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T-dualization of bosonic string and type IIB superstring in presence of coordinate dependent background fields

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T-dualizacija bozonske strune i tip IIB superstrune u prisustvu koorinatno zavisnih pozadinskih polja

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T-dualization of bosonic string and type IIB superstring in presence of coordinate dependent background fields

Abstract

Topic of this disertation is examination of non-commutative and non-associative properties that emerge in context of closed string theory. This examination will be carried out on two distinct models. One where we work with bosonic string and other where we work with type IIB superstring. Furthermore, both of these models will be analyzed in presence of coordinate dependent background fields. Subjecting these models to T-dualization we will be able to obtain both T-dual theories and transformation laws that connect coordinates of starting theory with T-dual one. Utilizing transformation laws and commutative relations of starting theory we will be able to deduce non-commutative properties of T-dual theories. Method for obtaining T-duality will be based on Buscher procedure and its extensions. Main idea of Buscher procedure lies in localization of translational symmetry by replacing partial derivatives and coordinates that appear in action with covariant derivatives and invariant coordinates. This substitution inevitably introduces additional degrees of freedom which are encoded in gauge fields. By elimination of newly introduced degrees of freedom with method of Lagrange multipliers and subsequently finding equations of motion for gauge fields we obtain transformation laws. Inserting these laws into the action we will obtain T-dual theory.

In examination of bosonic string theory, we will work with $3D$ space where Kalb-Ramond background field will have infinitesimal linear dependence on one coordinate, z coordinate. Dualization will be carried along two distinct chains, one where coordinate that appears in background fields will be dualized last and other where it will be dualized first. By comparing these two approaches we will be able do discern what are necessary components for emergence of non-commutative properties.

Second part of thesis will be concerned with T-duality of type II superstring that propagates in linearly coordinate dependent Ramond-Ramond field. Unlike previous case, this theory possesses both bosonic and fermionic coordinates, however background field will only depend on bosonic part. T-duality will first focus only on bosonic part and later we will also incorporate fermionic part. We will also present alternative chain of duality where first we dualize fermionic coordinates and later bosonic ones. It will be shown that both chains produce same non-commutative relations. Finally, at the end of the thesis, we will also make analysis of same case when we have more general Ramond-Ramond field.

Key words: String theory, non-commutativity, non-associativity, Buscher procedure
Scientific area: Physics
Scientific subfield: High energy theoretical physics

T-dualizacija bozonske strune i tip IIB superstrune u prisustvu koorinatno zavisnih pozadinskih polja

Sažetak

Tema ove disertacije je bazirana na izučavanju nekomutativnih i neasocijativnih osobina koje se javljaju u kontekstu teorije struna. Ovo izučavanje će biti obavljeno na dva različita modela. Prvi model sa kojim ćemo raditi je model bozonske strune dok je drugi model za tip IIB superstrunu. Oba modela će biti analizirana u prisustvu koordinatno zavisnih pozadinskih polja. Podvrgavanjem ovih modela T-dualizaciji bićemo u stanju da dobijemo T-dualne teorije i zakone transfromacija koji povezuju koordinate početnih i T-dualnih teorija. Korišćenjem datih zakona transformacije, kao i komutativnih osobina početnih teorija bićemo u stanju da dedukujemo nekomutativne osobine T-dualnih teorija. Metoda za dobijanje T-dualnosti je bazirana na Bušerovoj proceduri i njenim proširenjima. Glavna ideja Bušerove procedure leži u lokalizaciji translacione simetrije, gde mi zamenjujemo parcijalne izvode i koordinate koje se javljaju u dejstvu sa kovarijantnim izvodima i invarijantnim koordinatama. Ova smena sa sobom povlači i uvođenje dodatnih stepeni slobode koji su kodirani preko kalibracionih polja. Eliminacijom novih stepeni slobode preko metode Lagranževih množitelja a zatim pronalaženjem jednačina kretanja za kalibraciona polja, dobijamo zakone transformacija između koordinata. Ubacivanjem ovih zakona transformacija u dejstvo dobijamo T-dualnu teoriju.

U proučavanje bozonske teorije struna, radićemo sa $3D$ prostorom gde uzimamo da Kalb-Ramondovo pozadinsko polje ima infinitezimalu linearnu koordinatnu zavisnost od samo jedne koordinate, z koordinate. Dualizacija će biti sprovedena duž dva različita lanca, jedan gde tek na kraju dualizujemo duž koordinate koja se javlja u pozadinskom polju a druge gde ovu koordinatu prvu dualizujemo. Poređenjem ova dva pristupa bićemo u stanju da zaključimo koji su sastojci neophodni za javljanje nekomutativnih osobina.

Drugi deo disertacije tiče se T-dualizacije tip II superstrune koja se kreće u linearno koordinatno zavisnom Ramond-Ramon polju. Za razliku od plošlig slučaja, ova teorija poseduje i bozonske i fermionske koordinate, doduše pozadinska polja zavise samo od bozonskih koordinata. T-dualizacija će se prvo fokusirati samo na bozonski deo, nakon toga ćemo uključiti i fermionske koordinate. Kao i u prošlom slučaju, predstavićemo i jos jedan alternativan lanaz dualizacije, lanac gde prvo dualizujemo fermionske a zatim bozonske koordinate. Pokazaćemo da oba lanca vode do istih nekomutativnih relacija. Konačno, na kraju disertacije, izvršićemo analizu iste teorije ali sa opštijim slučajem Ramond-Ramond polja.

Ključne reči: Teroija struna, nekomutativnost, neasocijativnost, Bušerova procedura
Naučna oblast: Fizika
Uža naučna oblast: Teorijska fizika visokih energija

Contents

1. Introduction

In the first half ot the 20th century physics has been the subject of two massive developments. First of these developments was switch from deterministic description of nature to one that is probabilistic, giving rise to quantum mechanics. This change introduces many counter intuitive ideas into physics. Some of these changes are that energies of objects are no longer continuous, they are now discrete taking on only certain values. We have that objects are no longer described by their positions and momenta, they are now described with object called wave function in which every relevant quantity is encoded. Furthermore, even when position and momenta are obtained they no longer commute, making it impossible to measure both values to arbitrary high precision. Existence of wave function suggests that objects propagate as waves of probability until measured, then they coalese to one of available states that was allowed by the theory. Other development happened due to realization that speed of light is the same for all observers. This in turn forced us to reconsider nature of space and time itself. Now, passage of time or measurements of positions were no longer the same for different observers moving at different speeds. Such drastic change in our understanding of reality forced us to forgo of concepts of space and time as separate entities but to consider them as one unified object, called space-time and theory which described motion is it was named special relativity. This paradigm shift soon ushered in realization that our understanding of gravity is not compatible with space-time. Now, gravity should be understood, not as a force but as a curvature of four dimensional space-time. Theory that deals with these concepts is called general relativity.

These two developments marked the start of modern theoretical physics, however they themselves are not compatible. Work on incorporation of quantum mechanics and general relativity into one unifying framework started in second half of 20th century and it is still ongoing. It should be noted that some progress has been made, quantum mechanics has been meshed with special relativity resulting in quantum field theory. Theory where main objects are no longer point particles but fields of energy whose excitations should be considered as particles which we observe. Quantum field theory allowed us to construct standard model of particle physics which to this day is the most accurate model of subatomic world. This theory also allows us to examine how fields behave when we have fixed space-time curvature but theory that describes full dynamics between fields and curvature is only a distant dream. Main problem with joining of these two formalism lies in the fact that quantum mechanics is plagued with infinities when we deal with interacting particles. While these infinities can be removed for electrodynamics, weak and strong nuclear forces with process called renormalization, in case of gravity divergences are non removable. This forces us to be more creative if we ever wish to obtain theory of quantum gravity.

Even before advent of quantum field theory there were propositions on how to best deal with emergent divergences. One idea that was proposed was to impose non-commutativity between coordinates, reminiscent of non-commutativity between coordinates and momenta. This would in turn mean that there is minimal possible length in nature and that we can not measure position of particle with infinite precision. When quantum field theory and renormalization were developed, this idea was mostly forgotten. It was not until publishing of paper [1] that non-commutativity came into consideration again. Usually, space time is treated as continuum but in the case when we have non-commutativity between coordinates it is possible to construct Lorentz invariant space-time. After this introduction, there were many attempts to formulate physics through non-commutative formalism and today, quantum field theory constructed on non-commutative space-time is one of possible extensions of standard model [2, 3]. Along with this approach there are other possible approaches to solving problem of quantum gravity, where two most dominant theories are string theory and loop quantum gravity. Since topic of this thesis lies with string theory, we will discus only this approach.

String theory [4, 5, 6, 7, 8] was first developed in 60ies, where it was originally conceived not as a theory of quantum gravity but as a theory of strong nuclear interactions. Due to advent of quantum field theory, string theory was replaced as main contender for description of strong nuclear force. However, it was observed that theory posseses few interesting features to be scraped entirely. Of all features, bt far most important one was that in context of string theory gravity naturally emerges. There was no need to forcibly mesh quantum mechanics and general relativity. Because of this progress in string theory has switched from description of strong nuclear force to possible description of all reality. In years following its conceivement, theory has undergone two major revolutions. First of which was introduction of supersymmetry, making theory applicable to both bosonic and fermionic states. Second revolution occurred when it was noticed that by introducing supersymmetry we inevitably bring along two additional symmetries, S and T dualities, and that there are now finite many consistent string theories. Dualities that emerge now span a web that connects all possible supersymmetric string theories, hinting that there should exist one over encompassing theory. It is not yet know if any of superstring models fully describes our universe and there is still ongoing work to determine this.

Even though there are few ongoing directions which could result in theory of quantum gravity, it should be noted that these alternate approaches are not always mutually exclusive. For example, coordinate non-commutativity also appears in context of string theory [9]. Where it was shown that open strings endpoints which usually propagate along Dirichlet manifold, in certain conditions propagate along non-commutative manifold, in turn giving rise to non-commutative properties. Non-commutativity has also been observed in closed strings, although for different reasons [10]. In case of close string, non-commutativity arises only if we have theory with coordinate dependent fields. By applying T-duality to such models it has been shown that T-dual theories are non-commutative ones. Work until now has been done only on bosonic coordinates, it was not until 2008 that it was found that same duality can emerge in case of fermionic coordinates [11]. This has given rise to possibility that results that have been obtained for supersymmetric particle would naturally follow for supersymmetric string. One of these results [12] is that there would be emergence of non-commutativity between fermionic coordinates which is proportional to bosonic ones. It has been sugested that by working with special configuration of background fields same result would be obtainable in string theory [13]. For now, these connections between supersymmetric particle and string theory are only speculations.

In this thesis, we will focus on examining non-commutativity for closed strings. We fill focus on two models, one for bosonic and other for fermionic strings. Since for these string theories non-commutativity only emerges in T-dual theories if we have coordinate dependent background fields, we will incorporate these kind of fields into our examination. For bosonic string we will be working with coordinate dependent Kalb-Ramond field, while in case of superstring we will be working with coordinate dpendent Ramond-Ramond field. While our choice of bosonic string background field has already been examined [14], approach that we will undertake is novel and it is only based on utilization of T-duality. We will also be discussing possibility of alternative routes for obtaining non-commutativity. For superstring, we decidet to work with coordinate dpendent Ramond-Ramond field because this configuration of fields was used as a basis for speculation of existence of fermionic non-commutativity [13] and our intention is to test if there is any validity to this hypothesis. Just as in the case of bosonic string, we will test alternative routes for obtaining T-dual theories. One where we first perform dualization along bosonic and then fermionic coordinates and one where this direction is reversed. In both models non-commutativity will be analized by establishing a link between coordinates of starting theory and coordinates of T-dual theory. This link when paired with Poisson brackets of starting model will in turn give rise to T-dual Poisson brackets. At the end of the thesis, we will also give short examination of case when we have more general structure of Ramond-Ramond field. Where only T-duality of bosonic coordinates will be examined.

In order to not be overwhelmed by new ideas and concepts, for the rest of this chapter we decided to give short introduction to bosonic strings, superstrings and T-duality. Where most concepts that will later be utilized are developed.

1.1 Bosonic string theory in coordinate dependent background fields

To gain understanding of string theory in general it is useful to begin with examination of bosonic string theory. While this theory lacks many features that would make it suitable for description of real world phenomena, results that are obtained here can be easily generalized to more realistic cases. Since string theory has been historically developed as an extension of model that describes free bosonic particle [4, 5, 6, 7, 8], it is educational for us to start at the same point.

1.1.1 Relativistic point particle

We begin by examining the action of particle of mass m that propagates in D dimensional space-time where we also have gravitational field described by metric tensor $G_{\mu\nu}$. This particle traces out 1-dimensional trajectory denoted with $x^\mu(\tau)$ ($\mu = 0, 1, ..., D$) called "world line", where τ is some arbitrary parameter we call "proper time". Particles trajectory is geodesic thus its action has to be proportional to invariant length of world line

$$S = -m \int ds, \tag{1.1.1}$$

where invariant length is given as

$$ds = \sqrt{-G_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}d\tau. \tag{1.1.2}$$

Equations of motion for particle are obtained by variation of above action with respect to x^μ, thus obtaining

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}, \tag{1.1.3}$$

where we introduced Cristoffel symbol $\Gamma^\mu_{\nu\rho}$ as

$$\Gamma^\mu_{\nu\rho} = G^{\mu\sigma}(x)\frac{1}{2}\left(\partial_\nu G_{\sigma\rho}(x) + \partial_\rho G_{\nu\sigma}(x) - \partial_\sigma G_{\nu\rho}(x)\right). \tag{1.1.4}$$

Action that we introduced possesses one important quality and that is invariance under reparametrizations $\tau \to \tau' = f(\tau)$. However, this action also posses few negative qualities. First of which is presence of square root, in turn making any attempt at quantization quite difficult. Second negative quality is that this action only describes massive particles. Both of those negative qualities can be resolved by introducing additional auxiliary field $e(\tau)$ and by utilizing action that is equivalent to starting action at classical level

$$S = \frac{1}{2}\int\left(\frac{1}{e}G_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - em^2\right)d\tau. \tag{1.1.5}$$

In order to see that this new action is equivalent to the old one, we can find equation of motion for auxiliary field $e(\tau)$

$$G_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - e^2m^2 = 0, \tag{1.1.6}$$

and substitute it into the action (1.1.5), from this we recover starting action (1.1.1). This equation can be thought of as mass-shell condition for propagation in curved space-time. In order to see if new action is also invariant under reparametrization, let us consider infinitesimal transformation of type

$$\tau \to \tau' = \tau + \lambda(\tau). \tag{1.1.7}$$

Coordinates $x^\mu(\tau)$ transform as scalars under reparametrization, thus for them we have

$$\delta_0 x^\mu(\tau) = x^{\mu\prime}(\tau) - x^\mu(\tau) = -\lambda(\tau)\frac{dx^\mu}{d\tau}. \tag{1.1.8}$$

Transformation of auxiliary field can be obtained if we notice that second term in action (1.1.5) must transform according to

$$e(\tau)d\tau = e'(\tau')d\tau', \tag{1.1.9}$$

from which we can deduce following transformation rule

$$\delta_0 e(\tau) = e'(\tau) - e(\tau) = -\frac{d}{d\tau}(\lambda e). \tag{1.1.10}$$

Having obtained how all fields behave under reparametrization of parameter τ it is straightforward to check that whole action (1.1.5) is also invariant

$$\delta_0 S = \frac{1}{2}\int d\tau \left(\frac{2}{e}G_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{d\delta_0 x^\nu}{d\tau} + \frac{1}{e}\partial_\rho G_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta_0 x^\rho - \frac{1}{e^2}G_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta_0 e - m^2\delta_0 e\right) \tag{1.1.11}$$

Applying partial integration to first term, neglecting surface terms and by plugging in transformation laws it is easy to show that

$$\delta_0 S = 0. \tag{1.1.12}$$

This proves to us that by adding additional auxiliary field we can be sure that we are not changing any underlying physics of the theory.

1.1.2 Bosonic string theory

Main idea of bosonic string theory is really simple, instead of working with action that describes one dimensional world line in space-time we work with action that describes two dimensional sheet which propagates in space-time. By working in D dimensional space-time, we denote the location of the string by coordinates $x^\mu(\tau,\sigma)$ $(\mu = 0,1,...,D)$, due to increase in dimensionality of the object we are analyzing, we also have increase in number of parameters needed for its parameterization. Where we have that world-sheet surface Σ is parameterized by τ and σ (in future chapters we will also utilize be ξ^0 and ξ^1). Action is formed by maximization of surface and it has following form

$$S = k\int_\Sigma d\mu. \tag{1.1.13}$$

Here k is called tension of the brane and it has dimmensionality of the $(mass)^2$ or $(length)^{-2}$. Brane tension is also usually represented as $k = 1/(2\pi\alpha')$, where parameter α' is named Regge slope. We have that $d\mu$ represent surface element which, in order for action to be dimensionless, has dimensionality of $(length)^2$. Surface element it is given by

$$d\mu = \sqrt{-det\left(G_{\mu\nu}(x)\partial_m x^\mu \partial_n x^\nu\right)}d^2\xi, \quad (m,n = 0,1.). \tag{1.1.14}$$

Just like before, we have that $G_{\mu\nu}$ describes metric of D dimensional space-time. We have also took the liberty to denote $d\tau d\sigma$ as $d^2\xi$, while indices m and n are world-sheet indices.

Similarly, as in case of point particle, we would like to obtain action that does not posses square root. Again this can be accomplished by introducing additional auxiliary field, this time we have auxiliary metric g_{mn} that describes intrinsic geometry of the two dimensional manifold. It should also be noted that auxiliary metric has Lorenzian signature. Action for bosonic string can then be transcribed as

$$S = \frac{k}{2}\int_\Sigma d^2\xi\sqrt{-g}g^{mn}(\sigma)G_{\mu\nu}(x)\partial_m x^\mu \partial_n x^\nu, \tag{1.1.15}$$

where g^{mn} is inverse and g is determinant of g_{mn}.

In order to examine some interesting properties of this action we will switch to D dimensional Minkowski spacetime. In this case action (1.1.15) reduces to

$$S = \frac{k}{2}\int_\Sigma d^2\xi\sqrt{-g}g^{mn}(\sigma)(x)\partial_m x^\mu \partial_n x_\mu, \tag{1.1.16}$$

Above given action has two important local symmetries. First is invariance to reparametrizations of type

$$\delta_0 x^\mu = \lambda^m \partial_m x^\mu, \tag{1.1.17}$$
$$\delta_0 g^{mn} = \lambda^r \partial_r g^{mn} - \partial_r \lambda^m g^{rn} - \partial_r \lambda^n g^{mr}, \tag{1.1.18}$$
$$\delta_0 \sqrt{-g} = \partial_m(\lambda^m \sqrt{-g}). \tag{1.1.19}$$

Second local symmetry is invariance to Weyl scaling

$$\delta_0 g^{mn} = \Lambda(\xi) g^{mn}. \tag{1.1.20}$$

While now presented just as curiosity, Weyl invariance plays a major role in derivation of consistency equations for background fields. Both λ^m and Λ are functions of world-sheet coordinates ξ^0 and ξ^1.

In addition to these local symmetries, action also possess Poincaré invariance

$$\delta_0 x^\mu = \omega^\mu{}_\nu x^\nu + l^\mu, \tag{1.1.21}$$
$$\delta_0 g^{mn} = 0. \tag{1.1.22}$$

where $\omega^\mu{}_\nu$ is antisymmetric.

Due to presence of Weyl invariance in bosonic string action we have that trance of energy-momentum is zero, $g^{mn}T_{mn} = 0$. This tensor is given as a variation of action with respect to world-sheet metric tensor g^{mn}

$$T_{mn} = \frac{2}{k}\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{mn}} = \partial_m x^\mu \partial_n x_\mu - \frac{1}{2} g_{mn} g^{kl} \partial_k x^\mu \partial_l x_\mu \tag{1.1.23}$$

Since term $\delta S/\delta g^{mn}$ represents equation of motion for field g^{mn}, we than have that energy-momentum tensor is zero. With this result obtained, we return to examination of bosonic string in nontrivial background fields.

1.1.3 Inclusion of background fields

We have already seen how we can incorporate space-time metric in bosonic string theory (1.1.15).

$$S_1 = \frac{k}{2}\int_\Sigma d^2\xi\sqrt{-g}g^{mn}(\sigma)G_{\mu\nu}(x)\partial_m x^\mu \partial_n x^\nu. \tag{1.1.24}$$

Inclusion of this tensor does not change any of local symmetries that we had in Minkowski case. Similarly, energy-momentum tensor is also zero.

In addition to space-time metric, there are two more massless tensors that can be added into the theory. First of these tensors is antisymmetric Kalb-Ramond field $B_{\mu\nu}$. Action that contains this tensor is given by

$$S_2 = \frac{k}{2}\int_\Sigma d^2\xi \epsilon^{mn}(\sigma) B_{\mu\nu}(x)\partial_m x^\mu \partial_n x^\nu. \tag{1.1.25}$$

Where we needed to introduce two dimensional Levi-Civita tensor ϵ^{mn} with signature $\epsilon^{\tau\sigma} = -1$. Similarly to space-time metric tensor, Kalb-Ramond field does not break reparametrization invariance or Wayl symmetry.

Final massless tensor that can be added to theory is scalar dilaton field $\Phi(x)$. This field couples to world-sheet scalar curvature

$$S_3 = \frac{k}{2}\int_\Sigma d^2\xi \sqrt{-g}\Phi(x) R^{(2)}. \tag{1.1.26}$$

Including dilation field into bosonic string violates Weyl invariance of the action [15]. It is necessary for consistency of the theory, if we want to make sensible quantum theory, that action be locally scale invariant. That is, we need to have traceless world-sheet energy momentum tensor. Breakdown of scale invariance in quantum field theories is usually encapsulated in β functions, where these functions arise from unltaviolet divergences in Feynman diagrams.

For theory that contains all tree terms we have following action

$$S = k\int_\Sigma d^2\xi\sqrt{-g}\left\{\left[\frac{1}{2}g^{mn}G_{\mu\nu}(x) + \frac{\epsilon^{mn}}{\sqrt{-g}}B_{\mu\nu}(x)\right]\partial_m x^\mu \partial_n x^\nu + \Phi(x)R^{(2)}\right\}. \tag{1.1.27}$$

For this action, trace of energy-momentum tensor has following general structure

$$2\pi T^m_m = \beta^\Phi\sqrt{-g}R^{(2)} + \beta^G_{\mu\nu}\sqrt{-g}g^{mn}\partial_m x^\mu\partial_n x^\nu + \beta^b_{\mu\nu}\epsilon^{mn}\partial_m x^\mu \partial_n x^\nu, \tag{1.1.28}$$

where β^Φ, β^G and β^B are local functions of the coupling functions $\Phi(x)$, $G_{\mu\nu}(x)$ and $B_{\mu\nu}(x)$. Given β functions can be calculated in perturbation theory, where we find vacuum expectation values of vertex operators. By performing these calculations, which are far beyond the scope of this thesis, and demanding that trace of energy-momentum tensor be zero we arrive at following set of equations [16]

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4}B_{\mu\rho\sigma}B_\nu^{\rho\sigma} + 2D_\mu a_\nu = 0, \tag{1.1.29}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho_{\mu\nu} - 2a_\rho B^\rho_{\mu\nu} = 0, \tag{1.1.30}$$

$$\beta^\Phi \equiv 2\pi\kappa\frac{D-26}{6} - R - \frac{1}{24}B_{\mu\rho\sigma}B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = c, \tag{1.1.31}$$

Here c is undetermined constant called Schwinger term, D_μ are space-time covariant derivatives and $R_{\mu\nu}$ is space-time Ricci tensor [17, 18, 19, 20, 21, 22]. Expressions for

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \quad a_\mu = \partial_\mu \Phi, \tag{1.1.32}$$

represents field strength of Kalb-Ramond field [23, 24], while a_μ is dilaton gradient. First term in β^Φ comes from conformal gauge Faddeev-Popov determinant [25]. In addition to these equations we also have following relation

$$D^\nu\beta^G_{\nu\mu} + \partial_\mu\beta^\Phi = 0, \tag{1.1.33}$$

which allows us to set β^Φ to constant.

By solving equations (1.1.29), (1.1.30) and (1.1.31) we will be able to find coordinate dependent configuration of background fields that will be used in examination of bosonics string non-commutativity.

1.2 Supersymmetric string theory

Having developed action for bosonic string we now focus on more complex case of superstrings. This immediately raises the question, why? Answer lies in the fact that bosonic string theory has few irredeemable qualities which make it unfit to be considered theory of everything. First of these problems is the fact that theory requires 26 dimensional space-time to operate. Second problem is existence of tachyons, making vacuum unstable. While it is true that both of these problems could be solvable, first by compactification of extra dimensions and second could also be solvable by finding some other stable vacuum. Third problem that theory possesses is sadly the greatest one and it is not solvable, theory lacks fermionic states. If we ever wish to describe real world we must have theory that deals with both bosons and fermions.

Introduction of fermionic states can be done in few different way, however all are focused on incorporation of supersymmetry [4, 6, 7, 8]. This can be done by adding supersymmetry at world-sheet level or at space-time level producing two different formalism, Ramond-Neveu-Schwarz formalism and Green-Schwarz formalism respectively. While distinct, it can be shown that these two fromulations are equivalent. These two formulations have one major flaw and that is that they needlesly complicate introduction of nontrivial background fields. Fortunately in last few years there has been emergence of third formulation of superstring theory, pure spinor formulation [26, 27, 28, 29, 30]. While having more technical difficulties in obtaining the starting action, theory shines in generalization of flat space action to one with complex fields. Where all fields are incorporated by adding integrated massless vertex operator [31] to the action of flat theory. This section will focus on obtaining pure spinor superstring theory.

1.2.1 Supersymmetric point particle

Just as was case for bosonic strings, in order to obtain action for supersymmetric string theory we need to start with action for supersymmetric point particle. This action is in fact generalization of action for point particle that we had before and action (1.1.5) will be our starting point.

$$S = \int \frac{1}{2e}\dot{x}^2 d\tau, \tag{1.2.34}$$

where we will work with flat Minkowski space-time. Since the mass term is not relevant for examination of string theory we decided to set m to zero.

In order to obtain supersymmetry we need to expand the space from only including bosonic coordinates x^μ to also including fermionic $\theta^{A\alpha}$ ones, here index $A = 1, 2, ...N$ denotes the number of supersymmetry and in turn number of anticommutating spinor coordinates, since higher order of supersymmetry does not produce any additional insight we will be only interested in case where $A = 1$ and from now on we will be neglecting this index. Index α denotes spinor components and in general for Dirac spinor it is dependent on number of dimensions D of space-time, where we have $\alpha = 0, 1..., 2^{D/2}$. Supersymmetry is obtained by demanding that both bosonic and fermionic coordinates transform in certain way

$$\delta_0 \theta^\alpha = \epsilon^\alpha, \quad \delta_0 x^\mu = i\epsilon^\alpha (\Gamma^\mu)_{\alpha\beta}\theta^\beta, \quad \delta_0 e = 0. \tag{1.2.35}$$

Here ϵ^α is infinitesimal constant Grassmann parameter, $(\Gamma^\mu)_{\alpha\beta}$ are Dirac matrices. Since we are interested only in Majorana-Weyl spinors, above relations are written in form appropriate to these spinors. It should be noted that these transformation laws do not single out any specific action. While there are many different supersymmetric actions that can be written we are only interested in simplest one, this leaves us with

$$S = \int \frac{1}{2e}\left(\dot{x} - i\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\dot{\theta}^\beta\right)^2 d\tau. \tag{1.2.36}$$

This action is invariant to full super-Poincaré symmetry. Equations of motion are

$$\Pi^2 = 0, \quad \dot{\Pi}^\mu = 0, \quad (\Gamma^\mu)_{\alpha\beta}\Pi_\mu \dot{\theta}^\beta = 0. \tag{1.2.37}$$

where we denoted with Π^μ

$$\Pi^\mu = \dot{x}^\mu - i\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\dot{\theta}^\beta. \tag{1.2.38}$$

Since we have that θ^α always comes attached to term $\Gamma^\mu \Pi_\mu$ and $(\Gamma^\mu \Pi_\mu)^2 = -\Pi^2$, action possesses additional symmetry. This is local symmetry known in literature as κ symmetry. By denoting with $\kappa(\tau)$ infinitesimal Grassmann spinor parameters we can observe that action is invariant under

$$\delta_\kappa \theta^\alpha = (\Gamma^\mu)_{\alpha\beta}\Pi_\mu \kappa^\beta, \quad \delta_\kappa x^\mu = i\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\delta_\kappa\theta^\beta, \quad \delta_\kappa e = 4ie\dot{\theta}^\alpha \kappa_\alpha \tag{1.2.39}$$

To see that action (1.2.36) is invariant under these transformations we can start by examining how Π^μ and e^{-1} transform

$$\begin{aligned}
\delta_\kappa \Pi^\mu &= \delta_\kappa \dot{x}^\mu - i\delta_\kappa \theta^\alpha (\Gamma^\mu)_{\alpha\beta}\dot{\theta}^\beta - -i\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\delta_\kappa \dot{\theta}^\beta \\
&= i\dot{\theta}^\alpha (\Gamma^\mu)_{\alpha\beta}\delta_\kappa\theta^\beta + i\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\delta_\kappa\dot{\theta}^\beta - i\delta_\kappa\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\dot{\theta}^\beta - -i\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\delta_\kappa\dot{\theta}^\beta \\
&= 2i\dot{\theta}^\alpha (\Gamma^\mu)_{\alpha\beta}\delta_\kappa\theta^\beta. \qquad (1.2.40)\\
\delta_\kappa \Pi^2 &= 2\Pi_\mu \delta_\kappa \Pi^\mu = 4i\dot{\theta}^\alpha (\Gamma^\mu)_{\alpha\beta}\delta_\kappa\theta^\beta \Pi_\mu = 4i\Pi^2 \dot{\theta}^\alpha \kappa_\alpha \qquad (1.2.41)\\
\delta_\kappa e^{-1} &= -e^{-2}\delta_\kappa e = -4ie^{-1}\dot{\theta}^\alpha \kappa_\alpha. \qquad (1.2.42)
\end{aligned}$$

Finding variation of action we have

$$\delta_\kappa S = \frac{1}{2}\int \left(\delta_\kappa e^{-1}\Pi^2 + e^{-1}\delta_\kappa \Pi^2\right) = \frac{i}{2}\int \left(4e^{-1}\Pi^2\dot{\theta}^\alpha \kappa_\alpha - 4e^{-1}\Pi^2\dot{\theta}^\alpha \kappa_\alpha\right) = 0 \tag{1.2.43}$$

While here presented for sake of completeness, existence of κ symmetry is crucial for obtaining Green-Schwarz formulation of string theory. Now we will demonstrate procedure for obtaining pure spinor description of superparticle. Methods that we develop here will be exactly the same as ones needed for string.

1.2.2 Pure spinor formulation of supersymmetric point particle

Before proceeding with pure spinor formalism it should be noted that action (1.2.36) possesess constraints. This can easily be seen by finding conjugated momentum of fermionic coordinate

$$\pi_\alpha = \frac{\delta S}{\delta \dot{\theta}} = i\pi^\mu (\Gamma^\mu)_{\alpha\beta}\theta^\beta = i\left(\dot{x}^\mu - i\theta^{\alpha_1}(\Gamma^\mu)_{\alpha_1\beta_1}\dot{\theta}^{\beta_1}\right)(\Gamma^\mu)_{\alpha\beta}\theta^\beta. \tag{1.2.44}$$

where π_μ is conjugate momenta of bosonic coordinate x^μ. While expressions for π_μ and Π_μ are identical it should be noted that their interpretations are different, one is conjugate momenta and other is just combination of bosonic and fermionic coordinates that is invariant under supersymmetry transformations. If we decidet to take more complex supersymmetry invariant combination of coordinates, these terms would not coincide. From this we see that

$$d_\alpha = \pi_\alpha + i\dot{x}_\mu (\Gamma^\mu)_{\alpha\beta}\theta^\beta + (\Gamma^\mu)_{\alpha\beta}\theta^\beta\theta^{\alpha_1}(\Gamma_\mu)_{\alpha_1\beta_1}\dot{\theta}^{\beta_1} \tag{1.2.45}$$

represents constrain. Since fermionic coordinates θ^α and their conjugated momenta π_α satisfy following Poisson bracket

$$\{\pi_\alpha, \theta^\beta\} = \delta_\alpha^\beta \tag{1.2.46}$$

we have that constraint satisfy

$$\{d_\alpha, d_\beta\} = \pi_\mu (\Gamma^\mu)_{\alpha\beta}. \tag{1.2.47}$$

Starting point for pure spinor particle is by replacing action (1.2.36) with following quadratic action [32]

$$S = \int d\tau \left(\pi_\mu \dot{x}^\mu + \pi_\alpha \dot{\theta}^\alpha + \omega_\alpha \dot{\lambda}^\alpha - \frac{1}{2}\pi^\mu \pi_\mu\right) \tag{1.2.48}$$

where π_α are now independent variables [33] and λ^α are pure spinor ghost variables satisfying pure spinor constraints

$$\lambda^\alpha (\Gamma^\mu)_{\alpha\beta}\lambda^\beta = 0, \tag{1.2.49}$$

their conjugated momenta are given by ω_α and they are defined up to gauge transformation

$$\delta_0 \omega_\alpha = (\Gamma^\mu)_{\alpha\beta}\lambda^\beta \Lambda_\mu. \tag{1.2.50}$$

To obtain correct correct physical states, action (1.2.48) needs to be supplemented with BRST like operator

$$Q = \lambda^\alpha d_\alpha \tag{1.2.51}$$

where all physical states are in the cohomology of above given operator. We have that $Q^2 = 0$ and this operator carries ghost number $+1$, that is if we define that λ^α and ω_α carry ghost numbers $+1$ and -1 respectively. This operator posseses few interesting properties, for example in case of massless relativistic point particle we had following mass-shell relation $\pi_\mu \pi^\mu = 0$, here this is indirectly implied by operator Q. Furthermore, we had that supersymmetric particle action is invariant under κ symmetry, here we have that pure spinor action does not posses this symmetry but is in fact replaced by gauge invariance generated by Q.

Now we would like to examine how to obtain background fields in which superparticle can propagate. We will give example for case where ghost number is $+1$, then wave function can be described as

$$\Psi(x,\theta) = \lambda^\alpha A_\alpha(x,\theta), \tag{1.2.52}$$

here A_α is the superfield. By acting with Q on this wave function we obtain

$$Q\Psi(x,\theta) = \lambda^\alpha \lambda^\beta D_\alpha A_\beta = 0, \tag{1.2.53}$$

this relation implies

$$(\Gamma_{[\mu\nu\rho\delta\gamma]})^{\alpha\beta} D_\alpha A_\beta = 0, \tag{1.2.54}$$

where $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{i}{2}(\Gamma^\mu)_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^\mu}$ is supersymmetric derivative and $(\Gamma_{[\mu\nu\rho\delta\gamma]})^{\alpha\beta}$ is totally antisymmetric product of five gamma matrices. Imposing following variation of wave funtion

$$\delta_0 \Psi(x,\theta) = Q\Lambda = \lambda^\alpha D_\alpha \Lambda \tag{1.2.55}$$

we have following transformation for superfield

$$\delta_0 A_\alpha(x,\theta) = D_\alpha \Lambda(x,\theta). \tag{1.2.56}$$

This means that equation (1.2.54) and (1.2.56) are Maxwell equations of motion and gauge invariance for superfield A_α. It should also be noted that field A_α can be transcribed as expansion in fermionic coordinates as

$$A_\alpha(x,\theta) = f_\alpha(x) + f_{\alpha\beta}(x)\theta^\beta + f_{\alpha\beta\rho}(x)\theta^\beta\theta^\rho + ... \tag{1.2.57}$$

By using wave function with ghost numbers that are different from +1 we can obtain more background fields and their equations of motion, sadly we will omit analyzing such cases.

1.2.3 Superstring

Action for superstring follows the same philosophy as the one we had for superparticle. In short, we want to expand spacetime by introducing fermionic coordinates and by finding some combination of coordinates that is invariant to supersymmetry. This is accomplished by following combination

$$\Pi^\mu_m = \partial_m x^\mu - i\theta^{\alpha A}(\Gamma_\mu)_{\alpha\beta}\partial_m \theta^{\beta A} \tag{1.2.58}$$

Superstrings are defined in ten dimensional space time, therefore we have that $\mu = 0, 1...9$, while for spinors we have $\alpha = 1, 2, ..., 16$. Indices m and n are world-sheet indices as before. Depending on the type of superstring theory, we have different number for supersymmetry. We are mainly interested in type II superstrings, hence we will be working with $N = 2$ SUSY. This way we can transcribe above equation as

$$\Pi^\mu_m = \partial_m x^\mu - i\theta^\alpha(\Gamma_\mu)_{\alpha\beta}\partial_m\theta^\beta - i\bar{\theta}^\alpha(\Gamma_\mu)_{\alpha\beta}\partial_m\bar{\theta}^\beta. \tag{1.2.59}$$

Based on chirality of spinors, there is further subdivison of type II theory. We have type IIA for opposite chirality and type IIB superstring theory for same chirality. Having defined supersymmetry invariant combination of coordinates, action has following form

$$S = \frac{k}{2}\int_\Sigma d^2\xi\sqrt{-g}g^{mn}\Pi^\mu_m \Pi_{n\mu} \tag{1.2.60}$$

This action, just as one for superparticle, is invariant under reparametrization and κ transformations. Action also has following constraints

$$d_{m\alpha} = \pi_{m\alpha} + \left(i\partial_m x^\mu + \frac{1}{2}\theta^{\alpha_1}(\Gamma_\mu)_{\alpha_1\beta_1}\partial_m\theta^{\beta_1} + \frac{1}{2}\bar{\theta}^{\alpha_1}(\Gamma_\mu)_{\alpha_!\beta_1}\partial_m\bar{\theta}^{\beta_1} \right)(\Gamma_\mu)_{\alpha\beta}\theta^\beta, \tag{1.2.61}$$

$$\bar{d}_{m\alpha} = \bar{\pi}_{m\alpha} + \left(i\partial_m x^\mu + \frac{1}{2}\theta^{\alpha_1}(\Gamma_\mu)_{\alpha_1\beta_1}\partial_m\theta^{\beta_1} + \frac{1}{2}\bar{\theta}^{\alpha_1}(\Gamma_\mu)_{\alpha_!\beta_1}\partial_m\bar{\theta}^{\beta_1} \right)(\Gamma_\mu)_{\alpha\beta}\bar{\theta}^\beta. \tag{1.2.62}$$

For pure spinor formalism, we make a switch from action (1.2.59) to action for flat space-time that is quadratic [26, 27, 28, 29]

$$S = \int_\Sigma d^2\xi \left(\frac{\kappa}{2}\eta_{\mu\nu}\partial_m x^\mu \partial_n x^\nu \eta^{mn} - \pi_\alpha \partial_- \theta^\alpha + \partial_+\bar{\theta}^\alpha \bar{\pi}_\alpha + \omega_\alpha \partial_- \lambda^\alpha + \bar{\omega}_\alpha \partial_+ \bar{\lambda}^\alpha \right), \tag{1.2.63}$$

Here x^μ, θ^α, π_α, $\bar{\theta}^\alpha$ and $\bar{\pi}_\alpha$ are Green-Schwarz-Siegel matter variables where indices take range $\mu = 0, 1..9$, $\alpha = 1, 2, ...16$. Pure spinors are labeled with λ^α and $\bar{\lambda}^\alpha$, while their conjugated momenta are ω_α and $\bar{\omega}_\alpha$, respectively. We have that pure spinors satisfy pure spinor constraints

$$\lambda^\alpha(\Gamma^\mu)_{\alpha\beta}\lambda^\beta = \bar{\lambda}(\Gamma^\mu)_{\alpha\beta}\bar{\lambda}^\beta = 0\,. \tag{1.2.64}$$

Similarly as in the case of superparticle, in order to obtain physical states we need to introduce operator BRST like operator Q, however in order to do that we need to transcribe equations (1.2.61) and (1.2.62) into light-cone coordinates

$$d_\alpha = \pi_\alpha - \frac{1}{2}(\Gamma_\mu\theta)_\alpha \left[\partial_+ x^\mu + \frac{1}{4}(\theta\Gamma^\mu\partial_+\theta) \right], \tag{1.2.65}$$

$$\bar{d}_\alpha = \bar{\pi}_\alpha - \frac{1}{2}(\Gamma_\mu\bar{\theta})_\alpha \left[\partial_- x^\mu + \frac{1}{4}(\bar{\theta}\Gamma^\mu\partial_-\bar{\theta}) \right], \tag{1.2.66}$$

then we have

$$Q_L = \int d\xi^+ \lambda^\alpha d_\alpha, \quad Q_R = \int d\xi^- \bar{\lambda}^\alpha \bar{d}_\alpha, \tag{1.2.67}$$

where

$$Q_L^2 = -\int d\xi^+ \lambda^\alpha (\Gamma^\mu)_{\alpha\beta}\lambda^\beta \Pi_{+\mu}, \quad Q_R^2 = -\int d\xi^- \bar{\lambda}^\alpha (\Gamma^\mu)_{\alpha\beta}\bar{\lambda}^\beta \Pi_{-\mu}. \tag{1.2.68}$$

Due to pure spinor constraint equations (1.2.64) we have that Q_L and Q_R are nillpotent. We introduced following two combinations of coordinates written in light-cone coordinates

$$\Pi_+^\mu = \partial_+ x^\mu + \frac{1}{2}\theta^\alpha(\Gamma^\mu)_{\alpha\beta}\partial_+\theta^\beta, \quad \Pi_-^\mu = \partial_- x^\mu + \frac{1}{2}\bar{\theta}^\alpha(\Gamma^\mu)_{\alpha\beta}\partial_-\bar{\theta}^\beta. \tag{1.2.69}$$

For ghost number zero, massless super Yang-Mills states are obtained from following unintegrated and integrated vertex operators

$$V = \lambda^\alpha A_\alpha(x, \theta, \bar{\theta}),$$

$$\int d\xi^+ U = \int d\xi^+ \left(\partial_+\theta^\alpha A_\alpha(x,\theta,\bar{\theta}) + \Pi_+^\mu A_\mu(x,\theta,\bar{\theta}) + d_\alpha W^\alpha(x,\theta,\bar{\theta}) + N_{+\mu\nu}F^{\mu\nu}(x,\theta,\bar{\theta}) \right) \tag{1.2.70}$$

Here A_α and A_μ are gauge fields, while W^α and $F^{\mu\nu}$ are superfield-strengths for super Yang-Mills. We also have introduction of Lorentz currents for pure spinor variables, given as

$$N_+^{\mu\nu} = \frac{1}{2}\omega_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \lambda^\beta, \quad \bar{N}_-^{\mu\nu} = \frac{1}{2}\bar{\omega}_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \bar{\lambda}^\beta. \tag{1.2.71}$$

By acting with Q on vertex operator in both integrated and unintegrated form and by demanding that

$$QV = 0, \quad QU = \partial_+ V \tag{1.2.72}$$

we can obtain equations of motion for fields with ghost number zero. However, we are not only interested in background fields with ghost number zero, we are interested in all possible background fields that are allowed by the theory. Fortunately there is finite amount of fields that are allowed, they can be collected into a supermatrix A_{MN}, while integrated form of vertex operator is given by

$$V_{SG} = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar{X}^N. \tag{1.2.73}$$

$$X^M = \begin{pmatrix} \partial_+\theta^\alpha \\ \Pi_+^\mu \\ d_\alpha \\ \frac{1}{2}N_+^{\mu\nu} \end{pmatrix}, \quad \bar{X}^M = \begin{pmatrix} \partial_-\bar{\theta}^\lambda \\ \Pi_-^\mu \\ \bar{d}_\lambda \\ \frac{1}{2}\bar{N}_-^{\mu\nu} \end{pmatrix}, \quad A_{MN} = \begin{bmatrix} A_{\alpha\beta} & A_{\alpha\nu} & E_\alpha{}^\beta & \Omega_{\alpha,\mu\nu} \\ A_{\mu\beta} & A_{\mu\nu} & \bar{E}_\mu^\beta & \Omega_{\mu,\nu\rho} \\ E^\alpha{}_\beta & E_\nu^\alpha & P^{\alpha\beta} & C^\alpha{}_{\mu\nu} \\ \Omega_{\mu\nu,\beta} & \Omega_{\mu\nu,\rho} & \bar{C}^\beta{}_{\mu\nu} & S_{\mu\nu,\rho\sigma} \end{bmatrix}. \tag{1.2.74}$$

Here, fields $A_{\mu\nu}$, $\bar{E}_\mu^\alpha$, E_μ^α and $P^{\alpha\beta}$ are known as physical superfields, while superfields that are in the first row and the first column are known as auxiliary because they can be expressed in terms of physical ones [31]. Remaining superfields $\Omega_{\mu,\nu\rho}$ ($\Omega_{\mu\nu,\rho}$), $C^\alpha{}_{\mu\nu}$ ($\bar{C}^\beta{}_{\mu\nu}$) and $S_{\mu\nu,\rho\sigma}$, are curvatures (field strengths) for physical fields. This notation is in accordance with Ref [31]. By acting with BRST operators Q_L and Q_R on this vertex operator we obtain following equations for background fields.

$$
\begin{aligned}
&(1+\mathbf{D})A_{\alpha\beta} = (\Gamma^\mu\theta)_\alpha A_{\mu\beta}, && (1+\bar{\mathbf{D}})A_{\alpha\beta} = (\Gamma^\mu\bar{\theta})_\beta A_{\alpha\mu}, && (1.2.75)\\
&\mathbf{D}A_{\mu\beta} = (\Gamma_\mu\theta)_\alpha E^\alpha{}_\beta && \bar{\mathbf{D}}A_{\alpha\mu} = (\Gamma_\mu\bar{\theta})_\beta E_\alpha{}^\beta, && (1.2.76)\\
&\mathbf{D}E^\alpha{}_\beta = -\frac{1}{4}(\Gamma^{[\mu\nu]}\theta)^\alpha\Omega_{\mu\nu,\beta}, && \bar{\mathbf{D}}E_\alpha{}^\beta = -\frac{1}{4}(\Gamma^{[\mu\nu]}\bar{\theta})^\beta\Omega_{\alpha,\mu\nu}, && (1.2.77)\\
&\mathbf{D}\Omega_{\mu\nu,\beta} = -(\Gamma_{[\mu}\theta)_\alpha\partial_{\nu]}E^\alpha{}_\beta, && \bar{\mathbf{D}}\Omega_{\alpha,\mu\nu} = -(\Gamma_{[\mu}\bar{\theta})_\beta\partial_{\nu]}E_\alpha{}^\beta, && (1.2.78)
\end{aligned}
$$

$$
\begin{aligned}
&(1+\mathbf{D})A_{\alpha\nu} = (\Gamma^\mu\theta)_\alpha A_{\mu\nu}, && (1+\bar{\mathbf{D}})A_{\nu\beta} = (\Gamma^\mu\bar{\theta})_\beta A_{\nu\mu}, && (1.2.79)\\
&\mathbf{D}A_{\mu\nu} = (\Gamma_\mu\theta)_\alpha E^\alpha_\nu, && \bar{\mathbf{D}}A_{\nu\mu} = (\Gamma_\mu\bar{\theta})_\beta \bar{E}^\beta_\nu, && (1.2.80)\\
&\mathbf{D}E^\alpha_{\mu_1} = -\frac{1}{4}(\Gamma^{[\mu\nu]}\theta)^\alpha\Omega_{\mu\nu,\mu_1}, && \bar{\mathbf{D}}\bar{E}^\beta_{\mu_1} = -\frac{1}{4}(\Gamma^{[\mu\nu]}\bar{\theta})^\beta\Omega_{\mu_1,\mu\nu}, && (1.2.81)\\
&\mathbf{D}\Omega_{\mu\nu,\mu_1} = -(\Gamma_{[\mu}\theta)_\alpha\partial_{\nu]}E^\alpha_{\mu_1}, && \bar{\mathbf{D}}\Omega_{\mu_1,\mu\nu} = -(\Gamma_{[\mu}\bar{\theta})_\beta\partial_{\nu]}\bar{E}^\beta_{\mu_1}, && (1.2.82)
\end{aligned}
$$

$$
\begin{aligned}
&(1+\mathbf{D})E_\alpha{}^\beta = (\Gamma^\mu\theta)_\alpha E^\beta_\mu, && (1+\bar{\mathbf{D}})E^\alpha{}_\beta = (\Gamma^\mu\bar{\theta})_\beta \bar{E}^\alpha_\mu, && (1.2.83)\\
&\mathbf{D}E^\beta_\mu = (\Gamma_\mu\theta)_\alpha P^{\alpha\beta}, && \bar{\mathbf{D}}\bar{E}^\alpha_\mu = (\Gamma_\mu\bar{\theta})_\beta P^{\alpha\beta}, && (1.2.84)\\
&\mathbf{D}P^{\alpha\beta} = -\frac{1}{4}(\Gamma^{[\mu\nu]}\theta)^\alpha C^\beta{}_{\mu\nu}, && \bar{\mathbf{D}}P^{\alpha\beta} = -\frac{1}{4}(\Gamma^{[\mu\nu]}\bar{\theta})^\beta \bar{C}^\alpha{}_{\mu\nu}, && (1.2.85)\\
&\mathbf{D}C^\beta_{\mu\nu} = -(\Gamma_{[\mu}\theta)_\alpha\partial_{\nu]}P^{\alpha\beta}, && \bar{\mathbf{D}}\bar{C}^\alpha{}_{\mu\nu} = -(\Gamma_{[\mu}\bar{\theta})_\beta\partial_{\nu]}P^{\alpha\beta}, && (1.2.86)
\end{aligned}
$$

$$
\begin{aligned}
&(1+\mathbf{D})\Omega_{\alpha,\mu_1\nu} = (\Gamma^\mu\theta)_\alpha\Omega_{\mu,\mu_1\nu}, && (1+\bar{\mathbf{D}})\Omega_{\mu\nu,\alpha} = (\Gamma^{\mu_1}\bar{\theta})_\beta\Omega_{\mu\nu,\mu_1}, && (1.2.87)\\
&\mathbf{D}\Omega_{\mu,\mu_1\nu} = (\Gamma_\mu\theta)_\alpha C^\alpha{}_{\mu_1\nu}, && \bar{\mathbf{D}}\Omega_{\mu_1\nu,\mu} = (\Gamma_\mu\bar{\theta})_\beta \bar{C}^\beta{}_{\mu_1\nu}, && (1.2.88)\\
&\mathbf{D}C^\alpha{}_{\mu_1\nu_1} = -\frac{1}{4}(\Gamma^{[\mu\nu]}\theta)^\alpha S_{\mu\nu,\mu_1\nu_1}, && \bar{\mathbf{D}}\bar{C}^\beta{}_{\mu_1\nu_1} = -\frac{1}{4}(\Gamma^{[\mu\nu]}\bar{\theta})^\beta S_{\mu_1\nu_1,\mu\nu}, && (1.2.89)\\
&\mathbf{D}S_{\mu\nu,\mu_1\nu_1} = -(\Gamma_{[\mu}\theta)_\alpha\partial_{\nu]}C^\alpha{}_{\mu_1\nu_1}, && \bar{\mathbf{D}}S_{\mu_1\nu_1\mu\nu,} = -(\Gamma_{[\mu}\bar{\theta})_\beta\partial_{\nu]}\bar{C}^\beta{}_{\mu_1\nu_1}. && (1.2.90)
\end{aligned}
$$

Here operators $\mathbf{D}$ and $\bar{\mathbf{D}}$ are given as

$$
\mathbf{D} \equiv \theta^\alpha \frac{\partial}{\partial\theta^\alpha}, \quad \bar{\mathbf{D}} \equiv \bar{\theta}^\alpha \frac{\partial}{\partial\bar{\theta}^\alpha} \tag{1.2.91}
$$

Every superfield appears in two groups of equation, reason for this is that both θ and $\bar{\theta}$ components of the field need to be fixed. Inside each group there is iterative structure [34, 35] which allows us to solve field equations recursively for any initial conditions. Furthermore, there is hierarchical structure that governs these equations, making it possible to solve them subsequently. Solutions to these equations produce fields that can be transcribed as expansions in fermionic coordinates θ^α and $\bar{\theta}^\alpha$. We will be utilize equations of motion for background fields again in Chapter 4, where we will be discussing initial conditions that correspond to coordinate dependent Ramond-Ramond field $P^{\alpha\beta}$.

1.3 T-duality

We have briefly touched upon naming convention for type IIB supersrings in previous chapter, where we explained that II originates from $N = 2$ supersymmetry and B originates from the chirality that we imposed on spinors. Type IIA is theory with $N = 2$ supersymmetry were spinors have opposite chirality and Type IIB is theory with $N = 2$ supersymmetry were spinors have same chirality. We could have chosen to work with some other number of supersymmetry or by having theory which is also invariant to some other symmetry group. By being able to pick and chose with what kind of symmetry we are working, it would be expected that there are infinite many possible string theories. However this statement is not true, there are only five consistent string theories. These theories are: type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$ string theories.

No matter with what superstring theory we are working, they all only make sense if we work in ten dimensional space-time. From our everyday experience it is obvious that our reality has three spatial and one time dimension, this disparity between reality and theory forces us to find a way to deal with six extra spatial dimensions. One way to make string theory comply with observations is by curling up excess dimensions, where they are now circles with radius R. This process is known as compactification [4, 5, 6, 7, 8].

By compactifying dimensions there is emergence of new kind of symmetry. symmetry that connects theories where radii of compactification is R with ones where it is $1/R$. However this is not all, this symmetry also connects different types of superstrings. We call this symmetry T-duality [36, 37, 38, 39, 40, 41]. In addition to T-duality, string theory is also invariant under S-duality which connects theories where constant of interaction is α with ones where it is $1/\alpha$ [42]. By working in tandem these two dualities connect every possible superstring theory. This has given rise to speculation that there exists one theory, M theory, which works in eleven dimensional space-time from which all other types of superstrings stem. At the time of writing of this thesis, there has not been much progress in obtaining this elusive theory.

Our interest in T-duality comes from the wish to examine non-commutative properties of closed strings. Since this duality connects different types of string theory, our idea is to have one theory with standard Poisson brackets which we dualize and see what are the Poisson brackets of dual theory. In order to be successful in this task it would be helpful to devise procedure which would streamline this whole process. Fortunately, such procedure already exists and it is called Buscher procedure [36, 43, 44, 45]. While understanding of procedure is best accomplished by working on concrete examples, we would like to briefly present main steps this procedure entails.

1.3.1 Busher procedure

Idea that lies in the hearth of Buscher procedure is localization of some isometry direction, usually shift symmetry. Therefore, first step in procedure entails testing if the action is invariant under global translations

$$S = \int_\Sigma d^2\xi \mathcal{L}(x, \partial x), \quad \delta x^\mu = \lambda^\mu \quad \to \quad \delta S = 0. \tag{1.3.92}$$

Localization of symmetry is accomplished by substituting partial derivatives ∂x^μ with covariant ones Dx^μ. In case where we have coordinates that do not appear under partial derivatives, typically when we have coordinate dependent background fields, we also need to introduce invariant coordinate x^{inv} [46, 44, 47, 48]. Inclusion of invariant coordinate separates standard form generalized Buscher procedure [46, 44, 45, 49, 50].

$$\partial_m x^\mu \to D_m x^\mu = \partial_m x^\mu + v_m^\mu, \quad x^{inv} = \int_P d^m \xi D_m x^\mu, \quad S = \int_\Sigma d^2\xi \mathcal{L}(x^{inv}, Dx). \tag{1.3.93}$$

Introduction of covariant derivatives has inevitably introduced additional gauge fields v^μ into the theory. Since we demand that starting theory and T-dual one have same number of degrees of freedom we need to eliminate excess degrees of freedom. This is accomplished by introducing following term with Lagrange multiplier

$$S_{add} = \int_\Sigma d^2\xi y_\mu \epsilon^{mn} \partial_m v_n^\mu, \tag{1.3.94}$$

where y_μ is Lagrange multiplier and ϵ^{mn} is Levi Civita tensor.

Next step in procedure is utilization of gauge freedom to fix translation symmetry, this way action becomes only function of gauge fields and Lagrange multipliers

$$S + S_{add} = \int_\Sigma d^2 \mathcal{L}(x, \partial x, y, v\partial v), \quad x(\xi) = constant \quad \to \quad S + S_{add} = \int_\Sigma d^2 \mathcal{L}(y, v, \partial v) \tag{1.3.95}$$

Finally, last step focuses on finding equations of motion for gauge fields and Lagrange multipliers. First set of equations of motion, when inserted into gauge fixed action produces T-dual action, which is now only function of Lagrange multipliers and their derivatives. Inserting equations of motion for Lagrange multipliers into gauge fixed action eliminates all changes we made in previous steps and brings us back to the start. By combining these two sets of equations of motion we can obtain T-dual transformation laws, laws that connect starting coordinates with their T-dual counterparts.

It should be noted that this procedure can be applicable even to cases when we do not have translational symmetry [45]. This is accomplished by substituting starting action with one that has translation symmetry, form of this new action is the same as the form of action where we introduced covariant derivatives, invariant coordinates, Lagrange multipliers and fixed gauge. Legality of this step is assured only if we are able to salvage original action by inserting solutions to equations of motion for Lagrange multipliers into its substitute. It is also important to say that, in cases where we deal with invariant coordinate, we are essentially switching from local theory to non-local one. Recently non-locality has been become very important issue in the quantum mechanical considerations [53].

Having gained some insight into how Buscher procedure works, in the next chapters we will focus on applying this procedure to different types of string theory.

2. T-duality of closed bosonic string with H-flux

This chapter is based on work done in paper [54]

It has been know for some time now that non-commutativity can emerge in context of string theory [52, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65], however this emergence was only in the context of open string theory with constant background field. Geometric properties of open strings, their "openness", gave rise to boundary conditions that must be imposed as canonical constraints. In order for these constraints to be consistent we are led to equations that describe boundary conditions as functions of σ world-sheet coordinate. By solving these equations we find interesting conclusion. that is we can express initial coordinates of the theory as linear combinations of effective coordinates and effective momenta. Imposing standard Poisson brackets between effective coordinates and effective momenta, we find that coordinates of initial theory do not commute. This kind of non-commutativity is known as "canonical non-commutativity" and it is not only exclusive to the string theory. In fact canonical non-commutativity can trace it's origins to Yang-Mills theories [66, 67, 68, 69].

Because of their open nature, open strings have one additional peculiar property and that is the existence of gauge fields at their endpoints [70]. This fact combined with property of non-commutativity creates a natural bridge between string theory and non-commutative Yang-Mills theories. Where examining properties of non-commutative quantum field theory (renormalization [71]) or even obtaining experimental proof of particle decays that are unique to these theories [72, 73] would allow us to give more credence to the idea of one large encompassing theory, string theory.

On the other hand, closed strings do not posses the benefits of their open counterparts. Their lack of borders forces us to be more creative in order to extract any kind of non-commutative behaviour. Due to their geometric restriction, one interesting idea emerges why not examine closed strings in coordinate dependent background fields and instead of finding dependence of initial coordinates on effective coordinates and momenta we find dependence of T-dual coordinates on starting coordinates and momenta? While this idea is not new [10, 14, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83] and in fact case that is disscused in this chapter has already been examined in [14], in order to obtain non-commutative T-dual theory authors had to utilize combination of standard Buscher procedure and nontrivial winding conditions. Our goal is to accomplish the same thing by focusing only on T-duality. T-duality from one theory to another establishes link between coordinates of theories, this link with combination of canonical momenta of original theory makes it possible to write T-dual coordinate as combination of initial coordinates and momenta. By finding Poisson brackets between T-dual coordinates and by enforcing standard Poisson bracket structure on original theory fascinating result emerges, T-dual coordinates of closed bosonic string do not commute.

It should be noted that although these two types of non-commutativity are dominant in string theory, there is one additional type of non-commutativity present in physics. This third type of non-commutativity is based on Lie algebras, where commutator between two coordinates is proportional to coordinate. This kind of non-commutativity is encapsulated in κ-Minkowski space-time [84, 85, 86, 87, 88, 89]. While κ-Minkowski space-time is non-commutative and associative, this behaviour is more of the outlier than norm. In general if we have that commutator of coordinates is proportional to linear combination of coordinates, then we expect space to be non-associative because jacobiator and associator would be nonzero. These kind of spaces are closely linked with L_∞ algebras [90].

In this chapter we focus on obtaining T-dual theory, T-dual transofrmation laws and T-dual non-commutativity of simplest possible theory with coordinate dependent background fields, three torus with H flux. To be more precise, we will deal with bosonic string theory where space-time metric is constant and Kalb-Ramond field has only one non-zero component $B_{xy} = Hz$ which depends linearly on z coordinate with infinitesimal proportionality constant H. Like we said, this case has been examined before however we will depart from conventional method by utilizing Buscher procedure throughout all chapter. Since standard Buscher procedure applies only to isometry directions on which background fields do not depend, we will utilize combination of standard and generalized procedure in order to obtain T-dual theory.

While this theory is quite simple, it's usefulness comes from just that. It is a good testing ground for some basic ideas before embarking on more complex case.

2.1 Bosonic String Action and Choice of Background Fields

We begin with action for closed bosonic string in the presence of the space-time metric $G_{\mu\nu}(x)$, Kalb-Ramond antisymmetric field $B_{\mu\nu}(x)$, and dilaton scalar field $\Phi(x)$ where action is given as

$$S = k\int_\Sigma d^2\xi\sqrt{-g}\left\{\left[\frac{1}{2}g^{mn}G_{\mu\nu}(x) + \frac{\epsilon^{mn}}{\sqrt{-g}}B_{\mu\nu}(x)\right]\partial_m x^\mu\partial_n x^\nu + \Phi(x)R^{(2)}\right\}, \tag{2.1.1}$$

Where notation is indetical to one in Chapter 1.1, that is, Σ is the world-sheet surface parameterized by $\xi^m = (\tau, \sigma)$ $[(m = 0, 1), \sigma \in (0, \pi)]$, while the D-dimensional space-time os spanned by the coordinates x^μ $(\mu = 0, 1, 2, ..., D-1)$. Intrinsic world-sheet metric is labeled with g_{mn} and its corresponding scalar curvature is given as $R^{(2)}$.

We have seen in previous chapter that background fields for bosonic string must obey following equations of motion (here presented again for sake of clarity) [16]

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B^{\rho\sigma}_{\nu} + 2 D_\mu a_\nu = 0, \tag{2.1.2}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho_{\mu\nu} - 2 a_\rho B^\rho_{\mu\nu} = 0, \tag{2.1.3}$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - R - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = c, \tag{2.1.4}$$

Where we had that c is an arbitrary constant and function β^Φ could also be set to a constant because of the relation

$$D^\nu \beta^G_{\nu\mu} + \partial_\mu \beta^\Phi = 0. \tag{2.1.5}$$

Further, we also had that $R_{\mu\nu}$ and D_μ are Ricci tensors and covariant derivative with respect to the space-time metric $G_{\mu\nu}$, while

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \quad a_\mu = \partial_\mu \Phi, \tag{2.1.6}$$

were field strength for Kalb-Ramond field $B_{\mu\nu}$ and dilaton gradient, respectively. Simplest solution to these equations is when all three background fields are set to constants, however this case does not provide any new insight into closed string non-commutativity. Case that we will be examining in this chapter is one where every background field, except Kalb-Ramond field, is set to constant. For Kalb-Ramond field we want to have linear coordinate dependence. In order to see if our wishes are consistent with reality, we should examine equations of motion for background fields more closely. By setting dilaton field to constant and demanding linear coordinate dependence for Kalb-Ramond tensor, first equation (2.1.2) reduces to

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B^{\rho\sigma}_{\nu} = 0, \tag{2.1.7}$$

where field strength $B_{\mu\nu\rho}$ is constant. If we additionally assume that this is infinitesimal constant H, then we are free to set space-time metric $G_{\mu\nu}$ also to a constant in approximation linear in $B_{\mu\nu\rho}$. These assumptions satisfy all three space-time field equations, where the third equation (2.1.4) takes following form

$$2\pi\kappa \frac{D-26}{6} = c. \tag{2.1.8}$$

This expression allows us to chose the number of dimensions we want to work in. In order to explore main properties of closed bosonic string in presence of coordinate dependent background fields and in order to not make calculations needlessly complicated we will work in $D = 3$ dimensions with the following choice of background fields

$$G_{\mu\nu} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.1.9}$$

here R_μ $(\mu = 1, 2, 3)$ are radii of the compact dimensions. This configuration of background fields, while simple it is not arbitrary, it represents geometry of torus with flux (field strength) H [51]. Our choice of infinitesimal flux H can be understood in terms of the radii as

$$\left(\frac{H}{R_1 R_2 R_3} \right)^2 = 0. \tag{2.1.10}$$

Since H is infinitesimal this means that flux is "diluted" and that our torus is large. This gives us the freedom to rescale the coordinates

$$x^\mu \mapsto \frac{x^\mu}{R_\mu}, \tag{2.1.11}$$

which simplifies the form of the metric even more

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.1.12}$$

With this configuration of background fields, action for closed string takes following form

$$\begin{aligned} S &= k \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \\ &= k \int_\Sigma d^2\xi \left[\frac{1}{2} \left(\partial_+ x \partial_- x + \partial_+ y \partial_- y + \partial_+ z \partial_- z \right) + \partial_+ x H z \partial_- y - \partial_+ y H z \partial_- x \right]. \end{aligned} \tag{2.1.13}$$

Where we have transcribed action into light-cone coordinates (more detail in Appendix A). Metric tensor $G_{\mu\nu}$ and Kalb-Ramond antisymmetric tensor $B_{\mu\nu}$ are combined into new tensor $\Pi_{+\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$. We also took liberty to relable coordinates x^μ as

$$x^\mu = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2.1.14}$$

T-dualization of dilaton field is performed separately within quantum formalism, since that is not the focus of this thesis from now on we will omit the term containing this field.

