ИНСТИТУТ ЗА ФИЗИКУ | БЕОГРАД
УНИВЕРЗИТЕТ У БЕОГРАДУ
ИНСТИТУТ ОД НАЦИОНАЛНОГ ЗНАЧАЈА ЗА РЕПУБЛИКУ СРБИЈУ
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Број 0801-166/1
Датум 06 FEB 2024

НАУЧНОМ ВЕЋУ ИНСТИТУТА ЗА ФИЗИКУ

Предмет: Молба за покретање поступка за стицање звања научни сарадник

Молим Научно веће Института за Физику у Београду да у складу са Правилником о поступку, начину вредновања и квантитативном исказивању научно-истраживачких резултата истраживача покрене поступак за мој избор у звање научни сарадник.

У прилог достављам:

1. мишљење руководиоца лабораторије са предлогом комисије за избор у звање,
2. стручну биографију,
3. преглед научне активности,
4. елементе за квалитативну и квантитативну оцену научног доприноса,
5. списак и копије објављених радова и других публикација,
6. податке о цитираности радова,
7. уверење о одбрањеној докторској дисертацији.

У Београду, 5.2.2024

Данијел Обрић

Научном већу Института за физику у Београду

Предмет: Мишљење руководиоца лабораторије о избору Данијела Обрића у звање научни сарадник

Данијел Обрић , рођен 27.11.1992. године у Бенковцу, је уписао окторске академске студије Физичког факултета Универзитета у Београду у школској 2018/2019. години. Докторску дисертацију под насловом *Т-дуализација бозонске струне и тип II суперструне у присуству координатно зависних позадинских поља* одбранио је 18.09.2023. на Физичком факултету Универзитета у Београду. Ментор докторске дисертације је др Бојана Николић, виши научни сарадник Института за физику.

Данијел Обрић је од 2018. године запослен у Групи са физику гравитације, честица и поља Института за физику где се бави проблемима везаним за Т-дуализацију (теорија струна). Данијел Обрић је до сада објавио пет научних радова категорије М21. Као што се види из приложеног материјала, он задовољава све предвиђене услове у складу са Правилником о поступку, начину вредновања и квантитативном исказивању научно-истраживачких резултата истраживача Министарства просвете, науке и технолошког развоја Владе Републике Србије, за избор у звање научни сарадник, те предлажем Научном већу Института за физику да покрене избор Данијела Обрића у поменуто звање. За чланове комисије предлажем следеће истраживаче. За избор Данијела Обрића у звање научни сарадник предлажем комисију у следећем саставу:

1. др Бојан Николић, виши научни сарадник Института за физику
2. др Бранислав Саздовић, научни саветник Института за физику у пензији
3. проф. др Воја Радовановић, редовни професор Физичког факултета Универзитета у Београду

Бранислав Цветковић

др Бранислав Цветковић
научни саветник
Руководилац Групе за гравитацију, честице и поља

Београд, 5. фебруар 2024. године

Стручна биографија

Данијел Обрић је рођен 27.11.1992. године у Бенковцу, Републици Хрватској. У Вршцу је завршио основну школу и средњу техничку сколу "Никола Тесла". Основне академске студије је започео 2011. године на Физичком факултету, Универзитета у Београду, смер Теоријска и експериментална физика. Основне студије је завршио 2016. године са просечном оценом 8.16. Исте године започиње мастер академске студије на истом факултету, смер Теоријска и експериментална физика, које завршава 2017. године са просечном оценом 10, одбранивши мастер рад на тему "Некомутативнос и неасоцијативност затворене бозонске струне". Мастер рад је урађен под руководством др Бојана Николића, вишег научног сарадника Института за физику у Београду.

Докторске академске студије уписује 2018. године на Физичком факултету, Универзитета у Београду, ужа научна област квантна поља, честице и гравитација. Научноистраживачки рад наставља на темама из теорије струна и њене интеракције са некомутативним феноменима под менторством др Бојана Николића, у оквиру групе за Гравитацију честице и поља Института за физику у Београду. Докторске студије завршава 18. септембра 2023. године, одбраниши докторску дисертацију под насловом "T-dualization of bosonic string and type IIB superstring in presence of coordinate dependent background fields" ("Т-дуализација бозонске струне и тип IIB суперструне у присуству координатно зависних позадинских поља").

Од априла 2019. године Данијел Обрић је запослен на Институту за физику у Београду као истраживач приправник у групи за Гравитацију, честице и поља, чији је руководилац др Бранислав Цветковић. Звање истраживач сарадник стекао је децембра 2022. године.

Од почетка свог радног односа са Институтом за физику у Београду, прво као истраживач приправник а затим као истраживач сарадник, Данијел Обрић је био ангажован на два домаћа пројекта финансирана од стране Министарства просвете, науке и технолошког развоја и Фонда за науку Републике Србије. Од 2019. године био је ангажован на пројекту основних истраживања "Физичке импликације модификованог просторвремена" (ОН171031) Министарства просвете, науке и технолошког развоја Републике Србије, којим је руководила проф. др Маја Бурић, професор Физичког факултета Универзитета у Београду. Од 2022. године Данијел Обрић учествује на пројекту "Quantum Gravity from Higher Gauge Theory 2021" (7745968) програма "Идеје" Фонда за науку Републике Србије чији је руководилац др Марко Војиновић, виши научни сарадник на Институту за физику у Београду.

За време докторских академских студија, Данијел Обрић је био полазник неколико школа за студенте докторских студија: CERN-SEENET-MTP Balkan School on High Energy and Particle Physics: "Theory and Phenomenology" у Јањини (Грчка) 2019. године, COST Action CA18108 "Quantum gravity phenomenology in the multi-messenger approach" у Београду 2022. године и ICTP Workshop on String Theory, Holography, and Black Holes у Трсту (Италија) 2023. године.

Тренутно, Данијел Обрић је имао пар одржаних конференција које су биле задужене за десиминацију науке: Коларац "Зашто општа теорија релативности" у Београду 2023. године, "ICTP: Преглед дешавања у модерној теоријској физици" у Крагујевцу 2023. године, "Општа теорија релативности: модерна теорија простор-времена" у Крагујевцу 2023. године и "Квантна теорија поља: модерна теорија фундаменталних сила" у Крагујевцу 2023. године.

Преглед научне активности

Досадашњи научно истраживачки рад Данијела Обрића, може се класификовати у два основна правца

- Анализа бозонске теорије струна у присуству координатно зависних позадинских поља. Примена Т-дуалности на ову теорију и добијање некомутативне дуалне теорије.

Посматрана је теорија бозонске струне где смо имали константно гравитационо поље и координатно зависно Калб-Рамондово поље. Применом Бушерове процедуре за Т-дуализацију успели смо да добијемо дуалну теорију. Процедура је била прво спроведена на координате од којих поља не зависе а затим на координату која се јављала у Калб-Рамондовом пољу. Повезивањем структуре Поисонових заграда почетне теорије са дуалном теоријом, примећено је да координате у дуалној теорији некомутирају. Обртањем редоследа примене процедуре за Т-дуализацију успели смо да покажемо да постоји мрежа некомутирајућих теорија које настају из почетне теорије.

Bojan Nikolić, **Danijel Obrić**
Noncommutativity and Nonassociativity of Closed Bosonic String on T-dual Toroidal Backgrounds
Fortschritte der Physik, 66, 1800009 (2018) ИФ= 3.263 (за 2017. годину)
DOI: https://doi.org/10.1002/prop.201800009

Bojan Nikolić, **Danijel Obrić**
Directly from H-flux to the family of three nonlocal R-flux theories
Journal of High Energy Physics, 03, 136 (2019) ИФ= 5.875 (за 2019. годину)
DOI: https://doi.org/10.1007/JHEP03(2019)136

- Анализа тип IIB суперструне у присуству координатно зависног Рамонд-Рамонд поља. Примена процедуре за Т-дуалност на бозонске и на фермионске координате и добијање некомутативне теорије.

Други део научно истраживачког рада Данијела Обрића фокусирао се на детаљну анализу тип IIB суперструне у присуству координатно зависног Рамонд-Рамонд поља. Ово поље је зависило само од бозонских координата док су остала поља која теорија дозвољава били константе. Први део ове анализе фокусирао се само на Т-дуализацију бозонских координата, где је добијена парцијално дуализована теорија. Оваква теорија је испољавала некомутативне особине између бозонских координата али и између бозонских и фермионских координата. Други део анализе фокусирао се на дуализацију фермионских координата, након чега је добијена потпуно дуализована теорија. Показано је да оваква теорија не поседује некомутативне релације између фермионских координата. Поред ове стандардне анализе, разматрали смо случај и кад Рамонд-Рамонд поље поседује и симетричан део. Овај пут је теорија дуализована само дуж бозонских координата и примећено је да је дуална теорија иста. Модификације се једино виде у структури некомутативних релација које су сад много компликованије.

Bojan Nikolić, **Danijel Obrić**, Branislav Sazdović
Noncommutativity and Nonassociativity of Type II Superstring with Coordinate Dependent RR Field
Fortschritte der Physik, 70, 2200048 (2022) ИФ= 5.532 (за 2021. годину)
DOI: https://doi.org/10.1002/prop.202200048

Bojan Nikolić, **Danijel Obrić**
Combined Fermionic and Bosonic T-duality of Type II Superstring Theory with Coordinate Dependent RR Field
Fortschritte der Physik, 71, 2200160 (2023) ИФ= 3.9 (за 2022. годину)
DOI: https://doi.org/10.1002/prop.202200160

Bojan Nikolić, **Danijel Obrić**
Noncommutativity and nonassociativity of type II superstring with coordinate dependent RR field — the general case
Journal of High Energy Physics, 12, 078 (2022) ИФ=6.376 (за 2021. годину)
DOI: https://doi.org/10.1007/JHEP12(2022)078

Елементи за квалитативну оцену научног доприноса

1 Квалитет научних резултата

1.1 Научни ниво и значај резултата, утицај научних радова

У свом досадашњем раду, др Данијел Обрић је објавио укупно пет радова, свих пет радова су категорије М21 (врхунски међународни часопис).

Најзначајни рад кандидата је

> Bojan Nikolić, **Danijel Obrić**
> Noncommutativity and Nonassociativity of Closed Bosonic String on T-dual Toroidal Backgrounds
> Fortschritte der Physik, 66, 1800009 (2018) ИФ= 3.263 (за 2017. годину)
> DOI: https://doi.org/10.1002/prop.201800009

У овом раду разматрана је затворена бозонска струна у присуству константне метрике и Калб-Рамодовог поља које има једну ненулту компоненту, $B_{xy} = Hz$, где је јачина поља H инфинитезимално мала. Користећи Бушерову Т-дуализациону процедуру, извршили смо дуализацију дуж x и y праваца а користећи уопштење Т-дуализационе процедуре дуализујемо дуж z правца, где намећемо тривијане услове за намотавање струне. После прве две Т-дуализације добијамо теорију са Q флуксом која је локално добро дефинисана. Након дуализације дуж све три правца добијамо нелокалну теорију са R флуксом. Извор нелокалности је варијабла ΔV, дефинисинана као интеграл дуж линије, која се појављује као аргумент позадинских поља. Преписивањем Т-дуалних закона трансформације у канонски облик и користећи стандардну Поасонову алгебру, добијамо да је теорија са Q флуксом комутативна док је теорија са R флуском некомутативна и неасоцијативна. Последица овога је да за затворену струну постоји повезаност између нелокалности са једне стране и некомутатицности и неасоцијативности са друге стране.

