sigma), k^\nu(\bar\sigma)\} &= \kappa^2\theta^{\nu\sigma}(\bar\sigma)\delta^\mu_\sigma\delta'(\sigma-\bar\sigma) - \kappa^2\theta^{\mu\rho}\delta^\nu_{\ \rho}\partial_{\bar\sigma}\delta(\sigma-\bar\sigma) \\ &\quad + \kappa^2\theta^{\nu\sigma}\partial_\sigma\theta^{\mu\rho}\pi_\rho\delta(\sigma-\bar\sigma) - \kappa^2\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho}\pi_\rho\delta(\sigma-\bar\sigma) \\ &= \kappa^2\theta^{\nu\mu}\delta'(\sigma-\bar\sigma) + \kappa^2\partial_\rho\theta^{\nu\mu}x'^\rho\delta(\sigma-\bar\sigma) + \kappa^2\theta^{\mu\nu}\delta'(\sigma-\bar\sigma) \\ &\quad + \kappa^2\theta^{\nu\sigma}\partial_\sigma\theta^{\mu\rho}\pi_\rho\delta(\sigma-\bar\sigma) - \kappa^2\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho}\pi_\rho\delta(\sigma-\bar\sigma) \\ &= -\kappa Q_\rho^{\ \mu\nu}k^\rho\delta(\sigma-\bar\sigma) - \kappa^2R^{\mu\nu\rho}\pi_\rho\delta(\sigma-\bar\sigma)\,, \end{aligned} \tag{10.6} $$

where in the second step we applied two $\delta$-function identities (7.7) and (7.35), and in the last step we used the inverted relation of (10.5)

$$ \kappa x'^\mu = k^\mu - \kappa\theta^{\mu\nu}\pi_\nu\,, \tag{10.7} $$

and expressed the structure coefficients as non-geometric fluxes $Q$ and $R$, given by

$$ Q_\rho^{\ \mu\nu} = \partial_\rho\theta^{\mu\nu}, \qquad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}\,. \tag{10.8} $$

These fluxes can create a potential that stabilizes the vacuum expectation value and provides mass to the moduli [23, 24, 25]. Additionally, $Q$ flux is linked to string non-commutativity [71], while $R$ flux is linked to string non-associativity [72].

The other relevant algebra relation is as easily obtained

$$ \{k^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \kappa\delta^\mu_\nu\delta'(\sigma-\bar\sigma) + \kappa\partial_\nu\theta^{\mu\rho}\pi_\rho\delta(\sigma-\bar\sigma)\,. \tag{10.9} $$

We also rewrite the symmetry generator in a new non-canonical basis

$$ \mathcal{G}^{\hat\theta}_{\hat\Lambda} = \int d\sigma\langle\hat\Lambda, \hat{X}\rangle\,, \tag{10.10} $$

which is the same as the generator $\mathcal{G}_\Lambda$, when the following relation between the symmetry parameters stand

$$\hat{\Lambda}^M = (e^{\hat{\theta}})^M_{\ N}\, \Lambda^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu + \kappa\theta^{\mu\nu}\lambda_\nu \\ \lambda_\mu \end{pmatrix} \equiv \begin{pmatrix} \hat{\xi}^\mu \\ \lambda_\mu \end{pmatrix} . \tag{10.11}$$

The algebra of parameters and auxiliary currents is a straightforward application of (7.3), e.g.

$$\{\lambda_\mu(\sigma), k^\nu(\bar{\sigma})\} = \kappa\theta^{\nu\rho}\partial_\rho\lambda_\mu\delta(\sigma-\bar{\sigma})\,, \tag{10.12}$$

and similarly for other cases.

## 10.2 $\theta$-twisted Courant bracket

Like in the previous chapter, we want to obtain the algebra in the form

$$\left\{\mathcal{G}^{\hat{\theta}}_{\hat{\Lambda}_1}, \mathcal{G}^{\hat{\theta}}_{\hat{\Lambda}_2}\right\} = -\mathcal{G}^{\hat{\theta}}{}_{[\hat{\Lambda}_1,\hat{\Lambda}_2]_{\mathcal{C}_\theta}}\,, \tag{10.13}$$

where $[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_\theta}$ is the $\theta$-twisted Courant bracket. Again, we will do the termwise calculations. From the vector-vector contribution, we obtain the Lie bracket

$$\{\hat{\xi}^\mu_1\pi_\mu(\sigma), \hat{\xi}^\nu_2\pi_\nu(\bar{\sigma})\} = -(\hat{\xi}^\nu_1\partial_\nu\hat{\xi}^\mu_2 - \hat{\xi}^\nu_2\partial_\nu\hat{\xi}^\mu_1)\pi_\mu\delta(\sigma-\bar{\sigma})\,. \tag{10.14}$$

For the form-form bracket, using (10.6) and (10.12) we obtain

$$\begin{aligned}\{\lambda_{1\mu}k^\mu(\sigma), \lambda_{2\nu}k^\nu(\bar{\sigma})\} &= -\Big(\kappa\theta^{\nu\rho}(\lambda_{1\nu}\partial_\rho\lambda_{2\mu} - \lambda_{2\nu}\partial_\rho\lambda_{1\mu}) + \kappa\lambda_{1\rho}\lambda_{2\nu}Q_\mu^{\ \rho\nu}\Big)k^\mu\delta(\sigma-\bar{\sigma}) \\ &\quad -\kappa^2 R^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\pi_\mu\delta(\sigma-\bar{\sigma})\,. \end{aligned}\tag{10.15}$$

The direct calculations of the form-vector part gives

$$\begin{aligned}\{\lambda_{1\mu}k^\mu(\sigma), \hat{\xi}^\nu_2\pi_\nu(\bar{\sigma})\} &= \hat{\xi}^\nu_2\partial_\nu\lambda_{1\mu}k^\mu\delta(\sigma-\bar{\sigma}) - \kappa\lambda_{1\mu}\theta^{\mu\rho}\partial_\rho\hat{\xi}^\nu_2\pi_\nu\delta(\sigma-\bar{\sigma}) \\ &\quad +\kappa\lambda_{1\mu}(\sigma)\hat{\xi}^\mu_2(\bar{\sigma})\delta'(\sigma-\bar{\sigma}) + \kappa\lambda_{1\mu}\partial_\nu\theta^{\mu\rho}\pi_\rho\hat{\xi}^\nu_2\delta(\sigma-\bar{\sigma})\,. \end{aligned}\tag{10.16}$$

As in the previous chapters, the anomalous part can be further transformed. Using (7.7) and (7.35), we obtain

$$\begin{aligned}\kappa\lambda_{1\mu}(\sigma)\hat{\xi}^\mu_2(\bar{\sigma})\delta'(\sigma-\bar{\sigma}) &= \frac{\kappa}{2}\lambda_{1\mu}(\sigma)\hat{\xi}^\mu_2(\bar{\sigma})\Big(\delta'(\sigma-\bar{\sigma}) - \partial_{\bar{\sigma}}\delta(\sigma-\bar{\sigma})\Big) \\ &= \frac{\kappa}{2}\Big(\lambda_{1\mu}\partial_\nu\hat{\xi}^\mu_2 - \partial_\nu\lambda_{1\mu}\hat{\xi}^\mu_2\Big)x'^\nu\delta(\sigma-\bar{\sigma}) \\ &\quad +\frac{\kappa}{2}\Big(\lambda_{1\mu}(\sigma)\hat{\xi}^\mu_2(\sigma)\delta'(\sigma-\bar{\sigma}) - \lambda_{1\mu}(\bar{\sigma})\hat{\xi}^\mu_2(\bar{\sigma})\partial_{\bar{\sigma}}\delta(\sigma-\bar{\sigma})\Big)\,. \end{aligned}\tag{10.17}$$

Now the anomalous part goes to zero after the integration over both $\sigma$ and $\bar{\sigma}$, while the non-anomalous part can be transformed with the help of (10.7). After relabeling some dummy indices, the expressions for resulting symmetry parameters are

$$\begin{aligned}
\hat{\xi}^{\mu} \quad = \quad & \hat{\xi}_1^{\nu}\partial_{\nu}\hat{\xi}_2^{\mu} - \hat{\xi}_2^{\nu}\partial_{\nu}\hat{\xi}_1^{\mu} + \\
& +\kappa\theta^{\mu\nu}\Big(\hat{\xi}_1^{\rho}(\partial_{\rho}\lambda_{2\nu} - \partial_{\nu}\lambda_{2\rho}) - \hat{\xi}_2^{\rho}(\partial_{\rho}\lambda_{1\nu} - \partial_{\nu}\lambda_{1\rho}) + \frac{1}{2}\partial_{\nu}(\hat{\xi}_1\lambda_2 - \hat{\xi}_2\lambda_1)\Big) \\
& +\kappa\hat{\xi}_1^{\nu}\partial_{\nu}(\lambda_{2\rho}\theta^{\rho\mu}) - \kappa\hat{\xi}_2^{\nu}\partial_{\nu}(\lambda_{1\rho}\theta^{\rho\mu}) + \kappa(\lambda_{1\nu}\theta^{\nu\rho})\partial_{\rho}\hat{\xi}_2^{\mu} - \kappa(\lambda_{2\nu}\theta^{\nu\rho})\partial_{\rho}\hat{\xi}_1^{\mu} \\
& +\kappa^2 R^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\,, \\
\lambda_{\mu} \quad = \quad & \hat{\xi}_1^{\nu}(\partial_{\nu}\lambda_{2\mu} - \partial_{\mu}\lambda_{2\nu}) - \hat{\xi}_2^{\nu}(\partial_{\nu}\lambda_{1\mu} - \partial_{\mu}\lambda_{1\nu}) + \frac{1}{2}\partial_{\mu}(\hat{\xi}_1\lambda_2 - \hat{\xi}_2\lambda_1) \\
& +\kappa\theta^{\nu\rho}(\lambda_{1\nu}\partial_{\rho}\lambda_{2\mu} - \lambda_{2\nu}\partial_{\rho}\lambda_{1\mu}) + \kappa\lambda_{1\rho}\lambda_{2\nu}Q_{\mu}^{\ \rho\nu}\,.
\end{aligned} \tag{10.18}$$

In the coordinate free notation, the above expressions read

$$\begin{aligned}
\hat{\xi} =& [\hat{\xi}_1, \hat{\xi}_2]_L - \kappa[\hat{\xi}_2, \lambda_1\theta]_L + \kappa[\hat{\xi}_1, \lambda_2\theta]_L + \frac{\kappa^2}{2}[\theta,\theta]_S(\lambda_1, \lambda_2, .) \\
& - \kappa\theta\Big(\mathcal{L}_{\hat{\xi}_2}\lambda_1 - \mathcal{L}_{\hat{\xi}_1}\lambda_2 + \frac{1}{2}d(i_{\hat{\xi}_1}\lambda_2 - i_{\hat{\xi}_2}\lambda_1)\Big) \\
\lambda =& \mathcal{L}_{\hat{\xi}_1}\lambda_2 - \mathcal{L}_{\hat{\xi}_2}\lambda_1 - \frac{1}{2}d(i_{\hat{\xi}_1}\lambda_2 - i_{\hat{\xi}_2}\lambda_1) + \kappa[\lambda_1, \lambda_2]_{\theta}\,,
\end{aligned} \tag{10.19}$$

where $[\theta,\theta]_S$ represents the Schouten-Nijenhuis bracket (4.25) [47], and $[\lambda_1, \lambda_2]_{\theta}$ is the Koszul bracket (5.11) [53], and $\theta(\lambda)$ is defined as in (5.9).

## 10.3 Courant algebroid

The obtained bracket is the $\theta$-twisted Courant bracket, given by

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_\theta} = e^{\hat{\theta}}[e^{-\hat{\theta}}\Lambda_1, e^{-\hat{\theta}}\Lambda_2]_{\mathcal{C}}\,. \tag{10.20}$$

It defines the Courant algebroid with the following anchor

$$\rho^{(\hat{\theta})}(\Lambda) = \xi^{\mu} - \kappa\theta^{\mu\nu}\lambda_{\nu}\,, \tag{10.21}$$

from which we easily obtain the differential operator

$$D^{(\hat{\theta})}f = \begin{pmatrix} \kappa\theta^{\mu\nu}\partial_{\nu}f \\ \partial_{\mu}f \end{pmatrix} = \begin{pmatrix} d_{\theta}f \\ df \end{pmatrix}\,, \tag{10.22}$$

where $d_\theta$ (5.20) is the exterior derivative corresponding to the Koszul bracket. The structure $\Big(T\mathcal{M}\oplus T^\star\mathcal{M}, [,]_{\mathcal{C}_\theta}, \langle,\rangle, \rho^{(\theta)}\Big)$ is a Courant algebroid, with all compatibility conditions satisfied.

To obtain the Dirac structures, we firstly substitute the graph of a 2-form into the definition of the symmetry generator (10.10)

$$
\begin{aligned}
\mathcal{G}^{\hat\theta}_{\mathcal{V}_{B}(\Lambda)} &= \int d\sigma \Big( \hat\xi^\mu \pi_\mu + 2B_{\mu\nu}\hat\xi^\nu \kappa\theta^{\mu\rho}\pi_\rho + 2B_{\mu\nu}\kappa x'^\mu \hat\xi^\nu \Big) \\
&= \int d\sigma \Big( \hat\xi^\nu (\delta^\mu_\nu - 2\kappa B_{\nu\rho}\theta^{\rho\mu})\pi_\mu + 2B_{\mu\nu}\kappa x'^\nu \hat\xi^\nu \Big) .
\end{aligned}
\tag{10.23}
$$

Next, we can use the following identity

$$
\delta^\mu_\nu = 2\kappa\theta^{\mu\rho}B_{\rho\nu} + (G_E^{-1})^{\mu\rho}G_{\rho\nu} \,, \tag{10.24}
$$

which is easily obtained from (2.36) and (3.16). The generator (10.23) can be rewritten in canonical form as

$$
\mathcal{G}^{\hat\theta}_{\mathcal{V}_B(\Lambda)} = \int d\sigma \Big( \tilde\xi^\mu \pi_\mu + \tilde\lambda_\mu \kappa x'^\mu \Big) = \mathcal{G}_{\tilde\Lambda} \,, \quad \tilde\xi^\mu = \hat\xi^\mu - 2\kappa\theta^{\mu\nu}B_{\nu\rho}\hat\xi^\rho \,,\; \tilde\lambda_\mu = 2(BG^{-1}G_E)_{\mu\nu}\tilde\xi^\nu \,. \tag{10.25}
$$

The generator written like this will give rise to the Courant bracket. We can use the results from previous chapters, namely (6.34), to establish that the condition for $\mathcal{V}_B$ to be Dirac structure is given by

$$
d(BG^{-1}G_E) = dB - 4d(BG^{-1}BG^{-1}B) = 0 \,. \tag{10.26}
$$

On the other hand, when we substitute the graph of the bi-vector into (10.10), the generator becomes

$$
\mathcal{G}^{\hat\theta}_{\mathcal{V}_\theta(\Lambda)} = \int d\sigma \kappa x'^\mu \lambda_\mu \,, \tag{10.27}
$$

which does not depend on canonical momenta, and hence $\mathcal{V}_\theta$ is always Dirac structure. Moreover, we have

$$
[\mathcal{V}_\theta(\Lambda_1), \mathcal{V}_\theta(\Lambda_2)]_{\mathcal{C}_\theta} = 0 \,. \tag{10.28}
$$

This reflects the basic asymmetry of the Courant bracket in the way how it treats vectors and 1-forms. On vectors, it reduces to the Lie bracket, which happens when the $B$-twisted Courant bracket is considered on the subspace $\mathcal{V}_B$. On 1-forms, the Courant bracket is zero. The $\theta$-twisted Courant bracket becomes precisely the Courant bracket of 1-forms on $\mathcal{V}_\theta$.

## 10.4 Relation to $B$-twisted Courant bracket via self T-duality

Suppose we want to implement T-duality within the same phase space, without adding any new $D$ coordinates or their corresponding momenta. Such a transformation should swap the momenta $\pi_\mu$

with the coordinate $\sigma$-derivatives, as well as the background fields with their T-dual counterparts, i.e.

$$\pi_\mu \leftrightarrow \kappa x'^\mu \,, \quad 2B_{\mu\nu} \leftrightarrow \kappa\theta^{\mu\nu} \,. \tag{10.29}$$

We will use the term "self T-duality" to describe this concept. Under such transformation, auxiliary currents $i_\mu$ (9.5) and $k^\mu$ (10.5) transform one into another

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu} x'^\nu \leftrightarrow \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu = k^\mu \,. \tag{10.30}$$

We can conclude that the generators (9.7) and (10.10) are also related by self T-duality, and so is their algebra. This means that $B$-twisted Courant bracket and $\theta$-twisted Courant bracket are mutually related by self T-duality. The Courant algebroid with $H$-flux under the exchange of mutually T-dual variables becomes the Courant algebroid with $Q$ and $R$ fluxes.

The advantage of considering the T-duality and all generators in the same phase space is that we can easily express one in terms of the other by coordinate transformation. Inverting relations (9.8) and substituting it into (10.11), we obtain the following relation between the parameters

$$\hat{\Lambda}^M_{(\theta)} = (e^{\hat{\theta}} e^{-\hat{B}})^M_{\ N} \, \hat{\Lambda}^N_{(B)} \,, \tag{10.31}$$

where in order to differentiate between parameters (9.8) and (10.11), we added indices $B$ and $\theta$. We have obtained the isomorphism between these Courant algebroids

$$\varphi = (e^{\hat{\theta}} e^{-\hat{B}})^M_{\ N} = \begin{pmatrix} \delta^\mu_\nu - 2\kappa(\theta B)^\mu_{\ \nu} & \kappa\theta^{\mu\nu} \\ -2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} . \tag{10.32}$$

The isomorphism $\varphi$ satisfies the first rule of (6.27), simply from the fact that the inner product is invariant under the $O(D,D)$ transformations. The second property can be also easily verified. Firstly, from relation (9.10) we can derive

$$\varphi[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = e^{\hat{\theta}}[e^{-\hat{B}}\Lambda_1, e^{-\hat{B}}\Lambda_2]_{\mathcal{C}} \,, \tag{10.33}$$

and from (10.20) we obtain

$$[\varphi(\Lambda_1), \varphi(\Lambda_2)]_{\mathcal{C}_\theta} = e^{\hat{\theta}}[e^{-\hat{B}}\Lambda_1, e^{-\hat{B}}\Lambda_2]_{\mathcal{C}} \,, \tag{10.34}$$

and therefore we obtain the second requirement in (6.27)

$$\varphi[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = \Big([\varphi(\Lambda_1), \varphi(\Lambda_2)]_{\mathcal{C}_\theta}\Big) \,. \tag{10.35}$$

We demonstrated that the Courant algebroid relations that govern symmetry generator algebra in both the initial and self T-dual picture are isomorphic. This isomorphism is governed by the same properties (6.27) as the isomorphism that governs topological T-duality on backgrounds defined on tori [63]. As a result, we extended the idea of T-duality as a Courant algebroid isomorphism to include symmetry transformations as well.

# Chapter 11

# $B$-$\theta$ twisted Courant bracket

The focus of this chapter is on creating a Courant bracket that is twisted by both $B$ and $\theta$ fields. The first step is to construct a twisting matrix that includes both fields and is self T-dual. Next, we derive the necessary algebraic relations to obtain the complete bracket, along with all the corresponding generalized fluxes. We then explain how these fluxes can be interpreted in terms of deformations of Lie algebroids.

## 11.1 Twisting transformation

The $B$-twisted Courant bracket is obtained with the action of $e^{\hat{B}}$, while the $\theta$-twisted Courant bracket is obtained with the action of $e^{\hat{\theta}}$ transformation on the basis of a double generator. Initially, one might think to obtain the twist by $B$ and $\theta$ by using the product of transformations $e^{\hat{B}}$ and $e^{\hat{\theta}}$. This would indeed give rise to the twisted Courant bracket since the composition of two group elements remains in that group. However, the two transformations do not commute, i.e.

$$e^{\hat{B}}e^{\hat{\theta}} = \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \kappa\theta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \kappa\theta \\ 2B & 1+2\kappa B\theta \end{pmatrix} , \tag{11.1}$$

$$e^{\hat{\theta}}e^{\hat{B}} = \begin{pmatrix} 1 & \kappa\theta \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix} = \begin{pmatrix} 1+2\kappa\theta B & \kappa\theta \\ 2B & 1 \end{pmatrix} . \tag{11.2}$$

The Courant bracket twisted with the transformation (11.1) yields the Roytenberg bracket. This is an extension of the Courant bracket which includes all fluxes and has been derived several times [60, 73, 74]. However, it is unclear why that bracket is to be preferred to the Courant bracket twisted with the transformation (11.2). Moreover, neither of the brackets is invariant under T-duality. Our goal is to twist the bracket in a way that treats $B$ and $\theta$ equally and maintains the T-dual invariance of the bracket. This we will refer to as the simultaneous twisting of the Courant bracket by both $B$ and $\theta$ [3]. Let us define $\breve{B}$ by

$$\breve{B} = \hat{B} + \hat{\theta} = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 2B_{\mu\nu} & 0 \end{pmatrix} . \tag{11.3}$$

By construction, $\breve{B}$ is invariant under self T-duality and it treats the Kalb-Ramond field $B$ and the non-commutativity parameter $\theta$ equally. Therefore, we defined the Courant bracket twisted at the same time by a 2-form $B$ and by a bi-vector $\theta$ by

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_{\breve{B}}} = e^{\breve{B}}[e^{-\breve{B}}\Lambda_1, e^{-\breve{B}}\Lambda_2]_{\mathcal{C}} . \tag{11.4}$$

It is not as straightforward to derive the formula for the matrix $e^{\breve{B}}$ as it was in the previous cases. The reason for this is that while the matrices $\hat{B}$ and $\hat{\theta}$ have a squared value of zero, the same is not true for the matrix $\breve{B}$. Hence, the full Taylor expression

$$e^{\breve{B}} = \sum_{n=0}^{\infty} \frac{\breve{B}^n}{n!} \tag{11.5}$$

has to be obtained. The square of the matrix $\breve{B}$ is given by

$$\breve{B}^2 = \begin{pmatrix} 2\kappa(\theta B)^\mu{}_\nu & 0 \\ 0 & 2\kappa(B\theta)_\mu{}^\nu \end{pmatrix} = \begin{pmatrix} \alpha^\mu{}_\nu & 0 \\ 0 & (\alpha^T)_\mu{}^\nu \end{pmatrix} , \tag{11.6}$$

while its cube is given by

$$\breve{B}^3 = \begin{pmatrix} 0 & 2\kappa^2(\theta B\theta)^{\mu\nu} \\ 4\kappa(B\theta B)_{\mu\nu} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \kappa\alpha^\mu{}_\rho\theta^{\rho\nu} \\ 2B_{\mu\rho}\alpha^\rho{}_\nu & 0 \end{pmatrix} , \tag{11.7}$$

where we have marked

$$\alpha^\mu{}_\nu = 2\kappa\theta^{\mu\rho}B_{\rho\nu} . \tag{11.8}$$

The matrix $\alpha$ is defined for pure convenience, and it possesses a couple of useful properties. Firstly, it is a symmetric matrix

$$\alpha^\mu{}_\nu = 2\kappa\theta^{\mu\rho}B_{\rho\nu} = 2\kappa(-\theta^{\rho\mu})(-B_{\nu\rho}) = 2\kappa B_{\nu\rho}\theta^{\rho\mu} = (\alpha^T)_\nu{}^\mu , \tag{11.9}$$

and it transforms into its transpose under the self T-duality relations (10.29)

$$\alpha^\mu{}_\nu \leftrightarrow (\alpha^T)_\nu{}^\mu . \tag{11.10}$$

Moreover, we have

$$\begin{aligned}\theta^{\mu\rho}(\alpha^T)_\rho^{\ \nu} &= 2\kappa\theta^{\mu\rho}B_{\rho\sigma}\theta^{\sigma\nu} = \alpha^\mu_{\ \sigma}\theta^{\sigma\nu} \\ B_{\mu\rho}\alpha^\rho_{\ \nu} &= 2\kappa B_{\mu\rho}\theta^{\rho\sigma}B_{\sigma\nu} = (\alpha^T)_\mu^{\ \sigma}B_{\sigma\nu}\,.\end{aligned} \tag{11.11}$$

One easily observes the regularity that degrees of $\breve{B}$ possess. The even degrees are given by

$$\breve{B}^{2n} = \begin{pmatrix}(\alpha^n)^\mu_{\ \nu} & 0 \\ 0 & ((\alpha^T)^n)_\mu^{\ \nu}\end{pmatrix}\,, \tag{11.12}$$

while the odd degrees by

$$\breve{B}^{2n+1} = \begin{pmatrix}0 & \kappa(\alpha^n\theta)^{\mu\nu} \\ 2(B\alpha^n)_{\mu\nu} & 0\end{pmatrix}\,, \tag{11.13}$$

where we applied (11.11). We can now substitute (11.12) and (11.13) into (11.5), and write

$$e^{\breve{B}} = \begin{pmatrix}\Big(\sum_{n=0}^\infty \frac{\alpha^n}{(2n)!}\Big)^\mu_{\ \nu} & \kappa\Big(\sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\Big)^\mu_{\ \rho}\theta^{\rho\nu} \\ 2B_{\mu\rho}\Big(\sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\Big)^\rho_{\ \nu} & \Big(\sum_{n=0}^\infty \frac{(\alpha^T)^n}{(2n)!}\Big)_\mu^{\ \nu}\end{pmatrix}\,. \tag{11.14}$$

The terms in the twisting matrix can be simplified using the Taylor expressions for hyperbolic functions

$$\cosh\alpha = \sum_{n=0}^\infty \frac{\alpha^{2n}}{(2n)!}\,, \quad \sinh\alpha = \sum_{n=0}^\infty \frac{\alpha^{2n+1}}{(2n+1)!}\,, \tag{11.15}$$

from which we derive

$$\cosh\sqrt{\alpha} = \sum_{n=0}^\infty \frac{\alpha^n}{(2n)!}\,, \quad \frac{\sinh\sqrt{\alpha}}{\sqrt{\alpha}} = \sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\,. \tag{11.16}$$

For our convenience let us introduce $\mathcal{S}^\mu_{\ \nu}$ and $\mathcal{C}^\mu_{\ \nu}$ by

$$\mathcal{S}^\mu_{\ \nu} = \Big(\frac{\sinh\sqrt{\alpha}}{\sqrt{\alpha}}\Big)^\mu_{\ \nu}\,, \quad \mathcal{C}^\mu_{\ \nu} = \Big(\cosh\sqrt{\alpha}\Big)^\mu_{\ \nu}\,. \tag{11.17}$$

We can now rewrite the full transformation matrix $e^{\breve{B}}$ as

$$e^{\breve{B}} = \begin{pmatrix}\mathcal{C}^\mu_{\ \nu} & \kappa\mathcal{S}^\mu_{\ \rho}\theta^{\rho\nu} \\ 2B_{\mu\rho}\mathcal{S}^\rho_{\ \nu} & (\mathcal{C}^T)_\mu^{\ \nu}\end{pmatrix}\,. \tag{11.18}$$

Due to (11.9), the hyperbolic functions (11.17) are symmetric

$$\mathcal{S}^\mu_{\ \nu} = (\mathcal{S}^T)_\nu^{\ \mu}\,, \quad \mathcal{C}^\mu_{\ \nu} = (\mathcal{C}^T)_\nu^{\ \mu}\,. \tag{11.19}$$

Secondly, the relation (11.11) is easily generalized to higher orders of $\alpha$, from which we obtain

$$\mathcal{S}^\mu{}_\rho \theta^{\rho\nu} = \theta^{\mu\rho} (\mathcal{S}^T)_\rho{}^\nu \,, \quad \mathcal{C}^\mu{}_\rho \theta^{\rho\nu} = \theta^{\mu\rho} (\mathcal{C}^T)_\rho{}^\nu \,, \qquad (11.20)$$
$$B_{\mu\rho} \mathcal{S}^\rho{}_\nu = (\mathcal{S}^T)_\mu{}^\rho B_{\rho\nu} \,, \quad B_{\mu\rho} \mathcal{C}^\rho{}_\nu = (\mathcal{C}^T)_\mu{}^\rho B_{\rho\nu} \,.$$

Thirdly, the well-known hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$ can also be expressed in terms of newly defined tensors by

$$(\mathcal{C}^2)^\mu{}_\nu - \alpha^\mu{}_\rho (\mathcal{S}^2)^\rho{}_\nu = \delta^\mu_\nu \,. \qquad (11.21)$$

Lastly, from (11.10) we conclude

$$\mathcal{C} \leftrightarrow \mathcal{C}^T \,, \;\; \mathcal{S} \leftrightarrow \mathcal{S}^T \,. \qquad (11.22)$$

The transformation has been obtained, but we need to check whether it is an $O(D,D)$ transformation, and therefore suitable for twisting the Courant bracket and defining the Courant algebroid. The transpose of $e^{\breve{B}}$ is given by

$$(e^{\breve{B}})^T = \begin{pmatrix} (\mathcal{C}^T)_\nu{}^\mu & -2B_{\mu\rho}\mathcal{S}^\rho{}_\nu \\ -\kappa \mathcal{S}^\mu{}_\rho \theta^{\rho\nu} & \mathcal{C}^\mu{}_\nu \end{pmatrix} \,, \qquad (11.23)$$

and therefore it can be easily verified that

$$\begin{aligned} (e^{\breve{B}})^T \, \eta \, e^{\breve{B}} &= \begin{pmatrix} \mathcal{C}^T & -2B\mathcal{S} \\ -\kappa\mathcal{S}\theta & \mathcal{C} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{C} & \kappa\mathcal{S}\theta \\ 2B\mathcal{S} & \mathcal{C}^T \end{pmatrix} \\ &= \begin{pmatrix} 2\mathcal{C}^T B\mathcal{S} - 2B\mathcal{S}\mathcal{C} & (\mathcal{C}^T)^2 - 2\kappa B\mathcal{S}^2\theta \\ -2\kappa\mathcal{S}\theta B\mathcal{S} + \mathcal{C}^2 & -\kappa\mathcal{S}\theta\mathcal{C}^T + \kappa\mathcal{C}\mathcal{S}\theta \end{pmatrix} = \eta \,, \end{aligned} \qquad (11.24)$$

where $\eta$ is $O(D,D)$ invariant metric (6.4), and in the last step the properties (11.20) and (11.21) were used. The determinant of $e^{\breve{B}}$ is given by

$$\det(e^{\breve{B}}) = e^{Tr(\breve{B})} = 1 \,, \qquad (11.25)$$

and its inverse by

$$e^{-\breve{B}} = \begin{pmatrix} \mathcal{C}^\mu{}_\nu & -\kappa\mathcal{S}^\mu{}_\rho \theta^{\rho\nu} \\ -2B_{\mu\rho}\mathcal{S}^\rho{}_\nu & (\mathcal{C}^T)_\mu{}^\nu \end{pmatrix} \,, \qquad (11.26)$$

which is in the accordance with (B.4).