2.2 T-dualization of Bosonic Closed String Action

Having established our starting point in previous section, content of this section will be based on obtaining T-dual action and T-dual transformation laws. T-dualization will be performed one direction at the time by utilizing standard and ,when situation demands, generalized Buscher procedures. Results that are obtained here will be used in subsequent section as a way to obtain non-commutative properties of T-dual theory.

2.2.1 T-dualization along x Direction - from Torus with H Flux to Twisted Torus

We start our T-dualization journey by dualizing action (2.1.13) along x direction. Since this x direction is an isometry direction this means that action possesses global shift symmetry $x \to x + a$ and since background fields do not depend on this coordinate we can accomplish our goal by utilizing standard Buscher procedure [36]. We will go through all the steps that were highlighted in previous chapter. As it has been already explained, starting point of T-dualization procedure is based on localization of global shift symmetry, this is accomplished by introducing covariant derivatives that will replace partial derivatives

$$\partial_\pm \to D_\pm = \partial_\pm x + v_\pm, \tag{2.2.15}$$

where $v_\pm$ are gauge fields that transform as

$$\delta v_\pm = -\partial_\pm a, \tag{2.2.16}$$

under local translations. By adding term with Lagrange multiplier γ_1

$$S_{add} = \frac{k}{2}\int_\Sigma d^2\xi \gamma_1(\partial_+ v_- - \partial_- v_+), \tag{2.2.17}$$

to the action we are making newly added gauge fields unphysical degrees of freedom. After gauge fixing, $x = const$, the action takes the following form

$$\begin{aligned} S_{fix} =& k\int_\Sigma d^2\xi \Big[\frac{1}{2}\left(v_+ v_- + \partial_+ y \partial_- y + \partial_+ z \partial_- z\right) + v_+ H z \partial_- y - \partial_+ y H z v_- \\ &+ \frac{1}{2}\gamma_1\left(\partial_+ v_- - \partial_- v_+\right)\Big]. \end{aligned} \tag{2.2.18}$$

We can find equations of motion for Lagrange multiplier γ_1 which tell us that field strength for the gauge field $v_\pm$ vanishes

$$F_{+-} = \partial_+ v_- - \partial_- v_+ = 0. \tag{2.2.19}$$

Solving this equation we obtain following solution for gauge filed

$$v_\pm = \partial_\pm x. \tag{2.2.20}$$

If we wish to return to original action (2.1.13), we only need to plug these solution into gauge fixed action (2.2.18). Varying the gauge fixed action (2.2.18) with respect to the gauge fields v_+ and v_- we obtain following two equations

$$v_- = -\partial_- \gamma_1 - 2Hz\partial_- y, \tag{2.2.21}$$

$$v_+ = \partial_+ \gamma_1 + 2Hz\partial_+ y. \tag{2.2.22}$$

Utilizing these two expressions in a manner that has been described before and by neglecting all terms that are not linear in H we are left with following T-dual action

$${}_xS = k\int_\Sigma d^2\xi \partial_+ ({}_xX)^\mu {}_x\Pi_{+\mu\nu} \partial_- ({}_xX)^\nu, \tag{2.2.23}$$

where subscript $_x$ denotes quantity obtained after T-dualization along x direction and we have grouped coordinate into

$${}_xX^\mu = \begin{pmatrix} \gamma_1 \\ y \\ z \end{pmatrix}. \tag{2.2.24}$$

After first T-duality theory has new altered background fields

$$\begin{aligned} {}_x\Pi_{+\mu\nu} &= {}_xB_{\mu\nu} + \frac{1}{2}{}_xG_{\mu\nu}, \\ {}_xB_{\mu\nu} = 0, \quad {}_xG_{\mu\nu} &= \begin{pmatrix} 1 & 2Hz & 0 \\ 2Hz & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.2.25}$$

These background fields define what is known in literature as a "twisted torus" geometry. String theory after one T-dualization is geometrically well defined both globally and locally. Theory is geometrical where flux H plays the role of connection. Combining the solution of equation of motion for Lagrange multiplier (2.2.20) with equations of motion for gauge fields (2.2.21) and (2.2.22) we get the transformation laws connecting initial coordinates x^μ with T-dual coordinates ${}_xX^\mu$

$$\partial_\pm \cong \pm\partial_\pm\gamma_1 \pm 2Hz\partial_\pm y, \tag{2.2.26}$$

where $\cong$ denotes T-dual relation. The momentum π_x is canonically conjugated to the initial coordinate x. Using the initial action (2.1.13) we get

$$\pi_x = \frac{\delta S}{\delta \dot{x}} = k(\dot{x} - 2Hzy'), \tag{2.2.27}$$

where, as before, we have denoted $\dot{A} \equiv \partial_\tau A$ and $A' \equiv \partial_\sigma A$. Combining transformation laws for light-cone derivatives of x coordinate, we are able to obtain

$$\dot{x} \cong \gamma_1' + 2Hzy', \tag{2.2.28}$$

which when utilized with the expression for momentum π_x, gives us transformation law in canonical form

$$\pi_x \cong k\gamma_1'. \tag{2.2.29}$$

2.2.2 From Twisted Torus to Non-geometrical Q Flux

We continue T-dualization by performing T-dualization of action (2.2.23) along y direction. We repeat same procedure from previous subsection and form the gauge fixed action

$$\begin{aligned}{}_xS_{fix} =&k \int_\Sigma d^2\xi \Big[\frac{1}{2}\left(\partial_+\gamma_1\partial_-\gamma_1 + v_+v_- + \partial_+z\partial_-z\right) + \partial_+\gamma_1 Hzv_- + v_+Hz\partial_-\gamma_1 \\ &+ \frac{1}{2}\gamma_2\left(\partial_+v_- - \partial_-v_=\right)\Big].\end{aligned} \tag{2.2.30}$$

Equations of motion for Lagrange multiplier γ_2 produce

$$\partial_+v_- - \partial_-v_+ = 0 \to v_\pm = \partial_\pm y. \tag{2.2.31}$$

Inserting these solutions to equations of motion into gauge fixed action becomes it returns to its starting form (2.2.23). By varying the gauge fixed action with respect to the gauge fields we obtain

$$v_\pm = \pm\partial_\pm\gamma_2 - 2Hz\partial_\pm\gamma_1. \tag{2.2.32}$$

Inserting these equations equations into gauge fixed action and by keeping only terms linear in H, we obtain T-dual action

$${}_{xy}S = k\int_\Sigma d^2\xi\partial_+({}_{xy}X)^\mu{}_{xy}\Pi_{+\mu\nu}\partial_-({}_{xy}X)^\nu, \tag{2.2.33}$$

where

$$({}_{xy}X)^\mu = \begin{pmatrix}\gamma_1\\ \gamma_2\\ z\end{pmatrix}, \tag{2.2.34}$$

$$ {}_{xy}\Pi_{+\mu\nu} = {}_{xy}B_{\mu\nu} + \frac{1}{2}{}_{xy}G_{\mu\nu}, \quad {}_{xy}\Pi_{+\mu\nu} = \begin{pmatrix} \frac{1}{2} & -Hz & 0 \\ Hz & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{2.2.35} $$

Separating tensor ${}_{xy}\Pi_{+\mu\nu}$ into its constituent fields we obtain following expressions for background fields

$$ {}_{xy}B_{\mu\nu} = \begin{pmatrix} 0 & -Hz & 0 \\ Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -B_{\mu\nu}, \quad {}_{xy}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2.36} $$

Background fields that we obtained after two successive T-dualizations resemble background fields of the starting theory, torus with H flux, but they should only be considered locally. Their global properties are nontrivial and because of this the term "non-geometry" was introduced. This configuration of fields is known as torus with Q flux.

Combining the equations of motion for Lagrange multipliers y_2 and equations of motion for gauge fields $v_\pm$ we can deduce T-dual transformation laws

$$ \partial_\pm y \cong \pm\partial_\pm \gamma_2 - 2Hz\partial_\pm \gamma_1. \tag{2.2.37} $$

In order to obtain canonical form of transformation laws we again need to find canonical momentum. this time of y coordinate

$$ \pi^y = \frac{\delta S}{\delta \dot{y}} = k(\dot{y} + 2Hzx'). \tag{2.2.38} $$

Adding the transformation laws (2.2.37) for $\partial_+ y$ and $\partial_- y$ together and utilizing properties of light-cone derivatives, we obtain

$$ \dot{y} \cong \gamma_2' - 2Hz\dot{\gamma}_1, \tag{2.2.39} $$

Combining this expression with one for momentum π_y we obtain transformation law in canonical form

$$ \pi_y \cong k\gamma_2' \tag{2.2.40} $$

Having obtained two separate transformation laws in canonical form (2.2.29), (2.2.40) we can notice that T-dual coordinates γ_1 and γ_2 are still commutative. This is a consequence of a simple fact that variables of initial theory, which is geometrical one, satisfy standard Poisson algebra

$$ \{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu_\nu \delta(\sigma - \bar{\sigma}), \quad \{x^\mu, x^\nu\} = \{\pi_\mu, \pi_\nu\} = 0, \tag{2.2.41} $$

where

$$ \pi_\mu = \begin{pmatrix} \pi_x \\ \pi_y \\ \pi_z \end{pmatrix}. \tag{2.2.42} $$

2.2.3 From Q to R Flux - T-dualization Along z Coordinate

Having dualized starting action along both x and y direction, we are left only with T-dualization along z direction. Since background Kalb-Ramond field depends on this coordinate we will utilize generalized T-dualization procedure [46, 44, 45, 49, 50].

Starting point is the action we obtained after performing T-dualization along x and y coordinates (2.2.33). Since Kalb-Ramond field depends on z it seems that we are lacking isometry along z. However this is not the case, action is indeed invariant under global shift transformations of z coordinate. To see this let us assume for a moment that Kalb-Ramond field linearly depends on all coordinates $B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho$ and check if some global transformation can be treated as isometry one. We start with global shift transformation

$$\delta x^\mu = \lambda^\mu, \tag{2.2.43}$$

and make a variation of action

$$\delta S = \frac{k}{3}B_{\mu\nu\rho}\lambda^\mu \int_\Sigma d^2\xi \partial_+ x^\mu \partial_- x^\nu = \frac{2k}{3}B_{\mu\nu\rho}\lambda^\mu \epsilon^{mn} \int_\Sigma d^2\xi \left[\partial_m(x^\mu \partial_n x^\nu) - x^\mu(\partial_m\partial_n x^\nu)\right]. \tag{2.2.44}$$

The second term vanishes explicitly due to contraction of antisymmetric tensor ϵ^{mn} and $(\partial_m\partial_n)$ tensor, while the first term is surface one and in general this term is different from zero. However, expression for δS is topological invariant and if we have topologically trivial map from world-sheet onto D-dimensional space-time this expression is set to zero. This means that properties of field strength H do not play a role in ensuring our action has invariance under shift symmetry.

There is one additional, although more technical, explanation for vanishment of surface term which is more appropriate to the approximations used in this chapter. Because we chose to work in linear approximation of H terms, we have that our coordinates x^μ satisfy $\partial_+\partial_- x^\mu = 0$ equations of motion for constant $G_{\mu\nu}$ and $B_{\mu\nu}$ whose solutions are well documented in standard string theory textbooks. From these solutions we have that, if we hold initial τ_i and final τ_f fixed and if we work with trivial winding conditions (winding number is set to zero), coordinates x^μ satisfy $x^\mu(\sigma + 2\pi) = x^\mu(\sigma)$ which ensures disappearance of surface term. Consequently in the case of constant metric and linearly dependent Kalb-Ramon field, global shift transformation is an isometry transformation. This means that we can make T-dualization along z coordinate using generalized T-dualization procedure.

As has been described in previous chapter, difference between generalized [46] and standard Buscher procedure is only in one additional step seen in the introduction of invariant coordinate. We again start by localizing shift symmetry of the action (2.2.33) and by introducing covariant derivative

$$\partial_\pm z \to D_\pm z = \partial_\pm z + v_\pm. \tag{2.2.45}$$

Now into play comes introduction of invariant coordinate as line integral

$$z^{inv} = \int_P d\xi^m D_m z = \int_P d\xi^+ D_+ z + \int_P d\xi^- D_- z = z(\xi) - z(\xi_0) + \Delta V, \tag{2.2.46}$$

where

$$\Delta V = \int_P d\xi^m v_m = \int_P (d\xi^+ v_+ + d\xi^- v_-). \tag{2.2.47}$$

Here ξ and ξ_0 are the current and initial point of the world-sheet line P. At the end, as in the standard Buscher Procedure, in order to make $v_\pm$ unphysical degrees of freedom we add to the action term with Lagrange multiplier

$$S_{add} = \frac{k}{2}\int_\Sigma d^2\xi \gamma_3(\partial_+ v_- - \partial_- v_+). \tag{2.2.48}$$

With these additions final form of action is

$$
\begin{aligned}
{}_{xy}\bar{S} =& k\int_\Sigma d^2\xi\Big[\frac{1}{2}\left(\partial_+\gamma_1\partial_-\gamma_1+\partial_+\gamma_2\partial_-\gamma_2+D_+zD_-z\right)-Hz^{inv}(\partial_+\gamma_1\partial_-\gamma_2-\partial_+\gamma_2\partial_-\gamma_1)\\
&+\frac{1}{2}\gamma_3(\partial_+v_--\partial_-v_+)\Big]. \qquad (2.2.49)
\end{aligned}
$$

Because of the existing shift symmetry we fix the gauge, $z(\xi)=z(\xi_0)$, and then the gauge fixed action takes the form

$$
\begin{aligned}
{}_{xy}S_{fix} =& k\int_\Sigma d^2\xi\Big[\frac{1}{2}\left(\partial_+\gamma_1\partial_-\gamma_1+\partial_+\gamma_2\partial_-\gamma_2+v_+v_-\right)-H\Delta V(\partial_+\gamma_1\partial_-\gamma_2-\partial_+\gamma_2\partial_-\gamma_1)\\
&+\frac{1}{2}\gamma_3(\partial_+v_--\partial_-v_+)\Big]. \qquad (2.2.50)
\end{aligned}
$$

Equation of motion for Lagrange multiplier γ_3 gives us

$$
\partial_+v_--\partial_-v_+=0\to v_\pm=\partial_\pm z,\quad \Delta V=\Delta z, \qquad (2.2.51)
$$

while equations of motion for gauge fields $v_\pm$ are

$$
v_\pm=\pm\partial_\pm\gamma_3-2\beta^\mp, \qquad (2.2.52)
$$

functions $\beta^\pm$ are defined as

$$
\beta^\pm=\pm\frac{1}{2}H(\gamma_1\partial_\mp\gamma_2-\gamma_2\partial_\mp\gamma_1). \qquad (2.2.53)
$$

These functions are obtained as a result of the variation of the term containing ΔV (more detail in Appendix C)

$$
\delta_v\left(-2k\int_\Sigma d^2\xi\epsilon^{mn}H\partial_m\gamma_1\partial_n\gamma_2\Delta V\right)=k\int_\Sigma d^2\xi(\beta^+\delta v_++\beta^-\delta v_-), \qquad (2.2.54)
$$

using partial integration and the fact that $\partial_\pm V=v_\pm$. Inserting these relations into gauge fixed action and again keeping only terms that are linear in H, we obtain T-dual action

$$
{}_{xyz}S=k\int_\Sigma d^2\xi\partial_+({}_{xyz}X)^\mu\,{}_{xyz}\Pi_{+\mu\nu}\partial_-({}_{xyz}X)^\nu \qquad (2.2.55)
$$

where

$$
{}_{xyz}x^\mu=\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix},\quad {}_{xyz}\Pi_{+\mu\nu}={}_{xyz}B_{\mu\nu}+\frac{1}{2}{}_{xyz}G_{\mu\nu}, \qquad (2.2.56)
$$

$$
{}_{xyz}B_{\mu\nu}=\begin{pmatrix}0 & -H\Delta\bar{\gamma}_3 & 0\\ H\Delta\bar{\gamma}_3 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad {}_{xyz}G_{\mu\nu}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}. \qquad (2.2.57)
$$

Here we introduced double coordinate $\bar{\gamma}_3$, defined as

$$
\partial_\pm\gamma_3\equiv\pm\partial_\pm\bar{\gamma}_3. \qquad (2.2.58)
$$

One thing to note is that ΔV never stands alone it is always accompanied by field strength H, which implies that, according to the diluted flux approximation, we calculate ΔV in zeroth order in H

$$\Delta V = \int_P d\xi^+ \partial_+ \gamma_3 - \int_P d\xi^- \partial_- \gamma_3. \tag{2.2.59}$$

This expression makes it clear why we defined double coordinate $\bar{\gamma}_3$ as in Eq. (2.2.58). Presence of ΔV, which is defined as line integral, represents the source of non-locality of the T-dual theory. The resoult of three T-dualizations is a theory with R flux. Combining equations of motion for Lagrange multiplier (2.2.51) with equations of motion for gauge fields (2.2.52), we obtain the T-dual transformation law for z coordinate

$$\partial_\pm \cong \pm \partial_\pm \gamma_3 - 2\beta^\mp. \tag{2.2.60}$$

Combining $\partial_+ z$ and $\partial_- z$ we get transformation law for $\dot{z}$

$$\dot{z} \cong \gamma_3' + H(\gamma_1 \gamma_2' - \gamma_2 \gamma_1') \tag{2.2.61}$$

which enables us to write down the transformation law in the canonical form

$$\gamma_3' \cong \frac{1}{k}\pi_z - H(xy' - yx'). \tag{2.2.62}$$

Here we used the expression for the canonical momentum of the initial theory (2.1.13)

$$\pi_z = \frac{\delta S}{\delta \dot{z}} = k\dot{z}. \tag{2.2.63}$$

2.3 Noncommutativity and Nonassociativity Using T-duality

Having obtained transformation laws in canonical form for all coordinates

$$\gamma_1' \cong \frac{1}{k}\pi_x, \quad \gamma_2 \cong \frac{1}{k}\pi_y \quad \gamma_3' \cong \frac{1}{k}\pi_z - H(xy' - yx'). \tag{2.3.64}$$

we can start analyzing how Poisson brackets of T-dual theory differ from starting one. In order to find the Poisson brackets between T-dual coordinates γ_μ, we will make heavy use the algebra of the coordinates and momenta of the initial theory

$$\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu_\nu \delta(\sigma - \bar{\sigma}), \quad \{x^\mu, x^\nu\} = \{\pi_\mu, \pi_\nu\} = 0, \tag{2.3.65}$$

as well as results obtained in Appendix. By examining structure of transformation laws it is obvious that only nontrivial Poisson brackets will be $\{\gamma_1(\sigma), \gamma_3(\bar{\sigma})\}$ and $\{\gamma_2(\sigma), \gamma_3(\bar{\sigma})\}$.

2.3.1 Noncommutativity Relations

We start by examining Poisson brackets between σ derivatives of T-dual coordinates γ_μ using (2.3.64), where only nontrivial ones are

$$\{\gamma_1'(\sigma), \gamma_3'(\bar{\sigma})\} \cong \frac{2}{k} H y'(\sigma)\delta(\sigma - \bar{\sigma}) + \frac{1}{k} H y(\sigma)\delta'(\sigma - \bar{\sigma}), \tag{2.3.66}$$

$$\{\gamma_2'(\sigma), \gamma_3'(\bar{\sigma})\} \cong -\frac{2}{k} H x'(\sigma)\delta(\sigma - \bar{\sigma}) - \frac{1}{k} H x(\sigma)\delta'(\sigma - \bar{\sigma}). \tag{2.3.67}$$

We see that these Poisson brackets are of the form (B.0.1), so we can apply the result (B.0.9). Consequently, we get

$$\{\gamma_1(\sigma), \gamma_3(\bar{\sigma})\} \cong -\frac{H}{k}\left[2y(\sigma) - y(\bar{\sigma})\right]\bar{H}(\sigma - \bar{\sigma}), \tag{2.3.68}$$

$$\{\gamma_2(\sigma), \gamma_3(\bar{\sigma})\} \cong \frac{H}{k}\left[2x(\sigma) - x(\bar{\sigma})\right]\bar{H}(\sigma - \bar{\sigma}), \tag{2.3.69}$$

These two Poisson brackets are zero when $\sigma = \bar{\sigma}$ and/or field strength H is equal to zero. But if we take that $\sigma - \bar{\sigma} = 2\pi$ then we have $\bar{H}(2\pi) = 1$ and it follows

$$\{\gamma_1(\sigma + 2\pi), \gamma_3(\sigma)\} \cong -\frac{H}{k}\left[3\pi N_y + y(\sigma)\right], \tag{2.3.70}$$

$$\{\gamma_2(\sigma + 2\pi), \gamma_3(\sigma)\} \cong \frac{H}{k}\left[4\pi N_x + x(\sigma)\right]. \tag{2.3.71}$$

where N_x and N_y are winding numbers defined as

$$x(\sigma + 2\pi) - x(\sigma) = 2\pi N_x, \quad y(\sigma + 2\pi) - y(\sigma) = 2\pi N_y. \tag{2.3.72}$$

From these relations we can see that if we chose σ for which $x(\sigma) = 0$ and $y(\sigma) = 0$ then noncommutativity relation are proportional to winding numbers. On the other side, even in cases where winding numbers are equal to zero there is still noncommutativity between T-dual coordinates.

2.3.2 Nonassociativity

In order to calculate Jacobi identity of the T-dual coordinates we first have to find Poisson brackets $\{\gamma_1(\sigma), x(\bar{\sigma})\}$ which is presented in Appendix B, here we give only result

$$\{\gamma_2(\sigma), y(\bar{\sigma})\} \cong -\frac{1}{k}\bar{H}(\sigma - \bar{\sigma}). \tag{2.3.73}$$

The relation for $\{\gamma_2(\sigma), y(\bar{\sigma})\}$ is similar because the transformation law for y-direction is of the same form as for x-direction, the Poisson bracket is of the same form Calculating Jacobi identiry by using noncommutativity relations (2.3.68) and (2.3.69) and previous equation we have

$$\begin{aligned}
&\{\gamma_1(\sigma_1), \gamma_2(\sigma_2), \gamma_3(\sigma_3)\} \equiv \\
&\{\gamma_1(\sigma_1), \{\gamma_2(\sigma_2), \gamma_3(\sigma_3)\}\} + \{\gamma_2(\sigma_2), \{\gamma_3(\sigma_3), \gamma_1(\sigma_1)\}\} + \{\gamma_3(\sigma_3), \{\gamma_1(\sigma_1), \gamma_2(\sigma_2)\}\} \\
&\cong -\frac{2H}{k^2}\Big[\bar{H}(\sigma_1 - \sigma_2)\bar{H}(\sigma_2 - \sigma_3) + \bar{H}(\sigma_2 - \sigma_1)\bar{H}(\sigma_1 - \sigma_3) + \bar{H}(\sigma_1 - \sigma_3)\bar{H}(\sigma_3 - \sigma_2)\Big].
\end{aligned}$$

Jacobi identity is nonzero which means that theory with R-flux is nonassociative. For $\sigma_2 = \sigma_3 = \sigma$ and $\sigma_1 = \sigma + 2\pi$ we get

$$\{\gamma_1(\sigma + 2\pi), \gamma_2(\sigma), \gamma_3(\sigma)\} \cong \frac{2H}{k^2} \tag{2.3.74}$$

From the last two equations, general form of Jacobi identiy and Jacobi identity for special choice of σ's, we see that presence of the coordinate dependent Kalb-Ramond field is a source of non-commutativity and nonassociativity.

3. From H-flux to the family of three nonlocal R-flux theories

This chapter is based on work done in paper [91]

We have seen in previous chapter how T-duality affects theory that describes propagation of bosonic string in presence of coordinate dependent Kalb-Ramond field. T-dualization was performed first along x and y coordinates and finally along coordinate on which background fields depend, that is z coordinate. Result of such chain of T-dualizations was that we had emergence of non-locality and non-commutativity only at the end, after dualizing z coordinate. This posses natural question, could we obtained non-commutativity earlier if we had chosen to follow another T-duality chain? The answer to this question is positive. If we had, for example, conducted T-dualization along xzy chain non-locality as well as non-commutativity would arise after second T-dualization. This fact illuminates new found richness of this simple model, where slight adjustemnts to T-duality chain give rise to whole new family of theories. It should be noted however that final T-dual theory is unique and that all alternate T-duality chains coalesce on same final theory which, as we have seen, was non-local and non-commutative.

Focus of this chapter is again on examining closed bosonic string with coordinate dependent Kalb-Ramon field but along alternative T-duality chain, namely zyx chain. This order of dualization will produce non-local theory after first T-duality and subsequent dualizations will give us non-commutative relations. Since background fields depended only on z coordinate, these subsequent dualizations will not affect locality of theory. Just like before, main tools for this endeavour will consist of general and standard Buscher procedure for coordinates z and x $/y$, respectively.

3.1 Preliminary action and background fields

Starting point of this chapter is the same as of the previous one and we will not repeat its contents. We will however, for clarity sake, only list starting action transcribed in light-cone coordinates

$$\begin{aligned} S &= k\int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu}\partial_- x^\nu \\ &= k\int_\Sigma d^2\xi \left[\frac{1}{2}\left(\partial_+ x\partial_- x + \partial_+ y\partial_- y + \partial_+ z\partial_- z\right) + \partial_+ xHz\partial_- y - \partial_+ yHz\partial_- x\right], \end{aligned} \tag{3.1.1}$$

as well as starting background fields

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.1.2}$$

Notation is the same as before, Σ denotes world-sheet surface which is parametrized by τ and σ world-sheet coordinates or, as in the case of light-cone basis, by ξ^+ and ξ^-, while space-time coordinates are denoted with x, y and z. Space-time metric tensor $G_{\mu\nu}$ is constant and antisymmetric Kalb-Ramond tensor $B_{\mu\nu}$ has infinitesimal linear coordinate dependence on z, both of these tensors combine into tensor $\Pi_{+\mu\nu}$ as $\Pi_{+\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}$.

3.2 Family of three R flux non-local theories

In this section we will perform T-dualization along zyx coordinate chain. After every T-dualization we will write down T-dual transformation laws in canonical form and check if the theory has become non-commutative. Since T-duality procedure is the same as in chapter before, although in reverse, we will omit most details and only focus on main results.

3.2.1 T-dualization along z direction - shortcut to R-flux

We have already seen that theories with coordinate dependent Kalb-Ramond field are invariant under global shift symmetry with transformations of type $x^\mu \to x^\mu + \lambda^\mu$, where invariance is guaranteed duo to inherent antisymmetry of tensor as well as by trivial winding conditions. This fact makes it possible to utilize general Buscher procedure in first step of T-dualization.

By substituting partial derivatives with covariant derivatives $D_\pm z$

$$\partial_\pm z \longrightarrow D_\pm z = \partial_\pm z + v_\pm, \tag{3.2.3}$$

introducing invariant coordinate z^{inv}

$$z^{inv} = \int_P d\xi^\alpha D_\alpha z = \int_P d\xi^+ D_+ z + \int_P d\xi^- D_- z = z(\xi) - z(\xi_0) + \Delta V\,, \tag{3.2.4}$$

and adding term with Lagrange multiplier γ_3

$$S_{add} = \frac{\kappa}{2}\int_\Sigma d^2\xi \gamma_3(\partial_+ v_- - \partial_- v_+)\,. \tag{3.2.5}$$

to the action (3.1.1), as well as utilizing gauge freedom to fix $z(\xi) = z(\xi_0)$ we obtain gauge fixed action

$$\begin{aligned} S_{fix} =& \kappa \int_\Sigma d^2\xi \Big[H\Delta V(\partial_+ x\partial_- y - \partial_+ y\partial_- x) + \frac{1}{2}(\partial_+ x\partial_- x + \partial_+ y\partial_- y + v_+ v_-) \\ &+ \frac{1}{2}y_3(\partial_+ v_- - \partial_- v_+)\Big]\,. \end{aligned} \tag{3.2.6}$$

Where ΔV is given as

$$\Delta V = \int_P d\xi^\alpha v_\alpha = \int_P (d\xi^+ v_+ + d\xi^- v_-)\,. \tag{3.2.7}$$

The equation of motion for Lagrange multiplier y_3 obtained from above action (3.2.6) produces

$$\partial_+ v_- - \partial_- v_+ = 0 \to v_\pm = \partial_\pm z \,, \tag{3.2.8}$$

which drives us back to the initial action (3.1.1). On the other side, if we found equations of motion for gauge fields $v_\pm$, we get

$$v_\pm = \pm \partial_\pm y_3 - 2\beta^\mp \,, \tag{3.2.9}$$

where $\beta^\pm$ functions are obtained in exactly the same way as before however, they are now functions of initial coordinates x and y instead of T-dual ones and we do not need to make and substitutions

$$\beta^\pm = \mp \frac{1}{2} H (x \partial_\mp y - y \partial_\mp x) \,. \tag{3.2.10}$$

Plugging equations of motion for gauge fields $v_\pm$ into gauge fixed action and keeping only terms lienar in H, we are let to T-dual action

$${}_zS = \kappa \int_\Sigma d^2\xi \partial_+ ({}_zX)^\mu {}_z\Pi_{+\mu\nu} \partial_- ({}_zX)^\nu \,, \tag{3.2.11}$$

where we now have following coordinates and background fields

$${}_zX^\mu = \begin{pmatrix} x \\ y \\ \gamma_3 \end{pmatrix} , \quad {}_z\Pi_{+\mu\nu} = {}_zB_{\mu\nu} + \frac{1}{2} {}_zG_{\mu\nu} \,, \tag{3.2.12}$$

$${}_zB_{\mu\nu} = \begin{pmatrix} 0 & H\Delta V & 0 \\ -H\Delta V & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad {}_zG_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.2.13}$$

Since ΔV is represented as line integral which is the source of non-locality, we have that this theory is non-local.

By combining solutions to the equation of motion for Lagrange multiplier (3.2.8) with equations of motion for gauge fields (3.2.9) we find T-dual transformation laws

$$\partial_\pm z \cong \pm \partial_\pm y_3 \mp H (x \partial_\pm y - y \partial_\pm x). \tag{3.2.14}$$

Finding expression for $\dot{z}$ and combining it with canonical momentum of z coordinate from original theory

$$\pi_z = \frac{\delta S}{\delta \dot{z}} = \kappa \dot{z}, \tag{3.2.15}$$

we get canonical transformation law for σ derivative of T-dual coordinate γ_3

$$\gamma_3' \cong \frac{1}{\kappa} \pi_z + H (x y' - y x') \,, \tag{3.2.16}$$

which is of the same form as in the xyz case.

Unlike in previous chapter here we will keep the symbol ΔV throughout, instead of introducing double coordinate $\bar{\gamma}_3$.

Since we are dealing with exactly the same starting theory, initial coordinates satisfy standard Poisson algebra

$$\{x^\mu(\sigma), x^\nu(\bar{\sigma})\} = \{\pi_\mu(\sigma), \pi_\nu(\bar{\sigma})\} = 0 \,, \quad \{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu{}_\nu \delta(\sigma - \bar{\sigma}), \tag{3.2.17}$$

which when utilizes with expression for canonical transformation law (3.2.16) we are led to the conclusion that theory we have just obtained is commutative. As a consequence of this, it also follows that this theory is also associative. While this theory is also theory with R-flux as one we obtained at the end of xyz dualization chain, their commutative properties are different.

3.2.2 T-dualization along y direction

Starting from the action that was dualized along z direction (3.2.11), we continue along our T-dualization journey by focusing on y coordinate. In this case background fields are independent of coordinate in question therefore we will apply standard Buscher procedure. In order to not repeat unnecessary steps we immediately introduce gauge fixed action The gauge fixed action is of the form

$$
{}_zS_{fix} = \kappa \int_\Sigma d^2\xi \Big[\frac{1}{2} \left(\partial_+ x \partial_- x + v_+ v_- + \partial_+ \gamma_3 \partial_- \gamma_3 \right) + H\Delta V (v_- \partial_+ x - v_+ \partial_- x) \tag{3.2.18}
$$

$$
+ \frac{1}{2} \gamma_2 (\partial_+ v_- - \partial_- v_+) \Big] . \tag{3.2.19}
$$

Where solutions to equations for motion of Lagrange multiplier γ_2 are

$$
v_\pm = \partial_\pm y, \tag{3.2.20}
$$

while the equations of motion for gauge fields are

$$
v_\pm = \pm \partial_\pm \gamma_2 \mp 2H\Delta V \partial_\pm x \, . \tag{3.2.21}
$$

Inserting the expression for gauge fields (3.2.21) into gauge fixed action (3.2.18), we obtain the T-dual action

$$
{}_{zy}S = \kappa \int_\Sigma d^2\xi \, \partial_+ \left({}_{zy}X \right)^\mu \, {}_{zy}\Pi_{+\mu\nu} \, \partial_- \left({}_{zy}X \right)^\nu , \tag{3.2.22}
$$

where

$$
{}_{zy}X^\mu = \begin{pmatrix} x \\ \gamma_2 \\ \gamma_3 \end{pmatrix} , \quad {}_{zy}\Pi_{+\mu\nu} = {}_{zy}B_{\mu\nu} + \frac{1}{2} \, {}_{zy}G_{\mu\nu} \, , \tag{3.2.23}
$$

$$
{}_{zy}B_{\mu\nu} = 0 \, , \quad {}_{zy}G_{\mu\nu} = \begin{pmatrix} 1 & -2H\Delta V & 0 \\ -2H\Delta V & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.2.24}
$$

Comparing results from zyx and xyz chain after two successive T-dualizations, we see that fields are different. In fact this configuration of fields does not emerge at any point in previous chapter.

Continuing the procedure by finding canonical momenta of original theory

$$
\pi_y = \kappa(\dot{y} + 2Hzx'), \tag{3.2.25}
$$

and combining them with transformation laws

$$
\partial_\pm y \cong \pm \partial_\pm \gamma_2 \mp 2H\Delta V \partial_\pm x, \tag{3.2.26}
$$

we are left with transformation laws in canonical form

$$y_2' \cong \frac{1}{\kappa}\pi_y. \tag{3.2.27}$$

Which are the same as in xyz case.

Having obtained two different transformation laws we can immediately notice that this theory has emergent non-commutative properties. By utilizing standard Poisson algebra (3.2.17) we find new nonzero Poisson bracket

$$\{y_2'(\sigma), y_3'(\bar\sigma)\} \cong \frac{H}{\kappa}\left[2x'(\sigma)\delta(\sigma-\bar\sigma) + x(\sigma)\delta'(\sigma-\bar\sigma)\right], \tag{3.2.28}$$

where $\delta' \equiv \partial_\sigma\delta(\sigma-\bar\sigma)$. If we take following substitutions $A'(\sigma) = y_2'(\sigma)$, $B'(\bar\sigma) = y_3'(\bar\sigma)$, $U'(\sigma) = \frac{H}{\kappa}2x'(\sigma)$ and $V(\sigma) = \frac{H}{\kappa}x(\sigma)$, we notice that above relation takes the form (B.0.1). Utilizing result (B.0.9) from Appendix B, we find following Poisson bracket

$$\{y_2(\sigma), y_3(\bar\sigma)\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar\sigma)\right]\bar H(\sigma-\bar\sigma). \tag{3.2.29}$$

Examining case where string is winded around compactified coordinate $\sigma \to \sigma + 2\pi$ and $\bar\sigma \to \sigma$ gives us

$$\{y_2(\sigma+2\pi), y_3(\sigma)\} \cong -\frac{H}{\kappa}\left[x(\sigma) + 4\pi N_x\right], \tag{3.2.30}$$

Here N_x winding number defined exactly the same as in previous chapter (2.3.72). As we can see, the non-commutativity relation (3.2.29) is of κ-Minkowski type. Calculating Jacobi identity, it is straightforward to see that

$$\{x(\sigma_1),\{\gamma_2(\sigma_2),\gamma_3(\sigma_3)\}\} + \{\gamma_2(\sigma_2),\{\gamma_3(\sigma_3),x(\sigma_1)\}\} + \{\gamma_3(\sigma_3),\{x(\sigma_1),\gamma_2(\sigma_2)\}\} \cong 0. \tag{3.2.31}$$

Because the Jacobiator is zero, we conclude that this R-flux theory is non-commutative but associative.

3.2.3 T-dualization along x direction

In this section we finish zyx chain of T-dualization by dualizing remaining x coordinate. Since this section is mostly the same as one for y coordinate we will omit all but most important results.

We immediately begin with gauge fixed action obtained from (3.2.22)

$$ {}_{zy}S_{fix} = \kappa\int_\Sigma d^2\xi\left[\frac{1}{2}\left(v_+v_- + \partial_+\gamma_2\partial_-\gamma_2 + \partial_+\gamma_3\partial_-\gamma_3\right) - H\Delta V(v_+\partial_-\gamma_2 + \partial_+\gamma_2\, v_-) \tag{3.2.32}$$

$$ + \frac{1}{2}\gamma_1(\partial_+v_- - \partial_-v_+)\right]. \tag{3.2.33}$$

The equations of motion for Lagrange multiplier produces

$$v_\pm = \partial_\pm x\,, \tag{3.2.34}$$

while the equations of motion for gauge fields $v_\pm$ give

$$v_\pm = \pm\partial_\pm\gamma_1 + 2H\Delta V\,\partial_\pm\gamma_2\,. \tag{3.2.35}$$

Inserting expressions for $v_\pm$ into gauge fixed action we get the T-dual action

$$ {}_{zyx}S = \kappa \int_\Sigma d^2\xi \partial_+ ({}_{zyx}X)^\mu \ {}_{zyx}\Pi_{+\mu\nu} ({}_{zyx}X)^\nu \,, \tag{3.2.36} $$

where

$$ {}_{zyx}X^\mu = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}, \quad {}_{zyx}\Pi_{+\mu\nu} = {}_{zyx}B_{\mu\nu} + \frac{1}{2}\ {}_{zyx}G_{\mu\nu} \tag{3.2.37} $$

$$ {}_{zyx}B_{\mu\nu} = \begin{pmatrix} 0 & -H\Delta V & 0 \\ H\Delta V & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad {}_{zyx}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2.38} $$

These fields are exactly the same as the ones obtained at the end of xyz dualization, which is expected since T-dual theory is unique.

By combining equations of motion (3.2.35) with solutions for equation of motion of Lagrange multiplier γ_1 (3.2.34) and by substituting into this combination canonical momentum of original theory

$$ \pi_x = \kappa \dot{x} - 2\kappa H z y', \tag{3.2.39} $$

we are left with the canonical form of the T-dual transformation law

$$ \gamma_1' \cong \frac{1}{\kappa}\pi_x \,. \tag{3.2.40} $$

As we see the full set of T-dual transformation laws, (3.2.16), (3.2.27) and (3.2.40), are the same as in previous chapter (2.3.64), up to $H \to -H$. Since T-dualized theory is the same we find same non-commutative

$$ \{\gamma_1(\sigma), \gamma_3(\bar\sigma)\} \cong \frac{H}{\kappa}\left[2y(\sigma) - y(\bar\sigma)\right] \bar{H}(\sigma - \bar\sigma), \tag{3.2.41} $$

$$ \{\gamma_2(\sigma), \gamma_3(\bar\sigma)\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar\sigma)\right] \bar{H}(\sigma - \bar\sigma), \tag{3.2.42} $$

and non-associativite relations

$$ \begin{gathered} \{\gamma_1(\sigma_1), \gamma_2(\sigma_2), \gamma_3(\sigma_3)\} \equiv \\ \{\gamma_1(\sigma_1), \{\gamma_2(\sigma_2), \gamma_3(\sigma_3)\}\} + \{\gamma_2(\sigma_2), \{\gamma_3(\sigma_3), \gamma_1(\sigma_1)\}\} + \{\gamma_3(\sigma_3), \{\gamma_1(\sigma_1), \gamma_2(\sigma_2)\}\} \\ \cong \frac{2H}{k^2}\Big[\bar{H}(\sigma_1 - \sigma_2)\bar{H}(\sigma_2 - \sigma_3) + \bar{H}(\sigma_2 - \sigma_1)\bar{H}(\sigma_1 - \sigma_3) + \bar{H}(\sigma_1 - \sigma_3)\bar{H}(\sigma_3 - \sigma_2)\Big]. \end{gathered} $$

which can be obtained from the corresponding ones in xyz case by replacing $H \to -H$.

3.3 Quantum aspects of T-dualization in the weakly curved background

Both in this and previous chapter in order to prove isometry of z coordinate and to compute the $\beta^\pm$ functions we assumed the trivial topology and that surface terms, that occur after partial integration, vanish. Now we shift our focus on some quantum features of these problems that manifest in nontrivial topologies. Action that we will examine will be slightly modified, we are still examining bosonic string in presence of constant space-time metric tensor, but this time we take Kalb-Ramond field which depends on all coordinates with infinitesimal field strength. Previously discussed case, torus with H flux, is just a special case of this more general model.

Classic theory that we considered until now contained some problems. They all stem from generalized Buscher procedure which demanded the introduction of invariant coordinate x^μ_{inv}. Since invariant coordinate is multivalued and in order to prove that the gauged theory and initial one are the same we needed to consider global characteristics. By switching from classical theory to quantum one at higher genus, situation is additionally complicated by holonomies of the world-sheet gauge fields Fortunately, these problems can be resolved in Abelian case in the quantum theory [37, 92, 93].

Starting point of this disscussion is partition function for bosonic string in weakly curved background fields

$$Z = \sum_{g=0}^{\infty} \int \mathcal{D}\gamma \mathcal{D}v \, e^{i\frac{\kappa}{2}\int_\Sigma v\, G\star v + i\kappa\int_\Sigma vB[V]v + \frac{i\kappa}{2}\int_\Sigma v d\gamma} . \tag{3.3.1}$$

Where by making Wick rotation $\tau \to -i\tau$, we multiply term containing the metric tensor $G_{\mu\nu}$ by i, while the terms which contain Kalb-Ramond field $B_{\mu\nu}$ and Lagrange multiplier γ_μ stay unchanged. Then new form of partition function is

$$Z = \sum_{g=0}^{\infty} \int \mathcal{D}\gamma \mathcal{D}v \, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\star v + i\kappa\int_\Sigma vB[V]v + \frac{i\kappa}{2}\int_\Sigma v d\gamma} . \tag{3.3.2}$$

Here star represents the Hodge duality operator, while g denotes the genus of manifold. In order to pass main idea across and not be dragged down by index calculus we oped to use differential form notation and omit all space-time indices.

The first step in the calculation process is separation the one form Lagrange multiplier $d\gamma$ into the exact part $d\gamma_e$ (γ_e is single valued) and the harmonic part γ_h ($d\gamma_h = 0 = d^\dagger y_h$)

$$d\gamma = d\gamma_e + \gamma_h. \tag{3.3.3}$$

For the closed forms the co-exact term $d^\dagger\gamma_{co}$ in the Hodge decomposition is missing.

The path integral (3.3.2) goes over all degrees of freedom including local degrees of freedom as well as the sum over different topologies. According to the (3.3.3), we can split $\mathcal{D}\gamma$ into part containing path integral over γ_e and the sum over all possible topologically nontrivial states contained in γ_h (marked by H_γ)

$$\mathcal{D}\gamma \to \mathcal{D}\gamma_e \sum_{H_\gamma} . \tag{3.3.4}$$

By integrating over γ_e, and utilizing functional representation of dirac delta

$$\int \mathcal{D}\xi e^{i\int_\Sigma \xi\phi} = \delta(\phi), \tag{3.3.5}$$

we obtain that field strength vanishes

$$Z = \int \mathcal{D}v \, \delta(dv) \, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\star v + i\kappa\int_\Sigma vB[V]v} \sum_{H_\gamma} e^{\frac{i\kappa}{2}\int_\Sigma v\gamma_h}. \tag{3.3.6}$$

Same way as we had split Lagrange multiplier 1-form, we can split the v 1-form by expressing it as sum of exact, co-exact and the harmonic parts

$$v = dx + d^\dagger v_{ce} + v_h, \tag{3.3.7}$$

which means that

$$\mathcal{D}v \to \mathcal{D}x\mathcal{D}d^{\dagger}v_{ce}\, dH_v. \tag{3.3.8}$$

The functional integration over harmonic part v_h drives to the ordinary integration over topologically nontrivial periods (marked by symbol H_v). After integration over $d^{\dagger}v_{ce}$ we get

$$Z = \int \mathcal{D}x dH_v\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\, {}^\star v + i\kappa \int_\Sigma vB[V]v} \sum_{H_\gamma} e^{\frac{i\kappa}{2}\int_\Sigma v\gamma_h}. \tag{3.3.9}$$

The last term in the exponent is responsible for nontrivial holonomies. Eliminating v_{ce} part, the 1-form v becomes closed and the Riemann bilinear relation becomes usable

$$\int_\Sigma v\gamma_h = \sum_{i=1}^{g}\Big[\oint_{a_i} v \oint_{b_i} \gamma_h - \oint_{a_i}\gamma_h \oint_{b_i} v\Big]. \tag{3.3.10}$$

The symbols $a_i, b_i\, (i = 1, 2, \dots, g)$ represent the canonical homology basis for the world-sheet. Because of the periodicity of the Lagrange multiplier y, we have that these periods are just the winding numbers around cycles a_i and b_i

$$N_{a_i} = \oint_{a_i} \gamma_h, \quad N_{b_i} = \oint_{b_i} \gamma_h\,. \tag{3.3.11}$$

Denoting the periods with

$$A_i = \oint_{a_i} v, \quad B_i = \oint_{b_i} v\,, \tag{3.3.12}$$

and inserting these expressions into 3.3.10

$$\int_\Sigma v\gamma_h = \sum_{i=1}^{g}(N_{b_i}A_i - N_{a_i}B_i). \tag{3.3.13}$$

Now the partition function (3.3.9) gets the form

$$Z = \int \mathcal{D}x\, dA_i dB_i e^{-\frac{\kappa}{2}\int_\Sigma v\, G\, {}^\star v + i\kappa\int_\Sigma vB[V]v} \sum_{N_{a_i}, N_{b_i}\in Z} e^{\frac{i\kappa}{2}\sum_{i=1}^{g}(N_{b_i}A_i - N_{a_i}B_i)}. \tag{3.3.14}$$

The periodic delta function is defined as $\delta(x) = \frac{1}{2\pi}\sum_{n\in Z} e^{inx}$, which produces

$$Z = \int \mathcal{D}x\, dA_i dB_i \delta(\frac{\kappa}{2}A_i)\delta(\frac{\kappa}{2}B_i)\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\, {}^\star v + i\kappa\int_\Sigma vB[V]v}\,. \tag{3.3.15}$$

It is useful to examine the path dependence of the variable V^μ, whose form is now

$$V^\mu(\xi) = x^\mu(\xi) - x^\mu(\xi_0) + \int_P v_h^\mu. \tag{3.3.16}$$

Let us consider two paths, P_1 and P_2, with the same initial ξ_0^α and the final points ξ^α. Now we will subtract from the value of V^μ along P_1 the value along path P_2 and obtain the integral over closed curve $P_1P_2^{-1}$ of the harmonic form

$$V^\mu[P_1] - V^\mu[P_2] = \oint_{P_1P_2^{-1}} v_h^\mu. \tag{3.3.17}$$

Establishing the homology between the closed curve $P_1 P_2^{-1}$ and curve $\sum_i \left[n_i a_i + m_i b_i\right]$, $(n_i, m_i \in Z)$ we get finally

$$V^\mu[P_1] = V^\mu[P_2] + \sum_i (n_i A_i^\mu + m_i B_i^\mu). \tag{3.3.18}$$

The variable $V^\mu(\xi)$ in classical theory is path dependent if holonomies are nontrivial.

Integrating Eq.(3.3.15) over A_i and B_i implies that periods A_i and B_i are zero. Consequently

$$v = dx. \tag{3.3.19}$$

The variable V^μ becomes single valued, and the initial theory is restored

$$Z = \int \mathcal{D}x e^{-\frac{\kappa}{2}\int_\Sigma dx\, G\, {}^\star dx + i\kappa \int_\Sigma dx B[x] dx} = \int \mathcal{D}x e^{-\kappa \int_\Sigma d^2\xi \partial x \Pi_+[x] \bar{\partial} x}. \tag{3.3.20}$$

By proving that we could salvage initial theory from gauged fixed action of bosonic string in the weakly curved background in the presence of nontrivial topologies we showed that our choice of coordinate dependent Kalb-Ramond field is consistent with path integral quantization process.

It is useful to compare our examination of this model with similar results. During our work we were using only Abelian isometries with combination of standard and generalized Bushcer procedures. There has been work done in alternate approach, where nonAmbelian isometries were considered and only standard Buscher procedure was utilized [94]. There it was showed that spaces with isometry maps to the nonisometry spaces, while in this and previous chapters there was isometry in every T-dualization step. In paper [95], authors also utilized generalized Buscher pocedure with invariant coordinates, however dualization was again conducted along non isometry directions using extension of gauge symmetry. It is also useful to mention that this case can be treated in double space formalism [96]. This way T-duality is represented as rotation in space-time that is spanned by coordinates x^μ and their dual counterparts y_μ. Results that are obtained here are in accordance with results we obtained in this and previous chapters.

4. Bosonic T-duality of supersimetric string with coordinate dependent RR-field

This chapter is based on work done in paper [97]

Having analyzed bosonic string in sufficient detail, we will from now on focus on superstring. In bosonic string case we have seen that emergence of non-commutativity arises from the fact that background fields depended on coordinates. To be more precise, if we had original theory in which background fields depended on one specific coordinate we found out that T-dual theory now has one non-commutative coordinate which is dual to original. Similar situation happens in superstring case although on much more complex stage. While bosonic case had only three fields ($G_{\mu\nu}$ spacetime-metric, $B_{\mu\nu}$ Kalb-Ramond field and Φ dilaton field) which could depend on space-time coordinates, superstring theory has plethora of fields that can depend both on bosonic and fermionic coordinates, where dependence on fermionic part is given as an expansion. Even through T-duality procedures for obtaining T-dual theories are the same this complexity of fields makes it imposible to get any sensible result for any except simplest configurations. As we shall see, near the end of this thesis, even working with field that has both symmetric and antisymmetric part gives rise to plethora of problems.

In this chapter we will examine T-duality and subsequent non-commutativity of bosonic part of type II superstring in presence of coordinate dependent Ramond-Ramond (RR) field. Coordinate dependence will be linear and based only on bosonic part, furthermore just like in H-flux theory we will set work with infinitesimal coordinate dependent part, but in order to get $\beta^{\pm}$ functions we will also assume this part is antisymmetric. Motivation for this configuration of background fields comes from Ref. [31, 13] where it has been speculated that this exact configuration will give rise to non-commutativity of fermionic coordinates. Where these new fermionic non-commutative relations are expected to be proportional to bosonic coordinates, This hypothesis, if proven true, would suggest that all space-time has underlying fermionic structure. Having seen that in bosonic case non-commutativity arises only in coordinates on which fields in starting theory depend, does this fact prematurely kills this dream? Even through answer to this question is that there are no new fermionic non-commutativity relations (details are in following chapter), without doing calculations the answer is not so obvious. This stems from the fact that, since theory contains two sets of coordinates, we expect that fermionic coordinates emerge in bosonic T-dual transforamtion laws and vice versa.

As we have already said, this chapter will only focus on T-dualization of bosonic part of superstring theory. Method that we will utilized has already been seen in action in dualization of z coordinate, generalized Buscher procedure. Unlike previous case, T-duality here will not be carried one coordinate at the time, we will dualize all bosonic coordinates at once. When we obtain T-dual theory we will carry T-dualization once again to obtain starting theory.

While this step seems excessive, due to sheer complexity of the theory it is necessary as an additional check that we did not make mistakes during T-dualization. At the end of the chapter, by utulizing T-duality transformation laws, we will examine non-commutativity and non-associativity of the final theory.

4.1 General type II superstring action and choice of background fields

We begin by recalling type II superstring action in pure spinor formulation [26, 27, 28, 29] from Chapter 1.1. Also we will give detailed exposition and all needed assumptions before we begin T-dualization.

4.1.1 General form of the pure spinor type II superstring action

Sigma model of type IIB superstring has the following form [31]

$$S = S_0 + V_{SG}. \tag{4.1.1}$$

This general form of action is expressed as a sum of the part that describes the motion of string in flat background

$$S_0 = \int_\Sigma d^2\xi \left(\frac{\kappa}{2}\eta_{\mu\nu}\partial_m x^\mu \partial_n x^\nu \eta^{mn} - \pi_\alpha \partial_- \theta^\alpha + \partial_+ \bar{\theta}^\alpha \bar{\pi}_\alpha\right) + S_\lambda + S_{\bar{\lambda}}\,, \tag{4.1.2}$$

and part that governs the modifications to the background fields

$$V_{SG} = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar{X}^N. \tag{4.1.3}$$

The terms S_λ and $S_{\bar{\lambda}}$ in (4.1.2) are free-field actions for pure spinors

$$S_\lambda = \int_\Sigma d^2\xi \omega_\alpha \partial_- \lambda^\alpha, \quad S_{\bar{\lambda}} = \int_\Sigma d^2\xi \bar{\omega}_\alpha \partial_+ \bar{\lambda}^\alpha\,. \tag{4.1.4}$$

Here, λ^α and $\bar{\lambda}^\alpha$ are pure spinors whose canonically conjugated momenta are ω_α and $\bar{\omega}_\alpha$, respectively. Pure spinors satisfy pure spinor constraints

$$\lambda^\alpha (\Gamma^\mu)_{\alpha\beta} \lambda^\beta = \bar{\lambda}(\Gamma^\mu)_{\alpha\beta}\bar{\lambda}^\beta = 0\,. \tag{4.1.5}$$

In general case, vectors X^M and X^N as well as a supermatrix A_{MN} are given by

$$X^M = \begin{pmatrix} \partial_+\theta^\alpha \\ \Pi_+^\mu \\ d_\alpha \\ \frac{1}{2}N_+^{\mu\nu} \end{pmatrix}, \quad \bar{X}^M = \begin{pmatrix} \partial_-\bar{\theta}^\lambda \\ \Pi_-^\mu \\ \bar{d}_\lambda \\ \frac{1}{2}\bar{N}_-^{\mu\nu} \end{pmatrix}, \quad A_{MN} = \begin{bmatrix} A_{\alpha\beta} & A_{\alpha\nu} & E_\alpha{}^\beta & \Omega_{\alpha,\mu\nu} \\ A_{\mu\beta} & A_{\mu\nu} & \bar{E}_\mu^\beta & \Omega_{\mu,\nu\rho} \\ E^\alpha{}_\beta & E_\nu^\alpha & P^{\alpha\beta} & C^\alpha{}_{\mu\nu} \\ \Omega_{\mu\nu,\beta} & \Omega_{\mu\nu,\rho} & \bar{C}^\beta{}_{\mu\nu} & S_{\mu\nu,\rho\sigma} \end{bmatrix}, \tag{4.1.6}$$

where notation is the same as in Chapter 1.1. The components of matrix A_{MN} are generally functions of x^μ, θ^α and $\bar{\theta}^\alpha$. Components themselves are derived as expansions in powers of θ^α and $\bar{\theta}^\alpha$ (details are presented in Chapter 1.1 and consult [31]). The superfields $A_{\mu\nu}$, $\bar{E}_\mu^\alpha$, E_μ^α and

$P^{\alpha\beta}$ are known as physical superfields, while superfields that are in the first row and the first column are known as auxiliary because they can be expressed in terms of physical ones [31]. Remaining superfields $\Omega_{\mu,\nu\rho}$ ($\Omega_{\mu\nu,\rho}$), $C^{\alpha}{}_{\mu\nu}$ ($\bar{C}^{\beta}{}_{\mu\nu}$) and $S_{\mu\nu,\rho\sigma}$, are curvatures (field strengths) for physical fields. Components of vectors X^M and $\bar{X}^N$ are defined as

$$\Pi^{\mu}_{+} = \partial_{+}x^{\mu} + \frac{1}{2}\theta^{\alpha}(\Gamma^{\mu})_{\alpha\beta}\partial_{+}\theta^{\beta}, \quad \Pi^{\mu}_{-} = \partial_{-}x^{\mu} + \frac{1}{2}\bar{\theta}^{\alpha}(\Gamma^{\mu})_{\alpha\beta}\partial_{-}\bar{\theta}^{\beta}, \tag{4.1.7}$$

$$\begin{aligned} d_{\alpha} &= \pi_{\alpha} - \frac{1}{2}(\Gamma_{\mu}\theta)_{\alpha}\left[\partial_{+}x^{\mu} + \frac{1}{4}(\theta\Gamma^{\mu}\partial_{+}\theta)\right], \\ \bar{d}_{\alpha} &= \bar{\pi}_{\alpha} - \frac{1}{2}(\Gamma_{\mu}\bar{\theta})_{\alpha}\left[\partial_{-}x^{\mu} + \frac{1}{4}(\bar{\theta}\Gamma^{\mu}\partial_{-}\bar{\theta})\right], \end{aligned} \tag{4.1.8}$$

$$N^{\mu\nu}_{+} = \frac{1}{2}\omega_{\alpha}(\Gamma^{[\mu\nu]})^{\alpha}{}_{\beta}\lambda^{\beta}, \quad \bar{N}^{\mu\nu}_{-} = \frac{1}{2}\bar{\omega}_{\alpha}(\Gamma^{[\mu\nu]})^{\alpha}{}_{\beta}\bar{\lambda}^{\beta}. \tag{4.1.9}$$

The world sheet Σ is parameterized by $\xi^m = (\xi^0 = \tau, \xi^1 = \sigma)$ and world sheet light-cone partial derivatives are defines as $\partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}$. Superspace in which string propagates is spanned both by bosonic x^{μ} ($\mu = 0, 1, ..., 9$) and fermionic θ^{α}, $\bar{\theta}^{\alpha}$ ($\alpha = 1, 2, ..., 16$) coordinates. Variables π_{α} and $\bar{\pi}_{\alpha}$ represent canonically conjugated momenta of fermionic coordinates θ^{α} and $\bar{\theta}^{\alpha}$, respectively. Fermionic coordinates and their canonically conjugated momenta are Majorana-Weyl spinors. It means that each of these spinors has 16 independent real valued components.

4.1.2 Choice of the background fields

Background fields that we will work with are all constants except except RR field $P^{\alpha\beta}$. This field will have linear coordinate dependence only on bosonic coordinates x^{μ}. These fields can not be chosen at random, infact they must satisfy consistency relations outlined in Chapter 1.1. These consistency relations impose following form on supermatrix A_{MN}

$$A_{MN} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa(\frac{1}{2}g_{\mu\nu} + B_{\mu\nu}) & \bar{\Psi}^{\beta}_{\mu} & 0 \\ 0 & -\Psi^{\alpha}_{\nu} & \frac{2}{\kappa}(f^{\alpha\beta} + C^{\alpha\beta}_{\rho}x^{\rho}) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.1.10}$$

Here $g_{\mu\nu}$ is symmetric tensor, $B_{\mu\nu}$ is again Kalb-Ramond antisymmetric field, Ψ^{α}_{μ} and $\bar{\Psi}^{\alpha}_{\mu}$ are Mayorana-Weyl gravitino fields, and finally, $f^{\alpha\beta}$ and $C^{\alpha\beta}_{\rho}$ are constants. Dilaton field Φ is assumed to be constant, so, the factor e^{Φ} is included in $f^{\alpha\beta}$ and $C^{\alpha\beta}_{\rho}$. This will be a classical analysis and we will not calculate the dilaton shift under T-duality transformation. Based on the chirality of spinors, there are type IIA superstring theory for opposite chirality and type IIB superstring theory for same chirality.

Since background fields must satisfy consistency relations we have one additional restriction imposed

$$\Gamma^{\mu}_{\alpha\beta}C^{\beta\gamma}_{\mu} = 0, \quad \Gamma^{\mu}_{\alpha\beta}C^{\gamma\beta}_{\mu} = 0. \tag{4.1.11}$$

Remaining constraints [31] are trivial and applied only to non-physical fields.

In addition to choice of supermatrix, in order to simplify calculation of bosonic T-duality, because all background fields are expanded in powers of θ^α and $\bar\theta^\alpha$, all θ^α and $\bar\theta^\alpha$ non-linear terms in X^M and $\bar X^N$ will be neglected. This greatly simplifies components of these two vectors and they now have following form

$$\Pi^\mu_\pm \to \partial_\pm x^\mu, \quad d_\alpha \to \pi_\alpha, \quad \bar d_\alpha \to \bar\pi_\alpha. \tag{4.1.12}$$

Taking into account all these assumptions, the action (4.1.1) takes the form

$$\begin{aligned} S = \int_\Sigma d^2\xi \Big[\frac{\kappa}{2}\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu - \pi_\alpha(\partial_-\theta^\alpha + \Psi^\alpha_\nu \partial_- x^\nu) + (\partial_+\bar\theta^\alpha + \partial_+ x^\mu \bar\Psi^\alpha_\mu)\bar\pi_\alpha \\ + \frac{2}{\kappa}\pi_\alpha(f^{\alpha\beta} + C^{\alpha\beta}_\rho x^\rho)\bar\pi_\beta\Big]. \end{aligned} \tag{4.1.13}$$

Here, we combined flat space-time metric $\eta_{\mu\nu}$ with $g_{\mu\nu}$ to obtain metric tensor $G_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu}$. This tensor is again combinet with Kalb-Ramon field to obtain $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}$ which is the same tensor we had introduced when we dealt with bosonic string.

Fermionic momenta π_α and $\bar\pi_\alpha$ that appear in above action play the role of auxiliary fields which can be removed by finding their equations om motion and inserting them back into the action.

$$\bar\pi_\beta = \frac{\kappa}{2}\left(F^{-1}(x)\right)_{\beta\alpha}(\partial_-\theta^\alpha + \Psi^\alpha_\nu\partial_- x^\nu), \tag{4.1.14}$$

$$\pi_\alpha = -\frac{\kappa}{2}\left(\partial_+\bar\theta^\beta + \partial_+ x^\mu \bar\Psi^\beta_\mu\right)\left(F^{-1}(x)\right)_{\beta\alpha}, \tag{4.1.15}$$

where we took the liberty to denote two new substitutions $F^{\alpha\beta}(x)$ and $(F^{-1}(x))_{\alpha\beta}$ of the form

$$F^{\alpha\beta}(x) = f^{\alpha\beta} + C^{\alpha\beta}_\mu x^\mu, \quad (F^{-1}(x))_{\alpha\beta} = (f^{-1})_{\alpha\beta} - (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta}\, x^\mu. \tag{4.1.16}$$

If we wish to invert previous equations and T-dual transformation laws, as well as to simplify calculations, we must take into account two additional assumptions. First assumption has already been touched upon and that is need for infinitesimal $C^{\alpha\beta}_\mu$. This is reminiscent of diluted flux approximation for bosonic string and in fact this assumption plays exactly the same role as did assumption for H. Second assumption is that $(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}$ is antisymmetric under exchange of first and last index. In other words, tensor $(F^{-1}(x))_{\alpha\beta}$ has only antisymmetric part that depends on x^μ and it is infinitesimal. These two additional assumptions do not in any way, shape or form infringe on consistency relations for background fields or alter the properties of other fields.

Substituting equations (4.1.14) and (4.1.15) into (4.1.13) the final form of action is

$$S = \kappa \int_\Sigma d^2\xi \left[\Pi_{+\mu\nu}\partial_+ x^\mu\partial_- x^\nu + \frac{1}{2}(\partial_+\bar\theta^\alpha + \partial_+ x^\mu\bar\Psi^\alpha_\mu)\left(F^{-1}(x)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi^\beta_\nu\partial_- x^\nu)\right]. \tag{4.1.17}$$

Having obtained this action, we can safely proceed with T-dualization.

4.2 T-dualization

In this section T-duality will be performed along all bosonic coordinates in order to find relations that connect T-dual coordinates with coordinates and momenta of original theory. These transformation laws will then be used in subsequent chapters to find non-commutativity relations between coordinates of T-dual theory.

Starting point for considering T-duality will be generalized Buscher T-dualization procedure [44]. This procedure works when we have theories with coordinate dependent background fields. Standard Buscher procedure [36, 43], which we had partially utilized in previous two chapters, is designed to be applied along isometry directions on which background fields do not depend and is not applicable here. The shift symmetry in the generalized procedure is localized by introduction of covariant derivatives, invariant coordinates and additional gauge fields. These newly introduced gauge fields produce additional degrees of freedom. Since we expect that starting and T-dual theory have exactly the same number of degrees of freedom we need to eliminate all excessive degrees of freedom. This is accomplished by demanding that field strength of gauge fields ($F_{+-} = \partial_+ v_- - \partial_- v_+$) vanishes by addition of Lagrange multipliers. Next step in procedure is fixing the gauge symmetry such that starting coordinates are constant and action is only left with gauge fields and its derivatives. From this gauge fixed action, finding equations of motion for gauge fields, expressing gauge fields as function of Lagrange multipliers and inserting those equations into action we can obtain T-dual action, were Lagrange multipliers of original theory now play the role of T-dual coordinates.