1.2 Цитираност научних радова кандидата

Укупан број цитата кандидата на дан 5. фебруара 2024. године је по WoS бази 8, односно по Scopus бази 8, од тога је број хетероцитата по WoS бази

8, односно по Scopus бази 8. Према обе базе, Хиршхов индекс кандидата је исти и износи 2. Обе базе података имају пропусте и не дају потпуну слику цитираности кандидата. Кандидат поседује рад (B. Nikolić, D. Obrić, Fortschr. Phys. 2018, 66, 1800009. https://doi.org/10.1002/prop.201800009) који је цитиран као препринт умсето објављеног чланка и доставља рад који га је цитирао (R. Szabo, PoS(CORFU2017)151, https://doi.org/10.22323/1.318.0151) као доказ.

1.3 Параметри квалитета радова и часцописа

Кандидат, др Данијел Обрић је објавио укупно пет радова у међународним часописима и то:

- 3 рада у врхунском међународном часопису Fortschritte der Physik (ИФ = 3.263, СНИП = 0.8; ИФ = 5.532, СНИП = 1.293; ИФ = 3.9, СНИП = 1.224)
- 2 рада у врхунском међународном часопису Journal of High Energy Physics (ИФ = 5.875, СНИП = 1.295; ИФ = 6.376, СНИП = 1.322)

Библиометарски показатељи су сумирани у наредној табели

	ИФ	М	СНИП
Укупно	24.946	40	5.934
Усредњено по чланку	4.9892	8	1.1868
Усредњено по аутору	11.551	18.66	2.7515

2 Нормирање броја коауторских радова, патената и техничких резултата

Од 5 радова кандидата, 4 рада имају 2 аутора а 1 рад има 3 аутора. Радови кандидата припадају класи теоријских радова у оквиру природних наука и свих пет радова се признају са пуним бројем М поена.

3 Учешће у пројектима, потпројектима и пројектним задатцима

Кандидат је учествовао на следећим пројектима:

- пројекту Министарства просвете, науке и технолошког развоја Републике Србије ОН171031 „Физичке импликације модификованог просторвремена",
- пројекту „Quantum Gravity from Higher Gauge Theory 2021" (7745968), програма „Идеје" Фонда за науку Републике Србије од јануара 2021. године, са очекиваним крајем пројекта у децембру 2024. године.

4 Утицај научних резултата

Утицај научних резултата кандидата се огледа у броју цитата који су наведени у тачки 1.2 овог прилога, као и у прилогу о цитираности. Значај резултата кандидата је такође описан у тачки 1.1.

5 Активност у научним и научно-стручним друштвима

Кандидат је по позиву рецензирао рад у часопису Kragujevac Journal of Science

6 Конкретан допринос кандидата у реализацији радова у научним центрима у земљи и иностранству

Кандидат је све своје истраживачке активности реализовао у Институту за физику у Београду. Свој допринос током истраживања дао је у рачунању, интерпретацији и презентовању резултата, писању радова и комуникацији са рецезентима.

7 Уводна предавања на конференцијама, друга предавања и активности

Током докторских студија Данијел Обрић је своје истраживање представио на конференцији *Workshop on Gravity and String Theory: "New ideas for unsolved problems III* на Златибору 2018. године, предавањем "*Noncommutativity and nonassociativity of closed bosonic string on T-dual toroidal backgrounds*". Након завршетка студија кандидат је одржао и неколико научно популарних предавања намењених широј публици. Овој категорији предавања припадају: „Зашто општа релативност" као део циклуса предавања „Квантна гравитација - Свети грал савремене физике" у Задужбини Илије М. Коларца у Београду 2023. године, „ICTP Преглед дешавања у модерној теоријској физици" на Институту за Физику у Крагујевцу 2023. године, „Општа теорија релативности - Модерна теорија простор-времена" на Институту за Физику у Крагујевцу 2023. године и "Квантна теорија поља - Модерна теорија фундаменталних сила " на Институту за Физику у Крагујевцу 2023. године.

Елементи за квантитативну оцену научног доприноса

Остварнеи резултати

Категорија	**М бодова по раду**	**Број радова**	**Укупно М бодова**	**Укупно М бодова са нормирањем**
M21	8	5	40	40
M70	6	1	6	6

Поређење са минималним квантитативним условима за избор у звање научни сарадник

Минимални број М бодова		Остварнео М бодова без нормирања	Остварено М бодова са нормирањем
Укупно	**16**	46	46
M10+M20+M31+M32+M33+M41+M42	**10**	40	40
M11+M12+M21+M22+M23	**6**	40	40

Списак објављених радова

Радови у врхунским међународним часописима (M21) :

1. Bojan Nikolić, **Danijel Obrić**
   Noncommutativity and Nonassociativity of Closed Bosonic String on T-dual Toroidal Backgrounds
   Fortschritte der Physik, 66, 1800009 (2018)
   DOI: https://doi.org/10.1002/prop.201800009

2. Bojan Nikolić, **Danijel Obrić**
   Directly from H-flux to the family of three nonlocal R-flux theories
   Journal of High Energy Physics, 03, 136 (2019)
   DOI: https://doi.org/10.1007/JHEP03(2019)136

3. Bojan Nikolić, **Danijel Obrić**, Branislav Sazdović
   Noncommutativity and Nonassociativity of Type II Superstring with Coordinate Dependent RR Field
   Fortschritte der Physik, 70, 2200048 (2022)
   DOI: https://doi.org/10.1002/prop.202200048

4. Bojan Nikolić, **Danijel Obrić**
   Noncommutativity and nonassociativity of type II superstring with coordinate dependent RR field — the general case
   Journal of High Energy Physics, 12, 078 (2022)
   DOI: https://doi.org/10.1007/JHEP12(2022)078

5. Bojan Nikolić, **Danijel Obrić**
   Combined Fermionic and Bosonic T-duality of Type II Superstring Theory with Coordinate Dependent RR Field
   Fortschritte der Physik, 71, 2200160 (2023)
   DOI: https://doi.org/10.1002/prop.202200160

ORIGINAL PAPER

Fortschritte der Physik
Progress of Physics

www.fp-journal.org

Noncommutativity and Nonassociativity of Closed Bosonic String on T-dual Toroidal Backgrounds

B. Nikolić and D. Obrić*

In this article we consider closed bosonic string in the presence of constant metric and Kalb-Ramond field with one non-zero component, $B_{xy} = Hz$, where field strength H is infinitesimal. Using Buscher T-duality procedure we dualize along x and y directions and using generalized T-duality procedure along z direction imposing trivial winding conditions. After first two T-dualizations we obtain Q flux theory which is just locally well defined, while after all three T-dualizations we obtain nonlocal R flux theory. Origin of non-locality is variable ΔV defined as line integral, which appears as an argument of the background fields. Rewriting T-dual transformation laws in the canonical form and using standard Poisson algebra, we obtained that Q flux theory is commutative one and the R flux theory is noncommutative and nonassociative one. Consequently, there is a correlation between non-locality and closed string noncommutativity and nonassociativity.

1. Introduction

Coordinate noncommutativity means that there exists minimal possible length, which imposes natural UV cutoff. Idea of coordinate noncommutativity is very old. Heisenberg suggested coordinate noncommutativity to solve the problem of the occurrence of infinite quantities before renormalization procedure was developed and accepted. The first scientific paper considering this subject appeared 1947[1] where construction of discrete Lorentz invariant space-time is presented. Later in the period of 1980s A. Connes developed noncommutative geometry as a generalization of the standard commutative geometry.[2]

Noncommutativity became again interesting for particle physicists when the paper[3] appeared. In this article it is shown using propagators that open string endpoints in the presence of the constant metric and Kalb-Ramond field become noncommutative. D-brane on which the string endpoints are forced to move becomes noncommutative manifold. After this article many articles[4] appeared addressing the same subject but using different approaches - Fourier expansion, canonical methods, solving of boundary conditions etc.

In the last two articles of[4] the method of solving of boundary conditions is presented. The basic idea is that open string boundary condition is treated as canonical constraint. Investigating the consistency of the canonical constraint we obtained the σ dependent form of the boundary condition. Further, we can proceed twofold: to introduce Dirac brackets or solve the constraint. Solving the constraint, we obtained the initial coordinate as a linear combination of the effective coordinate and momenta. Consequently, initial coordinates are noncommutative and the main contribution to noncomutativity parameter comes from Kalb-Ramond field as it was expected.

Following the result of the article[5] it can be proven that gauge fields "live" at the open string endpoints. Consequently, many interesting papers concerning non-commutative Yang-Mills theories and their renormalisability appeared.[6] In the papers[7] cross sections for some decays, allowed in noncommutative Yang-Mills theories and forbidden in commutative ones, are calculated, which offers a possibility of the experimental check of the noncommutativity idea and further, indirectly, idea of strings.

It is obvious that closed bosonic string in the presence of constant background fields remains commutative. There are no boundaries and, consequently, boundary conditions constraining string dynamics. In the case of open string we obtained initial coordinate in the form of linear combination of effective coordinates and momenta using boundary condition. That is achieved in the closed string case[8] using T-duality procedure and coordinate dependent background.

T-duality as a fundamental feature of string theory,[9–15] unexperienced by point particle, makes that there is no physical difference between string theory compactified on a circle of radius R and circle of radius $1/R$. Buscher T-dualization procedure[10] represents a mathematical frame in which T-dualization is realized. If the background fields do not depend on some coordinates then those coordinates are isometry directions. Consequently, that symmetry can be localized replacing ordinary world-sheet derivatives $\partial_\pm$ by covariant ones $D_\pm x^\mu = \partial_\pm x^\mu + v_\pm^\mu$, where $v_\pm^\mu$ are gauge fields. In order to make T-dual theory has the same number of degrees of freedom, the new term with Lagrange multipliers is added to the action which forces the gauge fields to be unphysical degrees of freedom. Because of the shift symmetry, using gauge freedom we fix initial coordinates. Variation of this gauge fixed action with respect to the Lagrange multipliers

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DOI: 10.1002/prop.201800009

produces initial action and with respect to the gauge fields produces T-dual action.

Standard Buscher T-dualization was applied in closed string case in the papers.[8,16–19] In Ref. [16] authors consider 3-torus in the presence of constant metric and Kalb-Ramond field with one nonzero component $B_{xy} = Hz$, where field strength H is infinitesimal. They systematically apply Buscher procedure and, after two T-dualizations along isometry directions, obtain theory with Q flux which is noncommutative. In the calculations they used nontrivial boundary conditions (winding conditions). The result is that T-dual closed string coordinates are noncommutative for the same values of parameters $\sigma = \bar{\sigma}$ with noncommutativity parameter proportional to field strength H and N_3, winding number for z coordinate.

But, except this standard Buscher procedure, there is a generalized Buscher procedure dealing with background fields depending on all coordinates. The generalized procedure was applied to the case of bosonic string moving in the weakly curved background[20–22] and in the case where metric is quadratic in coordinates and Kalb-Ramond field is linear function of coordinates.[23] The generalized procedure enables us to make T-dualization in mentioned cases along arbitrary subset of coordinates.

Double space is one picturesque framework for representation of T-duality. Double space is introduced two to three decades ago.[24–28] It is spanned by double coordinates $Z^M = (x^\mu, y_\mu)$ $(\mu = 0, 1, 2, \ldots, D-1)$, where x^μ are the coordinates of the initial theory and y_μ are T-dual coordinates. In this space T-dualization is represented as $O(d, d)$ transformation.[29–33] Permutation of the appropriate subsets of the initial and T-dual coordinates is interpreted as partial T-dualization[34,35] expanding Duff's idea.[24] The newly invented intrinsic noncommutativity[36] is related to double space. Intrinsic noncommutativity exists in the constant background case because it is considered within double space framework.