## 11.2 Self T-dual generator

Having constructed the $O(D,D)$-invariant self T-dual twisting transformation with both $B$ and $\theta$, we can move forward with the approach outlined in Chapter 8 to derive the corresponding twisted Courant bracket using the generator algebra. The Poisson bracket representation of the Courant bracket twisted by $B$ and $\theta$ can be obtained from the generator written in the basis

$$\breve{X}^M = (e^{\breve{B}})^M_{\ N}\, X^N = \begin{pmatrix} \breve{k}^\mu \\ \breve{\imath}_\mu \end{pmatrix} , \tag{11.27}$$

where the new currents $\breve{\imath}_\mu$ and $\breve{k}^\mu$ are given by

$$\begin{aligned} \breve{k}^\mu &= \kappa \mathcal{C}^\mu_{\ \nu} x'^\nu + \kappa (\mathcal{S}\theta)^{\mu\nu} \pi_\nu \,, \\ \breve{\imath}_\mu &= 2(B\mathcal{S})_{\mu\nu} x'^\nu + (\mathcal{C}^T)_\mu^{\ \nu} \pi_\nu \,. \end{aligned} \tag{11.28}$$

These currents are mutually related by self T-duality (10.29), implying that the generator is invariant under self T-duality. Using (11.26), the relations (11.28) can easily be inverted

$$\begin{aligned} \kappa x'^\mu &= \mathcal{C}^\mu_{\ \nu} \breve{k}^\nu - \kappa (\mathcal{S}\theta)^{\mu\nu} \breve{\imath}_\nu \,, \\ \pi_\mu &= -2(B\mathcal{S})_{\mu\nu} \breve{k}^\nu + (\mathcal{C}^T)_\mu^{\ \nu} \breve{\imath}_\nu \,. \end{aligned} \tag{11.29}$$

The double generator is given by

$$\breve{\mathcal{G}}_{\breve{\Lambda}} = \int d\sigma \langle \breve{X}, \breve{\Lambda} \rangle \,, \tag{11.30}$$

and is equal to the generator $\mathcal{G}_\Lambda$ (7.33) for

$$\breve{\Lambda}^M = (e^{\breve{B}})^M_{\ N}\, \Lambda^N = \begin{pmatrix} \breve{\xi}^\mu \\ \breve{\lambda}_\mu \end{pmatrix} , \tag{11.31}$$

where

$$\begin{aligned} \breve{\xi}^\mu &= \mathcal{C}^\mu_{\ \nu} \xi^\nu + \kappa (\mathcal{S}\theta)^{\mu\nu} \lambda_\nu \,, \\ \breve{\lambda}_\mu &= 2(B\mathcal{S})_{\mu\nu} \xi^\nu + (\mathcal{C}^T)_\mu^{\ \nu} \lambda_\nu \,. \end{aligned} \tag{11.32}$$

We could proceed by directly computing the Poisson bracket between the generators $\breve{\mathcal{G}}_{\breve{\Lambda}}$. However, the interpretation of terms is easier if an auxiliary basis is introduced by

$$\mathring{X}^M = \begin{pmatrix} \mathcal{C}^\mu_{\ \nu} & 0 \\ 0 & ((\mathcal{C}^{-1})^T)_\mu^{\ \nu} \end{pmatrix} \begin{pmatrix} \breve{k}^\nu \\ \breve{\imath}_\nu \end{pmatrix} = \begin{pmatrix} \mathring{k}^\mu \\ \mathring{\imath}_\mu \end{pmatrix} , \tag{11.33}$$

where the auxiliary currents are

$$\begin{aligned} \mathring{k}^\mu &= \kappa x'^\mu + \kappa \mathring{\theta}^{\mu\nu} \mathring{\imath}_\nu \,, \\ \mathring{\imath}_\mu &= \pi_\mu + 2\kappa \mathring{B}_{\mu\nu} x'^\nu \,, \end{aligned} \tag{11.34}$$

and auxiliary fields are

$$\mathring{B}_{\mu\nu} = B_{\mu\rho}\mathcal{S}^\rho{}_\sigma(\mathcal{C}^{-1})^\sigma{}_\nu \,, \tag{11.35}$$

and

$$\mathring{\theta}^{\mu\nu} = \mathcal{C}^\mu{}_\rho\mathcal{S}^\rho{}_\sigma\theta^{\sigma\nu} \,. \tag{11.36}$$

The hyperbolic functions of the background fields are incorporated into the auxiliary background fields in the auxiliary basis. Additionally, the auxiliary currents (11.34) take the same form as the currents that generate the Roytenberg bracket [60], although they depend on a different set of fields. We can easily invert the relation (11.33), and obtain

$$\breve{k}^\mu = \mathcal{C}^\mu{}_\nu\mathring{k}^\mu \,, \quad \breve{\imath}_\mu = \mathring{\imath}_\nu\mathcal{C}^\nu{}_\mu \,. \tag{11.37}$$

Moreover, the coordinate $\sigma$-derivative is as easily expressed in terms of auxiliary currents by

$$\kappa x'^\mu = \mathring{k}^\mu - \kappa\mathring{\theta}^{\mu\nu}\mathring{\imath}_\nu \,, \tag{11.38}$$

all of which simplifies computations substantially.

We will find the algebra of the auxiliary currents with auxiliary fluxes as its structure functions. Using these relations and equation (11.37), we will then determine the necessary algebraic relations for the twisted Courant bracket with both $B$ and $\theta$ backgrounds.

## 11.3 Algebra of auxiliary currents

The Poisson bracket algebra of the auxiliary currents $\mathring{\imath}_\mu$ is given by

$$\{\mathring{\imath}_\mu(\sigma), \mathring{\imath}_\nu(\bar{\sigma})\} = -2\mathring{B}_{\mu\nu\rho}\mathring{k}^\rho\delta(\sigma-\bar{\sigma}) - \mathring{\mathcal{F}}_{\mu\nu}{}^\rho\,\mathring{\imath}_\rho\delta(\sigma-\bar{\sigma}) \,, \tag{11.39}$$

where

$$\mathring{B}_{\mu\nu\rho} = \partial_\mu\mathring{B}_{\nu\rho} + \partial_\nu\mathring{B}_{\rho\mu} + \partial_\rho\mathring{B}_{\mu\nu} \,, \tag{11.40}$$

and

$$\mathring{\mathcal{F}}_{\mu\nu}{}^\rho = -2\kappa\mathring{B}_{\mu\nu\sigma}\mathring{\theta}^{\sigma\rho} \,. \tag{11.41}$$

The algebra of currents $\mathring{k}^\mu$ is given by

$$\{\mathring{k}^\mu(\sigma), \mathring{k}^\nu(\bar{\sigma})\} = -\kappa\mathring{\mathcal{Q}}_\rho{}^{\mu\nu}\mathring{k}^\rho\delta(\sigma-\bar{\sigma}) - \kappa^2\mathring{\mathcal{R}}^{\mu\nu\rho}\mathring{\imath}_\rho\delta(\sigma-\bar{\sigma}) \,, \tag{11.42}$$

where

$$\mathring{\mathcal{Q}}_\mu{}^{\nu\rho} = \mathring{Q}_\mu{}^{\nu\rho} + 2\kappa\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\rho\tau}\mathring{B}_{\mu\sigma\tau} \,, \quad \mathring{Q}_\mu{}^{\nu\rho} = \partial_\mu\mathring{\theta}^{\nu\rho} \,, \tag{11.43}$$

and

$$\mathring{\mathcal{R}}^{\mu\nu\rho} = \mathring{R}^{\mu\nu\rho} + 2\kappa\mathring{\theta}^{\mu\lambda}\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\rho\tau}\mathring{B}_{\lambda\sigma\tau}\,, \quad \mathring{R}^{\mu\nu\rho} = \mathring{\theta}^{\mu\sigma}\partial_\sigma\mathring{\theta}^{\nu\rho} + \mathring{\theta}^{\nu\sigma}\partial_\sigma\mathring{\theta}^{\rho\mu} + \mathring{\theta}^{\rho\sigma}\partial_\sigma\mathring{\theta}^{\mu\nu}\,. \tag{11.44}$$

The remaining algebra of currents $\mathring{k}^\mu$ and $\mathring{\imath}_\mu$ can be as easily obtained

$$\{\mathring{\imath}_\mu(\sigma), \mathring{k}^\nu(\bar{\sigma})\} = \kappa\delta^\nu_\mu\delta'(\sigma-\bar{\sigma}) + \mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu}\,\mathring{k}^\rho\delta(\sigma-\bar{\sigma}) - \kappa\mathring{\mathcal{Q}}_\mu^{\ \nu\rho}\mathring{\imath}_\rho\delta(\sigma-\bar{\sigma})\,. \tag{11.45}$$

These algebra relations can be summarized in the double formalism as

$$\{\mathring{X}^M, \mathring{X}^N\} = -\mathring{F}^{MN}_{\ \ \ \ P}\,\mathring{X}^P\delta(\sigma-\bar{\sigma}) + \kappa\eta^{MN}\delta'(\sigma-\bar{\sigma})\,, \tag{11.46}$$

with

$$F^{MN\rho} = \begin{pmatrix} \kappa^2\mathring{\mathcal{R}}^{\mu\nu\rho} & -\kappa\mathring{\mathcal{Q}}_\nu^{\ \mu\rho} \\ \kappa\mathring{\mathcal{Q}}_\mu^{\ \nu\rho} & \mathring{\mathcal{F}}_{\mu\nu}^{\ \ \rho} \end{pmatrix}\,, \qquad F^{MN}{}_\rho = \begin{pmatrix} \kappa\mathring{\mathcal{Q}}_\rho^{\ \mu\nu} & \mathring{\mathcal{F}}_{\nu\rho}^{\ \ \mu} \\ -\mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu} & 2\mathring{B}_{\mu\nu\rho} \end{pmatrix}\,. \tag{11.47}$$

The terms appearing are generalized fluxes [20, 21, 22]. Now we can use these relations to obtain the fluxes related to the Courant bracket twisted with $B$ and $\theta$.

## 11.4 Fluxes of self T-dual currents

The algebra of auxiliary currents closes with auxiliary fluxes as its structure coefficients. We can now use the expression for self T-dual currents $\breve{k}$ and $\breve{\imath}$ in terms of their auxiliary counterparts (11.37) to compute the fluxes relevant for the Courant bracket twisted with both $B$ and $\theta$. Besides the relations (11.46), we will require the algebra relations in the form

$$\{\mathcal{C}^\mu_{\ \rho}(\sigma), \mathring{\imath}_\nu(\bar{\sigma})\} = \partial_\nu\mathcal{C}^\mu_{\ \rho}\delta(\sigma-\bar{\sigma})\,, \quad \{\mathcal{C}^\mu_{\ \rho}(\sigma), \mathring{k}^\nu(\bar{\sigma})\} = \kappa\mathring{\theta}^{\nu\sigma}\partial_\sigma\mathcal{C}^\mu_{\ \rho}\delta(\sigma-\bar{\sigma})\,, \tag{11.48}$$

and similar identities for algebra between variables that do not depend on momenta with currents that form the self T-dual basis.

Firstly, we write

$$\begin{aligned}
\{\breve{\imath}_\mu(\sigma), \breve{\imath}_\nu(\bar{\sigma})\} &= \mathcal{C}^\rho_{\ \mu}\mathcal{C}^\sigma_{\ \nu}\{\mathring{\imath}_\rho, \mathring{\imath}_\sigma\} + \mathcal{C}^\rho_{\ \mu}\mathring{\imath}_\sigma\{\mathring{\imath}_\rho, \mathcal{C}^\sigma_{\ \nu}\} + \mathring{\imath}_\rho\mathcal{C}^\sigma_{\ \nu}\{\mathcal{C}^\rho_{\ \mu}, \mathring{\imath}_\sigma\} \\
&= -2\mathcal{C}^\rho_{\ \mu}\mathcal{C}^\sigma_{\ \nu}\mathring{B}_{\rho\sigma\alpha}\mathring{k}^\alpha\delta(\sigma-\bar{\sigma}) - \mathcal{C}^\rho_{\ \mu}\mathcal{C}^\sigma_{\ \nu}\mathring{\mathcal{F}}_{\rho\sigma}^{\ \ \alpha}\,\mathring{\imath}_\alpha\delta(\sigma-\bar{\sigma}) \\
&\quad -\Big(\mathcal{C}^\rho_{\ \mu}\partial_\rho\mathcal{C}^\sigma_{\ \nu} - \mathcal{C}^\rho_{\ \nu}\partial_\rho\mathcal{C}^\sigma_{\ \mu}\Big)\mathring{\imath}_\sigma\delta(\sigma-\bar{\sigma}) \\
&= -2\breve{\mathcal{B}}_{\mu\nu\rho}\breve{k}^\rho\delta(\sigma-\bar{\sigma}) - \breve{\mathcal{F}}_{\mu\nu}^{\ \ \rho}\,\breve{\imath}_\rho\delta(\sigma-\bar{\sigma})\,,
\end{aligned} \tag{11.49}$$

where we firstly used (11.37), after which we substituted (11.39) and (11.48). The resulting algebra we expressed in terms of $\breve{\mathcal{B}}$ flux, given by

$$\breve{\mathcal{B}}_{\mu\nu\rho} = \mathcal{C}^{\alpha}_{\ \mu}\mathcal{C}^{\beta}_{\ \nu}\mathcal{C}^{\gamma}_{\ \rho}\mathring{B}_{\alpha\beta\gamma}\,, \tag{11.50}$$

and $\breve{\mathcal{F}}$ flux, given by

$$\breve{\mathcal{F}}_{\mu\nu}^{\ \ \rho} = \mathring{\mathcal{F}}_{\alpha\beta}^{\ \ \gamma}\mathcal{C}^{\alpha}_{\ \mu}\mathcal{C}^{\beta}_{\ \nu}(\mathcal{C}^{-1})^{\rho}_{\ \gamma} + (\mathcal{C}^{-1})^{\rho}_{\ \tau}\Big(\mathcal{C}^{\sigma}_{\ \mu}\partial_\sigma\mathcal{C}^{\tau}_{\ \nu} - \mathcal{C}^{\sigma}_{\ \nu}\partial_\sigma\mathcal{C}^{\tau}_{\ \mu}\Big)\,. \tag{11.51}$$

In order to simplify the expression for $\breve{\mathcal{F}}$ flux, let us introduce new sets of derivatives by

$$\hat{\partial}_\mu = (\mathcal{C}^T)_\mu^{\ \nu}\partial_\nu\,. \tag{11.52}$$

Here it is worth noting that in general there is no coordinate system $\hat{x}^\mu$ so that $\hat{\partial}_\mu$ are well-defined partial derivatives in that system. This would only be true if the matrix $\mathcal{C}^T$ defined the diffeomorphisms, i.e., if $(\mathcal{C}^T)_\mu^{\ \nu} = \frac{\partial x^\mu}{\partial \hat{x}^\nu}$, which would imply $\hat{\partial}_\mu(\mathcal{C}^T)_\nu^{\ \rho} - \hat{\partial}_\nu(\mathcal{C}^T)_\mu^{\ \rho} = 0$. However, the matrix $\mathcal{C}$ is defined in terms of the string fields $B$ and $\theta$, and this relation does not hold.

Furthermore, we introduce a new non-commutative field $\breve{\theta}$, which is given by

$$\breve{\theta}^{\mu\nu} = (\mathcal{S}\mathcal{C}^{-1})^{\mu}_{\ \rho}\theta^{\rho\nu} = (C^{-2})^{\mu}_{\ \rho}\mathring{\theta}^{\rho\nu}\,. \tag{11.53}$$

After substituting (11.41) and (11.53) into (11.51), one obtains

$$\breve{\mathcal{F}}_{\mu\nu}^{\ \ \rho} = \breve{f}_{\mu\nu}^{\ \ \rho} - 2\kappa\breve{\mathcal{B}}_{\mu\nu\sigma}\breve{\theta}^{\sigma\rho}\,, \tag{11.54}$$

where

$$\breve{f}_{\mu\nu}^{\ \ \rho} = (\mathcal{C}^{-1})^{\rho}_{\ \sigma}\Big(\hat{\partial}_\mu\mathcal{C}^{\sigma}_{\ \nu} - \hat{\partial}_\nu\mathcal{C}^{\sigma}_{\ \mu}\Big)\,. \tag{11.55}$$

Similarly, starting with (11.37), with the help of (11.42) and (11.48), we have

$$\begin{aligned}
\{\breve{k}^\mu(\sigma), \breve{k}^\nu(\bar{\sigma})\} &= (\mathcal{C}^{-1})^{\mu}_{\ \rho}(\mathcal{C}^{-1})^{\nu}_{\ \sigma}\{\mathring{k}^\rho, \mathring{k}^\sigma\} + (\mathcal{C}^{-1})^{\mu}_{\ \rho}\mathring{k}^\sigma\{\mathring{k}^\rho, (\mathcal{C}^{-1})^{\nu}_{\ \sigma}\} \\
&\quad + \mathring{k}^\rho\{(\mathcal{C}^{-1})^{\mu}_{\ \rho}, \mathring{k}^\sigma\}(\mathcal{C}^{-1})^{\nu}_{\ \sigma} \\
&= -\kappa(\mathcal{C}^{-1})^{\mu}_{\ \rho}(\mathcal{C}^{-1})^{\nu}_{\ \sigma}\mathring{Q}_\tau^{\ \rho\sigma}\mathring{k}^\tau\delta(\sigma-\bar{\sigma}) - \kappa^2(\mathcal{C}^{-1})^{\mu}_{\ \rho}(\mathcal{C}^{-1})^{\nu}_{\ \sigma}\mathring{R}^{\rho\sigma\tau}\mathring{\iota}_\tau\delta(\sigma-\bar{\sigma}) \\
&\quad -\kappa\Big(\breve{\theta}^{\mu\alpha}\hat{\partial}_\alpha(\mathcal{C}^{-1})^{\nu}_{\ \rho} - \breve{\theta}^{\nu\alpha}\hat{\partial}_\alpha(\mathcal{C}^{-1})^{\mu}_{\ \rho}\Big)\mathring{k}^\rho\delta(\sigma-\bar{\sigma}) \\
&= -\kappa\breve{\mathcal{Q}}_\rho^{\ \mu\nu}\breve{k}^\rho\delta(\sigma-\bar{\sigma}) - \kappa^2\breve{\mathcal{R}}^{\mu\nu\rho}\breve{\iota}_\rho\delta(\sigma-\bar{\sigma})\,,
\end{aligned} \tag{11.56}$$

where we used the relations (11.52) and (11.53) to simplify result. The fluxes that we obtained are

$$\breve{\mathcal{Q}}_\rho^{\ \mu\nu} = \mathcal{C}^{\alpha}_{\ \rho}(\mathcal{C}^{-1})^{\mu}_{\ \beta}(\mathcal{C}^{-1})^{\nu}_{\ \gamma}\mathring{\mathcal{Q}}_\alpha^{\ \beta\gamma} - \mathcal{C}^{\alpha}_{\ \rho}\Big(\breve{\theta}^{\nu\beta}\hat{\partial}_\beta(\mathcal{C}^{-1})^{\mu}_{\ \alpha} - \breve{\theta}^{\mu\beta}\hat{\partial}_\beta(\mathcal{C}^{-1})^{\nu}_{\ \alpha}\Big)\,, \tag{11.57}$$

and

$$\breve{\mathcal{R}}^{\mu\nu\rho} = \mathring{\mathcal{R}}^{\alpha\beta\gamma}(\mathcal{C}^{-1})^{\mu}{}_{\alpha}(\mathcal{C}^{-1})^{\nu}{}_{\beta}(\mathcal{C}^{-1})^{\rho}{}_{\gamma}\,. \tag{11.58}$$

These are expressions for analogs of non-geometric fluxes. We will now proceed to rewrite them in more recognizable forms. For $\breve{\mathcal{Q}}$-flux, substituting (11.43) into (11.57), we obtain

$$\breve{\mathcal{Q}}_{\rho}^{\ \mu\nu} = \breve{Q}_{\rho}^{\ \mu\nu} + 2\kappa\breve{\theta}^{\mu\alpha}\breve{\theta}^{\nu\beta}\breve{\mathcal{B}}_{\rho\alpha\beta}\,, \tag{11.59}$$

where

$$\breve{Q}_{\rho}^{\ \mu\nu} = \mathcal{C}^{\alpha}_{\ \rho}(\mathcal{C}^{-1})^{\mu}{}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\gamma}\mathring{Q}_{\alpha}^{\ \beta\gamma} - \mathcal{C}^{\alpha}_{\ \rho}\Big(\breve{\theta}^{\nu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \breve{\theta}^{\mu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\alpha}\Big)\,. \tag{11.60}$$

After expressing the previous relation in terms of $\breve{\theta}$ (11.53), we obtain

$$\begin{aligned}
\breve{Q}_{\rho}^{\ \mu\nu} &= \partial_{\alpha}(\mathcal{C}^2\breve{\theta})^{\beta\gamma}\mathcal{C}^{\alpha}_{\ \rho}(\mathcal{C}^{-1})^{\mu}{}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\gamma} - \mathcal{C}^{\alpha}_{\ \rho}\Big(\breve{\theta}^{\nu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \breve{\theta}^{\mu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\alpha}\Big) \\
&= \hat{\partial}_{\rho}\breve{\theta}^{\mu\nu} - \Big((\mathcal{C}\breve{\theta})^{\beta\nu}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\mu}{}_{\beta} + (\mathcal{C}\breve{\theta})^{\mu\gamma}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\nu}{}_{\gamma})\Big) - \mathcal{C}^{\alpha}_{\ \rho}\Big(\breve{\theta}^{\nu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \breve{\theta}^{\mu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\alpha}\Big) \\
&= \hat{\partial}_{\rho}\breve{\theta}^{\mu\nu} + \mathcal{C}^{\alpha}_{\ \beta}\breve{\theta}^{\nu\beta}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \mathcal{C}^{\alpha}_{\ \beta}\breve{\theta}^{\mu\beta}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\nu}{}_{\alpha} - \mathcal{C}^{\alpha}_{\ \rho}\Big(\breve{\theta}^{\nu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \breve{\theta}^{\mu\beta}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\alpha}\Big) \\
&= \hat{\partial}_{\rho}\breve{\theta}^{\mu\nu} + \breve{\theta}^{\nu\beta}\Big(\mathcal{C}^{\alpha}_{\ \beta}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\mu}{}_{\alpha} - \mathcal{C}^{\alpha}_{\ \rho}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\mu}{}_{\alpha}\Big) - \breve{\theta}^{\mu\beta}\Big(\mathcal{C}^{\alpha}_{\ \beta}\hat{\partial}_{\rho}(\mathcal{C}^{-1})^{\nu}{}_{\alpha} - \mathcal{C}^{\alpha}_{\ \rho}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\nu}{}_{\alpha}\Big) \\
&= \hat{\partial}_{\rho}\breve{\theta}^{\mu\nu} + \breve{f}_{\rho\sigma}^{\ \ \mu}\,\breve{\theta}^{\sigma\nu} - \breve{f}_{\rho\sigma}^{\ \ \nu}\,\breve{\theta}^{\sigma\mu}\,.
\end{aligned} \tag{11.61}$$

In the first step, we expressed the flux $\mathring{Q}$ using the non-commutative field $\breve{\theta}$. Then, in the second step, we used partial integration on the first term and rearranged the resulting expression using equation (11.52). In subsequent steps, we recognized that $(\mathcal{C}\breve{\theta})^{\beta\nu}$ can be expressed as $\mathcal{C}^{\beta}_{\ \sigma}\breve{\theta}^{\sigma\nu}$ and that $\mathcal{C}^{\mu}_{\ \beta}\partial_{\alpha}(\mathcal{C}^{-1})^{\beta}{}_{\sigma}$ equals $-\partial_{\alpha}\mathcal{C}^{\mu}_{\ \beta}(\mathcal{C}^{-1})^{\beta}{}_{\sigma}$. By relabeling some indices and using equation (11.55), we arrived at the final step of equation (11.61).