Action (4.1.17) is invariant to translation symmetry, by the virtue of antisymmetric part of $F^{-1}_{\alpha\beta}$, tensor $(f^{-1}C_\mu f^{-1})_{\alpha\beta}$. Following antisymmetricity of this tensor, we can rewrite the action (4.1.17) in the following way

$$S = \kappa \int_\Sigma d^2\xi \left[\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu + \frac{1}{2}\epsilon^{mn}\partial_m(\bar\theta^\alpha + x^\mu \bar\Psi^\alpha_\mu)\left(F^{-1}(x)\right)_{\alpha\beta}\partial_n(\theta^\beta + \Psi^\beta_\nu x^\nu)\right]. \tag{4.2.1}$$

Let us now consider the global shift symmetry $\delta x^\mu = \lambda^\mu$ and vary the action (4.2.1)

$$\delta S = -\frac{\kappa}{2}(f^{-1}C_\mu f^{-1})_{\alpha\beta}\lambda^\mu \int_\Sigma d^2\xi \epsilon^{mn}\partial_m(\bar\theta^\alpha + \bar\Psi^\alpha_\nu x^\nu)\partial_n(\theta^\beta + \Psi^\beta_\rho x^\rho)\,, \tag{4.2.2}$$

where m, n are indices of the twodimensional worldsheet. After one partial integration, we first obtain surface term, which we neglect because we are interested only in trivial topologies with trivial winding conditions. Second term which we obtain is identically zero because it is product of symmetric, $\partial_m\partial_n$, and antisymmetric, ϵ^{mn}, tensor. So, the shift isometry exists.

In order to find T-dual action we have to implement following substitutions

$$\partial_\pm x^\mu \to D_\pm x^\mu = \partial_\pm x^\mu + v^\mu_\pm, \tag{4.2.3}$$

$$x^\rho \quad \to x^\mu_{inv} = \int_P d\xi^m D_m x^\mu = x^\mu(\xi) - x^\mu(\xi_0) + \Delta V^\mu\,, \Delta V^\mu = \int_P d\xi^m v^\rho_m(\xi), \tag{4.2.4}$$

$$S \quad \to S + \frac{\kappa}{2}\int_\Sigma d^2\xi\,[v^\mu_+\partial_- y_\mu - v^\mu_-\partial_+ y_\mu]\,. \tag{4.2.5}$$

Here we decided that y_μ will play the role of Lagrange multiplier and subsequently the role of T-dual coordinate, which is different from bosonic string case where we had γ_μ occupying that role. Reason for this comes down to aesthetics. Because of the shift symmetry we are allowed to fix the gauge, $x^\mu(\xi) = x^\mu(\xi_0)$ and, inserting these substitutions into action (4.1.17), we obtain auxiliary action suitable for T-dualization

$$S_{fix} = \kappa \int_\Sigma d^2\xi \Big[\Pi_{+\mu\nu} v_+^\mu v_-^\nu + \frac{1}{2}(\partial_+\bar{\theta}^\alpha + v_+^\mu \bar{\Psi}_\mu^\alpha) \left(F^{-1}(\Delta V)\right)_{\alpha\beta} (\partial_-\theta^\beta + \Psi_\nu^\beta v_-^\nu) + \frac{1}{2}(v_+^\mu \partial_- y_\mu - v_-^\mu \partial_+ y_\mu) \Big]. \tag{4.2.6}$$

Similarly as before, working with invariant coordinate we have necessary introduction of non-locality into the theory. Also, path P that is taken in expression for ΔV^ρ goes from some starting point ξ_0 to end point ξ.

In order to check if substitutions we had introduced are valid and that they will lead to correct T-dual theory of starting action, we need to be able to obtain original action by finding solutions to equations of motion for Lagrange multipliers. Equations of motion for Lagrange multipliers give us

$$\partial_- v_+^\mu - \partial_+ v_-^\mu = 0 \quad \Rightarrow \quad v_\pm^\mu = \partial_\pm x^\mu. \tag{4.2.7}$$

Inserting this result into (4.2.4) we get the following

$$\Delta V^\rho = \int_P d\xi'^m \partial_m x^\rho(\xi') = x^\rho(\xi) - x^\rho(\xi_0) = \Delta x^\rho. \tag{4.2.8}$$

Since, we had shift symmetry in original action, we can let $x^\rho(\xi_0)$ be any arbitrary constant. Taking all this into account and inserting (4.2.7), (4.2.8) into (4.2.6) we obtain our starting action (4.1.17).

Before we obtain equations for motion for gauge fields, we would like to make following substitution in action

$$Y_{+\mu} = \partial_+ y_\mu - \partial_+\bar{\theta}^\alpha \left(F^{-1}(\Delta V)\right)_{\alpha\beta} \Psi_\mu^\beta, \qquad Y_{-\mu} = \partial_- y_\mu + \bar{\Psi}_\mu^\alpha \left(F^{-1}(\Delta V)\right)_{\alpha\beta} \partial_-\theta^\beta, \tag{4.2.9}$$

$$\bar{\Pi}_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar{\Psi}_\mu^\alpha \left(F^{-1}(\Delta V)\right)_{\alpha\beta} \Psi_\nu^\beta = \breve{\Pi}_{+\mu\nu} - \frac{1}{2}\bar{\Psi}_\mu^\alpha (f^{-1})_{\alpha\alpha_1} C_\rho^{\alpha_1\beta_1} (f^{-1})_{\beta_1\beta} \Psi_\nu^\beta \Delta V^\rho, \tag{4.2.10}$$

$$\breve{\Pi}_{+\mu\nu} \equiv \Pi_{+\mu\nu} + \frac{1}{2}\bar{\Psi}_\mu^\alpha (f^{-1})_{\alpha\beta} \Psi_\nu^\beta. \tag{4.2.11}$$

These substitutions allow us to write down gauge fixed action into more manageable form

$$S_{fix} = \kappa \int_\Sigma d^2\xi \left[\bar{\Pi}_{+\mu\nu} v_+^\mu v_-^\nu + \frac{1}{2} v_+^\mu Y_{-\mu} - \frac{1}{2} v_-^\mu Y_{+\mu} + \frac{1}{2}\partial_+\bar{\theta}^\alpha \left(F^{-1}(\Delta V)\right)_{\alpha\beta} \partial_-\theta^\beta \right]. \tag{4.2.12}$$

This action produces following equations of motion for gauge fields

$$\bar{\Pi}_{+\mu\nu} v_-^\nu = -(\frac{1}{2} Y_{-\mu} + \beta_\mu^+(V)), \qquad \bar{\Pi}_{+\mu\nu} v_+^\mu = \frac{1}{2} Y_{+\nu} - \beta_\nu^-(V). \tag{4.2.13}$$

Here, function $\beta^\pm(V)$ is obtained from variation of term containing ΔV^ρ in expression for $F^{-1}(\Delta V)$ (details are presented in Appendix C)

$$\beta^-_\mu(V) = \frac{1}{4}\partial_+\big[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\big](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\big[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\big]$$
$$-\frac{1}{4}\big[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\big](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_+\big[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\big], \tag{4.2.14}$$

$$\beta^+_\mu(V) = \frac{1}{4}\big[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\big](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_-\big[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\big]$$
$$-\frac{1}{4}\partial_-\big[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\big](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\big[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\big]. \tag{4.2.15}$$

Here we have took advantage of the fact that $\partial_\pm V^\mu = v^\mu_\pm$ (more details in Appendix C). Let us note that V^μ in the expressions for beta functions is actually $V^{(0)\mu}$ because it stands besides $C^{\alpha\beta}_\mu$. We omit index (0) just in order to simplify the form of the expressions.

In order to find how gauge fields depend on Lagrange multipliers, we need to invert equations of motion (4.2.13). Since $C^{\alpha\beta}_\mu$ is an infinitesimal constant, these equations can be inverted iteratively [14]. We separate variables into two parts, one finite and one proportional to $C^{\alpha\beta}_\mu$. After doing this we have

$$v^\nu_- = -\bar\Theta^{\nu\mu}_-\Big[\frac{1}{2}Y_{-\mu} + \beta^+_\mu(V^{(0)})\Big], \qquad v^\mu_+ = \Big[\frac{1}{2}Y_{+\nu} - \beta^-_\nu(V^{(0)})\Big]\bar\Theta^{\nu\mu}_-. \tag{4.2.16}$$

Functions $\beta_{\pm\mu}(V^{(0)})$ are obtained by substituting first order of expression for $v_\pm$ into $\beta_{\pm\mu}(V)$, where $V^{(0)}$ is given by

$$\Delta V^{(0)\rho} = \int_P d\xi^m v^{(0)\rho}_m$$
$$= \frac{1}{2}\int_P d\xi^+\breve\Theta^{\rho_1\rho}_-\left[\partial_+ y_{\rho_1} - \partial_+\bar\theta^\alpha(f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] - \frac{1}{2}\int_P d\xi^-\breve\Theta^{\rho\rho_1}_-\left[\partial_- y_{\rho_1} + \bar\Psi^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right]. \tag{4.2.17}$$

Where $\bar\Theta^{\mu\nu}_-$ is inverse tensor of $\bar\Pi_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar\Psi^\alpha_\mu\left(F^{-1}(\Delta V)\right)_{\alpha\beta}\Psi^\beta_\nu$, defined as

$$\bar\Theta^{\mu\nu}_-\bar\Pi_{+\nu\rho} = \delta^\mu_\rho, \tag{4.2.18}$$

where

$$\bar\Theta^{\mu\nu}_- = \breve\Theta^{\mu\nu}_- + \frac{1}{2}\breve\Theta^{\mu\mu_1}_-\bar\Psi^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho V^{(0)\rho}(f^{-1})_{\beta_1\beta}\Psi^{\beta_1}_{\nu_1}\breve\Theta^{\nu_1\nu}_-, \tag{4.2.19}$$

$$\breve\Theta^{\mu\nu}_-\breve\Pi_{\nu\rho} = \delta^\mu_\rho, \qquad \breve\Theta^{\mu\nu}_- = \Theta^{\mu\nu}_- - \frac{1}{2}\Theta^{\mu\mu_1}_-\bar\Psi^\alpha_{\mu_1}(\bar f^{-1})_{\alpha\beta}\Psi^\beta_{\nu_1}\Theta^{\nu_1\nu}_- \tag{4.2.20}$$

$$\bar f^{\alpha\beta} = f^{\alpha\beta} + \frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu, \tag{4.2.21}$$

$$\Theta^{\mu\nu}_-\Pi_{+\mu\rho} = \delta^\mu_\rho, \qquad \Theta_- = -4(G_E^{-1}\Pi_- G^{-1})^{\mu\nu}. \tag{4.2.22}$$

Tensor $G_{E\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$ is known in the literature as the effective metric.

Inserting equations (4.2.16) into (4.2.6), keeping only terms that are linear in $C^{\alpha\beta}_\mu$ we obtain T-dual action

$${}^bS = \frac{\kappa}{2}\int_\Sigma\Big[\frac{1}{2}\bar\Theta^{\mu\nu}_- Y_{+\mu}Y_{-\nu} + \partial_+\bar\theta^\alpha\left(F^{-1}(\Delta V)\right)_{\alpha\beta}\partial_-\theta^\beta\Big]. \tag{4.2.23}$$

Comparing starting action (4.1.17) with T-dual action, were we note that $\partial_\pm x^\mu$ transforms into $\partial_\pm y_\mu$ and x^μ transforms into $V^{(0)}$, we can deduce that T-dual action has following arguments.

$$ {}^b\bar{\Pi}_+^{\mu\nu} = \frac{1}{4}\bar{\Theta}_-^{\mu\nu}, \tag{4.2.24}$$

$$ \left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta} = \left(F^{-1}(V^{(0)})\right)_{\alpha\beta} - \frac{1}{2}\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi_\mu^{\alpha_1}\bar{\Theta}_-^{\mu\nu}\bar{\Psi}_\nu^{\beta_1}\left(F^{-1}(V^{(0)})\right)_{\beta_1\beta}, \tag{4.2.25}$$

$$ {}^b\bar{\Psi}^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta} = \frac{1}{2}\bar{\Theta}_-^{\mu\nu}\bar{\Psi}_\nu^\alpha\left(F^{-1}(V^{(0)})\right)_{\alpha\beta}, \tag{4.2.26}$$

$$ \left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}{}^b\Psi^{\nu\beta} = -\frac{1}{2}\left(F^{-1}(V^{(0)})\right)_{\alpha\beta}\Psi_\mu^\beta\bar{\Theta}_-^{\mu\nu}. \tag{4.2.27}$$

In order to express T-dual gravitino background fields in terms of its components, it is useful to calculate inverse of field ${}^bF^{-1}_{\alpha\beta}$

$$ {}^bF^{\alpha\beta}(V^{(0)}) = F^{\alpha\beta}(V^{(0)}) + \frac{1}{2}\Psi_\mu^\alpha\Theta_-^{\mu\nu}\bar{\Psi}_\nu^\beta. \tag{4.2.28}$$

With this equation at hand it is straightforward to obtain T-dual gravitino fields. Here we present T-dual gravitino fields expanded in terms of their components

$$ {}^b\bar{\Psi}^{\mu\alpha} = \frac{1}{2}\bar{\Theta}^{\mu\nu}\bar{\Psi}_\nu^\alpha + \frac{1}{4}\bar{\Theta}_-^{\mu\mu_1}\bar{\Psi}_{\mu_1}^\beta\left(F^{-1}(V^{(0)})\right)_{\beta\beta_1}\Psi_\nu^{\beta_1}\Theta_-^{\nu\nu_1}\bar{\Psi}_{\nu_1}^\alpha, \tag{4.2.29}$$

$$ {}^b\Psi^{\nu\beta} = -\frac{1}{2}\Psi_\mu^\beta\bar{\Theta}_-^{\mu\nu} - \frac{1}{2}\Psi_\mu^\beta\Theta_-^{\mu\mu_1}\bar{\Psi}_{\mu_1}^\alpha\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi_{\nu_1}^{\alpha_1}\bar{\Theta}_-^{\nu_1\nu}. \tag{4.2.30}$$

Here superscript ^{b} denotes bosonic T-duality. The general conclusion is that all background fields get the linear corrections in $C_\mu^{\alpha\beta}$ comparing with the results of the case with constant background fields [40]. Also the coordinate dependence is present in all T-dual background fields.

From the above equations we see how background fields of original theory transform under T-duality. It should be noted that these actions are of the same form taking into account that initial coordinates x^μ are replaced by y_μ after T-dualization.

4.3 T-dualization of T-dual theory

Requirement that original theory and T-dual one describe the same physics it should be possible that we can switch from one onto other by cycling T-duality. That is, applying T-duality twice does not introduce any changes. In this chapter we would like to do just that, we will apply T-duality procedure onto already dualized theory in order to get back to starting theory. This fact is actually a way to test if our calculations were correct.

When we started with T-duality in previous chapter, we started by testing if theory possessed translational invariance. Here, duo to presence of $\Delta V^{(0)}$, we do not need to conduct such a test, theory is invariant. This can be easily deduced by checking eq. (4.2.17) and taking notice that every instance of dual coordinate y_μ is accompanied with partial derivative. We begin T-dualization by implementing following substitutions

$$
\begin{aligned}
\partial_\pm y_\mu \to & D_\pm y_\mu = \partial_\pm y_\mu + u_{\pm\mu} \to D_\pm y_\mu = u_{\pm\mu}, && (4.3.1)\\
\Delta V^{(0)\rho} \to & \Delta U^{(0)\rho}, && (4.3.2)\\
\Delta U^{(0)\rho} = & \frac{1}{2}\int_P d\xi^+ \breve{\Theta}_-^{\rho_1\rho}\left[u_{+\rho_1} - \partial_+\bar{\theta}^\alpha (f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] \\
& -\frac{1}{2}\int_P d\xi^- \breve{\Theta}_-^{\rho\rho_1}\left[u_{-\rho_1} + \bar{\Psi}^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right], && (4.3.3)\\
Y_{+\mu} \to & U_{+\mu} = u_{+\mu} - \partial_+\bar{\theta}^\alpha \left(F^{-1}(\Delta U^{(0)})\right)_{\alpha\beta}\Psi^\beta_\mu && (4.3.4)\\
Y_{-\mu} \to & U_{-\mu} = u_{-\mu} + \bar{\Psi}^\alpha_\mu \left(F^{-1}(\Delta U^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta && (4.3.5)\\
{}^bS \to & {}^bS + \frac{\kappa}{2}\int_\Sigma d^2\xi (u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu)\,. && (4.3.6)
\end{aligned}
$$

We denoted newly introduced gauge fields with $u_\pm$ while x^μ are Lagrange multipliers. In first line we immediately fixed gauge by choosing $y(\xi) = const$. Inserting these substitutions into (4.2.23) we get

$$
{}^bS_{fix} = \frac{k}{2}\int_\Sigma d^2\xi\Big[\frac{1}{2}\bar{\Theta}_-^{\mu\nu}U_{+\mu}U_{-\nu} + \partial_+\bar{\theta}^\alpha \left(F^{-1}(\Delta U^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta + (u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu)\Big]. \quad (4.3.7)
$$

Finding equations of motion for Lagrange multipliers and inserting solution to those equations into gauge fixed action we return to the starting point of this chapter, T-dual action. On the other hand, finding equations of motion for gauge fields

$$
\begin{aligned}
u_{+\mu} &= 2\Big[\partial_+ x^\nu \bar{\Pi}_{+\nu\mu} + \beta^-_\mu(x)\Big] + \partial_+\bar{\theta}^\alpha \left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\mu, && (4.3.8)\\
u_{-\nu} &= -2\Big[\bar{\Pi}_{+\mu\nu}\partial_- x^\nu + \beta^+_\mu(x)\Big] - \bar{\Psi}^\alpha_\nu \left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta\,, && (4.3.9)
\end{aligned}
$$

and inserting these equations into the gauge fixed action, keeping all terms linear with respect to $C^{\mu\nu}_\rho$,, we obtain our original action (4.1.17). Here we use the freedom to choose $\Delta x^\mu = x(\xi) - x(\xi_0)$, with $x(\xi_0) = 0$.

4.4 Non-commutative relations

We have already seen is simpler case that T-dual transformation laws, that connect dual and original theory through their coordinates, along with Poisson bracket of original theory can be combined in such a way to generate Poisson algebra of T-dual theory. This chapter continues on this philosophy however we will be interested only in Poisson brackets of one half of the theory, bosonic half. We leave examination of completely dualized theory as well as examination of full Poisson algebra to following chapter.

Starting theory was geometric theory whose were space-time coordinates $x^\mu(\xi)$ and their conjugated momenta $\pi_\mu(\xi)$. It is natural to impose standard Poisson bracket structure on such a theory

$$
\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu_\nu \delta(\sigma - \bar{\sigma}), \quad \{x^\mu(\sigma), x^\nu(\bar{\sigma})\} = 0, \quad \{\pi_\mu(\sigma), \pi_\nu(\bar{\sigma})\} = 0. \quad (4.4.1)
$$

Since we have applied T-duality twice to this case we have two sets of transformation laws that connect gauge wields with Lagrange multipliers. First set was presented in (4.2.16) and other set was given by (4.3.8) and (4.3.9). These laws are equivalent and no matter the choice there is no difference in results. We will chose to start with relations (4.3.8) and (4.3.9) and using solutions to equations of motion for Lagrange multipliers x^μ, $u_{\pm\mu} = \partial_\pm y_\mu$, we obtain following T-dual transformation laws

$$\partial_+ y_\mu \cong 2\Big[\partial_+ x^\nu \bar{\Pi}_{+\nu\mu} + \beta^-_\mu(x)\Big] + \partial_+\bar{\theta}^\alpha \left(F^{-1}(x)\right)_{\alpha\beta} \Psi^\beta_\mu \,, \tag{4.4.2}$$

$$\partial_- y_\mu \cong -2\Big[\bar{\Pi}_{+\mu\nu}\partial_- x^\nu + \beta^+_\mu(x)\Big] - \bar{\Psi}^\alpha_\nu \left(F^{-1}(x)\right)_{\alpha\beta} \partial_-\theta^\beta \,, \tag{4.4.3}$$

where symbol $\cong$ denotes T-dual transformation. Subtracting these two equations and by utilizing properties of light-cone coordinates (Appendix A), we get

$$y'_\mu \cong \bar{\Pi}_{+\mu\nu}\partial_- x^\nu + \partial_+ x^\nu \bar{\Pi}_{+\nu\mu} + \beta^+_\mu + \beta^-_\mu + \frac{1}{2}\partial_+\bar{\theta}^\alpha \left(F^{-1}(x)\right)_{\alpha\beta} \Psi^\beta_\mu + \frac{1}{2}\bar{\Psi}^\alpha_\nu \left(F^{-1}(x)\right)_{\alpha\beta} \partial_-\theta^\beta \,. \tag{4.4.4}$$

Taking into account that bosonic momenta, π_μ of original theory are of the form

$$\pi_\mu = \kappa\Big[\bar{\Pi}_{+\mu\nu}\partial_- x^\nu + \partial_+ x^\nu \bar{\Pi}_{+\nu\mu} + \frac{1}{2}\bar{\Psi}^\alpha_\mu \left(F^{-1}(x)\right)_{\alpha\beta} \partial_-\theta^\beta + \frac{1}{2}\partial_+\bar{\theta}^\alpha \left(F^{-1}(x)\right)_{\alpha\beta} \Psi^\beta_\nu\Big], \tag{4.4.5}$$

and $\beta^0_\mu = \beta^+_\mu + \beta^-_\mu$, we obtain

$$y'_\mu \cong \frac{\pi_\mu}{\kappa} + \beta^0_\mu(x) \,. \tag{4.4.6}$$

Here $\beta^0_\mu(x)$ is given by

$$\begin{aligned}\beta^0_\mu(x) =& \frac{1}{2}\partial_\sigma\left[\bar{\theta}^\alpha + x^{\nu_1}\bar{\Psi}^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta}\left[\theta^\beta + \Psi^\beta_{\nu_2}x^{\nu_2}\right] \\ &- \frac{1}{2}\left[\bar{\theta}^\alpha + x^{\nu_1}\bar{\Psi}^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta}\partial_\sigma\left[\theta^\beta + \Psi^\beta_{\nu_2}x^{\nu_2}\right] \,.\end{aligned} \tag{4.4.7}$$

To find Poisson bracket between T-dual coordinates, we can start by finding Poisson bracket of sigma derivatives of T-dual coordinates and then integrating twice (see [13, 47, 54], Appendix B). Implementing this procedure we have that Poisson bracket is given as

$$\begin{aligned}&\{y_{\nu_1}(\sigma), y_{\nu_2}(\bar{\sigma})\} \cong \\ &\frac{1}{2k}[2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2} - \delta^{\mu_2}_{\nu_1}\delta^{\mu_1}_{\nu_2}]\Big[K_{\mu_1\mu_2}(\bar{\sigma}) + K_{\mu_2\mu_1}(\sigma)\Big]\bar{H}(\sigma-\bar{\sigma}) \,,\end{aligned} \tag{4.4.8}$$

where, for the sake of simplicity, we introduced

$$\begin{aligned}K_{\mu\nu}(\sigma) =& \Big(\bar{\theta}^\alpha(\sigma) + x^{\mu_1}(\sigma)\bar{\Psi}^\alpha_{\mu_1}\Big)(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta}\Psi^\beta_\nu \\ &- \bar{\Psi}^\alpha_\nu (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta}\Big(\theta^\beta(\sigma) + \Psi^\beta_{\nu_1}x^{\nu_1}(\sigma)\Big).\end{aligned} \tag{4.4.9}$$

Here, $\bar{H}(\sigma-\bar{\sigma})$ is same step function defined in Appendix B. Due to how we defined Heaviside step function $\bar{H}$ we have that these Poisson brackets are zero when $\sigma = \bar{\sigma}$. However, in cases where string in curled around compactified dimension, that is cases where $\sigma - \bar{\sigma} = 2\pi$, we have following situation

$$\{y_{\nu_1}(\sigma+2\pi), y_{\nu_2}(\sigma)\} \cong \frac{1}{2k}[2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2} - \delta^{\mu_2}_{\nu_1}\delta^{\mu_1}_{\nu_2}]\Big[K_{\mu_1\mu_2}(\sigma) + K_{\mu_2\mu_1}(\sigma)\Big]$$
$$+\frac{\pi}{k}N^\mu \bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_{\mu_2}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_3}[\delta^{\mu_1}_{\mu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} - \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\mu} + \delta^{\mu_1}_{\mu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\mu}]\,. \tag{4.4.10}$$

Here we used fact that $\bar{H}(2\pi) = 1$, while N^ρ is winding number around compactified coordinate defined as

$$x^\mu(\sigma+2\pi) - x^\mu(\sigma) = 2\pi N^\mu. \tag{4.4.11}$$

From this relation we can see that if we choose $x^\mu(\sigma) = 0$ than Poisson bracket has linear dependence on winding number. In cases where we do not have any winding number, we still have non-commutativity that is proportional to background fields.

Using the expression for sigma derivative of y_ν (4.4.6) and expression for Poisson bracket of T-dual coordinates (4.4.8), we can find non-associative relations. Procedure is the same as for finding Poisson brackets of T-dual theory, we find Poisson bracket of sigma derivative and integrate with respect to sigma coordinate, this time integration is done once. Going along with this procedure we have the final result

$$\{y_\nu(\sigma), \{y_{\nu_1}(\sigma_1), y_{\nu_2}(\sigma_2)\}\} \cong \frac{1}{2k}H(\sigma_1-\sigma_2)\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_{\mu_2}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_3}$$
$$\times\Big[H(\sigma_1-\sigma)[2\delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} - 2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu} - \delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} + \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu}]$$
$$+H(\sigma_2-\sigma)[2\delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} - 2\delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu} - \delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} + \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu}]\Big]\,. \tag{4.4.12}$$

Since Jacobi identity is non-zero for T-dual theory we have that coordinate dependent RR field produces non-associative theory. However putting $\sigma = \sigma_2 = \bar{\sigma}$ and $\sigma_1 = \bar{\sigma} + 2\pi$ we have following Jacobi identity

$$\{y_\nu(\bar{\sigma}), \{y_{\nu_1}(\bar{\sigma}+2\pi), y_{\nu_2}(\bar{\sigma})\}\} \cong$$
$$\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_{\mu_2}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_3}[2\delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} - 2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu} - \delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} + \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu}]\,. \tag{4.4.13}$$

Examining equation (4.4.6), we notice that $\partial_\sigma y_\mu$ is not only a linear combination of initial coordinate and its momenta but also has terms that are proportional to fermionic coordinates. This might lead us to believe that T-dual theory would have nontrivial Poisson bracket between T-dual coordinate and fermionic coordinates. However, this is not the case, and it can be directly calculated by finding Poisson bracket between sigma derivative of T-dual coordinate and fermion coordinates (more details in Appendix B).

$$\{\theta^\alpha(\sigma), y_\mu(\bar{\sigma})\} \cong 0, \qquad \{\bar{\theta}^\alpha(\sigma), y_\mu(\bar{\sigma})\} \cong 0. \tag{4.4.14}$$

We will examine if these Poisson brackets go through any change when we also dualize fermionic coordinates.

5. Fermionic T-duality of supersimetric string with coordinate dependent RR-field

This chapter is based on work done in paper [98]

Up until now we have only been interested in T-duality of bosonic coordinates, which is reminiscent of historical development of said topic. Even through T-duality was originally conceived with bosonic coordinates in mind [38] it is possible to extend it to fermionic coordinates also [11, 99, 100, 101]. Fermionic T-duality, just as was case before, maps supersymmetric background fields and fermionic coordinates of one theory to supersymmetric backgrounds and coordinates of other theory. While actors in this play are different it is surprising that method for obtaining fermionic T-duality is the same as before, Buscher procedure [36, 43]. Even in cases where background fields depend on fermionic coordinates generalized procedure is applicable. Since we had enough opportunity to see both standard and generalized procedures at work, this favorable circumstance greatly reduces difficulty of this chapter.

In this chapter we continue the work that has been started in previous chapter, we finish dualization of action that has been dualized along bosonic coordinates. Main reason we are interested in this endeavor is to find out if such a theory will give rise to non-commutative relations of the type $\{\bar{\theta}, \theta\} \sim x$. While it was already hinted that we will fail short in our quest, without doing explicit calculations this is not obvious. This raises one question, why is it not obvious that there will be no fermionic non-commutativity? Have we not seen that in bosonic case, when we do not have background fields that depend on coordinates we can not expect emergence of non-commutativity in dual theory no mater the order in which we chose to do T-dualization. Answer to this question lies in one little caveat of superstring case, while it is true that background fields of starting theory did not depend on fermionic coordinate, by performing bosonic T-duality we have introduced non-local part which was encoded in ΔV^{μ}. This term now possesses dependence on fermionic coordinate and appears in all backgroundfields of T-dual theory. This fact opens another question, if we had decided to perform fermionic T-duality first and then bosonic, then we would not have to worry about appearance of fermionic coordinates in background fields, does this mean that order of T-dualiy is important in superstring case? Sadly, answer to this question is again negative and throughout this chapter we will carry on explicit calculations that show this fact.

To summarise, this chapter will deal with fermionic T-duality of both theory that has already been dualized along bosonic coordinates and one which has not been dualized. By obtaining transforamtion laws for both cases we will examine non-commutative relations as we have done many times before.

5.1 Type II superstring action and action dualized along bosonic coordinates

Actions that we will work with are superstring action in pure spinor formalism with coordinate dependent RR field as well as its T-dual. Both of these actions have already been presented in previous chapter and here we will only give quick summary.

5.1.1 Type II superstring in pure spinor formulation

For the first action we have

$$S = k \int_\Sigma d^2\xi \left[\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu + \frac{1}{2}(\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu \bar{\Psi}^\alpha_\mu)\left(F^{-1}(x)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi^\beta_\nu \partial_- x^\nu)\right], \quad (5.1.1)$$

where again tensors that appear in above expression have following form

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}, \quad (5.1.2)$$

$$F^{\alpha\beta}(x) = f^{\alpha\beta} + C^{\alpha\beta}_\mu x^\mu\,, \quad (F^{-1}(x))_{\alpha\beta} = (f^{-1})_{\alpha\beta} - (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\rho x^\rho (f^{-1})_{\beta_1\beta}\,. \quad (5.1.3)$$

Since this is a logical follow up to previous chapter, properties of the $(F^{-1}(x))_{\alpha\beta}$ tensor are the same as before. We are working with tensor that has antisymmertic and infinitesimal coordinate dependent part. Rest of the symbols have following meaning: x^μ $(\mu = 0, 1, ..., 9)$ are bosonic coordinates, θ^α and $\bar{\theta}^\alpha$ are fermionic coordinates with 16 independent real components each which are Majorana-Wejl spinors, uperspace is parameterized by ξ^m $(\xi^0 = \tau, \xi^1 = \sigma)$ and light-cone partial derivatives $\partial_\pm = \partial_\tau \pm \partial_\sigma$.

5.1.2 Bosonic T-dual action

Action that is obtained after T-dualizing (5.1.1)

$$\begin{aligned} {}^bS = \frac{k}{2} \int_\Sigma d^2\xi \Big[&\frac{1}{2}\bar{\Theta}^{\mu\nu}\partial_+ y_\mu \partial_- y_\nu + \partial_+\bar{\theta}^\alpha \left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta \\ &+ \partial_+ y_\mu {}^b\bar{\Psi}^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta + \partial_+\bar{\theta}^\alpha \left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta} {}^b\Psi^{\nu\beta}\partial_- y_\nu \Big]. \end{aligned} \quad (5.1.4)$$

y_μ represents T-dual coordinate, left superscript ^{b} denotes bosonic T-duality and V^0 represents following integral

$$\Delta V^{(0)\rho} =$$
$$= \frac{1}{2}\int_P d\xi^+ \breve{\Theta}^{\rho_1\rho}_- \left[\partial_+ y_{\rho_1} - \partial_+\bar{\theta}^\alpha (f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] - \frac{1}{2}\int_P d\xi^- \breve{\Theta}^{\rho\rho_1}_- \left[\partial_- y_{\rho_1} + \bar{\Psi}^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right]. \quad (5.1.5)$$

As we can see $\Delta V^{(0)\rho}$ does indeed contain fermionic coordinates and by association so do background fields. This is the main differentiating fact between type II superstring theory and its bosonic counterpart.

T-dual tensors that appear in action have following interpretation: $\bar\Theta_-^{\mu\nu}$ is inverse tensor of $\bar\Pi_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar\Psi_\mu^\alpha \left(F^{-1}(x)\right)_{\alpha\beta} \Psi_\nu^\beta = \breve\Pi_{+\mu\nu} - \frac{1}{2}\bar\Psi_\mu^\alpha (f^{-1})_{\alpha\alpha_1} C_\rho^{\alpha_1\beta_1} x^\rho (f^{-1})_{\beta_1\beta}\Psi_\nu^\beta$, defined as

$$\bar\Theta_-^{\mu\nu}\bar\Pi_{+\nu\rho} = \delta_\rho^\mu, \tag{5.1.6}$$

Remaining properties are listed in great detain in previous chapter (4.2.19), (4.2.20), (4.2.21), (4.2.22). While background fields are given as

$$\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta} = \left(F^{-1}(V^{(0)})\right)_{\alpha\beta} - \frac{1}{2}\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1} \Psi_\mu^{\alpha_1}\bar\Theta_-^{\mu\nu}\bar\Psi_\nu^{\beta_1}\left(F^{-1}(V^{(0)})\right)_{\beta_1\beta}. \tag{5.1.7}$$

Tensor $\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}$ is T-dual to $\left(F^{-1}(x)\right)_{\alpha\beta}$ and T-dual gravitino fields are given as

$${}^b\bar\Psi^{\mu\alpha} = \frac{1}{2}\bar\Theta^{\mu\nu}\bar\Psi_\nu^\alpha + \frac{1}{4}\bar\Theta_-^{\mu\mu_1}\bar\Psi_{\mu_1}^\beta\left(F^{-1}(V^{(0)})\right)_{\beta\beta_1}\Psi_\nu^{\beta_1}\Theta_-^{\nu\nu_1}\bar\Psi_{\nu_1}^\alpha = \frac{1}{2}\Theta_-^{\mu\nu}\bar\Psi_\mu^\alpha, \tag{5.1.8}$$

$${}^b\Psi^{\nu\beta} = -\frac{1}{2}\Psi_\mu^\beta\bar\Theta_-^{\mu\nu} - \frac{1}{2}\Psi_\mu^\beta\Theta_-^{\mu\mu_1}\bar\Psi_{\mu_1}^\alpha\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi_{\nu_1}^{\alpha_1}\bar\Theta_-^{\nu_1\nu} = -\frac{1}{2}\Psi_\mu^\beta\Theta_-^{\mu\nu}. \tag{5.1.9}$$

Having presented these two theories we can focus on main part of this chapter, fermionic T-duality.

5.2 Fermionic T-duality

No matter on which theory we decide to dualize first we have to note one thing and that is that both actions (5.1.1) and (5.1.4) do not posses terms proportional to $\partial_+\theta^\alpha$ and $\partial_-\bar\theta^\alpha$. This means that our fermionic coordinates have following local symmetry

$$\delta\theta^\alpha = \epsilon^\alpha(\sigma^+), \quad \delta\bar\theta^\alpha = \bar\epsilon^\alpha(\sigma^-), \quad (\sigma^\pm = \tau \pm \sigma). \tag{5.2.1}$$

We need to fix this symmetry before obtaining T-dual theory, one way to do this is through BRST formalism. This symmetry has following corresponding BRST transformations for fermionic fields

$$s\theta^\alpha = c^\alpha(\sigma^+), \qquad s\bar\theta^\alpha = \bar c^\alpha(\sigma^-). \tag{5.2.2}$$

Here s is BRST nilpotent operator, c^α and $\bar c^\alpha$ represent ghost fields that correspond to gauge parameters ϵ^α and $\bar\epsilon^\alpha$ respectively. In addition to ghost fields we also have following BRST transformations

$$sC_\alpha = b_{+\alpha}, \quad s\bar C_\alpha = \bar b_{-\alpha}, \quad sb_{+\alpha} = 0, \quad s\bar b_{-\alpha} = 0. \tag{5.2.3}$$

where $\bar C_\alpha$ and C_α are anti-ghosts, $b_{+\alpha}$ and $\bar b_{-\alpha}$ are Nakanishi-Lautrup auxiliary fields.

Fixing of gauge symmetry is accomplished by introduction of gauge fermion, where we have decided to follow procedure that has been outlined in [102]

$$\Psi = \frac{k}{2}\int_\Sigma d^2\xi\Big[\bar C_\alpha\Big(\partial_+\theta^\alpha + \frac{1}{2}\alpha^{\alpha\beta}b_{+\beta}\Big) + \Big(\partial_-\bar\theta^\alpha + \frac{1}{2}\bar b_{-\beta}\alpha^{\beta\alpha}\Big)C_\alpha\Big], \tag{5.2.4}$$

here $\alpha^{\alpha\beta}$ is arbitrary invertible matrix.

Applying BRST transformation to gauge fermion we obtain gauge fixed action and Fadeev-Popov action

$$S_{gf} = \frac{k}{2}\int_\Sigma d^2\xi\Big[\bar{b}_{-\alpha}\partial_+\theta^\alpha + \partial_-\bar{\theta}^\alpha b_{+\alpha} + \bar{b}_\alpha \alpha^{\alpha\beta} b_{+\beta}\Big], \tag{5.2.5}$$

$$S_{F-P} = \frac{k}{2}\int_\Sigma d^2\xi\Big[\bar{C}_\alpha\partial_+ c^\alpha + (\partial_-\bar{c}^\alpha)C_\alpha\Big]. \tag{5.2.6}$$

Fadeev-Popov term contains only ghosts and anti-ghosts and it is decoupled from the actions (5.1.1) and (5.1.4). From this point on, this term will be ignored. Gauge fixing term contains auxiliary fields $\bar{b}_{-\alpha}$ and $b_{+\alpha}$ that can be removed with equations of motion

$$\bar{b}_{-\alpha} = -\partial_-\bar{\theta}^\beta(\alpha^{-1})_{\beta\alpha}, \qquad b_{+\alpha} = -(\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta, \tag{5.2.7}$$

giving us

$$S_{gf} = -\frac{k}{2}\int_\Sigma d^2\xi\partial_-\bar{\theta}^\alpha(\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta. \tag{5.2.8}$$

Inserting guage fixing term into (5.1.1) and (5.1.4) gives us actions that can be dualized with Buscher procedure.

5.2.1 Type II superstring - fermionic T-duality

In order to not be overwhelmed we will work with non dualized theory first. Buscher procedure that we apply here, after finding gauge fixing term, does not in any significant way differ from one we applied before.

Since both action (5.1.1) and gauge fixing term (5.2.8) are trivially invariant to global translations of fermionic coordinates, we localize this translational symmetry by replacing partial derivatives with covariant ones

$$\partial_\pm\theta^\alpha \to D_\pm\theta^\alpha = \partial_\pm\theta^\alpha + u^\alpha_\pm, \tag{5.2.9}$$

$$\partial_\pm\bar{\theta}^\alpha \to D_\pm\bar{\theta}^\alpha = \partial_\pm\bar{\theta}^\alpha + \bar{u}^\alpha_\pm. \tag{5.2.10}$$

New gauge fields $u^\alpha_\pm$ and $\bar{u}^\alpha_\pm$ introduce new degrees of freedom that are removed by addition of term

$$S_{add} = \frac{k}{2}\int_\Sigma d^2\xi\left[\bar{z}_\alpha(\partial_+u^\alpha_- - \partial_-u^\alpha_+) + (\partial_+\bar{u}^\alpha_- - \partial_-\bar{u}^\alpha_+)z_\alpha\right]. \tag{5.2.11}$$

Gauge freedom can be utilized to fix fermionic coordinates such that $\theta^\alpha = \theta^\alpha_0 = const$ and $\bar{\theta}^\alpha = \bar{\theta}^\alpha_0 = const$. This in turn reduces our covariant derivatives to

$$D_\pm\theta^\alpha \to u^\alpha_\pm, \qquad D_\pm\bar{\theta}^\alpha \to \bar{u}^\alpha_\pm. \tag{5.2.12}$$

With all this in mind, we have following action

$$S_{fix} = k \int_\Sigma d^2\xi \Big[\Pi_{+\mu\nu} \partial_+ x^\mu \partial_- x^\nu + \frac{1}{2} (\bar{u}_+^\alpha + \partial_+ x^\mu \bar{\Psi}_\mu^\alpha) \left(F^{-1}(x)\right)_{\alpha\beta} (u_-^\beta + \Psi_\nu^\beta \partial_- x^\nu) \\ - \frac{1}{2} \bar{u}_-^\alpha (\alpha^{-1})_{\alpha\beta} u_+^\beta + \frac{1}{2} \bar{z}_\alpha (\partial_+ u_-^\alpha - \partial_- u_+^\alpha) + \frac{1}{2} (\partial_+ \bar{u}_-^\alpha - \partial_- \bar{u}_+^\alpha) z_\alpha \Big]. \tag{5.2.13}$$

On one side we have equations of motion for Lagrange multipliers $\bar{\chi}_\alpha$ and χ_α

$$\partial_+ u_-^\alpha - \partial_- u_+^\alpha = 0 \quad \rightarrow \quad u_\pm^\alpha = \partial_\pm \theta^\alpha, \tag{5.2.14}$$

$$\partial_+ \bar{u}_-^\alpha - \partial_- \bar{u}_+^\alpha = 0 \quad \rightarrow \quad \bar{u}_\pm^\alpha = \partial_\pm \bar{\theta}^\alpha. \tag{5.2.15}$$

Inserting solutions for these equations into action (5.2.13) we obtain starting action plus gauge fixing term.

Variation of action with respect to gauge fields produces following set of equations of motion

$$u_-^\alpha = -\left(F^{\alpha\beta}(x) \partial_- z_\beta + \Psi_\mu^\alpha \partial_- x^\mu\right), \tag{5.2.16}$$

$$u_+^\alpha = -\alpha^{\alpha\beta} \partial_+ z_\beta, \tag{5.2.17}$$

$$\bar{u}_+^\alpha = \partial_+ \bar{z}_\beta F^{\beta\alpha}(x) - \partial_+ x^\mu \bar{\Psi}_\mu^\alpha, \tag{5.2.18}$$

$$\bar{u}_-^\alpha = \partial_- \bar{z}_\beta \alpha^{\beta\alpha}. \tag{5.2.19}$$

Utilizing these equations we can remove gauge fields from action, resulting in action that depends only on Lagrange multipliers and bosonic coordinates

$${}^f S = k \int_\Sigma d^2\xi \Big[\Pi_{+\mu\nu} \partial_+ x^\mu \partial_- x^\nu + \frac{1}{2} \partial_+ \bar{z}_\alpha \Psi_\mu^\alpha \partial_- x^\mu + \frac{1}{2} \partial_+ \bar{z}_\alpha F^{\alpha\beta}(x) \partial_- z_\beta \\ - \frac{1}{2} \partial_+ x^\mu \bar{\Psi}_\mu^\alpha \partial_- z_\alpha - \frac{1}{2} \partial_- \bar{z}_\alpha \alpha^{\alpha\beta} \partial_+ z_\beta \Big]. \tag{5.2.20}$$

Just like in the bosonic case, we have that left superscript ^{f} denotes fermionic T-duality. From here we can deduce background fields of feriomionic T-dual theory

$${}^f\bar{\Pi}_{+\mu\nu} = \Pi_{+\mu\nu}, \tag{5.2.21}$$

$${}^f \left(F^{-1}(x)\right)^{\alpha\beta} = F^{\alpha\beta}(x), \tag{5.2.22}$$

$${}^f\bar{\Psi}_{\mu\beta} \, {}^f \left(F^{-1}(x)\right)^{\beta\alpha} = -\bar{\Psi}_\mu^\alpha \quad \rightarrow \quad {}^f\bar{\Psi}_{\mu\beta} = -\bar{\Psi}_\mu^\alpha \left(F^{-1}(x)\right)_{\alpha\beta}, \tag{5.2.23}$$

$${}^f \left(F^{-1}(x)\right)^{\alpha\beta} \, {}^f\Psi_{\mu\beta} = \Psi_\mu^\alpha \quad \rightarrow \quad {}^f\Psi_{\mu\beta} = \left(F^{-1}(x)\right)_{\beta\alpha} \Psi_\mu^\alpha. \tag{5.2.24}$$

Unlike bosonic case, fermionic T-dual theory is local. This can be attributed to the fact that background fields do not depend on fermionic coordinates. This in turn means that theory is geometric and we should not expect emergence of non-commutative phenomena.

5.2.2 Fermionic T-duality of bosonic T-dual theory

To obtain fully dualized theory we start with action that is already T-dualized along bosonic coordinates (5.1.4). Procedure for fermionic T-duality is mostly the same as described before. The only difference comes from the fact that bosonic T-duality introduced non-local term V^0

which depends on θ^α and $\bar\theta^\alpha$ and now we need to introduce invariant fermionic coordinates in order for action to exhibit local shift symmetry

$$D_\pm\theta^\alpha = \partial_\pm\theta^\alpha + u^\alpha_\pm, \tag{5.2.25}$$
$$D_\pm\bar\theta^\alpha = \partial_\pm\bar\theta^\alpha + \bar u^\alpha_\pm, \tag{5.2.26}$$
$$\theta^\alpha_{inv} = \int_P d\xi^m D_m\theta^\alpha = \int_P d\xi^m(\partial_m\theta^\alpha + u^\alpha_m) = \Delta\theta^\alpha + \Delta U^\alpha, \tag{5.2.27}$$
$$\bar\theta^\alpha_{inv} = \int_P d\xi^m D_m\bar\theta^\alpha = \int_P d\xi^m(\partial_m\bar\theta^\alpha + \bar u^\alpha_m) = \Delta\bar\theta^\alpha + \Delta\bar U^\alpha. \tag{5.2.28}$$

Fixing gauge symmetry as before, setting fermionic coordinates to constants, we deduce following relations

$$D_\pm\theta^\alpha \to u^\alpha_\pm, \qquad D_\pm\bar\theta^\alpha \to \bar u^\alpha_\pm, \qquad \theta^\alpha_{inv} \to \Delta U^\alpha, \qquad \bar\theta^\alpha_{inv} \to \Delta\bar U^\alpha. \tag{5.2.29}$$

With these relations we obtain action that is only a function of gauge fields, lagrange multipliers and dual coordinates

$$\begin{aligned}{}^bS_{fix} = \frac{k}{2}\int_\Sigma d^2\xi\Big[\frac{1}{2}\bar\Theta^{\mu\nu}\partial_+y_\mu\partial_-y_\nu + \bar u^\alpha_+\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}u^\beta_- \\ +\partial_+y_\mu{}^b\bar\Psi^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}u^\beta_- + \bar u^\alpha_+\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}{}^b\Psi^{\nu\beta}\partial_-y_\nu - \bar u^\alpha_-(\alpha^{-1})_{\alpha\beta}u^\beta_+ \\ +\bar z_\alpha(\partial_+u^\alpha_- - \partial_-u^\alpha_+) + (\partial_+\bar u^\alpha_- - \partial_-\bar u^\alpha_+)z_\alpha\Big].\end{aligned} \tag{5.2.30}$$

In order to simplify calculations we introduce the following two substitutions

$$\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}{}^b\Psi^{\nu\beta}\partial_-y_\nu + \partial_-z_\alpha = Z_{-\alpha}, \qquad \partial_+y_\mu{}^b\bar\Psi^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta} - \partial_+\bar z_\beta = \bar Z_{+\beta}. \tag{5.2.31}$$

Now, our action can be expressed as

$$\begin{aligned}{}^bS_{fix} = \frac{k}{2}\int_\Sigma d^2\xi\Big[\frac{1}{2}\bar\Theta^{\mu\nu}\partial_+y_\mu\partial_-y_\nu + \bar u^\alpha_+\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}u^\beta_- + \bar Z_{+\beta}u^\beta_- + \bar u^\alpha_+Z_{-\alpha} \\ -\bar u^\alpha_-(\alpha^{-1})_{\alpha\beta}u^\beta_+ + \partial_-\bar z_\alpha u^\alpha_+ - \bar u^\alpha_-\partial_+z_\alpha.\Big]\end{aligned} \tag{5.2.32}$$

Similar to the first case, we can always revert to starting action by finding equations of motion for Lagrange multipliers and inserting their solutions into the action. In both cases equations of motion are the same so we take the freedom to omit them here.

Equations of motion for gauge fields differ in this case. Since we have that $V^{(0)}$ depends on fermionic coordinates, equations of motion have additional term that depends on invariant coordinate.

$$u^\alpha_+ = -(\alpha)^{\alpha\beta}\partial_+z_\beta, \qquad \bar u^\beta_- = \partial_-\bar z_\alpha(\alpha)^{\alpha\beta}, \tag{5.2.33}$$

$$\bar u^\alpha_+ = -\bar Z_{+\beta}\,{}^bF^{\beta\alpha}(V^{(0)}) - \beta^-_\nu(V^{(0)},U^{(0)})\,{}^b\bar\Psi^{\nu\alpha}, \tag{5.2.34}$$
$$u^\beta_- = -{}^bF^{\beta\alpha}(V^{(0)})Z_{-\alpha} - \beta^+_\mu(V^{(0)},U^{(0)})\,{}^b\Psi^{\mu\beta}. \tag{5.2.35}$$

The beta functions, $\beta^{\pm}_{\mu}(V^{(0)},U^{(0)})$, are obtained by varying $V^{(0)}$ (see [97] and Appendix C for more details). They are given as

$$\beta^{\pm}_{\mu}(V^{(0)},U^{(0)}) = \mp\frac{1}{8}\partial_{\mp}\big[\bar{U}^{\alpha}+V^{\nu_1}\bar{\Psi}^{\alpha}_{\nu_1}\big](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_{\mu}(f^{-1})_{\beta_1\beta}\big[U^{\beta}+\Psi^{\beta}_{\nu_2}V^{\nu_2}\big] \\ \pm\frac{1}{8}\big[\bar{U}^{\alpha}+V^{\nu_1}\bar{\Psi}^{\alpha}_{\nu_1}\big](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_{\mu}(f^{-1})_{\beta_1\beta}\partial_{\mp}\big[U^{\beta}+\Psi^{\beta}_{\nu_2}V^{\nu_2}\big]. \qquad (5.2.36)$$

Inserting equations of motion for gauge fields into action (5.2.32) and keeping only terms linear with respect to $C^{\alpha\beta}_{\mu}$, we obtain fully dualized action

$${}^{bf}S = \frac{k}{2}\int_{\Sigma}d^2\xi\Big[\frac{1}{2}\bar{\Theta}^{\mu\nu}_{-}\partial_{+}y_{\mu}\partial_{-}y_{\nu} - \bar{Z}_{+\alpha}{}^{b}F^{\alpha\beta}(V^{(0)})Z_{-\beta} - \partial_{-}\bar{z}_{\alpha}(\alpha)^{\alpha\beta}\partial_{+}z_{\beta}\Big]. \qquad (5.2.37)$$

Expanded, we have

$${}^{bf}S = k\int_{\Sigma}d^2\xi\Big[\frac{1}{4}\Theta^{\mu\nu}_{-}\partial_{+}y_{\mu}\partial_{-}y_{\nu} - \frac{1}{4}\partial_{+}y_{\mu}\Theta^{\mu\nu}_{-}\bar{\Psi}^{\alpha}_{\nu}\partial_{-}z_{\alpha} - \frac{1}{4}\partial_{+}\bar{z}_{\alpha}\Psi^{\alpha}_{\mu}\Theta^{\mu\nu}_{-}\partial_{-}y_{\nu} \\ +\frac{1}{2}\partial_{+}\bar{z}_{\alpha}{}^{b}F^{\alpha\beta}(V^{(0)})\partial_{-}z_{\beta} - \frac{1}{2}\partial_{-}\bar{z}_{\alpha}(\alpha)^{\alpha\beta}\partial_{+}z_{\beta}\Big]. \qquad (5.2.38)$$

From here, we can read background fields of T-dual theory

$${}^{bf}\bar{\Pi}^{\mu\nu}_{+} = \frac{1}{4}\bar{\Theta}^{\mu\nu}_{-} - \frac{1}{2}{}^{b}\bar{\Psi}^{\mu\alpha}\left({}^{b}F^{-1}(V^{(0)})\right)_{\alpha\beta}{}^{b}\Psi^{\nu\beta} = \Theta^{\mu\nu}_{-}, \\ {}^{bf}\left(F^{-1}(x)\right)^{\alpha\beta} = {}^{b}F^{\alpha\beta}(x) = F^{\alpha\beta}(x) + \frac{1}{2}\Psi^{\alpha}_{\mu}\Theta^{\mu\nu}_{-}\bar{\Psi}^{\beta}_{\nu}, \\ {}^{bf}\bar{\Psi}^{\mu}_{\alpha}\,{}^{bf}\left(F^{-1}(x)\right)^{\alpha\beta} = {}^{b}\bar{\Psi}^{\mu\beta} = \frac{1}{2}\Theta^{\mu\nu}_{-}\bar{\Psi}^{\beta}_{\nu} \quad\rightarrow\quad {}^{bf}\bar{\Psi}^{\mu}_{\alpha} = \frac{1}{2}\bar{\Theta}^{\mu\nu}_{-}\bar{\Psi}^{\beta}_{\nu}\left(F^{-1}(x)\right)_{\beta\alpha}, \\ {}^{bf}\left(F^{-1}(x)\right)^{\alpha\beta}\,{}^{bf}\Psi^{\nu}_{\beta} = {}^{b}\Psi^{\nu\alpha} = -\frac{1}{2}\Psi^{\alpha}_{\mu}\Theta^{\mu\nu}_{-} \quad\rightarrow\quad {}^{bf}\Psi^{\nu}_{\beta} = -\frac{1}{2}\left(F^{-1}(x)\right)_{\beta\alpha}\Psi^{\alpha}_{\mu}\bar{\Theta}^{\mu\nu}_{-}. \qquad (5.2.39)$$

Comparing background fields in different stages of T-dualization we notice that both fermionic T-duality and bosonic T-duality affect all field, where all T-dual theories now have coordinate dependent fields. It should also be noted that non-commutative relations in theory emerge only after performing bosonic T-duality. Fermionic T-dual coordinates are always only proportional to fermionic momenta therefore Poisson brackets between fermionic coordiantes always remain zero.

5.2.3 Bosonic T-duality of fermionic T-dual theory

For completion sake, we will also T-dualize fermionic T-dual action (5.2.20) along x^{μ} coordinates. In this specific case, where only RR field depends on bosonic coordinate, we expect that bosonic and ferionic T-dualities commute. Therefore, this section can be also thought of as a check for calculations from previous section.

We again start by localizing translational symmetry, inserting Lagrange multipliers and fixing gauge fields. This produces following gauge fixed action

$$
\begin{aligned}
{}^{f}S_{fix} &= \kappa \int d^2\xi \Big[v_+^\mu \Pi_{+\mu\nu} v_-^\nu + \frac{1}{2}\partial_+ \bar{z}_\alpha f^{\alpha\beta} \partial_- z_\beta + \frac{1}{2}\partial_+ \bar{z}_\alpha C_\mu^{\alpha\beta} \partial_- z_\beta \Delta V^\mu \\
&+ \frac{1}{2}\partial_+ \bar{z}_\alpha \Psi_\mu^\alpha v_-^\mu - \frac{1}{2} v_+^\mu \bar{\Psi}_\mu^\alpha \partial_- z_\alpha - \frac{1}{2}\partial_- \bar{z}_\alpha \alpha^{\alpha\beta} \partial_+ z_\beta + \frac{1}{2} y_\mu (\partial_+ v_-^\mu - \partial_- v_+^\mu) \Big] . \quad (5.2.40)
\end{aligned}
$$

Introducing the variables

$$
Y_{+\mu} = \partial_+ y_\mu - \partial_+ \bar{z}_\alpha \Psi_\mu^\alpha , \quad Y_{-\mu} = \partial_- y_\mu - \bar{\Psi}_\mu^\alpha \partial_- z_\alpha , \tag{5.2.41}
$$

the action (5.2.40) gets much simpler form

$$
\begin{aligned}
{}^{f}S_{fix} &= \kappa \int d^2\xi \Big[v_+^\mu \Pi_{+\mu\nu} v_-^\nu + \frac{1}{2}\partial_+ \bar{z}_\alpha f^{\alpha\beta} \partial_- z_\beta + \frac{1}{2}\partial_+ \bar{z}_\alpha C_\mu^{\alpha\beta} \partial_- z_\beta \Delta V^\mu \\
&- \frac{1}{2} Y_{+\mu} v_-^\mu + \frac{1}{2} v_+^\mu Y_{-\mu} \Big] . \quad (5.2.42)
\end{aligned}
$$

Varying the above action with respect to gauge fields v_+^μ and v_-^μ, we get, respectively,

$$
\Pi_{+\mu\nu} v_-^\nu = -\left(\frac{1}{2} Y_{-\mu} + \beta_{+\mu}(V) \right) , \tag{5.2.43}
$$

$$
v_+^\nu \Pi_{+\nu\mu} = \frac{1}{2} Y_{+\mu} - \beta_{-\mu}(V) , \tag{5.2.44}
$$

where $\beta_{\pm\mu}$ are the beta functions obtained from coordinate dependent term in the action

$$
\beta_{\pm\mu} = \mp \frac{1}{8} \left(\bar{z}_\alpha C_\mu^{\alpha\beta} \partial_\mp z_\beta - \partial_\mp \bar{z}_\alpha C_\mu^{\alpha\beta} z_\beta \right) . \tag{5.2.45}
$$

Inserting (5.2.43) and (5.2.44) into the auxiliary action (5.2.42), keeping the terms linear in $C_\mu^{\alpha\beta}$, we obtain fully T-dualized action (first fermionic, then bosonic T-dualization)

$$
{}^{fb}S = \kappa \int d^2\xi \left[\frac{1}{2}\partial_+ \bar{z}_\alpha F^{\alpha\beta}(\Delta V) \partial_- z_\beta + \frac{1}{4} Y_{+\mu} (\Pi_+^{-1})^{\mu\nu} Y_{-\nu} \right] . \tag{5.2.46}
$$

Expanding above action, it can easily be seen that it is identical to one given in (5.2.38).

5.3 Few notes on non-commutativity

We have already saw that bosonic T-duality produces non-commutative relations between bosonic T-dual coordinates. Now we want to see how these relations are modified by fermionic T-duality and if new ones emerge. Procedure for finding non-commutativity is exactly the same as in all the chapters leading to here, we impose standard Poisson bracket structure on starting theory and utilize T-dual transformation laws. Only difference here is that we have additional starting Poisson brackets, brackets containing fermionic coordinates

$$
\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta_\nu^\mu \delta(\sigma - \bar{\sigma}), \quad \{\theta^\alpha(\sigma), \pi_\beta(\bar{\sigma})\} = \{\bar{\theta}^\alpha(\sigma), \bar{\pi}_\beta(\bar{\sigma})\} = \delta_\beta^\alpha \delta(\sigma - \bar{\sigma}), \tag{5.3.1}
$$

where all other Poisson brackets vanish.

We start with case that has only been T-dualized along fermionic coordinates. To find how T-dual coordinates depend on starting ones and their momenta we can begin by finding fermionic momenta of starting theory. It is useful to remember that starting theory did not posses terms that are proportional to $\partial_+\theta^\alpha$ and $\partial_-\bar{\theta}^\alpha$ and that this symmetry was fixed with BRST formalism. Addition of gauge fixing term introduced modification to momenta of starting theory and to obtain correct non-commutative relations we should be working with theories that have gauge fixing term in them. With this in mind, it is easy to find fermionic momentum of original theory (5.2.13)

$$\pi_\beta = -\frac{k}{2}\Big[(\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu \bar{\Psi}^\alpha_\mu)\left(F^{-1}(x)\right)_{\alpha\beta} - \partial_-\bar{\theta}^\alpha(\alpha^{-1})_{\alpha\beta}\Big], \tag{5.3.2}$$

$$\bar{\pi}_\alpha = \frac{k}{2}\Big[\left(F^{-1}(x)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi^\beta_\nu \partial_- x^\nu) - (\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta\Big]. \tag{5.3.3}$$

Since we want to obtain Poisson brackets for equal τ we want to find σ partial derivatives of dual coordinate

$$\partial_\sigma z_\alpha = \frac{1}{2}(\partial_+ z_\alpha - \partial_- z_\alpha) \cong \frac{1}{k}\bar{\pi}_\alpha, \tag{5.3.4}$$

$$\partial_\sigma \bar{z}_\alpha = \frac{1}{2}(\partial_+ \bar{z}_\alpha - \partial_- \bar{z}_\alpha) \cong -\frac{1}{k}\pi_\alpha. \tag{5.3.5}$$

Fermionic Momenta of original theory commute with each other and with x^μ coordinates, therefore we deduce that there has been no change to geometric structure of this theory.

For fully dualized theory, transformation laws (5.2.33) (5.2.34) (5.2.35) all depend on dual bosonic coordinate however, when we insert transformation laws that connect original bosonic coordinates with T-dual ones

$$\partial_+ y_\mu \cong 2\Big[\partial_+ x^\nu \bar{\Pi}_{+\nu\mu} + \beta^-_\mu(x)\Big] + \partial_+\bar{\theta}^\alpha\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\mu, \tag{5.3.6}$$

$$\partial_- y_\nu \cong -2\Big[\bar{\Pi}_{+\mu\nu}\partial_- x^\nu + \beta^+_\mu(x)\Big] - \bar{\Psi}^\alpha_\nu\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta\,, \tag{5.3.7}$$

into transformation laws for fermionic coordinates (5.2.33), (5.2.34) and (5.2.35) we again obtain relations (5.3.4) and (5.3.5).

On a first glance it would seem that fermionic T-duality has not produced any new Poisson brackets, however this is not the case. While it is true that there are no modifications to Poisson brackets between fermions, we have new Poisson bracket structure between fermions and bosons. This can be seen from σ derivative of bosonic T-dual coordinate

$$y'_\mu \cong \frac{\pi_\mu}{\kappa} + \beta^0_\mu(x), \tag{5.3.8}$$

where $\beta^0_\mu(x)$ is combination $\beta^+_\mu(x) + \beta^-_\mu(x)$ we encountered before

$$\begin{aligned}\beta^0_\mu(x) =& \frac{1}{2}\partial_\sigma\left[\bar{\theta}^\alpha + x^{\nu_1}\bar{\Psi}^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left[\theta^\beta + \Psi^\beta_{\nu_2}x^{\nu_2}\right]\\ &-\frac{1}{2}\left[\bar{\theta}^\alpha + x^{\nu_1}\bar{\Psi}^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_\sigma\left[\theta^\beta + \Psi^\beta_{\nu_2}x^{\nu_2}\right].\end{aligned} \tag{5.3.9}$$

Finding Poisson brackets between σ derivatives of coordinates and integrating twice we obtain following relations

$$\{y_\mu(\sigma),\bar{z}_\beta(\bar{\sigma})\}\cong \\ \frac{1}{2k}\Big[\bar{\theta}^\alpha(\sigma)+x^{\nu_1}(\sigma)\bar{\Psi}^\alpha_{\nu_1}-2\big(\bar{\theta}^\alpha(\bar{\sigma})+x^{\nu_1}(\bar{\sigma})\bar{\Psi}^\alpha_{\nu_1}\big)\Big](f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\bar{H}(\sigma-\bar{\sigma}), \tag{5.3.10}$$

$$\{y_\mu(\sigma),z_\alpha(\bar{\sigma})\}\cong \\ \frac{1}{2k}(f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Big[\theta^\beta(\sigma)+\Psi^\beta_{\nu_2}x^{\nu_2}(\sigma)-2\big(\theta^\beta(\bar{\sigma})+\Psi^\beta_{\nu_2}x^{\nu_2}(\bar{\sigma})\big)\Big]\bar{H}(\sigma-\bar{\sigma}). \tag{5.3.11}$$

Having failed to obtain non-commutativity relations between fermionic coordinates, not to mention ones that are proportional to bosonic ones, we conclude that hypothesis made in [31, 13] is not correct. Furthermore, having gained some insight how T-duality affects non-commutativity, we suspect that fermionic non-commutativity relations are possible but only in case where background fields depend on those coordinates.

6. Bosonic T-duality of supersymmetric string with coordinate dependent RR field - general case

This chapter is based on work done in paper [103]

In previous two chapters we have seen how T-duality affects type II superstring theory with coordinate dependent Ramond-Ramond field. During that presentation our ability to obtain both T-dual theory and T-dual transformation laws depended on two premises, one was that coordinate dependent part of RR field was infinitesimal and other was that it is antisymmetric. First of these assumptions was necessary in order to invert transformation laws while second one was introduced only in order to obtain $\beta^{\pm}$ functions. It is sufficient to say that latter of these assumptions can be removed and since $\beta^{\pm}$ functions do not play any role in T-dual action we expect the form of action to remain the same.