In this article we will deal with closed bosonic string propagating in the constant metric and linear dependent Kalb-Ramond field with $B_{xy} = Hz$, the same background as in [16]. This configuration is known in literature as torus with H-flux. As in the Ref. [16] we will use approximation of diluted flux, which means that in all calculations we keep constant and linear terms in infinitesimal field strength H. Transformation laws, relations which connect initial and T-dual variables, we will write in canonical form expressing initial momenta in terms of the T-dual coordinates. Unlike Ref. [16], except T-dualization along two isometry directions, we will make one step more and T-dualize along z coordinate using generalized T-dualization procedure. During dualization procedure we will use trivial boundary (winding) conditions.

Transformation laws in canonical form enable us to express sigma derivative of the T-dual coordinate as a linear combination of the initial momenta and coordinates. Because initial theory is geometrical locally and globally, its coordinates and canonically conjugated momenta satisfy standard Poisson algebra. This fact means that we can calculate the Poisson brackets of the T-dual coordinates using technical instruction given in subsection 4.1.

After T-dualizations along isometry directions (along x and y) we obtain the same background as in Ref. [16] but, obtained Q flux theory, which is still locally well defined, is commutative. This is a consequence of the imposed trivial winding conditions. Having in mind the generalized T-duality procedure,[20,21,23] T-dualization along z coordinate produces R flux nonlocal theory because it depends on the variable ΔV which is defined as line integral. Calculating Poisson brackets of the T-dual coordinates we obtain two nonzero Poisson brackets and show that there is a correlation between non-locality and closed string noncommutativity.

The form of noncommutativity is such that it exists when arguments of the coordinates are different, $\sigma \neq \bar{\sigma}$. That is another difference with respect to the result of Ref. [16] but there is no contradiction because the origins of noncommutativity are different. In this article non-locality is related with noncommutativity of R flux theory under trivial winding conditions while in Ref. [16] it is about noncommutativity of Q flux theory under nontrivial winding conditions.

From the noncommutativity relations it follows that Jacobi identity is broken i.e. nonassociativity occurs. Nonassociativity parameter, R flux, is proportional to the field strength H. Using generalized T-duality[20,21,23] we obtain the concrete form of nonassociativity from string dynamics. Similar as noncommutativity, discovery of nonassociativity pushes the scientists to explore the effects of nonassociativity in the field of renormalisability of ϕ^4 theory[37] as well as formulation of nonassociative gravity.[38]

At the end we add an appendix containing some conventions used in the paper.

2. Bosonic String Action and Choice of Background Fields

The action of the closed bosonic string in the presence of the space-time metric $G_{\mu\nu}(x)$, Kalb-Ramond antisymmetric field $B_{\mu\nu}(x)$, and dilaton scalar field $\Phi(x)$ is given by the following expression[9]

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \times \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\}, \tag{2.1}$$

where Σ is the world-sheet surface parameterized by $\xi^\alpha = (\tau, \sigma)$ $[(\alpha = 0, 1),\ \sigma \in (0, \pi)]$, while the D-dimensional space-time is spanned by the coordinates x^μ $(\mu = 0, 1, 2, \ldots, D-1)$. We denote intrinsic world sheet metric with $g_{\alpha\beta}$, and the corresponding scalar curvature with $R^{(2)}$.

In order to keep conformal symmetry on the quantum level background fields must obey space-time field equations[39]

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2 D_\mu a_\nu = 0, \tag{2.2}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2 a_\rho B^\rho{}_{\mu\nu} = 0, \tag{2.3}$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - R - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = c, \tag{2.4}$$

where c is an arbitrary constant. The function β^Φ could be a constant because of the relation

$$D^\nu \beta^G_{\nu\mu} + \partial_\mu \beta^\Phi = 0. \tag{2.5}$$

Further, $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to the space-time metric $G_{\mu\nu}$, while

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \quad a_\mu = \partial_\mu \Phi, \tag{2.6}$$

are field strength for Kalb-Ramond field $B_{\mu\nu}$ and dilaton gradient, respectively. Trivial solution of these equations is that all three background fields are constant. This case was pretty exploited in the analysis of the open string noncommutativity.

The less trivial case would be a case where some background fields are coordinate dependent. If we choose Kalb-Ramond field to be linearly coordinate dependent and dilaton field to be constant then the first equation (2.2) becomes

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0. \tag{2.7}$$

The field strength $B_{\mu\nu\rho}$ is constant and, if we assume that it is infinitesimal, then we can take $G_{\mu\nu}$ to be constant in approximation linear in $B_{\mu\nu\rho}$. Consequently, all three space-time field equations are satisfied. Especially, the third one is of the form

$$2\pi\kappa \frac{D-26}{6} = c, \tag{2.8}$$

which enables us to work in arbitrary number of space-time dimensions.

In this article we will work in $D = 3$ dimensions with the following choice of background fields

$$G_{\mu\nu} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.9}$$

where $R_\mu (\mu = 1, 2, 3)$ are radii of the compact dimensions. This choice of background fields is known in geometry as torus with flux (field strength) H.[16] Our choice of infinitesimal H can be understood in terms of the radii as that

$$\left(\frac{H}{R_1 R_2 R_3}\right)^2 = 0. \tag{2.10}$$

This approximation is known in literature as the approximation of diluted flux. Physically, this means that we work with the torus which is sufficiently large. Consequently, we can rescale the coordinates

$$x^\mu \longmapsto \frac{x^\mu}{R_\mu}, \tag{2.11}$$

which simplifies the form of the metric

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

The final form of the closed bosonic string action is

$$\begin{aligned} S &= \kappa \int_\Sigma d^2\xi\, \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \\ &= \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}(\partial_+ x \partial_- x + \partial_+ y \partial_- y + \partial_+ z \partial_- z) \right. \\ &\quad \left. + \partial_+ x\, Hz \partial_- y - \partial_+ y\, Hz \partial_- x\right], \end{aligned} \tag{2.13}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ is world-sheet derivative with respect to the light-cone coordinates $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$ and

$$x^\mu = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2.14}$$

Let us note that we do not write dilaton term because its T-dualization is performed separately within quantum formalism and here will be skipped.

3. T-dualization of the Bosonic Closed String Action

In this section we will perform T-dualization along three directions, one direction at time. Our goal is to find the relations connecting initial variables with T-dual ones called transformation laws. Using transformation laws we will find noncommutativity and nonassociativity relations.

3.1. T-dualization Along x Direction – from Torus with H Flux to the Twisted Torus

Let us perform standard Buscher T-dualization[10] of action (2.13) along x direction. Note that x direction is an isometry direction which means that action has a global shift symmetry, $x \longrightarrow x + a$. In order to perform Buscher procedure, we have to localize this symmetry introducing covariant world-sheet derivatives instead of the ordinary ones

$$\partial_\pm x \longrightarrow D_\pm x = \partial_\pm x + v_\pm, \tag{3.1}$$

where $v_\pm$ are gauge fields which transform as $\delta v_\pm = -\partial_\pm a$. Because T-dual action must have the same number of degrees of freedom as initial one, we have to make these fields $v_\pm$ be unphysical degrees of freedom. This is accomplished by adding following term to the action

$$S_{add} = \frac{\kappa}{2} \int_\Sigma d^2\xi\, y_1 (\partial_+ v_- - \partial_- v_+), \tag{3.2}$$

where y_1 is a Lagrange multiplier. After gauge fixing, $x = const.$, the action gets the form

$$\begin{aligned} S_{fix} = \kappa \int d^2\xi &\left[\frac{1}{2}(v_+ v_- + \partial_+ y \partial_- y + \partial_+ z \partial_- z) + v_+ Hz \partial_- y \right. \\ &\left. - \partial_+ y\, Hz v_- + \frac{1}{2} y_1 (\partial_+ v_- - \partial_- v_+)\right]. \end{aligned} \tag{3.3}$$

From the equations of motion for y_1 we obtain that field strength for the gauge field $v_\pm$ is equal to zero

$$F_{+-} = \partial_+ v_- - \partial_- v_+ = 0, \tag{3.4}$$

which gives us the solution for gauge field

$$v_\pm = \partial_\pm x. \tag{3.5}$$

Inserting this solution for gauge field into gauge fixed action (3.3) we obtain initial action given by Eq. (2.13). Equations of motion for $v_\pm$ will lead to the T-dual action. Varying the gauge fixed action (3.3) with respect to the gauge field v_+ we get

$$v_- = -\partial_- y_1 - 2Hz\partial_- y, \tag{3.6}$$

while on the equation of motion for v_- it holds

$$v_+ = \partial_+ y_1 + 2Hz\partial_+ y. \tag{3.7}$$

Inserting relations (3.6) and (3.7) into expression for gauge fixed action (3.3), keeping terms linear in H, we obtain the T-dual action

$$_xS = \kappa \int_\Sigma d^2\xi\, \partial_+ ({}_xX)^\mu\, {}_x\Pi_{+\mu\nu} \partial_- ({}_xX)^\nu, \tag{3.8}$$

where subscript $_x$ denotes quantity obtained after T-dualization along x direction and

$$_xX^\mu = \begin{pmatrix} y_1 \\ y \\ z \end{pmatrix}. \tag{3.9}$$

Further we have the T-dual background fields

$$_x\Pi_{+\mu\nu} = {}_xB_{\mu\nu} + \frac{1}{2} {}_xG_{\mu\nu}, \quad {}_xB_{\mu\nu} = 0,$$
$$_xG_{\mu\nu} = \begin{pmatrix} 1 & 2Hz & 0 \\ 2Hz & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.10}$$

Obtained background fields (3.10) define that what is known in literature as *twisted torus geometry*. String theory after one T-dualization is geometrically well defined globally and locally or, simply, theory is geometrical (flux H takes the role of connection).

Combining the solutions of equations of motion for Lagrange multiplier (3.5) and for gauge fields, (3.6) and (3.7), we get the transformation laws connecting initial, x^μ, and T-dual, $_xX^\mu$, coordinates

$$\partial_\pm x \cong \pm\partial_\pm y_1 \pm 2Hz\partial_\pm y, \tag{3.11}$$

where $\cong$ denotes T-duality relation. The momentum π_x is canonically conjugated to the initial coordinate x. Using the initial action (2.13) we get

$$\pi_x = \frac{\delta S}{\delta \dot{x}} = \kappa(\dot{x} - 2Hzy'), \tag{3.12}$$

where $\dot{A} \equiv \partial_\tau A$ and $A' \equiv \partial_\sigma A$. From transformation law (3.11) it is straightforward to obtain

$$\dot{x} \cong y_1' + 2Hzy', \tag{3.13}$$

which, inserted in the expression for momentum π_x, gives transformation law in canonical form

$$\pi_x \cong \kappa y_1'. \tag{3.14}$$

3.2. From Twisted Torus to Non-geometrical Q Flux

In this subsection we will continue the T-dualization of action (3.8) along y direction. After x and y T-dualization we obtain the structure which has local geometrical interpretation but global omissions. Such structure is known in literature as non-geometry.

We repeat the procedure from the previous subsection and form the gauge fixed action

$$S_{fix} = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}(\partial_+ y_1 \partial_- y_1 + v_+ v_- + \partial_+ z \partial_- z) + \partial_+ y_1 Hz v_- \right.$$
$$\left. + v_+ Hz \partial_- y_1 + \frac{1}{2} y_2 (\partial_+ v_- - \partial_- v_+) \right]. \tag{3.15}$$

From the equation of motion for Lagrange multiplier y_2

$$\partial_+ v_- - \partial_- v_+ = 0 \longrightarrow v_\pm = \partial_\pm y, \tag{3.16}$$

gauge fixed action becomes initial one (3.8). Varying the gauge fixed action (3.15) with respect to the gauge fields we get

$$v_\pm = \pm\partial_\pm y_2 - 2Hz\partial_\pm y_1. \tag{3.17}$$

Inserting these expressions for gauge fields into gauge fixed action, keeping the terms linear in H, gauge fixed action is driven into T-dual action

$$_{xy}S = \kappa \int d^2\xi\, \partial_+ ({}_{xy}X)^\mu\, {}_{xy}\Pi_{+\mu\nu} \partial_- ({}_{xy}X)^\nu, \tag{3.18}$$

where

$$({}_{xy}X)^\mu = \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix},$$
$$_{xy}\Pi_{+\mu\nu} = {}_{xy}B_{\mu\nu} + \frac{1}{2} {}_{xy}G_{\mu\nu} = \begin{pmatrix} \frac{1}{2} & -Hz & 0 \\ Hz & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{3.19}$$

Explicit expressions for background fields are

$$_{xy}B_{\mu\nu} = \begin{pmatrix} 0 & -Hz & 0 \\ Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -B_{\mu\nu}, \quad {}_{xy}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.20}$$

Let us note that background fields obtained after two T-dualizations are similar to the geometric background of torus with H flux, but they should be considered only locally. Their global properties are non-trivial and because of that the term "non-geometry" is introduced.