Similarly, substituting (11.44) into (11.58), we obtain

$$\breve{\mathcal{R}}^{\mu\nu\rho} = \breve{R}^{\mu\nu\rho} + 2\kappa\breve{\theta}^{\mu\alpha}\breve{\theta}^{\nu\beta}\breve{\theta}^{\rho\gamma}\breve{\mathcal{B}}_{\alpha\beta\gamma}\,, \tag{11.62}$$

where

$$\breve{R}^{\mu\nu\rho} = \mathring{R}^{\alpha\beta\gamma}(\mathcal{C}^{-1})^{\mu}{}_{\alpha}(\mathcal{C}^{-1})^{\nu}{}_{\beta}(\mathcal{C}^{-1})^{\rho}{}_{\gamma} \tag{11.63}$$

The $\breve{R}$-flux can further be rewritten by

$$\begin{aligned}
\breve{R}^{\mu\nu\rho} &= (\mathcal{C}^2\breve{\theta})^{\alpha\sigma}\partial_{\sigma}(\mathcal{C}^2\breve{\theta})^{\beta\gamma}(\mathcal{C}^{-1})^{\mu}{}_{\alpha}(\mathcal{C}^{-1})^{\nu}{}_{\beta}(\mathcal{C}^{-1})^{\rho}{}_{\gamma} + cyclic \\
&= (\mathcal{C}\breve{\theta})^{\mu\sigma}\partial_{\sigma}\breve{\theta}^{\nu\rho} - (\mathcal{C}\breve{\theta})^{\mu\sigma}(\mathcal{C}\breve{\theta})^{\beta\gamma}(\partial_{\sigma}(\mathcal{C}^{-1})^{\nu}{}_{\beta}(\mathcal{C}^{-1})^{\rho}{}_{\gamma} + (\mathcal{C}^{-1})^{\nu}{}_{\beta}\partial_{\sigma}(\mathcal{C}^{-1})^{\rho}{}_{\gamma}) + cyclic \\
&= \breve{\theta}^{\mu\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\nu\rho} + \breve{\theta}^{\mu\alpha}\breve{\theta}^{\rho\beta}\hat{\partial}_{\alpha}(\mathcal{C}^{-1})^{\nu}{}_{\tau}\mathcal{C}^{\tau}{}_{\beta} - \breve{\theta}^{\mu\beta}\breve{\theta}^{\nu\alpha}\hat{\partial}_{\beta}(\mathcal{C}^{-1})^{\rho}{}_{\tau}\mathcal{C}^{\tau}{}_{\alpha} + cyclic \\
&= \breve{\theta}^{\mu\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\nu\rho} + \breve{\theta}^{\nu\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\rho\mu} + \breve{\theta}^{\rho\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\mu\nu} - \breve{\theta}^{\mu\alpha}\breve{\theta}^{\rho\beta}(\mathcal{C}^{-1})^{\nu}{}_{\tau}\hat{\partial}_{\alpha}\mathcal{C}^{\tau}{}_{\beta} + \breve{\theta}^{\mu\beta}\breve{\theta}^{\nu\alpha}(\mathcal{C}^{-1})^{\rho}{}_{\tau}\hat{\partial}_{\beta}\mathcal{C}^{\tau}{}_{\alpha} \\
&\quad -\breve{\theta}^{\nu\alpha}\breve{\theta}^{\mu\beta}(\mathcal{C}^{-1})^{\rho}{}_{\tau}\hat{\partial}_{\alpha}\mathcal{C}^{\tau}{}_{\beta} + \breve{\theta}^{\nu\beta}\breve{\theta}^{\rho\alpha}(\mathcal{C}^{-1})^{\mu}{}_{\tau}\hat{\partial}_{\beta}\mathcal{C}^{\tau}{}_{\alpha} - \breve{\theta}^{\rho\alpha}\breve{\theta}^{\nu\beta}(\mathcal{C}^{-1})^{\mu}{}_{\tau}\hat{\partial}_{\alpha}\mathcal{C}^{\tau}{}_{\beta} + \breve{\theta}^{\rho\beta}\breve{\theta}^{\mu\alpha}(\mathcal{C}^{-1})^{\nu}{}_{\tau}\hat{\partial}_{\beta}\mathcal{C}^{\tau}{}_{\alpha} \\
&= \breve{\theta}^{\mu\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\nu\rho} + \breve{\theta}^{\nu\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\rho\mu} + \breve{\theta}^{\rho\sigma}\hat{\partial}_{\sigma}\breve{\theta}^{\mu\nu} - (\breve{\theta}^{\mu\alpha}\breve{\theta}^{\rho\beta}\breve{f}_{\alpha\beta}^{\ \ \nu} + \breve{\theta}^{\nu\alpha}\breve{\theta}^{\mu\beta}\breve{f}_{\alpha\beta}^{\ \ \rho} + \breve{\theta}^{\rho\alpha}\breve{\theta}^{\nu\beta}\breve{f}_{\alpha\beta}^{\ \ \mu})\,.
\end{aligned} \tag{11.64}$$

First, we expressed the flux as a function of $\breve{\theta}$ according to equation (11.53). Then, we applied the chain rule and the derivative $\hat{\partial}$ (11.52). Finally, we applied the chain rule again to hyperbolic functions and used equation (11.55) to obtain the final expression.

Lastly, the remaining algebra between currents is obtained from (11.45) and (11.48) in a following way

$$
\begin{aligned}
\{\breve{\imath}_\mu(\sigma), \breve{k}^\nu(\bar{\sigma})\} &= \mathcal{C}^\sigma{}_\mu (\mathcal{C}^{-1})^\nu{}_\rho \{\mathring{\imath}_\sigma, \mathring{k}^\rho(\bar{\sigma})\} + \mathcal{C}^\sigma{}_\mu \{\mathring{\imath}_\sigma, (\mathcal{C}^{-1})^\nu{}_\rho\} \mathring{k}^\rho + \mathring{\imath}_\sigma \{\mathcal{C}^\sigma{}_\mu, \mathring{k}^\rho\} (\mathcal{C}^{-1})^\nu{}_\rho \\
&= \kappa \mathcal{C}^\sigma{}_\mu(\sigma)(\mathcal{C}^{-1})^\nu{}_\sigma(\bar{\sigma})\delta'(\sigma-\bar{\sigma}) + \Big( \mathring{\mathcal{F}}_{\rho\tau}{}^\sigma \, \mathcal{C}^\rho{}_\mu (\mathcal{C}^{-1})^\nu{}_\sigma - \hat{\partial}_\mu (\mathcal{C}^{-1})^\nu{}_\tau \Big) \mathring{k}^\tau \delta(\sigma-\bar{\sigma}) \\
&\quad + \Big( -\kappa \mathring{\mathcal{Q}}_\rho{}^{\sigma\tau} \mathcal{C}^\rho{}_\mu (\mathcal{C}^{-1})^\nu{}_\sigma + \kappa (\mathcal{C}^{-1})^\nu{}_\sigma \mathring{\theta}^{\sigma\rho} \partial_\rho \mathcal{C}^\tau{}_\mu \Big) \mathring{\imath}_\tau \delta(\sigma-\bar{\sigma}) \,.
\end{aligned} \tag{11.65}
$$

Using (7.7), the anomalous term becomes

$$
\begin{aligned}
\kappa \mathcal{C}^\sigma{}_\mu(\sigma)(\mathcal{C}^{-1})^\nu{}_\sigma(\bar{\sigma})\delta'(\sigma-\bar{\sigma}) &= \kappa \delta^\nu_\mu \delta'(\sigma-\bar{\sigma}) + \kappa \mathcal{C}^\sigma{}_\mu \partial_\rho (\mathcal{C}^{-1})^\nu{}_\sigma x'^\rho \delta(\sigma-\bar{\sigma}) \\
&= \kappa \delta^\nu_\mu \delta'(\sigma-\bar{\sigma}) + \mathcal{C}^\rho{}_\mu \partial_\sigma (\mathcal{C}^{-1})^\nu{}_\rho \mathring{k}^\sigma \delta(\sigma-\bar{\sigma}) \\
&\quad - \kappa \mathcal{C}^\rho{}_\mu \partial_\sigma (\mathcal{C}^{-1})^\nu{}_\rho \mathring{\theta}^{\sigma\tau} \mathring{\imath}_\tau \delta(\sigma-\bar{\sigma}) \,,
\end{aligned} \tag{11.66}
$$

where we also used (11.38). Substituting (11.66) into (11.65), we obtain

$$
\begin{aligned}
\{\breve{\imath}_\mu(\sigma), \breve{k}^\nu(\bar{\sigma})\} &= \kappa \delta^\nu_\mu \delta'(\sigma-\bar{\sigma}) \\
&\quad + \Big( \mathcal{C}^\rho{}_\mu (\partial_\sigma (\mathcal{C}^{-1})^\nu{}_\rho - \partial_\rho (\mathcal{C}^{-1})^\nu{}_\sigma) + \mathcal{C}^\rho{}_\mu (\mathcal{C}^{-1})^\nu{}_\tau \mathring{\mathcal{F}}_{\rho\sigma}{}^\tau \Big) \mathring{k}^\sigma \delta(\sigma-\bar{\sigma}) \\
&\quad + \kappa \Big( (\mathcal{C}^{-1})^\nu{}_\sigma \partial_\rho \mathcal{C}^\tau{}_\mu \mathring{\theta}^{\sigma\rho} - \mathcal{C}^\rho{}_\mu \partial_\sigma (\mathcal{C}^{-1})^\nu{}_\rho \mathring{\theta}^{\sigma\tau} - \mathcal{C}^\rho{}_\mu (\mathcal{C}^{-1})^\nu{}_\sigma \mathring{\mathcal{Q}}_\rho{}^{\sigma\tau} \Big) \mathring{\imath}_\tau \delta(\sigma-\bar{\sigma}) \,.
\end{aligned} \tag{11.67}
$$

Substituting relations between currents in the previous expression (11.37), we obtain

$$
\{\breve{\imath}_\mu(\sigma), \breve{k}^\nu(\bar{\sigma})\} = \kappa \delta^\nu_\mu \delta'(\sigma-\bar{\sigma}) + \breve{\mathcal{F}}_{\mu\rho}{}^\nu \, \breve{k}^\rho \delta(\sigma-\bar{\sigma}) - \kappa \breve{\mathcal{Q}}_\mu{}^{\nu\rho} \breve{\imath}_\rho \delta(\sigma-\bar{\sigma}) \,, \tag{11.68}
$$

All Poisson bracket relations between currents (11.49), (11.56), (11.68) can be summarized by

$$
\{\breve{X}^M, \breve{X}^N\} = -\breve{F}^{MN}{}_P \, \breve{X}^P \delta(\sigma-\bar{\sigma}) + \kappa \eta^{MN} \delta'(\sigma-\bar{\sigma}) \,, \tag{11.69}
$$

where

$$
\breve{F}^{MN\rho} = \begin{pmatrix} \kappa^2 \breve{\mathcal{R}}^{\mu\nu\rho} & -\kappa \breve{\mathcal{Q}}_\nu{}^{\mu\rho} \\ \kappa \breve{\mathcal{Q}}_\mu{}^{\nu\rho} & \breve{\mathcal{F}}_{\mu\nu}{}^\rho \end{pmatrix} , \qquad \breve{F}^{MN}{}_\rho = \begin{pmatrix} \kappa \breve{\mathcal{Q}}_\rho{}^{\mu\nu} & \breve{\mathcal{F}}_{\nu\rho}{}^\mu \\ -\breve{\mathcal{F}}_{\mu\rho}{}^\nu & 2\breve{\mathcal{B}}_{\mu\nu\rho} \end{pmatrix} . \tag{11.70}
$$

With basic algebra relations, we can now proceed with calculations of the Courant bracket twisted simultaneously by $B$ and $\theta$.

## 11.5 Full bracket

The full bracket can be obtained from Poisson bracket relations of the double generator (11.30)

$$\{\breve{\mathcal{G}}_{\Lambda_1}, \breve{\mathcal{G}}_{\Lambda_2}\} = -\breve{\mathcal{G}}_{[\Lambda_1,\Lambda_2]_{\mathcal{C}_{\breve{B}}}} . \tag{11.71}$$

We rewrite term-wise the left hand side of the previous relation

$$\begin{aligned}\{\breve{\mathcal{G}}_{\Lambda_1}, \breve{\mathcal{G}}_{\Lambda_2}\} &= \int d\sigma d\bar{\sigma}\Big(\{\xi_1^\mu(\sigma)\breve{\imath}_\mu(\sigma), \xi_2^\nu(\bar{\sigma})\breve{\imath}_\nu(\bar{\sigma})\} + \{\lambda_{1\mu}(\sigma)\breve{k}^\mu(\sigma), \lambda_{2\nu}(\bar{\sigma})\breve{k}^\nu(\bar{\sigma})\} \\ &\quad +\{\xi_1^\mu(\sigma)\breve{\imath}_\mu(\sigma), \lambda_{2\nu}(\bar{\sigma})\breve{k}^\nu(\bar{\sigma})\} + \{\lambda_{1\mu}(\sigma)\breve{k}^\mu(\sigma), \xi_2^\nu(\bar{\sigma})\breve{\imath}_\nu(\bar{\sigma})\}\Big) .\end{aligned} \tag{11.72}$$

Apart from the relations between currents, other Poisson bracket relations that we will use are

$$\{\xi_1^\mu(\sigma), \breve{\imath}_\nu(\bar{\sigma})\} = (\mathcal{C}^T)_\nu{}^\rho\partial_\rho\xi_1^\mu\delta(\sigma-\bar{\sigma}) = \hat{\partial}_\nu\xi_1^\mu\delta(\sigma-\bar{\sigma}) , \tag{11.73}$$

$$\{\xi_1^\mu(\sigma), \breve{k}^\nu(\bar{\sigma})\} = \kappa(\mathcal{C}\breve{\theta})^{\nu\rho}\partial_\rho\xi_1^\mu\delta(\sigma-\bar{\sigma}) = \kappa\breve{\theta}^{\nu\rho}\hat{\partial}_\rho\xi_1^\mu\delta(\sigma-\bar{\sigma}) . \tag{11.74}$$

Using (11.49) and (11.73), the first term of (11.72) becomes

$$\begin{aligned}\{\xi_1^\mu(\sigma)\breve{\imath}_\mu(\sigma), \xi_2^\nu(\bar{\sigma})\breve{\imath}_\nu(\bar{\sigma})\} &= (\xi_2^\nu\hat{\partial}_\nu\xi_1^\mu - \xi_1^\nu\hat{\partial}_\nu\xi_2^\mu - \breve{\mathcal{F}}_{\nu\rho}{}^\mu\,\xi_1^\nu\xi_2^\rho)\breve{\imath}_\mu\delta(\sigma-\bar{\sigma}) \\ &\quad -2\breve{\mathcal{B}}_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho\breve{k}^\mu\delta(\sigma-\bar{\sigma}) ,\end{aligned} \tag{11.75}$$

which after we substitute (11.54) becomes

$$\begin{aligned}\{\xi_1^\mu(\sigma)\breve{\imath}_\mu(\sigma), \xi_2^\nu(\bar{\sigma})\breve{\imath}_\nu(\bar{\sigma})\} &= (\xi_2^\nu\hat{\partial}_\nu\xi_1^\mu - \xi_1^\nu\hat{\partial}_\nu\xi_2^\mu - (\breve{f}_{\nu\rho}{}^\mu - 2\kappa\breve{\mathcal{B}}_{\nu\rho\sigma}\breve{\theta}^{\sigma\mu})\xi_1^\nu\xi_2^\rho)\breve{\imath}_\mu\delta(\sigma-\bar{\sigma}) \\ &\quad -2\breve{\mathcal{B}}_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho\breve{k}^\mu\delta(\sigma-\bar{\sigma}) ,\end{aligned} \tag{11.76}$$

and similarly, from (11.56) and (11.74), we obtain

$$\begin{aligned}\{\lambda_{1\mu}(\sigma)\breve{k}^\mu(\sigma), \lambda_{2\nu}(\bar{\sigma})\breve{k}^\nu(\bar{\sigma})\} &= -\kappa^2\breve{\mathcal{R}}^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\breve{\imath}_\mu\delta(\sigma-\bar{\sigma}) \\ &\quad -\kappa(\breve{\mathcal{Q}}_\mu{}^{\nu\rho}\lambda_{1\nu}\lambda_{2\rho} + \kappa\breve{\theta}^{\nu\rho}\hat{\partial}_\rho\lambda_{2\mu}\lambda_{1\nu} - \kappa\breve{\theta}^{\nu\rho}\hat{\partial}_\rho\lambda_{1\mu}\lambda_{2\nu})\breve{k}^\mu\delta(\sigma-\bar{\sigma}) .\end{aligned} \tag{11.77}$$

which after substituting (11.59) and (11.62) becomes

$$\begin{aligned}\{\lambda_{1\mu}(\sigma)\breve{k}^\mu(\sigma), \lambda_{2\nu}(\bar{\sigma})\breve{k}^\nu(\bar{\sigma})\} &= -\kappa^2\Big(\breve{R}^{\mu\nu\rho} + 2\kappa\breve{\theta}^{\mu\alpha}\breve{\theta}^{\nu\beta}\breve{\theta}^{\rho\gamma}\breve{\mathcal{B}}_{\alpha\beta\gamma}\Big)\lambda_{1\nu}\lambda_{2\rho}\breve{\imath}_\mu\delta(\sigma-\bar{\sigma}) \\ &\quad -\kappa\Big(\hat{\partial}_\mu\breve{\theta}^{\nu\rho}\lambda_{1\nu}\lambda_{2\rho} + \breve{\theta}^{\nu\rho}\hat{\partial}_\rho\lambda_{2\mu}\lambda_{1\nu} - \breve{\theta}^{\nu\rho}\hat{\partial}_\rho\lambda_{1\mu}\lambda_{2\nu} \\ &\quad +(\breve{f}_{\mu\sigma}{}^\nu\,\breve{\theta}^{\sigma\rho} - \breve{f}_{\mu\sigma}{}^\rho\,\breve{\theta}^{\sigma\nu} + 2\kappa\breve{\theta}^{\nu\alpha}\breve{\theta}^{\rho\beta}\breve{\mathcal{B}}_{\mu\alpha\beta})\lambda_{1\nu}\lambda_{2\rho}\Big)\breve{k}^\mu\delta(\sigma-\bar{\sigma}) .\end{aligned} \tag{11.78}$$

Lastly, using (11.68), (11.73), and (11.74), one obtains

$$\begin{aligned}\{\xi_1^\mu(\sigma)\breve{\iota}_\mu(\sigma),\lambda_{2\nu}(\bar\sigma)\breve{k}^\nu(\bar\sigma)\} &= \kappa\xi_1^\mu(\sigma)\lambda_{2\mu}(\bar\sigma)\delta'(\sigma-\bar\sigma) \\ &\quad +(\breve{\mathcal{F}}_{\nu\mu}{}^{\rho}\,\xi_1^\nu\lambda_{2\rho}-\xi_1^\nu\hat\partial_\nu\lambda_{2\mu})\breve{k}^\mu\delta(\sigma-\bar\sigma) \\ &\quad +(-\kappa\breve{\mathcal{Q}}_\rho{}^{\nu\mu}\xi_1^\rho\lambda_{2\nu}+\kappa\lambda_{2\nu}\breve\theta^{\nu\rho}\hat\partial_\rho\xi_1^\mu)\breve{\iota}_\mu\delta(\sigma-\bar\sigma)\,.\end{aligned} \tag{11.79}$$

The anomalous part depends on both $\sigma$ and $\bar\sigma$, so it should be further modified by

$$\begin{aligned}\kappa\xi_1^\mu(\sigma)\lambda_{2\mu}(\bar\sigma)\delta'(\sigma-\bar\sigma) &= \frac{\kappa}{2}\xi_1^\mu(\sigma)\lambda_{2\mu}(\bar\sigma)\delta'(\sigma-\bar\sigma)-\frac{\kappa}{2}\xi_1^\mu(\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma) \\ &= \frac{\kappa}{2}\xi_1^\mu\lambda_{2\mu}\delta'(\sigma-\bar\sigma)+\frac{\kappa}{2}\xi_1^\mu\partial_\nu\lambda_{2\mu}x'^\nu\delta(\sigma-\bar\sigma) \\ &\quad -\frac{\kappa}{2}\xi_1^\mu(\bar\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma)-\frac{\kappa}{2}\partial_\nu\xi_1^\mu\lambda_{2\mu}x'^\nu\delta(\sigma-\bar\sigma) \\ &= \frac{\kappa}{2}\xi_1^\mu\lambda_{2\mu}\delta'(\sigma-\bar\sigma)-\frac{\kappa}{2}\xi_1^\mu(\bar\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma) \\ &\quad +\frac{\kappa}{2}(\xi_1^\rho\hat\partial_\nu\lambda_{2\rho}\breve\theta^{\nu\mu}-\hat\partial_\nu\xi_1^\rho\lambda_{2\rho}\breve\theta^{\nu\mu})\breve{\iota}_\mu\delta(\sigma-\bar\sigma) \\ &\quad +\frac{1}{2}(\xi_1^\nu\hat\partial_\mu\lambda_{2\nu}-\hat\partial_\mu\xi_1^\nu\lambda_{2\nu})\breve{k}^\mu\delta(\sigma-\bar\sigma) \\ &= \frac{\kappa}{2}\xi_1^\mu\lambda_{2\mu}\delta'(\sigma-\bar\sigma)-\frac{\kappa}{2}\xi_1^\mu(\bar\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma) \\ &\quad +\kappa\breve\theta^{\mu\nu}(\xi_1^\rho\hat\partial_\nu\lambda_{2\rho}-\frac{1}{2}\hat\partial_\nu(\xi_1^\rho\lambda_{2\rho}))\breve{\iota}_\mu\delta(\sigma-\bar\sigma) \\ &\quad +(\xi_1^\nu\hat\partial_\mu\lambda_{2\nu}-\frac{1}{2}\hat\partial_\mu(\xi_1^\nu\lambda_{2\nu}))\breve{k}^\mu\delta(\sigma-\bar\sigma)\,.\end{aligned} \tag{11.80}$$

We used the property of the delta function (7.7) in the initial two steps, followed by using the relation (11.29) in the subsequent step, and eventually, we applied the chain rule in the final step. After substituting (11.80) into (11.79), we obtain

$$\begin{aligned}\{\xi_1^\nu(\sigma)\breve{\iota}_\nu(\sigma),\lambda_{2\mu}(\bar\sigma)\breve{k}^\mu(\bar\sigma)\} &= \frac{\kappa}{2}\xi_1^\mu\lambda_{2\mu}\delta'(\sigma-\bar\sigma)-\frac{\kappa}{2}\xi_1^\mu(\bar\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma) \\ &\quad +\Big(\breve{\mathcal{F}}_{\nu\mu}{}^{\rho}\,\xi_1^\nu\lambda_{2\rho}+\xi_1^\nu(\hat\partial_\mu\lambda_{2\nu}-\hat\partial_\nu\lambda_{2\mu})-\frac{1}{2}\hat\partial_\mu(\xi_1^\nu\lambda_{2\nu})\Big)\breve{k}^\mu\delta(\sigma-\bar\sigma) \\ &\quad +\Big(-\kappa\breve{\mathcal{Q}}_\rho{}^{\nu\mu}\xi_1^\rho\lambda_{2\nu}+\kappa\lambda_{2\nu}\breve\theta^{\nu\rho}\hat\partial_\rho\xi_1^\mu \\ &\qquad +\kappa\breve\theta^{\mu\nu}(\xi_1^\rho\hat\partial_\nu\lambda_{2\rho}-\frac{1}{2}\hat\partial_\nu(\xi_1^\rho\lambda_{2\rho}))\Big)\breve{\iota}_\mu\delta(\sigma-\bar\sigma)\,.\end{aligned} \tag{11.81}$$

To obtain more recognizable terms, we will use the chain rule in order to transform the term containing $\breve{\mathcal{Q}}$ flux (11.59)

$$\begin{aligned}-\kappa\breve{\mathcal{Q}}_\rho{}^{\nu\mu}\xi_1^\rho\lambda_{2\nu} &= -\kappa\Big(\hat\partial_\rho\breve\theta^{\nu\mu}+\breve{f}_{\rho\sigma}{}^{\nu}\breve\theta^{\sigma\mu}-\breve{f}_{\rho\sigma}{}^{\mu}\breve\theta^{\sigma\nu}+2\kappa^2\breve\theta^{\nu\alpha}\breve\theta^{\mu\beta}\breve{\mathcal{B}}_{\alpha\beta\rho}\Big)\xi_1^\rho\lambda_{2\nu} \\ &= -\kappa\xi_1^\rho\hat\partial_\rho(\lambda_{2\nu}\breve\theta^{\nu\mu})+\kappa\xi_1^\rho\hat\partial_\rho\lambda_{2\nu}\breve\theta^{\nu\mu} \\ &\quad -\kappa\Big(\breve{f}_{\rho\sigma}{}^{\nu}\breve\theta^{\sigma\mu}-\breve{f}_{\rho\sigma}{}^{\mu}\breve\theta^{\sigma\nu}+2\kappa^2\breve\theta^{\nu\alpha}\breve\theta^{\mu\beta}\breve{\mathcal{B}}_{\alpha\beta\rho}\Big)\xi_1^\rho\lambda_{2\nu}\,.\end{aligned} \tag{11.82}$$

Substituting (11.54) and (11.82) into (11.81), we obtain

$$\begin{aligned}
\{\xi_1^\nu \breve{\imath}_\nu, \lambda_{2\mu}(\bar\sigma)\breve{k}^\mu(\bar\sigma)\} \;=\; & \frac{\kappa}{2}\xi_1^\mu \lambda_{2\mu}\delta'(\sigma-\bar\sigma) - \frac{\kappa}{2}\xi_1^\mu(\bar\sigma)\lambda_{2\mu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma) \\
& + \Big(\xi_1^\nu(\hat\partial_\mu\lambda_{2\nu} - \hat\partial_\nu\lambda_{2\mu}) - \frac{1}{2}\hat\partial_\mu(\xi_1^\nu\lambda_{2\nu}) \\
& + (\breve{f}_{\nu\mu}^{\ \ \rho} - 2\kappa\breve{\mathcal{B}}_{\nu\mu\sigma}\breve\theta^{\sigma\rho})\xi_1^\nu\lambda_{2\rho}\Big)\breve{k}^\mu\delta(\sigma-\bar\sigma) \\
& + \Big[\kappa\breve\theta^{\mu\nu}\Big(\xi_1^\rho(\hat\partial_\nu\lambda_{2\rho} - \hat\partial_\rho\lambda_{2\nu}) - \frac{1}{2}\hat\partial_\nu(\xi_1^\rho\lambda_{2\rho})\Big) \\
& + \kappa\lambda_{2\nu}\breve\theta^{\nu\rho}\hat\partial_\rho\xi_1^\mu - \kappa\xi_1^\rho\hat\partial_\rho(\lambda_{2\nu}\breve\theta^{\nu\mu}) \\
& - \kappa\Big(\breve{f}_{\rho\sigma}^{\ \ \nu}\breve\theta^{\sigma\mu} - \breve{f}_{\rho\sigma}^{\ \ \mu}\breve\theta^{\sigma\nu} + 2\kappa^2\breve\theta^{\nu\alpha}\breve\theta^{\mu\beta}\breve{\mathcal{B}}_{\alpha\beta\rho}\Big)\xi_1^\rho\lambda_{2\nu}\Big]\breve{\imath}_\mu\delta(\sigma-\bar\sigma)\,.
\end{aligned} \tag{11.83}$$

Substituting (11.76), (11.78), and (11.83) into (11.72), with the help of (11.71), we obtain

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_{\breve B}} = \Lambda = \xi \oplus \lambda\,, \tag{11.84}$$

where

$$\begin{aligned}
\xi^\mu \;=\; & \xi_1^\nu\hat\partial_\nu\xi_2^\mu - \xi_2^\nu\hat\partial_\nu\xi_1^\mu + \breve{f}_{\nu\rho}^{\ \ \mu}\,\xi_1^\nu\xi_2^\rho \\
& + \kappa\breve\theta^{\mu\nu}\Big(\xi_1^\rho(\hat\partial_\rho\lambda_{2\nu} - \hat\partial_\nu\lambda_{2\rho}) - \xi_2^\rho(\hat\partial_\rho\lambda_{1\nu} - \hat\partial_\nu\lambda_{1\rho}) + \frac{1}{2}\hat\partial_\nu(\xi_1\lambda_2 - \xi_2\lambda_1) \\
& \qquad + \kappa\breve{f}_{\nu\rho}^{\ \ \sigma}(\xi_1^\rho\lambda_{2\nu} - \xi_2^\rho\lambda_{1\nu})\Big) \\
& + \kappa\xi_1^\nu\hat\partial_\nu(\lambda_{2\rho}\breve\theta^{\rho\mu}) - \kappa\xi_2^\nu\hat\partial_\nu(\lambda_{1\rho}\breve\theta^{\rho\mu}) - \kappa\lambda_{2\nu}\breve\theta^{\nu\rho}\hat\partial_\rho\xi_1^\mu + \kappa\lambda_{1\nu}\breve\theta^{\nu\rho}\hat\partial_\rho\xi_2^\mu \\
& \qquad + \kappa\breve{f}_{\rho\sigma}^{\ \ \nu}\,\breve\theta^{\sigma\mu}(\xi_1^\rho\lambda_{2\nu} - \xi_2^\rho\lambda_{1\nu}) \\
& + \kappa^2\breve{R}^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho} \\
& - 2\kappa\breve{B}_{\nu\rho\sigma}\breve\theta^{\sigma\mu}\xi_1^\nu\xi_2^\rho + 2\kappa^2\breve\theta^{\nu\alpha}\breve\theta^{\mu\beta}\breve{\mathcal{B}}_{\alpha\beta\rho}(\xi_1^\rho\lambda_{2\nu} - \xi_2^\rho\lambda_{1\nu}) + 2\kappa^3\breve\theta^{\mu\alpha}\breve\theta^{\nu\beta}\breve\theta^{\rho\gamma}\breve{\mathcal{B}}_{\alpha\beta\gamma}\lambda_{1\nu}\lambda_{2\rho}\,,
\end{aligned} \tag{11.85}$$

and

$$\begin{aligned}
\lambda_\mu \;=\; & \xi_1^\nu(\hat\partial_\nu\lambda_{2\mu} - \hat\partial_\mu\lambda_{2\nu}) - \xi_2^\nu(\hat\partial_\nu\lambda_{1\mu} - \hat\partial_\mu\lambda_{1\nu}) + \frac{1}{2}\hat\partial_\mu(\xi_1\lambda_2 - \xi_2\lambda_1) \\
& + \breve{f}_{\mu\nu}^{\ \ \rho}(\xi_1^\nu\lambda_{2\rho} - \xi_2^\nu\lambda_{1\rho}) \\
& + \kappa\breve\theta^{\nu\rho}(\lambda_{1\nu}\hat\partial_\rho\lambda_{2\mu} - \lambda_{2\nu}\hat\partial_\rho\lambda_{1\mu}) + \kappa\lambda_{1\rho}\lambda_{2\nu}\hat\partial_\mu\breve\theta^{\rho\nu} + \kappa(\breve{f}_{\mu\sigma}^{\ \ \nu}\,\breve\theta^{\sigma\rho} - \breve{f}_{\mu\sigma}^{\ \ \rho}\,\breve\theta^{\sigma\nu})\lambda_{1\nu}\lambda_{2\rho} \\
& + 2\breve{\mathcal{B}}_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho - 2\kappa\breve{\mathcal{B}}_{\mu\nu\sigma}\breve\theta^{\sigma\rho}(\xi_1^\nu\lambda_{2\rho} - \xi_2^\nu\lambda_{1\rho}) + 2\kappa^2\breve\theta^{\nu\alpha}\breve\theta^{\rho\beta}\breve{\mathcal{B}}_{\mu\alpha\beta}\lambda_{1\nu}\lambda_{2\rho}\,.
\end{aligned} \tag{11.86}$$

We grouped the terms in expressions (11.85) and (11.86) for future convenience.