Here we will deal with theory whose coordinate dependent part of Ramond-Ramond field has both symmetric and antisymmetric part, where we will only focus on bosonic T-duality. This simple modification produces more general forms of $\beta^{\pm}$ functions, $N^{\pm}$ functions. Where $\beta^{\pm}$ functions were dependent on some combination of coordinates and their derivatives, $N^{\pm}$ are functions that depend on path of path integral that has been introduced with invariant coordinate. We expect that this departure from antisymmetric tensor to general one would not affect theory that much, however this can not be further from the truth. Even though final T-dual theory is the same, transforamtion laws and non-commutative relations are drastically more complex. This rise in complexion makes it impossibile to deduce Poisson brackets of T-dual theory the same way as we did before.

Since we have already obtained both bosonic duality and full duality of type II superstring it is natural to wonder why go this extra step. The answer to this question lies in the fact that, if we ever wish to work with more complex background fields it will not be possible to impose antisymmetric restrictions on all fields. By working out the kinks of such approach on case when the fields are relatively simple we hope that all future extensions to background fields would follow the same procedure for obtaining T-dual transformation laws.

6.1 Action and choice of background fields

Since this is only generalization of work done before and action that we will work with has already been mentioned few times we will just quickly list it here again, without going in any detail what the symbols mean (for detailed exposition consult Chapter 4 and Chapter 5)

$$S = k\int_\Sigma d^2\xi\left[\Pi_{+\mu\nu}\partial_+ x^\mu\partial_- x^\nu + \frac{1}{2}(\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu\bar{\Psi}^\alpha_\mu)\left(F^{-1}(x)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi^\beta_\nu\partial_- x^\nu)\right]. \quad (6.1.1)$$

with

$$F^{\alpha\beta}(x) = f^{\alpha\beta} + C^{\alpha\beta}_\mu x^\mu\,, \quad (F^{-1}(x))_{\alpha\beta} = (f^{-1})_{\alpha\beta} - (f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho x^\rho (f^{-1})_{\beta_1\beta}\,. \quad (6.1.2)$$

where this time we do not impose any conditions on tensors $f^{\alpha\beta}$ and $C^{\alpha\beta}_\mu$, except that $C^{\alpha\beta}_\mu$ is infinitesimal. Having done this we can focus on main point of this chapter T-duality.

6.2 T-dualization - general case

This section will deal with problem of obtaining T-dual theory and transformation laws that connect dual and original theory. Where these laws will be used in subsequent sections.

6.2.1 Implementation of the generalized T-dualization procedure

In every chapter until now where we dealth with T-duality we relied on either standard [36, 43] or generalized [46] Buscher procedures. This stemmed from the fact that actions were invariant to shift symmetry. Which, in case with coordinate dependent background fields, was a consequence of antisymmetry. Because we decided to work with field that has symmetric part thus rendering any invariance of to translations invalid, we can not rely on dualization methods we utilized before.

Thankfully, there has been development on this front and there are methods for obtaining T-duality in the cases with absence of shift symmetry [45]. In fact method that has been developed is mostly identical to generalized Buscher procedure, where we replace original action with auxiliary one which does possess translation invariance. Shape of this action is exactly the same as the form of action where translation symmetry was localized and gauge fixed. In order for this action to produce correct T-dual theory we need to be able to salvaged original action from it.

Following this philosophy we insert following substitutions into action (6.1.1) in order to make it invariant to translations

$$\partial_\pm x^\mu \to v^\mu_\pm, \quad (6.2.1)$$

$$x^\rho \quad \to \Delta V^\rho = \int_P d\xi'^m v^\rho_m(\xi'), \quad (6.2.2)$$

$$S \quad \to S + \frac{k}{2}\int_\Sigma d^2\xi\left[v^\mu_+\partial_- y_\mu - v^\mu_-\partial_+ y_\mu\right], \quad (6.2.3)$$

The result is auxiliary action convenient for T-dualization procedure

$$S_{aux} = k\int_\Sigma d^2\xi\Big[\Pi_{+\mu\nu}v^\mu_+ v^\nu_- + \frac{1}{2}(\partial_+\bar{\theta}^\alpha + v^\mu_+\bar{\Psi}^\alpha_\mu)\left(F^{-1}(\Delta V)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi^\beta_\nu v^\nu_-) + \frac{1}{2}(v^\mu_+\partial_- y_\mu - v^\mu_-\partial_+ y_\mu)\Big]. \quad (6.2.4)$$

The shape of this action is exactly the same as one we had when we worked with antisymmetric field. Let us note that path P starts from ξ_0 and ends in ξ. In this way action becomes non-local.

Finding equations of motion for Lagrange multipliers

$$\partial_- v_+^\mu - \partial_+ v_-^\mu = 0 \quad v_\pm^\mu = \partial_\pm x^\mu, \tag{6.2.5}$$

and inserting them into (6.2.2) we have

$$\Delta V^\rho = \int_P d\xi'^m \partial_m x^\rho(\xi') = x^\rho(\xi) - x^\rho(\xi_0) = \Delta x^\rho. \tag{6.2.6}$$

In absence of translational symmetry, in order to extract starting action from auxiliary one, we impose $x^\rho(\xi_0) = 0$ as a constraint. Taking all this into account, we get the starting action (6.1.1).

Euler-Lagrange equations of motion for gauge fields $v_\pm(\kappa)$ give the following ones

$$\begin{aligned} -\frac{1}{2}\partial_- y_\mu(\kappa) &= \Pi_{+\mu\nu} v_-^\nu(\kappa) + \frac{1}{2}\bar\Psi_\mu^\alpha \left(F^{-1}(\Delta V)\right)_{\alpha\beta} \left(\partial_-\theta^\beta(\kappa) + \Psi_\nu^\beta v_-^\nu(\kappa)\right) \\ -\frac{1}{2}\int_\Sigma d^2\xi \big[\partial_+\bar\theta^\alpha(\xi) &+ v_+^{\nu_1}(\xi)\bar\Psi_{\nu_1}^\alpha\big](f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta} N(\kappa^+)\big[\partial_-\theta^\beta(\xi) + \Psi_{\nu_2}^\beta v_-^{\nu_2}(\xi)\big], \end{aligned} \tag{6.2.7}$$

$$\begin{aligned} \frac{1}{2}\partial_+ y_\mu(\kappa) &= \Pi_{+\nu\mu} v_+^\nu(\kappa) + \frac{1}{2}(\partial_+\bar\theta^\alpha(\kappa) + v_+^\nu(\kappa)\bar\Psi_\nu^\alpha)\left(F^{-1}(\Delta V)\right)_{\alpha\beta}\Psi_\mu^\beta \\ -\frac{1}{2}\int_\Sigma d^2\xi \big[\partial_+\bar\theta^\alpha(\xi) &+ v_+^{\nu_1}(\xi)\bar\Psi_{\nu_1}^\alpha\big](f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta} N(\kappa^-)\big[\partial_-\theta^\beta(\xi) + \Psi_{\nu_2}^\beta v_-^{\nu_2}(\xi)\big]. \end{aligned} \tag{6.2.8}$$

Here, function $N(\kappa^\pm)$ is obtained from variation of term containing ΔV^ρ in expression for $F^{-1}(\Delta V)$ (details are presented in Appendix D). They represent the generalization of beta functions introduced in Ref.[97]

$$N(\kappa^+) = \delta\Big(\xi'^-\left((\xi'^+)^{-1}(\kappa^+)\right) - \kappa^-\Big)\left[\bar H(\xi^+ - \kappa^+) - H(\xi_0^+ - \kappa^+)\right], \tag{6.2.9}$$

$$N(\kappa^-) = \delta\Big(\xi'^+\left((\xi'^-)^{-1}(\kappa^-)\right) - \kappa^+\Big)\left[\bar H(\xi^- - \kappa^-) - H(\xi_0^- - \kappa^-)\right], \tag{6.2.10}$$

where more details on Dirac delta function and step function are given in Appendix A. As we see the expressions for derivatives of y_μ are more complex comparing with those in Chapter 4, where translational symmetry is present.

Assuming that $C_\mu^{\alpha\beta}$ is an infinitesimal, we can iteratively invert equations of motion (6.2.7) and (6.2.8) [14]. Separating variables into two parts, one finite and one infinitesimal proportional to $C_\mu^{\alpha\beta}$, we have

$$v_-^\nu(\kappa) = -\frac{1}{2}\breve{\Theta}_-^{\nu\nu_1}\Bigg\{\partial_- y_{\nu_1}(\kappa) + \bar{\Psi}^\alpha_{\nu_1}(F^{-1}(\Delta V))_{\alpha\beta}\partial_-\theta^\beta(\kappa)$$
$$+\frac{1}{2}\Psi^\alpha_{\nu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\alpha_2}_\rho\Delta V^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_{\nu_2}\breve{\Theta}_-^{\nu_2\nu_3}\Big(\partial_- y_{\nu_3}(\kappa)+\bar{\Psi}^{\beta_1}_{\nu_3}(f^{-1})_{\beta_1\beta}\partial_-\theta^\beta(\kappa)\Big)$$
$$-\int_\Sigma d^2\xi\Big[\partial_+\bar{\theta}^\alpha(\xi)+\frac{1}{2}\Big(\partial_+ y_{\mu_1}(\xi)-\partial_+\bar{\theta}^{\gamma_1}(\xi)(f^{-1})_{\gamma_1\gamma_2}\Psi^{\gamma_2}_{\mu_1}\Big)\breve{\Theta}_-^{\mu_1\nu_1}\bar{\Psi}^\alpha_{\nu_1}\Big](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_{\nu_1}$$
$$\times(f^{-1})_{\beta_1\beta}N(\kappa^+)\Big[\partial_-\theta^\beta(\xi)-\frac{1}{2}\Psi^\beta_{\nu_2}\breve{\Theta}_-^{\nu_2\mu_2}\Big(\partial_- y_{\mu_2}(\xi)+\bar{\Psi}^{\gamma_3}_{\mu_2}(f^{-1})_{\gamma_3\gamma_4}\partial_-\theta^{\gamma_4}(\xi)\Big)\Big]\Bigg\}, \tag{6.2.11}$$

$$v_+^\mu(\kappa) = \frac{1}{2}\breve{\Theta}_-^{\mu_1\mu}\Bigg\{\partial_+ y_{\mu_1}(\kappa)-\partial_+\bar{\theta}^\alpha(\kappa)(F^{-1}(\Delta V))_{\alpha\beta}\Psi^\beta_{\mu_1}$$
$$+\frac{1}{2}\Big(\partial_+ y_{\mu_2}(\kappa)-\partial_+\bar{\theta}^\alpha(\kappa)(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_{\mu_2}\Big)\breve{\Theta}_-^{\mu_2\mu_3}\bar{\Psi}^{\beta_3}_{\mu_3}(f^{-1})_{\beta_3\beta_2}C^{\beta_2\beta_1}_\rho\Delta V^\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_1}$$
$$+\int_\Sigma d^2\xi\Big[\partial_+\bar{\theta}^\alpha(\xi)+\frac{1}{2}\Big(\partial_+ y_{\mu_2}(\xi)-\partial_+\bar{\theta}^{\gamma_1}(\xi)(f^{-1})_{\gamma_1\gamma_2}\Psi^{\gamma_2}_{\mu_2}\Big)\breve{\Theta}_-^{\mu_2\nu_1}\bar{\Psi}^\alpha_{\nu_1}\Big](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_{\mu_1}$$
$$\times(f^{-1})_{\beta_1\beta}N(\kappa^-)\Big[\partial_-\theta^\beta(\xi)-\frac{1}{2}\Psi^\beta_{\nu_2}\breve{\Theta}_-^{\nu_2\mu_3}\Big(\partial_- y_{\mu_3}(\xi)+\bar{\Psi}^{\gamma_3}_{\mu_3}(f^{-1})_{\gamma_3\gamma_4}\partial_-\theta^{\gamma_4}(\xi)\Big)\Big]\Bigg\}. \tag{6.2.12}$$

Tensor $\breve{\Theta}_-^{\mu\nu}$ is inverse tensor to $\breve{\Pi}_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar{\Psi}^\alpha_\mu(f^{-1})_{\alpha\beta}\Psi^\beta_\nu$, which satisfy $\breve{\Theta}_-^{\mu\nu}\breve{\Pi}_{+\nu\rho} = \delta^\mu_\rho$ Whose properties are given in detail in equations (4.2.20), (4.2.21), (4.2.22). In above expressions ΔV is a quantity in the zeroth order in $C^{\alpha\beta}_\mu$, which has the same shape as we have encountered before

$$\Delta V^{(0)\rho} = \int d\xi^+ v_+^\rho + \int d\xi^- v_-^\rho$$
$$= \frac{1}{2}\int_P d\xi^+\breve{\Theta}_-^{\rho_1\rho}\left[\partial_+ y_{\rho_1}-\partial_+\bar{\theta}^\alpha(f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right]-\frac{1}{2}\int_P d\xi^-\breve{\Theta}_-^{\rho\rho_1}\left[\partial_- y_{\rho_1}+\bar{\Psi}^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right]. \tag{6.2.13}$$

Using (6.2.7) and (6.2.8) and inserting them into (6.2.4), we get T-dual action

$$
\begin{aligned}
{}^bS = k\int_P d^2\xi \Bigg[& \frac{1}{4}\breve{\Theta}_-^{\mu\nu}\partial_+ y_\mu \partial_- y_\nu \\
& + \frac{1}{8}\breve{\Theta}_-^{\mu\mu_1}\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho \Delta V^\rho (f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_1}\breve{\Theta}_-^{\nu_1\nu}\partial_+ y_\mu\partial_- y_\nu \\
& + \frac{1}{2}\partial_+\bar{\theta}^\alpha\Big((F^{-1}(\Delta V))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\alpha_2}_\rho\Delta V^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_\mu\breve{\Theta}_-^{\mu\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta} \\
& - \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\nu}\bar{\Psi}^{\beta_3}_\nu(f^{-1})_{\beta_3\beta_2}C^{\beta_2\beta_1}_\rho\Delta V^\rho(f^{-1})_{\beta_1\beta} \\
& - \frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\mu_1}\bar{\Psi}^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C^{\alpha_3\beta_3}_\rho\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\breve{\Theta}_-^{\nu_1\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\Big)\partial_-\theta^\beta \\
& + \frac{1}{4}\partial_+ y_\mu\breve{\Theta}_-^{\mu\mu_1}\bar{\Psi}^\alpha_{\mu_1}\Big((F^{-1}(\Delta V))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_3}_\rho\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\breve{\Theta}_-^{\nu_1\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\Big) \\
& \times\partial_-\theta^\beta \\
& - \frac{1}{4}\partial_+\bar{\theta}^\alpha\Big((F^{-1}(\Delta V))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\mu_1}\bar{\Psi}^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C^{\alpha_3\beta_1}_\rho\Delta V^\rho(f^{-1})_{\beta_1\beta}\Big)\Psi^\beta_{\nu_1}\breve{\Theta}_-^{\nu_1\nu} \\
& \times\partial_- y_\nu\Bigg].
\end{aligned}
\tag{6.2.14}
$$

Let us note that above we kept terms up to to the first order in $C^{\alpha\beta}_\mu$.

T-dual action contains all terms as initial action (4.1.17) up to the change $x^\mu \to y_\mu$. Consequently, T-dual background fields are of the form

$$
\begin{aligned}
{}^b\Pi_+^{\mu\nu} = & \frac{1}{4}\breve{\Theta}_-^{\mu\nu} + \frac{1}{8}\breve{\Theta}_-^{\mu\mu_1}\bar{\Psi}^\alpha_{\mu_1}\Big[(F^{-1}(\Delta V))_{\alpha\beta} + (f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho\Delta V^\rho(f^{-1})_{\beta_1\beta} \\
& - \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_{\mu_2}\breve{\Theta}_-^{\mu_2\nu_2}\bar{\Psi}^{\beta_1}_{\nu_2}(f^{-1})_{\beta_1\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\alpha_2}_\rho\Delta V^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_{\mu_2}\breve{\Theta}_-^{\mu_2\nu_2}\bar{\Psi}^{\beta_1}_{\nu_2}(f^{-1})_{\beta_1\beta} \\
& + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_{\mu_2}\breve{\Theta}_-^{\mu_2\nu_2}\bar{\Psi}^{\beta_3}_{\nu_2}(f^{-1})_{\beta_3\beta_2}C^{\beta_2\beta_1}_\rho\Delta V^\rho(f^{-1})_{\beta_1\beta} \\
& - \frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_{\mu_2}\breve{\Theta}_-^{\mu_2\mu_3}\bar{\Psi}^{\alpha_2}_{\mu_3}(f^{-1})_{\alpha_2\alpha_3}C^{\alpha_3\beta_3}_\rho\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_3}\breve{\Theta}_-^{\nu_3\nu_2}\bar{\Psi}^{\beta_1}_{\nu_2}(f^{-1})_{\beta_1\beta}\Big]\Psi^\beta_{\nu_1}\breve{\Theta}_-^{\nu_1\nu},
\end{aligned}
\tag{6.2.15}
$$

$$
\begin{aligned}
{}^b(F^{-1}(x))_{\alpha\beta} = & (F^{-1}(\Delta V))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\alpha_2}_\rho\Delta V^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_\mu\breve{\Theta}_-^{\mu\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta} \\
& - \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\nu}\bar{\Psi}^{\beta_3}_\nu(f^{-1})_{\beta_3\beta_2}C^{\beta_2\beta_1}_\rho\Delta V^\rho(f^{-1})_{\beta_1\beta} \\
& - \frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\mu_1}\bar{\Psi}^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C^{\alpha_3\beta_3}_\rho\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\breve{\Theta}_-^{\nu_1\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta},
\end{aligned}
\tag{6.2.16}
$$

$$
{}^b\bar{\Psi}^{\mu\alpha} = \frac{1}{2}\Theta_-^{\mu\nu}\bar{\Psi}^\alpha_\nu, \qquad {}^b\Psi_{\nu\beta} = -\frac{1}{2}\Psi^\beta_\mu\Theta_-^{\mu\nu}. \tag{6.2.17}
$$

Comparing background field of T-dual theory with background fields from Chapter 4 we immediately notice that background fields have become more complex. However, this is just an illusion. In both cases background field are exactly the same only difference is that here we did not introduce tensor $\bar{\Pi}_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar{\Psi}^\alpha_\mu F^{-1}(\Delta V)_{\alpha\beta}\Psi^\beta_\nu$ and its inverse, therefore we are missing ingredients to express our fields in more compactified format.

6.2.2 T-dualization of T-dual theory - general case

Having obtained T-dual theory we would like to check if this result is correct, best way to do this is to apply T-duality procedure again. Since the initial theory was not symmetric under translations, we had to introduce auxiliary action (6.2.4) which was invariant. This action produced T-dual theory which is invariant to translations of T-dual coordinates. Because of this we can dualize T-dual theory by generalized Buscher procedure. We start with the introduction of following substitutions

$$
\begin{aligned}
\partial_\pm y_\mu \to & D_\pm y_\mu = \partial_\pm y_\mu + u_{\pm\mu} \to D_\pm y_\mu = u_{\pm\mu}, && (6.2.18)\\
\Delta V^\rho \to & \Delta U^\rho, && (6.2.19)\\
\Delta U^\rho = & \frac{1}{2}\int_P d\xi^+ \breve{\Theta}_-^{\rho_1\rho}\left[u_{+\rho_1} - \partial_+\bar\theta^\alpha (f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] \\
& -\frac{1}{2}\int_P d\xi^- \breve{\Theta}_-^{\rho\rho_1}\left[u_{-\rho_1} + \bar\Psi^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right], && (6.2.20)\\
S \to & S + \frac{1}{2}(u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu). && (6.2.21)
\end{aligned}
$$

From the first line we see that gauge is fixed by choosing $y(\xi) = const$. Inserting these substitutions into (6.2.14) we obtain

$$
\begin{aligned}
{}^bS_{fix} = \kappa \int_P d^2\xi \Bigg[& \frac{1}{4}\breve{\Theta}_-^{\mu\nu} u_{+\mu}u_{-\nu} \\
& + \frac{1}{8}\breve{\Theta}_-^{\mu\mu_1}\bar\Psi^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho \Delta U^\rho (f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_1}\breve{\Theta}_-^{\nu_1\nu}u_{+\mu}u_{-\nu} \\
& + \frac{1}{2}\partial_+\bar\theta^\alpha\Big((F^{-1}(\Delta U))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\alpha_2}_\rho\Delta U^\rho (f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_\mu \breve{\Theta}_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu (f^{-1})_{\beta_1\beta} \\
& - \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu (f^{-1})_{\beta_1\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\nu}\bar\Psi^{\beta_3}_\nu (f^{-1})_{\beta_3\beta_2}C^{\beta_2\beta_1}_\rho\Delta U^\rho (f^{-1})_{\beta_1\beta} \\
& - \frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\mu_1}\bar\Psi^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C^{\alpha_3\beta_3}_\rho\Delta U^\rho (f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\breve{\Theta}_-^{\nu_1\nu}\bar\Psi^{\beta_1}_\nu (f^{-1})_{\beta_1\beta}\Big)\partial_-\theta^\beta \\
& + \frac{1}{4}u_{+\mu}\breve{\Theta}_-^{\mu\mu_1}\bar\Psi^\alpha_{\mu_1}\Big((F^{-1}(\Delta U))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_3}_\rho\Delta U^\rho (f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\breve{\Theta}_-^{\nu_1\nu}\bar\Psi^{\beta_1}_\nu (f^{-1})_{\beta_1\beta}\Big) \\
& \times \partial_-\theta^\beta \\
& - \frac{1}{4}\partial_+\bar\theta^\alpha\Big((F^{-1}(\Delta U))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\breve{\Theta}_-^{\mu\mu_1}\bar\Psi^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C^{\alpha_3\beta_1}_\rho\Delta U^\rho (f^{-1})_{\beta_1\beta}\Big)\Psi^\beta_{\nu_1}\breve{\Theta}_-^{\nu_1\nu} \\
& \times u_{-\nu} \\
& + \frac{1}{2}(u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu)\Bigg].
\end{aligned}
\tag{6.2.22}
$$

Using equations of motion for Lagrange multipliers, we return to the T-dual action. Finding equations of motion for gauge fields, we have

$$
\begin{aligned}
u_{+\mu}(\kappa) = \; & 2\breve{\Pi}_{+\nu\mu}\partial_+ x^\nu(\kappa) - \partial_+ x^\nu(\kappa)\bar{\Psi}^\alpha_\nu (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\rho \Delta x^\rho (f^{-1})_{\beta_1\beta}\Psi^\beta_\mu \\
& + \partial_+\bar{\theta}^\alpha(\kappa)(F^{-1}(\Delta x))_{\alpha\beta}\Psi^\beta_\mu \\
& - \int_\Sigma d^2\xi \Big(\partial_+\bar{\theta}^\alpha(\xi) + \partial_+ x^{\mu_1}(\xi)\bar{\Psi}^\alpha_{\mu_1}\Big)(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta} N(\kappa^-) \\
& \times \Big(\partial_-\theta^\beta(\xi) + \Psi^\beta_\nu \partial_- x^\nu(\xi)\Big), \qquad (6.2.23) \\
u_{-\nu}(\kappa) = & - 2\breve{\Pi}_{+\nu\mu}\partial_- x^\mu(\kappa) + \bar{\Psi}^\alpha_\nu (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\rho \Delta x^\rho (f^{-1})_{\beta_1\beta}\Psi^\beta_\mu \partial_- x^\mu(\kappa) \\
& - \bar{\Psi}^\alpha_\nu (F^{-1}(\Delta x))_{\alpha\beta}\partial_-\theta^\beta(\kappa) \\
& + \int_\Sigma d^2\xi \Big(\partial_+\bar{\theta}^\alpha(\xi) + \partial_+ x^\mu(\xi)\bar{\Psi}^\alpha_\mu\Big)(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\nu (f^{-1})_{\beta_1\beta} N(\kappa^-) \\
& \times \Big(\partial_-\theta^\beta(\xi) + \Psi^\beta_{\nu_1}\partial_- x^{\nu_1}(\xi)\Big). \qquad (6.2.24)
\end{aligned}
$$

Here we have that $\Delta x^\mu = x(\xi) - x(\xi_0)$, and inserting these equations into the gauge fixed action, keeping all terms linear with respect to $C^{\mu\nu}_\rho$ and selecting ξ_0 such that $x(\xi_0) = 0$, we obtain our original action (4.1.17).

6.3 Non-commutative relations - general case

In order to find Poisson structure of T-dual theory we will be using relations (6.2.7) and (6.2.8) by expressing them in terms of coordinates and momenta of the initial theory. While this method worked flawlessly in previous chapters, here we would still be left with terms containing $\partial_\tau x^\mu(\xi)$ which come from function $N(\xi^\pm)$. This means that it is impossible to find same τ Poisson brackets. One way to circumvent this is by first using equations of motion for coordinate $x^\mu(\xi)$ and then replacing remaining $\partial_\tau x^\mu$ term with canonical momentum. By doing all the steps that were outlined, we have following relationship between T-dual coordinate and variables of starting theory

$$
\begin{aligned}
&\partial_\sigma y_\nu(\sigma) \cong 2B_{\nu\mu}\partial_\sigma x^\mu - G_{\nu\mu}(\breve{\Pi}_+ + \breve{\Pi}^T_+)^{-1\nu_1\mu}\Bigg[\frac{\pi_{\nu_1}}{k} - \frac{1}{2}\bar{\Psi}^\alpha_{\nu_1}\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta \\
&- \frac{1}{2}\partial_+\bar{\theta}^\alpha\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_{\nu_1} - \Big[\Pi_{+\mu_1\mu_2} + \frac{1}{2}\bar{\Psi}^\alpha_{\mu_1}\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_{\mu_2}\Big](\delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2})\partial_\sigma x^{\nu_2} \\
&+ \frac{1}{2}\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho x^\rho(\sigma)(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_2}(\delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2})(\breve{\Pi}_+ + \breve{\Pi}^T_+)^{-1\rho\nu_2} \\
&\times \left[\frac{\pi_\rho}{k} - \frac{1}{2}\bar{\Psi}^\gamma_\rho(f^{-1})_{\gamma\gamma_1}\partial_-\theta^{\gamma_1} - \frac{1}{2}\partial_+\bar{\theta}^\gamma(f^{-1})_{\gamma\gamma_1}\Psi^{\gamma_1}_\rho + \breve{\Pi}_{+\rho\rho_1}\partial_\sigma x^{\rho_1} - \breve{\Pi}_{+\rho_1\rho}\partial_\sigma x^{\rho_1}\right]\Bigg]. \qquad (6.3.1)
\end{aligned}
$$

To find Poisson bracket between T-dual coordinates, we can start by finding Poisson bracket of sigma derivatives of T-dual coordinates and then integrating twice (see Appendix B). Implementing this procedure and utilizing Poisson brackets of original theory

$$
\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu_\nu\delta(\sigma - \bar{\sigma}), \quad \{x^\mu(\sigma), x^\nu(\bar{\sigma})\} = 0, \quad \{\pi_\mu(\sigma), \pi_\nu(\bar{\sigma})\} = 0. \qquad (6.3.2)
$$

we have that Poisson bracket for sigma derivatives is given as

$$
\begin{aligned}
&\{\partial_{\sigma_1} y_{\nu_1}(\sigma_1), \partial_{\sigma_2} y_{\nu_2}(\sigma_2)\} \cong \\
&= \frac{2}{k}(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_1\mu_2}\Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}\partial_{\sigma_2}\delta(\sigma_1-\sigma_2) - B_{\nu_1\mu_1}G_{\nu_2\mu_2}\partial_{\sigma_1}\delta(\sigma_1-\sigma_2)\Big] \\
&+ \frac{1}{k}\bar{\Psi}^\alpha_{\nu_3}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_3\mu_1}(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_4\mu_2} \\
&\times \Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}x^\rho(\sigma_1)\partial_{\sigma_2}\delta(\sigma_1-\sigma_2) - B_{\nu_1\mu_1}G_{\nu_2\mu_2}x^\rho(\sigma_2)\partial_{\sigma_1}\delta(\sigma_1-\sigma_2)\Big].
\end{aligned}
\tag{6.3.3}
$$

Integrating with respect to σ_1 (σ_2), where we set boundaries as σ_0 ($\bar{\sigma}_0$) and σ ($\bar{\sigma}$). Extracting only Poisson bracket terms that contain σ and $\bar{\sigma}$, we have

$$
\begin{aligned}
&\{y_{\nu_1}(\sigma), y_{\nu_2}(\bar{\sigma})\} \cong \frac{2}{k}(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_1\mu_2}\Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2} + B_{\nu_1\mu_1}G_{\nu_2\mu_2}\Big]H(\sigma-\bar{\sigma}) \\
&+ \frac{1}{k}\bar{\Psi}^\alpha_{\nu_3}(f^{-1})_{\alpha\alpha_1}G_\rho^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_3\mu_1}(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_4\mu_2} \\
&\times \Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}x^\rho(\bar{\sigma}) + B_{\nu_1\mu_1}G_{\nu_2\mu_2}x^\rho(\sigma)\Big]\bar{H}(\sigma-\bar{\sigma}).
\end{aligned}
\tag{6.3.4}
$$

Here, $\bar{H}(\sigma-\bar{\sigma})$ is same step function defined in Appendix A. It follows from definition of step functions we have that Poisson brackets are zero for $\sigma=\bar{\sigma}$. However, in cases where string in curled around compactified dimension, that is cases where $\sigma-\bar{\sigma}=2\pi$, we have following situation

$$
\begin{aligned}
&\{y_{\nu_1}(\sigma+2\pi), y_{\nu_2}(\sigma)\} \cong \frac{2}{k}(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_1\mu_2}\Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2} + B_{\nu_1\mu_1}G_{\nu_2\mu_2}\Big] \\
&+ \frac{1}{k}\bar{\Psi}^\alpha_{\nu_3}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_3\mu_1}(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_4\mu_2} \\
&\times \Big[4\pi G_{\nu_1\mu_1}B_{\nu_2\mu_2}N^\rho + \big(G_{\nu_1\mu_1}B_{\nu_2\mu_2} + B_{\nu_1\mu_1}G_{\nu_2\mu_2}\big)x^\rho(\sigma)\Big].
\end{aligned}
\tag{6.3.5}
$$

We used fact that $\bar{H}(2\pi)=1$. As we had before, symbol N^μ denotes winding number around compactified coordinate, if is defined as

$$
x^\mu(\sigma+2\pi) - x^\mu(\sigma) = 2\pi N^\mu. \tag{6.3.6}
$$

Putting $x^\mu(\sigma)=0$ we have that Poisson bracket has linear dependence on winding number. In cases where we don't have any winding number, we still have non-commutativity that is proportional to background fields.

Using the expression for sigma derivative of y_ν (6.3.1) and expression for Poisson bracket of sigma derivatives (6.3.3), we can find non-associative relations. Procedure is the same as for finding Poisson brackets of T-dual theory, we find Poisson bracket of sigma derivatives and integrate with respect to sigma coordinate, this time integral is done trice. Going along with this procedure we have following final result

$$
\begin{aligned}
&\{y_\nu(\sigma), \{y_{\nu_1}(\sigma_1), y_{\nu_2}(\sigma_2)\}\} \cong \frac{G_{\nu\mu}}{k^2}(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\rho\mu} \\
&\times \bar{\Psi}^\alpha_{\nu_3}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_3\mu_1}(\breve{\Pi}_+ + \breve{\Pi}_+^T)^{-1\mu_4\mu_2} \\
&\times \Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}\bar{H}(\sigma-\sigma_2) + B_{\nu_1\mu_1}G_{\nu_2\mu_2}\bar{H}(\sigma-\sigma_1)\Big]\bar{H}(\sigma_1-\sigma_2).
\end{aligned}
\tag{6.3.7}
$$

Since Jacobi identity is non-zero for T-dual theory we have that coordinate dependent RR field produces non-associative theory. However putting $\sigma_1 = \sigma_2 = \bar{\sigma}$ and $\sigma = \bar{\sigma} + 2\pi$ we have that Jacobi identity disappears

$$\{y_\nu(\bar{\sigma} + 2\pi), \{y_{\nu_1}(\bar{\sigma}), y_{\nu_2}(\bar{\sigma})\}\} \cong 0. \tag{6.3.8}$$

Comparing these non-commutative relations with ones obtained in Chapter 4, we can observe that theory now has both different non-commutative and non-associative structure.

7. Conclusion

At the end of this thesis we would like to present general summary of the work that has been done thus far, as well as possibility of future extension of said work. It should also be stated that results that have been presented, while original, are natural progression of work that has been done by string field community and that they should not be examined in isolation. With such rich history and vastness that accompanies string theory, author hopes that work exhibited in this thesis was more or less self contained and that it did not leave too many readers confused.

Work done in this thesis, while covering few different topics, could best be examined by splitting them in two main groups, first group that focuses on work based on bosonic string and second where we focus on work based on type II supertring. Even though both groups are connected with overarching methodologies and some results can be carried over from one group to other, this split is best thought of as ideological one where we make separation between theory which is toy model and one which has a chance for describing real world. Methodology that connected these two groups was the Buscher [36, 43] procedure as well as its derivatives [46, 45]. This procedure was instrumental in obtaining T-dual theory as well as transformation laws that connected coordinates of original and dual theories. Tasked with such monumental task, it is a miracle that this procedure is rather simple, where whole procedure can be summarised in few steps. First step was to examine if action was invariant to translational symmetry and if it was then second step would be to localize this symmetry. Localization was done by interchanging all partial derivatives with covariant ones. This way we introduced additional gauge fields which, in order to obtain correct T-dual theory, would have to be eliminated. Elimination of gauge fields marks the beginning of third step and this is done by introducing Lagrange multiplier term into action. Following this, fourth step is utilizing gauge freedom to fix starting coordinates to constants which leaves action that only depends on gauge fields and Lagrange multipliers. Finding equations of motion for Lagrange multipliers and gauge fields, where inserting the latter into action, we obtain T-dual theory. There are two main extensions to this procedure, case when background fields depend on background fields and case when we do not have translational symmetry. First case can be dualized by intoducing invariant coordinate in step two, this coordinate is given as a path integral of covariant derivative and with its introduction T-dual theory becomes non-local. Second one of these extensions is accomplished by neglecting steps one, tow and three and introducing auxiliary action which is invariant to translations. By sheer luck, shape of auxiliary is the same one as of the action which has been gauge fixed. Both extensions to procedure produce non-local T-dual theory.

First major topic that was examined in this thesis was bosonic string with coordinate dependent Kalb-Ramond field. While this theory has been examined before by many different authors [10, 14, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83], it was not sufficiently examined with apparatus of Buscher procedure. Even though this analysis is original it should be best thought out as a stepping stone for more physically relevant case, stepping stone where we get to familiarize ourselves more with generalized Bushcer procedure. In Chapter 2 we began our work on this theory by performing T-duality first along coordinates on which background fields did not depend, namely x and y coordinates and finally dualizing along direction on which Kalb-Ramond field depended, z coordinate. By doing T-duality in this order we have gone through three distinct theories which had different geometric interpretations. After first dualization we obtained twisted torus, theory which was well defined both locally and globally. Second dualization produces theory which was named torus with Q flux and while this theory was locally well defined we could not say the same for its global structure. Final dualization produced theory which was non-local and non-commutative, theory with R flux. By examining transformation laws of final theory we were able to obtain Poisson brackets between T-dual coordinates which possessed both non-commutative and non-associative properties.

Having saw what properties emerge in fully dualized bosonic string with coordinate dependent Kalb-Ramond field, in Chapter 3 we examined if can obtain these properties earlier in dualization chain by altering order of T-duality. We focused on duality chain that starts with z coordinate and finishes with x coordinate. This way we showed that right from the first T-duality we obtain theory that is non-local, however this theory was still commutative. Only after second dualization, we obtained one half of non-commutative relations and after finding fully dualized theory all non-commutative relations were salvaged. By obtaining T-duality by two different directions we observed that non-commutativity is only possible after performing dualization along coordinate on which background fields depend. This way only T-dual coordinates of ones that appear in background fields are non-commutative.

Having exhausted bosonic string case, remainder of this thesis was focused on type II superstring in pure spinor formalism [26, 27, 28, 29] with coordinate dependent Ramond-Ramond field. All remaining fields were constants or set to zero. This choice of the fields was motivated by papers [31, 13] where it has been speculated that this exact combination of fields would produce non-commutative relations between fermionic coordinates that are proportional to bosonic ones. In order to find transformation laws and T-dual theory we utilized two additional assumptions, first was that Ramond-Ramond field demended only infinitesimally on coordinates and second one was that term which contained coordinate was antisymmetric. Whith all these assumptions we have conducted T-duality of bosonic coordinates in Chapter 4. Since background field depended on all bosonic coordinates we had to utilize generalized Buscher procedure and in turn obtained T-dual theory was non-local. Transformation laws that we obtained we combined with momenta of original theory in order to transcribe them in canonical form. Having transcribed dual coordiantes as linear combination of momenta and coordinates of original theory we found non-commutative relations of T-dual theory. By enforcing special conditions on world-sheet coordinate σ, we found out how non-commutativity depends on winding numbers N_μ. It should also be mentioned that in this chapter we subjected T-dual theory under dualization of dual coordinates, this way we were able to obtain original theory in turn giving us confirmation that T-dualization was carried on correctly.

After examining bosonic duality we focused on fermionic duality of type II superstring. Having seen, in case of bosonic string, that non-commutativity emerges only when we dualize along coordinate on which background fields depend we wanted to see if same rule applies in fermionic case. In Chapter 5 we first carried out fermionic T-duality of previously non-dualized theory where we found out that this new theory is commutative. Next, we dualized theory which has already been dualized along bosonic coordinates obtaining fully dual theory. In this case we had emergence of two new Poisson brackets, brackets between dual bosonic and dual fermionic coordinates. However we have not been able to prove that hypothesis from papers [31, 13] is correct. There were no additional Poisson brackets between two fermionic coordinates. Finally, for completion sake, we have also conducted bosonic dualization of theory that has been only dualized along bosonic coordinates, proving that these two T-dualities commute.

Final part of this thesis focused again on bosonic duality of type II superstring but with one less assumption imposed on Ramond-Ramond field. We examined how properties of transformation laws and dual theory when field has both symmetric and antisymmetric part. This small modification removed translational invariance of the theory and made us rely on extension of Buscher procedure that works on such cases. Overcoming this minor setback by working with auxiliary action, we proceed to find transformation laws. By introducing symmetric part of the field we were not able to obtain $\beta^{\pm}$ functions that have appeared in previous dualizations but have instead derived their generalization, $N(\kappa^{\pm})$ functions. These functions were not only dependent of coordinates and their derivatives like their predecessor but were also dependent on choice of path in line integral that appeared in definition of invariant coordinate. This sudden increase in complexity made it impossible to obtain canonical transformation laws in manner that was similar as before. We had to rely both on equations of motion as well as canonical momenta of original theory. Having obtained canonical transformation laws we were finally able to deduce non-commutative relations of T-dual theory which, when compared to case with antisymmetric field, are now different. On the other hand, since neither $\beta^{\pm}$ nor $N(\kappa^{\pm})$ functions do not play any role in T-dual theory, T-dual action that was obtained was the same as in Chapter 4. Similarly as we did with antisymmetric field, we also performed T-duality of already dualized theory where we were able to return to starting action.

Having done all this work, following question arises: What to do next? There are two main directions that extension to this work can follow. First extension focuses on bosonic string where instead of following standard wisdom and working only with coordinate dependent Kalb-Ramond field we could incorporate also coordinate dependent space-time metric. This extension would produce transformation laws that depend on $N(\kappa^{\pm})$ functions thus making it possible to obtain non-commutativity for such field configurations. Furthermore this line or research would be able to shed some light on type of changes that affect nontrivial space-time metric after T-duality. Other possible line of research is focused on type II superstring where we also include fermionic coordinates in background fields. Since background fields are all interconnected it is reasonable to expect that such inclusion would produce configuration where no field can remain constant. Having seen what are the requirements for non-commutativity, we wager that such configuration would certainly produce non-commutative relations between fermionic coordinates. However, we are not certain what would be the form these relations take.

A. Light-cone coordinates

Throughout this thesis we have often relied on usage of light-cone (lc) coordinates. Here we will give basic overview of both lc coordinates and some tensors expressed in this basis. We begin by defining lc coordinates as

$$\xi^{\pm} = \frac{1}{2}(\tau \pm \sigma). \tag{A.0.1}$$

This definition naturally lends itself to introduction of corresponding partial derivatives

$$\partial_{\pm} \equiv \frac{\partial}{\partial \xi^{\pm}} = \partial_\tau \pm \partial_\sigma \tag{A.0.2}$$

With light-cone coordinates and their partial derivatives defined we can cast our gaze on two tensors that endow string theory world-sheet, two dimensional Levi-Civita tensor and world-sheet metric tensor.

We begin first with two dimensional Levi-Civita tensor ϵ^{mn} which is defined in (τ, σ) basis as $\epsilon^{\tau\sigma} = -1$. Consequently, in light-cone basis the form of this tensor is

$$\epsilon_{lc} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{A.0.3}$$

On the other hand the flat world-sheet metric is of the following form in (τ, σ) and light-cone basis. respectively

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta_{lc} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{A.0.4}$$

In both cases subscript lc denotes light-cone basis.

B. Poisson brackets

Throughout this thesis, we have seen that T-dual transformation laws connect derivatives of T-dual coordinates with coordinates and momenta of initial theory. While initial theory satisfies standard Poisson brackets, in order to find Poisson brackets for T-dual theory, we first need to find Poisson brackets between σ derivatives of T-dual coordinates. This type of Poisson bracket will, in general case, be some function of initial coordinates, Dirac delta functions and their derivatives with respect to σ. Having this in mind, general case for our Poisson brackets will have following form

$$\{A'(\sigma), B'(\bar{\sigma})\} = U'(\sigma)\delta(\sigma-\bar{\sigma}) + V(\sigma)\delta'(\sigma-\bar{\sigma})\,, \tag{B.0.1}$$

where $\delta'(\sigma-\bar{\sigma}) \equiv \partial_\sigma \delta(\sigma-\bar{\sigma})$. For terms $A'(\sigma)$, $U'(\sigma)$ and $B'(\bar{\sigma})$, symbol $'$ stands for partial derivative with respect to σ and $\bar{\sigma}$, respectively. If we want to calculate the Poisson bracket

$$\{A(\sigma), B(\bar{\sigma})\}\,,$$

first we have to calculate the following one

$$\{\Delta A(\sigma,\sigma_0), \Delta B(\bar{\sigma},\bar{\sigma}_0)\}\,, \tag{B.0.2}$$

where

$$\Delta A(\sigma,\sigma_0) = \int_{\sigma_0}^{\sigma} dx A'(x) = A(\sigma) - A(\sigma_0)\,, \quad \Delta B(\bar{\sigma},\bar{\sigma}_0) = \int_{\bar{\sigma}_0}^{\bar{\sigma}} dx B'(x) = B(\bar{\sigma}) - B(\bar{\sigma}_0)\,. \tag{B.0.3}$$

Substituting the expressions (B.0.3) into (B.0.2), we have

$$\{\Delta A(\sigma,\sigma_0), \Delta B(\bar{\sigma},\bar{\sigma}_0)\} = \int_{\sigma_0}^{\sigma} dx \int_{\bar{\sigma}_0}^{\bar{\sigma}} dy \;\left[U'(x)\delta(x-y) + V(x)\delta'(x-y)\right]\,. \tag{B.0.4}$$

After integration over y we get

$$\begin{aligned}&\{\Delta A(\sigma,\sigma_0), \Delta B(\bar{\sigma},\bar{\sigma}_0)\} = \\ &= \int_{\sigma_0}^{\sigma} dx \{U'(x)\left[\bar{H}(x-\bar{\sigma}_0) - \bar{H}(x-\bar{\sigma})\right] + V(x)\left[\delta(x-\bar{\sigma}_0) - \delta(x-\bar{\sigma})\right]\},\end{aligned} \tag{B.0.5}$$

where we utilized following property of Dirac delta functions

$$\int_{\sigma_0}^{\sigma} d\eta f(\eta)\delta(\eta-\bar{\eta}) = f(\bar{\eta})\left[\bar{H}(\sigma-\bar{\eta}) - \bar{H}(\sigma_0-\bar{\eta})\right]\,, \tag{B.0.6}$$

and where $\bar{H}(x)$ is Heaviside step function defined as

$$\bar{H}(x) = \int_0^x d\eta \delta(\eta) = \frac{1}{2\pi}\left[x + 2\sum_{n\geq 1}\frac{1}{n}\sin(nx)\right] = \begin{cases} 0 & \text{if } x=0 \\ 1/2 & \text{if } 0<x<2\pi\,, \\ 1 & \text{if } x=2\pi \end{cases} \tag{B.0.7}$$

where $\delta(x)=\frac{1}{2\pi}\sum_{n\in Z}e^{inx}$. Finally, integrating over x, we obtain

$$
\begin{aligned}
&\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}= \\
&U(\sigma)[\bar H(\sigma-\bar\sigma_0)-\bar H(\sigma-\bar\sigma)]-U(\sigma_0)[\bar H(\sigma_0-\bar\sigma_0)-\bar H(\sigma_0-\bar\sigma)] \\
&-U(\bar\sigma_0)[\bar H(\sigma-\bar\sigma_0)-\bar H(\sigma_0-\bar\sigma_0)]+U(\bar\sigma)[\bar H(\sigma-\bar\sigma)-\bar H(\sigma_0-\bar\sigma)] \\
&+V(\bar\sigma_0)[\bar H(\sigma-\bar\sigma_0)-\bar H(\sigma_0-\bar\sigma_0]-V(\bar\sigma)[\bar H(\sigma-\bar\sigma)-\bar H(\sigma_0-\bar\sigma)].
\end{aligned}
\tag{B.0.8}
$$

From the last expression, using (B.0.3), we extract the searched Poisson bracket

$$
\{A(\sigma),B(\bar\sigma)\}=-[U(\sigma)-U(\bar\sigma)+V(\bar\sigma)]\bar H(\sigma-\bar\sigma)\,. \tag{B.0.9}
$$

In order to calculate Jacobiator we have to find Poisson brackets of type $\{y(\sigma),x(\bar\sigma)\}$, where $y(\sigma)$ is coordinate T-dual to initial one $x(\sigma)$. Having this in mind, we start with the following Poisson bracket

$$
\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}=\{\int_{\sigma_0}^{\sigma}d\eta y'(\eta),x(\bar\sigma)\}\,, \tag{B.0.10}
$$

and using T-dual transformation law in canonical form

$$
\pi\cong\kappa y'\,, \tag{B.0.11}
$$

we get

$$
\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}\cong\frac{1}{\kappa}\{\int_{\sigma_0}^{\sigma}d\eta\pi(\eta),x(\bar\sigma)\}\,, \tag{B.0.12}
$$

where $\pi(\sigma)$ is momentum canonically conjugated to the coordinate $x(\sigma)$. Initial theory is geometric one which variables satisfy standard Poisson algebra, so, the final result is of the form

$$
\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}\cong-\frac{1}{\kappa}\left[\bar H(\sigma-\bar\sigma)-\bar H(\sigma_0-\bar\sigma)\right]\to\{y(\sigma),x(\bar\sigma)\}\cong-\frac{1}{\kappa}\bar H(\sigma-\bar\sigma)\,. \tag{B.0.13}
$$

C. Obtaining $\beta_\mu^\pm$ terms

During our examination of both bosonic string theory and superstring theory we have several instances where we had to find variation of term that contained ΔV. Variation of this term gave rise to $\beta_\mu^\pm(V)$ functions in T-dual transformation laws. While In case of bosonic string theory we have slightly touched this issue, here we would like to give proper exhibition that this problem entails. Even through we focus only on superstring case procedure that will be presented is applicable to both cases, bosonic case and superstring case.

We decided to obtain $\beta_\mu^\pm(V)$ functions for superstring however, we will use following substitutions $\partial_+\bar\Theta^\alpha = \partial_+\bar\theta^\alpha + v_+^{\nu_1}\bar\Psi^\alpha_{\nu_1}$, $\partial_-\Theta^\beta = \partial_-\theta^\beta + \Psi^\beta_{\nu_2}v_-^{\nu_2}$, also we will use $F_{\alpha\beta\rho}$ to represent term containing infinitesimal constant, in order to bring this exposition as close as we can to bosonic string case.

$$\begin{aligned}
&\int_\Sigma d^2\xi\partial_+\bar\Theta^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_-\Theta^\beta = \int_\Sigma d^2\xi\epsilon^{mn}\partial_m\bar\Theta^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_n\Theta^\beta\\
&= \int_\Sigma d^2\xi\Big[\frac{1}{2}\epsilon^{mn}\partial_m\bar\Theta^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_n\Theta^\beta - \frac{1}{2}\epsilon^{mn}\partial_n\bar\Theta^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_m\Theta^\beta\Big]\\
&= -\frac{1}{2}\int_\Sigma d^2\xi\Big[\epsilon^{mn}\bar\Theta^\alpha F_{\alpha\beta\rho}\partial_m\Delta V^{(0)\rho}\partial_n\Theta^\beta - \epsilon^{mn}\partial_n\bar\Theta^\alpha F_{\alpha\beta\rho}\partial_m\Delta V^{(0)\rho}\Theta^\beta\Big]\\
&= -\frac{1}{2}\int_\Sigma d^2\xi\epsilon^{mn}\partial_m\Delta V^{(0)\rho}\Big[\bar\Theta^\alpha F_{\alpha\beta\rho}\partial_n\Theta^\beta - \partial_n\bar\Theta^\alpha F_{\alpha\beta\rho}\Theta^\beta\Big]\\
&= -\frac{1}{2}\int_\Sigma d^2\xi\epsilon^{mn}v_m^\rho\Big[\bar\Theta^\alpha F_{\alpha\beta\rho}\partial_n\Theta^\beta - \partial_n\bar\Theta^\alpha F_{\alpha\beta\rho}\Theta^\beta\Big] = \int_\Sigma d^2\xi v_m^\rho\beta_\rho^m.
\end{aligned} \tag{C.0.1}$$

Variation with respect to gauge field $v_\pm^\rho$, and setting $F_{\alpha\beta\rho} = -(f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}$ produces desired $\beta_\rho^\pm$ functions (4.2.14), (4.2.15) in equations of motion (4.2.13). On the other hand setting $F_{\alpha\beta\rho}$ to $2H$ (factor of 2 comes from the fact that term containing Kalb-Ramond field was already transcribed as sum of two parts, effectively having two identical terms to one given in first line of above equation), $\bar\Theta^\alpha$ to γ_1 and Θ^α to γ_2 we obtain $\beta^\pm$ functions for bosonic case (2.2.53). Fermionic beta functions are obtained in exactly the same manner where only difference stems from using gauge fields $\bar u_\pm$ and $u_\pm$ instead of $v_\pm$.

Here we have used the property that $F_{\alpha\beta\rho}$, is antisymmetric under exchange of α and β, this, in combination with the fact that we can express $\partial_+\bar\Theta\partial_-\Theta$ as $\epsilon^{nm}\partial_n\bar\Theta\partial_m\Theta$, removes all terms proportional to $\partial_+\partial_-$, using identity $\epsilon^{mn}\bar\Theta\partial_m\partial_n\Theta = 0$. While surface terms disappear from the requirement that we are working with trivial topology

It should be noted that $\beta_{\pm\mu}(V)$ functions are not unique, we could have obtained different function simply by not using using symmetrization in (C.0.1). In case of non-symmetric $\beta_{\pm\mu}(V)$, all results that have been obtained would take a simpler form. We have chosen to work with antisymmetric function because results that are deduced from this case can be easily reduced, by neglecting terms, to simpler case.

D. Obtaining $N(\kappa^{\pm})$ terms

Functions $N(\kappa^{\pm})$ emerged in Chapter 6 during calculation of T-dual transformation laws as a consequence of variation of term that was proportional to ΔV. Here we will present derivation of this function.

$$\begin{aligned}
&\frac{\delta\left(F^{-1}(\Delta V)\right)_{\alpha\beta}}{\delta v_{+}^{\mu}(\kappa)} = -(f^{-1})_{\alpha\alpha_1} C_{\rho}^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_P d\xi'^m \frac{\delta v_m^{\rho}(\xi')}{\delta v_{+}^{\mu}(\kappa)} = \\
&= -(f^{-1})_{\alpha\alpha_1} C_{\mu}^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_P d\xi'^{+}\delta(\xi'^{+}-\kappa^{+})\delta(\xi'^{-}-\kappa^{-}) \\
&= -(f^{-1})_{\alpha\alpha_1} C_{\mu}^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_{t_i}^{t_f} dt\frac{d\xi'^{+}}{dt}\delta(\xi'(t)^{+}-\kappa^{+})\delta(\xi'^{-}(t)-\kappa^{-}) \\
&= -(f^{-1})_{\alpha\alpha_1} C_{\mu}^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_{\xi_0{}^{+}}^{\xi^{+}} du\delta(u-\kappa^{+})\delta\Big(\xi'^{-}\left((\xi'^{+})^{-1}(u)\right)-\kappa^{-}\Big) \\
&= -(f^{-1})_{\alpha\alpha_1} C_{\mu}^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\delta\Big(\xi'^{-}\left((\xi'^{+})^{-1}(\kappa^{+})\right)-\kappa^{-}\Big)\left[\bar{H}(\xi^{+}-\kappa^{+})-\bar{H}(\xi_0^{+}-\kappa^{+})\right] \\
&= -(f^{-1})_{\alpha\alpha_1} G_{\mu}^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta} N(\kappa^{+}).
\end{aligned} \tag{D.0.1}$$

In third line we have parametrized the path with parameter t where $\xi'^{+}(t_i) = \xi_0{}^{+}$ and $\xi'^{+}(t_f) = \xi^{+}$. In fourth line we introduced substitution $u = \xi'^{+}(t)$, in delta function this substitute is inverted. Fifth line is obtained by using following integration rule for Dirac delta function

$$\int_{\sigma_0}^{\sigma} d\eta f(\eta)\delta(\eta-\bar{\eta}) = f(\bar{\eta})\left[\bar{H}(\sigma-\bar{\eta})-\bar{H}(\sigma_0-\bar{\eta})\right]. \tag{D.0.2}$$

Here, $\bar{H}(x)$ is a step function defined in Appendix B equation (B.0.7).

Procedure for obtaining $N(\kappa^{-})$ is similar.

D.1 Properties of $N(\kappa^{\pm})$ terms

Here we will list some properties of $N(\kappa^{\pm})$ function.

These functions can be combined in same way as $\beta^{\pm}$ functions in order to get τ and σ representations

$$N(\kappa^{+}) + N(\kappa^{-}) = N(\kappa^{0}), \tag{D.1.1}$$

$$N(\kappa^{+}) - N(\kappa^{-}) = N(\kappa^{1}), \tag{D.1.2}$$

where κ^0 and κ^1 represent τ and σ coordinates respectevly

Acting with partial derivatives on $N(\kappa^+)$ ($N(\kappa^-)$) and integrating over world-sheet we have following relations

$$\int_\Sigma d^2\xi \partial_+ N(\kappa^+) = 1, \qquad \int_\Sigma d^2\xi \partial_- N(\kappa^+) = 0, \tag{D.1.3}$$

$$\int_\Sigma d^2\xi \partial_- N(\kappa^-) = 1, \qquad \int_\Sigma d^2\xi \partial_+ N(\kappa^-) = 0. \tag{D.1.4}$$

These relationships can be checked directly by applying partial derivatives to expressions from D. Here we give explicit calculations to first of these relation

$$\int_\Sigma d^2\xi \partial_+ N(\kappa^+) = \int_\Sigma d^2\xi \delta\Big(\xi'^-\big((\xi'^+)^{-1}(\kappa^+)\big) - \kappa^-\Big)\partial_+ \left[\bar{H}(\xi^+ - \kappa^+) - \bar{H}(\xi_0^+ - \kappa^+)\right]$$

$$= \int_\Sigma d^2\xi \delta\Big(\xi'^-\big((\xi'^+)^{-1}(\kappa^+)\big) - \kappa^-\Big)\delta(\xi^+ - \kappa^+) = \int_\Sigma d\xi^- \delta\Big(\xi'^-\big((\xi'^+)^{-1}(\xi^+)\big) - \kappa^-\Big). \tag{D.1.5}$$

At the beginning of this Appendix we had following parametrisation of path P : $\xi'^+(t_i) = {\xi_0}^+$ and $\xi'^+(t_f) = \xi^+$. Applying inverse parametrisation we have $(\xi'^+)^{-1}(\xi_0^+) = t_i$ and $(\xi'^+)^{-1}(\xi^+) = t_f$. With these we have

$$\int_\Sigma d\xi^- \delta\Big(\xi'^-\big((\xi'^+)^{-1}(\xi^+)\big) - \kappa^-\Big) = \int_\Sigma d\xi^- \delta\Big(\xi'^-\big(t_f\big) - \kappa^-\Big)$$

$$= \int_\Sigma d\xi^- \delta\Big(\xi^- - \kappa^-\Big) = 1. \tag{D.1.6}$$

Same rules apply for $N(\kappa^-)$, $N(\kappa^0)$ and $N(\kappa^1)$. In cases where $F^{-1}(x)_{\alpha\beta}$ is antisymmetric we can transfer partial derivatives from $\partial_\pm V^\mu$ to $N(\kappa^\pm)$ and obtain standard $\beta^\pm$ functions.

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На основу члана 29 Закона о општем управном поступку («Службени гласник РС» број 18/2016 и 95/2018), и члана 149 Статута Универзитета у Београду - Физичког факултета, по захтеву ДАНИЈЕЛА ОБРИЋА, мастер физичара, издаје се следеће

УВЕРЕЊЕ

ДАНИЈЕЛ ОБРИЋ, мастер физичар, дана 18. септембра 2023. године, одбранио је докторску дисертацију под називом

"T-DUALIZATION OF BOSONIC STRING AND TYPE IIB SUPERSTRING IN PRESENCE OF COORDINATE DEPENDENT BACKGROUND FIELDS" (Т-дуализација бозонске струне и тип IIB суперструне у пристуству координатно зависних позадинских поља)

пред Комисијом Универзитета у Београду - Физичког факултета и тиме испунио све услове за промоцију у ДОКТОРА НАУКА – ФИЗИЧКЕ НАУКЕ.

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Noncommutativity and Nonassociativity of Closed Bosonic String on T-dual Toroidal Backgrounds

B. Nikolić and D. Obrić*

In this article we consider closed bosonic string in the presence of constant metric and Kalb-Ramond field with one non-zero component, $B_{xy} = Hz$, where field strength H is infinitesimal. Using Buscher T-duality procedure we dualize along x and y directions and using generalized T-duality procedure along z direction imposing trivial winding conditions. After first two T-dualizations we obtain Q flux theory which is just locally well defined, while after all three T-dualizations we obtain nonlocal R flux theory. Origin of non-locality is variable ΔV defined as line integral, which appears as an argument of the background fields. Rewriting T-dual transformation laws in the canonical form and using standard Poisson algebra, we obtained that Q flux theory is commutative one and the R flux theory is noncommutative and nonassociative one. Consequently, there is a correlation between non-locality and closed string noncommutativity and nonassociativity.

1. Introduction

Coordinate noncommutativity means that there exists minimal possible length, which imposes natural UV cutoff. Idea of coordinate noncommutativity is very old. Heisenberg suggested coordinate noncommutativity to solve the problem of the occurrence of infinite quantities before renormalization procedure was developed and accepted. The first scientific paper considering this subject appeared 1947[1] where construction of discrete Lorentz invariant space-time is presented. Later in the period of 1980s A. Connes developed noncommutative geometry as a generalization of the standard commutative geometry.[2]

Noncommutativity became again interesting for particle physicists when the paper[3] appeared. In this article it is shown using propagators that open string endpoints in the presence of the constant metric and Kalb-Ramond field become noncommutative. D-brane on which the string endpoints are forced to move becomes noncommutative manifold. After this article many articles[4] appeared addressing the same subject but using different approaches - Fourier expansion, canonical methods, solving of boundary conditions etc.

In the last two articles of[4] the method of solving of boundary conditions is presented. The basic idea is that open string boundary condition is treated as canonical constraint. Investigating the consistency of the canonical constraint we obtained the σ dependent form of the boundary condition. Further, we can proceed twofold: to introduce Dirac brackets or solve the constraint. Solving the constraint, we obtained the initial coordinate as a linear combination of the effective coordinate and momenta. Consequently, initial coordinates are noncommutative and the main contribution to noncomutativity parameter comes from Kalb-Ramond field as it was expected.

Following the result of the article[5] it can be proven that gauge fields "live" at the open string endpoints. Consequently, many interesting papers concerning non-commutative Yang-Mills theories and their renormalisability appeared.[6] In the papers[7] cross sections for some decays, allowed in noncommutative Yang-Mills theories and forbidden in commutative ones, are calculated, which offers a possibility of the experimental check of the noncommutativity idea and further, indirectly, idea of strings.

It is obvious that closed bosonic string in the presence of constant background fields remains commutative. There are no boundaries and, consequently, boundary conditions constraining string dynamics. In the case of open string we obtained initial coordinate in the form of linear combination of effective coordinates and momenta using boundary condition. That is achieved in the closed string case[8] using T-duality procedure and coordinate dependent background.

T-duality as a fundamental feature of string theory,[9–15] unexperienced by point particle, makes that there is no physical difference between string theory compactified on a circle of radius R and circle of radius $1/R$. Buscher T-dualization procedure[10] represents a mathematical frame in which T-dualization is realized. If the background fields do not depend on some coordinates then those coordinates are isometry directions. Consequently, that symmetry can be localized replacing ordinary world-sheet derivatives $\partial_\pm$ by covariant ones $D_\pm x^\mu = \partial_\pm x^\mu + v_\pm^\mu$, where $v_\pm^\mu$ are gauge fields. In order to make T-dual theory has the same number of degrees of freedom, the new term with Lagrange multipliers is added to the action which forces the gauge fields to be unphysical degrees of freedom. Because of the shift symmetry, using gauge freedom we fix initial coordinates. Variation of this gauge fixed action with respect to the Lagrange multipliers

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produces initial action and with respect to the gauge fields produces T-dual action.

Standard Buscher T-dualization was applied in closed string case in the papers.[8,16–19] In Ref. [16] authors consider 3-torus in the presence of constant metric and Kalb-Ramond field with one nonzero component $B_{xy} = Hz$, where field strength H is infinitesimal. They systematically apply Buscher procedure and, after two T-dualizations along isometry directions, obtain theory with Q flux which is noncommutative. In the calculations they used nontrivial boundary conditions (winding conditions). The result is that T-dual closed string coordinates are noncommutative for the same values of parameters $\sigma = \bar{\sigma}$ with noncommutativity parameter proportional to field strength H and N_3, winding number for z coordinate.

But, except this standard Buscher procedure, there is a generalized Buscher procedure dealing with background fields depending on all coordinates. The generalized procedure was applied to the case of bosonic string moving in the weakly curved background[20–22] and in the case where metric is quadratic in coordinates and Kalb-Ramond field is linear function of coordinates.[23] The generalized procedure enables us to make T-dualization in mentioned cases along arbitrary subset of coordinates.

Double space is one picturesque framework for representation of T-duality. Double space is introduced two to three decades ago.[24–28] It is spanned by double coordinates $Z^M = (x^\mu, y_\mu)$ $(\mu = 0, 1, 2, \ldots, D-1)$, where x^μ are the coordinates of the initial theory and y_μ are T-dual coordinates. In this space T-dualization is represented as $O(d, d)$ transformation.[29–33] Permutation of the appropriate subsets of the initial and T-dual coordinates is interpreted as partial T-dualization[34,35] expanding Duff's idea.[24] The newly invented intrinsic noncommutativity[36] is related to double space. Intrinsic noncommutativity exists in the constant background case because it is considered within double space framework.

In this article we will deal with closed bosonic string propagating in the constant metric and linear dependent Kalb-Ramond field with $B_{xy} = Hz$, the same background as in [16]. This configuration is known in literature as torus with H-flux. As in the Ref. [16] we will use approximation of diluted flux, which means that in all calculations we keep constant and linear terms in infinitesimal field strength H. Transformation laws, relations which connect initial and T-dual variables, we will write in canonical form expressing initial momenta in terms of the T-dual coordinates. Unlike Ref. [16], except T-dualization along two isometry directions, we will make one step more and T-dualize along z coordinate using generalized T-dualization procedure. During dualization procedure we will use trivial boundary (winding) conditions.

Transformation laws in canonical form enable us to express sigma derivative of the T-dual coordinate as a linear combination of the initial momenta and coordinates. Because initial theory is geometrical locally and globally, its coordinates and canonically conjugated momenta satisfy standard Poisson algebra. This fact means that we can calculate the Poisson brackets of the T-dual coordinates using technical instruction given in subsection 4.1.

After T-dualizations along isometry directions (along x and y) we obtain the same background as in Ref. [16] but, obtained Q flux theory, which is still locally well defined, is commutative. This is a consequence of the imposed trivial winding conditions. Having in mind the generalized T-duality procedure,[20,21,23] T-dualization along z coordinate produces R flux nonlocal theory because it depends on the variable ΔV which is defined as line integral. Calculating Poisson brackets of the T-dual coordinates we obtain two nonzero Poisson brackets and show that there is a correlation between non-locality and closed string noncommutativity.