Combining the equations of motion for Lagrange multiplier y_2 and for gauge fields $v_\pm$, we obtain T-dual transformation laws

$$\partial_\pm y \cong \pm\partial_\pm y_2 - 2Hz\partial_\pm y_1. \tag{3.21}$$

The y component of the initial canonical momentum π_y is a variation of the initial action with respect to the $\dot{y}$

$$\pi_y = \frac{\delta S}{\delta \dot{y}} = \kappa(\dot{y} + 2Hzx'). \tag{3.22}$$

Using T-dual transformation laws (3.21) we easily get

$$\dot{y} \cong y_2' - 2Hz\dot{y}_1, \tag{3.23}$$

while from the transformation law (3.11), at zeroth order in H, it holds $x' \cong \dot{y}_1$. Inserting last two expression into π_y we obtain transformation law in canonical form

$$\pi_y \cong \kappa y_2'. \tag{3.24}$$

After two T-dualizations along isometry directions, in the approximation of the diluted flux (keeping just terms linear in H), according to the canonical forms of the transformation laws (3.14) and (3.24), we see that T-dual coordinates y_1 and y_2 are still commutative. This is a consequence of the simple fact that variables of the initial theory, which is geometrical one, satisfy standard Poisson algebra

$$\left\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\right\} = \delta^\mu{}_\nu\delta(\sigma-\bar\sigma), \quad \{x^\mu, x^\nu\} = \left\{\pi_\mu, \pi_\nu\right\} = 0, \tag{3.25}$$

where

$$\pi_\mu = \begin{pmatrix} \pi_x \\ \pi_y \\ \pi_z \end{pmatrix}. \tag{3.26}$$

3.3. From *Q* to *R* Flux – T-dualization Along *z* Coordinate

In this subsection we will finalize the process of T-dualization dualizing along remaining z direction. For this purpose we will use generalized T-dualization procedure.[20,21,23] The result is a theory which is not well defined even locally and is known in literature as theory with R-flux.

We start with the action obtained after T-dualizations along x and y directions (3.18). The Kalb-Ramond field (3.20) depends on z and it seems that it is not possible to perform T-dualization. Let us assume that Kalb-Ramond field linearly depends on all coordinates, $B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho$ and check if some global transformation can be treated as isometry one. We start with global shift transformation

$$\delta x^\mu = \lambda^\mu, \tag{3.27}$$

and make a variation of action

$$\begin{aligned}\delta S &= \frac{\kappa}{3}B_{\mu\nu\rho}\lambda^\rho\int_\Sigma d^2\xi\,\partial_+x^\mu\partial_-x^\nu \\ &= \frac{2k}{3}B_{\mu\nu\rho}\lambda^\rho\epsilon^{\alpha\beta}\int_\Sigma d^2\xi[\partial_\alpha(x^\mu\partial_\beta x^\nu) - x^\mu(\partial_\alpha\partial_\beta x^\nu)].\end{aligned} \tag{3.28}$$

The second term vanishes explicitly, while the first term is surface one. Consequently, in the case of constant metric and linearly dependent Kalb-Ramond field, global shift transformation is an isometry transformation. This means that we can make T-dualization along z coordinate using generalized T-dualization procedure.

The generalized T-dualization procedure is presented in detail in Ref. [20]. In order to localize shift symmetry of the action (3.18) along z direction we introduce covariant derivative

$$\partial_\pm z \longrightarrow D_\pm z = \partial_\pm z + v_\pm, \tag{3.29}$$

which is a part of the standard Buscher procedure. The novelty is introduction of the invariant coordinate as line integral

$$\begin{aligned}z^{inv} &= \int_P d\xi^\alpha D_\alpha z \\ &= \int_P d\xi^+ D_+z + \int_P d\xi^- D_-z = z(\xi) - z(\xi_0) + \Delta V,\end{aligned} \tag{3.30}$$

where

$$\Delta V = \int_P d\xi^\alpha v_\alpha = \int_P (d\xi^+ v_+ + d\xi^- v_-). \tag{3.31}$$

Here ξ and ξ_0 are the current and initial point of the world-sheet line P. At the end, as in the standard Buscher procedure, in order to make $v_\pm$ to be unphysical degrees of freedom we add to the action term with Lagrange multiplier

$$S_{add} = \frac{\kappa}{2}\int_\Sigma d^2\xi\; y_3(\partial_+v_- - \partial_+v_-). \tag{3.32}$$

The final form of the action is

$$\begin{aligned}\bar{S} = \kappa\int_\Sigma d^2\xi\Big[&-Hz^{inv}(\partial_+y_1\partial_-y_2 - \partial_+y_2\partial_-y_1) \\ &+\frac{1}{2}(\partial_+y_1\partial_-y_1 + \partial_+y_2\partial_-y_2 + D_+zD_-z) \\ &+\frac{1}{2}\,y_3(\partial_+v_- - \partial_-v_+)\Big].\end{aligned} \tag{3.33}$$

Because of existing shift symmetry we fix the gauge, $z(\xi) = z(\xi_0)$, and then the gauge fixed action takes the form

$$S_{fix} = \kappa \int_\Sigma d^2\xi \Big[- H\Delta V(\partial_+ y_1 \partial_- y_2 - \partial_+ y_2 \partial_- y_1) + \frac{1}{2}(\partial_+ y_1 \partial_- y_1 + \partial_+ y_2 \partial_- y_2 + v_+ v_-) + \frac{1}{2} y_3 (\partial_+ v_- - \partial_- v_+)\Big]. \tag{3.34}$$

From the equation of motion for Lagrange multiplier y_3 we obtain

$$\partial_+ v_- - \partial_- v_+ = 0 \Longrightarrow v_\pm = \partial_\pm z, \quad \Delta V = \Delta z, \tag{3.35}$$

which drives back the gauge fixed action to the initial action (3.18). Varying the gauge fixed action (3.34) with respect to the gauge fields $v_\pm$ we get the following equations of motion

$$v_\pm = \pm \partial_\pm y_3 - 2\beta^\mp, \tag{3.36}$$

where $\beta^\pm$ functions are defined as

$$\beta^\pm = \pm \frac{1}{2} H(y_1 \partial_\mp y_2 - y_2 \partial_\mp y_1). \tag{3.37}$$

The $\beta^\pm$ functions are obtained as a result of the variation of the term containing ΔV

$$\delta_v \left(-2\kappa \int d^2\xi \varepsilon^{\alpha\beta} H \partial_\alpha y_1 \partial_\beta y_2 \Delta V \right) = \kappa \int d^2\xi \left(\beta^+ \delta v_+ + \beta^- \delta v_- \right), \tag{3.38}$$

using partial integration and the fact that $\partial_\pm V = v_\pm$. Inserting the relations (3.36) into the gauge fixed action, keeping linear terms in H, we obtain the T-dual action

$$_{xyz}S = \kappa \int_\Sigma d^2\xi \partial_{+\,xyz} X^\mu {}_{xyz}\Pi_{+\mu\nu} \partial_{-\,xyz} X^\nu, \tag{3.39}$$

where

$$_{xyz}X^\mu = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad {}_{xyz}\Pi_{+\mu\nu} = {}_{xyz}B_{\mu\nu} + \frac{1}{2} {}_{xyz}G_{\mu\nu}, \tag{3.40}$$

$$_{xyz}B_{\mu\nu} = \begin{pmatrix} 0 & -H\Delta\tilde{y}_3 & 0 \\ H\Delta\tilde{y}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad {}_{xyz}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.41}$$

Here we introduced double coordinate $\tilde{y}_3$ defined as

$$\partial_\pm y_3 \equiv \pm \partial_\pm \tilde{y}_3. \tag{3.42}$$

Let us note that ΔV stands beside field strength H, which implicates that, according to the diluted flux approximation, we calculate ΔV in the zeroth order in H

$$\Delta V = \int d\xi^+ \partial_+ y_3 - \int d\xi^- \partial_- y_3. \tag{3.43}$$

Having this into account it is clear why we defined double coordinate $\tilde{y}_3$ as in Eq. (3.42). Also it is useful to note that presence of ΔV, which is defined as line integral, represents the source of non-locality of the T-dual theory. the result of the three T-dualization is a theory with R flux as it is known in the literature.

Combining the equations of motion for Lagrange multiplier (3.35), $v_\pm = \partial_\pm z$, and equations of motion for gauge fields (3.36), we obtain the T-dual transformation law

$$\partial_\pm z \cong \pm \partial_\pm y_3 - 2\beta^\mp. \tag{3.44}$$

Adding transformation laws for $\partial_\pm z$ and $\partial_- z$ we get the transformation law for $\dot{z}$

$$\dot{z} \cong y_3' + H(y_1 y_2' - y_2 y_1'), \tag{3.45}$$

which enables us to write down the transformation law in the canonical form

$$y_3' \cong \frac{1}{\kappa}\pi_z - H(x y' - y x'). \tag{3.46}$$

Here we used the expression for the canonical momentum of the initial theory (2.13)

$$\pi_z = \frac{\delta S}{\delta \dot{z}} = \kappa \dot{z}. \tag{3.47}$$

4. Noncommutativity and Nonassociativity Using T-duality

In the open string case noncommutativity comes from the boundary conditions which makes that coordinates x^μ depend both on the effective coordinates and on the effective momenta.[4] Effective coordinates and momenta do not commute and, consequently, coordinates x^μ do not commute. In the closed bosonic string case the logic is the same but the execution is different. Using T-duality we obtained transformation laws, (3.11), (3.21) and (3.44), which relate T-dual coordinates with the initial coordinates and their canonically conjugated momenta. In this section we will use these relations to get noncommutativity and nonassociativity relations.