In the process of twisting the Courant bracket simultaneously by $B$ and $\theta$, we did not rely on the fact that these fields are T-dual background fields. Thus, the results should be valid regardless of this property. If we take that either of the fields $B$ and $\theta$ is zero, the derivative $\hat{\partial}_\mu$ reduces to the ordinary partial derivative $\partial_\mu$, and all the $\breve{f}$ flux terms become zero. Specifically, for $\theta = 0$ and an arbitrary $B$, we obtain $B$-twisted Courant bracket, while for $B = 0$ and an arbitrary $\theta$, we obtain the $\theta$-twisted Courant bracket.

## 11.6 Coordinate-free notation

In contrast to prior instances, obtaining the coordinate-free notation with a clear interpretation of its terms is not as trivial for this bracket. We will need to introduce novel brackets that will ultimately be identified as the brackets of Lie and quasi-Lie algebroids that have not been encountered previously.

### 11.6.1 Twisted Lie bracket

Firstly, we will seek the Lie algebroid with $\mathcal{C}$ as its anchor. This step will turn out to be crucial for interpreting many terms that appear in the expressions (11.85) and (11.86). The bracket of this Lie algebroid should be related to the Lie bracket by

$$\begin{aligned}\Big(\mathcal{C}[\xi_1,\xi_2]_{\hat{L}}\Big)^\mu &= \Big([\mathcal{C}\xi_1,\mathcal{C}\xi_2]_L\Big)^\mu = \mathcal{C}^\nu{}_\rho \xi_1^\rho \partial_\nu (\mathcal{C}^\mu{}_\sigma \xi_2^\sigma) - \mathcal{C}^\nu{}_\rho \xi_2^\rho \partial_\nu (\mathcal{C}^\mu{}_\sigma \xi_1^\sigma) \\ &= \mathcal{C}^\nu{}_\rho \mathcal{C}^\mu{}_\sigma \Big(\xi_1^\rho \partial_\nu \xi_2^\sigma - \xi_2^\rho \partial_\nu \xi_1^\sigma\Big) + \xi_1^\rho \xi_2^\sigma \Big(\mathcal{C}^\nu{}_\rho \partial_\nu \mathcal{C}^\mu{}_\sigma - \mathcal{C}^\nu{}_\sigma \partial_\nu \mathcal{C}^\mu{}_\rho\Big) \\ &= \mathcal{C}^\mu{}_\sigma \Big(\xi_1^\rho \hat{\partial}_\rho \xi_2^\sigma - \xi_2^\rho \hat{\partial}_\rho \xi_1^\sigma\Big) + \xi_1^\rho \xi_2^\sigma \Big(\hat{\partial}_\rho \mathcal{C}^\mu{}_\sigma - \hat{\partial}_\sigma \mathcal{C}^\mu{}_\rho\Big)\,,\end{aligned} \tag{11.87}$$

where we used (11.52) and relabeled some indices. Multiplying the previous relation with $\mathcal{C}^{-1}$ and taking into the account (11.55), we obtain

$$\Big([\xi_1,\xi_2]_{\hat{L}}\Big)^\mu = \xi_1^\nu \hat{\partial}_\nu \xi_2^\mu - \xi_2^\nu \hat{\partial}_\nu \xi_1^\mu + \breve{f}_{\nu\rho}{}^\mu\, \xi_1^\nu \xi_2^\rho\,, \tag{11.88}$$

which is exactly the first line of (11.85). Analogous to our notation for twisted Courant brackets, we will denote this bracket as the twisted Lie bracket, since

$$[\xi_1,\xi_2]_{\hat{L}} = \mathcal{C}^{-1}[\mathcal{C}\xi_1,\mathcal{C}\xi_2]_L\,. \tag{11.89}$$

In order for $\mathcal{C}$ to be a proper anchor of a Lie algebroid, the Leibniz rule has to be satisfied, i.e.

$$[\xi_1, f\xi_2]_{\hat{L}} = (\mathcal{L}_{\mathcal{C}\xi_1} f)\,\xi_2 + f[\xi_1,\xi_2]_{\hat{L}}\,, \tag{11.90}$$

from which we can derive the action of corresponding Lie derivative on functions

$$\hat{\mathcal{L}}_\xi\, f = \mathcal{L}_{\mathcal{C}\xi}\, f = \xi^\mu \hat{\partial}_\mu f\,. \tag{11.91}$$

Its action on vectors is simply given by the twisted Lie bracket. The Jacobi identity is also satisfied, since

$$\mathrm{Jac}_{\hat{L}}(\xi_1, \xi_2, \xi_3) = \mathcal{C}^{-1}[\mathcal{C}\xi_1, [\mathcal{C}\xi_2, \mathcal{C}\xi_3]_L]_L + cyclic = \mathcal{C}^{-1}\mathrm{Jac}_L(\mathcal{C}\xi_1, \mathcal{C}\xi_2, \mathcal{C}\xi_3) = 0\,. \tag{11.92}$$

To write the action of Lie derivative $\mathcal{L}_{\hat{\xi}}$ on 1-forms, we firstly apply the Leibniz rule (11.90) on 1-form-vector contraction

$$\begin{aligned} \hat{\mathcal{L}}_{\xi_1}(\xi_2^\mu \lambda_{2\mu}) &= (\hat{\mathcal{L}}_{\xi_1}\xi_2)^\mu \lambda_{2\mu} + \xi_2^\mu (\hat{\mathcal{L}}_{\xi_1}\lambda_2)_\mu \\ &= (\xi_1^\nu \hat{\partial}_\nu \xi_2^\mu - \xi_2^\nu \hat{\partial}_\nu \xi_1^\mu)\lambda_{2\mu} + \breve{f}_{\nu\rho}{}^\mu\, \xi_1^\nu \xi_2^\rho \lambda_{2\mu} + \xi_2^\mu (\hat{\mathcal{L}}_{\xi_1}\lambda_2)_\mu\,, \end{aligned} \tag{11.93}$$

and then (11.91) on that contraction, since it is effectively a scalar

$$\hat{\mathcal{L}}_{\xi_1}(\xi_2^\mu \lambda_{2\mu}) = \xi_1^\nu \hat{\partial}_\nu (\xi_2^\mu \lambda_{2\mu}) = \xi_1^\nu \hat{\partial}_\nu \xi_2^\mu \lambda_{2\mu} + \xi_1^\nu \xi_2^\mu \hat{\partial}_\nu \lambda_{2\mu}\,. \tag{11.94}$$

When we equate right-hand sides of previous two relations, we obtain

$$(\hat{\mathcal{L}}_{\xi_1}\lambda_2)_\mu = \hat{\partial}_\mu \xi_1^\nu \lambda_{2\nu} + \xi_1^\nu \hat{\partial}_\nu \lambda_{2\mu} + \breve{f}_{\mu\nu}{}^\rho\, \xi_1^\nu \lambda_{2\rho} = \xi_1^\nu (\hat{\partial}_\nu \lambda_{2\mu} - \hat{\partial}_\mu \lambda_{2\nu}) + \hat{\partial}_\mu (\xi_1^\nu \lambda_{2\nu}) + \breve{f}_{\mu\nu}{}^\rho\, \xi_1^\nu \lambda_{2\rho}\,. \tag{11.95}$$

The exterior algebra is easily derived from the relation (5.7). Let us explicitly obtain the action of exterior derivative on functions and 1-forms. Functions correspond to the case $p = 0$, so we have

$$\hat{d}f(\xi) = \mathcal{C}\xi(f) = \xi^\mu \hat{\partial}_\mu f\,, \quad (\hat{d}f)_\mu = \hat{\partial}_\mu f\,. \tag{11.96}$$

From (11.91), we see that the usual relation for the action of Lie derivatives on function $\hat{\mathcal{L}}_\xi f = i_\xi \hat{d} f$ still holds in the twisted case. On the other hand, 1-forms correspond to the case of $p = 1$ in (5.7), from which we obtain

$$\begin{aligned} \hat{d}\lambda(\xi_1, \xi_2) &= \mathcal{C}\xi_1(\lambda(\xi_2)) - \mathcal{C}\xi_2(\lambda(\xi_1)) - \lambda([\xi_1, \xi_2]_{\hat{L}}) \\ &= \xi_1^\mu \xi_2^\nu \Big(\hat{\partial}_\mu \lambda_\nu - \hat{\partial}_\nu \lambda_\mu - \hat{f}_{\mu\nu}{}^\rho \lambda_\rho\Big)\,, \\ (\hat{d}\lambda)_{\mu\nu} &= \hat{\partial}_\mu \lambda_\nu - \hat{\partial}_\nu \lambda_\mu - \hat{f}_{\mu\nu}{}^\rho \lambda_\rho\,. \end{aligned} \tag{11.97}$$

The Cartan formula $\hat{\mathcal{L}}_\xi \lambda = i_\xi \hat{d}\lambda + \hat{d} i_\xi \lambda$ can be easily demonstrated using (11.95) and (11.97), and holds true for any $p$-form $\lambda$.

We saw how the hyperbolic function $\mathcal{C}$ defines an anchor for the Lie algebroid defined with the twisted Lie bracket as its bracket. Various terms in the expressions (11.85) and (11.86) can be expressed in terms of the corresponding twisted Lie derivative.

### 11.6.2 Generalized H-flux

Let us explore the generalized $H$-flux $\breve{\mathcal{B}}$ (11.50). It has a structure of a 3-form, that when contracted with three vectors can be written by

$$\breve{\mathcal{B}}_{\mu\nu\rho}\xi_1^\mu\xi_2^\nu\xi_3^\rho = \mathring{B}_{\alpha\beta\gamma}\mathcal{C}^\alpha{}_\mu\xi_1^\mu\mathcal{C}^\beta{}_\nu\xi_2^\nu\mathcal{C}^\gamma{}_\rho\xi_3^\rho = d\mathring{B}(\mathcal{C}\xi_1,\mathcal{C}\xi_2,\mathcal{C}\xi_3)\,. \tag{11.98}$$

where $\mathring{B}_{\mu\nu\rho}$ is defined in (11.40). The right-hand side of the previous relation is expressed in a non coordinate notation, which using (4.11) can be further transformed by

$$\begin{aligned}(d\mathring{B})(\mathcal{C}\xi_1,\mathcal{C}\xi_2,\mathcal{C}\xi_3) &= \mathcal{C}\xi_1\Big(\mathring{B}(\mathcal{C}\xi_2,\mathcal{C}\xi_3)\Big) - \mathcal{C}\xi_2\Big(\mathring{B}(\mathcal{C}\xi_1,\mathcal{C}\xi_3)\Big) + \mathcal{C}\xi_3\Big(\mathring{B}(\mathcal{C}\xi_1,\mathcal{C}\xi_2)\Big) \\ &\quad -\mathring{B}\Big([\mathcal{C}\xi_1,\mathcal{C}\xi_2]_L,\mathcal{C}\xi_3\Big) + \mathring{B}\Big([\mathcal{C}\xi_1,\mathcal{C}\xi_3]_L,\mathcal{C}\xi_2\Big) - \mathring{B}\Big([\mathcal{C}\xi_2,\mathcal{C}\xi_3]_L,\mathcal{C}\xi_1\Big) \\ &= \mathcal{C}\xi_1\Big(\hat{B}(\xi_2,\xi_3)\Big) - \mathcal{C}\xi_2\Big(\hat{B}(\xi_1,\xi_3)\Big) + \mathcal{C}\xi_3\Big(\hat{B}(\xi_1,\xi_2)\Big) \\ &\quad -\hat{B}\Big([\xi_1,\xi_2]_{\hat{L}},\xi_3\Big) + \hat{B}\Big([\xi_1,\xi_3]_{\hat{L}},\xi_2\Big) - \hat{B}\Big([\xi_2,\xi_3]_{\hat{L}},\xi_1\Big) \\ &= \hat{d}\hat{B}(\xi_1,\xi_2,\xi_3)\,,\end{aligned} \tag{11.99}$$

where $\hat{B}$ is a new field that we defined by

$$\hat{B}_{\mu\nu} = \mathring{B}_{\alpha\beta}\mathcal{C}^\alpha{}_\mu\mathcal{C}^\beta{}_\nu = (B\mathcal{S}\mathcal{C})_{\mu\nu}\,, \tag{11.100}$$

and in the last step recognized the expression for twisted exterior derivative acting on a 2-form.

### 11.6.3 Twisted Koszul bracket

We define the twisted Koszul bracket by

$$[\lambda_1,\lambda_2]_{\breve{\theta}} = \hat{\mathcal{L}}_{\breve{\theta}(\lambda_1)}\lambda_2 - \hat{\mathcal{L}}_{\breve{\theta}(\lambda_2)}\lambda_1 - \hat{d}(\breve{\theta}(\lambda_1,\lambda_2))\,, \tag{11.101}$$

where $\breve{\theta}(\lambda_1)^\mu = \lambda_{1\nu}\breve{\theta}^{\nu\mu}$. This is an analogous definition to the one for the (non-twisted) Koszul bracket. In some local basis, its components are given by

$$\Big([\lambda_1,\lambda_2]_{\breve{\theta}}\Big)_\mu = \breve{\theta}^{\nu\rho}(\lambda_{1\nu}\hat{\partial}_\rho\lambda_{2\mu} - \lambda_{2\nu}\hat{\partial}_\rho\lambda_{1\mu}) + \hat{\partial}_\mu\breve{\theta}^{\nu\rho}\lambda_{1\nu}\lambda_{2\rho} + (\breve{f}_{\mu\nu}{}^\rho\,\breve{\theta}^{\nu\sigma} - \breve{f}_{\mu\nu}{}^\sigma\,\breve{\theta}^{\nu\rho})\lambda_{1\rho}\lambda_{2\sigma}\,. \tag{11.102}$$

This bracket can be related to the twisted Lie bracket. In order to do that, we firstly obtain

$$\begin{aligned}\breve{\theta}\Big([\lambda_1,\lambda_2]_{\breve{\theta}}\Big) &= \breve{\theta}^{\nu\mu}\Big(\breve{\theta}^{\sigma\rho}(\lambda_{1\sigma}\hat{\partial}_\rho\lambda_{2\nu} - \lambda_{2\sigma}\hat{\partial}_\rho\lambda_{1\nu}) + \hat{\partial}_\nu\breve{\theta}^{\sigma\rho}\lambda_{1\sigma}\lambda_{2\rho} \\ &\quad +(\breve{f}_{\nu\tau}{}^\rho\,\breve{\theta}^{\tau\sigma} - \breve{f}_{\nu\tau}{}^\sigma\,\breve{\theta}^{\tau\rho})\lambda_{1\rho}\lambda_{2\sigma}\Big)\,..\end{aligned} \tag{11.103}$$

Next, we obtain

$$
\begin{aligned}
[\breve{\theta}(\lambda_1), \breve{\theta}(\lambda_2)]_{\hat{L}} &= \breve{\theta}^{\nu\rho}\lambda_{1\rho}\hat{\partial}_\nu(\breve{\theta}^{\mu\sigma}\lambda_{2\sigma}) - \breve{\theta}^{\nu\rho}\lambda_{2\rho}\hat{\partial}_\nu(\breve{\theta}^{\mu\sigma}\lambda_{1\sigma}) + \breve{f}_{\nu\rho}^{\ \ \mu}\,\breve{\theta}^{\nu\sigma}\breve{\theta}^{\rho\tau}\lambda_{1\sigma}\lambda_{2\tau} \qquad (11.104)\\
&= \breve{\theta}^{\nu\rho}\breve{\theta}^{\mu\sigma}\Big(\lambda_{1\rho}\hat{\partial}_\nu\lambda_{2\sigma} - \lambda_{2\rho}\hat{\partial}_\nu\lambda_{1\sigma}\Big) + \breve{f}_{\nu\rho}^{\ \ \mu}\,\breve{\theta}^{\nu\sigma}\breve{\theta}^{\rho\tau}\lambda_{1\sigma}\lambda_{2\tau}\\
&\quad +(\breve{\theta}^{\rho\nu}\hat{\partial}_\nu\breve{\theta}^{\sigma\mu} + \breve{\theta}^{\sigma\nu}\hat{\partial}_\nu\breve{\theta}^{\mu\rho})\lambda_{1\rho}\lambda_{2\sigma}\,.
\end{aligned}
$$

After relabeling of some dummy indices, we obtain the relation

$$
\Big[\breve{\theta}\Big([\lambda_1,\lambda_2]_{\breve{\theta}}\Big)\Big]^\mu = \Big([\breve{\theta}(\lambda_1), \breve{\theta}(\lambda_2)]_{\hat{L}}\Big)^\mu + \breve{R}^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\,, \qquad (11.105)
$$

where $\breve{R}$ is defined in (11.64). We can use the definition of the twisted Lie bracket (11.89), to rewrite the previous relation as

$$
\Big[\mathcal{C}\breve{\theta}\Big([\lambda_1,\lambda_2]_{\breve{\theta}}\Big)\Big]^\mu = \Big([\mathcal{C}\breve{\theta}(\lambda_1), \mathcal{C}\breve{\theta}(\lambda_2)]_{L}\Big)^\mu + \mathcal{C}^\mu_{\ \sigma}\breve{R}^{\sigma\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\,. \qquad (11.106)
$$

The twisted Koszul bracket defines a quasi-Lie algebroid with anchor $\hat{\rho}_{\breve{\theta}} = \mathcal{C}\breve{\theta}$ and with the $\breve{R}$-flux as deformation from the Lie algebroid structure.

We can still define the exterior derivative associated with the quasi-Lie algebroid defined with the twisted Koszul bracket. On functions, its action is obtained from (5.7)

$$
\hat{d}_{\breve{\theta}}f(\lambda) = \breve{\theta}^{\mu\nu}\hat{\partial}_\nu f\lambda_\mu\,, \qquad (\hat{d}_{\breve{\theta}}f)^\mu = \breve{\theta}^{\mu\nu}\hat{\partial}_\nu f\,. \qquad (11.107)
$$

Similarly, on vectors it becomes

$$
\begin{aligned}
\hat{d}_{\breve{\theta}}\xi(\lambda_1,\lambda_2) &= (\lambda_{1\rho}\breve{\theta}^{\rho\nu})\hat{\partial}_\nu(\xi^\mu\lambda_{2\mu}) - (\lambda_{2\rho}\breve{\theta}^{\rho\nu})\hat{\partial}_\nu(\xi^\mu\lambda_{1\mu}) - \xi^\mu\Big([\lambda_1,\lambda_2]_{\breve{\theta}}\Big)_\mu \qquad (11.108)\\
&= \Big(\breve{\theta}^{\mu\rho}\hat{\partial}_\rho\xi^\nu - \breve{\theta}^{\nu\rho}\hat{\partial}_\rho\xi^\mu - \xi^\rho(\hat{\partial}_\rho\breve{\theta}^{\mu\nu} + \breve{f}_{\rho\sigma}^{\ \ \mu}\,\breve{\theta}^{\sigma\nu} - \breve{f}_{\rho\sigma}^{\ \ \nu}\,\breve{\theta}^{\sigma\mu})\Big)\lambda_{1\mu}\lambda_{2\nu}\,.
\end{aligned}
$$

The exterior derivative $\hat{d}_{\breve{\theta}}$ satisfies Leibniz rule, but is not idempotent, unless the $\breve{R}$-flux is zero.

### 11.6.4 Twisted Schouten-Nijenhuis bracket

Lastly, we are going to interpret the term containing the $\breve{R}$-flux in terms of newly defined quasi-Lie algebroid. Recall that in the previous chapters, we defined $R$-flux as the Schouten-Nijenhuis bracket, that can be written as $d_\theta\theta = [\theta,\theta]_S$. This motivates us to consider the action of exterior derivative $\hat{d}_{\breve{\theta}}$ on the bi-vector $\breve{\theta}$. From definition (5.7), we have

$$
\begin{aligned}
\hat{d}_{\breve{\theta}}\breve{\theta}(\lambda_1,\lambda_2,\lambda_3) &= \mathcal{C}\breve{\theta}(\lambda_1)\Big([\lambda_2,\lambda_3]_{\breve{\theta}}\Big) - \mathcal{C}\breve{\theta}(\lambda_2)\Big([\lambda_1,\lambda_3]_{\breve{\theta}}\Big) + \mathcal{C}\breve{\theta}(\lambda_3)\Big([\lambda_1,\lambda_2]_{\breve{\theta}}\Big)\\
&\quad -\breve{\theta}\Big([\lambda_1,\lambda_2]_{\breve{\theta}},\lambda_3\Big) + \breve{\theta}\Big([\lambda_1,\lambda_3]_{\breve{\theta}},\lambda_2\Big) - \breve{\theta}\Big([\lambda_2,\lambda_3]_{\breve{\theta}},\lambda_1\Big)\,. \qquad (11.109)
\end{aligned}
$$

There are two types of terms, so let us calculate the components of a representative of each type. Firstly, we have

$$\begin{aligned}\mathcal{C}\breve{\theta}(\lambda_1)\Big([\lambda_2,\lambda_3]_{\breve{\theta}}\Big) &= \lambda_{1\mu}\breve{\theta}^{\mu\sigma}\hat{\partial}_\sigma\Big(\breve{\theta}^{\nu\rho}\lambda_{2\nu}\lambda_{3\rho}\Big) \\ &= \breve{\theta}^{\mu\sigma}\hat{\partial}_\sigma\breve{\theta}^{\nu\rho}\lambda_{1\mu}\lambda_{2\nu}\lambda_{3\rho}+\breve{\theta}^{\mu\sigma}\breve{\theta}^{\nu\rho}\lambda_{1\mu}(\hat{\partial}_\sigma\lambda_{2\nu}\lambda_{3\rho}+\lambda_{2\nu}\hat{\partial}_\sigma\lambda_{3\rho})\,,\end{aligned} \tag{11.110}$$

and secondly

$$\begin{aligned}-\breve{\theta}\Big([\lambda_1,\lambda_2]_{\breve{\theta}},\lambda_3\Big) &= \lambda_{1\mu}\lambda_{2\nu}\lambda_{3\rho}\Big(\breve{\theta}^{\rho\sigma}\hat{\partial}_\sigma\breve{\theta}^{\mu\nu}+\breve{f}_{\sigma\tau}{}^{\mu}\,\breve{\theta}^{\nu\sigma}\breve{\theta}^{\rho\tau}-\breve{f}_{\sigma\tau}{}^{\nu}\,\breve{\theta}^{\mu\sigma}\breve{\theta}^{\rho\tau}\Big) \\ &\quad -\breve{\theta}^{\mu\rho}\breve{\theta}^{\nu\sigma}(\lambda_{1\nu}\hat{\partial}_\sigma\lambda_{2\mu}-\lambda_{2\nu}\hat{\partial}_\sigma\lambda_{1\mu})\lambda_{3\rho}\,.\end{aligned} \tag{11.111}$$

When (11.110) and (11.111) are substituted in (11.109), we obtain

$$\hat{d}_{\breve{\theta}}\breve{\theta}(\lambda_1,\lambda_2,\lambda_3)=2\breve{R}^{\mu\nu\rho}\lambda_{1\mu}\lambda_{2\nu}\lambda_{3\rho}\,, \tag{11.112}$$

which is exactly what we hoped for. We will therefore define the twisted Schouten-Nijenhuis bracket as

$$[\breve{\theta},\breve{\theta}]_{\hat{S}}=\hat{d}_{\breve{\theta}}\breve{\theta}\,. \tag{11.113}$$

## 11.7 Courant algebroid

We are now able to express all the terms in the Courant bracket twisted by $B$ and $\theta$ in terms of newly defined twisted brackets. In the coordinate free notation, the expression (11.85) is given by

$$\begin{aligned}\xi &= [\xi_1,\xi_2]_{\hat{L}}-\kappa\breve{\theta}\Big(\hat{\mathcal{L}}_{\xi_1}\lambda_2-\hat{\mathcal{L}}_{\xi_2}\lambda_1-\frac{1}{2}\hat{d}(i_{\xi_1}\lambda_2-i_{\xi_2}\lambda_1)\Big) \\ &\quad +[\xi_1,\kappa\breve{\theta}(\lambda_2)]_{\hat{L}}-[\xi_2,\kappa\breve{\theta}(\lambda_1)]_{\hat{L}}+\frac{\kappa^2}{2}[\breve{\theta},\breve{\theta}]_{\hat{S}}(\lambda_1,\lambda_2,.) \\ &\quad +2\kappa\breve{\theta}\;\hat{d}\hat{B}(.,\xi_1,\xi_2)-2\wedge^2\kappa\breve{\theta}\;\hat{d}\hat{B}(.,\lambda_1,\xi_2)+2\wedge^2\kappa\breve{\theta}\;\hat{d}\hat{B}(.,\lambda_2,\xi_1)+2\wedge^3\kappa\breve{\theta}\;\hat{d}\hat{B}(\lambda_1,\lambda_2,.)\,,\end{aligned} \tag{11.114}$$

and the expression (11.86) by

$$\begin{aligned}\lambda &= \hat{\mathcal{L}}_{\xi_1}\lambda_2-\hat{\mathcal{L}}_{\xi_2}\lambda_1+\frac{1}{2}\hat{d}(i_{\xi_1}\lambda_2-i_{\xi_2}\lambda_1)+\kappa[\lambda_1,\lambda_2]_{\breve{\theta}} \\ &\quad +2\hat{d}\hat{B}(\xi_1,\xi_2,.)-2\kappa\breve{\theta}\;\hat{d}\hat{B}(\lambda_2,.,\xi_1)+2\kappa\breve{\theta}\;\hat{d}\hat{B}(\lambda_1,.,\xi_2)+2\wedge^2\kappa\breve{\theta}\;\hat{d}\hat{B}(\lambda_1,\lambda_2,.)\,.\end{aligned} \tag{11.115}$$

The exponents on the wedge represent how many times a bi-vector is contracted with a 3-form, while the dot denotes the non-contracted index, e.g.

$$\Big(\wedge^2\kappa\breve{\theta}\;\hat{d}\hat{B}(.,\lambda_1,\xi_2)\Big)^\mu=\kappa^2\breve{\theta}^{\mu\alpha}\breve{\theta}^{\nu\beta}\breve{\mathcal{B}}_{\alpha\beta\rho}\lambda_{1\nu}\xi_2^\rho\,. \tag{11.116}$$

The Courant bracket twisted at the same time by $B$ and $\theta$ defines a Courant algebroid. The anchor is obtained from substituting $e^{-\breve{B}}$ (11.26) into (8.8)

$$\rho^{(\breve{B})}\Lambda = \mathcal{C}\xi - \kappa\mathcal{C}\breve{\theta}\lambda\,, \tag{11.117}$$

and similarly the differential operator from (11.18) and (8.10)

$$\mathcal{D}^{(\breve{B})}f = \begin{pmatrix} \kappa\mathcal{C}^\mu{}_\rho\breve{\theta}^{\rho\nu}\partial_\nu f \\ (\mathcal{C}^T)_\mu{}^\nu\partial_\nu f \end{pmatrix} = \begin{pmatrix} \hat{d}_{\breve{\theta}}f \\ \hat{d}f \end{pmatrix}\,. \tag{11.118}$$

Let us now obtain the Dirac structures for this Courant algebroid. Firstly, consider an isotropic space in the form of graph of $\mathring{B}$ over the tangent bundle

$$\mathcal{V}_{\mathring{B}}(\Lambda) = \xi^\mu \oplus 2\mathring{B}_{\mu\nu}\xi^\nu\,. \tag{11.119}$$

On this sub-bundle, the symmetry generator (11.30) becomes

$$\begin{aligned} \breve{\mathcal{G}}_{\mathcal{V}_{\mathring{B}}(\Lambda)} &= \int d\sigma\xi^\nu(\mathcal{C}_\nu{}^\mu - 2\kappa\mathring{B}_{\mu\nu}(\mathcal{S}\theta)^{\nu\rho})\pi_\mu \\ &= \int d\sigma\pi_\mu(\mathcal{C}^{-1})^\mu{}_\rho\Big((\mathcal{C}^2)^\rho{}_\nu - 2\kappa(\mathcal{S}^2\theta B)^\rho{}_\nu\Big)\xi^\nu \\ &= \int d\sigma\pi_\mu(\mathcal{C}^{-1})^\mu{}_\nu\xi^\nu\,, \end{aligned} \tag{11.120}$$

where we firstly used (11.35) and (11.8), and then (11.21). This is the generator of diffeomorphisms with the parameter $(\mathcal{C}^{-1})^\mu{}_\nu\xi^\nu$, which closes on the Lie bracket in the Poisson bracket algebra. Therefore, the sub-bundle $\mathcal{V}_{\mathring{B}}$ will be a Dirac structure and no restrictions on the $\mathring{B}$-field have to be imposed.