The form of noncommutativity is such that it exists when arguments of the coordinates are different, $\sigma \neq \bar{\sigma}$. That is another difference with respect to the result of Ref. [16] but there is no contradiction because the origins of noncommutativity are different. In this article non-locality is related with noncommutativity of R flux theory under trivial winding conditions while in Ref. [16] it is about noncommutativity of Q flux theory under nontrivial winding conditions.

From the noncommutativity relations it follows that Jacobi identity is broken i.e. nonassociativity occurs. Nonassociativity parameter, R flux, is proportional to the field strength H. Using generalized T-duality[20,21,23] we obtain the concrete form of nonassociativity from string dynamics. Similar as noncommutativity, discovery of nonassociativity pushes the scientists to explore the effects of nonassociativity in the field of renormalisability of ϕ^4 theory[37] as well as formulation of nonassociative gravity.[38]

At the end we add an appendix containing some conventions used in the paper.

2. Bosonic String Action and Choice of Background Fields

The action of the closed bosonic string in the presence of the space-time metric $G_{\mu\nu}(x)$, Kalb-Ramond antisymmetric field $B_{\mu\nu}(x)$, and dilaton scalar field $\Phi(x)$ is given by the following expression[9]

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \times \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\}, \tag{2.1}$$

where Σ is the world-sheet surface parameterized by $\xi^\alpha = (\tau, \sigma)$ $[(\alpha = 0, 1),\ \sigma \in (0, \pi)]$, while the D-dimensional space-time is spanned by the coordinates x^μ $(\mu = 0, 1, 2, \ldots, D-1)$. We denote intrinsic world sheet metric with $g_{\alpha\beta}$, and the corresponding scalar curvature with $R^{(2)}$.

In order to keep conformal symmetry on the quantum level background fields must obey space-time field equations[39]

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2 D_\mu a_\nu = 0, \tag{2.2}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2 a_\rho B^\rho{}_{\mu\nu} = 0, \tag{2.3}$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - R - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = c, \tag{2.4}$$

where c is an arbitrary constant. The function β^Φ could be a constant because of the relation

$$D^\nu \beta^G_{\nu\mu} + \partial_\mu \beta^\Phi = 0. \tag{2.5}$$

Further, $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to the space-time metric $G_{\mu\nu}$, while

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \quad a_\mu = \partial_\mu \Phi, \tag{2.6}$$

are field strength for Kalb-Ramond field $B_{\mu\nu}$ and dilaton gradient, respectively. Trivial solution of these equations is that all three background fields are constant. This case was pretty exploited in the analysis of the open string noncommutativity.

The less trivial case would be a case where some background fields are coordinate dependent. If we choose Kalb-Ramond field to be linearly coordinate dependent and dilaton field to be constant then the first equation (2.2) becomes

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0. \tag{2.7}$$

The field strength $B_{\mu\nu\rho}$ is constant and, if we assume that it is infinitesimal, then we can take $G_{\mu\nu}$ to be constant in approximation linear in $B_{\mu\nu\rho}$. Consequently, all three space-time field equations are satisfied. Especially, the third one is of the form

$$2\pi\kappa \frac{D-26}{6} = c, \tag{2.8}$$

which enables us to work in arbitrary number of space-time dimensions.

In this article we will work in $D = 3$ dimensions with the following choice of background fields

$$G_{\mu\nu} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.9}$$

where $R_\mu (\mu = 1, 2, 3)$ are radii of the compact dimensions. This choice of background fields is known in geometry as torus with flux (field strength) H.[16] Our choice of infinitesimal H can be understood in terms of the radii as that

$$\left(\frac{H}{R_1 R_2 R_3}\right)^2 = 0. \tag{2.10}$$

This approximation is known in literature as the approximation of diluted flux. Physically, this means that we work with the torus which is sufficiently large. Consequently, we can rescale the coordinates

$$x^\mu \longmapsto \frac{x^\mu}{R_\mu}, \tag{2.11}$$

which simplifies the form of the metric

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

The final form of the closed bosonic string action is

$$\begin{aligned} S &= \kappa \int_\Sigma d^2\xi\, \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \\ &= \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}(\partial_+ x \partial_- x + \partial_+ y \partial_- y + \partial_+ z \partial_- z) \right. \\ &\quad \left. + \partial_+ x Hz \partial_- y - \partial_+ y Hz \partial_- x\right], \end{aligned} \tag{2.13}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ is world-sheet derivative with respect to the light-cone coordinates $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$ and

$$x^\mu = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2.14}$$

Let us note that we do not write dilaton term because its T-dualization is performed separately within quantum formalism and here will be skipped.

3. T-dualization of the Bosonic Closed String Action

In this section we will perform T-dualization along three directions, one direction at time. Our goal is to find the relations connecting initial variables with T-dual ones called transformation laws. Using transformation laws we will find noncommutativity and nonassociativity relations.

3.1. T-dualization Along x Direction – from Torus with H Flux to the Twisted Torus

Let us perform standard Buscher T-dualization[10] of action (2.13) along x direction. Note that x direction is an isometry direction which means that action has a global shift symmetry, $x \longrightarrow x + a$. In order to perform Buscher procedure, we have to localize this symmetry introducing covariant world-sheet derivatives instead of the ordinary ones

$$\partial_\pm x \longrightarrow D_\pm x = \partial_\pm x + v_\pm, \tag{3.1}$$

where $v_\pm$ are gauge fields which transform as $\delta v_\pm = -\partial_\pm a$. Because T-dual action must have the same number of degrees of freedom as initial one, we have to make these fields $v_\pm$ be unphysical degrees of freedom. This is accomplished by adding following term to the action

$$S_{add} = \frac{\kappa}{2} \int_\Sigma d^2\xi\, y_1 (\partial_+ v_- - \partial_- v_+), \tag{3.2}$$

where y_1 is a Lagrange multiplier. After gauge fixing, $x = const.$, the action gets the form

$$\begin{aligned} S_{fix} = \kappa \int d^2\xi &\left[\frac{1}{2}(v_+ v_- + \partial_+ y \partial_- y + \partial_+ z \partial_- z) + v_+ Hz \partial_- y \right. \\ &\left. - \partial_+ y Hz v_- + \frac{1}{2} y_1 (\partial_+ v_- - \partial_- v_+)\right]. \end{aligned} \tag{3.3}$$

From the equations of motion for y_1 we obtain that field strength for the gauge field $v_\pm$ is equal to zero

$$F_{+-} = \partial_+ v_- - \partial_- v_+ = 0, \tag{3.4}$$

which gives us the solution for gauge field

$$v_\pm = \partial_\pm x. \tag{3.5}$$

Inserting this solution for gauge field into gauge fixed action (3.3) we obtain initial action given by Eq. (2.13). Equations of motion for $v_\pm$ will lead to the T-dual action. Varying the gauge fixed action (3.3) with respect to the gauge field v_+ we get

$$v_- = -\partial_- y_1 - 2Hz\partial_- y, \tag{3.6}$$

while on the equation of motion for v_- it holds

$$v_+ = \partial_+ y_1 + 2Hz\partial_+ y. \tag{3.7}$$

Inserting relations (3.6) and (3.7) into expression for gauge fixed action (3.3), keeping terms linear in H, we obtain the T-dual action

$${}_xS = \kappa \int_\Sigma d^2\xi\, \partial_+ ({}_xX)^\mu\, {}_x\Pi_{+\mu\nu} \partial_- ({}_xX)^\nu, \tag{3.8}$$

where subscript $_x$ denotes quantity obtained after T-dualization along x direction and

$${}_xX^\mu = \begin{pmatrix} y_1 \\ y \\ z \end{pmatrix}. \tag{3.9}$$

Further we have the T-dual background fields

$${}_x\Pi_{+\mu\nu} = {}_xB_{\mu\nu} + \frac{1}{2}\, {}_xG_{\mu\nu}, \quad {}_xB_{\mu\nu} = 0,$$

$${}_xG_{\mu\nu} = \begin{pmatrix} 1 & 2Hz & 0 \\ 2Hz & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.10}$$

Obtained background fields (3.10) define that what is known in literature as *twisted torus geometry*. String theory after one T-dualization is geometrically well defined globally and locally or, simply, theory is geometrical (flux H takes the role of connection).

Combining the solutions of equations of motion for Lagrange multiplier (3.5) and for gauge fields, (3.6) and (3.7), we get the transformation laws connecting initial, x^μ, and T-dual, ${}_xX^\mu$, coordinates

$$\partial_\pm x \cong \pm\partial_\pm y_1 \pm 2Hz\partial_\pm y, \tag{3.11}$$

where $\cong$ denotes T-duality relation. The momentum π_x is canonically conjugated to the initial coordinate x. Using the initial action (2.13) we get

$$\pi_x = \frac{\delta S}{\delta \dot{x}} = \kappa(\dot{x} - 2Hzy'), \tag{3.12}$$

where $\dot{A} \equiv \partial_\tau A$ and $A' \equiv \partial_\sigma A$. From transformation law (3.11) it is straightforward to obtain

$$\dot{x} \cong y_1' + 2Hzy', \tag{3.13}$$

which, inserted in the expression for momentum π_x, gives transformation law in canonical form

$$\pi_x \cong \kappa y_1'. \tag{3.14}$$

3.2. From Twisted Torus to Non-geometrical Q Flux

In this subsection we will continue the T-dualization of action (3.8) along y direction. After x and y T-dualization we obtain the structure which has local geometrical interpretation but global omissions. Such structure is known in literature as non-geometry.

We repeat the procedure from the previous subsection and form the gauge fixed action

$$S_{fix} = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}(\partial_+ y_1 \partial_- y_1 + v_+ v_- + \partial_+ z \partial_- z) + \partial_+ y_1 Hz v_- \right.$$
$$\left. + v_+ Hz \partial_- y_1 + \frac{1}{2} y_2 (\partial_+ v_- - \partial_- v_+) \right]. \tag{3.15}$$

From the equation of motion for Lagrange multiplier y_2

$$\partial_+ v_- - \partial_- v_+ = 0 \longrightarrow v_\pm = \partial_\pm y, \tag{3.16}$$

gauge fixed action becomes initial one (3.8). Varying the gauge fixed action (3.15) with respect to the gauge fields we get

$$v_\pm = \pm\partial_\pm y_2 - 2Hz\partial_\pm y_1. \tag{3.17}$$

Inserting these expressions for gauge fields into gauge fixed action, keeping the terms linear in H, gauge fixed action is driven into T-dual action

$${}_{xy}S = \kappa \int d^2\xi\, \partial_+ ({}_{xy}X)^\mu\, {}_{xy}\Pi_{+\mu\nu} \partial_- ({}_{xy}X)^\nu, \tag{3.18}$$

where

$$({}_{xy}X)^\mu = \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix},$$

$${}_{xy}\Pi_{+\mu\nu} = {}_{xy}B_{\mu\nu} + \frac{1}{2}\, {}_{xy}G_{\mu\nu} = \begin{pmatrix} \frac{1}{2} & -Hz & 0 \\ Hz & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{3.19}$$

Explicit expressions for background fields are

$${}_{xy}B_{\mu\nu} = \begin{pmatrix} 0 & -Hz & 0 \\ Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -B_{\mu\nu}, \quad {}_{xy}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.20}$$

Let us note that background fields obtained after two T-dualizations are similar to the geometric background of torus with H flux, but they should be considered only locally. Their global properties are non-trivial and because of that the term "non-geometry" is introduced.

Combining the equations of motion for Lagrange multiplier y_2 and for gauge fields $v_\pm$, we obtain T-dual transformation laws

$$\partial_\pm y \cong \pm\partial_\pm y_2 - 2Hz\partial_\pm y_1. \tag{3.21}$$

The y component of the initial canonical momentum π_y is a variation of the initial action with respect to the $\dot{y}$

$$\pi_y = \frac{\delta S}{\delta \dot{y}} = \kappa(\dot{y} + 2Hzx'). \tag{3.22}$$

Using T-dual transformation laws (3.21) we easily get

$$\dot{y} \cong y_2' - 2Hz\dot{y}_1, \tag{3.23}$$

while from the transformation law (3.11), at zeroth order in H, it holds $x' \cong \dot{y}_1$. Inserting last two expression into π_y we obtain transformation law in canonical form

$$\pi_y \cong \kappa y_2'. \tag{3.24}$$

After two T-dualizations along isometry directions, in the approximation of the diluted flux (keeping just terms linear in H), according to the canonical forms of the transformation laws (3.14) and (3.24), we see that T-dual coordinates y_1 and y_2 are still commutative. This is a consequence of the simple fact that variables of the initial theory, which is geometrical one, satisfy standard Poisson algebra

$$\left\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\right\} = \delta^\mu{}_\nu\delta(\sigma - \bar{\sigma}), \quad \{x^\mu, x^\nu\} = \left\{\pi_\mu, \pi_\nu\right\} = 0, \tag{3.25}$$

where

$$\pi_\mu = \begin{pmatrix} \pi_x \\ \pi_y \\ \pi_z \end{pmatrix}. \tag{3.26}$$

3.3. From Q to R Flux – T-dualization Along z Coordinate

In this subsection we will finalize the process of T-dualization dualizing along remaining z direction. For this purpose we will use generalized T-dualization procedure.[20,21,23] The result is a theory which is not well defined even locally and is known in literature as theory with R-flux.

We start with the action obtained after T-dualizations along x and y directions (3.18). The Kalb-Ramond field (3.20) depends on z and it seems that it is not possible to perform T-dualization. Let us assume that Kalb-Ramond field linearly depends on all coordinates, $B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho$ and check if some global transformation can be treated as isometry one. We start with global shift transformation

$$\delta x^\mu = \lambda^\mu, \tag{3.27}$$

and make a variation of action

$$\delta S = \frac{\kappa}{3}B_{\mu\nu\rho}\lambda^\rho\int_\Sigma d^2\xi\,\partial_+x^\mu\partial_-x^\nu$$
$$= \frac{2k}{3}B_{\mu\nu\rho}\lambda^\rho\epsilon^{\alpha\beta}\int_\Sigma d^2\xi[\partial_\alpha(x^\mu\partial_\beta x^\nu) - x^\mu(\partial_\alpha\partial_\beta x^\nu)]. \tag{3.28}$$

The second term vanishes explicitly, while the first term is surface one. Consequently, in the case of constant metric and linearly dependent Kalb-Ramond field, global shift transformation is an isometry transformation. This means that we can make T-dualization along z coordinate using generalized T-dualization procedure.

The generalized T-dualization procedure is presented in detail in Ref. [20]. In order to localize shift symmetry of the action (3.18) along z direction we introduce covariant derivative

$$\partial_\pm z \longrightarrow D_\pm z = \partial_\pm z + v_\pm, \tag{3.29}$$

which is a part of the standard Buscher procedure. The novelty is introduction of the invariant coordinate as line integral

$$z^{inv} = \int_P d\xi^\alpha D_\alpha z$$
$$= \int_P d\xi^+ D_+z + \int_P d\xi^- D_-z = z(\xi) - z(\xi_0) + \Delta V, \tag{3.30}$$

where

$$\Delta V = \int_P d\xi^\alpha v_\alpha = \int_P (d\xi^+ v_+ + d\xi^- v_-). \tag{3.31}$$

Here ξ and ξ_0 are the current and initial point of the world-sheet line P. At the end, as in the standard Buscher procedure, in order to make $v_\pm$ to be unphysical degrees of freedom we add to the action term with Lagrange multiplier

$$S_{add} = \frac{\kappa}{2}\int_\Sigma d^2\xi\; y_3(\partial_+v_- - \partial_+v_-). \tag{3.32}$$

The final form of the action is

$$\bar{S} = \kappa\int_\Sigma d^2\xi\Big[-Hz^{inv}(\partial_+y_1\partial_-y_2 - \partial_+y_2\partial_-y_1)$$
$$+\frac{1}{2}(\partial_+y_1\partial_-y_1 + \partial_+y_2\partial_-y_2 + D_+zD_-z)$$
$$+\frac{1}{2}\; y_3(\partial_+v_- - \partial_-v_+)\Big]. \tag{3.33}$$

Because of existing shift symmetry we fix the gauge, $z(\xi) = z(\xi_0)$, and then the gauge fixed action takes the form

$$S_{fix} = \kappa \int_\Sigma d^2\xi \Big[- H\Delta V(\partial_+ y_1 \partial_- y_2 - \partial_+ y_2 \partial_- y_1) + \frac{1}{2}(\partial_+ y_1 \partial_- y_1 + \partial_+ y_2 \partial_- y_2 + v_+ v_-) + \frac{1}{2} y_3 (\partial_+ v_- - \partial_- v_+) \Big]. \tag{3.34}$$

From the equation of motion for Lagrange multiplier y_3 we obtain

$$\partial_+ v_- - \partial_- v_+ = 0 \Longrightarrow v_\pm = \partial_\pm z, \quad \Delta V = \Delta z, \tag{3.35}$$

which drives back the gauge fixed action to the initial action (3.18). Varying the gauge fixed action (3.34) with respect to the gauge fields $v_\pm$ we get the following equations of motion

$$v_\pm = \pm\partial_\pm y_3 - 2\beta^\mp, \tag{3.36}$$

where $\beta^\pm$ functions are defined as

$$\beta^\pm = \pm\frac{1}{2} H(y_1 \partial_\mp y_2 - y_2 \partial_\mp y_1). \tag{3.37}$$

The $\beta^\pm$ functions are obtained as a result of the variation of the term containing ΔV

$$\delta_v \left(-2\kappa \int d^2\xi \varepsilon^{\alpha\beta} H \partial_\alpha y_1 \partial_\beta y_2 \Delta V \right) = \kappa \int d^2\xi \left(\beta^+ \delta v_+ + \beta^- \delta v_-\right), \tag{3.38}$$

using partial integration and the fact that $\partial_\pm V = v_\pm$. Inserting the relations (3.36) into the gauge fixed action, keeping linear terms in H, we obtain the T-dual action

$$_{xyz}S = \kappa \int_\Sigma d^2\xi \, \partial_{+\, xyz}X^\mu {}_{xyz}\Pi_{+\mu\nu} \partial_{-\, xyz}X^\nu, \tag{3.39}$$

where

$$_{xyz}X^\mu = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad {}_{xyz}\Pi_{+\mu\nu} = {}_{xyz}B_{\mu\nu} + \frac{1}{2} {}_{xyz}G_{\mu\nu}, \tag{3.40}$$

$$_{xyz}B_{\mu\nu} = \begin{pmatrix} 0 & -H\Delta\tilde{y}_3 & 0 \\ H\Delta\tilde{y}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad {}_{xyz}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.41}$$

Here we introduced double coordinate $\tilde{y}_3$ defined as

$$\partial_\pm y_3 \equiv \pm\partial_\pm \tilde{y}_3. \tag{3.42}$$

Let us note that ΔV stands beside field strength H, which implicates that, according to the diluted flux approximation, we calculate ΔV in the zeroth order in H

$$\Delta V = \int d\xi^+ \partial_+ y_3 - \int d\xi^- \partial_- y_3. \tag{3.43}$$

Having this into account it is clear why we defined double coordinate $\tilde{y}_3$ as in Eq. (3.42). Also it is useful to note that presence of ΔV, which is defined as line integral, represents the source of non-locality of the T-dual theory. the result of the three T-dualization is a theory with R flux as it is known in the literature.

Combining the equations of motion for Lagrange multiplier (3.35), $v_\pm = \partial_\pm z$, and equations of motion for gauge fields (3.36), we obtain the T-dual transformation law

$$\partial_\pm z \cong \pm\partial_\pm y_3 - 2\beta^\mp. \tag{3.44}$$

Adding transformation laws for $\partial_\pm z$ and $\partial_- z$ we get the transformation law for $\dot{z}$

$$\dot{z} \cong y_3' + H(y_1 y_2' - y_2 y_1'), \tag{3.45}$$

which enables us to write down the transformation law in the canonical form

$$y_3' \cong \frac{1}{\kappa}\pi_z - H(xy' - yx'). \tag{3.46}$$

Here we used the expression for the canonical momentum of the initial theory (2.13)

$$\pi_z = \frac{\delta S}{\delta \dot{z}} = \kappa \dot{z}. \tag{3.47}$$

4. Noncommutativity and Nonassociativity Using T-duality

In the open string case noncommutativity comes from the boundary conditions which makes that coordinates x^μ depend both on the effective coordinates and on the effective momenta.[4] Effective coordinates and momenta do not commute and, consequently, coordinates x^μ do not commute. In the closed bosonic string case the logic is the same but the execution is different. Using T-duality we obtained transformation laws, (3.11), (3.21) and (3.44), which relate T-dual coordinates with the initial coordinates and their canonically conjugated momenta. In this section we will use these relations to get noncommutativity and nonassociativity relations.

4.1. Noncommutativity Relations

Let us start with the Poisson bracket of the σ derivatives of two arbitrary coordinates in the form

$$\{A'(\sigma), B'(\bar{\sigma})\} = U'(\sigma)\delta(\sigma - \bar{\sigma}) + V(\sigma)\delta'(\sigma - \bar{\sigma}), \tag{4.1}$$

where $\delta'(\sigma-\bar\sigma)\equiv\partial_\sigma\delta(\sigma-\bar\sigma)$. In order to find the form of the Poisson bracket

$$\{A(\sigma),\ B(\bar\sigma)\},$$

we have to find the form of the Poisson bracket

$$\{\Delta A(\sigma,\sigma_0),\ \Delta B(\bar\sigma,\bar\sigma_0)\},$$

where

$$\Delta A(\sigma,\sigma_0)=\int_{\sigma_0}^{\sigma}dx A'(x)=A(\sigma)-A(\sigma_0),$$
$$\Delta B(\bar\sigma,\bar\sigma_0)=\int_{\bar\sigma_0}^{\bar\sigma}dx B'(x)=B(\bar\sigma)-B(\bar\sigma_0). \tag{4.2}$$

Now we have

$$\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=\int_{\sigma_0}^{\sigma}dx\int_{\bar\sigma_0}^{\bar\sigma}dy\ \left[U'(x)\delta(x-y)+V(x)\delta'(x-y)\right]. \tag{4.3}$$

After integration over y we get

$$\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=\int_{\sigma_0}^{\sigma}dx\left\{U'(x)\left[\theta(x-\bar\sigma_0)-\theta(x-\bar\sigma)\right]+V(x)\left[\delta(x-\bar\sigma_0)-\delta(x-\bar\sigma)\right]\right\}, \tag{4.4}$$

where function $\theta(x)$ is defined as

$$\theta(x)=\int_0^x d\eta\delta(\eta)=\frac{1}{2\pi}\left[x+2\sum_{n\geq1}\frac{1}{n}\sin(nx)\right]=\begin{cases}0 & \text{if } x=0\\ 1/2 & \text{if } 0<x<2\pi.\\ 1 & \text{if } x=2\pi\end{cases} \tag{4.5}$$

Integrating over x using partial integration finally we obtain

$$\begin{aligned}\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}&=U(\sigma)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma-\bar\sigma)]\\&-U(\sigma_0)[\theta(\sigma_0-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma)]-U(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0)]\\&+U(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)]+V(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0]\\&-V(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)].\end{aligned} \tag{4.6}$$

From the last expression, using the right-hand sides of the expressions in Eq. (4.2), we extract the desired Poisson bracket

$$\{A(\sigma),\ B(\bar\sigma)\}=-[U(\sigma)-U(\bar\sigma)+V(\bar\sigma)]\theta(\sigma-\bar\sigma). \tag{4.7}$$

Let us rewrite the canonical forms of the transformation laws, (3.14), (3.24) and (3.46), in the following way

$$y_1'\cong\frac{1}{\kappa}\pi_x,\quad y_2'\cong\frac{1}{\kappa}\pi_y,\quad y_3'\cong\frac{1}{\kappa}\pi_z-H(xy'-yx'). \tag{4.8}$$

In order to find the Poisson brackets between T-dual coordinates y_μ we will use the algebra of the coordinates and momenta of the initial theory (3.25). It is obvious that only nontrivial Poisson brackets will be $\{y_1(\sigma),\ y_3(\bar\sigma)\}$ and $\{y_2(\sigma),\ y_3(\bar\sigma)\}$.

Let us first write the corresponding Poisson brackets of the sigma derivatives of T-dual coordinates y_μ using (4.8)

$$\{y_1'(\sigma),\ y_3'(\bar\sigma)\}\cong\frac{2}{\kappa}Hy'(\sigma)\delta(\sigma-\bar\sigma)+\frac{1}{\kappa}Hy(\sigma)\delta'(\sigma-\bar\sigma), \tag{4.9}$$

$$\{y_2'(\sigma),\ y_3'(\bar\sigma)\}\cong-\frac{2}{\kappa}Hx'(\sigma)\delta(\sigma-\bar\sigma)-\frac{1}{\kappa}Hx(\sigma)\delta'(\sigma-\bar\sigma), \tag{4.10}$$

while all other Poisson brackets are zero. We see that these Poisson brackets are of the form (4.1), so, we can apply the result (4.7). Consequently, we get

$$\{y_1(\sigma),\ y_3(\bar\sigma)\}\cong-\frac{H}{\kappa}\left[2y(\sigma)-y(\bar\sigma)\right]\theta(\sigma-\bar\sigma), \tag{4.11}$$

$$\{y_2(\sigma),\ y_3(\bar\sigma)\}\cong\frac{H}{\kappa}\left[2x(\sigma)-x(\bar\sigma)\right]\theta(\sigma-\bar\sigma), \tag{4.12}$$

where function $\theta(x)$ is defined in (4.5). Let us note that these two Poisson brackets are zero when $\sigma=\bar\sigma$ and/or field strength H is equal to zero. But if we take that $\sigma-\bar\sigma=2\pi$ then we have $\theta(2\pi)=1$ and it follows

$$\{y_1(\sigma+2\pi),\ y_3(\sigma)\}\cong-\frac{H}{\kappa}\left[4\pi N_y+y(\sigma)\right], \tag{4.13}$$

$$\{y_2(\sigma+2\pi),\ y_3(\sigma)\}\cong\frac{H}{\kappa}\left[4\pi N_x+x(\sigma)\right], \tag{4.14}$$

where N_x and N_y are winding numbers defined as

$$x(\sigma+2\pi)-x(\sigma)=2\pi N_x,\quad y(\sigma+2\pi)-y(\sigma)=2\pi N_y. \tag{4.15}$$

From these relations we can see that if we choose such σ for which $x(\sigma)=0$ and $y(\sigma)=0$ then noncommutativity relations are proportional to winding numbers. On the other side, for winding numbers which are equal to zero there is still noncommutativity between T-dual coordinates.

4.2. Nonassociativity

In order to calculate Jacobi identity of the T-dual coordinates we first have to find Poisson brackets $\{y_1(\sigma),\ x(\bar\sigma)\}$ as well as $\{y_2(\sigma),\ y(\bar\sigma)\}$. We start with

$$\{\Delta y_1(\sigma,\sigma_0),\ x(\bar\sigma)\}=\left\{\int_{\sigma_0}^{\sigma}d\eta y_1'(\eta),\ x(\bar\sigma)\right\}, \tag{4.16}$$

and then use the T-dual transformation for x-direction in canonical form

$$\pi_x\cong\kappa y_1'. \tag{4.17}$$

From these two equations it follows

$$\{\Delta y_1(\sigma,\sigma_0), x(\bar\sigma)\} \cong \frac{1}{\kappa}\left\{\int_{\sigma_0}^{\sigma} d\eta \pi_x(\eta), x(\bar\sigma)\right\}, \tag{4.18}$$

which, using the standard Poisson algebra, produces

$$\{\Delta y_1(\sigma,\sigma_0), x(\bar\sigma)\} \cong -\frac{1}{\kappa}\left[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)\right]$$

$$\Longrightarrow \{y_1(\sigma), x(\bar\sigma)\} \cong -\frac{1}{\kappa}\theta(\sigma-\bar\sigma). \tag{4.19}$$

The relation $\{y_2(\sigma), y(\bar\sigma)\}$ can be obtained in the same way. Because the transformation law for y-direction is of the same form as for x-direction, the Poisson bracket is of the same form

$$\{y_2(\sigma), y(\bar\sigma)\} \cong -\frac{1}{\kappa}\theta(\sigma-\bar\sigma). \tag{4.20}$$

Now we can calculate Jacobi identity using noncommutativity relations (4.11) and (4.12) and above two Poisson brackets

$$\{y_1(\sigma_1), y_2(\sigma_2), y_3(\sigma_3)\} \equiv \{y_1(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\}$$
$$+\{y_2(\sigma_2), \{y_3(\sigma_3), y_1(\sigma_1)\}\} + \{y_3(\sigma_3), \{y_1(\sigma_1), y_2(\sigma_2)\}\}$$
$$\cong -\frac{2H}{\kappa^2}\left[\theta(\sigma_1-\sigma_2)\theta(\sigma_2-\sigma_3)\right.$$
$$\left. + \theta(\sigma_2-\sigma_1)\theta(\sigma_1-\sigma_3) + \theta(\sigma_1-\sigma_3)\theta(\sigma_3-\sigma_2)\right]. \tag{4.21}$$

Jacobi identity is nonzero which means that theory with R-flux is nonassociative. For $\sigma_2=\sigma_3=\sigma$ and $\sigma_1=\sigma+2\pi$ we get

$$\{y_1(\sigma+2\pi), y_2(\sigma), y_3(\sigma)\} \cong \frac{2H}{\kappa^2}. \tag{4.22}$$

From the last two equations, general form of Jacobi identity and Jacobi identity for special choice of σ's, we see that presence of the coordinate dependent Kalb-Ramond field is a source of noncommutativity and nonassociativity.

5. Conclusion

In this article we have considered the closed bosonic string propagating in the three-dimensional constant metric and Kalb-Ramond field with just one nonzero component $B_{xy}=Hz$. This choice of background is in accordance with consistency conditions in the sense that all calculations were made in approximation linear in Kalb-Ramond field strength H. Geometrically, this settings corresponds to the torus with H flux. Then we performed standard Buscher T-dualization procedure along isometry directions, first along x and then along y direction. At the end we performed generalized T-dualization procedure along z direction and obtained nonlocal theory with R flux. Using the relations between initial and T-dual variables, called T-dual transformation laws, in canonical form we find the noncommutativity and nonassociativity relations between T-dual coordinates.

After T-dualization along x direction we obtained theory embedded in geometry known in literature as twisted torus geometry. The relation between initial and T-dual variables is trivial, $\pi_x \cong \kappa y_1'$, where π_x is x component of the canonical momentum of the initial theory and y_1 is coordinate T-dual to x. Consequently, flux H takes a role of connection, obtained theory is globally and locally well defined and commutative, because the coordinates and their canonically conjugated momenta satisfy the standard Poisson algebra (3.25).

The second T-dualization, along y direction, produces nongeometrical theory, in literature known as Q flux theory. The metric is the same as initial one and Kalb-Ramond field have the same form as initial up to minus sign. But, this theory has just local geometrical interpretation. We obtained that, in approximation linear in H, the transformation law in canonical form is again trivial, $\pi_y \cong \kappa y_2'$, where π_y is y component od the canonical momentum of the initial theory and y_2 is coordinate T-dual to y. As a consequence of the standard Poisson algebra (3.25), we conclude that Q flux theory is still commutative. This result seems to be opposite from the result of the reference [16] where in detailed calculation it is shown that Q flux theory is noncommutative. The difference is in the so called boundary condition i.e. winding condition. In the Ref. [16] they imposed nontrivial winding condition which mixes the coordinates and their T-dual partners (condition given in Eq. (C.18) of Ref. [16]) and the result is noncommutativity. In this article the trivial winding condition is imposed on x and y coordinates. The consequence is that Q flux theory is commutative. But as it is written in Ref. [16] on page 42, "a priori other reasonings could as well be pursued".

T-dualizing along coordinate z using the machinery of the generalized T-dualization procedure[20,21,23] we obtain the nonlocal theory (theory with R flux) and nontrivial transformation law in canonical form. Non-locality stems from the fact that background fields are expressed in terms of the variable ΔV which is defined as line integral. On the other side, dependence of the Kalb-Ramond field on z coordinate produces the $\beta^{\pm}(x,y)$ functions and nontrivial transformation law for π_z. Consequently, coordinate dependent background gives non-locality and, further, nonzero Poisson brackets of the T-dual coordinates. We can claim that there is a correlation between non-locality (R-flux theory) and closed string noncommutativity and nonassociativity. In addition, nonzero Poisson bracket implies nonzero Jacobi identity which is a signal of nonassociativity.

From the expressions (4.11), (4.12) and (4.21) it follows that parameters of noncommutativity and nonassociativity are proportional to the field strength H. That means that closed string noncommuatativity and nonassociativity are consequence of the fact that Kalb-Ramond field is coordinate dependent, $B_{xy}=Hz$, where H is an infinitesimal parameter according to the approximation of diluted flux. Using T-duality and trivial winding conditions we obtained noncommutativity relations. The noncommutativity relations are zero if $\sigma=\bar\sigma$ because in noncommuatativity relations function $\theta(\sigma-\bar\sigma)$ is present, which is zero if its argument is zero. This is also at the first glance opposite to the result of Ref. [16], but, having in mind that origin of noncommutativity is not same, this difference is not surprising. If we made a round in sigma choosing $\sigma\to\sigma+2\pi$ and $\bar\sigma\to\sigma$, because of $\theta(2\pi)=1$, we obtained nonzero Poisson brackets. From the relations (4.13) and (4.14) we see that noncommutativity exists even in the case when winding numbers are zero, noncommutativity relations still stand unlike the result in [16]. Consequently, we can speak about some essential noncommutativity originating from non-locality.

We showed that in *ordinary* space coordinate dependent background is a sufficient condition for closed string noncommutativity. Some papers[36] show that noncommutativity is possible even in the constant background case. But that could be realized using the *double space formalism*. At the zeroth order the explanation follows from the fact that transformation law in canonical form is of the form $\pi_\mu \cong \kappa y'_\mu$, where y_μ is T-dual coordinate. Forming double space spanned by $Z^M = (x^\mu, y_\mu)$, we obtained noncommuative (double) space. In literature this kind of noncommutativity is called intrinsic one.

Appendix: Light-Cone Coordinates

In the paper we often use light-cone coordinates defined as

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma). \tag{A.1}$$

The corresponding partial derivatives are

$$\partial_\pm \equiv \frac{\partial}{\partial \xi^\pm} = \partial_\tau \pm \partial_\sigma. \tag{A.2}$$

Two dimensional Levi-Civita $\varepsilon^{\alpha\beta}$ is chosen in (τ, σ) basis as $\varepsilon^{\tau\sigma} = -1$. Consequently, in the light-cone basis the form of tensor is

$$\varepsilon_{lc} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{A.3}$$

The flat world-sheet metric is of the form in (τ, σ) and light-cone basis, respectively

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta_{lc} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{A.4}$$

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Conflict of Interest

The authors have declared no conflict of interest.

Keywords

closed string, nonassociativity, noncommutativity, non-locality

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Noncommutativity and nonassociativity of type II superstring with coordinate dependent RR field — the general case

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Abstract: In this paper we consider non-commutativity that arises from T-duality of bosonic coordinates of type II superstring in presence of coordinate dependent Ramond-Ramond field. Action with such choice of the background fields is not translational invariant. Consequently, we will employ generalization of Buscher procedure that can be applied to cases that have coordinate dependent fields and that do not possess translational isometry. Bosonic part of newly obtained T-dual theory is non-local and defined in non-geometric double space spanned by Lagrange multipliers y_μ and double coordinate ΔV^μ. We will apply Buscher procedure once more on T-dual theory to check if original theory can be salvaged. Finally, we will use T-dual transformation laws along with Poisson brackets of original theory to derive Poisson bracket structure of T-dual theory.

Keywords: String Duality, Superstrings and Heterotic Strings

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Contents

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1 Introduction

String theory as a possible candidate for unification of all known interactions offers a framework for description of both gauge interactions and gravity. Analyzing the relation between world-sheet diffeomorphisms and transformations of the background fields for open bosonic string [1–3] it is concluded that Kalb-Ramond field gets one additional term that is in fact a field strength of some gauge field. In sigma-model action it looks like that gauge fields are attached at the string endpoints moving along Dp-brane. Finiteness of the gauge theories (UV cutoff) demands existence of some minimal length. Consequently, noncommutativity naturally arose in open bosonic string theory in the presence of the constant background fields [4–7].

The fact that noncommutativity appears together with gauge theory produces a new line of investigation in quantum field theory — noncommutative gauge theories [8–10], but not noncommutative gravity.

The open bosonic string in the presence of the constant background fields gives constant noncommutativity [4–7], but, consequently, Jacobi identity is zero and associativity is not broken. Closed string in the presence of the constant background fields remains commutative.

Noncommutativity in open string theory comes from the boundary conditions (see [5–7]). Coordinates and their canonically conjugated momenta are mixed in the boundary conditions, and because they obey standard Poisson algebra, at the end we get noncommutativity of the initial coordinates. All these facts tell us about the way how we can reach noncommutativity in the bosonic closed string case. Kalb-Ramond field must be, at least, linearly coordinate dependent, and the generalized T-dualization procedure is a machinery [11–19]. The noncommutativity relations are coordinate dependent and produce nonzero Jacobi identity — nonassociativity appears in the closed string theory [13, 18, 20–22].

The noncommutativity and nonassociativity can be considered also within superstring theory [23, 24]. In ref. [25] we considered one special case of the type II superstring theory in pure spinor formulation [26–30] — all physical background fields are constant except Ramond-Ramond (RR) field strength. The RR field strength consists of the constant part and linearly coordinate dependent one, which is infinitesimal. In accordance with consistency conditions, we have chosen constant part of RR field strength to be symmetric and cooordinate dependent part to be antisymmetric tensor.

The motivation for this choice of background fields is the quest for the anticommutation relation between fermionic coordinates suggested in [30, 31]. Formally, this case is similar to the bosonic string case with coordinate dependent Kalb-Ramond field (weakly curved background). The difference is that in the superstring case noncommutativity parameter depends both on the bosonic and fermionic coordinates. Also we obtained that Jacobiator is nonzero. Both noncommutativity and nonassociativity parameters are proportional to the infinitesimal tensor from RR field strength.

In this article we consider the same action as in [25], but we will not imply the additional restrictions on the constant and coordinate dependent part of RR field strength as in [25]. The fundamental difference in relation to the choice of background field in [25] is in the fact that action with RR field strength without restrictions does not possess translational isometry. In that sense this case can be considered as general one comparing with [25].

We will use the generalized T-dualization procedure [16, 19] along bosonic directions. Because this general case cannot be deduced to the form of the bosonic string with linearly dependent Kalb-Ramond field, as it could in the case [25], we obtained more complicated form of T-dual transformation laws and, consequently, the generalization of β_μ functions in the form of $N(\xi)$ functions. Besides the complexity of the T-dual transformation laws we succeeded to find expressions for noncommutativity and nonassociativity as well as the form of the T-dual theory.

At the end we give some concluding remarks. In the appendices we present the derivation of $N(\xi)$ functions and show their properties.

2 General type II superstring action and choice of background fields

In this section we will shortly present how we derive the action of type II superstring in pure spinor formulation with all constant background fields except RR field strength from the general form of that action given in [26–30].

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2.1 General form of the pure spinor type II superstring action

The general form of the type II superstring action in pure spinor formalism is derived and given in [30]. It consists of two parts and can be represent as their sum

$$S = S_0 + V_{SG}\,, \tag{2.1}$$

where S_0 describes the motion of string in flat background

$$S_0 = \int_\Sigma d^2\xi \left(\frac{k}{2}\eta_{\mu\nu}\partial_m x^\mu \partial_n x^\nu \eta^{mn} - \pi_\alpha \partial_- \theta^\alpha + \partial_+ \bar{\theta}^\alpha \bar{\pi}_\alpha\right) + S_\lambda + S_{\bar{\lambda}}\,, \tag{2.2}$$

while the second one contains all possible interactions

$$V_{SG} = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar{X}^N. \tag{2.3}$$

The second part of the action is expressed in terms of the integrated form of massless type II supergravity vertex operator V_{SG}. The actions S_λ and $S_{\bar{\lambda}}$ in (2.2) are free-field actions for pure spinors

$$S_\lambda = \int d^2\xi \omega_\alpha \partial_- \lambda^\alpha, \quad S_{\bar{\lambda}} = \int d^2\xi \bar{\omega}_\alpha \partial_+ \bar{\lambda}^\alpha\,. \tag{2.4}$$

Here, λ^α and $\bar{\lambda}^\alpha$ are pure spinors whose canonically conjugated momenta are ω_α and $\bar{\omega}_\alpha$, respectively.

The vectors X^M and X^N and matrix A_{MN} are of the form

$$X^M = \begin{pmatrix} \partial_+\theta^\alpha \\ \Pi_+^\mu \\ d_\alpha \\ \frac{1}{2}N_+^{\mu\nu} \end{pmatrix}, \quad \bar{X}^M = \begin{pmatrix} \partial_-\bar{\theta}^\lambda \\ \Pi_-^\mu \\ \bar{d}_\lambda \\ \frac{1}{2}\bar{N}_-^{\mu\nu} \end{pmatrix}, \quad A_{MN} = \begin{bmatrix} A_{\alpha\beta} & A_{\alpha\nu} & E_\alpha{}^\beta & \Omega_{\alpha,\mu\nu} \\ A_{\mu\beta} & A_{\mu\nu} & \bar{E}_\mu^\beta & \Omega_{\mu,\nu\rho} \\ E^\alpha{}_\beta & E_\nu^\alpha & P^{\alpha\beta} & C^\alpha{}_{\mu\nu} \\ \Omega_{\mu\nu,\beta} & \Omega_{\mu\nu,\rho} & \bar{C}^\beta{}_{\mu\nu} & S_{\mu\nu,\rho\sigma} \end{bmatrix}, \tag{2.5}$$

where notation is taken from refs. [25, 30]. Every component of the matrix A_{MN} is function of bosonic, x^μ, and fermionic, θ^α and $\bar{\theta}^\alpha$, coordinates. For more details about derivation of the components consult [30]. The superfields $A_{\mu\nu}$, $\bar{E}_\mu^\alpha$, E_μ^α and $P^{\alpha\beta}$ are known as physical superfields, superfields that are in the first row and the first column are known as auxiliary because they can be expressed in terms of physical ones [30]. Remaining superfields $\Omega_{\mu,\nu\rho}$ ($\Omega_{\mu\nu,\rho}$), $C^\alpha{}_{\mu\nu}$ ($\bar{C}^\beta{}_{\mu\nu}$) and $S_{\mu\nu,\rho\sigma}$, are curvatures (field strengths) for physical fields. Components of X^M and $\bar{X}^N$ are of the form

$$\Pi_+^\mu = \partial_+ x^\mu + \frac{1}{2}\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\partial_+\theta^\beta, \qquad \Pi_-^\mu = \partial_- x^\mu + \frac{1}{2}\bar{\theta}^\alpha (\Gamma^\mu)_{\alpha\beta}\partial_-\bar{\theta}^\beta, \tag{2.6}$$

$$d_\alpha = \pi_\alpha - \frac{1}{2}(\Gamma_\mu\theta)_\alpha \left[\partial_+ x^\mu + \frac{1}{4}(\theta\Gamma^\mu\partial_+\theta)\right],$$
$$\bar{d}_\alpha = \bar{\pi}_\alpha - \frac{1}{2}(\Gamma_\mu\bar{\theta})_\alpha \left[\partial_- x^\mu + \frac{1}{4}(\bar{\theta}\Gamma^\mu\partial_-\bar{\theta})\right], \tag{2.7}$$

$$N_+^{\mu\nu} = \frac{1}{2}\omega_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \lambda^\beta, \qquad \bar{N}_-^{\mu\nu} = \frac{1}{2}\bar{\omega}_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \bar{\lambda}^\beta. \tag{2.8}$$

The world-sheet is spanned by $\xi^m = (\xi^0 = \tau, \xi^1 = \sigma)$, while world-sheet light-cone partial derivatives are defined as $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Superspace contains bosonic x^μ ($\mu = 0,1,\ldots,9$) and fermionic θ^α, $\bar{\theta}^\alpha$ ($\alpha = 1,2,\ldots,16$) coordinates. Variables π_α and $\bar{\pi}_\alpha$ are canonically conjugated momenta to the fermionic coordinates θ^α and $\bar{\theta}^\alpha$, respectively.

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2.2 Choice of the background fields

In this particular case we will use the supermatrix A_{MN} where all physical background fields, except RR field strength $P^{\alpha\beta}$, are constant. RR fields strength will have linear coordinate dependence on bosonic coordinate x^μ. Consequently, supermatrix A_{MN} is of the following form

$$A_{MN}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & k(\frac{1}{2}g_{\mu\nu}+B_{\mu\nu}) & \bar{\Psi}^\beta_\mu & 0\\ 0 & -\Psi^\alpha_\nu & \frac{2}{k}(f^{\alpha\beta}+C^{\alpha\beta}_\rho x^\rho) & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \tag{2.9}$$

where $g_{\mu\nu}$ is symmetric tensor, $B_{\mu\nu}$ is Kalb-Ramond antisymmetric field, Ψ^α_μ and $\bar{\Psi}^\alpha_\mu$ are Mayorana-Weyl gravitino fields and $f^{\alpha\beta}$ and $C^{\alpha\beta}_\rho$ are constant tensors. Let us stress this will be a classical analysis and we will not calculate the dilaton shift under T-duality transformation.

From the consistency conditions given in ref. [30], following this choice of background fields, it follows

$$\gamma^\mu_{\alpha\beta}C^{\beta\gamma}_\mu=0,\quad \gamma^\mu_{\alpha\beta}C^{\gamma\beta}_\mu=0. \tag{2.10}$$

Because all background fields are expanded in powers of θ^α and $\bar{\theta}^\alpha$, θ^α and $\bar{\theta}^\alpha$ terms in X^M and $\bar{X}^N$ will be neglected. Taking into account all imposed assumptions and approximations, the full action S is getting the form

$$\begin{aligned}S=\int_\Sigma d^2\xi\Big[&\frac{k}{2}\Pi_{+\mu\nu}\partial_+x^\mu\partial_-x^\nu-\pi_\alpha(\partial_-\theta^\alpha+\Psi^\alpha_\nu\partial_-x^\nu)+(\partial_+\bar{\theta}^\alpha+\partial_+x^\mu\bar{\Psi}^\alpha_\mu)\bar{\pi}_\alpha\\ &+\frac{2}{k}\pi_\alpha(f^{\alpha\beta}+C^{\alpha\beta}_\rho x^\rho)\bar{\pi}_\beta\Big],\end{aligned} \tag{2.11}$$

where $\Pi_{\pm\mu\nu}=B_{\mu\nu}\pm\frac{1}{2}G_{\mu\nu}$, and $G_{\mu\nu}=\eta_{\mu\nu}+g_{\mu\nu}$ is metric tensor. The actions S_λ and $S_{\bar{\lambda}}$ are fully decoupled from the rest and they will not be analyzed from now on.

It is easy to notice that fermionic momenta play the roles of the auxiliary fields in full action. They can be integrated out finding equations for motion for both π_α and $\bar{\pi}_\alpha$

$$\bar{\pi}_\beta=\frac{k}{2}\left(F^{-1}(x)\right)_{\beta\alpha}(\partial_-\theta^\alpha+\Psi^\alpha_\nu\partial_-x^\nu), \tag{2.12}$$

$$\pi_\alpha=-\frac{k}{2}\left(\partial_+\bar{\theta}^\beta+\partial_+x^\mu\bar{\Psi}^\beta_\mu\right)\left(F^{-1}(x)\right)_{\beta\alpha}, \tag{2.13}$$

where $F^{\alpha\beta}(x)$ and $(F^{-1}(x))_{\alpha\beta}$ are of the form

$$F^{\alpha\beta}(x)=f^{\alpha\beta}+C^{\alpha\beta}_\mu x^\mu,\quad (F^{-1}(x))_{\alpha\beta}=(f^{-1})_{\alpha\beta}-(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho x^\rho(f^{-1})_{\beta_1\beta}. \tag{2.14}$$

For practical reasons, we assume that ${C^{\alpha\beta}}_\mu$ is infinitesimal. This assumption is in accordance with constraints (2.10). Substituting equations (2.12) and (2.13) into (2.11) the final form of action is

$$S=k\int_\Sigma d^2\xi\left[\Pi_{+\mu\nu}\partial_+x^\mu\partial_-x^\nu+\frac{1}{2}(\partial_+\bar{\theta}^\alpha+\partial_+x^\mu\bar{\Psi}^\alpha_\mu)\left(F^{-1}(x)\right)_{\alpha\beta}(\partial_-\theta^\beta+\Psi^\beta_\nu\partial_-x^\nu)\right]. \tag{2.15}$$

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Let us note that we did not impose any conditions on tensors $f^{\alpha\beta}$ and $C_\mu^{\alpha\beta}$ as we did in ref. [25]. The case considered in this article is more general than the case studied in [25], because action does not possess translational symmetry.

3 T-dualization

Here we will make T-dualization of all bosonic directions aiming to find T-dual transformation laws — relations between T-dual coordinates and canonical variables of the original theory. The T-dual transformation laws will be used to calculate Poisson brackets of the T-dual coordinates.

3.1 Implementation of the generalized T-dualization procedure

In implementing of T-dualization procedure we will use *generalized Buscher T-dualization procedure* [16]. The standard Buscher procedure [14, 15] is made to be used along directions on which background fields do not depend (isometry directions), while generalized Buscher procedure can be applied to theories with coordinate dependent background fields along all directions. The generalized T-dualization procedure follows three steps — introduction of covariant derivatives, invariant coordinates and additional gauge fields, which produces additional degrees of freedom. The starting and T-dual theory must have the same number of degrees of freedom. In order to achieve that we eliminate all excessive degrees of freedom demanding that field strength of gauge fields ($F_{+-} = \partial_+ v_- - \partial_- v_+$) vanishes by addition of Lagrange multipliers. Then we fix the gauge symmetry (shift symmetry) and action is left with gauge fields and their derivatives. Finding equations of motion for gauge fields, expressing in terms of the Lagrange multipliers and inserting those equations into action we obtain T-dual action, where Lagrange multipliers have roles of T-dual coordinates.

T-duality can be performed also in the cases of the absence of shift symmetry [19]. Then we replace original action with translation invariant auxiliary action. Form of the auxiliary action is exactly the same as the form of action where translation symmetry was localized and gauge fixed. It produces correct T-dual theory only if original action can be salvaged from it.

Action (2.15) is not translational invariant. Consequently, we make the following substitutions

$$\partial_\pm x^\mu \to v_\pm^\mu, \tag{3.1}$$

$$x^\rho \to \Delta V^\rho = \int_P d\xi'^m v_m^\rho(\xi'), \tag{3.2}$$

$$S \to S + \frac{k}{2}\int_\Sigma d^2\xi \left[v_+^\mu \partial_- y_\mu - v_-^\mu \partial_+ y_\mu\right], \tag{3.3}$$

and insert them in action (2.15). The result is auxiliary action convenient for T-dualization procedure

$$S_{aux} = k\int_\Sigma d^2\xi \Big[\Pi_{+\mu\nu} v_+^\mu v_-^\nu + \frac{1}{2}(\partial_+\bar\theta^\alpha + v_+^\mu \bar\Psi_\mu^\alpha)\left(F^{-1}(\Delta V)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi_\nu^\beta v_-^\nu) + \frac{1}{2}(v_+^\mu \partial_- y_\mu - v_-^\mu \partial_+ y_\mu)\Big]. \tag{3.4}$$

Let us note that path P starts from ξ_0 and ends in ξ. In this way action becomes non-local.

Finding equations of motion for Lagrange multipliers

$$\partial_- v_+^\mu - \partial_+ v_-^\mu = 0 \quad v_\pm^\mu = \partial_\pm x^\mu, \tag{3.5}$$

and inserting them into (3.2) we have

$$\Delta V^\rho = \int_P d\xi'^m \partial_m x^\rho(\xi') = x^\rho(\xi) - x^\rho(\xi_0) = \Delta x^\rho. \tag{3.6}$$

In absence of translational symmetry, in order to extract starting action from auxiliary one, we impose $x^\rho(\xi_0) = 0$ as a constraint. Taking all this into account, we get the starting action (2.15).

Euler-Lagrange equations of motion for gauge fields $v_\pm(\kappa)$ give the following ones

$$-\frac{1}{2}\partial_- y_\mu(\kappa) = \Pi_{+\mu\nu} v_-^\nu(\kappa) + \frac{1}{2}\bar{\Psi}_\mu^\alpha \left(F^{-1}(\Delta V)\right)_{\alpha\beta} (\partial_-\theta^\beta(\kappa) + \Psi_\nu^\beta v_-^\nu(\kappa)) \tag{3.7}$$
$$-\frac{1}{2}\int_\Sigma d^2\xi [\partial_+\bar{\theta}^\alpha(\xi) + v_+^{\nu_1}(\xi)\bar{\Psi}_{\nu_1}^\alpha](f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1} (f^{-1})_{\beta_1\beta} N(\kappa^+)[\partial_-\theta^\beta(\xi) + \Psi_{\nu_2}^\beta v_-^{\nu_2}(\xi)],$$
$$\frac{1}{2}\partial_+ y_\mu(\kappa) = \Pi_{+\nu\mu} v_+^\nu(\kappa) + \frac{1}{2}(\partial_+\bar{\theta}^\alpha(\kappa) + v_+^\nu(\kappa)\bar{\Psi}_\nu^\alpha) \left(F^{-1}(\Delta V)\right)_{\alpha\beta} \Psi_\mu^\beta \tag{3.8}$$
$$-\frac{1}{2}\int_\Sigma d^2\xi [\partial_+\bar{\theta}^\alpha(\xi) + v_+^{\nu_1}(\xi)\bar{\Psi}_{\nu_1}^\alpha](f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1} (f^{-1})_{\beta_1\beta} N(\kappa^-)[\partial_-\theta^\beta(\xi) + \Psi_{\nu_2}^\beta v_-^{\nu_2}(\xi)].$$

Here, function $N(\kappa^\pm)$ is obtained from variation of term containing ΔV^ρ in expression for $F^{-1}(\Delta V)$ (details are presented in appendix A). They represent the generalization of beta functions introduced in ref. [25]

$$N(\kappa^+) = \delta\Big(\xi'^-((\xi'^+)^{-1}(\kappa^+)) - \kappa^-\Big)\left[H(\xi^+ - \kappa^+) - H(\xi_0^+ - \kappa^+)\right], \tag{3.9}$$
$$N(\kappa^-) = \delta\Big(\xi'^+((\xi'^-)^{-1}(\kappa^-)) - \kappa^+\Big)\left[H(\xi^- - \kappa^-) - H(\xi_0^- - \kappa^-)\right], \tag{3.10}$$

where more details on Dirac delta function and step function are given in appendix A. As we see the expressions for derivatives of y_μ are more complex comparing with those in [25], where translational symmetry is present.

Assuming that $C_\mu^{\alpha\beta}$ is an infinitesimal, we can iteratively invert equations of motion (3.7) and (3.8) [20]. Separating variables into two parts, one finite and one infinitesimal proportional to $C_\mu^{\alpha\beta}$, we have

$$
\begin{aligned}
v_-^\nu(\kappa) = &-\frac{1}{2}\bar{\Theta}_-^{\nu\nu_1}\Bigg\{\partial_- y_{\nu_1}(\kappa) + \bar{\Psi}_{\nu_1}^\alpha (F^{-1}(\Delta V))_{\alpha\beta}\partial_-\theta^\beta(\kappa) \\
&+\frac{1}{2}\Psi_{\nu_1}^\alpha (f^{-1})_{\alpha\alpha_1} C_\rho^{\alpha_1\alpha_2}\Delta V^\rho (f^{-1})_{\alpha_2\alpha_3}\Psi_{\nu_2}^{\alpha_3}\bar{\Theta}_-^{\nu_2\nu_3}\Big(\partial_- y_{\nu_3}(\kappa) + \bar{\Psi}_{\nu_3}^{\beta_1}(f^{-1})_{\beta_1\beta}\partial_-\theta^\beta(\kappa)\Big) \\
&-\int_\Sigma d^2\xi\Big[\partial_+\bar{\theta}^\alpha(\xi) + \frac{1}{2}\Big(\partial_+ y_{\mu_1}(\xi) - \partial_+\bar{\theta}^{\gamma_1}(\xi)(f^{-1})_{\gamma_1\gamma_2}\Psi_{\mu_1}^{\gamma_2}\Big)\bar{\Theta}_-^{\mu_1\nu_1}\bar{\Psi}_{\nu_1}^\alpha\Big](f^{-1})_{\alpha\alpha_1}C_{\nu_1}^{\alpha_1\beta_1} \\
&\times (f^{-1})_{\beta_1\beta}N(\kappa^+)\Big[\partial_-\theta^\beta(\xi) - \frac{1}{2}\Psi_{\nu_2}^\beta\bar{\Theta}_-^{\nu_2\mu_2}\Big(\partial_- y_{\mu_2}(\xi) + \bar{\Psi}_{\mu_2}^{\gamma_3}(f^{-1})_{\gamma_3\gamma_4}\partial_-\theta^{\gamma_4}(\xi)\Big)\Big]\Bigg\},
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
v_+^\mu(\kappa) = &\frac{1}{2}\bar{\Theta}_-^{\mu_1\mu}\Bigg\{\partial_+ y_{\mu_1}(\kappa) - \partial_+\bar{\theta}^\alpha(\kappa)(F^{-1}(\Delta V))_{\alpha\beta}\Psi_{\mu_1}^\beta \\
&+\frac{1}{2}\Big(\partial_+ y_{\mu_2}(\kappa) - \partial_+\bar{\theta}^\alpha(\kappa)(f^{-1})_{\alpha\alpha_1}\Psi_{\mu_2}^{\alpha_1}\Big)\bar{\Theta}_-^{\mu_2\mu_3}\bar{\Psi}_{\mu_3}^{\beta_3}(f^{-1})_{\beta_3\beta_2}C_\rho^{\beta_2\beta_1}\Delta V^\rho (f^{-1})_{\beta_1\beta}\Psi_{\mu_1}^\beta \\
&+\int_\Sigma d^2\xi\Big[\partial_+\bar{\theta}^\alpha(\xi) + \frac{1}{2}\Big(\partial_+ y_{\mu_2}(\xi) - \partial_+\bar{\theta}^{\gamma_1}(\xi)(f^{-1})_{\gamma_1\gamma_2}\Psi_{\mu_2}^{\gamma_2}\Big)\bar{\Theta}_-^{\mu_2\nu_1}\bar{\Psi}_{\nu_1}^\alpha\Big](f^{-1})_{\alpha\alpha_1}C_{\mu_1}^{\alpha_1\beta_1} \\
&\times (f^{-1})_{\beta_1\beta}N(\kappa^-)\Big[\partial_-\theta^\beta(\xi) - \frac{1}{2}\Psi_{\nu_2}^\beta\bar{\Theta}_-^{\nu_2\mu_3}\Big(\partial_- y_{\mu_3}(\xi) + \bar{\Psi}_{\mu_3}^{\gamma_3}(f^{-1})_{\gamma_3\gamma_4}\partial_-\theta^{\gamma_4}(\xi)\Big)\Big]\Bigg\}.
\end{aligned}
\tag{3.12}
$$

Tensor $\bar{\Theta}_-^{\mu\nu}$ is inverse tensor to $\bar{\Pi}_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar{\Psi}_\mu^\alpha (f^{-1})_{\alpha\beta}\Psi_\nu^\beta$

$$
\bar{\Theta}_-^{\mu\nu}\bar{\Pi}_{+\nu\rho} = \delta_\rho^\mu, \tag{3.13}
$$

where

$$
\bar{\Theta}_-^{\mu\nu} = \Theta_-^{\mu\nu} - \frac{1}{2}\Theta_-^{\mu\mu_1}\bar{\Psi}_{\mu_1}^\alpha (\bar{f}^{-1})_{\alpha\beta}\Psi_{\nu_1}^\beta\Theta_-^{\nu_1\nu}, \tag{3.14}
$$

$$
\bar{f}^{\alpha\beta} = f^{\alpha\beta} + \frac{1}{2}\Psi_\mu^\alpha\Theta_-^{\mu\nu}\bar{\Psi}_\nu^\beta, \tag{3.15}
$$

$$
\Theta_-^{\mu\nu}\Pi_{+\mu\rho} = \delta_\rho^\mu, \qquad \Theta_- = -4(G_E^{-1}\Pi_- G^{-1})^{\mu\nu}. \tag{3.16}
$$

Effective metric tensor is defined as $G_{E\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$.

In above expressions ΔV is a quantity in the zeroth order in $C_\mu^{\alpha\beta}$

$$
\begin{aligned}
\Delta V^\rho = &\int d\xi^+ v_+^\rho + \int d\xi^- v_-^\rho \\
= &\frac{1}{2}\int_P d\xi^+\bar{\Theta}_-^{\rho_1\rho}\Big[\partial_+ y_{\rho_1} - \partial_+\bar{\theta}^\alpha (f^{-1})_{\alpha\beta}\Psi_{\rho_1}^\beta\Big] - \frac{1}{2}\int_P d\xi^-\bar{\Theta}_-^{\rho\rho_1}\Big[\partial_- y_{\rho_1} + \bar{\Psi}_{\rho_1}^\alpha (f^{-1})_{\alpha\beta}\partial_-\theta^\beta\Big].
\end{aligned}
\tag{3.17}
$$

Using (3.7) and (3.8) and inserting them into (3.4), we get T-dual action

$$
\begin{aligned}
S_{T-dual} = k\int_P d^2\xi\bigg[&\frac{1}{4}\bar\Theta_-^{\mu\nu}\partial_+ y_\mu\partial_- y_\nu \qquad (3.18)\\
&+\frac{1}{8}\bar\Theta_-^{\mu\mu_1}\bar\Psi^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_1}\bar\Theta_-^{\nu_1\nu}\partial_+ y_\mu\partial_- y_\nu\\
&+\frac{1}{2}\partial_+\bar\theta^\alpha\Big((F^{-1}(\Delta V))_{\alpha\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\alpha_2}\Delta V^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\\
&\quad-\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_3}_\nu(f^{-1})_{\beta_3\beta_2}C_\rho^{\beta_2\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta}\\
&\quad-\frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\mu_1}\bar\Psi^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_3}\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\bar\Theta_-^{\nu_1\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\Big)\partial_-\theta^\beta\\
&+\frac{1}{4}\partial_+ y_\mu\bar\Theta_-^{\mu\mu_1}\bar\Psi^\alpha_{\mu_1}\Big((F^{-1}(\Delta V))_{\alpha\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_3}\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\bar\Theta_-^{\nu_1\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\Big)\\
&\quad\times\partial_-\theta^\beta\\
&-\frac{1}{4}\partial_+\bar\theta^\alpha\Big((F^{-1}(\Delta V))_{\alpha\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\mu_1}\bar\Psi^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta}\Big)\Psi^\beta_{\nu_1}\bar\Theta_-^{\nu_1\nu}\\
&\quad\times\partial_- y_\nu\bigg].
\end{aligned}
$$

Let us note that above we kept terms up to the first order in $C_\mu^{\alpha\beta}$.

T-dual action contains all terms as initial action (2.15) up to the change $x^\mu\to y_\mu$. Consequently, T-dual background fields are of the form

$$
\begin{aligned}
{}^\star\Pi_+^{\mu\nu}=&\frac{1}{4}\bar\Theta_-^{\mu\nu}+\frac{1}{8}\bar\Theta_-^{\mu\mu_1}\bar\Psi^\alpha_{\mu_1}\Big[(F^{-1}(\Delta V))_{\alpha\beta}+(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta} \qquad (3.19)\\
&-\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_{\mu_2}\bar\Theta_-^{\mu_2\nu_2}\bar\Psi^{\beta_1}_{\nu_2}(f^{-1})_{\beta_1\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\alpha_2}\Delta V^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_{\mu_2}\bar\Theta_-^{\mu_2\nu_2}\bar\Psi^{\beta_1}_{\nu_2}(f^{-1})_{\beta_1\beta}\\
&+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_{\mu_2}\bar\Theta_-^{\mu_2\nu_2}\bar\Psi^{\beta_3}_{\nu_2}(f^{-1})_{\beta_3\beta_2}C_\rho^{\beta_2\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta}\\
&-\frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_{\mu_2}\bar\Theta_-^{\mu_2\mu_3}\bar\Psi^{\alpha_2}_{\mu_3}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_3}\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_3}\bar\Theta_-^{\nu_3\nu_2}\bar\Psi^{\beta_1}_{\nu_2}(f^{-1})_{\beta_1\beta}\Big]\Psi^\beta_{\nu_1}\bar\Theta_-^{\nu_1\nu},\\
{}^\star(F^{-1}(x))_{\alpha\beta}=&(F^{-1}(\Delta\bar y))_{\alpha\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\alpha_2}\Delta V^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta} \qquad (3.20)\\
&-\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_3}_\nu(f^{-1})_{\beta_3\beta_2}C_\rho^{\beta_2\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta}\\
&-\frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\mu_1}\bar\Psi^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_3}\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\bar\Theta_-^{\nu_1\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta},\\
{}^\star\bar\Psi^{\mu\alpha}=&\frac{1}{2}\Theta_-^{\mu\nu}\bar\Psi^\alpha_\nu,\qquad {}^\star\Psi_{\nu\beta}=-\frac{1}{2}\Psi^\beta_\mu\Theta_-^{\mu\nu}. \qquad (3.21)
\end{aligned}
$$

Comparing background field of T-dual theory with background fields from [25] we immediately notice that background fields have become more complex. However, this is just an illusion. In both cases background field are exactly the same only difference is that here we did not introduce tensor $\breve\Pi_{+\mu\nu}=\Pi_{+\mu\nu}+\frac{1}{2}\bar\Psi^\alpha_\mu F^{-1}(\Delta V)_{\alpha\beta}\Psi^\beta_\nu$ and its inverse, therefore we are missing ingredients to express our fields in more compactified format.