4.1. Noncommutativity Relations

Let us start with the Poisson bracket of the σ derivatives of two arbitrary coordinates in the form

$$\{A'(\sigma), B'(\bar{\sigma})\} = U'(\sigma)\delta(\sigma - \bar{\sigma}) + V(\sigma)\delta'(\sigma - \bar{\sigma}), \tag{4.1}$$

where $\delta'(\sigma-\bar\sigma)\equiv\partial_\sigma\delta(\sigma-\bar\sigma)$. In order to find the form of the Poisson bracket

$$\{A(\sigma),\, B(\bar\sigma)\},$$

we have to find the form of the Poisson bracket

$$\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\},$$

where

$$\Delta A(\sigma,\sigma_0)=\int_{\sigma_0}^{\sigma}dx A'(x)=A(\sigma)-A(\sigma_0),$$
$$\Delta B(\bar\sigma,\bar\sigma_0)=\int_{\bar\sigma_0}^{\bar\sigma}dx B'(x)=B(\bar\sigma)-B(\bar\sigma_0). \tag{4.2}$$

Now we have

$$\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\} = \int_{\sigma_0}^{\sigma}dx\int_{\bar\sigma_0}^{\bar\sigma}dy\,\left[U'(x)\delta(x-y)+V(x)\delta'(x-y)\right]. \tag{4.3}$$

After integration over y we get

$$\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=\int_{\sigma_0}^{\sigma}dx\left\{U'(x)\left[\theta(x-\bar\sigma_0)-\theta(x-\bar\sigma)\right]\right. \left.+\,V(x)\left[\delta(x-\bar\sigma_0)-\delta(x-\bar\sigma)\right]\right\}, \tag{4.4}$$

where function $\theta(x)$ is defined as

$$\theta(x)=\int_0^x d\eta\delta(\eta)=\frac{1}{2\pi}\left[x+2\sum_{n\geq1}\frac{1}{n}\sin(nx)\right] =\begin{cases}0 & \text{if } x=0\\ 1/2 & \text{if } 0<x<2\pi.\\ 1 & \text{if } x=2\pi\end{cases} \tag{4.5}$$

Integrating over x using partial integration finally we obtain

$$\begin{aligned}\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}&=U(\sigma)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma-\bar\sigma)]\\ &-U(\sigma_0)[\theta(\sigma_0-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma)]-U(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0)]\\ &+U(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)]+V(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0]\\ &-V(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)].\end{aligned} \tag{4.6}$$

From the last expression, using the right-hand sides of the expressions in Eq. (4.2), we extract the desired Poisson bracket

$$\{A(\sigma),\, B(\bar\sigma)\}=-[U(\sigma)-U(\bar\sigma)+V(\bar\sigma)]\theta(\sigma-\bar\sigma). \tag{4.7}$$

Let us rewrite the canonical forms of the transformation laws, (3.14), (3.24) and (3.46), in the following way

$$y_1'\cong\frac{1}{\kappa}\pi_x,\quad y_2'\cong\frac{1}{\kappa}\pi_y,\quad y_3'\cong\frac{1}{\kappa}\pi_z-H(xy'-yx'). \tag{4.8}$$

In order to find the Poisson brackets between T-dual coordinates y_μ we will use the algebra of the coordinates and momenta of the initial theory (3.25). It is obvious that only nontrivial Poisson brackets will be $\{y_1(\sigma),\, y_3(\bar\sigma)\}$ and $\{y_2(\sigma),\, y_3(\bar\sigma)\}$.

Let us first write the corresponding Poisson brackets of the sigma derivatives of T-dual coordinates y_μ using (4.8)

$$\{y_1'(\sigma),\, y_3'(\bar\sigma)\}\cong\frac{2}{\kappa}Hy'(\sigma)\delta(\sigma-\bar\sigma)+\frac{1}{\kappa}Hy(\sigma)\delta'(\sigma-\bar\sigma), \tag{4.9}$$

$$\{y_2'(\sigma),\, y_3'(\bar\sigma)\}\cong-\frac{2}{\kappa}Hx'(\sigma)\delta(\sigma-\bar\sigma)-\frac{1}{\kappa}Hx(\sigma)\delta'(\sigma-\bar\sigma), \tag{4.10}$$

while all other Poisson brackets are zero. We see that these Poisson brackets are of the form (4.1), so, we can apply the result (4.7). Consequently, we get

$$\{y_1(\sigma),\, y_3(\bar\sigma)\}\cong-\frac{H}{\kappa}\left[2y(\sigma)-y(\bar\sigma)\right]\theta(\sigma-\bar\sigma), \tag{4.11}$$

$$\{y_2(\sigma),\, y_3(\bar\sigma)\}\cong\frac{H}{\kappa}\left[2x(\sigma)-x(\bar\sigma)\right]\theta(\sigma-\bar\sigma), \tag{4.12}$$

where function $\theta(x)$ is defined in (4.5). Let us note that these two Poisson brackets are zero when $\sigma=\bar\sigma$ and/or field strength H is equal to zero. But if we take that $\sigma-\bar\sigma=2\pi$ then we have $\theta(2\pi)=1$ and it follows

$$\{y_1(\sigma+2\pi),\, y_3(\sigma)\}\cong-\frac{H}{\kappa}\left[4\pi N_y+y(\sigma)\right], \tag{4.13}$$

$$\{y_2(\sigma+2\pi),\, y_3(\sigma)\}\cong\frac{H}{\kappa}\left[4\pi N_x+x(\sigma)\right], \tag{4.14}$$

where N_x and N_y are winding numbers defined as

$$x(\sigma+2\pi)-x(\sigma)=2\pi N_x,\qquad y(\sigma+2\pi)-y(\sigma)=2\pi N_y. \tag{4.15}$$

From these relations we can see that if we choose such σ for which $x(\sigma)=0$ and $y(\sigma)=0$ then noncommutativity relations are proportional to winding numbers. On the other side, for winding numbers which are equal to zero there is still noncommutativity between T-dual coordinates.

4.2. Nonassociativity

In order to calculate Jacobi identity of the T-dual coordinates we first have to find Poisson brackets $\{y_1(\sigma),\, x(\bar\sigma)\}$ as well as $\{y_2(\sigma),\, y(\bar\sigma)\}$. We start with

$$\{\Delta y_1(\sigma,\sigma_0),\, x(\bar\sigma)\}=\left\{\int_{\sigma_0}^{\sigma}d\eta y_1'(\eta),\, x(\bar\sigma)\right\}, \tag{4.16}$$

and then use the T-dual transformation for x-direction in canonical form

$$\pi_x\cong\kappa y_1'. \tag{4.17}$$

From these two equations it follows

$$\{\Delta y_1(\sigma,\sigma_0), x(\bar\sigma)\} \cong \frac{1}{\kappa}\left\{\int_{\sigma_0}^{\sigma} d\eta \pi_x(\eta), x(\bar\sigma)\right\}, \tag{4.18}$$

which, using the standard Poisson algebra, produces

$$\{\Delta y_1(\sigma,\sigma_0), x(\bar\sigma)\} \cong -\frac{1}{\kappa}\left[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)\right]$$

$$\Longrightarrow \{y_1(\sigma), x(\bar\sigma)\} \cong -\frac{1}{\kappa}\theta(\sigma-\bar\sigma). \tag{4.19}$$

The relation $\{y_2(\sigma), y(\bar\sigma)\}$ can be obtained in the same way. Because the transformation law for y-direction is of the same form as for x-direction, the Poisson bracket is of the same form

$$\{y_2(\sigma), y(\bar\sigma)\} \cong -\frac{1}{\kappa}\theta(\sigma-\bar\sigma). \tag{4.20}$$

Now we can calculate Jacobi identity using noncommutativity relations (4.11) and (4.12) and above two Poisson brackets

$$\{y_1(\sigma_1), y_2(\sigma_2), y_3(\sigma_3)\} \equiv \{y_1(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\}$$
$$+\{y_2(\sigma_2), \{y_3(\sigma_3), y_1(\sigma_1)\}\} + \{y_3(\sigma_3), \{y_1(\sigma_1), y_2(\sigma_2)\}\}$$
$$\cong -\frac{2H}{\kappa^2}\big[\theta(\sigma_1-\sigma_2)\theta(\sigma_2-\sigma_3)$$
$$+\theta(\sigma_2-\sigma_1)\theta(\sigma_1-\sigma_3)+\theta(\sigma_1-\sigma_3)\theta(\sigma_3-\sigma_2)\big]. \tag{4.21}$$

Jacobi identity is nonzero which means that theory with R-flux is nonassociative. For $\sigma_2=\sigma_3=\sigma$ and $\sigma_1=\sigma+2\pi$ we get

$$\{y_1(\sigma+2\pi), y_2(\sigma), y_3(\sigma)\} \cong \frac{2H}{\kappa^2}. \tag{4.22}$$

From the last two equations, general form of Jacobi identity and Jacobi identity for special choice of σ's, we see that presence of the coordinate dependent Kalb-Ramond field is a source of noncommutativity and nonassociativity.

5. Conclusion

In this article we have considered the closed bosonic string propagating in the three-dimensional constant metric and Kalb-Ramond field with just one nonzero component $B_{xy}=Hz$. This choice of background is in accordance with consistency conditions in the sense that all calculations were made in approximation linear in Kalb-Ramond field strength H. Geometrically, this settings corresponds to the torus with H flux. Then we performed standard Buscher T-dualization procedure along isometry directions, first along x and then along y direction. At the end we performed generalized T-dualization procedure along z direction and obtained nonlocal theory with R flux. Using the relations between initial and T-dual variables, called T-dual transformation laws, in canonical form we find the noncommutativity and nonassociativity relations between T-dual coordinates.

After T-dualization along x direction we obtained theory embedded in geometry known in literature as twisted torus geometry. The relation between initial and T-dual variables is trivial, $\pi_x \cong \kappa y_1'$, where π_x is x component of the canonical momentum of the initial theory and y_1 is coordinate T-dual to x. Consequently, flux H takes a role of connection, obtained theory is globally and locally well defined and commutative, because the coordinates and their canonically conjugated momenta satisfy the standard Poisson algebra (3.25).

The second T-dualization, along y direction, produces nongeometrical theory, in literature known as Q flux theory. The metric is the same as initial one and Kalb-Ramond field have the same form as initial up to minus sign. But, this theory has just local geometrical interpretation. We obtained that, in approximation linear in H, the transformation law in canonical form is again trivial, $\pi_y \cong \kappa y_2'$, where π_y is y component od the canonical momentum of the initial theory and y_2 is coordinate T-dual to y. As a consequence of the standard Poisson algebra (3.25), we conclude that Q flux theory is still commutative. This result seems to be opposite from the result of the reference [16] where in detailed calculation it is shown that Q flux theory is noncommutative. The difference is in the so called boundary condition i.e. winding condition. In the Ref. [16] they imposed nontrivial winding condition which mixes the coordinates and their T-dual partners (condition given in Eq. (C.18) of Ref. [16]) and the result is noncommutativity. In this article the trivial winding condition is imposed on x and y coordinates. The consequence is that Q flux theory is commutative. But as it is written in Ref. [16] on page 42, "a priori other reasonings could as well be pursued".

T-dualizing along coordinate z using the machinery of the generalized T-dualization procedure[20,21,23] we obtain the nonlocal theory (theory with R flux) and nontrivial transformation law in canonical form. Non-locality stems from the fact that background fields are expressed in terms of the variable ΔV which is defined as line integral. On the other side, dependence of the Kalb-Ramond field on z coordinate produces the $\beta^{\pm}(x,y)$ functions and nontrivial transformation law for π_z. Consequently, coordinate dependent background gives non-locality and, further, nonzero Poisson brackets of the T-dual coordinates. We can claim that there is a correlation between non-locality (R-flux theory) and closed string noncommutativity and nonassociativity. In addition, nonzero Poisson bracket implies nonzero Jacobi identity which is a signal of nonassociativity.

From the expressions (4.11), (4.12) and (4.21) it follows that parameters of noncommutativity and nonassociativity are proportional to the field strength H. That means that closed string noncommuatativity and nonassociativity are consequence of the fact that Kalb-Ramond field is coordinate dependent, $B_{xy}=Hz$, where H is an infinitesimal parameter according to the approximation of diluted flux. Using T-duality and trivial winding conditions we obtained noncommutativity relations. The noncommutativity relations are zero if $\sigma=\bar\sigma$ because in noncommuatativity relations function $\theta(\sigma-\bar\sigma)$ is present, which is zero if its argument is zero. This is also at the first glance opposite to the result of Ref. [16], but, having in mind that origin of noncommutativity is not same, this difference is not surprising. If we made a round in sigma choosing $\sigma\to\sigma+2\pi$ and $\bar\sigma\to\sigma$, because of $\theta(2\pi)=1$, we obtained nonzero Poisson brackets. From the relations (4.13) and (4.14) we see that noncommutativity exists even in the case when winding numbers are zero, noncommutativity relations still stand unlike the result in [16]. Consequently, we can speak about some essential noncommutativity originating from non-locality.