Similarly, we seek Dirac structures in the form of graphs of $\breve{\theta}$, i.e.

$$\mathcal{V}_{\breve{\theta}}(\Lambda) = \breve{\theta}^{\mu\nu}\lambda_\nu \oplus \lambda_\mu\,. \tag{11.121}$$

The generator (11.30) becomes

$$\breve{\mathcal{G}}_{\mathcal{V}_{\breve{\theta}}(\Lambda)} = \int d\sigma\lambda_\mu\mathcal{C}^\mu{}_\nu\kappa x'^\nu\,, \tag{11.122}$$

We encountered this case at the end of the previous chapter - this is a generator that does not depend on $\pi$ and therefore gives zero bracket in its Poisson bracket algebra. The graph $\mathcal{V}_{\breve{\theta}}$ will be a Dirac structure, regardless of $\breve{\theta}$. Once again, we do not need to impose any restrictions on fluxes on the Dirac structures. Therefore, the Courant bracket twisted at the same time by $B$ and $\theta$ defines a Courant algebroid, such that on its Dirac structures all fluxes can exist without restrictions.

# Part IV

# Double theory

# Chapter 12

# Double theory action

In this chapter, we will introduce the basic notions of a string theory defined in a phase space that is a direct sum of the initial and T-dual phase space, which we call double theory. We will introduce the Lagrangian of the double theory, derive its canonical momenta and Hamiltonian, and extend the Poisson bracket relations to the double phase space.

## 12.1 Lagrangian and Hamiltonian in double formalism

The idea behind the double theory [75, 76, 77, 78, 79] is the unification of the $D$-dimensional initial and its corresponding T-dual theory into a single theory defined in $2D$ dimensions. This theory should incorporate T-duality as its symmetry, and both the initial and T-dual theory should be obtained after projection to a suitable $D$-coordinate subspace. One of the most straightforward justifications for the double theory may be observed in the scenario of a closed string in which certain dimensions are compactified, enabling it to wrap around these compact dimensions. The winding number, which is to say the number of times a string curls around the compactified dimension, can be associated with the set of T-dual momenta, as it was previously demonstrated. The coordinates conjugate to these T-dual momenta $y_\mu$ are additional degrees of freedom, so the full description of the string theory should incorporate them as well.

In order to write Lagrangian, we firstly define a double coordinate $X^M$, defined in a direct sum of the initial coordinate space, characterized by $x^\mu$, and T-dual coordinate space, characterized by $y_\mu$, i.e.

$$X^M = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix} , \tag{12.1}$$

where $\mu = 0, 1...D-1,\ M = 0, 1...2D-1,\ D = 26.$ We assume that the generalized metric has the same form as in the single theory, but with all fields, in general, depending on the double set of coordinates $x^\mu$ and $y_\mu$

$$H_{MN} = \begin{pmatrix} G^E_{\mu\nu}(x,y) & -2B_{\mu\rho}(x,y)(G^{-1})^{\rho\nu}(x,y) \\ 2(G^{-1})^{\mu\rho}(x,y)B_{\rho\nu}(x,y) & (G^{-1})^{\mu\nu}(x,y) \end{pmatrix} . \tag{12.2}$$

The Lagrangian density is taken in the same form as in the initial theory

$$\mathcal{L} = \frac{\kappa}{2}\partial_+ X^M H_{MN}\partial_- X^N . \tag{12.3}$$

The equations of motions from the variation of the Lagrangian (12.3) become

$$\partial_+(H_{MN}\partial_- X^N) + \partial_-(H_{MN}\partial_+ X^N) = 0 . \tag{12.4}$$

In the case of constant background fields, these relations simplify to

$$\partial_+\partial_- x^\mu = 0 , \quad \partial_+\partial_- y_\mu = 0 , \tag{12.5}$$

which are well-known equations of motion for the initial and T-dual theories, respectively. The relation (12.4) is also known as the Bianchi identity. We see that the Bianchi identities and equations of motion are united into a single relation in double formalism.

The double set of coordinates is accompanied by the double set of momenta conjugate to them. It can be easily obtained by varying the Lagrangian (12.3) with respect to $\dot{X}^M$

$$\Pi_M = \frac{\delta\mathcal{L}}{\delta\dot{X}^M} = \kappa H_{MN}\dot{X}^N , \tag{12.6}$$

which can be written in the component notation as

$$\Pi_M = \begin{pmatrix} \pi_\mu \\ {}^\star\pi^\mu \end{pmatrix} , \tag{12.7}$$

where

$$\pi_\mu = G^E_{\mu\nu}\dot{x}^\nu - 2(BG^{-1})_\mu{}^\nu\dot{y}_\nu , \tag{12.8}$$

and

$${}^\star\pi^\mu = (G^{-1})^{\mu\nu}\dot{y}_\nu + 2(G^{-1}B)^\mu{}_\nu\dot{x}^\nu . \tag{12.9}$$

We can easily obtain the inverse of the relation (12.6)

$$\dot{X}^M = \frac{1}{\kappa}H^{MN}\Pi_N , \tag{12.10}$$

where $H^{MN}$ is the inverse of the generalized metric, given by

$$H^{MN} = \eta^{MK}H_{KL}\eta^{LN} . \tag{12.11}$$

Now we can apply the Legendre transformation of the Lagrangian (12.3), in order to obtain the canonical Hamiltonian

$$\mathcal{H}_{\mathcal{C}} = \Pi_M \dot{X}^M - \mathcal{L} = \frac{1}{2\kappa}\Pi_M H^{MN}\Pi_N + \frac{\kappa}{2}X'^M H_{MN}X'^N \,, \tag{12.12}$$

where we used (12.10).

## 12.2 T-duality

Our goal is to rewrite the Buscher T-duality transformation laws (3.22) and (3.25) in the double formalism. We note that by separating terms that change sign with those that do not, the T-duality relations can be rewritten as

$$\begin{aligned} \pm\partial_\pm y_\mu &\simeq G^E_{\mu\nu}\partial_\pm x^\nu - 2(BG^{-1})_\mu{}^\nu\partial_\pm y_\nu \,, \\ \pm\partial_\pm x^\mu &\simeq 2(G^{-1}B)^\mu{}_\nu\partial_\pm x^\nu + (G^{-1})^{\mu\nu}\partial_\pm y_\nu \,, \end{aligned} \tag{12.13}$$

which can be easily integrated into a single relation

$$\partial_\pm X^M \simeq \pm\eta^{MN}H_{NK}\,\partial_\pm X^K \,. \tag{12.14}$$

To obtain the canonical form of T-duality relations, using (2.19) we rewrite (12.14)

$$\dot{X}^M \pm X'^M \simeq \eta^{MN}H_{NK}X'^K \pm \eta^{MN}H_{NK}\dot{X}^K \,, \tag{12.15}$$

or equivalently

$$\dot{X}^M \simeq \eta^{MN}H_{NK}X'^K \,, \quad X'^M \simeq \eta^{MN}H_{NK}\dot{X}^K \,, \tag{12.16}$$

which using (12.6) can be expressed as

$$\Pi_M \simeq \kappa\,\eta_{MN}X'^M \,. \tag{12.17}$$

After application of (12.14) to (12.3), we have

$$\frac{\kappa}{2}\partial_+ X^M H_{MN}\partial_- X^N \simeq -\frac{\kappa}{2}\eta^{MK}H_{KL}\partial_+ X^L H_{MN}\eta^{NP}H_{PQ}\partial_- X^Q = -\mathcal{L} \,. \tag{12.18}$$

The Lagrangian is (up to a sign) invariant under T-duality. The change of sign does not matter, since equations of the motion will remain the same, and we will have exactly the same theory.

## 12.3 Poisson bracket relations in double formalism

As double theory should generalize both initial and T-dual theory, we assume the standard Poisson bracket relations within the initial and T-dual phase spaces

$$\{x'^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu \delta'(\sigma-\bar\sigma)\,, \qquad \{y'_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = \delta^\nu_\mu \delta'(\sigma-\bar\sigma)\,, \tag{12.19}$$

with other brackets of canonical variables within the same phase space being zero, i.e.

$$\{\kappa x'^\mu(\sigma), \kappa x'^\nu(\bar\sigma)\} = \{\kappa y'_\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} = 0\,, \tag{12.20}$$

and

$$\{\pi_\mu(\sigma), \pi_\nu(\bar\sigma)\} = \{{}^\star\pi^\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = 0\,. \tag{12.21}$$

These relations have to be extended so that they include relations of phase space variables from mutually T-dual phase spaces, which we will do using T-duality. Let us firstly apply T-duality along all coordinates $y_\mu$ to the Poisson bracket relation between coordinate derivatives in mutually T-dual phase spaces

$$\{\kappa x'^\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} \simeq \{\kappa x'^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \kappa\delta^\mu_\nu \delta'(\sigma-\bar\sigma)\,. \tag{12.22}$$

Relations (12.20) and (12.22) can be rewritten in terms of double coordinates as

$$\{\kappa X'^M(\sigma), \kappa X'^N(\bar\sigma)\} \simeq \kappa\,\eta_{MN}\delta'(\sigma-\bar\sigma)\,. \tag{12.23}$$

If we were to obtain the Poisson bracket relation between double coordinates, rather than their derivatives, we could integrate the previous relation along both $\sigma$ and $\bar\sigma$, and obtain

$$\{\kappa X^M(\sigma), \kappa X^N(\bar\sigma)\} = -\kappa\eta^{MN}\theta(\sigma-\bar\sigma)\,, \tag{12.24}$$

where $\theta$ is Heavyside step function. The relation (12.24) is determined up to boundary conditions, that can be set with different choice of Heavyside step function.

Secondly, we apply T-dualization along all $y_\mu$ coordinates to the Poisson bracket of momenta from mutually T-dual spaces, and obtain

$$\{\pi_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} \simeq \kappa\{\pi_\mu(\sigma), x'^\nu(\bar\sigma)\} = \kappa\delta^\nu_\mu \delta'(\sigma-\bar\sigma)\,. \tag{12.25}$$

Relations (12.21) and (12.25) nicely combine into

$$\{\Pi_M(\sigma), \Pi_N(\bar\sigma)\} \simeq \kappa\,\eta_{MN}\delta'(\sigma-\bar\sigma)\,. \tag{12.26}$$

Lastly, we once again T-dualize along all the initial coordinates $x^\mu$ to obtain the remaining bracket

$$\{\kappa x'^\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} \simeq \{{}^\star\pi^\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = 0\,, \tag{12.27}$$

which with the other brackets (12.19) can be written as

$$\{X'^M(\sigma), \Pi_N(\bar{\sigma})\} = \delta^M_N \delta'(\sigma - \bar{\sigma}) . \tag{12.28}$$

Some Poisson bracket relations are written as T-duality relations, emphasizing that double theory intrinsically incorporates T-duality.

## 12.4 Restricted fields

While it is true that background fields depend on both initial and T-dual coordinates, in order to achieve invariance under both diffeomorphisms and T-dual diffeomorphisms, specific constraints must be imposed to the background fields. Firstly, we will demand that all fields are annihilated by the operator

$$\Delta = \eta^{MN} \partial_M \partial_N = \partial^M \partial_M = 0 , \tag{12.29}$$

where $\partial^M$ are the derivatives in a double theory, given by

$$\partial_M = \begin{pmatrix} \partial_\mu \\ \tilde{\partial}^\mu \end{pmatrix} , \qquad \left( \partial_\mu \equiv \frac{\partial}{\partial x^\mu} , \quad \tilde{\partial}^\mu \equiv \frac{\partial}{\partial y_\mu} \right) . \tag{12.30}$$

Moreover, we will require the so-called strong constraints, in which all the products of any two fields $\phi$ and $\psi$ are also annihilated by (12.29), i.e.

$$\partial^M \partial_M (\phi\psi) = (\partial^M \partial_M \phi)\, \psi + 2\partial^M \phi\, \partial_M \psi + \phi\, \partial^M \partial_M \psi = 2\partial^M \phi\, \partial_M \psi = 0 . \tag{12.31}$$

These conditions appear also from the Virasoro conditions [80, 81]. Without strong constraints, the symmetry algebra would not close.

# Chapter 13

# $C$-bracket

In this chapter, we provide the world-sheet derivation of the $C$-bracket, which is the double theory generalization of the Lie bracket. We will present the double generator of diffeomorphisms and show that the $C$-bracket appears in its algebra. We will end this chapter by considering the projection of the $C$-bracket to the initial and T-dual phase spaces and show that it reduces to the Courant bracket.

## 13.1 Generator of diffeomorphisms in double theory

Previously, we saw that the diffeomorphisms are generated by momenta $\pi_\mu$, and we expect that T-dual diffeomorphisms are generated by T-dual momenta ${}^\star\pi^\mu$. In double theory, these momenta are integrated into a double momentum $\Pi_M$ (12.7). We will construct the double generator which is a sum of generators of initial and T-dual diffeomorphisms. It can be written as the $O(D,D)$ invariant inner product

$$\mathcal{G}_\Lambda = \int d\sigma \langle \Lambda, \Pi \rangle \,, \tag{13.1}$$

where $\Lambda^M$ are the symmetry parameters, which can be expressed by

$$\Lambda^M(X) = \begin{pmatrix} \xi^\mu(x,y) \\ \lambda_\mu(x,y) \end{pmatrix} \,. \tag{13.2}$$

The parameters $\xi^\mu$ are associated with initial diffeomorphisms, while the parameters $\lambda_\mu$ are associated with T-dual diffeomorphisms. Both parameters depend on all initial coordinates $x^\mu$ and all T-dual coordinates $y_\mu$.

We want to obtain the Poisson bracket relations of a double generator. We have

$$\begin{aligned}\{\mathcal{G}_{\Lambda_1}(\sigma),\mathcal{G}_{\Lambda_2}(\bar\sigma)\} &= \int d\sigma d\bar\sigma\Big(\Lambda_1^M(\sigma)\{\Pi_M(\sigma),\Pi_N(\bar\sigma)\}\Lambda_2^N(\bar\sigma) \\ &\quad +\Lambda_1^M(\sigma)\{\Pi_M(\sigma),\Lambda_2^N(\bar\sigma)\}\Pi_N(\bar\sigma)+\Pi_M(\sigma)\{\Lambda_1^M(\sigma),\Pi_N(\bar\sigma)\}\Lambda_2^N(\bar\sigma)\Big)\,.\end{aligned} \tag{13.3}$$

We did not write the term $\Pi_M(\sigma)\{\Lambda_1^M(\sigma),\Lambda_2^N(\bar\sigma)\}\Pi_N(\bar\sigma)$, since it is zero after applying the strong constraint (12.31) and the chain rule

$$\{\Lambda_1^M(\sigma),\Lambda_2^N(\bar\sigma)\} = -\frac{1}{\kappa}\partial^P\Lambda_1^M\partial_P\Lambda_2^N\theta(\sigma-\bar\sigma) = -\frac{1}{\kappa}\Big(\Delta(\Lambda_1^M\Lambda_2^N)-\Delta\Lambda_1^M\Lambda_2^N\Big)\theta(\sigma-\bar\sigma)=0\,. \tag{13.4}$$

Without this condition, the generator algebra would be anomalous. In fact, there were successful attempts to find constraints that are weaker than the strong constraint that we imposed, in which case anomalous part in the algebra contributes to the trivial transformation [82]. We are primarily interested in constructing the $C$-bracket, and for this purpose, it is sufficient to assume strong constraints (12.31).

To the first term of (13.3), we apply the relation (12.26)

$$\kappa\Lambda_1^M(\sigma)\{\Pi_M(\sigma),\Pi_N(\bar\sigma)\}\Lambda_2^N(\bar\sigma)\delta'(\sigma-\bar\sigma)\simeq\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma)\,. \tag{13.5}$$

After applying (7.7) on the right-hand side of the previous relation, we obtain

$$\begin{aligned}\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma) &= \frac{\kappa}{2}\Big(\langle\Lambda_1,\Lambda_2'\rangle-\langle\Lambda_1',\Lambda_2\rangle\Big)\delta(\sigma-\bar\sigma)\\ &\quad+\frac{\kappa}{2}\Big(\langle\Lambda_1,\Lambda_2\rangle+\langle\Lambda_1,\Lambda_2\rangle(\bar\sigma)\Big)\delta'(\sigma-\bar\sigma)\,,\end{aligned} \tag{13.6}$$

where parameters depend on $\sigma$ unless otherwise explicitly expressed. With the help of the chain rule

$$\kappa\Lambda'^M=\kappa\,X'^N\partial_N\Lambda^M\,, \tag{13.7}$$

the relation (13.6) further transforms into

$$\begin{aligned}\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma) = &\quad \frac{1}{2}\eta_{PQ}\left(\Lambda_1^P\partial_N\Lambda_2^Q-\Lambda_2^P\partial_N\Lambda_1^Q\right)X'^N\delta(\sigma-\bar\sigma)\\ &+\frac{\kappa}{2}\Big(\langle\Lambda_1,\Lambda_2\rangle(\sigma)+\langle\Lambda_1,\Lambda_2\rangle(\bar\sigma)\Big)\delta'(\sigma-\bar\sigma)\,.\end{aligned} \tag{13.8}$$

The anomaly disappears after the integration with respect to $\sigma$ and $\bar\sigma$. The first line in (13.8) contributes to the $C$-bracket expression. We apply the T-duality relations (12.17) to it, and obtain

$$\frac{1}{2}\eta_{PQ}\left(\Lambda_1^P\partial_N\Lambda_2^Q-\Lambda_2^P\partial_N\Lambda_1^Q\right)X'^N\simeq\frac{1}{2}\eta_{PQ}\,\eta^{MN}\Big(\Lambda_1^P\partial_N\Lambda_2^Q-\Lambda_2^P\partial_N\Lambda_1^Q\Big)\Pi_M\,. \tag{13.9}$$

Note that we applied T-duality twice - in (13.5) and (13.9), and consequentially we can write

$$\Lambda_1^M(\sigma)\{\Pi_M(\sigma),\Pi_N(\bar\sigma)\}\Lambda_2^N(\bar\sigma)=\frac{1}{2}\eta_{PQ}\,\eta^{MN}\Big(\Lambda_1^P\partial_N\Lambda_2^Q-\Lambda_2^P\partial_N\Lambda_1^Q\Big)\Pi_M\delta(\sigma-\bar\sigma)\,. \tag{13.10}$$

After relabeling of some dummy indices, the remaining terms in (13.3) can be written as

$$-\Pi_M(\Lambda_1^N\partial_N\Lambda_2^M - \Lambda_2^N\partial_N\Lambda_1^M)\delta(\sigma-\bar{\sigma}) \tag{13.11}$$

From relations (13.10) and (13.11), we can express the generator algebra relations

$$\{\mathcal{G}_{\Lambda_1}, \mathcal{G}_{\Lambda_2}\} = -\mathcal{G}_{[\Lambda_1,\Lambda_2]_{\mathbf{C}}}\,, \tag{13.12}$$

where $[\Lambda_1,\Lambda_2]_{\mathbf{C}}$ is the $C$-bracket, given by

$$\Big([\Lambda_1,\Lambda_2]_{\mathbf{C}}\Big)^M = \Lambda_1^N\partial_N\Lambda_2^M - \Lambda_2^N\partial_N\Lambda_1^M - \frac{1}{2}\Big(\Lambda_1^N\partial^M\Lambda_{2N} - \Lambda_2^N\partial^M\Lambda_{1N}\Big)\,. \tag{13.13}$$

The $C$-bracket was firstly obtained by Siegel [80, 81]. It is a generalization of the Lie bracket to double space. One can introduce the double Lie derivative $\hat{\mathcal{L}}_\Lambda$, that acts on all indices as if they were both covariant and contravariant and its algebra will give $C$-bracket. For example, its action on the generalized metric is given by

$$\hat{\mathcal{L}}_\Lambda H^{MN} = \Lambda^P\partial_P H^{MN} + (\partial^M\Lambda_P - \partial_P\Lambda^M)H^{PN} + (\partial^N\Lambda_P - \partial_P\Lambda^N)H^{MP}\,. \tag{13.14}$$

If no dependence on T-dual momenta and T-dual coordinates exists, the generator $\mathcal{G}_\Lambda$ would be just the generator of diffeomorphisms. Its algebra is known to close and produces the Lie bracket.

## 13.2 Projections to the initial and T-dual phase spaces

Let us consider projections of the $C$-bracket to the initial and T-dual phase spaces. Firstly, we will demand that all parameters in (13.13) depend exclusively on the initial coordinates $x^\mu$. In that case, the double derivative $\partial^M$ reduces to the derivative along $x^\mu$, i.e.

$$\partial^M \to \begin{pmatrix} 0 \\ \partial_\mu \end{pmatrix}\,. \tag{13.15}$$

The terms in the $C$-bracket also simplify

$$\Lambda_1^N\partial_N\Lambda_2^M \to \begin{pmatrix} \xi_1^\nu\partial_\nu\xi_2^\mu \\ \xi_1^\nu\partial_\nu\lambda_{2\mu} \end{pmatrix}\,, \quad \Lambda_1^N\partial^M\Lambda_{2N} \to \begin{pmatrix} 0 \\ \lambda_{1\nu}\partial_\mu\xi_2^\nu + \xi_1^\nu\partial_\mu\lambda_{2\nu} \end{pmatrix}\,, \tag{13.16}$$

where parameters depend only on $x$. Substituting the previous relation into (13.13), we obtain the projection of the $C$-bracket to the initial phase space

$$\begin{aligned}
[\Lambda_1,\Lambda_2]_{\mathbf{C}} \to \quad & \begin{pmatrix} \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu \\ \xi_1^\nu\partial_\nu\lambda_{2\mu} - \xi_2^\nu\partial_\nu\lambda_{1\mu} - \frac{1}{2}(\lambda_{1\nu}\partial_\mu\xi_2^\nu - \lambda_{2\nu}\partial_\mu\xi_1^\nu + \xi_1^\nu\partial_\mu\lambda_{2\nu} - \xi_2^\nu\partial_\mu\lambda_{1\nu}) \end{pmatrix} \\
= & \begin{pmatrix} \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu \\ \xi_1^\nu(\partial_\nu\lambda_{2\mu} - \partial_\mu\lambda_{2\nu}) - \xi_2^\nu(\partial_\nu\lambda_{1\mu} - \partial_\mu\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1^\nu\lambda_{2\nu} - \xi_2^\nu\lambda_{1\nu}) \end{pmatrix}\,,
\end{aligned}$$

where we used the chain rule, in order to recognize the result as the Courant bracket. By projecting the $C$-bracket to the initial theory, we obtained the standard Courant bracket.

Secondly, by ignoring all dependence on $x^\mu$, we will obtain the $C$-bracket projection to the T-dual phase space. Then, the double derivative reduces to the derivative along T-dual coordinates $y_\mu$

$$\partial^M \to \begin{pmatrix} \tilde{\partial}^\mu \\ 0 \end{pmatrix} , \tag{13.17}$$

and similarly, we obtain

$$\Lambda_1^N \partial_N \Lambda_2^M \to \begin{pmatrix} \lambda_{1\nu} \tilde{\partial}^\nu \xi_2^\mu \\ \lambda_{1\nu} \tilde{\partial}^\nu \lambda_{2\mu} \end{pmatrix} , \quad \Lambda_1^N \partial^M \Lambda_{2N} \to \begin{pmatrix} \xi_1^\nu \tilde{\partial}^\mu \lambda_{2\nu} + \lambda_{1\nu} \tilde{\partial}^\mu \xi_2^\nu \\ 0 \end{pmatrix} . \tag{13.18}$$

Now all parameters depend solely on T-dual coordinates $y_\mu$. Substituting (13.18) into (13.13), one obtains

$$\begin{aligned} [\Lambda_1, \Lambda_2]_{\mathbf{C}} \to & \begin{pmatrix} \lambda_{1\nu} \tilde{\partial}^\nu \xi_2^\mu - \lambda_{2\nu} \tilde{\partial}^\nu \xi_1^\mu - \frac{1}{2} (\xi_1^\nu \tilde{\partial}^\mu \lambda_{2\nu} + \lambda_{1\nu} \tilde{\partial}^\mu \xi_2^\nu - \xi_2^\nu \tilde{\partial}^\mu \lambda_{1\nu} - \lambda_{2\nu} \tilde{\partial}^\mu \xi_1^\nu) \\ \lambda_{1\nu} \tilde{\partial}^\nu \lambda_{2\mu} - \lambda_{2\nu} \tilde{\partial}^\nu \lambda_{1\mu} \end{pmatrix} \\ = & \begin{pmatrix} \lambda_{1\nu} (\tilde{\partial}^\nu \xi_2^\mu - \tilde{\partial}^\mu \xi_2^\nu) - \lambda_{2\nu} (\tilde{\partial}^\nu \xi_1^\mu - \tilde{\partial}^\mu \xi_1^\nu) + \frac{1}{2} \tilde{\partial}^\mu (\lambda_{1\nu} \xi_2^\nu - \xi_1^\nu \lambda_{2\nu}) \\ \lambda_{1\nu} \tilde{\partial}^\nu \lambda_{2\mu} - \lambda_{2\nu} \tilde{\partial}^\nu \lambda_{1\mu} \end{pmatrix} . \end{aligned}$$

We applied the chain rule in this instance as well. The resulting bracket is again the Courant bracket. This time, the symmetry parameters $\xi^\mu$ and $\lambda_\mu$ have swapped their roles.

Both in the initial and T-dual theory, the $C$-bracket reduces to the Courant bracket. This way, the invariance of the Courant bracket under T-duality is shown once again. The $C$-bracket is a double theory extension of the Courant bracket.

# Chapter 14

# $B$-twisted $C$-bracket

We are going to obtain the $B$-twisted $C$-bracket, together with its corresponding flux. Subsequently, we will consider this bracket's projection to the initial and T-dual phase space and show that in the former it produces the $B$-twisted Couran bracket, while in the latter, it produces the $\theta$-twisted Courant bracket.