3.2 T-dualization of T-dual theory

Since the initial theory is not symmetric under translations, T-dual action that is obtained from auxiliary action (3.4) and it is now invariant to translations of T-dual coordinates. Consequently, we can dualize T-dual theory by generalized Buscher procedure. We start with the introduction of following substitutions

$$\partial_\pm y_\mu \to D_\pm y_\mu = \partial_\pm y_\mu + u_{\pm\mu} \to D_\pm y_\mu = u_{\pm\mu}, \tag{3.22}$$

$$\Delta\bar{y}^\rho \to \Delta\bar{u}^\rho, \tag{3.23}$$

$$\begin{aligned}\Delta\bar{u}^\rho = &\frac{1}{2}\int_P d\xi^+ \bar\Theta_-^{\rho_1\rho}\left[u_{+\rho_1} - \partial_+\bar\theta^\alpha (f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right]\\ &-\frac{1}{2}\int_P d\xi^- \bar\Theta_-^{\rho\rho_1}\left[u_{-\rho_1} + \bar\Psi^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right],\end{aligned} \tag{3.24}$$

$$S \to S + \frac{1}{2}(u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu). \tag{3.25}$$

From the first line we see that gauge is fixed choosing $y(\xi) = const.$ Inserting these substitutions into (3.18) we obtain

$$\begin{aligned}S_{\text{gauge fix}} = \kappa\int_P d^2\xi\Bigg[&\frac{1}{4}\bar\Theta_-^{\mu\nu}u_{+\mu}u_{-\nu}\\
&+\frac{1}{8}\bar\Theta_-^{\mu\mu_1}\bar\Psi^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}\Delta\bar{u}^\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_1}\bar\Theta_-^{\nu_1\nu}u_{+\mu}u_{-\nu}\\
&+\frac{1}{2}\partial_+\bar\theta^\alpha\Big((F^{-1}(\Delta\bar{u}))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\alpha_2}\Delta\bar{u}^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\\
&-\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_3}_\nu(f^{-1})_{\beta_3\beta_2}C_\rho^{\beta_2\beta_1}\Delta\bar{u}^\rho(f^{-1})_{\beta_1\beta}\\
&-\frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\mu_1}\bar\Psi^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_3}\Delta\bar{u}^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\bar\Theta_-^{\nu_1\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\Big)\partial_-\theta^\beta\\
&+\frac{1}{4}u_{+\mu}\bar\Theta_-^{\mu\mu_1}\bar\Psi^\alpha_{\mu_1}\left((F^{-1}(\Delta\bar{u}))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_3}\Delta\bar{u}^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\bar\Theta_-^{\nu_1\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\right)\\
&\times\partial_-\theta^\beta\\
&-\frac{1}{4}\partial_+\bar\theta^\alpha\Big((F^{-1}(\Delta\bar{u}))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\mu_1}\bar\Psi^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_1}\Delta\bar{u}^\rho(f^{-1})_{\beta_1\beta}\Big)\Psi^\beta_{\nu_1}\bar\Theta_-^{\nu_1\nu}\\
&\times u_{-\nu}\\
&+\frac{1}{2}(u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu)\Bigg].\end{aligned} \tag{3.26}$$

Using equations of motion for Lagrange multipliers, we return to the T-dual action. Finding equations of motion for gauge fields, we have

$$\begin{aligned}u_{+\mu}(\kappa) = &\;2\bar\Pi_{+\nu\mu}\partial_+ x^\nu(\kappa) - \partial_+ x^\nu(\kappa)\bar\Psi^\alpha_\nu(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}\Delta x^\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_\mu\\
&+\partial_+\bar\theta^\alpha(\kappa)(F^{-1}(\Delta\bar{x}))_{\alpha\beta}\Psi^\beta_\mu\\
&-\int_\Sigma d^2\xi\Big(\partial_+\bar\theta^\alpha(\xi) + \partial_+ x^{\mu_1}(\xi)\bar\Psi^\alpha_{\mu_1}\Big)(f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}N(\kappa^-)\\
&\times\Big(\partial_-\theta^\beta(\xi) + \Psi^\beta_\nu\partial_- x^\nu(\xi)\Big),\end{aligned} \tag{3.27}$$

$$
\begin{aligned}
u_{-\nu}(\kappa) = &-2\bar{\Pi}_{+\nu\mu}\partial_- x^\mu(\kappa) + \bar{\Psi}^\alpha_\nu (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\rho \Delta x^\rho (f^{-1})_{\beta_1\beta}\Psi^\beta_\mu \partial_- x^\mu(\kappa) \\
&- \bar{\Psi}^\alpha_\nu (F^{-1}(\Delta\bar{x}))_{\alpha\beta}\partial_-\theta^\beta(\kappa) \\
&+ \int_\Sigma d^2\xi \Big(\partial_+\bar{\theta}^\alpha(\xi) + \partial_+ x^\mu(\xi)\bar{\Psi}^\alpha_\mu\Big)(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\nu (f^{-1})_{\beta_1\beta} N(\kappa^-) \\
&\times \Big(\partial_-\theta^\beta(\xi) + \Psi^\beta_{\nu_1}\partial_- x^{\nu_1}(\xi)\Big)\,.
\end{aligned}
\tag{3.28}
$$

Here we have that $\Delta x^\mu = x(\xi) - x(\xi_0)$, and inserting these equations into the gauge fixed action, keeping all terms linear with respect to $C^{\mu\nu}_\rho$ and selecting ξ_0 such that $x(\xi_0) = 0$, we obtain our original action (2.15).

4 Non-commutative relations

In this section we will establish a relationship between Poisson brackets of original and T-dual theory using results from the previous one. Original theory is a geometric one, which means that canonical variables $x^\mu(\xi)$ and $\pi_\mu(\xi)$ satisfy standard Poisson algebra

$$
\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu_\nu \delta(\sigma - \bar{\sigma}), \quad \{x^\mu(\sigma), x^\nu(\bar{\sigma})\} = 0, \quad \{\pi_\mu(\sigma), \pi_\nu(\bar{\sigma})\} = 0. \tag{4.1}
$$

We will find Poisson structure of T-dual theory using relations (3.7) and (3.8) and expressing them in terms of the coordinates and momenta of the initial theory. Replacing gauge fields with solutions of equations of motion for Lagrange multipliers, we get T-dual transformation laws in Lagrangian form. Because we implement here canonical approach, the next step is removing of all terms that are proportional to $\partial_\tau x^\mu(\xi)$. The most of the terms of this type will get incorporated into expression for canonical momenta $\pi_\mu(\xi)$, but term that is non-local and which is dependent on function $N(\xi^\pm)$ remains. One way of removing this term is to first use equations of motion for coordinate $x^\mu(\xi)$, and then replace remaining $\partial_\tau x^\mu$ term with canonical momentum. By doing all the steps that were outlined, we have following relationship between T-dual coordinate and variables of starting theory

$$
\begin{aligned}
\partial_\sigma y_\nu(\sigma) = &\, 2B_{\nu\mu}\partial_\sigma x^\mu - G_{\nu\mu}(\bar{\Pi}_+ + \bar{\Pi}^T_+)^{-1\nu_1\mu}\Bigg[\frac{\pi_{\nu_1}}{k} - \frac{1}{2}\bar{\Psi}^\alpha_{\nu_1}\Big(F^{-1}(x)\Big)_{\alpha\beta}\partial_-\theta^\beta \\
&- \frac{1}{2}\partial_+\bar{\theta}^\alpha\Big(F^{-1}(x)\Big)_{\alpha\beta}\Psi^\beta_{\nu_1} - \Big[\Pi_{+\mu_1\mu_2} + \frac{1}{2}\bar{\Psi}^\alpha_{\mu_1}\Big(F^{-1}(x)\Big)_{\alpha\beta}\Psi^\beta_{\mu_2}\Big](\delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2})\partial_\sigma x^{\nu_2} \\
&+ \frac{1}{2}\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho x^\rho(\sigma)(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_2}(\delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2})(\bar{\Pi}_+ + \bar{\Pi}^T_+)^{-1\rho\nu_2} \\
&\times \Big[\frac{\pi_\rho}{k} - \frac{1}{2}\bar{\Psi}^\gamma_\rho(f^{-1})_{\gamma\gamma_1}\partial_-\theta^{\gamma_1} - \frac{1}{2}\partial_+\bar{\theta}^\gamma(f^{-1})_{\gamma\gamma_1}\Psi^{\gamma_1}_\rho + \bar{\Pi}_{+\rho\rho_1}\partial_\sigma x^{\rho_1} - \bar{\Pi}_{+\rho_1\rho}\partial_\sigma x^{\rho_1}\Big]\Bigg].
\end{aligned}
\tag{4.2}
$$

To find Poisson bracket between T-dual coordinates, we can start by finding Poisson bracket of sigma derivatives of T-dual coordinates and then integrating twice (see [25]).

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Implementing this procedure we have that Poisson bracket for sigma derivatives is given as

$$
\begin{aligned}
&\{\partial_{\sigma_1} y_{\nu_1}(\sigma_1), \partial_{\sigma_2} y_{\nu_2}(\sigma_2)\} = \\
&= \frac{2}{k}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_1\mu_2}\Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}\partial_{\sigma_2}\delta(\sigma_1-\sigma_2) - B_{\nu_1\mu_1}G_{\nu_2\mu_2}\partial_{\sigma_1}\delta(\sigma_1-\sigma_2)\Big] \\
&\quad + \frac{1}{k}\bar{\Psi}^{\alpha}_{\nu_3}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Psi^{\beta}_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_3\mu_1}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_4\mu_2} \\
&\quad \times \Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}x^\rho(\sigma_1)\partial_{\sigma_2}\delta(\sigma_1-\sigma_2) - B_{\nu_1\mu_1}G_{\nu_2\mu_2}x^\rho(\sigma_2)\partial_{\sigma_1}\delta(\sigma_1-\sigma_2)\Big].
\end{aligned}
\tag{4.3}
$$

Then we integrate with respect to σ_1 (σ_2), where we set boundaries as σ_0 ($\bar{\sigma}_0$) and σ ($\bar{\sigma}$). Extracting only Poisson bracket terms that contain σ and $\bar{\sigma}$, we have

$$
\begin{aligned}
&\{y_{\nu_1}(\sigma), y_{\nu_2}(\bar{\sigma})\} = \frac{2}{k}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_1\mu_2}\Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2} + B_{\nu_1\mu_1}G_{\nu_2\mu_2}\Big]H(\sigma-\bar{\sigma}) \\
&+ \frac{1}{k}\bar{\Psi}^{\alpha}_{\nu_3}(f^{-1})_{\alpha\alpha_1}G_\rho^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Psi^{\beta}_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_3\mu_1}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_4\mu_2} \\
&\times \Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}x^\rho(\bar{\sigma}) + B_{\nu_1\mu_1}G_{\nu_2\mu_2}x^\rho(\sigma)\Big]H(\sigma-\bar{\sigma}).
\end{aligned}
\tag{4.4}
$$

Here, $H(\sigma-\bar{\sigma})$ is same step function defined in appendix A. It should be noted that these Poisson brackets are zero when $\sigma = \bar{\sigma}$. However, in cases where string in curled around compactified dimension, that is cases where $\sigma - \bar{\sigma} = 2\pi$, we have following situation

$$
\begin{aligned}
&\{y_{\nu_1}(\sigma+2\pi), y_{\nu_2}(\sigma)\} = \frac{2}{k}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_1\mu_2}\Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2} + B_{\nu_1\mu_1}G_{\nu_2\mu_2}\Big] \\
&+ \frac{1}{k}\bar{\Psi}^{\alpha}_{\nu_3}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Psi^{\beta}_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_3\mu_1}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_4\mu_2} \\
&\times \Big[4\pi G_{\nu_1\mu_1}B_{\nu_2\mu_2}N^\rho + (G_{\nu_1\mu_1}B_{\nu_2\mu_2} + B_{\nu_1\mu_1}G_{\nu_2\mu_2})x^\rho(\sigma)\Big].
\end{aligned}
\tag{4.5}
$$

We used fact that $H(2\pi) = 1$. The symbol N^μ denotes winding number around compactified coordinate, if is defined as

$$
x^\mu(\sigma+2\pi) - x^\mu(\sigma) = 2\pi N^\mu. \tag{4.6}
$$

Let us note that if we choose $x^\mu(\sigma) = 0$ than Poisson bracket has linear dependence on winding number. In cases where we don't have any winding number, we still have non-commutativity that is proportional to background fields.

Using the expression for sigma derivative of y_ν (4.2) and expression for Poisson bracket of sigma derivatives (4.3), we can find non-associative relations. Procedure is the same as for finding Poisson brackets of T-dual theory, we find Poisson bracket of sigma derivatives and integrate with respect to sigma coordinate, this time integral is done trice. Going along with this procedure we have following final result

$$
\begin{aligned}
&\{y_\nu(\sigma), \{y_{\nu_1}(\sigma_1), y_{\nu_2}(\sigma_2)\}\} = \frac{G_{\nu\mu}}{k^2}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\rho\mu} \\
&\times \bar{\Psi}^{\alpha}_{\nu_3}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Psi^{\beta}_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_3\mu_1}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_4\mu_2} \\
&\times \Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}H(\sigma-\sigma_2) + B_{\nu_1\mu_1}G_{\nu_2\mu_2}H(\sigma-\sigma_1)\Big]H(\sigma_1-\sigma_2).
\end{aligned}
\tag{4.7}
$$

Since Jacobi identity is non-zero for T-dual theory we have that coordinate dependent RR field produces non-associative theory. However putting $\sigma_1 = \sigma_2 = \bar{\sigma}$ and $\sigma = \bar{\sigma} + 2\pi$ we have that Jacobi identity disappears

$$\{y_\nu(\bar{\sigma} + 2\pi), \{y_{\nu_1}(\bar{\sigma}), y_{\nu_2}(\bar{\sigma})\}\} = 0. \tag{4.8}$$

5 Conclusion

In this article we examined type II superstring propagating in presence of coordinate dependent RR field. This choice of background was in accordance with consistency conditions for background field and all calculations were made in approximation that are linear with respect to coordinate dependent part of RR field. We have also excluded parts that were non-linear in fermionic coordinates and neglected pure spinor actions. Using equations of motion for fermionic momenta we obtained action that was expressed in terms of bosonic coordinates, their derivatives and derivatives of fermionic coordinates. Unlike [25] we do not impose any conditions on the constant and coordinate dependent part of RR field strength, so, it is not possible to deduce this case to the form of the weakly curved background one.

Action with our choice of background fields did not possess translation symmetry, therefore we needed to use Buscher procedure that was extended to such cases. By substituting starting action with auxiliary action we gave up on locality in order to be able to find T-dual theory. Finding equations of motion of newly introduced Lagrange multipliers we were able to salvage starting action giving us assurance that auxiliary action we selected would produce correct T-dual theory. After this we found equations of motion for gauge fields and by inserting them into action, we found T-dual theory.

Having found T-dual theory, we applied T-dual procedure once again as a more thorough way of checking if action we obtained was in fact correct T-dual of starting action. Unlike starting action, T-dual action possessed translation symmetry and was non-local from the start by virtue of having dual coordinate ΔV^μ. Applying steps of generalized Buscher procedure [16, 19] we obtained starting action, again confirming that our choice of auxiliary action was correct.

We obtained non-commutativity relations in context of T-dual theory, where we used T-dual transformation laws as a bridge between Poisson brackets of starting theory and T-dual theory. T-dual transformation laws were expressed as functions of coordinates and momenta of original theory and using their standard Poisson algebra, we got non-commutativity in T-dual theory. From expression for Poisson brackets (4.4) we can see that non-commutativity is proportional to infinitesimal part of RR field as well as to symmetrised inverse of field $\bar{\Pi}$. Non-commutativity relations are zero in case when $\sigma = \bar{\sigma}$, while in case where $\sigma = \bar{\sigma} + 2\pi$ we see the emergence of winding numbers.

Taking into account Poisson brackets of sigma derivatives and expression for sigma derivative of T-dual coordinate we were able to find non-associative relation for T-dual theory. In general case this relation was non-zero and it was proportional to infinitesimal constant $C^{\mu\nu}_\rho$. In special case when we put $\sigma_1 = \sigma_2 = \bar{\sigma}$ and $\sigma = \bar{\sigma} + 2\pi$ we noticed that non-associativity relation disappears. During the implementation of the T-dualization procedure and calculations, we obtained generalization of β_μ functions in the form of the N-functions.

It should be noted that since we did not preform T-dualization along fermionic coordinates their Poisson structure would remain the same as in original theory. Furthermore, since background fields do not depend on fermionic coordinates it should be expected, as in the case of bosonic coordinates [18], that T-duality would leave Poisson brackets between fermionic fields the same. We expect that, if proposed non-commutative relations from [30, 31] are even possible, we would need at least RR field that depends both on fermionic and bosonic coordinates.

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A Obtaining $N(\kappa^\pm)$ terms

In this paper function $N(\kappa^\pm)$ emerged in T-dual transformation laws as a consequence of variation of term that was proportional to ΔV. Here we will present derivation of this function.

$$\begin{aligned}
\frac{\delta\left(F^{-1}(\Delta V)\right)_{\alpha\beta}}{\delta v_+^\mu(\kappa)} &= -(f^{-1})_{\alpha\alpha_1} C_l^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_P d\xi'^m \frac{\delta v_m^\rho(\xi')}{\delta v_+^\mu(\kappa)} = \\
&= -(f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_P d\xi'^+\delta(\xi'^+-\kappa^+)\delta(\xi'^- -\kappa^-) \\
&= -(f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_{t_i}^{t_f} dt\frac{d\xi'^+}{dt}\delta(\xi'(t)^+-\kappa^+)\delta(\xi'^-(t)-\kappa^-) \\
&= -(f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_{\xi_0{}^+}^{\xi^+} du\delta(u-\kappa^+)\delta\Big(\xi'^-((\xi'^+)^{-1}(u))-\kappa^-\Big) \\
&= -(f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\delta\Big(\xi'^-((\xi'^+)^{-1}(\kappa^+))-\kappa^-\Big)\Big[H(\xi^+-\kappa^+)-H(\xi_0^+-\kappa^+)\Big] \\
&= -(f^{-1})_{\alpha\alpha_1} G_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta} N(\kappa^+).
\end{aligned} \tag{A.1}$$

In third line we have parametrized the path with parameter t where $\xi'^+(t_i) = \xi_0{}^+$ and $\xi'^+(t_f) = \xi^+$. In fourth line we introduced substitution $u = \xi'^+(t)$, in delta function this substitute is inverted. Fifth line is obtained by using following integration rule for Dirac delta function

$$\int_{\sigma_0}^{\sigma} d\eta f(\eta)\delta(\eta-\bar{\eta}) = f(\bar{\eta})\left[H(\sigma-\bar{\eta})-H(\sigma_0-\bar{\eta})\right]. \tag{A.2}$$

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Here, $H(x)$ is a step function defined as

$$H(x) = \int_0^x d\eta\delta(\eta) = \frac{1}{2\pi}\Big[x + \sum_{n\geq 2}\frac{1}{n}\sin(nx)\Big]$$
$$= \begin{cases} 0 & \text{if} \quad x = 0 \\ 1/2 & \text{if} \quad 0 < x < 2\pi \\ 1 & \text{if} \quad x = 2\pi \end{cases} . \tag{A.3}$$

Procedure for obtaining $N(\kappa^-)$ is similar.

B Properties of $N(\kappa^\pm)$ terms

Here ve will list some properties of $N(\kappa^\pm)$ function.

$$N(\kappa^+) + N(\kappa^-) = N(\kappa^0), \tag{B.1}$$
$$N(\kappa^+) - N(\kappa^-) = N(\kappa^1), \tag{B.2}$$

Where κ^0 and κ^1 represent τ and σ coordinates respectevly

$$\int_\Sigma d^2\xi\partial_+ N(\kappa^+) = 1, \qquad \int_\Sigma d^2\xi\partial_- N(\kappa^+) = 0, \tag{B.3}$$
$$\int_\Sigma d^2\xi\partial_- N(\kappa^-) = 1, \qquad \int_\Sigma d^2\xi\partial_+ N(\kappa^-) = 0. \tag{B.4}$$

These relationships can be checked directly by applying partial derivatives to expressions from A.

$$\int_\Sigma d^2\xi\partial_+ N(\kappa^+) = \int_\Sigma d^2\xi\delta\Big(\xi'^-((\xi'^+)^{-1}(\kappa^+)) - \kappa^-\Big)\partial_+\Big[H(\xi^+ - \kappa^+) - H(\xi_0^+ - \kappa^+)\Big]$$
$$= \int_\Sigma d^2\xi\delta\Big(\xi'^-((\xi'^+)^{-1}(\kappa^+)) - \kappa^-\Big)\delta(\xi^+ - \kappa^+) = \int d\xi^-\delta\Big(\xi'^-((\xi'^+)^{-1}(\xi^+)) - \kappa^-\Big). \tag{B.5}$$

In appendix A we had following parametrisation of path P: $\xi'^+(t_i) = \xi_0{}^+$ and $\xi'^+(t_f) = \xi^+$. Applying inverse parametrisation we have $(\xi'^+)^{-1}(\xi_0^+) = t_i$ and $(\xi'^+)^{-1}(\xi^+) = t_f$. With these we have

$$\int d\xi^-\delta\Big(\xi'^-((\xi'^+)^{-1}(\xi^+)) - \kappa^-\Big) = \int_\Sigma d\xi^-\delta\Big(\xi'^-(t_f) - \kappa^-\Big)$$
$$= \int_\Sigma d\xi^-\delta\Big(\xi^- - \kappa^-\Big) = 1. \tag{B.6}$$

Same rules apply for $N(\kappa^-)$, $N(\kappa^0)$ and $N(\kappa^1)$. In cases where $F^{-1}(x)_{\alpha\beta}$ is antisymmetric we can transfer partial derivatives from $\partial_\pm V^m u$ to $N(\kappa^\pm)$ and obtain standard $\beta^\pm$ functions.

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Directly from H-flux to the family of three nonlocal R-flux theories

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Abstract: In this article we consider T-dualization of the 3D closed bosonic string in the weakly curved background — constant metric and Kalb-Ramond field with one non-zero component, $B_{xy} = Hz$, where field strength H is infinitesimal. We use standard and generalized Buscher T-dualization procedure and make T-dualization starting from coordinate z, via y and finally along x coordinate. All three theories are *nonlocal*, because variable ΔV, defined as line integral, appears as an argument of background fields. After the first T-dualization we obtain commutative and associative theory, while after we T-dualize along y, we get, κ-Minkowski-like, noncommutative and associative theory. At the end of this T-dualization chain we come to the theory which is both noncommutative and nonassociative. The form of the final T-dual action does not depend on the order of T-dualization while noncommutativity and nonassociativity relations could be obtained from those in the $x \to y \to z$ case by replacing $H \to -H$.

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Contents

1 Introduction

Noncommutativity of coordinates has come into focus of physics about hundred years ago when the problem with infinite value of physical quantities occurred. The solution was proposed by Heisenberg in the form of noncommutative coordinates. But after developing of renormalization procedure coordinate noncommutativity was forgotten as a tool for cancelling of infinities.

Commuting of coordinates means that there is no minimal possible length in Nature i.e. that we can measure the position of particle with infinite precision. The return of noncommutativity into physics starts with the article of Hartland Snyder [1]. Usually we treat space-time as continuum but Snyder showed that there is Lorentz invariant discrete space-time. Consequently, this means that commutator of coordinates is nonzero, and noncommutativity parameter dictates the scale at which noncommutativity exists.

In the paper [2] existence of noncommutative manifold was shown using propagators in open bosonic string theory with constant metric and constant Kalb-Ramond field. This result is proven in many articles [3–12] after that but using different mathematical methods. Obtained noncommutativity with constant noncommutativity parameter is known in literature as canonical noncommutativity. Consequently, canonical noncommutativity implies that theory is still associative one.

One of the first application of canonical noncommutativity was in Yang-Mills (YM) theories [13–16]. Noncommutative YM theories are constructed and their renormalisability properties are analyzed. It turned out that some processes forbidden in commutative YM are allowed in noncommutative YM theories. Consequently, cross sections for those decays and processes are calculated [17, 18]. Such predictions offer the possibility of indirect check of idea of noncommutativity.

The next type of noncommuatativity which is considered in literature is Lie-algebraic one, which means that commutator of two coordinates is proportional to the coordinate. The κ-Minkowski space-time is an example of this kind of noncommutativity and it is considered in various contexts [19–24]. The κ-Minkowski space is noncommutative but it is easy to check that is associative one. But, in general, if the commutator of the coordinates is proportional to the some linear combination of coordinates, then the space is nonassociative because jacobiator and associator are nonzero. For example, such spaces are closely related to the L_∞ algebra [25].

The mathematical framework for T-dualization is standard Buscher procedure [26, 27]. It consists of the localization of the shift symmetry and adding a term with Lagrange multiplier in order to make gauge fields unphysical degrees of freedom. Also there is an improvement of standard Buscher procedure developed and applied in refs. [28–31], generalized Buscher procedure. In the application of the generalized procedure of T-dualization there is one additional step with respect to the standard one. We introduce invariant coordinate in order to localize shift symmetry in the coordinate dependent backgrounds.

The first articles addressing the subject of coordinate dependent backgrounds appear in the last ten years [32–43]. A 3-torus with constant metric and Kalb-Ramond field with just one nonzero component, $B_{xy} = Hz$, was considered within standard Buscher procedure [33]. Authors made two successive T-dualzation along isometry directions x and y, and, using nontrivial winding conditions, obtained noncommutativity with parameter proportional to field strength H and winding number N_3.

Using generalized T-duality procedure [30, 44] we obtained coordinate dependent noncommutativity and, consequently, nonassociativity. Also it is shown that final theory is nonlocal. In ref. [30] the bosonic string is considered in the weakly curved background — constant metric and linearly coordinate dependent Kalb-Ramond field with infinitesimal field strength, while in [44] we consider the same model as in [33], but T-dualizing along all three directions and imposing trivial winding conditions. Obtained nonlocality comes from the coordinate dependent background, or more precisely, from invariant coordinates. At the end of T-dualization procedure background fields depend on ΔV, defined as line integral. Nonlocality has been become very important issue in the quantum mechanical considerations [45].

In this article we will deal with closed bosonic string propagating in the constant metric and linear dependent Kalb-Ramond field with $B_{xy} = Hz$, the same background as in [33, 44]. But our goal here is to examine the influence of order of T-dualizations. In ref. [44] we T-dualize first along isometry directions, first along x and then along y, and at the end, along direction z. The first T-dualization produces configuration known as twisted torus which is commutative, and it is globally and locally well defined. After second T-dualization we obtained nongeometric theory with Q flux which is still locally well defined and it is commutative. The final T-dualization along z direction produces nonlocal theory which is noncommutative and nonassociative one. This line of T-dualizations we will call xyz one.

But what it will happen, if we change the order of T-dualizations, regrading (non)locality issue as well as (non)commutativity and (non)associativity? It is quite obvious that nothing will be changed if we T-dualize along line yxz, because the first two directions, which are T-dualized, are isometry ones. Some nontrivial issues could be expected if we T-dualize first along z direction. In this article we will present T-dualization of the model from [33, 44] along the T-dualization line zyx. After every step of T-dualization we will rewrite the T-dual transformation law in canonical form using the expressions for canonical momenta of the initial theory. Also we will check whether the obtained theory is commutative or not and, consequently, we will see whether it is associative or not.

The fact which is quite sure is that all three theories which we will obtain from the T-dualization line zyx are *nonlocal*. The explanation comes from the fact that background field $B_{\mu\nu}$ is z dependent and according to the generalized T-dualization procedure, after T-dualization along z, we obtain quantity ΔV which is defined as line integral. Consequently, the theory is nonlocal. But because y and x T-dualizations do not affect ΔV, all three theories obtained in zyx T-dualization line are nonlocal. That is a difference with respect to the xyz T-dualization line considered in [44].

The interesting thing is that transformation laws can be obtained from the corresponding ones in [44] by replacing $H \to -H$, but because in this article we T-dualize in the opposite direction, that produces theories of the different commutative and associative features with respect to [44]. After first T-dualization we get commutative and associative theory which is the same as in xyz case from [44]. But the second T-dualization here produces *noncommutative* and associative theory of κ-Minkowski type. That is different with respect to the xyz case, where second theory in the line is both commutative and associative. At the end we obtain the same theory as in [44] which is nonassociative and noncommutative. The noncommutativity and nonassociativity parameters have one additional "$-$" sign comparing with the corresponding ones in [44]. In this article as well as in [44], we impose trivial winding conditions which means $x^\mu(\sigma + 2\pi) = x^\mu(\sigma) + 2\pi N^\mu$, where N^μ is a winding number.

At the end we comment some quantum aspects of the problem and add two appendices. The first one contains conventions regarding light-cone coordinates, while the second one is related to the mathematical details concerning derivation of two kinds of Poisson brackets appearing in the article.

2 Bosonic string action and choice of background fields

In this section we will introduce the action for bosonic string propagating in 3D space with constant metric and Kalb-Ramond field which single component is different from zero,

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$B_{xy} = Hz$. This model is well known in literature as torus with H-flux. Since we are working with the same model as in [33, 44], for completeness we will repeat most of the steps from introductory part in the [44].

The closed bosonic string which propagates in the presence of the space-time metric $G_{\mu\nu}(x)$, Kalb-Ramond field $B_{\mu\nu}(x)$, and dilaton field $\Phi(x)$ is described by action [46–48]

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\} , \tag{2.1}$$

where world-sheet surface Σ is parameterized by $\xi^\alpha = (\tau\,, \sigma)$ $[(\alpha = 0\,,1),\ \sigma \in (0\,,\pi)]$, while x^μ $(\mu = 0, 1, 2, \ldots, D-1)$ are space-time coordinates. Intrinsic world sheet metric is denoted by $g_{\alpha\beta}$, and the corresponding scalar curvature with $R^{(2)}$.

Conformal symmetry on the quantum level is not preserved for any choice of background fields. If we want to keep conformal symmetry on the quantum level, background fields must obey the space-time field equations [49]

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2 D_\mu a_\nu = 0\,, \tag{2.2}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2 a_\rho B^\rho{}_{\mu\nu} = 0\,, \tag{2.3}$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - R - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = c\,, \tag{2.4}$$

where c is an arbitrary constant. From

$$D^\nu \beta^G_{\nu\mu} + \partial_\mu \beta^\Phi = 0\,, \tag{2.5}$$

it follows that third beta function, β^Φ, is equal to an arbitrary constant. Here $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to the space-time metric $G_{\mu\nu}$. Field strength for Kalb-Ramond field $B_{\mu\nu}$ and dilaton gradient are defined as

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\,, \quad a_\mu = \partial_\mu \Phi\,. \tag{2.6}$$

One of the solutions of these equations which is important for us here is the solution where some background fields are coordinate dependent. Let us choose Kalb-Ramond field to be linearly coordinate dependent and dilaton field to be constant. The equation (2.2) turns into

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0\,. \tag{2.7}$$

If we assume that field strength is infinitesimal, then we take $G_{\mu\nu}$ to be constant in approximation linear in $B_{\mu\nu\rho}$. Consequently, the third equation (2.4) is of the form

$$2\pi\kappa \frac{D-26}{6} = c\,. \tag{2.8}$$

The constant c is arbitrary, and fixing its value at $c = -\frac{23\pi\kappa}{3}$, we obtain $D = 3$, dimension of the space in which we will work further.

The choice of background fields in the case we will consider is

$$G_{\mu\nu} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.9}$$

where $R_\mu(\mu = 1, 2, 3)$ are radii of the compact dimensions. In terms of radii, the imposed condition that H is infinitesimal, can be rewritten as

$$\left(\frac{H}{R_1 R_2 R_3}\right)^2 = 0\,. \tag{2.10}$$

Physically, infinitesimality of H means that we work with sufficiently large torus (diluted flux approximation). If we rescale the coordinates

$$x^\mu \longmapsto x'^\mu = R_\mu x^\mu\,, \tag{2.11}$$

where indices on the right hand-side of equation are not summed, the form of the metric simplifies

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

Taking all assumption into consideration, the action is of the form

$$\begin{aligned} S &= \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \\ &= \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\left(\partial_+ x \partial_- x + \partial_+ y \partial_- y + \partial_+ z \partial_- z\right) + \partial_+ x H z \partial_- y - \partial_+ y H z \partial_- x\right], \end{aligned} \tag{2.13}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ is world-sheet derivative with respect to the light-cone coordinates $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$ and

$$x^\mu = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2.14}$$

T-dualization of dilaton is done within quantum formalism and here it will not be presented.

3 Family of three *R* flux non-local theories

In this section we will perform T-dualization of closed bosonic string equipped by H-flux torus background fields, one direction at time. T-dualization procedure will go along zyx line. We will show that all three theories are nonlocal with R-flux. Also we will find expressions connecting initial and T-dual variables, so called T-dual transformation laws. Using transformation laws in canonical form, we will check after every step whether obtained theory is (non)commutative and/or (non)associative.

3.1 T-dualization along z direction — shortcut to R-flux

Unlike the cases considered in [33, 44], where T-dualization drives along xyz line, let us do that in opposite direction and perform generalized T-dualization [28] of action (2.13) along z direction.

3.1.1 T-dualization procedure

It looks like that this direction is not isometry one. But we can show that it can be treated like isometry direction. Let us consider the global transformation

$$\delta x^\mu = \lambda^\mu \,, \tag{3.1}$$

and vary the action with respect to this transformation

$$\delta S = \frac{\kappa}{3} B_{\mu\nu\rho}\lambda^\rho \int_\Sigma d^2\xi \partial_+ x^\mu \partial_- x^\nu = \frac{2k}{3} B_{\mu\nu\rho}\lambda^\rho \epsilon^{\alpha\beta} \int_\Sigma d^2\xi [\partial_\alpha (x^\mu \partial_\beta x^\nu) - x^\mu (\partial_\alpha \partial_\beta x^\nu)] \,. \tag{3.2}$$

The second term vanishes as a consequence of contraction of antisymmetric ($\epsilon^{\alpha\beta}$) and symmetric ($\partial_\alpha \partial_\beta$) tensors, while the first one, surface term, survives, and it is, in general, different from zero. But, the expression δS is an topological invariant, so it vanishes if the map from the world-sheet to D-dimensional space-time is topologically trivial. Essentially, infinitesimal field strength H does not affect the vanishing of the surface term.

There is one more explanation of vanishing of this surface term. It is more technical and adjusted to the approximation we used in this article which essence is the explanation in paragraph above. Because we work in the approximation up to the linear terms in H, x^μ satisfies equation of motion for constant $G_{\mu\nu}$ and $B_{\mu\nu}$, $\partial_+\partial_- x^\mu = 0$, which solution is well known in literature. If the winding number is equal to zero, it holds $x^\mu(2\pi + \sigma) = x^\mu(\sigma)$, and since the configuration in the initial τ_i and final moment τ_f is fixed, the surface term vanishes.

So, in the weakly curved background case (H-flux torus background is such like that), z direction is an isometry one. Localization of the shift symmetry of the action (2.13) along z starts with introducing the covariant derivative

$$\partial_\pm z \longrightarrow D_\pm z = \partial_\pm z + v_\pm \,, \tag{3.3}$$

where $v_\pm$ is a gauge field. In order to make gauge fields unphysical ones, we introduce term with Lagrange multiplier

$$S_{add} = \frac{\kappa}{2} \int_\Sigma d^2\xi y_3 (\partial_+ v_- - \partial_- v_+) \,. \tag{3.4}$$

These two steps are the part of the standard Buscher procedure. Because of coordinate dependent background field $B_{\mu\nu}$, generalized T-dualization procedure has an additional step, introducing of an invariant coordinate

$$z^{inv} = \int_P d\xi^\alpha D_\alpha z = \int_P d\xi^+ D_+ z + \int_P d\xi^- D_- z = z(\xi) - z(\xi_0) + \Delta V \,, \tag{3.5}$$

where

$$\Delta V = \int_P d\xi^\alpha v_\alpha = \int_P (d\xi^+ v_+ + d\xi^- v_-)\,. \tag{3.6}$$

The form of the action is now

$$\bar{S} = \kappa \int_\Sigma d^2\xi \left[H z^{inv}(\partial_+ x \partial_- y - \partial_+ y \partial_- x) + \frac{1}{2}(\partial_+ x \partial_- x + \partial_+ y \partial_- y + D_+ z D_- z) \right. \\ \left. + \frac{1}{2}\, y_3(\partial_+ v_- - \partial_- v_+) \right]. \tag{3.7}$$

Fixing the gauge, $z(\xi) = z(\xi_0)$, we get gauged fixed action in the form

$$S_{fix} = \kappa \int_\Sigma d^2\xi \left[H\Delta V(\partial_+ x \partial_- y - \partial_+ y \partial_- x) + \frac{1}{2}(\partial_+ x \partial_- x + \partial_+ y \partial_- y + v_+ v_-) \right. \\ \left. + \frac{1}{2} y_3(\partial_+ v_- - \partial_- v_+) \right]. \tag{3.8}$$

The equation of motion for Lagrange multiplier y_3 obtained from above action (3.8) produces

$$\partial_+ v_- - \partial_- v_+ = 0 \Longrightarrow v_\pm = \partial_\pm z\,, \tag{3.9}$$

which drives us back to the initial action (2.13). On the other side, if we found equations of motion for gauge fields $v_\pm$, we get

$$v_\pm = \pm \partial_\pm y_3 - 2\beta^\mp\,, \tag{3.10}$$

where $\beta^\pm$ functions are defined as

$$\beta^\pm = \mp \frac{1}{2} H(x \partial_\mp y - y \partial_\mp x)\,. \tag{3.11}$$

The $\beta^\pm$ functions stem from the variation of the term containing ΔV. The derivation of beta functions $\beta^\pm$ is based on the relation $\partial_\pm \Delta V = v_\pm$. In the derivation of the beta functions there is one nontrivial technical point and that is vanishing of the surface term after one partial integration. That surface term is of the same form as in eq. (3.2), so the same reasons for surface term vanishing hold here. Mathematical details regarding derivation of $\beta^\pm$ functions can be found in refs. [28–31, 44].

Inserting the relations (3.10) into the gauge fixed action, keeping linear terms in H, we obtain the T-dual action

$${}_zS = \kappa \int_\Sigma d^2\xi \partial_+ {}_zX^\mu {}_z\Pi_{+\mu\nu} \partial_- {}_zX^\nu\,, \tag{3.12}$$

where

$${}_zX^\mu = \begin{pmatrix} x \\ y \\ y_3 \end{pmatrix}, \qquad {}_z\Pi_{+\mu\nu} = {}_zB_{\mu\nu} + \frac{1}{2}{}_zG_{\mu\nu}\,, \tag{3.13}$$

$${}_zB_{\mu\nu} = \begin{pmatrix} 0 & H\Delta V & 0 \\ -H\Delta V & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad {}_zG_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.14}$$

Let us note that presence of ΔV, defined as line integral, represents the source of nonlocality of the T-dual theory.

3.1.2 T-dual transformation law

Combining the equations of motion for Lagrange multiplier (3.9) and for gauge fields (3.10), we obtain T-dual transformation laws

$$\partial_\pm z \cong \pm\partial_\pm y_3 \mp H(x\partial_\pm y - y\partial_\pm x)\,, \tag{3.15}$$

where $\cong$ is used here to mark T-dual relation. Momentum of the initial theory (2.13) canonically conjugated to the coordinate z is of the form

$$\pi_z = \frac{\partial\mathcal{L}}{\partial\dot{z}} = \kappa\dot{z}\,, \tag{3.16}$$

where $\mathcal{L}$ is a Lagrangian density defined as $S = \int_\Sigma d^2\xi\mathcal{L}$. Calculating $\dot{z}$ using T-dual transformation law (3.15), we get the T-dual transformation law in canonical form

$$y_3' \cong \frac{1}{\kappa}\pi_z + H(xy' - yx')\,, \tag{3.17}$$

which is of the same form as in the xyz case.

In all further expressions we will keep the symbol ΔV, but we must have in mind that we used equations of motion for Lagrange multipliers (3.9) at the end of T-dulization procedure along z coordinate, so, having in mind (3.6) and (3.15), we get

$$\Delta V = \Delta z \cong \int d\xi^+\partial_+ y_3 - \int d\xi^-\partial_- y_3 \equiv \tilde{y}_3\,. \tag{3.18}$$

The variable ΔV is multiplied by infinitesimal field strength H, so, in the above expression we used $\partial_\pm z \cong \pm\partial_\pm y_3$, as a consequence of diluted flux approximation.

3.1.3 (Non)commutativity and (non)associativity

The initial theory is geometric one and its variables satisfy the standard Poisson algebra

$$\{x^\mu(\sigma), x^\nu(\bar\sigma)\} = \{\pi_\mu(\sigma), \pi_\nu(\bar\sigma)\} = 0\,, \quad \{x^\mu, \pi_\nu(\bar\sigma)\} = \delta^\mu{}_\nu\delta(\sigma - \bar\sigma)\,, \tag{3.19}$$

where x^μ are the coordinates of the initial theory, while π_μ are their canonically conjugated momenta. Using expression (3.17) and standard Poisson algebra (3.19), we obtain that coordinates of the theory obtained after one T-dualization, ${}_zX^\mu$, are commutative. Consequently, Jacobiator is equal to zero, which means that theory is associative.

Summarizing this first step of T-dualization, obtained theory is *commutative and associative nonlocal R-flux theory*. Comparing with the results of the ref. [44] after first T-dualization, qualitatively we obtain the same result, but with the essential difference that here obtained theory is nonlocal R-flux theory unlike that in [44] which is geometrical one, locally and globally well defined.

3.2 Step 2 — T-dualization along y direction

Our starting point is the action given in eq. (3.12). The background fields are independent of y, so, we apply standard Buscher procedure. This means that, unlike the previous case, we perform just first two steps in T-dualization procedure and skip the third one — introducing of invariant coordinate. The T-dualization procedure is already presented, so, we will skip explaining procedure steps further.

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3.2.1 T-dualization procedure

The gauge fixed action is of the form

$$S_{fix} = \kappa \int_{\Sigma} d^2\xi \left[\frac{1}{2} (\partial_+ x \partial_- x + v_+ v_- + \partial_+ y_3 \partial_- y_3) + H\Delta V (v_- \partial_+ x - v_+ \partial_- x) \right] + \frac{\kappa}{2} \int_{\Sigma} d^2\xi y_2 (\partial_+ v_- - \partial_- v_+) \,. \tag{3.20}$$

Varying with respect to the Lagrange multiplier y_2 we get

$$v_\pm = \partial_\pm y \,, \tag{3.21}$$

while the equations of motion for gauge fields are

$$v_\pm = \pm \partial_\pm y_2 \mp 2H\Delta V \partial_\pm x \,. \tag{3.22}$$

Inserting the expression for gauge fields (3.22) into gauge fixed action (3.20), we obtain the T-dual action

$${}_{zy}S = \kappa \int_{\Sigma} d^2\xi \; \partial_+ \, {}_{zy}X^\mu \; {}_{zy}\Pi_{+\mu\nu} \; \partial_- \, {}_{zy}X^\nu \,, \tag{3.23}$$

where

$${}_{zy}X^\mu = \begin{pmatrix} x \\ y_2 \\ y_3 \end{pmatrix} , \qquad {}_{zy}\Pi_{+\mu\nu} = {}_{zy}B_{\mu\nu} + \frac{1}{2} \, {}_{zy}G_{\mu\nu} \,, \tag{3.24}$$

$${}_{zy}B_{\mu\nu} = 0 \,, \qquad {}_{zy}G_{\mu\nu} = \begin{pmatrix} 1 & -2H\Delta V & 0 \\ -2H\Delta V & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.25}$$

Let us note that after two T-dualizations in the xyz case in [44] we also obtained that T-dual Kalb-Ramond field is zero.

3.2.2 T-dual transformation law

Combining equations of motion (3.21) and (3.22) we get the corresponding transformation law

$$\partial_\pm y \cong \pm \partial_\pm y_2 \mp 2H\Delta V \partial_\pm x \,. \tag{3.26}$$

Let us now prescribe the transformation law in canonical form. The momentum canonically conjugated to the initial coordinate y is obtained by variation of the initial action (2.13) with respect to the $\dot{y}$ and it is of the form

$$\pi_y = \kappa(\dot{y} + 2Hzx') \,, \tag{3.27}$$

while from transformation law (3.26) we have

$$\dot{y} \cong y_2' - 2H\Delta V x' \,. \tag{3.28}$$

Combining last two equations and using the fact that, in the approximation linear in H, ΔV and z are T-dual to each other, we get

$$y_2' \cong \frac{1}{\kappa} \pi_y \,. \tag{3.29}$$

As we see the transformation law is the same as in the xyz case.

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3.2.3 (Non)commutativity and (non)associativity

In this paragraph we will calculate Poisson brackets of the coordinates ${}_{zy}X^\mu$ using transformation laws in canonical form given by eqs. (3.17) and (3.29).

With the help of the standard Poisson algebra (3.19) and instructions from appendix B, it is easy to see that

$$\{x(\sigma), x(\bar\sigma)\} = \{y_2(\sigma), y_2(\bar\sigma)\} = \{y_3(\sigma), y_3(\bar\sigma)\} = \{x(\sigma), y_2(\bar\sigma)\} = \{x(\sigma), y_3(\bar\sigma)\} = 0\,. \tag{3.30}$$

The only non-zero Poisson bracket is

$$\{y_2'(\sigma), y_3'(\bar\sigma)\} \cong \frac{H}{\kappa}\left[2x'(\sigma)\delta(\sigma-\bar\sigma) + x(\sigma)\delta'(\sigma-\bar\sigma)\right]\,, \tag{3.31}$$

where $\delta' \equiv \partial_\sigma\delta(\sigma-\bar\sigma)$. This result is obtained by straightforward calculation using T-dual transformation laws, (3.17) and (3.29), and standard Poisson algebra (3.19). The relation (3.31) is of the form (B.1), where $A'(\sigma) = y_2'(\sigma)$, $B'(\bar\sigma) = y_3'(\bar\sigma)$, $U'(\sigma) = \frac{H}{\kappa}2x'(\sigma)$ and $V(\sigma) = \frac{H}{\kappa}x(\sigma)$. With these substitutions in mind, we have that final expression is of the form (B.8)

$$\{y_2(\sigma), y_3(\bar\sigma)\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar\sigma)\right]\theta(\sigma-\bar\sigma)\,. \tag{3.32}$$

For $\sigma \to \sigma + 2\pi$ and $\bar\sigma \to \sigma$ we have

$$\{y_2(\sigma+2\pi), y_3(\sigma)\} \cong -\frac{H}{\kappa}\left[x(\sigma) + 4\pi N_x\right]\,, \tag{3.33}$$

because $\theta(2\pi) = 1$ (B.6), while N_x is winding number for x coordinate

$$x(\sigma+2\pi) - x(\sigma) = 2\pi N_x\,. \tag{3.34}$$

As we can see the noncommutativity relation (3.32) is of κ-Minkowski type. It is straightforward to see that

$$\{x(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\} + \{y_2(\sigma_2), \{y_3(\sigma_3), x(\sigma_1)\}\} + \{y_3(\sigma_3), \{x(\sigma_1), y_2(\sigma_2)\}\} \cong 0\,. \tag{3.35}$$

Because the Jacobiator is zero, we conclude that this R-flux theory is **noncommutative** and **associative** one.

3.3 Step 3 — T-dualization along x direction

In this subsection we will finish T-dualization procedure not repeating the mathematical details, but giving just the important equations and results.

The gauge fixed action is given by the following equation

$$\begin{aligned} S_{fix} = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}(v_+v_- + \partial_+y_2\partial_-y_2 + \partial_+y_3\partial_-y_3) - H\Delta V(v_+\partial_-y_2 + \partial_+y_2\, v_-)\right] \\ + \frac{\kappa}{2}\int_\Sigma d^2\xi y_1(\partial_+v_- - \partial_-v_+)\,. \end{aligned} \tag{3.36}$$

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The equations of motion for Lagrange multiplier produces

$$v_\pm = \partial_\pm x\,, \tag{3.37}$$

while the equations of motion for gauge fields $v_\pm$ give

$$v_\pm = \pm\partial_\pm y_1 + 2H\Delta V\ \partial_\pm y_2\,. \tag{3.38}$$

Inserting expressions for $v_\pm$ into gauge fixed action we get the T-dual action

$${}_{zyx}S = \kappa \int_\Sigma d^2\xi \partial_+\ {}_{zyx}X^\mu\ {}_{zyx}\Pi_{+\mu\nu}\ {}_{zyx}\ X^\nu\,, \tag{3.39}$$

where

$${}_{zyx}X^\mu = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad {}_{zyx}\Pi_{+\mu\nu} = {}_{zyx}B_{\mu\nu} + \frac{1}{2}\ {}_{zyx}G_{\mu\nu} \tag{3.40}$$

$${}_{zyx}B_{\mu\nu} = \begin{pmatrix} 0 & -H\Delta V & 0 \\ H\Delta V & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad {}_{zyx}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.41}$$

Combining the equations of motion (3.37) and (3.38) we obtain the T-dual transformation law

$$\partial_\pm x \cong \pm\partial_\pm y_1 + 2H\Delta V \partial_\pm y_2\,. \tag{3.42}$$

It directly follows that

$$\dot{x} \cong y_1' + 2H\Delta V \dot{y}_2\,. \tag{3.43}$$

From the initial action (2.13) it is obvious that momentum canonically conjugated to x is of the form

$$\pi_x = \kappa\dot{x} - 2\kappa H z y'\,. \tag{3.44}$$

The T-dual transformation law for y (3.26), in the approximation linear in H, produces that $y' \cong \dot{y}_2$. Taking into account the relation (3.43), we get the canonical form of the T-dual transformation law

$$y_1' \cong \frac{1}{\kappa}\pi_x\,. \tag{3.45}$$

As we see the full set of T-dual transformation laws, (3.17), (3.29) and (3.45), are the same as in the case where T-dualization was along xyz line [44] up to $H \to -H$. The full T-dualized theory is of the same form as in [44] with the expressions for **noncommutativity**

$$\{y_1(\sigma), y_3(\bar\sigma)\} \cong \frac{H}{\kappa}\left[2y(\sigma) - y(\bar\sigma)\right]\theta(\sigma-\bar\sigma)\,, \tag{3.46}$$

$$\{y_2(\sigma), y_3(\bar\sigma)\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar\sigma)\right]\theta(\sigma-\bar\sigma)\,, \tag{3.47}$$

and **nonassociativity**

$$\begin{aligned}
&\{y_1(\sigma_1), y_2(\sigma_2), y_3(\sigma_3)\} \equiv \\
&\{y_1(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\} + \{y_2(\sigma_2), \{y_3(\sigma_3), y_1(\sigma_1)\}\} + \{y_3(\sigma_3), \{y_1(\sigma_1), y_2(\sigma_2)\}\} \cong \\
&\frac{2H}{\kappa^2}\left[\theta(\sigma_1-\sigma_2)\theta(\sigma_2-\sigma_3) + \theta(\sigma_2-\sigma_1)\theta(\sigma_1-\sigma_3) + \theta(\sigma_1-\sigma_3)\theta(\sigma_3-\sigma_2)\right],
\end{aligned} \tag{3.48}$$

which can be obtained from the corresponding ones in xyz case [44] by replacing $H \to -H$.

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4 Quantum aspects of T-dualization in the weakly curved background

In proving isometry and computing the $\beta^{\pm}$ functions we assumed the trivial topology and the surface term occurring there vanishes. Now we want to discuss some quantum aspects of the considered problems in nontrivial topologies. We will consider the action for bosonic string in the weakly curved background — constant metric and Kalb-Ramond field depending on all coordinates and with infinitesimal field strength. Torus with infinitesimal H-flux is special case of this model.

On th classical level there are a few problems in the theory. In order to perform the generalized T-dualization procedure the invariant coordinate x^{μ}_{inv} is introduced. But it is multivalued and the proof of equivalence of gauged and initial theories needs the part considering global characteristics. Moreover, in the quantum theory at higher genus, the holonomies of the world-sheet gauge fields complicate the situation a little bit. Fortunately, these problems can be resolved in Abelian case in the quantum theory [50–52].

First, we make Wick rotation $\tau \to -i\tau$, which makes the term which contains metric tensor $G_{\mu\nu}$ gets multiplier i, while the terms which contain Kalb-Ramond field $B_{\mu\nu}$ and Lagrange multiplier y_μ stay unchanged. Then the partition function is of the form

$$Z = \sum_{g=0}^{\infty} \int \mathcal{D}y\mathcal{D}v \, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\, {}^\star v + i\kappa \int_\Sigma vB[V]v + \frac{i\kappa}{2}\int_\Sigma v dy} \,. \tag{4.1}$$

We use differential forms and omit the space-time indices to simplify writing of equations. The Hodge duality operator is denoted by star. The index g denotes the genus of manifold.

The first step in the calculation process is separation the one form dy into the exact part dy_e (y_e is single valued) and the harmonic part y_h ($dy_h = 0 = d^\dagger y_h$)

$$dy = dy_e + y_h. \tag{4.2}$$

For the closed forms the co-exact term $d^\dagger y_{co}$ in the Hodge decomposition is missing.

The path integral (4.1) goes over all degrees of freedom including local degrees of freedom as well as the sum over different topologies. Consequently, according to the (4.2), we substitute $\mathcal{D}y$ with the path integral over y_e and the sum over all possible topologically nontrivial states contained in y_h (marked by H_y)

$$\mathcal{D}y \to \mathcal{D}y_e \sum_{H_y} . \tag{4.3}$$

The integration over y_e induces vanishing of the field strength

$$Z = \int \mathcal{D}v \, \delta(dv) \, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\, {}^\star v + i\kappa\int_\Sigma vB[V]v} \sum_{H_y} e^{\frac{i\kappa}{2}\int_\Sigma v y_h}. \tag{4.4}$$

The 1-form v can be expressed as sum of exact, co-exact and the harmonic parts

$$v = dx + d^\dagger v_{ce} + v_h, \tag{4.5}$$

which means that

$$\mathcal{D}v \to \mathcal{D}x\mathcal{D}d^\dagger v_{ce}\, dH_v. \tag{4.6}$$

The functional integration over harmonic part v_h drives to the ordinary integration over topologically nontrivial periods (marked by symbol H_v). After integration over $d^\dagger v_{ce}$ we get

$$Z = \int \mathcal{D}x dH_v\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G^\star v + i\kappa \int_\Sigma vB[V]v} \sum_{H_y} e^{\frac{i\kappa}{2}\int_\Sigma v y_h}. \tag{4.7}$$

The last term in the exponent is responsible for nontrivial holonomies. Eliminating v_{ce} part, the 1-form v becomes closed and the Riemann bilinear relation becomes usable

$$\int_\Sigma v y_h = \sum_{i=1}^{g} \Big[\oint_{a_i} v \oint_{b_i} y_h - \oint_{a_i} y_h \oint_{b_i} v \Big]. \tag{4.8}$$

The symbols $a_i, b_i\, (i = 1, 2, \ldots, g)$ represent the canonical homology basis for the world-sheet. Because of the periodicity of the Lagrange multiplier y, its periods are just the winding numbers around cycles a_i and b_i

$$N_{a_i} = \oint_{a_i} y_h, \quad N_{b_i} = \oint_{b_i} y_h\, . \tag{4.9}$$

Denoting the periods with

$$A_i = \oint_{a_i} v, \quad B_i = \oint_{b_i} v\, , \tag{4.10}$$

we get

$$\int_\Sigma v y_h = \sum_{i=1}^{g} (N_{b_i} A_i - N_{a_i} B_i). \tag{4.11}$$

Now the partition function (4.7) gets the form

$$Z = \int \mathcal{D}x\, dA_i dB_i e^{-\frac{\kappa}{2}\int_\Sigma v\, G^\star v + i\kappa\int_\Sigma vB[V]v} \sum_{N_{a_i}, N_{b_i} \in Z} e^{\frac{i\kappa}{2}\sum_{i=1}^{g}(N_{b_i}A_i - N_{a_i}B_i)}. \tag{4.12}$$

The periodic delta function is defined as $\delta(x) = \frac{1}{2\pi}\sum_{n\in Z} e^{inx}$, which produces

$$Z = \int \mathcal{D}x\, dA_i dB_i \delta\left(\frac{\kappa}{2}A_i\right)\delta\left(\frac{\kappa}{2}B_i\right)\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G^\star v + i\kappa\int_\Sigma vB[V]v}\, . \tag{4.13}$$

It is useful to examine the path dependence of the variable V^μ, which form is now

$$V^\mu(\xi) = x^\mu(\xi) - x^\mu(\xi_0) + \int_P v_h^\mu. \tag{4.14}$$

Let us consider two paths, P_1 and P_2, with the same initial ξ_0^α and the final points ξ^α. Now we will subtract from the value of V^μ along P_1 the value along path P_2 and obtain the integral over closed curve $P_1 P_2^{-1}$ of the harmonic form

$$V^\mu[P_1] - V^\mu[P_2] = \oint_{P_1 P_2^{-1}} v_h^\mu. \tag{4.15}$$

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Establishing the homology between the closed curve $P_1P_2^{-1}$ and curve $\sum_i \left[n_i a_i + m_i b_i\right]$, $(n_i, m_i \in Z)$ we get finally

$$V^\mu[P_1] = V^\mu[P_2] + \sum_i (n_i A_i^\mu + m_i B_i^\mu). \tag{4.16}$$

The variable $V^\mu(\xi)$ in classical theory is path dependent if holonomies are nontrivial.

Integrating eq. (4.13) over A_i and B_i implies that periods A_i and B_i are zero. Consequently

$$v = dx. \tag{4.17}$$

The variable V^μ becomes single valued, and the initial theory is restored

$$Z = \int \mathcal{D}x e^{-\frac{\kappa}{2}\int_\Sigma dx\, G\, {}^\star dx + i\kappa\int_\Sigma dx B[x]dx} = \int \mathcal{D}x e^{-\kappa\int_\Sigma d^2\xi \partial x \Pi_+[x]\bar{\partial}x}. \tag{4.18}$$

Consequently, starting with partition function of the gauged fixed action of bosonic string in the weakly curved background, within path integral formalism and in the presence of nontrivial topologies, we came to the partition function of the initial theory. That means that introducing coordinate dependent Kalb-Ramond field is consistent with path integral quantization process.

5 Conclusion

In this article we studied the 3D closed bosonic string propagating in the geometry known as torus with H-flux — constant metric and Kalb-Ramond field with just one nonzero component, $B_{xy} = -B_{yx} = Hz$. The choice of background fields is consistent with the consistency conditions if we work in the diluted flux approximation which assumes that in all calculations we keep just the constant terms and those linear in the infinitesimal field strength H. Our goal was to study the T-dualization line which goes in the opposite direction from the standard one. First, we T-dualize z direction, then y and at the end along x direction — so-called zyx T-dualization line. We analyzed in every step the (non)commutativity and (non)associativity of the obtained theory and made comparisons with the case of xyz T-dualization line considered in [33, 44].

The common fact for all three theories obtained in the process of T-dualization step by step is that all three ones are nonlocal R-flux theories. The nonlocality comes as a result of the first step in T-dualization procedure, T-dualization along z direction. Generalized T-dualization procedure has one additional step with respect to the standard Buscher procedure and that is introduction of invariant coordinate. In the process of T-dualization invariant coordinate turns into variable ΔV which is defined as line integral. Consequently, this means that obtained theory is nonlocal. Further T-dualizations does not affect ΔV and, all three theories are nonlocal ones. As we know, in the case of xyz T-dualization line [44], we obtained three different theories in geometrical sense — twisted torus, Q-flux theory (which is local) and nonlocal R-flux theory.

The dualization along z direction produces nonlocal R-flux theory unlike the xyz case [33, 44] where the theory obtained after first T-dualization is locally and globally well defined. Because initial theory is geometrical one, its variables satisfy standard Poisson algebra (3.19). Using (3.19) and T-dual transformation law written in the canonical form (3.17), we showed that theory obtained after T-dualization along z coordinate (using generalized T-dualization procedure) is *commutative* and, consequently, *associative* one as in [44].

The second step in T-dualization is T-dualization along y direction. Using standard Buscher procedure, we obtained the form of the T-dual theory and the corresponding T-dual transformation law, which is rewritten in the canonical form (3.29) in terms of the coordinates and momenta of the initial theory. Using standard Poisson algebra (3.19) and T-dual transformation laws in canonical form, (3.17) and (3.29), we easily proved that theory after two T-dualizations is *noncommutative*, but it is still *associative* one. In this article we used trivial winding condition (3.34) and showed that T-dual coordinates $y_2(\sigma)$ and $y_3(\bar{\sigma})$ are commutative for equal arguments, $\sigma = \bar{\sigma}$, but they are noncommutative if $\sigma - \bar{\sigma} = 2\pi$. The result is qualitatively similar to the result of [33], where after two T-dualizations the obtained theory is noncommutative one. But, the difference is in the winding condition which is nontrivial in [33], mixing different coordinates. The different winding condition induces the noncommutativity for $\sigma = \bar{\sigma}$ (for more details see [33]). On the other hand in the analysis presented in [44] (xyz T-dualization line) the theory obtained after two T-dualizations is commutative under trivial winding condition.

The final step in T-dualization procedure is T-dualization along x direction. The theory after full T-dualization is the same as in xyz case [44] with the noncommutativity and nonassociativity parameters which can be obtained from those in xyz case [44] adding "$-$" sign. This is a consequence of the fact that the full set of T-dual transformation laws is the same as in [44] up to the replacing $H \to -H$. This difference up to the "$-$" sign stems from the initial actions. In this article we start from (2.13), while in [44] the starting action for z T-dualization is Q-flux action, formally the same as (2.13) up to the replacing $H \to -H$.

Finishing the discussion of the results obtained in this paper it is interesting to make comparison with some similar efforts. We studied the abelian isometries using both standard and generalized T-duality procedure, while in the paper [53] nonabelian isometries using standard Buscher procedure are considered. The authors of [53] showed that spaces with isometry maps to the nonisometry spaces, while in this paper there is isometry in every T- dualization step. One of their conclusions that T-dual transformations are more than continuous isometry can be added to the concluding remarks of this paper. In the ref. [43] generalized T-duality and nongeometric background are considered, but using low energy effective action, unlike here, where we used sigma model action. The paper [54] deals with T-dualizations along nonisometry directions like in [31], using extension of gauge symmetry, while the authors of [31] use the generalized T-dualization procedure introducing invariant coordinates (in [54] they call them "covariant" coordinates). In this paper we use this generalized T-dualization procedure but all directions considered here are isometry ones. It is useful to mention that in the paper [55] bosonic string in the presence of the weakly curved backgrounds is considered using double space formalism as well as the influence of the order of T-dualizations. The double space formalism gives the result which is in accordance with the result of the current paper.

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Consequently, we conclude that in the case of the full T-dualization the form of the T-dual theory do not depend on the order of T-dualization, while parameters of noncommutativity and nonassociativity change sign.

A Light-cone coordinates

In the paper we often use light-cone coordinates defined as

$$\xi^{\pm} = \frac{1}{2}(\tau \pm \sigma) . \tag{A.1}$$

The corresponding partial derivatives are

$$\partial_{\pm} \equiv \frac{\partial}{\partial \xi^{\pm}} = \partial_\tau \pm \partial_\sigma . \tag{A.2}$$

Two dimensional Levi-Civita $\varepsilon^{\alpha\beta}$ is chosen in (τ, σ) basis as $\varepsilon^{\tau\sigma} = -1$. Consequently, in the light-cone basis the form of tensor is

$$\varepsilon^{\alpha\beta}_{lc} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} . \tag{A.3}$$

The flat world-sheet metric is of the form in (τ, σ) and light-cone basis, respectively

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \eta^{lc}_{\alpha\beta} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} . \tag{A.4}$$

Let us stress that in whole article we use standard notation for τ and σ derivatives — $\dot{A} \equiv \partial_\tau A$ and $A' \equiv \partial_\sigma A$, where A is an arbitrary variable.