We showed that in *ordinary* space coordinate dependent background is a sufficient condition for closed string noncommutativity. Some papers[36] show that noncommutativity is possible even in the constant background case. But that could be realized using the *double space formalism*. At the zeroth order the explanation follows from the fact that transformation law in canonical form is of the form $\pi_\mu \cong \kappa y'_\mu$, where y_μ is T-dual coordinate. Forming double space spanned by $Z^M = (x^\mu, y_\mu)$, we obtained noncommuative (double) space. In literature this kind of noncommutativity is called intrinsic one.

Appendix: Light-Cone Coordinates

In the paper we often use light-cone coordinates defined as

$$\xi^{\pm} = \frac{1}{2}(\tau \pm \sigma). \tag{A.1}$$

The corresponding partial derivatives are

$$\partial_{\pm} \equiv \frac{\partial}{\partial \xi^{\pm}} = \partial_\tau \pm \partial_\sigma. \tag{A.2}$$

Two dimensional Levi-Civita $\varepsilon^{\alpha\beta}$ is chosen in (τ, σ) basis as $\varepsilon^{\tau\sigma} = -1$. Consequently, in the light-cone basis the form of tensor is

$$\varepsilon_{lc} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{A.3}$$

The flat world-sheet metric is of the form in (τ, σ) and light-cone basis, respectively

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta_{lc} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{A.4}$$

Acknowledgements

Work supported in part by the Serbian Ministry of Education, Science and Technological Development, under contract No. 171031. I also want to thank to Prof. Dr. Branislav Sazdović and Dr. Ljubica Davidović from Institute of Physics Belgrade for useful discussions.

Conflict of Interest

The authors have declared no conflict of interest.

Keywords

closed string, nonassociativity, noncommutativity, non-locality

Received: January 29, 2018

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Combined Fermionic and Bosonic T-duality of Type II Superstring Theory with Coordinate Dependent RR Field

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We investigate effects of fermionic T-duality on type II superstring in presence of Ramond-Ramond (RR) field that has infinitesimal linear dependence on bosonic coordinate x^μ. Other fields are assumed to be constant. Procedure that we employ for obtaining fermionic T-dual theory is Buscher procedure, where we will consider two distinct cases. One, where action has not been T-dualized along bosonic coordinates and other where it has. By analyzing these two cases, their actions and T-dual transformation laws, we obtain some insight into how background fields transform and what are necessary ingredients for emergence of fermionic non-commutativity.

1. Introduction

T-duality represents a map that connects different superstring theories, mapping geometry and topology from one theory to another.[1] This symmetry was originally developed with bosonic coordinates in mind, where two theories are connected by transformation laws that establish a link between coordinates.[2] It was not until 2008 that it has been noticed that the same duality can emerge in case of fermionic coordinates. In their paper[3] Berkovits and Maldacena showed that tree level superstring theories in presence of supersymmetric background fields posses new kind of symmetry. Symmetry that maps supersymmetric background fields of one theory to supersymmetric backgrounds of other theory, where dilaton and RR fields are now different. Just like in case of bosonic T-duality, mathematical machinery for obtaining T-dual theories is Buscher procedure[4,5] applied to fermionic coordinates θ^α and $\bar{\theta}^\alpha$.

Busher T-dualization procedure including its extension to fermionic coordinates[6–9] and its generalizations[10–12] mainly follow the same steps. We notice some global symmetry in the theory, usually shift symmetry, which is then localized by replacing partial derivatives with covariant ones. Covariant derivatives come with new gauge fields that insert new degrees of freedom into the theory, these degrees of freedom are eliminated with help of Lagrange multipliers. Next step is utilizing gauge freedom to fix starting coordinates. After that, finding equations of motion for gauge fields and inserting their solutions into the action we obtain T-dual theory. Extension of procedure for fermionic coordinates does not introduce any new steps into the play. However, extending the procedure to include coordinate dependent background fields does introduce one additional step. Namely, we need to replace all coordinates with invariant ones constructed as integrals of covariant derivatives. This step is necessary in order to preserve local shift symmetry.

In our previous paper[13] we demonstrated that T-dual of type II superstring which is moving in coordinate dependent RR field possesses non-commutative Poisson brackets. Since T-duality was performed only along bosonic coordinates it produced non-commutativity only between bosonic T-dual coordinates. In addition to this, we had that background fields that were constants in original theory became functions of both bosonic and fermionic coordinates in dual theory. This has left us with one open question: Would fermionic T-duality of starting theory or even theory that has been dualized along bosonic coordinates produce non-commutative relations between fermionic coordinates? While it has been shown that, in case of closed bosonic string, non-commutativity arises only in coordinates that had appeared in background fields of starting theory[14], it is not clear if that is the case for fermionic coordinates, especially when we have emergence of new coordinate dependence in background fields after bosonic T-duality.

In this article, our goal is to find what effects fermionic T-duality has on action where RR field has dependence on bosonic coordinates and do these effects change for fully dualized action. By obtaining background fields in different stages of T-duality we can determine how geometry of theory changes and when the theory makes the switch from being local to non-local one. At the end we provide few notes on how fermionic T-duality interacts with bosonic T-duality in providing new non-commutative relations.

2. Type II Superstring, Choice of Fields and Bosonic T-Duality

In this section we will present action for type II superstring in pure spinor formulation. We will also define background fields in which string propagates. Finally, we present action that has been T-dualized along bosonic coordinates.

2.1. Type II Superstring in Pure Spinor Formulation

The most general form of type II superstring action in pure spinor formalism[15–19] is given as

$$S = S_0 + V_{SG}. \qquad (2.1)$$

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DOI: 10.1002/prop.202200160

First term is action for string that propagates in flat background fields.

$$S_0 = \int_\Sigma d^2\xi \left(\frac{k}{2}\eta_{\mu\nu}\partial_m x^\mu \partial_n x^\nu \eta^{mn} - \pi_\alpha \partial_- \theta^\alpha + \partial_+ \bar{\theta}^\alpha \bar{\pi}_\alpha \right) + S_\lambda + S_{\bar{\lambda}}, \tag{2.2}$$

where terms S_λ and $S_{\bar{\lambda}}$represent actions that are composed of pure spinors and their canonical momenta. The pure spinors satisfy pure spinor constraints

$$\lambda^\alpha (\Gamma^\mu)_{\alpha\beta}\lambda^\beta = \bar{\lambda}(\Gamma^\mu)_{\alpha\beta}\bar{\lambda}^\beta = 0. \tag{2.3}$$

All modifications to flat background fields are accomplished by introducing second term in equation (2.1). This term is an integrated vertex operator for massless type II supergravity

$$V_{SG} = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar{X}^N. \tag{2.4}$$

In general case matrix A_{MN} is composed of physical fields, their curvatures (field strengths) and auxiliary fields that can be expressed with physical ones. These fields are some functions of both bosonic coordinates x^μ and fermionic coordinates θ^α and $\bar{\theta}^\alpha$. Dependence of fields on fermionic coordinates is given as expansion in powers of θ^α and $\bar{\theta}^\alpha$. In our particular case, we will set all background fields except RR field to be constant. Further more, in order to simplify calculations, all terms that are nonlinear in fermionic coordinates θ^α and $\bar{\theta}^\alpha$ will be neglected. With these assumptions in mind we have that vectors X^M and $\bar{X}^M$ and matrix A_{MN} have following form

$$X^M = \begin{pmatrix} \partial_+\theta^\alpha \\ \partial_+ x^\mu \\ \pi_\alpha \\ \frac{1}{2}N_+^{\mu\nu} \end{pmatrix}, \ \bar{X}^M = \begin{pmatrix} \partial_-\bar{\theta}^\lambda \\ \partial_- x^\mu \\ \bar{\pi}_\lambda \\ \frac{1}{2}\bar{N}_-^{\mu\nu} \end{pmatrix},$$

$$A_{MN} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k(\frac{1}{2}g_{\mu\nu} + B_{\mu\nu}) & \bar{\Psi}_\mu^\beta & 0 \\ 0 & -\Psi_\nu^\alpha & \frac{2}{k}(f^{\alpha\beta} + C_\rho^{\alpha\beta}x^\rho) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.5}$$

Pure spinor contribution to vectors X^M and $\bar{X}^M$ are encoded in

$$N_+^{\mu\nu} = \frac{1}{2}\omega_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \lambda^\beta, \quad \bar{N}_-^{\mu\nu} = \frac{1}{2}\bar{\omega}_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \bar{\lambda}^\beta. \tag{2.6}$$

Since this term does not contribute to the vertex operator, we have that pure spinor actions are decoupled from the rest. This allows us to neglect pure spinor parts from now on.

Our choice of matrix A_{MN} is composed of following fields: symmetric tensor $g_{\mu\nu}$, Kalb-Ramon antisymmetric tensor $B_{\mu\nu}$, Mayorana-Weyl gravitino fields Ψ_μ^α and $\bar{\Psi}_\mu^\alpha$, Ramond-Ramond field $\frac{2}{k}(f^{\alpha\beta} + C_\rho^{\alpha\beta}x^\rho)$ where $f^{\alpha\beta}$ and $C_\rho^{\alpha\beta}$ are constant tensors. We have also assumed that dilaton field Φ is constant. This means that factor e^Φ is included in constants $f^{\alpha\beta}$ and $C_\rho^{\alpha\beta}$. This choice of background fields is accompanied with following condition

$$\gamma_{\alpha\beta}^\mu C_\mu^{\beta\gamma} = 0, \quad \gamma_{\alpha\beta}^\mu C_\mu^{\gamma\beta} = 0. \tag{2.7}$$

String propagates in superspace spanned by bosonic coordinates x^μ ($\mu = 0, 1, ..., 9$) and fermionic ones θ^α, $\bar{\theta}^\alpha$ with 16 independent real components each. Fermionic coordinates are accompanied by their canonically conjugated momenta π_α and $\bar{\pi}_\alpha$. Both fermionic coordiantes and their momenta are given as Majorana-Wejl spinors. World sheet Σ that string sweeps in this superspace is parameterized by ξ^m ($\xi^0 = \tau$, $\xi^1 = \sigma$). By combining these parameters we can define light-cone parametrization $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$ and light-cone partial derivatives $\partial_\pm = \partial_\tau \pm \partial_\sigma$.

Inserting all these assumptions into action (2.1) and integrating out fermionic momenta, we are left with following expression

$$S = k\int_\Sigma d^2\xi \Big[\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu + \frac{1}{2}(\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu \bar{\Psi}_\mu^\alpha)\left(F^{-1}(x)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi_\nu^\beta \partial_- x^\nu)\Big], \tag{2.8}$$

where we have introduced following tensors

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}, \tag{2.9}$$

$$F^{\alpha\beta}(x) = f^{\alpha\beta} + C_\mu^{\alpha\beta}x^\mu,$$

$$(F^{-1}(x))_{\alpha\beta} = (f^{-1})_{\alpha\beta} - (f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}x^\rho (f^{-1})_{\beta_1\beta}. \tag{2.10}$$

To obtain meaningful T-dual transformation laws we need to assume that x^μ dependent part of tensor $(F^{-1}(x))_{\alpha\beta}$ is antisymmetric and infinitesimal. This additional assumption does not infringe on constraint (2.7).,[15]

Having obtained one of relevant actions , we will now focus on bosonic T-dualization of (2.8) to obtain our second action of interest.

2.2. Bosonic T-dualization

Bosonic T-dualization of action (2.8) is given in detail in [13]. Here we will only summarize the most important results.