## 14.1 Non-canonical basis and basic algebra relations

The generator in the double theory also has a form of the $O(D,D)$ invariant inner product, allowing us to generalize the procedure of twisting the Courant bracket for twisting the $C$-bracket in the double theory. Following the path we took in Chapter 9, we define a diagonal generalized metric $G_{MN}$ by

$$G_{MN} = \begin{pmatrix} G_{\mu\nu}(x,y) & 0 \\ 0 & (G^{-1})^{\mu\nu}(x,y) \end{pmatrix}, \tag{14.1}$$

which by the action of $B$-transformation produces the generalized metric $H_{MN}$

$$((e^{\hat{B}})^T)_M^{\ K}\, G_{KL}(e^{\hat{B}})^L_{\ N} = H_{MN}\,, \quad (e^{\hat{B}})^M_{\ N} = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu}(x,y) & \delta_\mu^\nu \end{pmatrix}. \tag{14.2}$$

The $B$-transformation has exact same form as we encountered before (6.5), with the only difference that the Kalb-Ramond field now depends both on the initial coordinates $x^\mu$ and the T-dual coordinates $y_\mu$. When relation (14.2) is substituted into (12.12) we obtain the free-form expression of the canonical Hamiltonian

$$\mathcal{H}_{\mathcal{C}} = \frac{1}{2\kappa}\hat{\Pi}_M G^{MN}\hat{\Pi}_N + \frac{\kappa}{2}\hat{X}'^M G_{MN}\hat{X}'^N\,, \tag{14.3}$$

where we introduced the non-canonical momenta $\hat{\Pi}$ by

$$\hat{\Pi}^M = (e^{\hat{B}})^M{}_N \Pi^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} {}^\star\pi^\nu \\ \pi_\nu \end{pmatrix} = \begin{pmatrix} {}^\star\pi^\mu \\ \pi_\mu + 2B_{\mu\nu}{}^\star\pi^\nu \end{pmatrix} \equiv \begin{pmatrix} {}^\star\pi^\mu \\ \hat{\pi}_\mu \end{pmatrix}, \tag{14.4}$$

and non-canonical coordinates $\sigma$-derivatives $\hat{X}'$ by

$$\hat{X}'^M = (e^{\hat{B}})^M{}_N X'^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} x'^\nu \\ y'_\nu \end{pmatrix} = \begin{pmatrix} x'^\mu \\ y'_\mu + 2B_{\mu\nu}x'^\nu \end{pmatrix} \equiv \begin{pmatrix} x'^\mu \\ \hat{y}'_\mu \end{pmatrix}. \tag{14.5}$$

The generator $\mathcal{G}_\Lambda$ (13.1) can be expressed in terms of non-canonical momenta $\hat{\Pi}$ by

$$\hat{\mathcal{G}}_{\hat{\Lambda}} = \int d\sigma \langle \hat{\Lambda}, \hat{\Pi} \rangle, \tag{14.6}$$

where we introduced a new symmetry parameter $\hat{\Lambda}$, related to the parameter $\Lambda$ (13.2) by

$$\hat{\Lambda}^M = (e^{\hat{B}})^M{}_N \Lambda^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu \\ \lambda_\mu + 2B_{\mu\nu}\xi^\nu \end{pmatrix} \equiv \begin{pmatrix} \xi^\mu \\ \hat{\lambda}_\mu \end{pmatrix}. \tag{14.7}$$

Using the fact that $e^{\hat{B}}$ is an $O(D,D)$ transformation, the generator algebra (13.12) when expressed in terms of generator $\hat{\mathcal{G}}_\Lambda$ takes the form

$$\{\hat{\mathcal{G}}_{\hat{\Lambda}_1}(\sigma), \hat{\mathcal{G}}_{\hat{\Lambda}_2}(\bar{\sigma})\} = -\hat{\mathcal{G}}_{[\hat{\Lambda}_1,\hat{\Lambda}_2]_{\mathbf{C}_B}}(\sigma)\delta(\sigma-\bar{\sigma}), \tag{14.8}$$

where $[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathbf{C}_B}$ we define as the $B$-twisted $C$-bracket, given by

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathbf{C}_B} = e^{\hat{B}}[e^{-\hat{B}}\hat{\Lambda}_1, e^{-\hat{B}}\hat{\Lambda}_2]_{\mathbf{C}}. \tag{14.9}$$

In order to obtain the $B$-twisted $C$-bracket from the Poisson bracket algebra, we require the Poisson bracket relations between non-canonical momenta $\hat{\Pi}$. Using (14.4) we write

$$\begin{aligned} \{\hat{\Pi}^M(\sigma), \hat{\Pi}^N(\bar{\sigma})\} &= \{(e^{\hat{B}}\Pi)^M(\sigma), (e^{\hat{B}}\Pi)^N(\bar{\sigma})\} \\ &= (e^{\hat{B}})^M_J(\sigma)(e^{\hat{B}})^N_K(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} \\ &\quad -(e^{\hat{B}})^M_J\partial^J(e^{\hat{B}})^N_K\Pi^K\delta(\sigma-\bar{\sigma}) + (e^{\hat{B}})^N_J\partial^J(e^{\hat{B}})^M_K\Pi^K\delta(\sigma-\bar{\sigma}). \end{aligned} \tag{14.10}$$

Next, using the T-duality relations (12.26) on the first term of the right-hand side of the previous expression, we obtain

$$(e^{\hat{B}})^M_J(\sigma)(e^{\hat{B}})^N_K(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} \simeq \kappa\left[e^B(\sigma)\eta(e^B)^T(\bar{\sigma})\right]^{MN}\delta'(\sigma-\bar{\sigma}), \tag{14.11}$$

which can be further tranformed by

$$\Big[e^{B}(\sigma)\eta(e^{B})^T(\bar\sigma)\Big]^{MN}\delta'(\sigma-\bar\sigma) = \kappa\eta^{MN}\delta'(\sigma-\bar\sigma) + \kappa(e^{\hat B})^M_{\ P}\eta^{PR}\partial_Q((e^{\hat B})^T)_R^{\ N}X'^Q\delta(\sigma-\bar\sigma)\,, \tag{14.12}$$

where we have used (7.7) and (6.3) for $B$-shifts. After applying the T-dual relations (12.17) to the non-anomalous part of (14.12), we obtain

$$\kappa(e^{\hat B})^M_{\ P}\,\eta^{PR}\partial_Q((e^{\hat B})^T)_R^{\ N}X'^Q\delta(\sigma-\bar\sigma) \simeq (e^{\hat B})^M_{\ P}\,\partial_Q\hat B^{PN}\Pi^Q\delta(\sigma-\bar\sigma)\,. \tag{14.13}$$

We note the following properties of matrix $\hat B^M_{\ N}$ (6.5)

$$\hat B^M_{\ K}\hat B^K_{\ N}=0\,,\qquad \hat B^M_{\ K}\partial^Q\hat B^K_{\ N}=0\,,\qquad (e^{\hat B})^M_{\ N}=\delta^M_{\ N}+\hat B^M_{\ N}\,, \tag{14.14}$$

and rewrite the relation (14.10) as

$$\{\hat\Pi^M(\sigma),\hat\Pi^N(\bar\sigma)\} = -\hat B^{MNQ}\,\hat\Pi_Q\delta(\sigma-\bar\sigma)+A^{MN}(\sigma-\bar\sigma)\,. \tag{14.15}$$

With $A^{MN}$ we have marked the anomalous term, given by

$$A^{MN}(\sigma-\bar\sigma)\simeq\kappa\eta^{MN}\delta'(\sigma-\bar\sigma)\,, \tag{14.16}$$

and with $\hat B^{MNQ}$ the double flux, where

$$\begin{aligned}\hat B^{MNQ} &= B^{MNQ}+S^{MNQ}\\ B^{MNQ} &= \partial^M\hat B^{NQ}+\partial^N\hat B^{QM}+\partial^Q\hat B^{MN}\\ S^{MNQ} &= \hat B^M_{\ K}\partial^K\hat B^{NQ}+\hat B^N_{\ K}\partial^K\hat B^{QM}+\hat B^Q_{\ K}\partial^K\hat B^{MN}\,.\end{aligned} \tag{14.17}$$

Flux can be written in a more compact manner

$$\hat B^{MNQ}=\hat\partial^M\hat B^{NQ}+\hat\partial^N\hat B^{QM}+\hat\partial^Q\hat B^{MN}\,, \tag{14.18}$$

where $\hat\partial$ is a new double derivative, given by

$$\hat\partial^M=(e^{\hat B})^M_{\ K}\partial^K=\partial^M+\hat B^M_{\ K}\,\partial^K\,. \tag{14.19}$$

Appart from the relation (14.15), we will also need another basic Poisson relation

$$\{\hat\Lambda^M(\sigma),\hat\Pi^N(\bar\sigma)\}=\hat\partial^N\hat\Lambda^M\delta(\sigma-\bar\sigma)\,, \tag{14.20}$$

and note that the bracket between parameters is zero, i.e.

$$\{\hat\Lambda^M(\sigma),\hat\Lambda^N(\bar\sigma)\}=0\,, \tag{14.21}$$

as a direct consequence of (12.29) and (12.31) (see discussion below (13.4) for more details).

## 14.2 Derivation of $B$-twisted $C$-bracket

Substituting (14.15), (14.20) and (14.21) into (14.8), we obtain

$$\begin{aligned}\{\hat{\mathcal{G}}_{\hat{\Lambda}_1}(\sigma),\hat{\mathcal{G}}_{\hat{\Lambda}_2}(\bar\sigma)\} &= \hat\Lambda_1^M(\sigma)\hat\Lambda_2^N(\bar\sigma)A_{MN} - \hat\Lambda_{1M}\hat\Lambda_{2N}\hat B^{MNQ}\hat\Pi_Q\delta(\sigma-\bar\sigma) \\ &+\hat\Pi_Q\Big[\hat\Lambda_2^N\hat\partial_N\hat\Lambda_1^Q - \hat\Lambda_1^N\hat\partial_N\hat\Lambda_2^Q\Big]\delta(\sigma-\bar\sigma)\,.\end{aligned} \tag{14.22}$$

The first term containing anomaly is transformed with the help of (7.7) and (14.16) by

$$\begin{aligned}&\hat\Lambda_1^M(\sigma)\hat\Lambda_2^N(\bar\sigma)A_{MN}(\sigma-\bar\sigma) \\ &\simeq \kappa\langle\hat\Lambda_1(\sigma),\hat\Lambda_2(\sigma)\rangle\delta'(\sigma-\bar\sigma) + \kappa\langle\hat\Lambda_1(\sigma),\hat\Lambda_2'(\sigma)\rangle\delta(\sigma-\bar\sigma) \\ &= \frac{\kappa}{2}\Big(2\langle\hat\Lambda_1,\hat\Lambda_2\rangle\delta'(\sigma-\bar\sigma) + \langle\hat\Lambda_1,\hat\Lambda_2\rangle'\delta(\sigma-\bar\sigma)\Big) + \frac{\kappa}{2}\Big(\langle\hat\Lambda_1,\hat\Lambda_2'\rangle - \langle\hat\Lambda_1',\hat\Lambda_2\rangle\Big)\delta(\sigma-\bar\sigma) \\ &= \frac{\kappa}{2}\Big(\langle\hat\Lambda_1,\hat\Lambda_2\rangle(\sigma) + \langle\hat\Lambda_1,\hat\Lambda_2\rangle(\bar\sigma)\Big)\delta'(\sigma-\bar\sigma) + \frac{\kappa}{2}\Big(\langle\hat\Lambda_1,\hat\Lambda_2'\rangle - \langle\hat\Lambda_1',\hat\Lambda_2\rangle\Big)\delta(\sigma-\bar\sigma)\,,\end{aligned} \tag{14.23}$$

resulting in two terms. The first term is anomalous and dissappears after the integration. On the second term, the T-duality relations (12.17) can be applied, after which one obtains

$$\begin{aligned}\frac{\kappa}{2}\Big(\langle\hat\Lambda_1,\hat\Lambda_2'\rangle - \langle\hat\Lambda_1',\hat\Lambda_2\rangle\Big) &= \frac{\kappa}{2}\eta_{MN}\Big(\hat\Lambda_1^M\partial_Q\hat\Lambda_2^N - \hat\Lambda_2^M\partial_Q\hat\Lambda_1^N\Big)X'^Q \\ &\simeq \frac{1}{2}\eta_{MN}\Big(\hat\Lambda_1^M\partial^Q\hat\Lambda_2^N - \hat\Lambda_2^M\partial^Q\hat\Lambda_1^N\Big)\Pi_Q \\ &= \frac{1}{2}\eta_{MN}\Big(\hat\Lambda_1^M\hat\partial^Q\hat\Lambda_2^N - \hat\Lambda_2^M\hat\partial^Q\hat\Lambda_1^N\Big)\hat\Pi_Q\,,\end{aligned} \tag{14.24}$$

where we used (14.19) and (14.4). Note that this is a second application of T-duality, which acts as equality. The substitution of (14.23) and (14.24) into (14.22) results in the final expression for the $B$-twisted $C$-bracket

$$\begin{aligned}\Big([\hat\Lambda_1,\hat\Lambda_2]_{\mathbf{C}_B}\Big)^M &= \hat\Lambda_1^N\hat\partial_N\hat\Lambda_2^M - \hat\Lambda_2^N\hat\partial_N\hat\Lambda_1^M \\ &-\frac{1}{2}\Big(\hat\Lambda_1^N\hat\partial^M\hat\Lambda_{2N} - \hat\Lambda_2^N\hat\partial^M\hat\Lambda_{1N}\Big) + \hat\Lambda_{1N}\hat\Lambda_{2Q}\hat B^{MNQ}\,.\end{aligned} \tag{14.25}$$

With the substitution of (14.19) into its expression, $B$-twisted $C$-bracket becomes

$$\begin{aligned}\Big([\hat\Lambda_1,\hat\Lambda_2]_{\mathbf{C}_B}\Big)^M &= \hat\Lambda_1^N\partial_N\hat\Lambda_2^M - \hat\Lambda_2^N\partial_N\hat\Lambda_1^M - \frac{1}{2}\Big(\hat\Lambda_1^N\partial^M\hat\Lambda_{2N} - \hat\Lambda_2^N\partial^M\hat\Lambda_{1N}\Big) \\ &+\hat B^N{}_R\Big(\hat\Lambda_{1N}\partial^R\hat\Lambda_2^M - \hat\Lambda_{2N}\partial^R\hat\Lambda_1^M\Big) - \frac{1}{2}\hat B^M{}_R\Big(\hat\Lambda_{1N}\partial^R\hat\Lambda_2^N - \hat\Lambda_{2N}\partial^R\hat\Lambda_1^N\Big) \\ &+\hat\Lambda_{1N}\hat\Lambda_{2Q}\hat B^{MNQ}\,.\end{aligned} \tag{14.26}$$

We can see that the first line is the $C$-bracket, while the other two lines are corrections due to twisting. If the Kalb-Ramond field is zero, the second and the third lines become zero and the bracket reduces to the $C$-bracket.

## 14.3 Projections to the initial and T-dual phase space

Firstly, let us consider the $B$-twisted $C$-bracket projected to the initial phase space. It can be obtained by demanding that all gauge fields depend solely on the initial coordinates $x^\mu$. In that case, the derivatives $\hat{\partial}^M$ become just derivatives along the initial coordinates $x^\mu$

$$\hat{\partial}^M \to \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} 0 \\ \partial_\nu \end{pmatrix} = \begin{pmatrix} 0 \\ \partial_\mu \end{pmatrix} . \tag{14.27}$$

The terms from the bracket simplify as

$$\hat{\Lambda}_1^N \hat{\partial}_N \hat{\Lambda}_2^M \to \begin{pmatrix} \xi_1^\nu \partial_\nu \xi_2^\mu \\ \xi_1^\nu \partial_\nu \hat{\lambda}_{2\mu} \end{pmatrix} , \tag{14.28}$$

and

$$\hat{\Lambda}_1^N \hat{\partial}^M \hat{\Lambda}_{2N} \to \begin{pmatrix} 0 \\ \hat{\lambda}_{1\nu} \partial_\mu \xi_2^\nu + \xi_1^\nu \partial_\mu \hat{\lambda}_{2\nu} \end{pmatrix} , \tag{14.29}$$

while the flux $\hat{B}^{MNQ}$ reduces to the standard $H$-flux, i.e.

$$\hat{B}^{MNQ} \hat{\Lambda}_{1N} \hat{\Lambda}_{2Q} \to \begin{pmatrix} 0 \\ 2B_{\mu\nu\rho} \xi_1^\nu \xi_2^\rho \end{pmatrix} . \tag{14.30}$$

Combining previous relations and using the chain rule, we obtain

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathbf{C}_B} \to [\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_B} = \hat{\Lambda} \equiv \begin{pmatrix} \xi \\ \hat{\lambda} \end{pmatrix} , \tag{14.31}$$

where

$$\begin{aligned} \xi^\mu &= \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu , \\ \hat{\lambda}_\mu &= \xi_1^\nu (\partial_\nu \hat{\lambda}_{2\mu} - \partial_\mu \hat{\lambda}_{2\nu}) - \xi_2^\nu (\partial_\nu \hat{\lambda}_{1\mu} - \partial_\mu \hat{\lambda}_{1\nu}) + \frac{1}{2} \partial_\mu (\xi_1 \hat{\lambda}_2 - \xi_2 \hat{\lambda}_1) + 2B_{\mu\nu\rho} \xi_1^\nu \xi_2^\rho . \end{aligned} \tag{14.32}$$

The $B$-twisted $C$-bracket becomes $B$-twisted Courant bracket in the initial theory.

Secondly, let us obtain the projection of $B$-twisted $C$-bracket to the T-dual phase space, by demanding that all variables depend solely on T-dual coordinates $y_\mu$. In this case, the derivative $\hat{\partial}^M$ becomes

$$\hat{\partial}^M \to \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \tilde{\partial}^\nu \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\partial}^\mu \\ 2B_{\mu\nu} \tilde{\partial}^\nu \end{pmatrix} , \tag{14.33}$$

so that the bracket terms reduce to

$$\hat{\Lambda}_1^N \partial_N \hat{\Lambda}_2^M \to \begin{pmatrix} \hat{\lambda}_{1\nu}\tilde{\partial}^\nu \xi_2^\mu + 2B_{\nu\rho}\xi_1^\rho \tilde{\partial}^\nu \xi_2^\mu \\ \hat{\lambda}_{1\nu}\tilde{\partial}^\nu \hat{\lambda}_{2\mu} + 2B_{\nu\rho}\xi_1^\rho \tilde{\partial}^\nu \hat{\lambda}_{2\mu} \end{pmatrix} , \tag{14.34}$$

and

$$\hat{\Lambda}_1^N \tilde{\partial}^M \hat{\Lambda}_{2N} \to \begin{pmatrix} \hat{\lambda}_{1\nu}\tilde{\partial}^\mu \xi_2^\nu + \xi_1^\nu \tilde{\partial}^\mu \hat{\lambda}_{2\nu} \\ 2\hat{\lambda}_{1\nu} B_{\mu\rho}\tilde{\partial}^\rho \xi_2^\nu + 2\xi_1^\nu B_{\mu\rho}\tilde{\partial}^\rho \hat{\lambda}_{2\nu} \end{pmatrix} . \tag{14.35}$$

The term containing flux $\hat{B}^{MNQ}$ becomes

$$\hat{B}^{MNQ}\hat{\Lambda}_{1N}\hat{\Lambda}_{2Q} \to \begin{pmatrix} \kappa\, {}^\star Q^\mu{}_{\nu\rho}\xi_1^\nu \xi_2^\rho \\ \kappa\, {}^\star Q^\nu{}_{\rho\mu}(\xi_1^\rho \hat{\lambda}_{2\nu} - \xi_2^\rho \hat{\lambda}_{1\nu}) + \kappa^{2\star} R_{\mu\nu\rho}\xi_1^\nu \xi_2^\rho \end{pmatrix} , \tag{14.36}$$

where we have marked the non-geometric fluxes in T-dual theory as a function of the T-dual non-commutative parameter ${}^\star\theta_{\mu\nu} = \frac{2}{\kappa} B_{\mu\nu}$ by

$$\kappa\, {}^\star Q^\mu{}_{\nu\rho} = 2\tilde{\partial}^\mu B_{\nu\rho} = \kappa\, \tilde{\partial}^\mu\, {}^\star\theta_{\nu\rho} , \tag{14.37}$$

and

$$\begin{aligned} \kappa^{2\star} R_{\mu\nu\rho} &= 4B_{\mu\sigma}\tilde{\partial}^\sigma B_{\nu\rho} + 4B_{\nu\sigma}\tilde{\partial}^\sigma B_{\rho\mu} + 4B_{\rho\sigma}\tilde{\partial}^\sigma B_{\mu\nu} , \\ &= \kappa^2\, {}^\star\theta_{\mu\sigma}\tilde{\partial}^\sigma\, {}^\star\theta_{\nu\rho} + \kappa^2\, {}^\star\theta_{\nu\sigma}\tilde{\partial}^\sigma\, {}^\star\theta_{\rho\mu} + \kappa^2\, {}^\star\theta_{\rho\sigma}\tilde{\partial}^\sigma\, {}^\star\theta_{\mu\nu} . \end{aligned} \tag{14.38}$$

Combining previous relations and using the chain rule, the $B$-twisted $C$-bracket becomes

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathbf{C}_B} \to [\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_B} = \hat{\Lambda} \equiv \begin{pmatrix} \xi \\ \hat{\lambda} \end{pmatrix} , \tag{14.39}$$

where

$$\begin{aligned} \xi^\mu &= \hat{\lambda}_{1\nu}(\tilde{\partial}^\nu \xi_2^\mu - \tilde{\partial}^\mu \xi_2^\nu) - \hat{\lambda}_{2\nu}(\tilde{\partial}^\nu \xi_1^\mu - \tilde{\partial}^\mu \xi_1^\nu) + \tilde{\partial}^\mu(\xi_1 \hat{\lambda}_2 - \xi_2 \hat{\lambda}_1) \\ &\quad 2B_{\nu\rho}(\xi_1^\nu \tilde{\partial}^\rho \xi_2^\mu - \xi_2^\nu \tilde{\partial}^\rho \xi_1^\mu) + 2\tilde{\partial}^\mu B_{\nu\rho}\xi_1^\nu \xi_2^\rho , \\ \hat{\lambda}_\mu &= \hat{\lambda}_{1\nu}\tilde{\partial}^\nu \hat{\lambda}_{2\mu} - \hat{\lambda}_{2\nu}\tilde{\partial}^\nu \hat{\lambda}_{1\mu} \\ &\quad -2B_{\mu\nu}\Big(\hat{\lambda}_{1\rho}(\tilde{\partial}^\nu \xi_2^\rho - \tilde{\partial}^\rho \xi_2^\nu) - \hat{\lambda}_{2\rho}(\tilde{\partial}^\nu \xi_1^\rho - \tilde{\partial}^\rho \xi_1^\nu) - \frac{1}{2}\tilde{\partial}^\nu(\hat{\lambda}_1 \xi_2 - \hat{\lambda}_2 \xi_1)\Big) \\ &\quad +2\hat{\lambda}_{1\nu}\tilde{\partial}^\nu(\xi_2^\rho B_{\rho\mu}) - 2\hat{\lambda}_{2\nu}\tilde{\partial}^\nu(\xi_1^\rho B_{\rho\mu}) + 2(\xi_1^\nu B_{\nu\rho})\tilde{\partial}^\rho \hat{\lambda}_{2\mu} - 2(\xi_2^\nu B_{\nu\rho})\tilde{\partial}^\rho \hat{\lambda}_{1\mu} \\ &\quad +4\Big(B_{\mu\sigma}\tilde{\partial}^\sigma B_{\nu\rho} + B_{\nu\sigma}\tilde{\partial}^\sigma B_{\rho\mu} + B_{\rho\sigma}\tilde{\partial}^\sigma B_{\mu\nu}\Big)\xi_1^\nu \xi_2^\rho . \end{aligned} \tag{14.40}$$

With the following change of variables

$${}^\star\hat{\lambda}^\mu = \xi^\mu , \quad {}^\star\xi_\mu = \hat{\lambda}_\mu , \tag{14.41}$$

the $B$-twisted $C$-bracket becomes ${}^\star\theta$-twisted Courant bracket in the T-dual phase space (14.40).

These are very interesting results that show the attractiveness of the double theory. When we considered only the initial theory and symmetries therein, we needed to act with different transformations on the generator's basis to obtain the $B$- and $\theta$-twisted Courant bracket. However, in double theory, they are both easily obtained from the projection of the $B$-twisted $C$-bracket to the relevant subspaces. Moreover, these projections reduce the double flux $\hat{B}^{MNQ}$ to the geometric $H$-flux, and also non-geometric $Q$ and $R$-fluxes, depending on the phase space to which we project it. Lastly, we discussed T-duality in the context of isomorphism between Courant algebroids and showed that with the exchange of background fields with their T-duals according to the Buscher rules, together with momenta and coordinate $\sigma$-derivatives, $B$-twisted and $\theta$-twisted Courant algebroids transform into each other. This Courant algebroid isomorphism becomes manifest in a double theory, as both algebroids can be obtained from the single bracket defined in a double space.

# Chapter 15

# $\theta$-twisted $C$-bracket

This chapter we devote to the derivation of the $\theta$-twisted $C$-bracket and its corresponding double flux. They have the same form as their $B$-twisted counterparts, which was not the case with their respective Courant brackets. We consider the projection of this bracket to the mutually T-dual phase spaces and obtain the $\theta$-twisted $C$-bracket in the initial, and $B$-twisted $C$-bracket in the T-dual phase space.

## 15.1 Non-canonical basis and basic algebra relations

In the analogy with the derivation of the $\theta$-twisted Courant bracket, we consider the string moving in a double space-time characterized by the T-dual metric

$$^\star G_{MN} = \begin{pmatrix} ^\star G^{-1}_{\mu\nu}(x,y) & 0 \\ 0 & ^\star G^{\mu\nu}(x,y) \end{pmatrix} = \begin{pmatrix} G^E_{\mu\nu}(x,y) & 0 \\ 0 & (G_E^{-1})^{\mu\nu}(x,y) \end{pmatrix} , \tag{15.1}$$

where $G_E$ is defined in (2.36). The generalized metric can be obtained from the action of $\theta$-transformation $e^{\hat{\theta}}$ (6.7)

$$^\star H_{MN} = ((e^{\hat{\theta}})^T)_M^{\ L}\, {}^\star G_{LK} (e^{\hat{\theta}})^K_{\ N} = \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix} , \tag{15.2}$$

which is exactly equal to the generalized metric (2.38), with the difference of background fields depending also on T-dual coordinates $y_\mu$. In terms of $^\star G_{MN}$ (15.1), the canonical Hamiltonian (12.12) is written in the form of a free Hamiltonian as

$$\mathcal{H}_{\mathcal{C}} = \frac{1}{2\kappa} \breve{\Pi}_M {}^\star G^{MN} \breve{\Pi}_N + \frac{\kappa}{2} \breve{X}'^M {}^\star G_{MN} \breve{X}'^N , \tag{15.3}$$

where the new non-canonical double coordinates $\sigma$-derivatives are given by

$$\breve{X}'^M = (e^{\hat{\theta}})^M_{\ N} X'^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta_\mu^\nu \end{pmatrix} \begin{pmatrix} x'^\nu \\ y'_\nu \end{pmatrix} = \begin{pmatrix} x'^\mu + \kappa\theta^{\mu\nu} y'_\nu \\ y'_\mu \end{pmatrix} \equiv \begin{pmatrix} \breve{x}'^\mu \\ y'_\mu \end{pmatrix} , \tag{15.4}$$

and new non-canonical double momenta by

$$\breve{\Pi}^M = (e^{\hat{\theta}})^M{}_N \Pi^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} {}^\star\pi^\nu \\ \pi_\nu \end{pmatrix} = \begin{pmatrix} {}^\star\pi^\mu + \kappa\theta^{\mu\nu}\pi_\nu \\ \pi_\mu \end{pmatrix} \equiv \begin{pmatrix} {}^\star\breve{\pi}^\mu \\ \pi_\mu \end{pmatrix} . \tag{15.5}$$

In this basis, the symmetry generator is given by

$$\breve{\mathcal{G}}_{\breve{\Lambda}} = \int d\sigma \langle \breve{\Lambda}, \breve{\Pi} \rangle \,, \tag{15.6}$$

where

$$\breve{\Lambda}^M = (e^{\hat{\theta}})^M{}_N \Lambda^N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu + \kappa\theta^{\mu\nu}\lambda_\nu \\ \lambda_\mu \end{pmatrix} \equiv \begin{pmatrix} \breve{\xi}^\mu \\ \lambda_\mu \end{pmatrix} . \tag{15.7}$$

The $\theta$-twisted $C$-bracket appears in the algebra of generators (15.6), via relation

$$\{\breve{\mathcal{G}}_{\breve{\Lambda}_1}(\sigma), \breve{\mathcal{G}}_{\breve{\Lambda}_2}(\bar{\sigma})\} = -\breve{\mathcal{G}}_{[\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta}}(\sigma)\delta(\sigma - \bar{\sigma}) \,, \tag{15.8}$$

where

$$[\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta} = e^{\hat{\theta}}[e^{-\hat{\theta}}\breve{\Lambda}_1, e^{-\hat{\theta}}\breve{\Lambda}_2]_{\mathbf{C}} \,. \tag{15.9}$$

In order to compute this bracket, we need to obtain the algebra between non-canonical momenta, which is expanded as

$$\begin{aligned} \{\breve{\Pi}^M(\sigma), \breve{\Pi}^N(\bar{\sigma})\} &= \{(e^{\hat{\theta}}\Pi)^M(\sigma), (e^{\hat{\theta}}\Pi)^N(\bar{\sigma})\} \\ &= (e^{\hat{\theta}})^M_J(\sigma)(e^{\hat{\theta}})^N_K(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} \\ &\quad -(e^{\hat{\theta}})^M_J \partial^J (e^{\hat{\theta}})^N_Q \Pi^Q \delta(\sigma - \bar{\sigma}) + (e^{\hat{\theta}})^N_J \partial^J (e^{\hat{\theta}})^M_Q \Pi^Q \delta(\sigma - \bar{\sigma}) \,. \end{aligned} \tag{15.10}$$

Using (12.26) and (7.7), we obtain

$$(e^{\hat{\theta}})^M_J(\sigma)(e^{\hat{\theta}})^N_K(\bar{\sigma})\{\Pi^J(\sigma), \Pi^K(\bar{\sigma})\} = A^{MN}(\sigma - \bar{\sigma}) + (e^{\hat{\theta}})^M_P \partial_Q \hat{\theta}^{PN} \Pi^Q \delta(\sigma - \bar{\sigma}) \,, \tag{15.11}$$

where $A^{MN}$ is the same anomaly defined in (14.16). Substituting (15.11) and (15.5) into (15.10), we obtain

$$\{\breve{\Pi}^M(\sigma), \breve{\Pi}^N(\bar{\sigma})\} = -\breve{\Theta}^{MNQ}\breve{\Pi}_Q \delta(\sigma - \bar{\sigma}) + A^{MN}(\sigma - \bar{\sigma}) \,, \tag{15.12}$$

where

$$\begin{aligned} \breve{\Theta}^{MNQ} &= \Theta^{MNQ} + R^{MNQ} \\ \Theta^{MNQ} &= \partial^M \hat{\theta}^{NQ} + \partial^N \hat{\theta}^{QM} + \partial^Q \hat{\theta}^{MN} \\ R^{MNQ} &= \hat{\theta}^M{}_K \partial^K \hat{\theta}^{NQ} + \hat{\theta}^N{}_K \partial^K \hat{\theta}^{QM} + \hat{\theta}^Q{}_K \partial^K \hat{\theta}^{MN} \,. \end{aligned} \tag{15.13}$$

In a similar manner as when twisting by $B$, we introduce derivatives $\breve{\partial}^M$ by

$$\breve{\partial}^M = (e^{\hat{\theta}})^M_{\ N}\, \partial^N = \partial^M + \hat{\theta}^M_{\ N}\, \partial^N\,, \tag{15.14}$$

and express the flux in a more compact form

$$\breve{\Theta}^{MNR} = \breve{\partial}^M \hat{\theta}^{NR} + \breve{\partial}^N \hat{\theta}^{RM} + \breve{\partial}^R \hat{\theta}^{MN}\,. \tag{15.15}$$

From definition of $\breve{\Pi}^M$ one easily obtains the relation

$$\{\breve{\Lambda}^M(\sigma), \breve{\Pi}^N(\bar{\sigma})\} = \breve{\partial}^N \breve{\Lambda}^M \delta(\sigma - \bar{\sigma})\,, \tag{15.16}$$

and from the strong constraints (12.31), the algebra between symmetry parameters is zero.