B Two types of Poisson brackets used in the paper

In this paper, we have seen that T-dual transformation laws connect derivatives of T-dual coordinates with coordinates and momenta of initial theory. While initial theory satisfies standard Poisson brackets, in order to find Poisson brackets for T-dual theory, we first need to find Poisson brackets between σ derivatives of T-dual coordinates. This type of Poisson bracket will, in general case, be some function of initial coordinates, Dirac delta functions and their derivatives with respect to σ. Having this in mind, general case for our Poisson brackets will have following form

$$\{A'(\sigma), B'(\bar{\sigma})\} = U'(\sigma)\delta(\sigma - \bar{\sigma}) + V(\sigma)\delta'(\sigma - \bar{\sigma}) , \tag{B.1}$$

where $\delta'(\sigma - \bar{\sigma}) \equiv \partial_\sigma \delta(\sigma - \bar{\sigma})$. For terms $A'(\sigma)$, $U'(\sigma)$ and $B'(\bar{\sigma})$, symbol $'$ stands for partial derivative with respect to σ and $\bar{\sigma}$, respectively. If we want to calculate the Poisson bracket

$$\{A(\sigma), B(\bar{\sigma})\} ,$$

first we have to calculate the following one

$$\{\Delta A(\sigma, \sigma_0), \Delta B(\bar{\sigma}, \bar{\sigma}_0)\} , \tag{B.2}$$

where

$$\Delta A(\sigma,\sigma_0)=\int_{\sigma_0}^{\sigma}dxA'(x)=A(\sigma)-A(\sigma_0)\,,\quad \Delta B(\bar\sigma,\bar\sigma_0)=\int_{\bar\sigma_0}^{\bar\sigma}dxB'(x)=B(\bar\sigma)-B(\bar\sigma_0)\,. \tag{B.3}$$

Substituting the expressions (B.3) into (B.2), we have

$$\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=\int_{\sigma_0}^{\sigma}dx\int_{\bar\sigma_0}^{\bar\sigma}dy\,\left[U'(x)\delta(x-y)+V(x)\delta'(x-y)\right]\,. \tag{B.4}$$

After integration over y we get

$$\begin{aligned}&\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=\\ &=\int_{\sigma_0}^{\sigma}dx\{U'(x)\left[\theta(x-\bar\sigma_0)-\theta(x-\bar\sigma)\right]+V(x)\left[\delta(x-\bar\sigma_0)-\delta(x-\bar\sigma)\right]\},\end{aligned} \tag{B.5}$$

where $\theta(x)$ is defined as

$$\theta(x)=\int_0^x d\eta\delta(\eta)=\frac{1}{2\pi}\left[x+2\sum_{n\geq 1}\frac{1}{n}\sin(nx)\right]=\begin{cases}0 & \text{if } x=0\\ 1/2 & \text{if } 0<x<2\pi\,,\\ 1 & \text{if } x=2\pi\end{cases} \tag{B.6}$$

where $\delta(x)=\frac{1}{2\pi}\sum_{n\in Z}e^{inx}$. Finally, integrating over x, we obtain

$$\begin{aligned}\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=&\,U(\sigma)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma-\bar\sigma)]-U(\sigma_0)[\theta(\sigma_0-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma)]\\ &-U(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0)]+U(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)]\\ &+V(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0]-V(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)].\end{aligned} \tag{B.7}$$

From the last expression, using (B.3), we extract the searched Poisson bracket

$$\{A(\sigma),B(\bar\sigma)\}=-[U(\sigma)-U(\bar\sigma)+V(\bar\sigma)]\theta(\sigma-\bar\sigma)\,. \tag{B.8}$$

In order to calculate Jacobiator we have to find Poisson brackets of type $\{y(\sigma),x(\bar\sigma)\}$, where $y(\sigma)$ is coordinate T-dual to initial one $x(\sigma)$. Having this in mind, we start with the following Poisson bracket

$$\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}=\left\{\int_{\sigma_0}^{\sigma}d\eta y'(\eta),x(\bar\sigma)\right\}\,, \tag{B.9}$$

and using T-dual transformation law in canonical form

$$\pi\cong\kappa y'\,, \tag{B.10}$$

we get

$$\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}\cong\frac{1}{\kappa}\left\{\int_{\sigma_0}^{\sigma}d\eta\pi(\eta),x(\bar\sigma)\right\}\,, \tag{B.11}$$

where $\pi(\sigma)$ is momentum canonically conjugated to the coordinate $x(\sigma)$. Initial theory is geometric one which variables satisfy standard Poisson algebra, so, the final result is of the form

$$\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}\cong-\frac{1}{\kappa}\left[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)\right]\quad\Longrightarrow\{y(\sigma),x(\bar\sigma)\}\cong-\frac{1}{\kappa}\theta(\sigma-\bar\sigma)\,. \tag{B.12}$$

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RESEARCH ARTICLE

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Combined Fermionic and Bosonic T-duality of Type II Superstring Theory with Coordinate Dependent RR Field

*B. Nikolić and D. Obrić**

We investigate effects of fermionic T-duality on type II superstring in presence of Ramond-Ramond (RR) field that has infinitesimal linear dependence on bosonic coordinate x^μ. Other fields are assumed to be constant. Procedure that we employ for obtaining fermionic T-dual theory is Buscher procedure, where we will consider two distinct cases. One, where action has not been T-dualized along bosonic coordinates and other where it has. By analyzing these two cases, their actions and T-dual transformation laws, we obtain some insight into how background fields transform and what are necessary ingredients for emergence of fermionic non-commutativity.

1. Introduction

T-duality represents a map that connects different superstring theories, mapping geometry and topology from one theory to another.[1] This symmetry was originally developed with bosonic coordinates in mind, where two theories are connected by transformation laws that establish a link between coordinates.[2] It was not until 2008 that it has been noticed that the same duality can emerge in case of fermionic coordinates. In their paper[3] Berkovits and Maldacena showed that tree level superstring theories in presence of supersymmetric background fields posses new kind of symmetry. Symmetry that maps supersymmetric background fields of one theory to supersymmetric backgrounds of other theory, where dilaton and RR fields are now different. Just like in case of bosonic T-duality, mathematical machinery for obtaining T-dual theories is Buscher procedure[4,5] applied to fermionic coordinates θ^α and $\bar{\theta}^\alpha$.

Busher T-dualization procedure including its extension to fermionic coordinates[6–9] and its generalizations[10–12] mainly follow the same steps. We notice some global symmetry in the theory, usually shift symmetry, which is then localized by replacing partial derivatives with covariant ones. Covariant derivatives come with new gauge fields that insert new degrees of freedom into the theory, these degrees of freedom are eliminated with help of Lagrange multipliers. Next step is utilizing gauge freedom to fix starting coordinates. After that, finding equations of motion for gauge fields and inserting their solutions into the action we obtain T-dual theory. Extension of procedure for fermionic coordinates does not introduce any new steps into the play. However, extending the procedure to include coordinate dependent background fields does introduce one additional step. Namely, we need to replace all coordinates with invariant ones constructed as integrals of covariant derivatives. This step is necessary in order to preserve local shift symmetry.

In our previous paper[13] we demonstrated that T-dual of type II superstring which is moving in coordinate dependent RR field possesses non-commutative Poisson brackets. Since T-duality was performed only along bosonic coordinates it produced non-commutativity only between bosonic T-dual coordinates. In addition to this, we had that background fields that were constants in original theory became functions of both bosonic and fermionic coordinates in dual theory. This has left us with one open question: Would fermionic T-duality of starting theory or even theory that has been dualized along bosonic coordinates produce non-commutative relations between fermionic coordinates? While it has been shown that, in case of closed bosonic string, non-commutativity arises only in coordinates that had appeared in background fields of starting theory[14], it is not clear if that is the case for fermionic coordinates, especially when we have emergence of new coordinate dependence in background fields after bosonic T-duality.

In this article, our goal is to find what effects fermionic T-duality has on action where RR field has dependence on bosonic coordinates and do these effects change for fully dualized action. By obtaining background fields in different stages of T-duality we can determine how geometry of theory changes and when the theory makes the switch from being local to non-local one. At the end we provide few notes on how fermionic T-duality interacts with bosonic T-duality in providing new non-commutative relations.

2. Type II Superstring, Choice of Fields and Bosonic T-Duality

In this section we will present action for type II superstring in pure spinor formulation. We will also define background fields in which string propagates. Finally, we present action that has been T-dualized along bosonic coordinates.

2.1. Type II Superstring in Pure Spinor Formulation

The most general form of type II superstring action in pure spinor formalism[15–19] is given as

$$S = S_0 + V_{SG}. \tag{2.1}$$

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First term is action for string that propagates in flat background fields.

$$S_0 = \int_\Sigma d^2\xi \left(\frac{k}{2}\eta_{\mu\nu}\partial_m x^\mu \partial_n x^\nu \eta^{mn} - \pi_\alpha \partial_- \theta^\alpha + \partial_+ \bar{\theta}^\alpha \bar{\pi}_\alpha \right) + S_\lambda + S_{\bar{\lambda}}, \tag{2.2}$$

where terms S_λ and $S_{\bar{\lambda}}$represent actions that are composed of pure spinors and their canonical momenta. The pure spinors satisfy pure spinor constraints

$$\lambda^\alpha (\Gamma^\mu)_{\alpha\beta} \lambda^\beta = \bar{\lambda}(\Gamma^\mu)_{\alpha\beta}\bar{\lambda}^\beta = 0. \tag{2.3}$$

All modifications to flat background fields are accomplished by introducing second term in equation (2.1). This term is an integrated vertex operator for massless type II supergravity

$$V_{SG} = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar{X}^N. \tag{2.4}$$

In general case matrix A_{MN} is composed of physical fields, their curvatures (field strengths) and auxiliary fields that can be expressed with physical ones. These fields are some functions of both bosonic coordinates x^μ and fermionic coordinates θ^α and $\bar{\theta}^\alpha$. Dependence of fields on fermionic coordinates is given as expansion in powers of θ^α and $\bar{\theta}^\alpha$. In our particular case, we will set all background fields except RR field to be constant. Further more, in order to simplify calculations, all terms that are nonlinear in fermionic coordinates θ^α and $\bar{\theta}^\alpha$ will be neglected. With these assumptions in mind we have that vectors X^M and $\bar{X}^M$ and matrix A_{MN} have following form

$$X^M = \begin{pmatrix} \partial_+\theta^\alpha \\ \partial_+ x^\mu \\ \pi_\alpha \\ \frac{1}{2}N_+^{\mu\nu} \end{pmatrix}, \; \bar{X}^M = \begin{pmatrix} \partial_-\bar{\theta}^\lambda \\ \partial_- x^\mu \\ \bar{\pi}_\lambda \\ \frac{1}{2}\bar{N}_-^{\mu\nu} \end{pmatrix},$$

$$A_{MN} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k(\frac{1}{2}g_{\mu\nu} + B_{\mu\nu}) & \bar{\Psi}_\mu^\beta & 0 \\ 0 & -\Psi_\nu^\alpha & \frac{2}{k}(f^{\alpha\beta} + C_\rho^{\alpha\beta}x^\rho) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.5}$$

Pure spinor contribution to vectors X^M and $\bar{X}^M$ are encoded in

$$N_+^{\mu\nu} = \frac{1}{2}\omega_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \lambda^\beta, \quad \bar{N}_-^{\mu\nu} = \frac{1}{2}\bar{\omega}_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \bar{\lambda}^\beta. \tag{2.6}$$

Since this term does not contribute to the vertex operator, we have that pure spinor actions are decoupled from the rest. This allows us to neglect pure spinor parts from now on.

Our choice of matrix A_{MN} is composed of following fields: symmetric tensor $g_{\mu\nu}$, Kalb-Ramon antisymmetric tensor $B_{\mu\nu}$, Mayorana-Weyl gravitino fields Ψ_μ^α and $\bar{\Psi}_\mu^\alpha$, Ramond-Ramond field $\frac{2}{k}(f^{\alpha\beta} + C_\rho^{\alpha\beta}x^\rho)$ where $f^{\alpha\beta}$ and $C_\rho^{\alpha\beta}$ are constant tensors. We have also assumed that dilaton field Φ is constant. This means that factor e^Φ is included in constants $f^{\alpha\beta}$ and $C_\rho^{\alpha\beta}$. This choice of background fields is accompanied with following condition

$$\gamma_{\alpha\beta}^\mu C_\mu^{\beta\gamma} = 0, \quad \gamma_{\alpha\beta}^\mu C_\mu^{\gamma\beta} = 0. \tag{2.7}$$

String propagates in superspace spanned by bosonic coordinates x^μ ($\mu = 0, 1, ..., 9$) and fermionic ones θ^α, $\bar{\theta}^\alpha$ with 16 independent real components each. Fermionic coordinates are accompanied by their canonically conjugated momenta π_α and $\bar{\pi}_\alpha$. Both fermionic coordiantes and their momenta are given as Majorana-Wejl spinors. World sheet Σ that string sweeps in this superspace is parameterized by ξ^m ($\xi^0 = \tau$, $\xi^1 = \sigma$). By combining these parameters we can define light-cone parametrization $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$ and light-cone partial derivatives $\partial_\pm = \partial_\tau \pm \partial_\sigma$.

Inserting all these assumptions into action (2.1) and integrating out fermionic momenta, we are left with following expression

$$S = k\int_\Sigma d^2\xi \Big[\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu + \frac{1}{2}(\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu \bar{\Psi}_\mu^\alpha)\big(F^{-1}(x)\big)_{\alpha\beta}(\partial_-\theta^\beta + \Psi_\nu^\beta \partial_- x^\nu)\Big], \tag{2.8}$$

where we have introduced following tensors

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}, \tag{2.9}$$

$$F^{\alpha\beta}(x) = f^{\alpha\beta} + C_\mu^{\alpha\beta}x^\mu,$$

$$(F^{-1}(x))_{\alpha\beta} = (f^{-1})_{\alpha\beta} - (f^{-1})_{\alpha\alpha_1} C_\rho^{\alpha_1\beta_1} x^\rho (f^{-1})_{\beta_1\beta}. \tag{2.10}$$

To obtain meaningful T-dual transformation laws we need to assume that x^μ dependent part of tensor $(F^{-1}(x))_{\alpha\beta}$ is antisymmetric and infinitesimal. This additional assumption does not infringe on constraint (2.7).,[15]

Having obtained one of relevant actions , we will now focus on bosonic T-dualization of (2.8) to obtain our second action of interest.

2.2. Bosonic T-dualization

Bosonic T-dualization of action (2.8) is given in detail in [13]. Here we will only summarize the most important results.

One way to obtain T-duality is by Buscher procedure. This procedure is based on localization of translation symmetry. When we localize symmetry we replace all partial derivatives with covariant ones, while in cases where background fields depend on coordinates we also need to introduce invariant coordinate. Invariant coordinate is non-local addition to action and it is the sole reason for emergence of non-commutative behavior in closed strings. Introduction of covariant derivatives and invariant coordinates produces additional gauge fields in action, which in turn add new degrees of freedom to the theory. T-dual and original theory represent same physical system and we expect that those two theories carry exact same degrees of freedom. Because of this, we remove all newly introduced degrees of freedom with Lagrange multipliers. By utilizing gauge freedom of action we can fix bosonic coordinates to be some constant, in essence removing them from action. This gauge fixed action is only a function of gauge fields and Lagrange multipliers. Finding equation of motion for Lagrange multipliers and inserting them into action we can restore original action. On the other hand, finding equations of motion for gauge fields and inserting them into action we obtain T-dual action.

Action (2.8), due to antisymmetric part of tensor $F^{-1}_{\alpha\beta}(x)$ is invariant under global translations of bosonic coordinates. Following steps of Busher procedure, described in preceding paragraph, we obtain following T-dual action

$$ {}^bS = \frac{k}{2}\int_\Sigma d^2\xi\Big[\frac{1}{2}\bar{\Theta}^{\mu\nu}\partial_+y_\mu\partial_-y_\nu + \partial_+\bar{\theta}^\alpha\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta + \partial_+y_\mu\,{}^b\bar{\Psi}^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta + \partial_+\bar{\theta}^\alpha\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}\,{}^b\Psi^{\nu\beta}\partial_-y_\nu\Big]. \tag{2.11}$$

Here, y_μ is a dual coordinate, left superscript ^{b} denotes bosonic T-duality and V^0 represents following integral

$$\Delta V^{(0)\rho} = \frac{1}{2}\int_P d\xi^+\breve{\Theta}_-^{\rho_1\rho}\left[\partial_+y_{\rho_1} - \partial_+\bar{\theta}^\alpha(f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] - \frac{1}{2}\int_P d\xi^-\breve{\Theta}_-^{\rho\rho_1}\left[\partial_-y_{\rho_1} + \bar{\Psi}^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right]. \tag{2.12}$$

T-dual tensors that appear in action have following interpretation: $\bar{\Theta}_-^{\mu\nu}$ is inverse tensor of $\bar{\Pi}_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar{\Psi}^\alpha_\mu(F^{-1}(x))_{\alpha\beta}\Psi^\beta_\nu = \breve{\Pi}_{+\mu\nu} - \frac{1}{2}\bar{\Psi}^\alpha_\mu(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho x^\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_\nu$, defined as

$$\bar{\Theta}_-^{\mu\nu}\bar{\Pi}_{+\nu\rho} = \delta^\mu_\rho, \tag{2.13}$$

where

$$\bar{\Theta}_-^{\mu\nu} = \breve{\Theta}_-^{\mu\nu} + \frac{1}{2}\breve{\Theta}_-^{\mu\mu_1}\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho V^{(0)\rho}(f^{-1})_{\beta_1\beta}\Psi^{\beta_1}_{\nu_1}\breve{\Theta}_-^{\nu_1\nu}, \tag{2.14}$$

$$\breve{\Theta}_-^{\mu\nu}\breve{\Pi}_{+\nu\rho} = \delta^\mu_\rho, \qquad \breve{\Theta}_-^{\mu\nu} = \Theta_-^{\mu\nu} - \frac{1}{2}\Theta_-^{\mu\mu_1}\bar{\Psi}^\alpha_{\mu_1}(\breve{f}^{-1})_{\alpha\beta}\Psi^\beta_{\nu_1}\Theta_-^{\nu_1\nu} \tag{2.15}$$

$$\breve{f}^{\alpha\beta} = f^{\alpha\beta} + \frac{1}{2}\Psi^\alpha_\mu\Theta_-^{\mu\nu}\bar{\Psi}^\beta_\nu, \tag{2.16}$$

$$\Theta_-^{\mu\nu}\Pi_{+\mu\rho} = \delta^\mu_\rho, \qquad \Theta_- = -4(G_E^{-1}\Pi_-G^{-1})^{\mu\nu}. \tag{2.17}$$

Tensor $({}^bF^{-1}\$(V^{(0)})\$)_{\alpha\beta}$ is T-dual to $(F^{-1}(x))_{\alpha\beta}$,

$$\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta} = \left(F^{-1}(V^{(0)})\right)_{\alpha\beta} - \frac{1}{2}\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar{\Theta}_-^{\mu\nu}\bar{\Psi}^{\beta_1}_\nu \times\left(F^{-1}(V^{(0)})\right)_{\beta_1\beta}. \tag{2.18}$$

Finally, ${}^b\bar{\Psi}^{\mu\alpha}$ and ${}^b\Psi^{\nu\beta}$ are T-dual gravitino fields, given as

$$ {}^b\bar{\Psi}^{\mu\alpha} = \frac{1}{2}\bar{\Theta}^{\mu\nu}\bar{\Psi}^\alpha_\nu + \frac{1}{4}\bar{\Theta}_-^{\mu\mu_1}\bar{\Psi}^\beta_{\mu_1}\left(F^{-1}(V^{(0)})\right)_{\beta\beta_1}\Psi^{\beta_1}_\nu\Theta_-^{\nu\nu_1}\bar{\Psi}^\alpha_{\nu_1} = \frac{1}{2}\Theta_-^{\mu\nu}\bar{\Psi}^\alpha_\mu, \tag{2.19}$$

$$ {}^b\Psi^{\nu\beta} = -\frac{1}{2}\Psi^\beta_\mu\bar{\Theta}_-^{\mu\nu} - \frac{1}{2}\Psi^\beta_\mu\Theta_-^{\mu\mu_1}\bar{\Psi}^\alpha_{\mu_1}\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi^{\alpha_1}_{\nu_1}\bar{\Theta}_-^{\nu_1\nu} = -\frac{1}{2}\Psi^\beta_\mu\Theta_-^{\mu\nu}. \tag{2.20}$$

Having obtained actions (2.8) and (2.11), we can now consider dualization along fermionic coordinates.

3. Fermionic T-duality

In this section the objectives are to find fermionic T-dual transformation laws and actions that have been T-dualized along fermionic coordinates for case where we performed bosonic T-duality and case where we have not.

Bosonic T-duality relies on utilization of symmetries of action to produce T-dual action and T-dual transformation laws. This task is usually accomplished by utilizing Busher procedure.[4,5,10–12] The main idea of fermionic T-duality is essentially the same, we utilize isometries of fermionic coordinates to generate T-dual action and T-dual transformation laws.[3,6–8] Just like in bosonic case, we localize translational symmetry by introducing covariant derivatives and, in cases where necessary, invariant coordinates. After this we introduce term that eliminates additional degrees of freedom and gauge fix existing symmetry. From this point on, finding equations of motion for gauge fields and inserting those equations of motion into gauge fixed action we obtain T-dual action.

Before proceeding with fermionic variant of Buscher procedure, we can notice that our actions (2.8) and (2.11) do not posses terms proportional to $\partial_+\theta^\alpha$ and $\partial_-\bar{\theta}^\alpha$. This means that our fermionic coordinates have following local symmetry

$$\delta\theta^\alpha = \epsilon^\alpha(\sigma^+), \quad \delta\bar{\theta}^\alpha = \bar{\epsilon}^\alpha(\sigma^-), \quad (\sigma^\pm = \tau \pm \sigma). \tag{3.1}$$

We need to fix this symmetry before obtaining T-dual theory, one way to do this is through BRST formalism. This symmetry has following corresponding BRST transformations for fermionic fields

$$s\theta^\alpha = c^\alpha(\sigma^+), \qquad s\bar{\theta}^\alpha = \bar{c}^\alpha(\sigma^-). \tag{3.2}$$

Here s is BRST nilpotent operator, c^α and $\bar{c}^\alpha$ represent ghost fields that correspond to gauge parameters ϵ^α and $\bar{\epsilon}^\alpha$ respectively. In addition to ghost fields we also have following BRST transformations

$$sC_\alpha = b_{+\alpha}, \quad s\bar{C}_\alpha = \bar{b}_{-\alpha}, \quad sb_{+\alpha} = 0, \quad s\bar{b}_{-\alpha} = 0. \tag{3.3}$$

where $\bar{C}_\alpha$ and C_α are anti-ghosts, $b_{+\alpha}$ and $\bar{b}_{-\alpha}$ are Nakanishi-Lautrup auxiliary fields.

Fixing of gauge symmetry is accomplished by introduction of gauge fermion, where we have decided to follow in the same choice as[9]

$$\Psi = \frac{k}{2}\int_\Sigma d^2\xi\left[\bar{C}_\alpha\left(\partial_+\theta^\alpha + \frac{1}{2}\alpha^{\alpha\beta}b_{+\beta}\right) + \left(\partial_-\bar{\theta}^\alpha + \frac{1}{2}\bar{b}_{-\beta}\alpha^{\beta\alpha}\right)C_\alpha\right], \tag{3.4}$$

here $\alpha^{\alpha\beta}$ is arbitrary invertible matrix.

Applying BRST transformation to gauge fermion we obtain gauge fixed action and Fadeev-Popov action

$$S_{gf} = \frac{k}{2}\int_\Sigma d^2\xi\left[\bar{b}_{-\alpha}\partial_+\theta^\alpha + \partial_-\bar{\theta}^\alpha b_{+\alpha} + \bar{b}_\alpha\alpha^{\alpha\beta}b_{+\beta}\right], \tag{3.5}$$

$$S_{F-P} = \frac{k}{2}\int_\Sigma d^2\xi\left[\bar{C}_\alpha\partial_+c^\alpha + (\partial_-\bar{c}^\alpha)C_\alpha\right]. \tag{3.6}$$

Fadeev-Popov term contains only ghosts and anti-ghosts and it is decoupled from the actions (2.8) and (2.11). From this point on, this term will be ignored. Gauge fixing term contains auxiliary fields $\bar{b}_{-\alpha}$ and $b_{+\alpha}$ that can be removed with equations of motion

$$\bar{b}_{-\alpha} = -\partial_-\bar{\theta}^\beta(\alpha^{-1})_{\beta\alpha}, \qquad b_{+\alpha} = -(\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta, \tag{3.7}$$

giving us

$$S_{gf} = -\frac{k}{2}\int_\Sigma d^2\xi\partial_-\bar{\theta}^\alpha(\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta. \tag{3.8}$$

Inserting guage fixing term into (2.8) and (2.11) gives us actions that can be dualized with Buscher procedure.

3.1. Type II superstring - Fermionic T-Duality

Since both action (2.8) and gauge fixing term (3.8) are trivially invariant to global translations of fermionic coordinates, we localize this translational symmetry by replacing partial derivatives with covariant ones

$$\partial_\pm\theta^\alpha \to D_\pm\theta^\alpha = \partial_\pm\theta^\alpha + u^\alpha_\pm, \tag{3.9}$$
$$\partial_\pm\bar{\theta}^\alpha \to D_\pm\bar{\theta}^\alpha = \partial_\pm\bar{\theta}^\alpha + \bar{u}^\alpha_\pm. \tag{3.10}$$

New gauge fields $u^\alpha_\pm$ and $\bar{u}^\alpha_\pm$ introduce new degrees of freedom that are removed by addition of term

$$S_{add} = \frac{k}{2}\int_\Sigma d^2\xi\left[\bar{z}_\alpha(\partial_+u^\alpha_- - \partial_-u^\alpha_+) + (\partial_+\bar{u}^\alpha_- - \partial_-\bar{u}^\alpha_+)z_\alpha\right]. \tag{3.11}$$

Gauge freedom can be utilized to fix fermionic coordinates such that $\theta^\alpha = \theta^\alpha_0 = const$ and $\bar{\theta}^\alpha = \bar{\theta}^\alpha_0 = const$. This in turn reduces our covariant derivatives to

$$D_\pm\theta^\alpha \to u^\alpha_\pm, \qquad D_\pm\bar{\theta}^\alpha \to \bar{u}^\alpha_\pm. \tag{3.12}$$

With all this in mind, we have following action

$$S_{gf} = k\int_\Sigma d^2\xi\Big[\Pi_{+\mu\nu}\partial_+x^\mu\partial_-x^\nu + \frac{1}{2}(\bar{u}^\alpha_+ + \partial_+x^\mu\bar{\Psi}^\alpha_\mu)\left(F^{-1}(x)\right)_{\alpha\beta} \times(u^\beta_- + \Psi^\beta_\nu\partial_-x^\nu) - \frac{1}{2}\bar{u}^\alpha_-(\alpha^{-1})_{\alpha\beta}u^\beta_+ + \frac{1}{2}\bar{z}_\alpha(\partial_+u^\alpha_- - \partial_-u^\alpha_+) + \frac{1}{2}(\partial_+\bar{u}^\alpha_- - \partial_-\bar{u}^\alpha_+)z_\alpha\Big]. \tag{3.13}$$

On one side we have equations of motion for Lagrange multipliers $\bar{\chi}_\alpha$ and χ_α

$$\partial_+u^\alpha_- - \partial_-u^\alpha_+ = 0 \quad\to\quad u^\alpha_\pm = \partial_\pm\theta^\alpha, \tag{3.14}$$
$$\partial_+\bar{u}^\alpha_- - \partial_-\bar{u}^\alpha_+ = 0 \quad\to\quad \bar{u}^\alpha_\pm = \partial_\pm\bar{\theta}^\alpha. \tag{3.15}$$

Inserting solutions for these equations into action (3.13) we obtain starting action plus gauge fixing term. Variation of action with respect to gauge fields produces following set of equations of motion

$$u^\alpha_- = -\left(F^{\alpha\beta}(x)\partial_-z_\beta + \Psi^\alpha_\mu\partial_-x^\mu\right), \tag{3.16}$$

$$u^\alpha_+ = -\alpha^{\alpha\beta}\partial_+z_\beta, \tag{3.17}$$

$$\bar{u}^\alpha_+ = \partial_+\bar{z}_\beta F^{\beta\alpha}(x) - \partial_+x^\mu\bar{\Psi}^\alpha_\mu, \tag{3.18}$$

$$\bar{u}^\alpha_- = \partial_-\bar{z}_\beta\alpha^{\beta\alpha}. \tag{3.19}$$

Utilizing these equations we can remove gauge fields from action, resulting in action that depends only on Lagrange multipliers and bosonic coordinates

$${}^fS = k\int_\Sigma d^2\xi\Big[\Pi_{+\mu\nu}\partial_+x^\mu\partial_-x^\nu + \frac{1}{2}\partial_+\bar{z}_\alpha\Psi^\alpha_\mu\partial_-x^\mu + \frac{1}{2}\partial_+\bar{z}_\alpha F^{\alpha\beta}(x)\partial_-z_\beta - \frac{1}{2}\partial_+x^\mu\bar{\Psi}^\alpha_\mu\partial_-z_\alpha - \frac{1}{2}\partial_-\bar{z}_\alpha\alpha^{\alpha\beta}\partial_+z_\beta\Big]. \tag{3.20}$$

Just like in the bosonic case, we have that left superscript ^{f} denotes fermionic T-duality. From here we can deduce background fields of feriomionic T-dual theory

$${}^f\Pi_{+\mu\nu} = \Pi_{+\mu\nu}, \tag{3.21}$$

$${}^f\left(F^{-1}(x)\right)^{\alpha\beta} = F^{\alpha\beta}(x), \tag{3.22}$$

$${}^f\bar{\Psi}_{\mu\beta}\,{}^f\left(F^{-1}(x)\right)^{\beta\alpha} = -\bar{\Psi}^\alpha_\mu \quad\to\quad {}^f\bar{\Psi}_{\mu\beta} = -\bar{\Psi}^\alpha_\mu\left(F^{-1}(x)\right)_{\alpha\beta}, \tag{3.23}$$

$${}^f\left(F^{-1}(x)\right)^{\alpha\beta}\,{}^f\Psi_{\mu\beta} = \Psi^\alpha_\mu \quad\to\quad {}^f\Psi_{\mu\beta} = \left(F^{-1}(x)\right)_{\beta\alpha}\Psi^\alpha_\mu. \tag{3.24}$$

Unlike bosonic case, fermionic T-dual theory is local. This can be attributed to the fact that background fields do not depend on fermionic coordinates. This in turn means that theory is geometric and we should not expect emergence of non-commutative phenomena.

3.2. Type II Superstring - Full T-Duality

To obtain fully dualized theory we start with action that is already T-dualized along bosonic coordinates (2.11). Procedure for fermionic T-duality is mostly the same as described before. The only difference comes from the fact that bosonic T-duality introduced non-local term V^0 which depends on θ^α and $\bar{\theta}^\alpha$ and now we need to introduce invariant fermionic coordinates in order for action to exhibit to local shift symmetry

$$D_\pm\theta^\alpha = \partial_\pm\theta^\alpha + u^\alpha_\pm, \tag{3.25}$$
$$D_\pm\bar{\theta}^\alpha = \partial_\pm\bar{\theta}^\alpha + \bar{u}^\alpha_\pm, \tag{3.26}$$
$$\theta^\alpha_{inv} = \int_P d\xi^m D_m\theta^\alpha = \int_P d\xi^m(\partial_m\theta^\alpha + u^\alpha_m) = \Delta\theta^\alpha + \Delta U^\alpha, \tag{3.27}$$
$$\bar{\theta}^\alpha_{inv} = \int_P d\xi^m D_m\bar{\theta}^\alpha = \int_P d\xi^m(\partial_m\bar{\theta}^\alpha + \bar{u}^\alpha_m) = \Delta\bar{\theta}^\alpha + \Delta\bar{U}^\alpha. \tag{3.28}$$

Fixing gauge symmetry as before, setting fermionic coordinates to constants, we deduce following relations

$$D_\pm\theta^\alpha \to u^\alpha_\pm, \qquad D_\pm\bar\theta^\alpha \to \bar u^\alpha_\pm, \qquad \theta^\alpha_{inv} \to \Delta U^\alpha, \qquad \bar\theta^\alpha_{inv} \to \Delta \bar U^\alpha. \tag{3.29}$$

With these relations we obtain action that is only a function of gauge fields, lagrange multipliers and dual coordinates

$$\begin{aligned}{}^bS_{gf} = \frac{k}{2}\int_\Sigma d^2\xi\Big[&\frac{1}{2}\bar\Theta^{\mu\nu}\partial_+\gamma_\mu\partial_-\gamma_\nu + \bar u^\alpha_+\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}u^\beta_-\\ &+\partial_+\gamma_\mu{}^b\bar\Psi^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}u^\beta_- + \bar u^\alpha_+\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}{}^b\Psi^{\nu\beta}\partial_-\gamma_\nu\\ &-\bar u^\alpha_-(\alpha^{-1})_{\alpha\beta}u^\beta_+ + \bar z_\alpha(\partial_+u^\alpha_- - \partial_-u^\alpha_+) + (\partial_+\bar u^\alpha_- - \partial_-\bar u^\alpha_+)z_\alpha\Big].\end{aligned} \tag{3.30}$$

In order to simplify calculations we introduce the following two substitutions

$$\begin{aligned}&\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}{}^b\Psi^{\nu\beta}\partial_-\gamma_\nu + \partial_-z_\alpha = Z_{-\alpha},\\ &\partial_+\gamma_\mu{}^b\bar\Psi^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta} - \partial_+\bar z_\beta = \bar Z_{+\beta}.\end{aligned} \tag{3.31}$$

Now, our action can be expressed as

$$\begin{aligned}{}^bS_{gf} = \frac{k}{2}\int_\Sigma d^2\xi\Big[&\frac{1}{2}\overline{\Theta}^{\mu\nu}\partial_+\gamma_\mu\partial_-\gamma_\nu + \bar u^\alpha_+\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}u^\beta_- + \bar Z_{+\beta}u^\beta_-\\ &+\bar u^+_\alpha Z_{-\alpha} - \bar u^-_\alpha(\alpha^{-1})_{\alpha\beta}u^+_\beta + \partial_-\bar z_\alpha u^+_\alpha - \bar u^-_\alpha\partial_+z_\alpha\Big].\end{aligned} \tag{3.32}$$

Similar to the first case, we can always revert to starting action by finding equations of motion for Lagrange multipliers and inserting their solutions into the action. In both cases equations of motion are the same so we take the freedom to omit them here.

Equations of motion for gauge fields differ in this case. Since we have that $V^{(0)}$ depends on fermionic coordinates, equations of motion have additional term that depends on invariant coordinate.

$$u^\alpha_+ = -(\alpha)^{\alpha\beta}\partial_+z_\beta, \qquad \bar u^\beta_- = \partial_-\bar z_\alpha(\alpha)^{\alpha\beta}, \tag{3.33}$$

$$\bar u^\alpha_+ = -\bar Z_{+\beta}{}^bF^{\beta\alpha}(V^{(0)}) - \beta^-_\nu(V^{(0)}, U^{(0)})\,{}^b\bar\Psi^{\nu\alpha}, \tag{3.34}$$

$$u^\beta_- = -{}^bF^{\beta\alpha}(V^{(0)})Z_{-\alpha} - \beta^+_\mu(V^{(0)}, U^{(0)})\,{}^b\Psi^{\mu\beta}. \tag{3.35}$$

The beta functions, $\beta^\pm_\mu(V^{(0)}, U^{(0)})$, are obtained by varying $V^{(0)}$ (see[13] for more details). They are given as

$$\begin{aligned}&\beta^\pm_\mu(V^{(0)}, U^{(0)})\\ &= \mp\frac{1}{8}\partial_\mp\left[\bar U^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left[U^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right]\\ &\quad\pm\frac{1}{8}\left[\bar U^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_\mp\left[U^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right].\end{aligned} \tag{3.36}$$

Inserting equations of motion for gauge fields into action (3.32) and keeping only terms linear with respect to $C^{\alpha\beta}_\mu$, we obtain fully dualized action

$$\begin{aligned}{}^{bf}S = \frac{k}{2}\int_\Sigma d^2\xi\\ \times\left[\frac{1}{2}\bar\Theta^{\mu\nu}_-\partial_+\gamma_\mu\partial_-\gamma_\nu - \bar Z_{+\alpha}{}^bF^{\alpha\beta}(V^{(0)})Z_{-\beta} - \partial_-\bar z_\alpha(\alpha)^{\alpha\beta}\partial_+z_\beta\right].\end{aligned} \tag{3.37}$$

Expanded, we have

$$\begin{aligned}{}^{bf}S = k\int_\Sigma d^2\xi\Big[&\frac{1}{4}\Theta^{\mu\nu}_-\partial_+\gamma_\mu\partial_-\gamma_\nu - \frac{1}{4}\partial_+\gamma_\mu\Theta^{\mu\nu}_-\bar\Psi^\alpha_\nu\partial_-z_\alpha\\ &-\frac{1}{4}\partial_+\bar z_\alpha\Psi^\alpha_\mu\Theta^{\mu\nu}_-\partial_-\gamma_\nu + \frac{1}{2}\partial_+\bar z_\alpha{}^bF^{\alpha\beta}(V^{(0)})\partial_-z_\beta\\ &-\frac{1}{2}\partial_-\bar z_\alpha(\alpha)^{\alpha\beta}\partial_+z_\beta\Big].\end{aligned} \tag{3.38}$$

From here, we can read background fields of T-dual theory

$$\begin{aligned}{}^{bf}\bar\Pi^{\mu\nu}_+ &= \frac{1}{4}\bar\Theta^{\mu\nu}_- - \frac{1}{2}{}^b\bar\Psi^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}{}^b\Psi^{\nu\beta} = \Theta^{\mu\nu}_-,\\ {}^{bf}\left(F^{-1}(x)\right)^{\alpha\beta} &= {}^bF^{\alpha\beta}(x) = F^{\alpha\beta}(x) + \frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu,\\ {}^{bf}\bar\Psi^\mu_\alpha{}^{bf}\left(F^{-1}(x)\right)^{\alpha\beta} &= {}^b\bar\Psi^{\mu\beta} = \frac{1}{2}\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu\\ &\to\quad {}^{bf}\bar\Psi^\mu_\alpha = \frac{1}{2}\bar\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu\left(F^{-1}(x)\right)_{\beta\alpha},\\ {}^{bf}\left(F^{-1}(x)\right)^{\alpha\beta}{}^{bf}\Psi^\nu_\beta &= {}^b\Psi^{\nu\alpha} = -\frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_-\\ &\to\quad {}^{bf}\Psi^\nu_\beta = -\frac{1}{2}\left(F^{-1}(x)\right)_{\beta\alpha}\Psi^\alpha_\mu\bar\Theta^{\mu\nu}_-.\end{aligned} \tag{3.39}$$

Comparing background fields in different stages of T-dualization we notice that both fermionic T-duality and bosonic T-duality affect all field, where all T-dual theories now have coordinate dependent fields. It should also be noted that non-commutative relations in theory emerge only after performing bosonic T-duality. Fermionic T-dual coordinates are always only proportional to fermionic momenta therefore Poisson brackets between fermionic coordiantes always remain zero.

3.3. Bosonic T-Duality of Fermionic T-Dual Theory

For completion sake, we will also T-dualize fermionic T-dual action (3.20) along x^μ coordinates. In this specific case, where only RR field depends on bosonic coordinate, we expect that bosonic and ferionic T-dualities commute. Therefore, this section can be thought of as a check for calculations from previous section.

Bosonic T-duality is mostly the same as fermionic one,[10–13] where only difference is the lack of introduction of Fadeev-Popov and gauge fixing actions. We again start by localizing translational symmetry, inserting Lagrange multipliers and fixing gauge fields. This produces following auxiliary action

$$
{}^{f}S_{aux} = \kappa \int d^2\xi \Big[v_+^\mu \Pi_{+\mu\nu} v_-^\nu + \frac{1}{2}\partial_+\bar{z}_\alpha f^{\alpha\beta}\partial_- z_\beta + \frac{1}{2}\partial_+\bar{z}_\alpha C_\mu^{\alpha\beta}\partial_- z_\beta \Delta V^\mu + \frac{1}{2}\partial_+\bar{z}_\alpha \Psi_\mu^\alpha v_-^\mu - \frac{1}{2} v_+^\mu \bar{\Psi}_\mu^\alpha \partial_- z_\alpha - \frac{1}{2}\partial_-\bar{z}_\alpha \alpha^{\alpha\beta}\partial_+ z_\beta + \frac{1}{2}y_\mu(\partial_+ v_-^\mu - \partial_- v_+^\mu) \Big]. \tag{3.40}
$$

Introducing the variables

$$
Y_{+\mu} = \partial_+ y_\mu - \partial_+\bar{z}_\alpha \Psi_\mu^\alpha , \quad Y_{-\mu} = \partial_- y_\mu - \bar{\Psi}_\mu^\alpha \partial_- z_\alpha , \tag{3.41}
$$

the action (3.40) gets much simpler form

$$
{}^{f}S_{aux} = \kappa \int d^2\xi \Big[v_+^\mu \Pi_{+\mu\nu} v_-^\nu + \frac{1}{2}\partial_+\bar{z}_\alpha f^{\alpha\beta}\partial_- z_\beta + \frac{1}{2}\partial_+\bar{z}_\alpha C_\mu^{\alpha\beta}\partial_- z_\beta \Delta V^\mu - \frac{1}{2} Y_{+\mu} v_-^\mu + \frac{1}{2} v_+^\mu Y_{-\mu} \Big]. \tag{3.42}
$$

Varying the above action with respect to gauge fields v_+^μ and v_-^μ, we get, respectively,

$$
\Pi_{+\mu\nu} v_-^\nu = -\Big(\frac{1}{2}Y_{-\mu} + \beta_{+\mu}(V)\Big) , \tag{3.43}
$$

$$
v_+^\nu \Pi_{+\nu\mu} = \frac{1}{2}Y_{+\mu} - \beta_{-\mu}(V) , \tag{3.44}
$$

where $\beta_{\pm\mu}$ are the beta functions obtained from coordinate dependent term in the action

$$
\beta_{\pm\mu} = \mp\frac{1}{8}\Big(\bar{z}_\alpha C_\mu^{\alpha\beta}\partial_\mp z_\beta - \partial_\mp \bar{z}_\alpha C_\mu^{\alpha\beta} z_\beta\Big). \tag{3.45}
$$

Inserting (3.43) and (3.44) into the auxiliary action (3.42), keeping the terms linear in $C_\mu^{\alpha\beta}$, we obtain fully T-dualized action (first fermionic, then bosonic T-dualization)

$$
{}^{fb}S = \kappa \int d^2\xi \Big[\frac{1}{2}\partial_+\bar{z}_\alpha F^{\alpha\beta}(\Delta V)\partial_- z_\beta + \frac{1}{4} Y_{+\mu}(\Pi_+^{-1})^{\mu\nu} Y_{-\nu} \Big] . \tag{3.46}
$$

Expanding above action we prove that it is identical to one given in (3.38).

4. Few Notes on Non-commutativity

In paper[13] it has been shown that bosonic T-duality produces non-commutative relations between bosonic T-dual coordinates. With this in mind, following question naturally arises: can we expect emergence of same behavior for fermionic coordinates after fermionic T-dualization? To get the answer for this question we have to express fermionic T-dual coordinates as some combination of starting coordinates and their momenta and connect T-dual Poisson brackets with Poisson brackets of original theory. Original theory is geometric theory with regular Poisson structure

$$
\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta_\nu^\mu \delta(\sigma - \bar{\sigma}),
$$

$$
\{\theta^\alpha(\sigma), \pi_\beta(\bar{\sigma})\} = \{\bar{\theta}^\alpha(\sigma), \bar{\pi}_\beta(\bar{\sigma})\} = \delta_\beta^\alpha \delta(\sigma - \bar{\sigma}), \tag{4.1}
$$

where all other Poisson brackets vanish.

We start with case that has only been T-dualized along fermionic coordinates. To find how T-dual coordinates depend on starting ones and their momenta we can begin by finding fermionic momenta of starting theory. It is useful to remember that starting theory did not posses terms that are proportional to $\partial_+\theta^\alpha$ and $\partial_-\bar{\theta}^\alpha$ and that this symmetry was fixed with BRST formalism. Addition of gauge fixing term introduced modification to momenta of starting theory and to obtain correct non-commutative relations we should be working with theories that have gauge fixing term in them. With this in mind, it is easy to find fermionic momentum of original theory (3.13)

$$
\pi_\beta = -\frac{k}{2}\Big[(\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu \bar{\Psi}_\mu^\alpha)\big(F^{-1}(x)\big)_{\alpha\beta} - \partial_-\bar{\theta}^\alpha(\alpha^{-1})_{\alpha\beta} \Big], \tag{4.2}
$$

$$
\bar{\pi}_\alpha = \frac{k}{2}\Big[\big(F^{-1}(x)\big)_{\alpha\beta}(\partial_-\theta^\beta + \Psi_\nu^\beta \partial_- x^\nu) - (\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta \Big]. \tag{4.3}
$$

Since we want to obtain Poisson brackets for equal τ we want to find σ partial derivatives of dual coordinate

$$
\partial_\sigma z_\alpha = \partial_+ z_\alpha - \partial_- z_\alpha = \frac{2}{k}\bar{\pi}_\alpha, \tag{4.4}
$$

$$
\partial_\sigma \bar{z}_\alpha = \partial_+ \bar{z}_\alpha - \partial_- \bar{z}_\alpha = -\frac{2}{k}\pi_\alpha. \tag{4.5}
$$

Momenta of original theory commute with each other and with x^μ coordinates, therefore we deduce that there has been no change to geometric structure of this theory.

For fully dualized theory, transformation laws (3.33) (3.34) (3.35) all depend on dual bosonic coordinate however, when we insert transformation laws that connect original bosonic coordinates with T-dual ones (more details in[13])

$$
\partial_+ y_\mu = 2\Big[\partial_+ x^\nu \bar{\Pi}_{+\nu\mu} + \beta_\mu^-(x)\Big] + \partial_+\bar{\theta}^\alpha\big(F^{-1}(x)\big)_{\alpha\beta}\Psi_\mu^\beta, \tag{4.6}
$$

$$
\partial_- y_\nu = -2\Big[\bar{\Pi}_{+\mu\nu}\partial_- x^\nu + \beta_\mu^+(x)\Big] - \bar{\Psi}_\nu^\alpha\big(F^{-1}(x)\big)_{\alpha\beta}\partial_-\theta^\beta , \tag{4.7}
$$

into transformation laws for fermionic coordinates (3.33), (3.34) and (3.35) we again obtain relations (4.4) and (4.5).

On a first glance it would seem that fermionic T-duality has not produced any new Poisson brackets, however this is not the case. While it is true that there are no modifications to Poisson brackets between fermions, we have new Poisson bracket structure between fermions and bosons. This can be seen from σ derivative of bosonic T-dual coordinate

$$
y'_\mu \cong \frac{\pi_\mu}{\kappa} + \beta_\mu^0(x), \tag{4.8}
$$

where $\beta_\mu^0(x)$ is combination $\beta_\mu^+(x) + \beta_\mu^-(x)$ given as

$$
\beta_\mu^0(x) = \frac{1}{2}\partial_\sigma\Big[\bar{\theta}^\alpha + x^{\nu_1}\bar{\Psi}_{\nu_1}^\alpha\Big](f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Big[\theta^\beta + \Psi_{\nu_2}^\beta x^{\nu_2}\Big] - \frac{1}{2}\Big[\bar{\theta}^\alpha + x^{\nu_1}\bar{\Psi}_{\nu_1}^\alpha\Big](f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\partial_\sigma\Big[\theta^\beta + \Psi_{\nu_2}^\beta x^{\nu_2}\Big]. \tag{4.9}
$$

Finding Poisson brackets between σ derivatives of coordinates and integrating twice we obtain following relations

$$\{y_\mu(\sigma), \bar{z}_\beta(\bar{\sigma})\} = \frac{1}{k}\Big[\bar{\theta}^\alpha(\sigma) + x^{\nu_1}(\sigma)\bar{\Psi}^\alpha_{\nu_1} - 2\Big(\bar{\theta}^\alpha(\bar{\sigma}) + x^{\nu_1}(\bar{\sigma})\bar{\Psi}^\alpha_{\nu_1}\Big)\Big] \times (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta} H(\sigma - \bar{\sigma}), \tag{4.10}$$

$$\{y_\mu(\sigma), z_\alpha(\bar{\sigma})\} = \frac{1}{k}(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta} \times \Big[\theta^\beta(\sigma) + \Psi^\beta_{\nu_2} x^{\nu_2}(\sigma) - 2\Big(\theta^\beta(\bar{\sigma}) + \Psi^\beta_{\nu_2} x^{\nu_2}(\bar{\sigma})\Big)\Big] \times H(\sigma - \bar{\sigma}). \tag{4.11}$$

5. Conclusions

In this article we examined effects of fermionic T-duality performed on action of type II superstring in pure spinor formalism. We carried out our investigation in two cases, one where we performed fermionic T-duality on previously non dualized action and in second case where we had action that was already dualized along bosonic coordinates. Starting (non dualized) action that we worked with described closed string that propagates in presence of Ramond-Ramond field with linear coordinate dependence. We made a decision to only consider dependence on bosonic coordinates, furthermore this dependence was tied to infinitesimal antisymmetric term $C^{\alpha\beta}_\mu$. Rest of the background fields were held constant. Terms in action that were non-linearly dependent on fermionic coordinates were neglected. These choices were in accordance with consistency conditions for background fields and were made in order to keep calculations manageable.

On the other hand, bosonic T-duality of starting action provided us with theory that was non-local. Unlike starting theory that only dependent on coordinates through RR field, this theory manifested coordinate dependence on all background fields. Furthermore, bosonic T-dual coordinates now exhibit non-commutative properties.

Before we could start with T-dualization we noticed that both cases posses additional local symmetry which removed terms proportional to $\partial_+\theta^\alpha$ and $\partial_-\bar{\theta}^\alpha$. In order to obtain correct T-dual theory this symmetry was fixed through BRST formalism. In both cases procedure for obtaining fermionic T-duality was the same, we employed Buscher T-dualizing procedure. Procedure is based on localization of translational symmetry where we replace partial derivatives with covariant ones. Introduction of covariant derivatives carries with itself new degrees of freedom in shape of gauge fields. By demanding that starting and T-dual theory give description of same physical system we inevitably demand for both theories to posses same degrees of freedom. Thus, all additional degrees of freedom must be removed with Lagrange multipliers. By utilizing gauge freedom we can also remove all instances of fermionic coordinates in action obtaining action that is only a function of gaguge fields and Lagrange multipliers. Finding equations of motion for gauge fields of this gauge fixed action and inserting their solutions into the action we obtain T-dual theory.

Carrying Buscher procedure for fermionic coordinates of non dualized action we obtain local theory where all fields depend on bosonic coordinates. This theory is commutative, its Poisson brackets are identical to Poisson brackets of starting theory.

Buscher procedure in case of theory that has been dualizied along bosonic coordinates does not change coordinate dependence of the background fields. All fields are still dependent on both bosonic and fermionic coordinates and theory is still non-local. However, this theory posseses two additional non-trivial Poisson brackets. We have emergence of non-commutativity between bosonic and fermionic coordinates, where non-commutativity is proportional to infinitesimal constant $C^{\alpha\beta}_\mu$.

Same result is obtained even in case where we first perform fermionic and then bosonic T-duality. Commutativity between different dualities was expected since fully T-dual theory must be unique. Only distinction between different paths of T-dualization procedures can be noticed in intermediate theories, where most important change is transition of theory from being local to non-local.

We suspect that it is possible to obtain T-dual theory that is fully non-commutative, theory that has non-commutativity even between fermionic coordinates, but we would need starting theory that has background fields that depend on both bosonic and fermionic coordinates.

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Conflict of Interest

The authors have declared no conflict of interest.

Keywords

T-duality, Buscher procedure, non-commutativity, fermionic T-duality, string theory

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Noncommutativity and Nonassociativity of Type II Superstring with Coordinate Dependent RR Field

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In this paper we will consider noncommutativity that arises from bosonic T-dualization of type II superstring in presence of Ramond-Ramond (RR) field, which linearly depends on the bosonic coordinates x^μ. The derivative of the RR field $C_\mu^{\alpha\beta}$ is infinitesimal. We will employ generalized Buscher procedure that can be applied to cases that have coordinate dependent background fields. Bosonic part of newly obtained T-dual theory is non-local. It is defined in non-geometric space spanned by Lagrange multipliers y_μ. We will apply generalized Buscher procedure once more on T-dual theory and prove that original theory can be salvaged. Finally, we will use T-dual transformation laws along with Poisson brackets of original theory to derive Poisson bracket structure of T-dual theory and nonassociativity relation. Noncommutativity parameter depends on the supercoordinates x^μ, θ^α and $\bar\theta^\alpha$, while nonassociativity parameter is a constant tensor containing infinitesimal $C_\mu^{\alpha\beta}$.

1. Introduction

In 1982 emerged a model[1] that would offer the possibility of obtaining bosonic coordinates of space-time as emergent properties of more fundamental fermionic coordinates. While this model worked with supersymmetric particle, this approach suggested that maybe we can express bosonic coordinates as a Poisson bracket of fermionic coordinates. In addition to these Poisson brackets, Poisson brackets between bosonic coordinates as well as between bosonic and fermionic coordinates would remain zero. Later, in paper,[2] it has been suggested that the same result can be obtained in context of string theory in case of coordinate dependent RR field. It is also suggested that unlike supersymmetric particle model, coordinate dependent RR field would produce full spectrum of non-commutative relations, bosonic coordinates themselves would become non-commutative. In this paper our goal is to determine if coordinate dependent RR field, while remaining background fields are as simple as consistency relations allow, can produce suggested non-commutative relations.

Superstring theory, as a theory of extended objects propagating in space-time, is defined in 10 dimensions.[3,4] In order to establish link between this mathematical model and real world observations, surplus space-like dimensions are compactified on circles of radius R. From this kind of geometry arises new kind of symmetry, T-duality, that links theories that have radii of compactification R with ones that have radii of compactification α'/R.[5,6] Existence of T-duality between different theories implies that those theories are physically equivalent and it gives us a way to explore how geometry and topology of one theory is connected to other. This connection between different geometries makes T-duality a useful tool in examining emergence of non-commutativity in context of closed strings.[7]

While in string theory both open and closed strings, under certain conditions, exhibit emergence of non-commutativity, mechanisms that enable this emergence are different. In case of open string, we have that endpoints of string that propagates in presence of constant metric and Kalb-Ramond field become non-commutative.[8] Basic idea of open string non-commutativity is that initial coordinates can be expressed as linear combination of effective coordinates and momenta by employing boundary conditions. In case of closed string, we do not have string endpoints therefore we don't have emergence of non-commutativity when string propagates in presence of constant background fields. In order to achieve same effect as in case of open string, we have to use coordinate dependent background fields. By finding T-dual of theories with this kind of geometry we obtain T-dual theory in non-geometric background, where T-dual coordinates are expressed as linear combinations of original coordinates and their conjugated momenta.

Mathematical framework for obtaining T-dual theories is standard Buscher procedure.[9,10] Procedure is based on existence of shift symmetry in relevant action and its implementation can be summarized in few steps. First step is localization of translational symmetry by introduction of covariant derivatives and introduction of Lagrange multipliers that make newly introduced gauge fields nonphysical. By gauge fixing and finding equations of motion of both gauge fields and Lagrange multipliers we obtain T-dual transformation laws. These transformation laws inserted into gauge fixed action produce T-dual action. For cases where we have coordinate dependent background fields there exist generalized Buscher procedure,[11–15] this extension has one additional step, replacement of all initial coordinates with invariant coordinates. Further extension of generalized Buscher procedure is possible[16] and it is applicable to theories that do not posses shift symmetry.

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In this article we will deal with closed superstring propagating in presence of linearly coordinate dependent Ramond-Ramond (RR) field using type II superstring model in pure spinor formulation. All calculations we will do in approximation of diluted flux, which means that in all calculations we keep constant and linear terms in infinitesimal derivative of the RR field strength. Rest of the fields, metric, Kalb-Ramond and gravitino fields are constant. Furthermore all dependence of background field on fermionic terms will be neglected for mathematical simplicity. This choice of field configuration is in full accordance with consistency equations for background fields.[17]

Because we are currently only interested in non-commutative relations between bosonic coordinates, T-dualization procedure will be applied only on bosonic part of action. To find T-dual action and T-dual transformation laws we will employ extension of generalized Buscher procedure that works with coordinate dependent background fields.[11] After finding T-dual theory, we will apply Buscher procedure once more to see if we can obtain original theory.

Transformation laws that connect variables from initial with variables from T-dual theory will be written in canonical form, where initial momenta are expressed in terms of the T-dual coordinates. By inverting these transformation laws we obtain how sigma derivatives of T-dual theory depend on linear combinations of coordinates and momenta of original theory. Taking into account that original theory is geometrical, both locally and globally, we have that its coordinate and conjugated momenta satisfy standard Poisson brackets. By using this fact we are able to find Poisson structure of sigma derivatives of T-dual coordinates and by doing integration, Poisson structure of T-dual coordinates is obtained.

The form of obtained non-commutativity is such that non-commutativity exists when arguments are different, $\sigma \neq \bar{\sigma}$. Imposing trivial winding conditions, we obtain string winding numbers from Poisson brackets.

In the end, we give conclusions and in appendix we present some technical details regarding derivation of $\beta^{\pm}_{\mu}$ functions.

2. General Type II Superstring Action and Choice of Background Fields

Starting point of this investigation will be action of type II superstring theory in pure spinor formulation.[18–21] We will present and explain assumed approximations in order to obtain type II pure spinor action with non-constant RR field-strength. It turns out that ghost fields are neglected and only quadratic terms are considered. Final form of this kind of action will be used in subsequent sections.

2.1. General Form of the Pure Spinor Type II Superstring Action

Sigma model of type IIB superstring has the following form[17]

$$S = S_0 + V_{SG}. \tag{2.1}$$

This general form of action is expressed as a sum of the part that describes the motion of string in flat background

$$S_0 = \int_\Sigma d^2\xi\Big(\frac{\kappa}{2}\eta_{\mu\nu}\partial_m x^\mu \partial_n x^\nu \eta^{mn} - \pi_\alpha \partial_- \theta^\alpha + \partial_+\bar{\theta}^\alpha \bar{\pi}_\alpha\Big) + S_\lambda + S_{\bar{\lambda}}\,, \tag{2.2}$$

and part that governs the modifications to the background fields

$$V_{SG} = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar{X}^N. \tag{2.3}$$

Modifications to the flat background are introduced by integrated form of massless type II supergravity vertex operator V_{SG}. The terms S_λ and $S_{\bar{\lambda}}$ in (2.2) are free-field actions for pure spinors

$$S_\lambda = \int_\Sigma d^2\xi \omega_\alpha \partial_- \lambda^\alpha, \quad S_{\bar{\lambda}} = \int_\Sigma d^2\xi \bar{\omega}_\alpha \partial_+ \bar{\lambda}^\alpha\,. \tag{2.4}$$

Here, λ^α and $\bar{\lambda}^\alpha$ are pure spinors whose canonically conjugated momenta are ω_α and $\bar{\omega}_\alpha$, respectively. Pure spinors satisfy pure spinor constraints

$$\lambda^\alpha (\Gamma^\mu)_{\alpha\beta}\lambda^\beta = \bar{\lambda}(\Gamma^\mu)_{\alpha\beta}\bar{\lambda}^\beta = 0\,. \tag{2.5}$$

In general case, vectors X^M and X^N as well as a supermatrix A_{MN} are given by

$$X^M = \begin{pmatrix} \partial_+\theta^\alpha \\ \Pi^\mu_+ \\ d_\alpha \\ \frac{1}{2}N^{\mu\nu}_+ \end{pmatrix}, \quad \bar{X}^M = \begin{pmatrix} \partial_-\bar{\theta}^\lambda \\ \Pi^\mu_- \\ \bar{d}_\lambda \\ \frac{1}{2}\bar{N}^{\mu\nu}_- \end{pmatrix},$$

$$A_{MN} = \begin{bmatrix} A_{\alpha\beta} & A_{\alpha\nu} & E_\alpha{}^\beta & \Omega_{\alpha,\mu\nu} \\ A_{\mu\beta} & A_{\mu\nu} & \bar{E}^\beta_\mu & \Omega_{\mu,\nu\rho} \\ E^\alpha{}_\beta & E^\alpha_\nu & P^{\alpha\beta} & C^\alpha{}_{\mu\nu} \\ \Omega_{\mu\nu,\beta} & \Omega_{\mu\nu,\rho} & \bar{C}^\beta{}_{\mu\nu} & S_{\mu\nu,\rho\sigma} \end{bmatrix}, \tag{2.6}$$

where notation is in accordance with Ref [17]. The components of matrix A_{MN} are generally functions of x^μ, θ^α and $\bar{\theta}^\alpha$. Components themselves are derived as expansions in powers of θ^α and $\bar{\theta}^\alpha$ (for details consult[17]). The superfields $A_{\mu\nu}$, $\bar{E}^\alpha_\mu$, E^α_μ and $P^{\alpha\beta}$ are known as physical superfields, while superfields that are in the first row and the first column are known as auxiliary because they can be expressed in terms of physical ones.[17] Remaining superfields $\Omega_{\mu,\nu\rho}$ ($\Omega_{\mu\nu,\rho}$), $C^\alpha{}_{\mu\nu}$ ($\bar{C}^\beta{}_{\mu\nu}$) and $S_{\mu\nu,\rho\sigma}$, are curvatures (field strengths) for physical fields. Components of vectors X^M and $\bar{X}^N$ are defined as

$$\Pi^\mu_+ = \partial_+ x^\mu + \frac{1}{2}\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\partial_+\theta^\beta, \quad \Pi^\mu_- = \partial_- x^\mu + \frac{1}{2}\bar{\theta}^\alpha (\Gamma^\mu)_{\alpha\beta}\partial_-\bar{\theta}^\beta, \tag{2.7}$$

$$d_\alpha = \pi_\alpha - \frac{1}{2}(\Gamma_\mu\theta)_\alpha\Big[\partial_+ x^\mu + \frac{1}{4}(\theta\Gamma^\mu\partial_+\theta)\Big],$$

$$\bar{d}_\alpha = \bar{\pi}_\alpha - \frac{1}{2}(\Gamma_\mu\bar{\theta})_\alpha\Big[\partial_- x^\mu + \frac{1}{4}(\bar{\theta}\Gamma^\mu\partial_-\bar{\theta})\Big], \tag{2.8}$$

$$N_+^{\mu\nu} = \frac{1}{2}\omega_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \lambda^\beta, \quad \bar{N}_-^{\mu\nu} = \frac{1}{2}\bar{\omega}_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \bar{\lambda}^\beta. \tag{2.9}$$

The world sheet Σ is parameterized by $\xi^m = (\xi^0 = \tau, \xi^1 = \sigma)$ and world sheet light-cone partial derivatives are defines as $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Superspace in which string propagates is spanned both by bosonic x^μ $(\mu = 0, 1, \dots, 9)$ and fermionic θ^α, $\bar{\theta}^\alpha$ $(\alpha = 1, 2, \dots, 16)$ coordinates. Variables π_α and $\bar{\pi}_\alpha$ represent canonically conjugated momenta of fermionic coordinates θ^α and $\bar{\theta}^\alpha$, respectively. Fermionic coordinates and their canonically conjugated momenta are Majorana-Weyl spinors. It means that each of these spinors has 16 independent real valued components.

2.2. Choice of the Background Fields

In this particular case we will work with the supermatrix A_{MN} where all background fields, except RR field strength $P^{\alpha\beta}$, are constants. RR field strength will have linear coordinate dependence on bosonic coordinate x^μ. With these restrictions in mind, supermatrix A_{MN} has the following form

$$A_{MN} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa\left(\frac{1}{2}g_{\mu\nu} + B_{\mu\nu}\right) & \bar{\Psi}_\mu^\beta & 0 \\ 0 & -\Psi_\nu^\alpha & \frac{2}{\kappa}(f^{\alpha\beta} + C_\rho^{\alpha\beta} x^\rho) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.10}$$

Here $g_{\mu\nu}$ is symmetric tensor, $B_{\mu\nu}$ is Kalb-Ramond antisymmetric field, Ψ_μ^α and $\bar{\Psi}_\mu^\alpha$ are Mayorana-Weyl gravitino fields, and finally, $f^{\alpha\beta}$ and $C_\rho^{\alpha\beta}$ are constants. Let us stress that dilaton field Φ is assumed to be constant, so, the factor e^Φ is included in $f^{\alpha\beta}$ and $C_\rho^{\alpha\beta}$. This will be a classical analysis and we will not calculate the dilaton shift under T-duality transformation. Based on the chirality of spinors, there are type IIA superstring theory for opposite chirality and type IIB superstring theory for same chirality.

This particular choice of supermatrix imposes following restriction on background fields

$$\gamma^\mu_{\alpha\beta} C_\mu^{\beta\gamma} = 0, \quad \gamma^\mu_{\alpha\beta} C_\mu^{\gamma\beta} = 0. \tag{2.11}$$

Remaining constraints[17] are trivial and applied only to nonphysical fields.

In addition to choice of supermatrix, in order to simplify calculation of bosonic T-duality, because all background fields are expanded in powers of θ^α and $\bar{\theta}^\alpha$, all θ^α and $\bar{\theta}^\alpha$ non-linear terms in X^M and $\bar{X}^N$ will be neglected. With this in mind, components of these two vectors reduce into the following form

$$\Pi_\pm^\mu \to \partial_\pm x^\mu, \quad d_\alpha \to \pi_\alpha, \quad \bar{d}_\alpha \to \bar{\pi}_\alpha. \tag{2.12}$$

Taking into account all these assumptions, the action (2.1) takes the form

$$S = \int_\Sigma d^2\xi \left[\frac{\kappa}{2}\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu - \pi_\alpha(\partial_-\theta^\alpha + \Psi_\nu^\alpha \partial_- x^\nu) + (\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu \bar{\Psi}_\mu^\alpha)\bar{\pi}_\alpha + \frac{2}{\kappa}\pi_\alpha(f^{\alpha\beta} + C_\rho^{\alpha\beta}x^\rho)\bar{\pi}_\beta\right]. \tag{2.13}$$

Here, new tensor $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}$ is introduced, where $G_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu}$ is metric tensor. Terms for S_λ and $S_{\bar{\lambda}}$ are fully decoupled from action and they will not be considered from now on.