One way to obtain T-duality is by Buscher procedure. This procedure is based on localization of translation symmetry. When we localize symmetry we replace all partial derivatives with covariant ones, while in cases where background fields depend on coordinates we also need to introduce invariant coordinate. Invariant coordinate is non-local addition to action and it is the sole reason for emergence of non-commutative behavior in closed strings. Introduction of covariant derivatives and invariant coordinates produces additional gauge fields in action, which in turn add new degrees of freedom to the theory. T-dual and original theory represent same physical system and we expect that those two theories carry exact same degrees of freedom. Because of this, we remove all newly introduced degrees of freedom with Lagrange multipliers. By utilizing gauge freedom of action we can fix bosonic coordinates to be some constant, in essence removing them from action. This gauge fixed action is only a function of gauge fields and Lagrange multipliers. Finding equation of motion for Lagrange multipliers and inserting them into action we can restore original action. On the other hand, finding equations of motion for gauge fields and inserting them into action we obtain T-dual action.

Action (2.8), due to antisymmetric part of tensor $F^{-1}_{\alpha\beta}(x)$ is invariant under global translations of bosonic coordinates. Following steps of Busher procedure, described in preceding paragraph, we obtain following T-dual action

$$
\begin{aligned}
{}^bS = \frac{k}{2}\int_\Sigma d^2\xi \Big[&\frac{1}{2}\bar{\Theta}^{\mu\nu}_-\partial_+ y_\mu \partial_- y_\nu + \partial_+\bar{\theta}^\alpha \left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta \\
&+ \partial_+ y_\mu {}^b\bar{\Psi}^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta \\
&+ \partial_+\bar{\theta}^\alpha\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}{}^b\Psi^{\nu\beta}\partial_- y_\nu \Big].
\end{aligned} \tag{2.11}
$$

Here, y_μ is a dual coordinate, left superscript ^{b} denotes bosonic T-duality and V^0 represents following integral

$$
\begin{aligned}
\Delta V^{(0)\rho} = &\frac{1}{2}\int_P d\xi^+ \breve{\Theta}^{\rho_1\rho}_-\left[\partial_+ y_{\rho_1} - \partial_+\bar{\theta}^\alpha (f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] \\
&- \frac{1}{2}\int_P d\xi^- \breve{\Theta}^{\rho\rho_1}_-\left[\partial_- y_{\rho_1} + \bar{\Psi}^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right].
\end{aligned} \tag{2.12}
$$

T-dual tensors that appear in action have following interpretation: $\bar{\Theta}^{\mu\nu}_-$ is inverse tensor of $\bar{\Pi}_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar{\Psi}^\alpha_\mu (F^{-1}(x))_{\alpha\beta}\Psi^\beta_\nu = \breve{\Pi}_{+\mu\nu} - \frac{1}{2}\bar{\Psi}^\alpha_\mu (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\rho x^\rho (f^{-1})_{\beta_1\beta}\Psi^\beta_\nu$, defined as

$$
\bar{\Theta}^{\mu\nu}_-\bar{\Pi}_{+\nu\rho} = \delta^\mu_\rho, \tag{2.13}
$$

where

$$
\bar{\Theta}^{\mu\nu}_- = \breve{\Theta}^{\mu\nu}_- + \frac{1}{2}\breve{\Theta}^{\mu\mu_1}_-\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho V^{(0)\rho}(f^{-1})_{\beta_1\beta}\Psi^{\beta_1}_{\nu_1}\breve{\Theta}^{\nu_1\nu}_-, \tag{2.14}
$$

$$
\breve{\Theta}^{\mu\nu}_-\breve{\Pi}_{+\nu\rho} = \delta^\mu_\rho, \qquad \breve{\Theta}^{\mu\nu}_- = \Theta^{\mu\nu}_- - \frac{1}{2}\Theta^{\mu\mu_1}_-\bar{\Psi}^\alpha_{\mu_1}(\bar{f}^{-1})_{\alpha\beta}\Psi^\beta_{\nu_1}\Theta^{\nu_1\nu}_- \tag{2.15}
$$

$$
\bar{f}^{\alpha\beta} = f^{\alpha\beta} + \frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_-\bar{\Psi}^\beta_\nu, \tag{2.16}
$$

$$
\Theta^{\mu\nu}_-\Pi_{+\mu\rho} = \delta^\mu_\rho, \qquad \Theta_- = -4(G_E^{-1}\Pi_- G^{-1})^{\mu\nu}. \tag{2.17}
$$

Tensor $({}^bF^{-1}\$(V^{(0)})\$)_{\alpha\beta}$ is T-dual to $(F^{-1}(x))_{\alpha\beta}$,

$$
\begin{aligned}
\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta} = &\left(F^{-1}(V^{(0)})\right)_{\alpha\beta} - \frac{1}{2}\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar{\Theta}^{\mu\nu}_-\bar{\Psi}^{\beta_1}_\nu \\
&\times\left(F^{-1}(V^{(0)})\right)_{\beta_1\beta}.
\end{aligned} \tag{2.18}
$$

Finally, ${}^b\bar{\Psi}^{\mu\alpha}$ and ${}^b\Psi^{\nu\beta}$ are T-dual gravitino fields, given as

$$
\begin{aligned}
{}^b\bar{\Psi}^{\mu\alpha} &= \frac{1}{2}\bar{\Theta}^{\mu\nu}_-\bar{\Psi}^\alpha_\nu + \frac{1}{4}\bar{\Theta}^{\mu\mu_1}_-\bar{\Psi}^\beta_{\mu_1}\left(F^{-1}(V^{(0)})\right)_{\beta\beta_1}\Psi^{\beta_1}_\nu\Theta^{\nu\nu_1}_-\bar{\Psi}^\alpha_{\nu_1} \\
&= \frac{1}{2}\Theta^{\mu\nu}_-\bar{\Psi}^\alpha_\mu,
\end{aligned} \tag{2.19}
$$

$$
\begin{aligned}
{}^b\Psi^{\nu\beta} &= -\frac{1}{2}\Psi^\beta_\mu\bar{\Theta}^{\mu\nu}_- - \frac{1}{2}\Psi^\beta_\mu\Theta^{\mu\mu_1}_-\bar{\Psi}^\alpha_{\mu_1}\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi^{\alpha_1}_{\nu_1}\bar{\Theta}^{\nu_1\nu}_- \\
&= -\frac{1}{2}\Psi^\beta_\mu\Theta^{\mu\nu}_-.
\end{aligned} \tag{2.20}
$$

Having obtained actions (2.8) and (2.11), we can now consider dualization along fermionic coordinates.

3. Fermionic T-duality

In this section the objectives are to find fermionic T-dual transformation laws and actions that have been T-dualized along fermionic coordinates for case where we performed bosonic T-duality and case where we have not.

Bosonic T-duality relies on utilization of symmetries of action to produce T-dual action and T-dual transformation laws. This task is usually accomplished by utilizing Busher procedure.[4,5,10–12] The main idea of fermionic T-duality is essentially the same, we utilize isometries of fermionic coordinates to generate T-dual action and T-dual transformation laws.[3,6–8] Just like in bosonic case, we localize translational symmetry by introducing covariant derivatives and, in cases where necessary, invariant coordinates. After this we introduce term that eliminates additional degrees of freedom and gauge fix existing symmetry. From this point on, finding equations of motion for gauge fields and inserting those equations of motion into gauge fixed action we obtain T-dual action.

Before proceeding with fermionic variant of Buscher procedure, we can notice that our actions (2.8) and (2.11) do not posses terms proportional to $\partial_+\theta^\alpha$ and $\partial_-\bar{\theta}^\alpha$. This means that our fermionic coordinates have following local symmetry

$$
\delta\theta^\alpha = \epsilon^\alpha(\sigma^+), \quad \delta\bar{\theta}^\alpha = \bar{\epsilon}^\alpha(\sigma^-), \quad (\sigma^\pm = \tau \pm \sigma). \tag{3.1}
$$

We need to fix this symmetry before obtaining T-dual theory, one way to do this is through BRST formalism. This symmetry has following corresponding BRST transformations for fermionic fields

$$
s\theta^\alpha = c^\alpha(\sigma^+), \qquad s\bar{\theta}^\alpha = \bar{c}^\alpha(\sigma^-). \tag{3.2}
$$

Here s is BRST nilpotent operator, c^α and $\bar{c}^\alpha$ represent ghost fields that correspond to gauge parameters ϵ^α and $\bar{\epsilon}^\alpha$ respectively. In addition to ghost fields we also have following BRST transformations

$$
sC_\alpha = b_{+\alpha}, \quad s\bar{C}_\alpha = \bar{b}_{-\alpha}, \quad sb_{+\alpha} = 0, \quad s\bar{b}_{-\alpha} = 0. \tag{3.3}
$$

where $\bar{C}_\alpha$ and C_α are anti-ghosts, $b_{+\alpha}$ and $\bar{b}_{-\alpha}$ are Nakanishi-Lautrup auxiliary fields.

Fixing of gauge symmetry is accomplished by introduction of gauge fermion, where we have decided to follow in the same choice as[9]

$$
\Psi = \frac{k}{2}\int_\Sigma d^2\xi\left[\bar{C}_\alpha\left(\partial_+\theta^\alpha + \frac{1}{2}\alpha^{\alpha\beta}b_{+\beta}\right) + \left(\partial_-\bar{\theta}^\alpha + \frac{1}{2}\bar{b}_{-\beta}\alpha^{\beta\alpha}\right)C_\alpha\right], \tag{3.4}
$$

here $\alpha^{\alpha\beta}$ is arbitrary invertible matrix.

Applying BRST transformation to gauge fermion we obtain gauge fixed action and Fadeev-Popov action

$$
S_{gf} = \frac{k}{2}\int_\Sigma d^2\xi\left[\bar{b}_{-\alpha}\partial_+\theta^\alpha + \partial_-\bar{\theta}^\alpha b_{+\alpha} + \bar{b}_\alpha\alpha^{\alpha\beta}b_{+\beta}\right], \tag{3.5}
$$

$$
S_{F-P} = \frac{k}{2}\int_\Sigma d^2\xi\left[\bar{C}_\alpha\partial_+ c^\alpha + (\partial_-\bar{c}^\alpha)C_\alpha\right]. \tag{3.6}
$$

Fadeev-Popov term contains only ghosts and anti-ghosts and it is decoupled from the actions (2.8) and (2.11). From this point on, this term will be ignored. Gauge fixing term contains auxiliary fields $\bar{b}_{-\alpha}$ and $b_{+\alpha}$ that can be removed with equations of motion

$$\bar{b}_{-\alpha} = -\partial_-\bar{\theta}^\beta(\alpha^{-1})_{\beta\alpha}, \qquad b_{+\alpha} = -(\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta, \tag{3.7}$$

giving us

$$S_{gf} = -\frac{k}{2}\int_\Sigma d^2\xi \partial_-\bar{\theta}^\alpha(\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta. \tag{3.8}$$

Inserting guage fixing term into (2.8) and (2.11) gives us actions that can be dualized with Buscher procedure.