We note that the algebra relations between non-canonical momenta $\hat{\Pi}^M$ (14.15) and parameters $\hat{\Lambda}^M$ (14.20) on the one side, and non-canonical momenta $\breve{\Pi}^M$ (15.12) and parameters $\breve{\Lambda}^M$ (15.16) on the other side, have the exact same form. The difference is that the former basic relations are expressed in terms of derivatives $\hat{\partial}^M$ (14.19) and flux $\hat{B}^{MNR}$ (14.18), and the latter in terms of derivatives $\breve{\partial}^M$ (15.14) and flux $\breve{\Theta}^{MNR}$ (15.15). Therefore, the $\theta$-twisted $C$-bracket can be obtained from relation (14.25), simply by substituting the relevant expressions with their analogons. We obtain

$$\begin{aligned}\Big([\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta}\Big)^M &= \breve{\Lambda}_1^N \breve{\partial}_N \breve{\Lambda}_2^M - \breve{\Lambda}_2^N \breve{\partial}_N \breve{\Lambda}_1^M \\ &\quad -\frac{1}{2}\Big(\breve{\Lambda}_1^N \breve{\partial}^M \breve{\Lambda}_{2N} - \breve{\Lambda}_2^N \breve{\partial}^M \breve{\Lambda}_{1N}\Big) + \breve{\Lambda}_{1N}\breve{\Lambda}_{2Q}\breve{\Theta}^{MNQ}\,,\end{aligned} \tag{15.17}$$

which once the expression (15.14) is substituted becomes

$$\begin{aligned}\Big([\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta}\Big)^M &= \breve{\Lambda}_1^N \partial_N \breve{\Lambda}_2^M - \breve{\Lambda}_2^N \partial_N \breve{\Lambda}_1^M - \frac{1}{2}\Big(\breve{\Lambda}_1^N \partial^M \breve{\Lambda}_{2N} - \breve{\Lambda}_2^N \partial^M \breve{\Lambda}_{1N}\Big) \\ &\quad +\hat{\theta}^N_{\ R}\Big(\breve{\Lambda}_{1N}\partial^R \breve{\Lambda}_2^M - \breve{\Lambda}_{2N}\partial^R \breve{\Lambda}_1^M\Big) - \frac{1}{2}\hat{\theta}^M_{\ R}\Big(\breve{\Lambda}_{1N}\partial^R \breve{\Lambda}_2^N - \breve{\Lambda}_{2N}\partial^R \breve{\Lambda}_1^N\Big) \\ &\quad +\breve{\Lambda}_{1N}\breve{\Lambda}_{2Q}\breve{\Theta}^{MNQ}\,.\end{aligned} \tag{15.18}$$

The first line is the $C$-bracket, while the remaining terms are contributions due to its twisting by $\theta$.

## 15.2 Projections to the initial and T-dual phase space

We conclude this chapter with the projections of the $\theta$-twisted $C$-bracket to the initial and T-dual phase spaces. In the former case, all fields and parameters will only depend on the initial coordinates $x^\mu$. The derivative $\breve{\partial}^M$ becomes

$$\breve{\partial}^M \to \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta_\mu^\nu \end{pmatrix} \begin{pmatrix} 0 \\ \partial_\nu \end{pmatrix} = \begin{pmatrix} \kappa\theta^{\mu\nu}\partial_\nu \\ \partial_\mu \end{pmatrix}\,, \tag{15.19}$$

and moreover

$$\breve{\Lambda}_1^N \breve{\partial}_N \breve{\Lambda}_2^M \to \begin{pmatrix} \lambda_{1\nu}\kappa\theta^{\nu\rho}\partial_\rho\breve{\xi}_2^\mu + \breve{\xi}_1^\nu\partial_\nu\breve{\xi}_2^\mu \\ \lambda_{1\nu}\kappa\theta^{\nu\rho}\partial_\rho\lambda_{2\mu} + \breve{\xi}_1^\nu\partial_\nu\lambda_{2\mu} \end{pmatrix} , \tag{15.20}$$

and

$$\breve{\Lambda}_1^N \breve{\partial}^M \breve{\Lambda}_{2N} \to \begin{pmatrix} \kappa\theta^{\mu\nu}(\breve{\xi}_1^\rho\partial_\nu\lambda_{2\rho} + \lambda_{1\rho}\partial_\nu\breve{\xi}_2^\rho) \\ \breve{\xi}_1^\rho\partial_\mu\lambda_{2\rho} + \lambda_{1\rho}\partial_\mu\breve{\xi}_2^\rho \end{pmatrix} . \tag{15.21}$$

The flux term is given by

$$\breve{\Lambda}_{1N}\breve{\Lambda}_{2Q}\breve{\Theta}^{MNQ} \to \begin{pmatrix} \kappa^2 R^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho} + \kappa Q_\nu^{\ \rho\mu}(\breve{\xi}_1^\nu\lambda_{2\rho} - \breve{\xi}_2^\nu\lambda_{1\rho}) \\ \kappa Q_\mu^{\ \rho\nu}\lambda_{1\rho}\lambda_{2\nu} \end{pmatrix} , \tag{15.22}$$

where $Q$ and $R$ are non-geometric fluxes (10.8).

Substituting (15.20), (15.21) and (15.22) into (15.17) we obtain the projection of the $\theta$-twisted $C$-bracket

$$[\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta} \to [\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathcal{C}_\theta} = \breve{\Lambda} \equiv \begin{pmatrix} \breve{\xi} \\ \lambda \end{pmatrix} , \tag{15.23}$$

where

$$\begin{aligned} \breve{\xi}^\mu \;=\;& \breve{\xi}_1^\nu\partial_\nu\breve{\xi}_2^\mu - \breve{\xi}_2^\nu\partial_\nu\breve{\xi}_1^\mu + \\ & -\kappa\theta^{\mu\nu}\Big(\breve{\xi}_1^\rho(\partial_\nu\lambda_{2\rho} - \partial_\rho\lambda_{2\nu}) - \breve{\xi}_2^\rho(\partial_\nu\lambda_{1\rho} - \partial_\rho\lambda_{1\nu}) - \frac{1}{2}\partial_\nu(\breve{\xi}_1\lambda_2 - \breve{\xi}_2\lambda_1)\Big) \\ & +\kappa\breve{\xi}_1^\nu\partial_\nu(\lambda_{2\rho}\theta^{\rho\mu}) - \kappa\breve{\xi}_2^\nu\partial_\nu(\lambda_{1\rho}\theta^{\rho\mu}) + \kappa(\lambda_{1\nu}\theta^{\nu\rho})\partial_\rho\breve{\xi}_2^\mu - \kappa(\lambda_{2\nu}\theta^{\nu\rho})\partial_\rho\breve{\xi}_1^\mu \\ & +\kappa^2(\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu})\lambda_{1\nu}\lambda_{2\rho} , \\ \lambda_\mu \;=\;& \breve{\xi}_1^\nu(\partial_\nu\lambda_{2\mu} - \partial_\mu\lambda_{2\nu}) - \breve{\xi}_2^\nu(\partial_\nu\lambda_{1\mu} - \partial_\mu\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\breve{\xi}_1\lambda_2 - \breve{\xi}_2\lambda_1) \\ & +\kappa\theta^{\nu\rho}(\lambda_{1\nu}\partial_\rho\lambda_{2\mu} - \lambda_{2\nu}\partial_\rho\lambda_{1\mu}) + \kappa\lambda_{1\rho}\lambda_{2\nu}\partial_\mu\theta^{\rho\nu} . \end{aligned} \tag{15.24}$$

These are relations defining the $\theta$-twisted Courant bracket.

On the other hand, the projection to the T-dual phase space is obtained by keeping only the terms with the T-dual coordinates $y_\mu$. The double derivatives are just derivatives along the T-dual coordinates $y_\mu$, i.e.

$$\breve{\partial}^M \to \begin{pmatrix} \delta_\nu^\mu & \kappa\theta^{\mu\nu} \\ 0 & \delta_\mu^\nu \end{pmatrix} \begin{pmatrix} \tilde{\partial}^\nu \\ 0 \end{pmatrix} = \begin{pmatrix} \tilde{\partial}^\mu \\ 0 \end{pmatrix} . \tag{15.25}$$

Furthermore, we have

$$\breve{\Lambda}_1^N \breve{\partial}_N \breve{\Lambda}_2^M \to \begin{pmatrix} \lambda_{1\nu}\tilde{\partial}^\nu\breve{\xi}_2^\mu \\ \lambda_{1\nu}\tilde{\partial}^\nu\lambda_{2\mu} \end{pmatrix} \tag{15.26}$$

and

$$\breve{\Lambda}_1^N \breve{\partial}^M \breve{\Lambda}_{2N} \to \begin{pmatrix} \lambda_{1\nu}\tilde{\partial}^\mu \breve{\xi}_2^\nu + \breve{\xi}_1^\nu \tilde{\partial}^\mu \lambda_{2\nu} \\ 0 \end{pmatrix} , \tag{15.27}$$

while the flux term is simply given by

$$\breve{\Lambda}_{1N}\breve{\Lambda}_{2Q}\breve{\Theta}^{MNQ} \to \begin{pmatrix} \kappa\, {}^\star B^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho} \\ 0 \end{pmatrix} , \tag{15.28}$$

where ${}^\star B^{\mu\nu\rho}$ is the $H$ flux in T-dual theory

$$\begin{aligned} \kappa^\star B^{\mu\nu\rho} &= \kappa\, \tilde{\partial}^\mu \theta^{\nu\rho} + \kappa\, \tilde{\partial}^\nu \theta^{\rho\mu} + \kappa\, \tilde{\partial}^\rho \theta^{\mu\nu} \\ &= 2\tilde{\partial}^\mu\, {}^\star B^{\nu\rho} + 2\tilde{\partial}^\nu\, {}^\star B^{\rho\mu} + 2\tilde{\partial}^\rho\, {}^\star B^{\mu\nu} . \end{aligned} \tag{15.29}$$

The expression for $\theta$-twisted $C$-bracket projected to the T-dual phase space is given by

$$[\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathbf{C}_\theta} \to [\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathcal{C}_\theta} = \breve{\Lambda} \equiv \begin{pmatrix} \breve{\xi} \\ \lambda \end{pmatrix} , \tag{15.30}$$

where

$$\begin{aligned} \breve{\xi}^\mu &= \lambda_{1\nu}(\tilde{\partial}^\nu \breve{\xi}_2^\mu - \tilde{\partial}^\mu \breve{\xi}_2^\nu) - \lambda_{2\nu}(\tilde{\partial}^\nu \breve{\xi}_1^\mu - \tilde{\partial}^\mu \breve{\xi}_1^\nu) + \frac{1}{2}\tilde{\partial}^\mu(\breve{\xi}_1\lambda_2 - \breve{\xi}_2\lambda_1) \\ &\quad + \kappa\, {}^\star B^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho} , \\ \lambda_\mu &= \lambda_{1\nu}\tilde{\partial}^\nu \lambda_{2\mu} - \lambda_{2\nu}\tilde{\partial}^\nu \lambda_{1\mu} . \end{aligned} \tag{15.31}$$

This is the Courant bracket twisted by a 2-form ${}^\star B$.

In the case of $\theta$-twisted $C$-bracket, we see that both the Courant bracket twisted by $B$ and by $\theta$ can be obtained from it, depending on which phase space we project. The isomorphism between two Courant algebroids appears naturally as a T-duality transformation between different projections of the bracket, in the same way as in the case of $B$-twisted $C$-bracket. It also features the flux that in different projections contains both $H$-flux and non-geometric $Q$ and $R$ fluxes.

# Part V

# Conclusions

In this thesis, we considered the application of generalized geometry to bosonic string theory and obtained various Courant algebroid structures in the symmetry algebra relations. Primarily we focused on the algebroid brackets and their properties, obtaining different fluxes in the algebroid relations. Moreover, we established relations of these brackets with T-duality, in both the single and double theory approach. For each algebroid, we obtained its Dirac structures and the constraints they impose on string fluxes.

The bosonic string $\sigma$-model is invariant under two groups of symmetries - diffeomorphisms and local gauge transformations. The generators of these transformations are self T-dual, so we united these generators into a single generator. It can be expressed as the $O(D, D)$ invariant inner product of two generalized vectors, one of which is the double symmetry parameter, a direct sum of diffeomorphism and local gauge transformations parameters, and another one is the double canonical variable, a direct sum of the coordinate $\sigma$ derivative and canonical momenta. We obtained the Poisson bracket algebra relations of these generators and showed that it closes on another generator parametrized with the Courant bracket of two double symmetry parameters. The Courant bracket is a well-known bracket on the generalized tangent bundle. We showed that it is in fact the self T-dual extension of the Lie bracket. The Courant bracket defines the standard Courant algebroid, which consists of the generalized tangent bundle as its vector bundle, the $O(D, D)$ invariant inner product, and the projection to the tangent bundle as its anchor. The Dirac structures related to the standard Courant algebroid are a symplectic manifold and a Poisson manifold. Translated into the language of string fluxes, these are spaces in which the $H$-flux and $R$-flux have to be zero, respectively.

Afterward, we developed a method of obtaining the twisted Courant bracket by an arbitrary $O(D, D)$ transformation. The method consists of choosing a different basis in which the generator is written, obtained by the action of the $O(D, D)$ transformation on a double canonical variable. If the symmetry parameter is transformed with the same transformation, the generator will remain the same, due to it being the $O(D, D)$ invariant inner product. We demonstrated that in the Poisson bracket algebra of such a generator, the twisted Courant bracket appears. Moreover, there is a natural way to define the Courant algebroid, consisting of the generalized tangent bundle, the twisted Courant bracket, the $O(D, D)$ invariant inner product, and the anchor defined as a composition of the natural projection to the tangent bundle and the inverse of the $O(D, D)$ transformation used for twisting. We showed that all five Courant algebroid conditions are apriori satisfied.

We chose three transformations relevant to string theory and twisted the Courant bracket by them, using the aforementioned method. Firstly, we considered $B$-transformations and with it acted on the double canonical variable. The resulting generalized vector consists of coordinates $\sigma$ derivatives as its vector and auxiliary currents $i_\mu$ as its 1-form components. This is a non-canonical basis, but when expressed in it, the Hamiltonian has a form of a free Hamiltonian, written in terms of diagonal generalized metric. The structure function of the Poisson bracket algebra of auxiliary currents is the

Kalb-Ramond field strength, i.e. the $H$-flux. We expressed the symmetry generator in this basis and obtained its algebra, where the Courant bracket twisted by $B$ appeared. The bracket differs from the Courant bracket by a term containing $H$-flux. We obtained Dirac structures corresponding to this bracket on which an arbitrary $H$-flux can exist. On the other set of Dirac structures, written in the form of a graph of bi-vector over a cotangent bundle, we showed that the generalized $R$-flux has to be zero.

Secondly, we considered the background characterized only by the effective metric, which is the T-dual metric. We acted with the $\theta$-transformation as a similarity transformation and obtained the generalized metric. It is possible to express this Hamiltonian in terms of the new non-canonical basis obtained with the action of the $\theta$-transformation to the double canonical variable. The resulting basis consists of a new set of auxiliary currents $k^\mu$ and canonical momenta. In the algebra of auxiliary currents, the non-geometric $Q-$ and $R-$fluxes appear as structure functions. We obtained the $\theta$-twisted Courant bracket in the Poisson algebra of this generator. Some of the terms in the $\theta$-twisted Courant bracket include the Koszul bracket (the Lie bracket generalization to the cotangent bundle) and Schouten-Nijenhuis bracket (the Lie bracket generalization to the space of multivectors). We showed that on Dirac structures related to the Courant algebroid with $\theta$-twisted Courant bracket, $R$-flux can exist without restrictions on the non-commutativity parameter.

We derived the $B$-twisted and $\theta$-twisted Courant brackets in [1, 2]. What we found as a peculiar property is their relation via T-duality. The T-duality is a known string phenomenon where winding and momenta numbers are interchanged. The former are obtained when the coordinate $\sigma$ derivative is integrated around the compact dimension, and the latter when the canonical momenta are integrated. Moreover, the non-commutativity parameter and effective metric are T-duals of the Kalb-Ramond field and metric tensor, respectively. The T-duality can be realized in the same phase space, by interchanging canonical momenta and coordinate $\sigma$ derivatives, together with the interchange of background fields with their T-duals. We coined this term self T-duality and showed that it directly relates two generators - one giving rise to the $B$-twisted Courant bracket and another giving rise to the $\theta$-twisted Courant bracket in the Poisson bracket algebra. Because we were working in the same phase space, we were able to obtain the coordinate transformation that takes the parameters of the one generator and results in the parameters of the other generator. We showed that this transformation defines the isomorphism between two Courant algebroids. This way, we demonstrated that $B$-twisted and $\theta$-twisted Courant brackets are self T-dual.

Thirdly, we obtained the Courant bracket that was simultaneously twisted both by $B$ and $\theta$. This bracket was first obtained in [3]. Beforehand, only the successive twists were considered, in which case the Courant bracket twisted firstly by $B$ and afterward by $\theta$ was obtained. This bracket, sometimes referred to as the Roytenberg bracket, contains all generalized fluxes, but the bracket itself is not invariant under T-duality. This is due to the fact that $B$-shifts and $\theta$-transformations do not commute.

Instead, we considered the matrix $\breve{B}$, which is a sum of $\hat{B}$ and $\hat{\theta}$, exponents of which govern twists of the Courant bracket by $B$ and $\theta$, respectively. By construction, this transformation is invariant under T-duality. The price we paid is that the square of the matrix $\breve{B}$ is not zero, and therefore all terms in Taylor's expansion had to be obtained. The full twisting matrix contained hyperbolic functions of the matrix $\alpha^\mu{}_\nu = 2\kappa\theta^{\mu\rho}B_{\rho\nu}$.

Computing the $B-\theta$-twisted Courant bracket was not an easy task. On the first hand, it seemed to produce a meaningless conundrum, with the appearance of a plethora of terms with no obvious interpretation. Luckily, we were able to overcome this obstacle by considering another twist, which was related to the simultaneous twist by $B$ and $\theta$ by a simple coordinate transformation. This auxiliary twist gave rise to the currents in a simpler form, such that it was possible to obtain the fluxes, which were then related to the fluxes of the $B-\theta$-twisted Courant bracket by an inverse of the above-mentioned twist.

We showed that this bracket contains all generalized fluxes. The $H$-flux is defined as a field strength of an antisymmetric field defined on the Lie algebroid, with the twisted Lie bracket as its bracket, while the $R$-flux we expressed as the twisted Schouten-Nijenhuis bracket of new bi-vectors $\breve{\theta}$. The bi-vector $\breve{\theta}$ is in general not the Poisson one, so it defines the quasi-Lie algebroid with the twisted Koszul bracket as its bracket. It is possible to define the non-nilpotent exterior derivative corresponding to the twisted Koszul bracket. Its action on the bi-vector $\breve{\theta}$ gives the $R$-flux and defines the twisted Schouten-Nijenhuis bracket. We found an interesting result when we computed the Dirac structures of the $B-\theta$-twisted Courant bracket: all generalized fluxes can exist on Dirac structures, with no restrictions imposed on them.

In the end, we generalized results to the case of double theory, in which all fields depend on both initial and T-dual coordinates. We considered diffeomorphisms, generated by canonical momenta, and T-dual diffeomorphisms, generated by T-dual canonical momenta. The parameters were taken to depend on both the initial and T-dual coordinates. We extended the Poisson bracket relations to the double space, taking into account that they should commute with T-duality relations. The generator governing both diffeomorphisms and T-dual diffeomorphisms was written in the form of an $O(D,D)$ invariant inner product. It has been shown it gives rise to the $C$-bracket, which was published in [1]. The $C$-bracket reduces to the Courant bracket when either all the initial coordinates or all the T-dual coordinates are projected out.

In addition, we twisted the $C$-bracket in the same way as the Courant bracket. We first considered the Hamiltonian with the generalized metric in the diagonal form, containing only metric tensor, and the Kalb-Ramond field appearing only through a flux in the non-canonical variables algebra. In its generator algebra, we obtained the $B$-twisted $C$-bracket. It extends the $C$-bracket with additional terms due to twisting, including the double theory flux. When dependence on T-dual coordinates is neglected, the bracket reduces to the $B$-twisted Courant bracket. On the other hand, when dependence on the initial coordinates is neglected, the bracket becomes the $\theta$-twisted Courant bracket. In a similar manner, we twisted the $C$-bracket with the $\theta$-transformation, obtained from the generator in double theory expressed in the non-canonical basis, in which Hamiltonian is diagonal, expressed in terms of T-dual metric. The $\theta$-twisted $C$-bracket was obtained in the algebra of this generator. The bracket has exactly the same form as the $B$-twisted $C$-bracket, which was not the case for their analogous twisted Courant brackets. When we neglected all the T-dual coordinates in the expression for the $\theta$-twisted $C$-bracket, the $\theta$-twisted Courant bracket was obtained, while when we neglected all the initial coordinates, the $B$-twisted Courant bracket was obtained. We showed that in both twisted $C$-brackets, the isomorphism between mutually T-dual Courant algebroids is naturally included. We obtained the results regarding the twisted $C$-brackets and their derivations in [4].

The explanation of T-duality in terms of generalized geometry is still a work in progress, and there is a lot more work to be done. For instance, there are solved cases of equations of motions on background fields that from a simple geometric theory produce the T-dual theory that is not local. It would be important to see how the understanding of T-duality as the Courant algebroid isomorphism would generalize to such cases. The non-locality of the T-dual theories poses a challenge to understanding and interpreting the symmetries of their conformal field theory.

Additionally, there are challenges in the description of open string T-duality in terms of generalized geometry apparatus that were not touched upon in this dissertation. The open string action has to be extended with the terms related to the boundary conditions, which also change the symmetry generator. There was some work in literature with the aim to interpret the $D$-branes as Dirac structures [83, 84]. Hopefully, our results related to Dirac structures of various Courant algebroids and the fluxes on them might find the purpose in the challenges related to the open strings.

In the end, the description of Nature in terms of strings is contingent on the formulation of M-theory that, on one hand, gives effective action that describes gravity, while on the other hand, connects to a myriad of realizations of supersymmetric string theories. The fact that many of the superstring theories are connected by T-duality makes understanding it a priority. Therefore, further work will have to include the supersymmetry and see if isomorphism between Courant algebroids is still a valid description of T-duality.

# Part VI

# Appendix

# Appendix A

# Poisson manifolds

Let $\mathcal{M}$ be a manifold, and $C^\infty(\mathcal{M})$ the vector space of real valued functions on $\mathcal{M}$. A Poisson bracket on $\mathcal{M}$ is a map $\{,\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ that satisfies:

1. Skew-symmetry: $\{f, g\} = -\{g, f\}$ ;
2. Leibniz rule: $\{f, gh\} = \{f, g\}h + \{f, h\}g$ ;
3. Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.

The Poisson bracket can be defined with the bi-vector $\theta$ by

$$\{f, g\} = \theta(df, dg)\,, \tag{A.1}$$

if the bi-vector satisfies the condition $[\theta, \theta]_S = 0$ (5.10). Two definitions are equivalent. The condition (5.10) ensures that the Jacobi identity is satisfied. The structure $(\mathcal{M}, \theta)$ is then called the Poisson manifold.

# Appendix B

# O(D, D) group

Indefinite orthogonal group $O(D,D)$ [85, 86] is defined as the Lie group of all linear transformations $\mathcal{O}$ of a $2D$-dimensional real vector space that leave invariant a non-degenerate symmetric bilinear form of signature $(D,D)$

$$\langle \mathcal{O}\Lambda_1, \mathcal{O}\Lambda_2\rangle = \langle \Lambda_1, \Lambda_2\rangle \,. \tag{B.1}$$

Let us express the general form of an $O(D,D)$ transformation as

$$\mathcal{O} = \begin{pmatrix} P^\mu_{\ \nu} & Q^{\mu\nu} \\ R_{\mu\nu} & S_\mu^{\ \nu} \end{pmatrix} \,, \tag{B.2}$$

where $P,\ Q,\ R,\ S$ are $D\times D$ matrices. Substituting (B.2) into (6.3), we obtain the constraint on these matrices:

$$P^T R + R^T P = 0\,, \quad P^T S + R^T Q = I\,, \quad Q^T S + S^T Q = 0\,, \tag{B.3}$$

where by $I$ we denoted the $D \times D$ identity matrix.

From relation (6.3) we easily obtain that the inverse of the matrix $\mathcal{O}$ is given by

$$\mathcal{O}^{-1} = \eta^{-1}\mathcal{O}^T\eta\,, \tag{B.4}$$

or

$$\mathcal{O}^{-1} = \begin{pmatrix} S^T & Q^T \\ R^T & P^T \end{pmatrix} \,. \tag{B.5}$$

From the requirement $\mathcal{O}\mathcal{O}^{-1} = I$, we obtain another set of conditions on $P,\ Q,\ R,\ S$

$$PQ^T + QP^T = 0\,, \quad PS^T + QR^T = I\,, \quad RS^T + SR^T = 0\,. \tag{B.6}$$

The generators of $O(D,D)$ group include the following elements:

$$\mathcal{O}_A = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^T \end{pmatrix} , \tag{B.7}$$

$$\mathcal{O}_B = \begin{pmatrix} 1 & 0 \\ 2B & 1 \end{pmatrix} , \tag{B.8}$$

and

$$\mathcal{O}_{\pm i} = \begin{pmatrix} 1 - E_i & \pm E_i \\ \pm E_i & 1 - E_i \end{pmatrix} , \tag{B.9}$$

where $(E_i)_{jk} = \delta^i_j \delta^i_k$. All other elements can be obtained from these generators.

# Appendix C

# Standard Courant algebroid

In this Appendix, we provide the proof for a claim that the structure $\Big(T\mathcal{M} \oplus T^\star\mathcal{M}, \langle , \rangle, [,]_\mathcal{C}, \pi\Big)$ consisting of the generalized tangent bundle over a smooth manifold, the natural inner product (6.2), and the Courant bracket is the Courant algebroid. Firstly, let us obtain the Courant algebroid differential operator (6.21), which we mark by

$$(\mathcal{D}f)^M = \begin{pmatrix} (\mathcal{D}^{(0)}f)^\mu \\ (\mathcal{D}^{(0)}f)_\mu \end{pmatrix} . \tag{C.1}$$

The right hand side of (6.21) becomes

$$\mathcal{L}_{\pi(\Lambda)} f = i_\xi \, df = \xi^\mu \partial_\mu f \, . \tag{C.2}$$

The left-hand side of (6.21) becomes

$$\langle \Lambda, \mathcal{D}^{(0)} f \rangle = \xi^\mu (\mathcal{D}^{(0)}f)_\mu + (\mathcal{D}^{(0)}f)^\mu \lambda_\mu \, . \tag{C.3}$$

Equating (C.2) and (C.3), we obtain

$$\mathcal{D}^{(0)} f = \begin{pmatrix} 0 \\ df \end{pmatrix} . \tag{C.4}$$

The differential operator $\mathcal{D}^{(0)}$ is basically just the exterior derivative $d$, but we chose the above notation so that its action on function gives generalized vector explicitly.