Before considering T-duality, we can notice that fermionic momenta act as auxiliary fields in full actions. These fields can be integrated out and final action will be function of only coordinates and their derivatives. Finding equations for motion for both π_α and $\bar{\pi}_\alpha$ we get following two equations

$$\bar{\pi}_\beta = \frac{\kappa}{2}\left(F^{-1}(x)\right)_{\beta\alpha}\left(\partial_-\theta^\alpha + \Psi_\nu^\alpha \partial_- x^\nu\right), \tag{2.14}$$

$$\pi_\alpha = -\frac{\kappa}{2}\left(\partial_+\bar{\theta}^\beta + \partial_+ x^\mu \bar{\Psi}_\mu^\beta\right)\left(F^{-1}(x)\right)_{\beta\alpha}, \tag{2.15}$$

where $F^{\alpha\beta}(x)$ and $(F^{-1}(x))_{\alpha\beta}$ are of the form

$$F^{\alpha\beta}(x) = f^{\alpha\beta} + C_\mu^{\alpha\beta} x^\mu,$$
$$(F^{-1}(x))_{\alpha\beta} = (f^{-1})_{\alpha\beta} - (f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1} (f^{-1})_{\beta_1\beta}\, x^\mu. \tag{2.16}$$

In order to invert previous equations and T-dual transformation laws, as well as to simplify calculations, we take two additional assumptions. First assumption is that $C_\mu^{\alpha\beta}$ is infinitesimal. Second assumption is that $(f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1} (f^{-1})_{\beta_1\beta}$ is antisymmetric under exchange of first and last index. In other words, tensor $(F^{-1}(x))_{\alpha\beta}$ has only antysimetric part that depends on x^μ and it is infinitesimal. These assumptions are in full accordance with constraints.[17]

Substituting equations (2.14) and (2.15) into (2.13) the final form of action is

$$S = \kappa \int_\Sigma d^2\xi \left[\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu + \frac{1}{2}(\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu \bar{\Psi}_\mu^\alpha)\left(F^{-1}(x)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi_\nu^\beta \partial_- x^\nu)\right]. \tag{2.17}$$

In the following sections, this form of action will be used for investigation of bosonic T-duality and for obtaining transformation laws between starting and T-dual coordinates.

3. T-dualization

In this section T-duality will be performed along all bosonic coordinates in order to find relations that connect T-dual coordinates with coordinates and momenta of original theory. These transformation laws will then be used in subsequent chapters to find non-commutativity relations between coordinates of T-dual theory.

Starting point for considering T-duality will be generalized Buscher T-dualization procedure.[11] Standard Buscher procedure[9,10] is designed to be applied along isometry directions on which background fields do not depend. Generalized Buscher procedure can be applied to theories with coordinate dependent background fields. The shift symmetry in the generalized procedure is localized by introduction of covariant derivatives, invariant coordinates and additional gauge fields. These newly introduced gauge fields produce additional degrees of freedom. Since we expect that starting and T-dual theory have exactly the same number of degrees of freedom we need to eliminate all excessive degrees of freedom. This is accomplished by demanding that field strength of gauge fields ($F_{+-} = \partial_+ v_- - \partial_- v_+$) vanishes by addition of Lagrange multipliers. Next step in procedure is fixing the gauge symmetry such that starting coordinates are constant and action is only left with gauge fields and its derivatives. From this gauge fixed action, finding equations of motion for gauge fields, expressing gauge fields as function of Lagrange multipliers and inserting those equations into action we can obtain T-dual action, were Lagrange multipliers of original theory now play the role of T-dual coordinates.

In cases where shift symmetry is absent, T-duality can still be performed by extending generalized Buscher procedure.[16] This extension is based on replacing original action with translation invariant auxiliary action. Form of this auxiliary action is exactly the same as the form of action where translation symmetry was localized and gauged fixed, that is, derivatives have been replaced with gauge fields and coordinates with integrals of gauge fields. Auxiliary action gives correct T-dual theory only if original action can be salvaged from it. In cases where this is possible, original theory is obtained by finding equations of motion with respect to Lagrange multipliers and inserting their solutions into auxiliary action.

Action (2.17) is invariant to translation symmetry, by the virtue of antisymmetric part of $F^{-1}_{\alpha\beta}$, tensor $(f^{-1}C_\mu f^{-1})_{\alpha\beta}$. Following antisymmetricity of this tensor, we can rewrite the action (2.17) in the following way

$$S = \kappa \int_\Sigma d^2\xi \Big[\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu + \frac{1}{2}\epsilon^{mn}\partial_m(\bar\theta^\alpha + x^\mu \bar\Psi^\alpha_\mu)\big(F^{-1}(x)\big)_{\alpha\beta}\partial_n(\theta^\beta + \Psi^\beta_\nu x^\nu)\Big]. \tag{3.1}$$

Let us now consider the global shift symmetry $\delta x^\mu = \lambda^\mu$ and vary the action (3.1)

$$\delta S = -\frac{\kappa}{2}(f^{-1}C_\mu f^{-1})_{\alpha\beta}\lambda^\mu \int_\Sigma d^2\xi \epsilon^{mn}\partial_m(\bar\theta^\alpha + \bar\Psi^\alpha_\nu x^\nu)\partial_n(\theta^\beta + \Psi^\beta_\rho x^\rho), \tag{3.2}$$

where m, n are indices of the twodimensional worldsheet. After one partial integration, we obtain one surface term and one term which is identically zero because it is summation of symmetric, $\partial_m\partial_n$, and antisymmetric, ϵ^{mn}, tensor. The surface term is zero for trivial topology. So, the shift isometry exists.

In order to find T-dual action we have to implement following substitutions

$$\partial_\pm x^\mu \to D_\pm x^\mu = \partial_\pm x^\mu + v^\mu_\pm, \tag{3.3}$$

$$x^\rho \to x^\mu_{inv} = \int_P d\xi^m D_m x^\mu = x^\mu(\xi) - x^\mu(\xi_0) + \Delta V^\mu,$$
$$\Delta V^\mu = \int_P d\xi^m v^\rho_m(\xi), \tag{3.4}$$

$$S \to S + \frac{\kappa}{2}\int_\Sigma d^2\xi \big[v^\mu_+\partial_- y_\mu - v^\mu_-\partial_+ y_\mu\big]. \tag{3.5}$$

Because of the shift symmetry we fix the gauge, $x^\mu(\xi) = x^\mu(\xi_0)$ and, inserting these substitutions into action (2.17), we obtain auxiliary action suitable for T-dualization

$$S_{aux} = \kappa \int_\Sigma d^2\xi \Big[\Pi_{+\mu\nu}v^\mu_+ v^\nu_- + \frac{1}{2}(\partial_+\bar\theta^\alpha + v^\mu_+\bar\Psi^\alpha_\mu)\big(F^{-1}(\Delta V)\big)_{\alpha\beta} \times (\partial_-\theta^\beta + \Psi^\beta_\nu v^\nu_-) + \frac{1}{2}(v^\mu_+\partial_- y_\mu - v^\mu_-\partial_+ y_\mu)\Big]. \tag{3.6}$$

It should be noted that, path P that is taken in expression for ΔV^ρ goes from some starting point ξ_0 to end point ξ. Introduction of this element makes this action non-local, however, this is a necessary step in order to find T-dual theory of coordinate dependent background fields.[11]

In order to check if substitutions we had introduced are valid and that they will lead to correct T-dual theory of starting action, we need to be able to obtain original action by finding solutions to equations of motion for Lagrange multipliers. Equations of motion for Lagrange multipliers give us

$$\partial_- v^\mu_+ - \partial_+ v^\mu_- = 0 \quad \Rightarrow \quad v^\mu_\pm = \partial_\pm x^\mu. \tag{3.7}$$

Inserting this result into (3.4) we get the following

$$\Delta V^\rho = \int_P d\xi'^m \partial_m x^\rho(\xi') = x^\rho(\xi) - x^\rho(\xi_0) = \Delta x^\rho. \tag{3.8}$$

Since, we had shift symmetry in original action, we can let $x^\rho(\xi_0)$ be any arbitrary constant. Taking all this into account and inserting (3.7), (3.8) into (3.6) we obtain our starting action (2.17).

Before we obtain equations for motion for gauge fields, we would like to make following substitution in action

$$Y_{+\mu} = \partial_+ y_\mu - \partial_+\bar\theta^\alpha\big(F^{-1}(\Delta V)\big)_{\alpha\beta}\Psi^\beta_\mu,$$
$$Y_{-\mu} = \partial_- y_\mu + \bar\Psi^\alpha_\mu\big(F^{-1}(\Delta V)\big)_{\alpha\beta}\partial_-\theta^\beta, \tag{3.9}$$

$$\bar\Pi_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar\Psi^\alpha_\mu\big(F^{-1}(\Delta V)\big)_{\alpha\beta}\Psi^\beta_\nu = \check\Pi_{+\mu\nu} - \frac{1}{2}\bar\Psi^\alpha_\mu(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_\nu\Delta V^\rho, \tag{3.10}$$

$$\check\Pi_{+\mu\nu} \equiv \Pi_{+\mu\nu} + \frac{1}{2}\bar\Psi^\alpha_\mu(f^{-1})_{\alpha\beta}\Psi^\beta_\nu. \tag{3.11}$$

With these substitutions in mind we have that auxiliary action takes the following form

$$S_{aux} = \kappa \int_\Sigma d^2\xi \Big[\bar\Pi_{+\mu\nu}v^\mu_+ v^\nu_- + \frac{1}{2}v^\mu_+ Y_{-\mu} - \frac{1}{2}v^\mu_- Y_{+\mu} + \frac{1}{2}\partial_+\bar\theta^\alpha\big(F^{-1}(\Delta V)\big)_{\alpha\beta}\partial_-\theta^\beta\Big]. \tag{3.12}$$

This action produces following equations of motion for gauge fields

$$\bar\Pi_{+\mu\nu}v^\nu_- = -(\frac{1}{2}Y_{-\mu} + \beta^+_\mu(V)), \qquad \bar\Pi_{+\mu\nu}v^\mu_+ = \frac{1}{2}Y_{+\nu} - \beta^-_\nu(V). \tag{3.13}$$

Here, function $\beta^\pm(V)$ is obtained from variation of term containing ΔV^ρ in expression for $F^{-1}(\Delta V)$ (details are presented in

Appendix A)

$$\beta^-_\mu(V) = \frac{1}{4}\partial_+\left[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right] - \frac{1}{4}\left[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_+\left[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right], \tag{3.14}$$

$$\beta^+_\mu(V) = \frac{1}{4}\left[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_-\left[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right] - \frac{1}{4}\partial_-\left[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right]. \tag{3.15}$$

Here we have took advantage of the fact that $\partial_\pm V^\mu = v^\mu_\pm$ (more details in Appendix A). Let us note that V^μ in the expressions for beta functions is actually $V^{(0)\mu}$ because it stands besides $C^{\alpha\beta}_\mu$. We omit index (0) just in order to simplify the form of the expressions.

In order to find how gauge fields depend on Lagrange multipliers, we need to invert equations of motion (3.13). Since $C^{\alpha\beta}_\mu$ is an infinitesimal constant, these equations can be inverted iteratively.[22] We separate variables into two parts, one finite and one proportional to $C^{\alpha\beta}_\mu$. After doing this we have

$$v^\nu_- = -\bar\Theta^{\nu\mu}_-\left[\frac{1}{2}Y_{-\mu} + \beta^+_\mu(V^{(0)})\right], \qquad v^\mu_+ = \left[\frac{1}{2}Y_{+\nu} - \beta^-_\nu(V^{(0)})\right]\bar\Theta^{\nu\mu}_-. \tag{3.16}$$

Functions $\beta_{\pm\mu}(V^{(0)})$ are obtained by substituting first order of expression for $v_\pm$ into $\beta_{\pm\mu}(V)$, where $V^{(0)}$ is given by

$$\Delta V^{(0)\rho} = \int_P d\xi^m v^{(0)\rho}_m = \frac{1}{2}\int_P d\xi^+\breve\Theta^{\rho_1\rho}_-\left[\partial_+ y_{\rho_1} - \partial_+\bar\theta^\alpha(f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] - \frac{1}{2}\int_P d\xi^-\breve\Theta^{\rho\rho_1}_-\left[\partial_- y_{\rho_1} + \bar\Psi^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right]. \tag{3.17}$$

Where $\bar\Theta^{\mu\nu}_-$ is inverse tensor of $\bar\Pi_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar\Psi^\alpha_\mu(F^{-1}(\Delta V))_{\alpha\beta}\Psi^\beta_\nu$, defined as

$$\bar\Theta^{\mu\nu}_-\bar\Pi_{+\nu\rho} = \delta^\mu_\rho, \tag{3.18}$$

where

$$\bar\Theta^{\mu\nu}_- = \breve\Theta^{\mu\nu}_- + \frac{1}{2}\breve\Theta^{\mu\mu_1}_-\bar\Psi^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho V^{(0)\rho}(f^{-1})_{\beta_1\beta}\Psi^{\beta_1}_{\nu_1}\breve\Theta^{\nu_1\nu}_-, \tag{3.19}$$

$$\breve\Theta^{\mu\nu}_-\breve\Pi_{\nu\rho} = \delta^\mu_\rho, \qquad \breve\Theta^{\mu\nu}_- = \Theta^{\mu\nu}_- - \frac{1}{2}\Theta^{\mu\mu_1}_-\bar\Psi^\alpha_{\mu_1}(\bar f^{-1})_{\alpha\beta}\Psi^\beta_{\nu_1}\Theta^{\nu_1\nu}_- \tag{3.20}$$

$$\bar f^{\alpha\beta} = f^{\alpha\beta} + \frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu, \tag{3.21}$$

$$\Theta^{\mu\nu}_-\Pi_{+\mu\rho} = \delta^\mu_\rho, \qquad \Theta_- = -4(G_E^{-1}\Pi_- G^{-1})^{\mu\nu}. \tag{3.22}$$

Tensor $G_{E\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$ is known in the literature as the effective metric.

Inserting equations (3.16) into (3.6), keeping only terms that are linear in $C^{\alpha\beta}_\mu$ we obtain T-dual action

$$S^\star = \frac{\kappa}{2}\int_\Sigma\left[\frac{1}{2}\bar\Theta^{\mu\nu}_- Y_{+\mu}Y_{-\nu} + \partial_+\bar\theta^\alpha\left(F^{-1}(\Delta V)\right)_{\alpha\beta}\partial_-\theta^\beta\right]. \tag{3.23}$$

Comparing starting action (2.17) with T-dual action, were we note that $\partial_\pm x^\mu$ transforms into $\partial_\pm y_\mu$ and x^μ transforms into $V^{(0)}$, we can deduce that T-dual action has following arguments.

$$^\star\bar\Pi^{\mu\nu}_+ = \frac{1}{4}\bar\Theta^{\mu\nu}_-, \tag{3.24}$$

$$\left(^\star F^{-1}(V^{(0)})\right)_{\alpha\beta} = \left(F^{-1}(V^{(0)})\right)_{\alpha\beta} - \frac{1}{2}\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta^{\mu\nu}_-\bar\Psi^{\beta_1}_\nu\left(F^{-1}(V^{(0)})\right)_{\beta_1\beta}, \tag{3.25}$$

$$^\star\bar\Psi^{\mu\alpha}\left(^\star F^{-1}(V^{(0)})\right)_{\alpha\beta} = \frac{1}{2}\bar\Theta^{\mu\nu}_-\bar\Psi^\alpha_\nu\left(F^{-1}(V^{(0)})\right)_{\alpha\beta}, \tag{3.26}$$

$$\left(^\star F^{-1}(V^{(0)})\right)_{\alpha\beta}{}^\star\Psi^{\nu\beta} = -\frac{1}{2}\left(F^{-1}(V^{(0)})\right)_{\alpha\beta}\Psi^\beta_\mu\bar\Theta^{\mu\nu}_-. \tag{3.27}$$

In order to express T-dual gravitino background fields in terms of its components, it is useful to calculate inverse of field $^\star F^{-1}_{\alpha\beta}$

$$^\star F^{\alpha\beta}(V^{(0)}) = F^{\alpha\beta}(V^{(0)}) + \frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu. \tag{3.28}$$

With this equation at hand it is straightforward to obtain T-dual gravitino fields. Here we present T-dual gravitino fields expanded in terms of their components

$$^\star\bar\Psi^{\mu\alpha} = \frac{1}{2}\bar\Theta^{\mu\nu}\bar\Psi^\alpha_\nu + \frac{1}{4}\bar\Theta^{\mu\mu_1}_-\bar\Psi^\beta_{\mu_1}\left(F^{-1}(V^{(0)})\right)_{\beta\beta_1}\Psi^{\beta_1}_\nu\Theta^{\nu\nu_1}_-\bar\Psi^\alpha_{\nu_1}, \tag{3.29}$$

$$^\star\Psi^{\nu\beta} = -\frac{1}{2}\Psi^\beta_\mu\bar\Theta^{\mu\nu}_- - \frac{1}{2}\Psi^\beta_\mu\Theta^{\mu\mu_1}_-\bar\Psi^\alpha_{\mu_1}\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi^{\alpha_1}_{\nu_1}\bar\Theta^{\nu_1\nu}_-. \tag{3.30}$$

The general conclusion is that all background fields get the linear corrections in $C^{\alpha\beta}_\mu$ comparing with the results of the case with constant background fields.[23] Also the coordinate dependence is present in all T-dual background fields.

From the above equations we see how background fields of original theory transform under T-duality. It should be noted that these actions are of the same form taking into account that initial coordinates x^μ are replaced by y_μ after T-dualization.

4. T-dualization of T-dual Theory

From requirement that original theory and T-dual theory be physically equivalent, it should be possible to obtain original theory from T-dual one by applying T-duality procedure a second time. Since original action possessed translation symmetry, we have that this symmetry is inherited by T-dual action. T-dual theory is invariant to translations of T-dual coordinate. However, even in cases where starting action is not invariant to translation symmetry we can expect emergence of this symmetry in T-dual theory. This is a natural consequence of introducing $\Delta V^{(0)}$ and of the fact that T-dual theory is intrinsically a non-local one. T-dualization of T-dual theory is obtained with generalized Buscher procedure and steps are identical as before.

$$\partial_\pm y_\mu \to D_\pm y_\mu = \partial_\pm y_\mu + u_{\pm\mu} \to D_\pm y_\mu = u_{\pm\mu}, \tag{4.1}$$

$$\Delta V^{(0)\rho} \to \Delta U^{(0)\rho}, \tag{4.2}$$

$$\Delta U^{(0)\rho} = \frac{1}{2}\int_P d\xi^+ \breve{\Theta}_-^{\rho_1\rho}\left[u_{+\rho_1} - \partial_+\bar\theta^\alpha (f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] - \frac{1}{2}\int_P d\xi^- \breve{\Theta}_-^{\rho\rho_1}\left[u_{-\rho_1} + \bar\Psi^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right], \tag{4.3}$$

$$Y_{+\mu} \to U_{+\mu} = u_{+\mu} - \partial_+\bar\theta^\alpha\left(F^{-1}(\Delta U^{(0)})\right)_{\alpha\beta}\Psi^\beta_\mu \tag{4.4}$$

$$Y_{-\mu} \to U_{-\mu} = u_{-\mu} + \bar\Psi^\alpha_\mu\left(F^{-1}(\Delta U^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta \tag{4.5}$$

$$^\star S \to {}^\star S + \frac{\kappa}{2}\int_\Sigma d^2\xi (u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu). \tag{4.6}$$

In first line we immediately fixed gauge by choosing $y(\xi) = const.$ Inserting these substitutions into (3.23) we get

$$^\star S_{gfix} = \frac{k}{2}\int_\Sigma d^2\xi\left[\frac{1}{2}\bar\Theta_-^{\mu\nu}U_{+\mu}U_{-\nu} + \partial_+\bar\theta^\alpha\left(F^{-1}(\Delta U^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta + (u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu)\right]. \tag{4.7}$$

Finding equations of motion for Lagrange multipliers and inserting solution to those equations into gauge fixed action we return to the starting point of this chapter, T-dual action. On the other hand, finding equations of motion for gauge fields

$$u_{+\mu} = 2\left[\partial_+ x^\nu \bar\Pi_{+\nu\mu} + \beta^-_\mu(x)\right] + \partial_+\bar\theta^\alpha\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\mu, \tag{4.8}$$

$$u_{-\nu} = -2\left[\bar\Pi_{+\mu\nu}\partial_- x^\nu + \beta^+_\mu(x)\right] - \bar\Psi^\alpha_\nu\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta, \tag{4.9}$$

and inserting these equations into the gauge fixed action, keeping all terms linear with respect to $C^{\mu\nu}_\rho$, we obtain our original action (2.17). Here we use the freedom to choose $\Delta x^\mu = x(\xi) - x(\xi_0)$, with $x(\xi_0) = 0$.

5. Non-commutative Relations

Having found T-dual action and equations that link T-dual coordinate with original coordinates in previous chapters, in this chapter we will focus on establishing a relationship between Poisson brackets of original and T-dual theory. Furthermore, we will mainly focus on Poisson brackets between bosonic variables and their momenta. Original theory is a geometric one with variables $x^\mu(\xi)$ and $\pi_\mu(\xi)$. Therefore, it is natural to impose standard Poisson structure on original theory

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu\delta(\sigma-\bar\sigma), \quad \{x^\mu(\sigma), x^\nu(\bar\sigma)\} = 0, \quad \{\pi_\mu(\sigma), \pi_\nu(\bar\sigma)\} = 0. \tag{5.1}$$

In order to find Poisson brackets of T-dual theory, we need to find T-dual transformation laws which connect the initial and T-dual coordinates. Starting with relations (4.8) and (4.9) and using equations of motion for Lagrange multipliers x^μ, $u_{\pm\mu} = \partial_\pm y_\mu$, we obtain T-dual transformation laws

$$\partial_+ y_\mu \cong 2\left[\partial_+ x^\nu \bar\Pi_{+\nu\mu} + \beta^-_\mu(x)\right] + \partial_+\bar\theta^\alpha\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\mu, \tag{5.2}$$

$$\partial_- y_\mu \cong -2\left[\bar\Pi_{+\mu\nu}\partial_- x^\nu + \beta^+_\mu(x)\right] - \bar\Psi^\alpha_\nu\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta, \tag{5.3}$$

where symbol $\cong$ denotes T-dual transformation. Subtracting these two equations, we get

$$y'_\mu \cong \bar\Pi_{+\mu\nu}\partial_- x^\nu + \partial_+ x^\nu\bar\Pi_{+\nu\mu} + \beta^+_\mu + \beta^-_\mu + \frac{1}{2}\partial_+\bar\theta^\alpha\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\mu + \frac{1}{2}\bar\Psi^\alpha_\nu\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta. \tag{5.4}$$

Taking into account that bosonic momenta, π_μ of original theory are of the form

$$\pi_\mu = \kappa\left[\bar\Pi_{+\mu\nu}\partial_- x^\nu + \partial_+ x^\nu\bar\Pi_{+\nu\mu} + \frac{1}{2}\bar\Psi^\alpha_\mu\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta + \frac{1}{2}\partial_+\bar\theta^\alpha\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\nu\right], \tag{5.5}$$

and $\beta^0_\mu = \beta^+_\mu + \beta^-_\mu$, we obtain

$$y'_\mu \cong \frac{\pi_\mu}{\kappa} + \beta^0_\mu(x). \tag{5.6}$$

Here $\beta^0_\mu(x)$ is given by

$$\beta^0_\mu(x) = \frac{1}{2}\partial_\sigma\left[\bar\theta^\alpha + x^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left[\theta^\beta + \Psi^\beta_{\nu_2}x^{\nu_2}\right] - \frac{1}{2}\left[\bar\theta^\alpha + x^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_\sigma\left[\theta^\beta + \Psi^\beta_{\nu_2}x^{\nu_2}\right]. \tag{5.7}$$

To find Poisson bracket between T-dual coordinates, we can start by finding Poisson bracket of sigma derivatives of T-dual coordinates and then integrating twice (see [13, 24, 25], B). Implementing this procedure we have that Poisson bracket is given as

$$\{y_{\nu_1}(\sigma), y_{\nu_2}(\bar\sigma)\} \cong \frac{1}{2k}[2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2} - \delta^{\mu_2}_{\nu_1}\delta^{\mu_1}_{\nu_2}]\left[K_{\mu_1\mu_2}(\bar\sigma) + K_{\mu_2\mu_1}(\sigma)\right]H(\sigma-\bar\sigma), \tag{5.8}$$

where, for the sake of simplicity, we introduced

$$K_{\mu\nu}(\sigma) = \left(\bar\theta^\alpha(\sigma) + x^{\mu_1}(\sigma)\bar\Psi^\alpha_{\mu_1}\right)(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\Psi^\beta_\nu - \bar\Psi^\alpha_\nu(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left(\theta^\beta(\sigma) + \Psi^\beta_{\nu_1}x^{\nu_1}(\sigma)\right). \tag{5.9}$$

Here, $H(\sigma-\bar\sigma)$ is same step function defined in Appendix B. It should be noted that these Poisson brackets are zero when $\sigma = \bar\sigma$. However, in cases where string in curled around compactified dimension, that is cases where $\sigma - \bar{\sigma} = 2\pi$, we have following situation

$$
\begin{aligned}
&\{\gamma_{\nu_1}(\sigma + 2\pi), \gamma_{\nu_2}(\sigma)\} \\
&\cong \frac{1}{2k}[2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2} - \delta^{\mu_2}_{\nu_1}\delta^{\mu_1}_{\nu_2}]\left[K_{\mu_1\mu_2}(\sigma) + K_{\mu_2\mu_1}(\sigma)\right] \\
&\quad + \frac{\pi}{k} N^\mu \bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_{\mu_2}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_3}[\delta^{\mu_1}_{\mu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} - \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\mu} \\
&\quad + \delta^{\mu_1}_{\mu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\mu}].
\end{aligned} \tag{5.10}
$$

Here we used fact that $H(2\pi) = 1$, while N^ρ is winding number around compactified coordinate defined as

$$
x^\mu(\sigma + 2\pi) - x^\mu(\sigma) = 2\pi N^\mu. \tag{5.11}
$$

From this relation we can see that if we choose $x^\mu(\sigma) = 0$ than Poisson bracket has linear dependence on winding number. In cases where we do not have any winding number, we still have non-commutativity that is proportional to background fields.

Using the expression for sigma derivative of γ_ν (5.6) and expression for Poisson bracket of T-dual coordinates (5.8), we can find non-associative relations. Procedure is the same as for finding Poisson brackets of T-dual theory, we find Poisson bracket of sigma derivative and integrate with respect to sigma coordinate, this time integration is done once. Going along with this procedure we have the final result

$$
\begin{aligned}
&\{\gamma_\nu(\sigma), \{\gamma_{\nu_1}(\sigma_1), \gamma_{\nu_2}(\sigma_2)\}\} \\
&\cong \frac{1}{2k} H(\sigma_1 - \sigma_2)\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_{\mu_2}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_3} \\
&\times\Big[H(\sigma_1 - \sigma)[2\delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} - 2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu} - \delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} + \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu}] \\
&+H(\sigma_2 - \sigma)[2\delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} - 2\delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu} - \delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} + \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu}]\Big].
\end{aligned} \tag{5.12}
$$

Since Jacobi identity is non-zero for T-dual theory we have that coordinate dependent RR field produces non-associative theory. However putting $\sigma = \sigma_2 = \bar{\sigma}$ and $\sigma_1 = \bar{\sigma} + 2\pi$ we have following Jacobi identity

$$
\begin{aligned}
&\{\gamma_\nu(\bar{\sigma}), \{\gamma_{\nu_1}(\bar{\sigma} + 2\pi), \gamma_{\nu_2}(\bar{\sigma})\}\} \\
&\cong \bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_{\mu_2}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_3}[2\delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} - 2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu} \\
&\quad - \delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} + \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu}].
\end{aligned} \tag{5.13}
$$

Examining equation (5.6), we notice that $\partial_\sigma \gamma_\mu$ is not only a linear combination of initial coordinate and its momenta but also has terms that are proportional to fermionic coordinates. This might lead us to believe that T-dual theory would have nontrivial Poisson bracket between T-dual coordinate and fermionic coordinates. However, this is not the case, and it can be directly calculated by finding Poisson bracket between sigma derivative of T-dual coordinate and fermion coordinates (more details in B).

$$
\{\theta^\alpha(\sigma), \gamma_\mu(\bar{\sigma})\} \cong 0, \qquad \{\bar{\theta}^\alpha(\sigma), \gamma_\mu(\bar{\sigma})\} \cong 0. \tag{5.14}
$$

6. Conclusion

In this article we examined type II superstring propagating in presence of coordinate dependent RR field. This choice of background was in accordance with consistency conditions for background field and all calculations were made in approximation that are linear with respect to the space-time derivative of the RR field, $C^{\alpha\beta}_\mu$, which is infinitesimal one. We have also excluded parts that were non-linear in fermionic coordinates and neglected pure spinor actions. Using equations of motion for fermionic momenta we obtained action that was expressed in terms of bosonic coordinates, their derivatives and derivatives of fermionic coordinates.

Action with our choice of background fields possessed translation symmetry, therefore we use generalized Buscher procedure that was developed for such cases. By substituting starting action with auxiliary action we gave up on locality in order to be able to find T-dual theory. Finding equations of motion of newly introduced Lagrange multipliers we were able to salvage starting action giving us assurance that auxiliary action we selected would produce correct T-dual theory. After this we found equations of motion for gauge fields and by inserting them into action, we found T-dual theory.

Having found T-dual theory, we applied T-dual procedure once again as a more thorough way of checking if action we obtained was in fact correct T-dual of starting action. Unlike starting action, T-dual action was non-local from the start by virtue of containing $V^{(0)}$ term. Applying steps of generalized Buscher procedure we obtained starting action, again confirming that our choice of auxiliary action was correct.

We obtained non-commutativity relations in context of T-dual theory, where we used T-dual transformation laws as a bridge between Poisson brackets of starting theory and T-dual theory. T-dual transformation laws were expressed in terms of coordinates and momenta of original theory, which produced non-commutativity in T-dual theory. From expression for Poisson brackets (5.8) we can see that non-commutativity is proportional to infinitesimal part of RR field. Non-commutativity relations are zero in case when $\sigma = \bar{\sigma}$, while in case where $\sigma = \bar{\sigma} + 2\pi$ we see the emergence of winding numbers. Noncommutativity parameters are linearly dependent on bosonic coordinates x^μ as well as on fermionic ones, θ^α and $\bar{\theta}^\alpha$.

Taking into account Poisson brackets of T-dual coordinates and expression for sigma derivative of T-dual coordinate we were able to find non-associative relation for T-dual theory. In general case this relation was non-zero and it was proportional to infinitesimal constant, which is proportional to $C^{\alpha\beta}_\mu$. In special case when we put $\sigma_1 = \sigma_2 = \bar{\sigma}$ and $\sigma_3 = \bar{\sigma} + 2\pi$ we noticed that non-associativity relation remains constant.

It should be noted that since we did not preform T-dualization along fermionic coordinates their Poisson structure would remain the same as in original theory. However, unlike original theory, T-dual coordinates depend on sigma derivatives of fermionic coordinates. This dependence does not affect the Poisson brackets of the T-dual coordinates and fermionic coordinates (5.14). So, T-dual SUSY algebra has non zero Poisson bracket of the bosonic coordinates, while the rest ones are zero. In further investigation we will study fermionic T-dualization and we expect the effect on the algebra of the fermionic coordinates.

Appendix A: Obtaining $\beta^{\pm}_{\mu}$ Terms

In this paper function $\beta^{\pm}_{\mu}(V)$ emerged in T-dual transformation laws as a consequence of variation of term that was proportional to ΔV. Here we will present derivation of this function.

Here we will use substitutions $\partial_+\bar{\Theta}^\alpha = \partial_+\bar{\theta}^\alpha + v_+^{\nu_1}\bar{\Psi}^\alpha_{\nu_1}$, $\partial_-\Theta^\beta = \partial_-\theta^\beta + \Psi^\beta_{\nu_2} v_-^{\nu_2}$, also we will use $F_{\alpha\beta\rho}$ to represent term containing infinitesimal constant

$$\begin{aligned}
\int_\Sigma d^2\xi \partial_+\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_-\Theta^\beta &= \int_\Sigma d^2\xi\epsilon^{mn}\partial_m\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_n\Theta^\beta \\
&= \int_\Sigma d^2\xi\Big[\frac{1}{2}\epsilon^{mn}\partial_m\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_n\Theta^\beta \\
&\quad - \frac{1}{2}\epsilon^{mn}\partial_n\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_m\Theta^\beta\Big] \\
&= -\frac{1}{2}\int_\Sigma d^2\xi\Big[\epsilon^{mn}\bar{\Theta}^\alpha F_{\alpha\beta\rho}\partial_m\Delta V^{(0)\rho}\partial_n\Theta^\beta \\
&\quad +\frac{1}{2}\epsilon^{mn}\partial_n\bar{\Theta}^\alpha F_{\alpha\beta\rho}\partial_m\Delta V^{(0)\rho}\Theta^\beta\Big] \\
&= -\frac{1}{2}\int_\Sigma d^2\xi\epsilon^{mn}\partial_m\Delta V^{(0)\rho}\left[\bar{\Theta}^\alpha F_{\alpha\beta\rho}\partial_n\Theta^\beta - \partial_n\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Theta^\beta\right] \\
&= -\frac{1}{2}\int_\Sigma d^2\xi\epsilon^{mn}v^\rho_m\left[\bar{\Theta}^\alpha F_{\alpha\beta\rho}\partial_n\Theta^\beta - \partial_n\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Theta^\beta\right] \\
&= \int_\Sigma d^2\xi v^\rho_m\beta^m_\rho.
\end{aligned} \tag{A.1}$$

Variation with respect to gauge field $v^\rho_\pm$, and setting $F_{\alpha\beta\rho} = -(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}$ produces desired $\beta^\pm_\rho$ functions (3.14), (3.15) in equations of motion (3.13). Here we have used the property that $(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}$, ie. $F_{\alpha\beta\rho}$, is antisymmetric under exchange of α and β, this, in combination with the fact that we can express $\partial_+\bar{\Theta}\partial_-\Theta$ as $\epsilon^{nm}\partial_n\bar{\Theta}\partial_m\Theta$, removes all terms proportional to $\partial_+\partial_-$, using identity $\epsilon^{mn}\bar{\Theta}\partial_m\partial_n\Theta = 0$.

It should be noted that $\beta_{\pm\mu}(V)$ functions are not unique, we could have obtained different function simply by not using using symmetrization in (A.1). In case of non-symmetric $\beta_{\pm\mu}(V)$, all results that have been obtained would take a simpler form. We have chosen to work with symmetric function because results that are deduced from this case can be easily reduced, by neglecting terms, to simpler case.

Appendix B: Poisson Bracket Between Sigma Derivatives of T-dual Coordinate

In this article, in order to find Poisson brackets of the T-dual coordinates we had to find first Poisson brackets of the sigma derivative of T-dual coordinates, $y'_\mu \equiv \partial_\sigma y_\mu(\sigma)$. In this section we will demonstrate how to obtain Poisson brackets from Poisson brackets that contain sigma derivatives. We will use canonical form of the T-dual transformation law (5.6) and standard Poisson algebra, because the initial theory is geometric one. First, we have to calculate the following Poisson bracket

$$\begin{aligned}
&\{\partial_{\sigma_1}y_{\nu_1}(\sigma_1), \partial_{\sigma_2}y_{\nu_2}(\sigma_2)\} \\
&= \frac{1}{2k}\Big[K_{\nu_2\nu_1}(\sigma_2)\partial_{\sigma_2}\delta(\sigma_2-\sigma_1) - K_{\nu_1\nu_2}(\sigma_1)\partial_{\sigma_1}\delta(\sigma_1-\sigma_2) \\
&\quad + \partial_{\sigma_1}K_{\nu_1\nu_2}(\sigma_1)\delta(\sigma_1-\sigma_2) - \partial_{\sigma_2}K_{\nu_2\nu_1}(\sigma_2)\delta(\sigma_2-\sigma_1)\Big],
\end{aligned} \tag{B.1}$$

where $K_{\mu\nu}(\sigma)$ is given by (5.9).

On the other side we have

$$\begin{aligned}
&\{\Delta y_{\nu_1}(\sigma_0,\sigma), \Delta y_{\nu_2}(\bar{\sigma}_0,\bar{\sigma})\} \\
&= \int_{\sigma_0}^{\sigma} d\sigma_1\int_{\bar{\sigma}_0}^{\bar{\sigma}} d\sigma_2\{\partial_{\sigma_1}y_{\nu_1}(\sigma_1), \partial_{\sigma_2}y_{\nu_2}(\sigma_2)\} \\
&= \{y_{\nu_1}(\sigma), y_{\nu_2}(\bar{\sigma})\} - \{y_{\nu_1}(\sigma), y_{\nu_2}(\bar{\sigma}_0)\} - \{y_{\nu_1}(\sigma_0), y_{\nu_2}(\bar{\sigma})\} \\
&\quad + \{y_{\nu_1}(\sigma_0), y_{\nu_2}(\bar{\sigma}_0)\},
\end{aligned} \tag{B.2}$$

where

$$\Delta y_\mu(\sigma_0,\sigma) \equiv \int_{\sigma_0}^{\sigma} d\sigma_1\partial_{\sigma_1}y_\mu(\sigma_1) = y_\mu(\sigma) - y_\mu(\sigma_0). \tag{B.3}$$

Combining the equations (B.1) and (B.2) we have

$$\begin{aligned}
&\{\Delta y_{\nu_1}(\sigma_0,\sigma), \Delta y_{\nu_2}(\bar{\sigma}_0,\bar{\sigma})\} = \frac{1}{2k}\int_{\sigma_0}^{\sigma} d\sigma_1\int_{\bar{\sigma}_0}^{\bar{\sigma}} d\sigma_2 \\
&\times\Big[K_{\nu_2\nu_1}(\sigma_2)\partial_{\sigma_2}\delta(\sigma_2-\sigma_1) - K_{\nu_1\nu_2}(\sigma_1)\partial_{\sigma_1}\delta(\sigma_1-\sigma_2) \\
&\quad + \partial_{\sigma_1}K_{\nu_1\nu_2}(\sigma_1)\delta(\sigma_1-\sigma_2) - \partial_{\sigma_2}K_{\nu_2\nu_1}(\sigma_2)\delta(\sigma_2-\sigma_1)\Big].
\end{aligned} \tag{B.4}$$

By applying partial integration, it is straightforward to extract the Poisson bracket of T-dual coordinates given by (5.8).

In paper[13] it has been shown that Poisson brackets between σ derivatives of coordinates have following form

$$\begin{aligned}
\{\partial_{\sigma_1}X_\mu(\sigma_1), \partial_{\sigma_2}Y_\nu(\sigma_2)\} &\cong \partial_{\sigma_1}K_{\mu\nu}(\sigma_1)\delta(\sigma_1-\sigma_2) \\
&\quad + L_{\mu\nu}(\sigma_1)\partial_{\sigma_1}\delta(\sigma_1-\sigma_2).
\end{aligned} \tag{B.5}$$

Applying integrating twice and using partial integration this equation reduces to

$$\{X_\mu(\sigma_1), Y_\nu(\sigma_2)\} \cong -\left[K_{\mu\nu}(\sigma_1) - K_{\mu\nu}(\sigma_2) + L_{\mu\nu}(\sigma_2)\right]\delta(\sigma_1-\sigma_2). \tag{B.6}$$

In our case, we can bring equation (B.1) to the form of equation (B.5) by making following substitutions

$$
\begin{aligned}
\partial_{\sigma_1} K_{\nu_1\nu_2}(\sigma_1) &= \partial_{\sigma_1} K_{\nu_1\nu_2}(\sigma_1), \qquad \partial_{\sigma_2} K_{\nu_2\nu_1}(\sigma_2) = \partial_{\sigma_2} K_{\nu_2\nu_1}(\sigma_2), \\
K_{\nu_1\nu_2}(\sigma_1) &= -L_{\nu_1\nu_2}(\sigma_1), \qquad K_{\nu_2\nu_1}(\sigma_2) = -L_{\nu_2\nu_1}(\sigma_2).
\end{aligned} \tag{B.7}
$$

Because we chose to work with symmetric $\beta^{\pm}_{\mu}$ function we obtain duplicated terms in (B.5).

Same procedure can be applied to find Poisson bracket between T-dual coordinate and fermionic momenta. That is, we start from Poisson bracket for sigma derivative of T-dual coordinate and fermionic momenta, then integrate once and compare left and right hand sides.

The step function $H(x)$ is defined as

$$
H(x) = \int_0^x ds\delta(s) = \frac{1}{2\pi}\Sigma_{n\in Z}\int_0^x e^{ins} = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1/2 & \text{if} \quad 0 < x < 2\pi \\ 1 & \text{if} \quad x = 2\pi \end{cases}. \tag{B.8}
$$

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Keywords

string theory, noncommutativity, nonassociativity, T-duality

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Higher category theory and n-groups as gauge symmetries for quantum gravity

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Higher category theory and n-groups as gauge symmetries for quantum gravity

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Abstract. Higher category theory can be employed to generalize the notion of a gauge group to the notion of a gauge n-group. This novel algebraic structure is designed to generalize notions of connection, parallel transport and holonomy from curves to manifolds of dimension higher than one. Thus it generalizes the concept of gauge symmetry, giving rise to a topological action called nBF action, living on a corresponding n-principal bundle over a spacetime manifold. Similarly as for the Plebanski action, one can deform the topological nBF action by adding appropriate simplicity constraints, in order to describe the correct dynamics of both gravity and matter fields. Specifically, one can describe the whole Standard Model coupled to gravity as a constrained $3BF$ or $4BF$ action. The split of the full action into a topological sector and simplicity constraints sector is adapted to the spinfoam quantization technique, with the aim to construct a full model of quantum gravity with matter. In addition, the properties of the gauge n-group structure open up a possibility of a nontrivial unification of all fields. An n-group naturally contains additional novel gauge groups which specify the spectrum of matter fields present in the theory, in a similar way to the ordinary gauge group that prescribes the spectrum of gauge vector bosons in the Yang-Mills theory. The presence and the properties of these new gauge groups has the potential to explain fermion families, and other structure in the matter spectrum of the theory.

1. Introduction

The formulation of a quantum theory of gravity represents one of the fundamental open problems in modern theoretical physics. Among the many approaches to this problem, some have developed into vast research frameworks, such as Loop Quantum Gravity, which aims to formulate a model of quantum gravity (QG) in a nonperturbative fashion, both canonically and covariantly [1, 2, 3]. The covariant approach aims to give a tentative rigorous definition of the path integral for the gravitational field,

$$Z = \int \mathcal{D}g \; e^{iS[g]} \,. \tag{1}$$

One of the essential assumptions is a triangulation of a spacetime manifold, and the path integral is introduced as a discrete state sum of the gravitational field configurations, living on the simplicial complex structure. This approach to quantization of gravity is usually called the *spinfoam* quantization method. It is performed via the following three steps:

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(1) one reformulates the classical action $S[g]$ as a constrained BF action, separating the topological BF part and the constraint part of the action;
(2) one employs the underlying Lie group structure of the BF sector of the action, in order to define a triangulation-independent state sum Z;
(3) finally, one deforms the topological state sum by applying the simplicity constraints, and therefore redefining it into a triangulation-dependent state sum, which plays the role of a definition for the path integral (1).

This type of quantization prescription has been implemented in a number of cases, for various choices of the gravitational action, of the Lie group, and of the spacetime dimension. Historically the first spinfoam model was the Ponzano-Regge model [4], defined in 3 spacetime dimensions. In 4 dimensions multiple models have been formulated, differing in the choice of the Lie group and the way one imposes the simplicity constraints [5, 6, 7, 8, 9]. While all these models do represent definitions of the gravitational path integral, none of them are able to include matter fields in a seamless way. Introducing the latter into a spinfoam QG model has so far had only limited success [10], predominantly due to the lack of the tetrad fields in the topological part of the model.

Recently, a new approach has been developed to address the issue of matter fields, which employs the framework of *higher gauge theory* (see [11] for a review). Specifically, one uses the notion of a *categorical ladder* to generalize the BF action (based on a Lie group) to a $2BF$ action (based on the so-called 2-group structure), and further to a $3BF$ action (based on a 3-group structure). A convenient choice of the *Poincaré 2-group* gives rise to the needed tetrad fields in the topological sector of the action [12], while an additional extension to the 3-group naturally introduces the matter fields (fermions and scalars) into the model [13]. The steps of the categorical ladder and their corresponding structures are summarized as follows:

categorical structure	algebraic structure	linear structure	topological action	degrees of freedom
Lie group	Lie group	Lie algebra	BF theory	gauge fields
Lie 2-group	Lie crossed module	differential Lie crossed module	$2BF$ theory	tetrad fields
Lie 3-group	Lie 2-crossed module	differential Lie 2-crossed module	$3BF$ theory	scalar and fermion fields

The main aim of this work is to provide a short review of the classical pure BF, $2BF$ and $3BF$ actions, in order to demonstrate the categorical ladder procedure and the construction of higher gauge theories. In other words, we mainly focus on the step 1 of the spinfoam quantization programme, with a very short review of step 2 of the programme.

The layout of the paper is as follows. Section 2 deals with first three examples of nBF theories, namely BF, $2BF$ and $3BF$ actions, and their construction using the categorical ladder. After this, in Section 3 we briefly present an application of a $3BF$ theory to the Standard Model of elementary particles coupled to Einstein-Cartan gravity. As it turns out, the scalar and fermion fields are *naturally associated to a new gauge group*, generalizing the role of an ordinary gauge group in the Yang-Mills theory. This opens up a possibility of an algebraic classification of matter fields, and (more speculatively) a possibility of the explanation of the three fermion families. Finally, Section 4 contains some discussion and our conclusions.

The notation and conventions are as follows. Spacetime indices are denoted by the Greek letters $\mu, \nu, \dots$, and are raised and lowered by the spacetime metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ are the tetrad fields. The inverse tetrad is denoted as $e^\mu{}_a$. The local Lorentz indices are denoted by the Latin letters $a, b, c, \dots$, take values $0, 1, 2, 3$, and are raised and lowered using the Minkowski metric η_{ab} with signature $(-, +, +, +)$. All other indices that appear in the paper depend on the context, and their use is explicitly defined in the text where they appear. We work in the natural system of units where $c = \hbar = 1$, and $G = l_p^2$, where l_p is the Planck length. The exterior product in the space of differential forms is denoted with the standard "wedge" symbol, $\wedge$.

2. nBF theories

We begin by giving a short review of nBF theories, for $n = 1, 2, 3$, which represent the most interesting cases for physics.

2.1. BF theory

A BF theory and its various applications in physics are already well known in the literature, see for example [14, 15, 16], so here we merely give a brief definition. Given a Lie group G, and its corresponding Lie algebra as $\mathfrak{g}$, one defines the BF action in the form (we discuss only the 4-dimensional spacetime manifolds $\mathcal{M}_4$):

$$S_{BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} \,. \tag{2}$$

Here, $\mathcal{F} \equiv \mathrm{d}\alpha + \alpha \wedge \alpha$ is the curvature 2-form for the $\mathfrak{g}$-valued connection 1-form $\alpha \in \Lambda^1(\mathcal{M}_4) \otimes \mathfrak{g}$, while $B \in \Lambda^2(\mathcal{M}_4) \otimes \mathfrak{g}$ is a $\mathfrak{g}$-valued Lagrange multiplier 2-form. Also, $\langle \_ , \_ \rangle_{\mathfrak{g}}$ denotes a G-invariant nondegenerate symmetric bilinear form over $\mathfrak{g}$.

Varying the action (2) with respect to B and α, one obtains the equations of motion:

$$\mathcal{F} = 0 \,, \qquad \nabla B \equiv \mathrm{d}B + \alpha \wedge B = 0 \,. \tag{3}$$

The first equation implies that α is a flat connection, in the sense that $\alpha = 0$ up to gauge transformations. The second equation then implies that B is covariantly constant. From these one can deduce that there are no local propagating degrees of freedom, and therefore the theory is said to be *topological*.

2.2. 2BF theory

Once we have introduced the BF model, we procceed to first step of the *categorical ladder*, generalizing the algebraic notion of a group to the notion of a 2-group. This leads to the generalization of the BF theory to the $2BF$ theory, also sometimes called $BFCG$ theory [11, 17, 18, 19].

The categorical ladder is a procedure of generalizing various notions in mathematics, using the framework of category theory, and works as follows. One starts from the notion of a group as an algebraic structure, and notes that it can be understood as a category with only one object and invertible morphisms [11]. Then, one employs the fundamental idea that a category can be generalized to the so-called *higher categories*, which have not only objects and morphisms, but also 2-morphisms (maps between morphisms), 3-morphisms (maps between 2-morphisms), and so on. This tower of n-categories is known as the *categorical ladder*. Applying the construction to groups, it is straightforward to introduce the notion of a 2-*group* as a 2-category consisting of only one object, where all the morphisms and all 2-morphisms are invertible. It was demonstrated that every strict 2-group is equivalent to a *crossed module* $(H \stackrel{\partial}{\to} G \,, \triangleright)$, see [13] for detailed definitions. Here G and H are groups, ∂ is a homomorphism from H to G, while $\triangleright : G \times H \to H$ is an action of G on H.

Just like an ordinary Lie group G has a naturally associated connection α and gives rise to a BF theory, a Lie 2-group has a naturally associated 2-connection $(\alpha \,, \beta)$, described by the usual $\mathfrak{g}$-valued 1-form $\alpha \in \Lambda^1(\mathcal{M}_4) \otimes \mathfrak{g}$ and an $\mathfrak{h}$-valued 2-form $\beta \in \Lambda^2(\mathcal{M}_4) \otimes \mathfrak{h}$, where $\mathfrak{h}$ is a

Lie algebra of the Lie group H. This 2-connection gives rise to the so-called *fake 2-curvature* $(\mathcal{F},\mathcal{G})$, defined as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = d\beta + \alpha \wedge^{\triangleright} \beta\,. \tag{4}$$

Here $\alpha \wedge^{\triangleright} \beta$ means that α and β are multiplied as forms using $\wedge$, and simultaneously multiplied as algebra elements using $\triangleright$, see [13]. The curvature pair $(\mathcal{F},\mathcal{G})$ is called "fake" due of the presence of the additional term $\partial\beta$ in the definition of $\mathcal{F}$ [11].

Using the structure of a 2-group, or equivalently the crossed module, one can introduce the so-called $2BF$ action, as a generalization of the BF action, as follows [17, 18]:

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F}\rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G}\rangle_{\mathfrak{h}}\,. \tag{5}$$

Here the 2-form $B \in \Lambda^2(\mathcal{M}_4) \otimes \mathfrak{g}$ and the 1-form $C \in \Lambda^1(\mathcal{M}_4) \otimes \mathfrak{h}$ are Lagrange multipliers. Also, $\langle \_\,,\_\rangle_{\mathfrak{g}}$ and $\langle \_\,,\_\rangle_{\mathfrak{h}}$ denote the G-invariant nondegenerate symmetric bilinear forms over the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. As a consequence of the axiomatic structure of a crossed module (see [13]), the bilinear form $\langle \_\,,\_\rangle_{\mathfrak{h}}$ is H-invariant as well. See [17, 18] for review and references.

The equations of motion for a $2BF$ theory are an extension of the equations of motion of a BF theory. Varying with respect to B and C one obtains

$$\mathcal{F}^{\alpha} = 0\,, \qquad \mathcal{G}^{a} = 0\,, \tag{6}$$

while varying with respect to α and β one obtains the equations for the multipliers,

$$\nabla B + C \wedge^{\mathcal{T}} \beta = 0\,, \qquad \nabla C - \partial B = 0\,. \tag{7}$$

Here the map $\mathcal{T}$ is defined in [13]. A rigorous Hamiltonian analysis of the model demonstrates that in this case as well there are no local propagating degrees of freedom [20, 21] (see also [22]). Therefore the $2BF$ theory is also topological.

2.3. 3BF theory

When constructing more realistic (nontopological) models by adding constraints to BF and $2BF$ models, it becomes apparent that the group G with a constrained BF action can successfully describe ordinary gauge vector bosons, while the so-called Poincaré 2-group with a constrained $2BF$ action can successfully describe general relativity. However, neither of these can suitably accomodate matter fields, such as fermions or scalars. Nevertheless, it turns out that this can be remedied if we make one further step in the categorical ladder, passing from the notion of a 2-group to the notion of a 3-group. As we shall see in the next Section, the notion of a 3-group will prove to be an excellent structure for the description of all fields that are present in the Standard Model, coupled to Einstein-Cartan gravity. Moreover, a 3-group contains one more gauge group, which is novel and specifies the spectrum of scalar and fermion fields present in the theory. This is an unexpected and beautiful result, absent from ordinary gauge theory.

Applying the categorical ladder once more, one can introduce the notion of a 3-group in the framework of higher category theory, as a 3-category with only one object where all the morphisms, 2-morphisms and 3-morphisms are invertible. Also, the equivalence between a 2-group and a crossed module has been generalized to the equivalence between a strict 3-group and a 2-*crossed module* [23]. A Lie 2-crossed module, denoted as $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\})$, is an algebraic structure specified by three Lie groups G, H and L, together with the homomorphisms δ and ∂, an action $\triangleright$ of the group G on all three groups, and a G-equivariant map

$$\{\_\,,\_\} : H \times H \to L\,.$$

called the Peiffer lifting. The maps ∂, δ, $\triangleright$ and the Peiffer lifting satisfy certain axioms, so that the resulting structure is equivalent to a 3-group [13].

Based on a given 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,,\triangleright\,,\{\_\,,\_\})$, one can introduce a gauge invariant topological $3BF$ action over the manifold $\mathcal{M}_4$ as follows. Denoting $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ as Lie algebras corresponding to the groups G, H and L, respectively, the Lie 3-group structure allows one to introduce a 3-connection (α, β, γ) given by the algebra-valued differential forms $\alpha \in \Lambda^1(\mathcal{M}_4) \otimes \mathfrak{g}$, $\beta \in \Lambda^2(\mathcal{M}_4) \otimes \mathfrak{h}$ and $\gamma \in \Lambda^3(\mathcal{M}_4) \otimes \mathfrak{l}$. The corresponding fake 3-curvature $(\mathcal{F}\,, \mathcal{G}\,, \mathcal{H})$ is then defined as

$$\begin{gathered}\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \\ \mathcal{H} = \mathrm{d}\gamma + \alpha \wedge^{\triangleright} \gamma + \{\beta \wedge \beta\}\,,\end{gathered} \tag{8}$$

see [23, 24] for details. Note that γ is a 3-form, while its corresponding field strength $\mathcal{H}$ is a 4-form, requiring that the spacetime manifold be at least 4-dimensional. Also, for this reason, going beyond 3-groups and 4-groups in the categorical ladder does not have many applications in realistic 4-dimensional physics. A $3BF$ action is defined as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}\,, \tag{9}$$

where $B \in \Lambda^2(\mathcal{M}_4) \otimes \mathfrak{g}$, $C \in \Lambda^1(\mathcal{M}_4) \otimes \mathfrak{h}$ and $D \in \Lambda^0(\mathcal{M}_4) \otimes \mathfrak{l}$ are Lagrange multipliers valued in the respective algebras. Note that exclusively in 4 spacetime dimensions the Lagrange multiplier D corresponding to $\mathcal{H}$ is a 0-form, i.e. a scalar function. As before, the bilinear forms $\langle\_\,,\_\rangle_{\mathfrak{g}}$, $\langle\_\,,\_\rangle_{\mathfrak{h}}$ and $\langle\_\,,\_\rangle_{\mathfrak{l}}$ are G-invariant, nondegenerate and symmetric, over the algebras $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, respectively.

The equations of motion can be obtained by varying the action with respect to the multipliers B, C and D,

$$\mathcal{F} = 0\,, \qquad \mathcal{G} = 0\,, \qquad \mathcal{H} = 0\,, \tag{10}$$

and by varying with respect to the connections α, β and γ,

$$\nabla B + C \wedge^{\mathcal{T}} \beta - D \wedge^{\mathcal{S}} \gamma = 0\,, \qquad \nabla C - \partial B - D \wedge^{(\mathcal{X}_1 + \mathcal{X}_2)} \beta = 0\,, \qquad \nabla D + \delta C = 0\,. \tag{11}$$

See [13] for the detailed definitions of the maps $\mathcal{T}$, $\mathcal{S}$, $\mathcal{X}_1$ and $\mathcal{X}_2$.

3. The Standard Model 3-group

At this point we are finally ready to construct a realistic classical action, featuring the full Standard Model of elementary particles coupled to Einstein-Cartan gravity. The action is based on a so-called Standard Model 3-group, which is a 2-crossed module ($L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G$, $\triangleright$, $\{\_\,,\_\}$) with a following choices for the Lie groups:

$$G = SO(3,1) \times SU(3) \times SU(2) \times U(1)\,, \qquad H = \mathbb{R}^4\,,$$

$$L = \mathbb{C}^4 \times \mathbb{G}^{64} \times \mathbb{G}^{64} \times \mathbb{G}^{64}\,.$$

We choose the group G as a product of the Lorentz group and the usual internal gauge symmetry group of the Standard Model. The group H is chosen to be the group of spacetime translations, motivated by the Poincaré 2-group construction [12]. Finally, we choose the group L as a product of $\mathbb{C}^4$ accouning for the doublet of complex scalar fields, and three copies of the 64-dimensional Grassmann algebra $\mathbb{G}^{64}$, representing three families of fermions. The maps δ, ∂ and $\{\_\,,\_\}$ are trivial, while the map $\triangleright$ is chosen in a natural way, in accord with the usual action of the gauge group G onto translations and various components of matter fields. It is defined in detail in [13].

Once the 3-group has been completely specified, the corresponding action can be written as a $3BF$ action with suitable constraint terms, as follows:

$$
\begin{array}{rcll}
S &=& \displaystyle\int \overbrace{B_\alpha \wedge F^\alpha + B^{[ab]} \wedge R_{[ab]}}^{\langle B\wedge\mathcal{F}\rangle} + \overbrace{e_a \wedge \nabla\beta^a}^{\langle C\wedge\mathcal{G}\rangle} + \overbrace{\phi^A (\nabla\gamma)_A + \bar\psi_A (\overset{\rightarrow}{\nabla}\gamma)^A - (\bar\gamma \overset{\leftarrow}{\nabla})_A \psi^A}^{\langle D\wedge\mathcal{H}\rangle} & 3BF \\
&-& \displaystyle\int \lambda_{[ab]} \wedge \left(B^{[ab]} - \frac{1}{16\pi l_p^2}\, \varepsilon^{[ab]cd}\, e_c \wedge e_d \right) + \frac{1}{96\pi l_p^2}\, \Lambda\, \varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d & \text{GR and CC} \\
&+& \displaystyle\int \lambda^\alpha \wedge \left(B_\alpha - 12\, C_\alpha{}^\beta M_{\beta ab}\, e^a \wedge e^b \right) + \zeta^{\alpha ab} \left(M_{\alpha ab}\, \varepsilon_{cdef}\, e^c \wedge e^d \wedge e^e \wedge e^f - F_\alpha \wedge e_a \wedge e_b \right) & \text{YM} \\
&+& \displaystyle\int \lambda^A \wedge \left(\gamma_A - H_{abcA}\, e^a \wedge e^b \wedge e^c \right) + \Lambda^{abA} \wedge \left(H_{abcA}\, \varepsilon^{cdef}\, e_d \wedge e_e \wedge e_f - (\nabla\phi)_A \wedge e_a \wedge e_b \right) & \text{Higgs} \\
&-& \displaystyle\int \frac{1}{12}\, \chi \left(\phi^A \phi_A - v^2 \right)^2 \varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d & \text{Higgs potential} \\
&+& \displaystyle\int \bar\lambda_A \wedge \left(\gamma^A + \frac{i}{6}\, \varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \left(\gamma^d \psi \right)^A \right) - \lambda^A \wedge \left(\bar\gamma_A - \frac{i}{6}\, \varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \left(\bar\psi \gamma^d \right)_A \right) & \text{Dirac} \\
&-& \displaystyle\int \frac{1}{12}\, Y_{ABC}\, \bar\psi^A \psi^B \phi^C\, \varepsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d & \text{Yukawa} \\
&+& \displaystyle\int 2\pi i\, l_p^2\, \bar\psi_A \gamma_5 \gamma^a \psi^A\, \varepsilon_{abcd}\, e^b \wedge e^c \wedge \beta^d\,. & \text{spin-torsion}
\end{array}
$$

Here the first row represents the topological $3BF$ part, while the remaining rows represent various constraint terms, each corresponding to one sector of the theory. Taking all together, the equations of motion obtained from the action S are equivalent to the full set of equations of motion for all Standard Model fields, coupled to the Einstein-Cartan theory of gravity.

The key novelty of the above structure is the role of the group L, which prescribes the spectrum of scalar and fermion fields present in the theory, via the $\langle D\wedge\mathcal{H}\rangle$ term in the topological sector of the action.

4. Conclusions

Let us summarize the results of the paper. In Section 2 we have introduced the nBF theories for $n = 1, 2, 3$, and explanied in brief terms how the categorical ladder procedure can be applied to generalize the notion of a group to the notions of a 2-group and a 3-group, which represent more powerful ways to describe the gauge symmetry of a physical theory. These structures were employed in Section 3 to construct the constrained $3BF$ action for the Standard Model of elementary particles coupled to the Einstein-Cartan gravity in the usual way. Within that framework, the spectrum of scalar and fermion fields happens to be determined by a *new gauge group*, in a way similar to that of the ordinary gauge group determining the spectrum of gauge vector bosons in Yang-Mills theory. This opens up a very interesting possibility of applying the structure of a 3-group to classify matter fields, and possibly gain some insight into why there are three families of fermions.

These results complete the first step of the spinfoam quantization programme, as outlined in the Introduction. The second step has also been performed in [25], for a general case of a Lie 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\})$. The resulting state sum is a novel topological invariant of a 4-dimensional manifold, and has the following form:

$$
\begin{aligned}
Z = {}& |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}\,|L|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|+|\Lambda_3|-|\Lambda_4|} \\
& \times \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell} \prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m} \\
& \times \prod_{(jk\ell)\in\Lambda_2} \delta_G\Big(\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk}\, g_{j\ell}^{-1}\Big) \prod_{(jk\ell m)\in\Lambda_3} \delta_H\Big(\delta(l_{jk\ell m}) h_{j\ell m}\, (g_{\ell m} \triangleright h_{jk\ell})\, h_{k\ell m}^{-1}\, h_{jkm}^{-1}\Big) \\
& \times \prod_{(jk\ell mn)\in\Lambda_4} \delta_L\Big(l_{j\ell mn}^{-1}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn} g_{\ell m)} \triangleright h_{jk\ell}\}_{\mathrm{p}}\, l_{jk\ell n}^{-1} (h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m})\Big).
\end{aligned}
\tag{12}
$$

Here $g_{ij}, h_{ijk}, l_{ijkl}$ are elements from groups G, H, L, respectively, which are assigned to simplices of the triangulation whose vertices are numerated by indices $i, j, \dots$ In other words, g_{ij} are assigned to edges, h_{ijk} are assigned to triangles, and l_{ijkl} are assigned to tetrahedra of the simplicial complex representing a compact 4-manifold, which has a total number of Λ_0 vertices, Λ_1 edges, Λ_2 triangles, Λ_3 tetrahedra, and Λ_4 4-simplices.

Of course, when building a realistic theory, we are in fact not interested in a topological theory, but instead in a theory which contains local propagating degrees of freedom. Thus the state sum Z should be appropriately deformed. This is the task of step 3 of the spinfoam quantization programme, by imposing the simplicity constraints on Z. The classical action from Section 3 manifestly distinguishes the topological sector from the simplicity constraints. Imposing those constraints should thus complete the spinfoam quantization programme, and would ultimately lead us to a tentative model of quantum gravity with matter, by providing a rigorous definition for the path integral

$$
Z = \int \mathcal{D}g \int \mathcal{D}\phi\; e^{iS[g,\phi]}\,, \tag{13}
$$

which is a generalization of (1) in the sense that it contains matter fields as well as gravity, at the quantum level.

In addition to the construction of a full quantum theory of gravity, there are also many additional possible studies of the classical constrained $3BF$ action. For example, a full Hamiltonian analysis of the $3BF$ action has been done for the example of scalar electrodynamics [26], and then also for a general choice of a Lie 3-group [27], and the complete gauge symmetry group has been discussed in detail [27, 28]. Also, it is worth looking into the idea of imposing the simplicity constraints using a spontaneous symmetry breaking mechanism, and some work has already begun in this area. Finally, one can also study in more depth the mathematical structure and properties of the simplicity constraints. The list is not conclusive, and there may be many other interesting topics to study.

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