The first property (6.22) is evident when we act with the projection $\pi$ to the definition of the Courant bracket (6.12), obtaining

$$\pi[\Lambda_1, \Lambda_2]_\mathcal{C} = [\xi_1, \xi_2]_L = [\pi(\Lambda_1), \pi(\Lambda_2)]_L \, . \tag{C.5}$$

To prove the second property, it is convenient to separate the vector and 1-form part of the left hand side of (6.23). The vector part becomes

$$[\xi_1, f\xi_2]_L = f[\xi_1, \xi_2]_L + (\mathcal{L}_{\xi_1} f)\, \xi_2 \,, \tag{C.6}$$

which is just the Leibniz rule for the Lie bracket (4.6). The 1-form part gives

$$\begin{aligned}
&\mathcal{L}_{\xi_1}(f\lambda_2) - \mathcal{L}_{(f\xi_2)}\lambda_1 - \frac{1}{2}d(i_{\xi_1}(f\lambda_2) - i_{(f\xi_2)}\lambda_1) = \\
&f\mathcal{L}_{\xi_1}\lambda_2 + (\mathcal{L}_{\xi_1} f)\, \lambda_2 - f\mathcal{L}_{\xi_2}\lambda_1 - df i_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}(f\lambda_2) - i_{(f\xi_2)}\lambda_1) = \\
&f\Big(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big) + (\mathcal{L}_{\xi_1} f)\, \lambda_2 - \frac{1}{2}\langle \Lambda_1, \Lambda_2\rangle df \,,
\end{aligned} \tag{C.7}$$

where in the second line we applied (4.6) to the term $\mathcal{L}_{\xi_1}(f\lambda_2)$, and (4.18) to the term $\mathcal{L}_{(f\xi_2)}\lambda_1$. In the last line the Leibniz property for exterior derivative $d$ was used. Combining relations (C.6) and (C.7) we obtain

$$[\Lambda_1, f\Lambda_2]_{\mathcal{C}} = f[\Lambda_1, \Lambda_2]_{\mathcal{C}} + (\mathcal{L}_{\pi(\Lambda_1)} f)\, \Lambda_2 - \frac{1}{2}\langle \Lambda_1, \Lambda_2\rangle \mathcal{D}^{(0)} f \,, \tag{C.8}$$

and therefore the second property (6.23) is satisfied.

For the third property (6.23) , we start from the first term on the right-hand side of it and write

$$\begin{aligned}
\langle [\Lambda_1, \Lambda_2] + \frac{1}{2}\mathcal{D}^{(0)}\langle \Lambda_1, \Lambda_2\rangle, \Lambda_3\rangle &= \langle [\xi_1, \xi_2]_L \oplus (\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 + d i_{\xi_2}\lambda_1), \xi_3 \oplus \lambda_3\rangle \\
&= i_{[\xi_1,\xi_2]_L}\lambda_3 + i_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - i_{\xi_2} d\lambda_1) \\
&= \mathcal{L}_{\xi_1} i_{\xi_2}\lambda_3 - i_{\xi_2}\mathcal{L}_{\xi_1}\lambda_3 + i_{\xi_3}(\mathcal{L}_{\xi_1}\lambda_2 - i_{\xi_2} d\lambda_1) \,,
\end{aligned} \tag{C.9}$$

where we firstly used the definition of the inner product (6.2) and Courant bracket (6.12), and afterwards the identity (4.20). Because we are working with a symmetric inner product, the second term of the right-hand side of (6.24) can be obtained from the previous relations by swapping $2 \leftrightarrow 3$

$$\langle \Lambda_2, [\Lambda_1, \Lambda_3] + \frac{1}{2}\mathcal{D}^{(0)}\langle \Lambda_1, \Lambda_3\rangle\rangle = \mathcal{L}_{\xi_1} i_{\xi_3}\lambda_2 - i_{\xi_3}\mathcal{L}_{\xi_1}\lambda_2 + i_{\xi_2}(\mathcal{L}_{\xi_1}\lambda_3 - i_{\xi_3} d\lambda_1) \,. \tag{C.10}$$

Adding (C.9) and (C.10), we obtain

$$\begin{aligned}
&\langle [\Lambda_1, \Lambda_2] + \frac{1}{2}\mathcal{D}^{(0)}\langle \Lambda_1, \Lambda_2\rangle, \Lambda_3\rangle + \langle \Lambda_2, [\Lambda_1, \Lambda_3] + \frac{1}{2}\mathcal{D}^{(0)}\langle \Lambda_1, \Lambda_3\rangle\rangle \\
&= \mathcal{L}_{\xi_1}(i_{\xi_2}\lambda_3 + i_{\xi_3}\lambda_2) - (i_{\xi_3} i_{\xi_2} + i_{\xi_2} i_{\xi_3}) d\lambda_1 \\
&= \mathcal{L}_{\pi(\Lambda_1)}\langle \Lambda_2, \Lambda_3\rangle \,.
\end{aligned} \tag{C.11}$$

Here, we used (6.14) and (6.2), as well as (4.19). The third condition (6.24) has therefore been proven.

The fourth property (6.25) is evident from the fact that the inner product (6.2) of pure 1-forms is zero, i.e.

$$\langle \mathcal{D}^{(0)} f, \mathcal{D}^{(0)} g \rangle = \langle 0 \oplus df, 0 \oplus dg \rangle = 0 \,. \tag{C.12}$$

The Jacobiator (6.19) for the Courant bracket can be easily obtained from definition of the Courant bracket. Firstly, we start with

$$\begin{aligned}
\text{Jac}(\Lambda_1, \Lambda_2, \Lambda_3) &= \xi \oplus \lambda \\
\xi &= [[\xi_1, \xi_2]_L, \xi_3]_L + cyclic = 0 \\
\lambda &= \mathcal{L}_{[\xi_1,\xi_2]_L} \lambda_3 - \mathcal{L}_{\xi_3} (\mathcal{L}_{\xi_1} \lambda_2 - \mathcal{L}_{\xi_2} \lambda_1 - \frac{1}{2} d(i_{\xi_1} \lambda_2 - i_{\xi_2} \lambda_1)) \\
&\quad - \frac{1}{2} d(i_{[\xi_1,\xi_2]_L} \lambda_3 - i_{\xi_3} (\mathcal{L}_{\xi_1} \lambda_2 - \mathcal{L}_{\xi_2} \lambda_1 - \frac{1}{2} d(i_{\xi_1} \lambda_2 - i_{\xi_2} \lambda_1))) + cyclic \,.
\end{aligned} \tag{C.13}$$

The vector part is just the Jacobi identity for the Lie bracket (4.7), and is therefore zero. The 1-form part is complicated, and it requires some more work, in order to be transformed properly. We use the Cartan formula (4.17) and nilpotency of the exterior derivative, as well as (4.20) to write

$$\begin{aligned}
\lambda &= \frac{1}{2} \Big( \mathcal{L}_{[\xi_1,\xi_2]_L} \lambda_3 - \mathcal{L}_{\xi_3} (\mathcal{L}_{\xi_1} \lambda_2 - \mathcal{L}_{\xi_2} \lambda_1 - \frac{1}{2} d(i_{\xi_1} \lambda_2 - i_{\xi_2} \lambda_1)) \\
&\quad + i_{[\xi_1,\xi_2]_L} d\lambda_3 - i_{\xi_3} d(\mathcal{L}_{\xi_1} \lambda_2 - \mathcal{L}_{\xi_2} \lambda_1 - \frac{1}{2} d(i_{\xi_1} \lambda_2 - i_{\xi_2} \lambda_1)) \Big) + cyclic \\
&= \frac{1}{2} \Big( \mathcal{L}_{[\xi_1,\xi_2]_L} \lambda_3 - \mathcal{L}_{\xi_3} (\mathcal{L}_{\xi_1} \lambda_2 - \mathcal{L}_{\xi_2} \lambda_1) + \frac{1}{2} d i_{\xi_3} d(i_{\xi_1} \lambda_2 - i_{\xi_2} \lambda_1)) \\
&\quad + (\mathcal{L}_{\xi_1} i_{\xi_2} - i_{\xi_2} \mathcal{L}_{\xi_1}) d\lambda_3 - i_{\xi_3} d(i_{\xi_1} d\lambda_2 - i_{\xi_2} d\lambda_1) \Big) + cyclic \\
&= \frac{1}{2} \Big( \mathcal{L}_{[\xi_1,\xi_2]_L} \lambda_3 - \mathcal{L}_{\xi_3} (\mathcal{L}_{\xi_1} \lambda_2 - \mathcal{L}_{\xi_2} \lambda_1) + \frac{1}{2} d i_{\xi_3} d(i_{\xi_1} \lambda_2 - i_{\xi_2} \lambda_1)) \\
&\quad + (i_{\xi_1} d i_{\xi_2} + d i_{\xi_1} i_{\xi_2} - i_{\xi_2} d i_{\xi_1}) d\lambda_3 - i_{\xi_3} d(i_{\xi_1} d\lambda_2 - i_{\xi_2} d\lambda_1) \Big) + cyclic \,.
\end{aligned} \tag{C.14}$$

Firstly, using the definition of the Lie bracket (4.4), we conclude that

$$\begin{aligned}
&\mathcal{L}_{[\xi_1,\xi_2]_L} \lambda_3 - \mathcal{L}_{\xi_3} (\mathcal{L}_{\xi_1} \lambda_2 - \mathcal{L}_{\xi_2} \lambda_1) + cyclic = \\
&(\mathcal{L}_{\xi_1} \mathcal{L}_{\xi_2} - \mathcal{L}_{\xi_2} \mathcal{L}_{\xi_1}) \lambda_3 + (\mathcal{L}_{\xi_2} \mathcal{L}_{\xi_3} - \mathcal{L}_{\xi_3} \mathcal{L}_{\xi_2}) \lambda_1 + (\mathcal{L}_{\xi_3} \mathcal{L}_{\xi_1} - \mathcal{L}_{\xi_1} \mathcal{L}_{\xi_3}) \lambda_2 \\
&- \mathcal{L}_{\xi_3} (\mathcal{L}_{\xi_1} \lambda_2 - \mathcal{L}_{\xi_2} \lambda_1) - \mathcal{L}_{\xi_1} (\mathcal{L}_{\xi_2} \lambda_3 - \mathcal{L}_{\xi_3} \lambda_2) - \mathcal{L}_{\xi_2} (\mathcal{L}_{\xi_3} \lambda_1 - \mathcal{L}_{\xi_1} \lambda_3) = 0 \,.
\end{aligned} \tag{C.15}$$

Secondly, we obtain

$$\begin{aligned}
&(i_{\xi_1} d i_{\xi_2} - i_{\xi_2} d i_{\xi_1}) d\lambda_3 - i_{\xi_3} d(i_{\xi_1} d\lambda_2 - i_{\xi_2} d\lambda_1) + cyclic = \\
&i_{\xi_1} d i_{\xi_2} d\lambda_3 + i_{\xi_2} d i_{\xi_3} d\lambda_1 + i_{\xi_3} d i_{\xi_1} d\lambda_2 - i_{\xi_2} d i_{\xi_1} d\lambda_3 - i_{\xi_3} d i_{\xi_2} d\lambda_1 - i_{\xi_1} d i_{\xi_3} d\lambda_2 \\
&- i_{\xi_3} d i_{\xi_1} d\lambda_2 - i_{\xi_1} d i_{\xi_2} d\lambda_3 - i_{\xi_2} d i_{\xi_3} d\lambda_1 + i_{\xi_3} d i_{\xi_2} d\lambda_1 + i_{\xi_1} d i_{\xi_3} d\lambda_2 + i_{\xi_2} d i_{\xi_1} d\lambda_3 = 0 \,.
\end{aligned} \tag{C.16}$$

Thirdly, we have

$$\begin{aligned}&di_{\xi_3}(di_{\xi_1}\lambda_2 - di_{\xi_2}\lambda_1) + cyclic = \\ &di_{\xi_3}di_{\xi_1}\lambda_2 + di_{\xi_1}di_{\xi_2}\lambda_3 + di_{\xi_2}di_{\xi_3}\lambda_1 - di_{\xi_3}di_{\xi_2}\lambda_1 - di_{\xi_1}di_{\xi_3}\lambda_2 - di_{\xi_2}di_{\xi_1}\lambda_3 = \\ &(di_{\xi_2}di_{\xi_3} - di_{\xi_3}di_{\xi_2})\lambda_1 + (di_{\xi_3}di_{\xi_1} - di_{\xi_1}di_{\xi_3})\lambda_2 + (di_{\xi_1}di_{\xi_2} - di_{\xi_2}di_{\xi_1})\lambda_3\,,\end{aligned} \tag{C.17}$$

and lastly

$$di_{\xi_1}i_{\xi_2}d\lambda_3 + cyclic = di_{\xi_1}i_{\xi_2}d\lambda_3 + di_{\xi_2}i_{\xi_3}d\lambda_1 + di_{\xi_3}i_{\xi_1}d\lambda_2\,. \tag{C.18}$$

Substituting (C.15), (C.16), (C.17) and (C.18) into (C.14), we obtain the Jacobiator of the Courant bracket

$$\begin{aligned}\mathrm{Jac}(\Lambda_1,\Lambda_2,\Lambda_3) &= \frac{1}{2}\Big(di_{\xi_1}i_{\xi_2}d\lambda_3 + di_{\xi_2}i_{\xi_3}d\lambda_1 + di_{\xi_3}i_{\xi_1}d\lambda_2\Big) \\ &+\frac{1}{4}\Big((di_{\xi_2}di_{\xi_3} - di_{\xi_3}di_{\xi_2})\lambda_1 + (di_{\xi_3}di_{\xi_1} - di_{\xi_1}di_{\xi_3})\lambda_2 \\ &\qquad +(di_{\xi_1}di_{\xi_2} - di_{\xi_2}di_{\xi_1})\lambda_3\Big)\,.\end{aligned} \tag{C.19}$$

Now in order to obtain the Nijenhuis operator, we substitute (6.12) and (6.2) in (6.20), and note that

$$\begin{aligned}\mathrm{Nij}(\Lambda_1,\Lambda_2,\Lambda_3) &= \frac{1}{6}\langle[\Lambda_1,\Lambda_2]_C,\Lambda_3\rangle + cyclic\,, \\ \langle[\Lambda_1,\Lambda_2]_C,\Lambda_3\rangle &= i_{[\xi_1,\xi_2]_L}\lambda_3 + i_{\xi_3}\Big(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big)\,.\end{aligned} \tag{C.20}$$

If we take the exterior derivative of the above relation, using (4.17) and (4.20), we obtain

$$\begin{aligned}d\langle[\Lambda_1,\Lambda_2]_C,\Lambda_3\rangle &= di_{[\xi_1,\xi_2]_L}\lambda_3 + di_{\xi_3}\Big(\mathcal{L}_{\xi_1}\lambda_2 - \mathcal{L}_{\xi_2}\lambda_1 - \frac{1}{2}d(i_{\xi_1}\lambda_2 - i_{\xi_2}\lambda_1)\Big) \\ &= di_{\xi_1}di_{\xi_2}\lambda_3 - di_{\xi_2}di_{\xi_1}\lambda_3 - di_{\xi_2}i_{\xi_1}d\lambda_3 + di_{\xi_3}i_{\xi_1}d\lambda_2 - di_{\xi_3}i_{\xi_2}d\lambda_1 \\ &\quad +\frac{1}{2}(di_{\xi_3}di_{\xi_1}\lambda_2 - di_{\xi_3}di_{\xi_2}\lambda_1)\,.\end{aligned} \tag{C.21}$$

Again, we can easily add the cyclic permutations od terms that have similar form. For instance, we have

$$\begin{aligned}&di_{\xi_1}di_{\xi_2}\lambda_3 - di_{\xi_2}di_{\xi_1}\lambda_3 + \frac{1}{2}(di_{\xi_3}di_{\xi_1}\lambda_2 - di_{\xi_3}di_{\xi_2}\lambda_1) + cyclic = \\ &di_{\xi_1}di_{\xi_2}\lambda_3 - di_{\xi_2}di_{\xi_1}\lambda_3 + di_{\xi_2}di_{\xi_3}\lambda_1 - di_{\xi_3}di_{\xi_2}\lambda_1 + di_{\xi_3}di_{\xi_1}\lambda_2 - di_{\xi_1}di_{\xi_3}\lambda_2 \\ &+\frac{1}{2}\Big(di_{\xi_3}di_{\xi_1}\lambda_2 - di_{\xi_3}di_{\xi_2}\lambda_1 + di_{\xi_1}di_{\xi_2}\lambda_3 - di_{\xi_1}di_{\xi_3}\lambda_2 + di_{\xi_2}di_{\xi_3}\lambda_1 - di_{\xi_2}di_{\xi_1}\lambda_3\Big) = \\ &\frac{3}{2}\Big((di_{\xi_2}di_{\xi_3} - di_{\xi_3}di_{\xi_2})\lambda_1 + (di_{\xi_3}di_{\xi_1} - di_{\xi_1}di_{\xi_3})\lambda_2 + (di_{\xi_1}di_{\xi_2} - di_{\xi_2}di_{\xi_1})\lambda_3\Big)\,.\end{aligned} \tag{C.22}$$

The remaining terms from (C.21) become

$$
\begin{aligned}
&-di_{\xi_2}i_{\xi_1}d\lambda_3 + di_{\xi_3}i_{\xi_1}d\lambda_2 - di_{\xi_3}i_{\xi_2}d\lambda_1 + cyclic = \\
&-di_{\xi_2}i_{\xi_1}d\lambda_3 + di_{\xi_3}i_{\xi_1}d\lambda_2 - di_{\xi_3}i_{\xi_2}d\lambda_1 - di_{\xi_3}i_{\xi_2}d\lambda_1 + di_{\xi_1}i_{\xi_2}d\lambda_3 - di_{\xi_1}i_{\xi_3}d\lambda_2 \\
&-di_{\xi_1}i_{\xi_3}d\lambda_2 + di_{\xi_2}i_{\xi_3}d\lambda_1 - di_{\xi_2}i_{\xi_1}d\lambda_3 = \\
&3\Big(di_{\xi_1}i_{\xi_2}d\lambda_3 + di_{\xi_2}i_{\xi_3}d\lambda_1 + di_{\xi_3}i_{\xi_1}d\lambda_2\Big)\,,
\end{aligned}
\tag{C.23}
$$

where we used the interior product property (4.19). The derivative $\mathcal{D}^{(0)}$of the Nijenhuis operator for the Courant bracket is obtained by substituting (C.22) and (C.23) in (C.21)

$$
\begin{aligned}
\mathcal{D}^{(0)}\text{Nij}(\Lambda_1,\Lambda_2,\Lambda_3) \;=\; & \frac{1}{2}\Big(di_{\xi_1}i_{\xi_2}d\lambda_3 + di_{\xi_2}i_{\xi_3}d\lambda_1 + di_{\xi_3}i_{\xi_1}d\lambda_2\Big) \\
& +\frac{1}{4}\Big((di_{\xi_2}di_{\xi_3} - di_{\xi_3}di_{\xi_2})\lambda_1 + (di_{\xi_3}di_{\xi_1} - di_{\xi_1}di_{\xi_3})\lambda_2 \\
& \quad +(di_{\xi_1}di_{\xi_2} - di_{\xi_2}di_{\xi_1})\lambda_3\Big)\,.
\end{aligned}
\tag{C.24}
$$

Comparing relations (C.19) and (C.24), we finally prove the last Courant algebroid compatibility condition (6.26)

$$
\text{Jac}(\Lambda_1,\Lambda_2,\Lambda_3) = \mathcal{D}^{(0)}\text{Nij}(\Lambda_1,\Lambda_2,\Lambda_3)\,. \tag{C.25}
$$

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# Curriculum vitae

Ilija Ivanišević was born on July 17, 1991, in Mostar. He completed his secondary education at the Mathematical High School in 2010. In 2014, he graduated from the Faculty of Physics with an average grade of 9.66/10. The following year, he obtained his Master's degree at the same faculty with a thesis entitled "T-dualization in curved space" under the supervision of Dr. Ljubica Davidović. During his studies, he was a scholarship recipient of the City of Belgrade (2007-2010), the Ministry of Education, Science and Technological Development of the Republic of Serbia (2010-2013, 2015-2016), and the Fund for Young Talents (2014-2015).

He enrolled in doctoral studies in the field of Quantum Fields, Particles, and Gravity in 2015, and passed his doctoral exams with an average grade of 9/10. Since 2018, he has been employed at the Institute of Physics, where he has been working under the mentorship of Dr. Ljubica Davidović. His area of interest is string theory, specifically T-duality and its relation to generalized geometry. He has published four peer-reviewed papers in top-tier international journals whose contents provide the basis for the present thesis. Ilija presented his work in student talks at the 10th Mathematical Physics Meeting: School and Conference on Modern Mathematical Physics in 2019 in Belgrade, and at the 17th DIAS-TH Winter School "Supersymmetry and Integrability" organized by the Bogoliubov Laboratory of Theoretical Physics, JINR (Dubna, Russia).

During his studies, Ilija represented the University of Belgrade in international university debate competitions, achieving numerous successes, winning over ten competitions, and advancing to the finals of the European University Debate Championship in Tallinn in 2017.

# Изјава о ауторству

Име и презиме аутора **Илија Иванишевић**

Број индекса **8016/2015**

**Изјављујем**

да је докторска дисертација под насловом

**Courant algebroids in bosonic string theory**

**(Курантови алгеброиди у бозонској теорији струна)**

- резултат сопственог истраживачког рада;
- да дисертација у целини ни у деловима није била предложена за стицање друге дипломе према студијским програмима других високошколских установа;
- да су резултати коректно наведени и
- да нисам кршио/ла ауторска права и користио/ла интелектуалну својину других лица.

**Потпис аутора**

У Београду, **21.04.2023**

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# Изјава о истоветности штампане и електронске верзије докторског рада

Име и презиме аутора **Илија Иванишевић**

Број индекса **8016/2015**

Студијски програм **квантна поља, честице и гравитација**

Наслов рада **Courant algebroids in bosonic string theory** (**Курантови алгеброиди у бозонској теорији струна)**

Ментор **др. Љубица Давидовић**

Изјављујем да је штампана верзија мог докторског рада истоветна електронској верзији коју сам предао/ла ради похрањена у **Дигиталном репозиторијуму Универзитета у Београду.**

Дозвољавам да се објаве моји лични подаци везани за добијање академског назива доктора наука, као што су име и презиме, година и место рођења и датум одбране рада.

Ови лични подаци могу се објавити на мрежним страницама дигиталне библиотеке, у електронском каталогу и у публикацијама Универзитета у Београду.

**Потпис аутора**

У Београду, **21.04.2023**

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# Изјава о коришћењу

Овлашћујем Универзитетску библиотеку „Светозар Марковић“ да у Дигитални репозиторијум Универзитета у Београду унесе моју докторску дисертацију под насловом:

**Courant algebroids in bosonic string theory**

(Курантови алгеброиди у бозонској теорији струна)

која је моје ауторско дело.

Дисертацију са свим прилозима предао/ла сам у електронском формату погодном за трајно архивирање.

Моју докторску дисертацију похрањену у Дигиталном репозиторијуму Универзитета у Београду и доступну у отвореном приступу могу да користе сви који поштују одредбе садржане у одабраном типу лиценце Креативне заједнице (Creative Commons) за коју сам се одлучио/ла.

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**6. Ауторство – делити под истим условима (CC BY-SA)**

(Молимо да заокружите само једну од шест понуђених лиценци.
Кратак опис лиценци је саставни део ове изјаве).

**Потпис аутора**

У Београду, **21.04.2023**

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1. **Ауторство**. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, и прераде, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце, чак и у комерцијалне сврхе. Ово је најслободнија од свих лиценци.

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3. **Ауторство – некомерцијално – без прерада**. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, без промена, преобликовања или употребе дела у свом делу, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце. Ова лиценца не дозвољава комерцијалну употребу дела. У односу на све остале лиценце, овом лиценцом се ограничава највећи обим права коришћења дела.

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5. **Ауторство – без прерада**. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, без промена, преобликовања или употребе дела у свом делу, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце. Ова лиценца дозвољава комерцијалну употребу дела.

6. **Ауторство – делити под истим условима**. Дозвољавате умножавање, дистрибуцију и јавно саопштавање дела, и прераде, ако се наведе име аутора на начин одређен од стране аутора или даваоца лиценце и ако се прерада дистрибуира под истом или сличном лиценцом. Ова лиценца дозвољава комерцијалну употребу дела и прерада. Слична је софтверским лиценцама, односно лиценцама отвореног кода.

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Датум 27-03-2024

# Предлог за студентску награду

Са великим задовољством за овогодишњу студентску награду Института за физику предлажем докторску дисертацију др Илије Иванишевића под називом:

"COURANT ALGEBROIDS IN BOSONIC STRING THEORY",
„КУРАНТОВИ АЛГЕБРОИДИ У БОЗОНСКОЈ ТЕОРИЈИ СТРУНА",

одбрањену на Физичком факултету Универзитета у Београду **15. септембра 2023.** године.

Илија Иванишевић рођен је 17. јула 1991. године у Мостару. Дипломирао је на Физичком факултету Универзитета у Београду 3. октобра 2014. године на смеру теоријска и експериментална физика, са просечном оценом 9,66. На истом факултету је 16. октобра 2015. године одбранио мастер тезу под називом *„Т-дуалност у закривљеном простору"*. Током студија Илија Иванишевић је био стипендиста Фонда за младе таленте и Министарства просвете, науке и технолошког развоја као један од најбољих студената. По завршетку студија као стипендиста Министарства науке долази на Институт за физику. Илија Иванишевић је био сарадник у настави Физичког факултета на предмету Основи математичке физике за студенте примењене физике, метеорологије и астрофизике, као и на предмету Методе математичке физике за студенте опште физике. Поред физике Илија Иванишевић наставља усавршавање у академским дебатама, јавном говорништву и страним језицима (напредни ниво енглеског, немачког и француског језика).

Истраживање обухваћено докторском дисертацијом, укључује садржај објављених радова

1. I. Ivanišević, Lj. Davidović, B. Sazdović, *"Courant bracket found out to be T-dual to Roytenberg one"*, *Eur. Phys. J.* **C 80**, (2020) 571, IF 4.59,

2. Lj. Davidović, I. Ivanišević, B. Sazdović, *"Courant bracket as T-dual invariant extension of Lie bracket"*, *JHEP* **03** (2021) 109, arxiv: 2010.10662, IF 6.379,

3. Lj. Davidović, I. Ivanišević, B. Sazdović, *"Courant bracket twisted both by a 2-form B and by a bi-vector $\theta$"*, *Eur. Phys. J.* **C 81** 685 (2021), arxiv:2103.09585, IF 4.991,

4. Lj. Davidović, I. Ivanišević, B. Sazdović, *"Twisted C bracket"*, *Fortschritte der Physik* **71** (2023), arxiv:2202.03227, IF 3.9,

5. Lj. Davidović, I. Ivanišević, B. Sazdović, *Courant and Roytenberg bracket and their relation via T-duality*, SFIN XXXIII Series A: Conferences, no.A1 (2020) 87-96,

као и рада на рецензији

6. Lj. Davidović, I. Ivanišević, B. Sazdović, *Fluxes of Courant bracket twisted at the same time by $B$ and $\theta$*, e-Print: 2312.11268 [hep-th].

Радови су објављени у врхунским међународним часописима у врху листе часописа у области физике високих енергија. Колега Иванишевић је при писању радова показао самосталност и склоност ка математичкој физици и усклађивању језика развијених у две области при истраживању теорије струна и сродних теорија. Дисертација је донела помак у разумевању упаривања параметара генералисаних струја и генератора симетрија бозонских струна у различитим репрезентацијама теорије, укључујући Т-дуалност и дупле теорије, и проналажењу веза између за физику релевантних математичких структура.

У оквиру редовног семинара групе за Гравитацију, честице и поља, петком у 11h, Илија Иванишевић је одржао два предавања 18. и 25. фебруара 2022. године. Увидом у правилник Института за физику о додели годишњих награда, констатујем да испуњава све услове за кандидату.

У Београду,
27. марта 2024.

Ментор:
Љубица Давидовић
др Љубица Давидовић,
виши научни сарадник
Института за физику