3.1. Type II superstring - Fermionic T-Duality

Since both action (2.8) and gauge fixing term (3.8) are trivially invariant to global translations of fermionic coordinates, we localize this translational symmetry by replacing partial derivatives with covariant ones

$$\partial_\pm\theta^\alpha \to D_\pm\theta^\alpha = \partial_\pm\theta^\alpha + u^\alpha_\pm, \tag{3.9}$$
$$\partial_\pm\bar{\theta}^\alpha \to D_\pm\bar{\theta}^\alpha = \partial_\pm\bar{\theta}^\alpha + \bar{u}^\alpha_\pm. \tag{3.10}$$

New gauge fields $u^\alpha_\pm$ and $\bar{u}^\alpha_\pm$ introduce new degrees of freedom that are removed by addition of term

$$S_{add} = \frac{k}{2}\int_\Sigma d^2\xi\left[\bar{z}_\alpha(\partial_+u^\alpha_- - \partial_-u^\alpha_+) + (\partial_+\bar{u}^\alpha_- - \partial_-\bar{u}^\alpha_+)z_\alpha\right]. \tag{3.11}$$

Gauge freedom can be utilized to fix fermionic coordinates such that $\theta^\alpha = \theta^\alpha_0 = const$ and $\bar{\theta}^\alpha = \bar{\theta}^\alpha_0 = const$. This in turn reduces our covariant derivatives to

$$D_\pm\theta^\alpha \to u^\alpha_\pm, \qquad D_\pm\bar{\theta}^\alpha \to \bar{u}^\alpha_\pm. \tag{3.12}$$

With all this in mind, we have following action

$$\begin{aligned}S_{gf} = k\int_\Sigma d^2\xi\Big[&\Pi_{+\mu\nu}\partial_+x^\mu\partial_-x^\nu + \frac{1}{2}(\bar{u}^\alpha_+ + \partial_+x^\mu\bar{\Psi}^\alpha_\mu)\left(F^{-1}(x)\right)_{\alpha\beta}\\ &\times(u^\beta_- + \Psi^\beta_\nu\partial_-x^\nu) - \frac{1}{2}\bar{u}^\alpha_-(\alpha^{-1})_{\alpha\beta}u^\beta_+ + \frac{1}{2}\bar{z}_\alpha(\partial_+u^\alpha_- - \partial_-u^\alpha_+)\\ &+\frac{1}{2}(\partial_+\bar{u}^\alpha_- - \partial_-\bar{u}^\alpha_+)z_\alpha\Big].\end{aligned} \tag{3.13}$$

On one side we have equations of motion for Lagrange multipliers $\bar{z}_\alpha$ and z_α

$$\partial_+u^\alpha_- - \partial_-u^\alpha_+ = 0 \quad\to\quad u^\alpha_\pm = \partial_\pm\theta^\alpha, \tag{3.14}$$
$$\partial_+\bar{u}^\alpha_- - \partial_-\bar{u}^\alpha_+ = 0 \quad\to\quad \bar{u}^\alpha_\pm = \partial_\pm\bar{\theta}^\alpha. \tag{3.15}$$

Inserting solutions for these equations into action (3.13) we obtain starting action plus gauge fixing term. Variation of action with respect to gauge fields produces following set of equations of motion

$$u^\alpha_- = -\left(F^{\alpha\beta}(x)\partial_-z_\beta + \Psi^\alpha_\mu\partial_-x^\mu\right), \tag{3.16}$$

$$u^\alpha_+ = -\alpha^{\alpha\beta}\partial_+z_\beta, \tag{3.17}$$

$$\bar{u}^\alpha_+ = \partial_+\bar{z}_\beta F^{\beta\alpha}(x) - \partial_+x^\mu\bar{\Psi}^\alpha_\mu, \tag{3.18}$$

$$\bar{u}^\alpha_- = \partial_-\bar{z}_\beta\alpha^{\beta\alpha}. \tag{3.19}$$

Utilizing these equations we can remove gauge fields from action, resulting in action that depends only on Lagrange multipliers and bosonic coordinates

$$\begin{aligned}{}^fS = k\int_\Sigma d^2\xi\Big[&\Pi_{+\mu\nu}\partial_+x^\mu\partial_-x^\nu + \frac{1}{2}\partial_+\bar{z}_\alpha\Psi^\alpha_\mu\partial_-x^\mu\\ &+\frac{1}{2}\partial_+\bar{z}_\alpha F^{\alpha\beta}(x)\partial_-z_\beta - \frac{1}{2}\partial_+x^\mu\bar{\Psi}^\alpha_\mu\partial_-z_\alpha - \frac{1}{2}\partial_-\bar{z}_\alpha\alpha^{\alpha\beta}\partial_+z_\beta\Big].\end{aligned} \tag{3.20}$$

Just like in the bosonic case, we have that left superscript ^{f} denotes fermionic T-duality. From here we can deduce background fields of feriomionic T-dual theory

$${}^f\bar{\Pi}_{+\mu\nu} = \Pi_{+\mu\nu}, \tag{3.21}$$

$${}^f\left(F^{-1}(x)\right)^{\alpha\beta} = F^{\alpha\beta}(x), \tag{3.22}$$

$${}^f\bar{\Psi}_{\mu\beta}\,{}^f\left(F^{-1}(x)\right)^{\beta\alpha} = -\bar{\Psi}^\alpha_\mu \quad\to\quad {}^f\bar{\Psi}_{\mu\beta} = -\bar{\Psi}^\alpha_\mu\left(F^{-1}(x)\right)_{\alpha\beta}, \tag{3.23}$$

$${}^f\left(F^{-1}(x)\right)^{\alpha\beta}\,{}^f\Psi_{\mu\beta} = \Psi^\alpha_\mu \quad\to\quad {}^f\Psi_{\mu\beta} = \left(F^{-1}(x)\right)_{\beta\alpha}\Psi^\alpha_\mu. \tag{3.24}$$

Unlike bosonic case, fermionic T-dual theory is local. This can be attributed to the fact that background fields do not depend on fermionic coordinates. This in turn means that theory is geometric and we should not expect emergence of non-commutative phenomena.

3.2. Type II Superstring - Full T-Duality

To obtain fully dualized theory we start with action that is already T-dualized along bosonic coordinates (2.11). Procedure for fermionic T-duality is mostly the same as described before. The only difference comes from the fact that bosonic T-duality introduced non-local term V^0 which depends on θ^α and $\bar{\theta}^\alpha$ and now we need to introduce invariant fermionic coordinates in order for action to exhibit to local shift symmetry

$$D_\pm\theta^\alpha = \partial_\pm\theta^\alpha + u^\alpha_\pm, \tag{3.25}$$
$$D_\pm\bar{\theta}^\alpha = \partial_\pm\bar{\theta}^\alpha + \bar{u}^\alpha_\pm, \tag{3.26}$$

$$\theta^\alpha_{inv} = \int_P d\xi^m D_m\theta^\alpha = \int_P d\xi^m(\partial_m\theta^\alpha + u^\alpha_m) = \Delta\theta^\alpha + \Delta U^\alpha, \tag{3.27}$$

$$\bar{\theta}^\alpha_{inv} = \int_P d\xi^m D_m\bar{\theta}^\alpha = \int_P d\xi^m(\partial_m\bar{\theta}^\alpha + \bar{u}^\alpha_m) = \Delta\bar{\theta}^\alpha + \Delta\bar{U}^\alpha. \tag{3.28}$$

Fixing gauge symmetry as before, setting fermionic coordinates to constants, we deduce following relations

$$D_\pm\theta^\alpha \to u^\alpha_\pm, \qquad D_\pm\bar\theta^\alpha \to \bar u^\alpha_\pm, \qquad \theta^\alpha_{inv} \to \Delta U^\alpha, \qquad \bar\theta^\alpha_{inv} \to \Delta \bar U^\alpha. \tag{3.29}$$

With these relations we obtain action that is only a function of gauge fields, lagrange multipliers and dual coordinates

$$\begin{aligned} {}^bS_{gf} = \frac{k}{2}\int_\Sigma d^2\xi \Big[&\frac{1}{2}\bar\Theta^{\mu\nu}_-\partial_+\gamma_\mu\partial_-\gamma_\nu + \bar u^\alpha_+ \left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta} u^\beta_- \\ &+\partial_+\gamma_\mu {}^b\bar\Psi^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}u^\beta_- + \bar u^\alpha_+\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}{}^b\Psi^{\nu\beta}\partial_-\gamma_\nu \\ &-\bar u^\alpha_-(\alpha^{-1})_{\alpha\beta}u^\beta_+ + \bar z_\alpha(\partial_+u^\alpha_- - \partial_-u^\alpha_+) + (\partial_+\bar u^\alpha_- - \partial_-\bar u^\alpha_+)z_\alpha\Big]. \end{aligned} \tag{3.30}$$

In order to simplify calculations we introduce the following two substitutions

$$\begin{aligned} &\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}{}^b\Psi^{\nu\beta}\partial_-\gamma_\nu + \partial_-z_\alpha = Z_{-\alpha}, \\ &\partial_+\gamma_\mu {}^b\bar\Psi^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta} - \partial_+\bar z_\beta = \bar Z_{+\beta}. \end{aligned} \tag{3.31}$$

Now, our action can be expressed as

$$\begin{aligned} {}^bS_{gf} = \frac{k}{2}\int_\Sigma d^2\xi\Big[&\frac{1}{2}\overline{\Theta}^{\mu\nu}_-\partial_+\gamma_\mu\partial_-\gamma_\nu + \bar u^\alpha_+\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}u^\beta_- + \bar Z_{+\beta}u^\beta_- \\ &+\bar u^+_\alpha Z_{-\alpha} - \bar u^-_\alpha(\alpha^{-1})_{\alpha\beta}u^+_\beta + \partial_-\bar z_\alpha u^+_\alpha - \bar u^-_\alpha\partial_+z_\alpha\Big]. \end{aligned} \tag{3.32}$$

Similar to the first case, we can always revert to starting action by finding equations of motion for Lagrange multipliers and inserting their solutions into the action. In both cases equations of motion are the same so we take the freedom to omit them here.

Equations of motion for gauge fields differ in this case. Since we have that $V^{(0)}$ depends on fermionic coordinates, equations of motion have additional term that depends on invariant coordinate.

$$u^\alpha_+ = -(\alpha)^{\alpha\beta}\partial_+z_\beta, \qquad \bar u^\beta_- = \partial_-\bar z_\alpha(\alpha)^{\alpha\beta}, \tag{3.33}$$

$$\bar u^\alpha_+ = -\bar Z_{+\beta}\,{}^bF^{\beta\alpha}(V^{(0)}) - \beta^-_\nu(V^{(0)}, U^{(0)})\,{}^b\bar\Psi^{\nu\alpha}, \tag{3.34}$$

$$u^\beta_- = -{}^bF^{\beta\alpha}(V^{(0)})Z_{-\alpha} - \beta^+_\mu(V^{(0)}, U^{(0)})\,{}^b\Psi^{\mu\beta}. \tag{3.35}$$

The beta functions, $\beta^\pm_\mu(V^{(0)}, U^{(0)})$, are obtained by varying $V^{(0)}$ (see[13] for more details). They are given as

$$\begin{aligned} &\beta^\pm_\mu(V^{(0)}, U^{(0)}) \\ &\quad = \mp\frac{1}{8}\partial_\mp\left[\bar U^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left[U^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right] \\ &\quad\quad \pm\frac{1}{8}\left[\bar U^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_\mp\left[U^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right]. \end{aligned} \tag{3.36}$$

Inserting equations of motion for gauge fields into action (3.32) and keeping only terms linear with respect to $C^{\alpha\beta}_\mu$, we obtain fully dualized action

$$\begin{aligned} {}^{bf}S = \frac{k}{2}\int_\Sigma d^2\xi \\ \times\left[\frac{1}{2}\bar\Theta^{\mu\nu}_-\partial_+\gamma_\mu\partial_-\gamma_\nu - \bar Z_{+\alpha}{}^bF^{\alpha\beta}(V^{(0)})Z_{-\beta} - \partial_-\bar z_\alpha(\alpha)^{\alpha\beta}\partial_+z_\beta\right]. \end{aligned} \tag{3.37}$$

Expanded, we have

$$\begin{aligned} {}^{bf}S = k\int_\Sigma d^2\xi\Big[&\frac{1}{4}\Theta^{\mu\nu}_-\partial_+\gamma_\mu\partial_-\gamma_\nu - \frac{1}{4}\partial_+\gamma_\mu\Theta^{\mu\nu}_-\bar\Psi^\alpha_\nu\partial_-z_\alpha \\ &-\frac{1}{4}\partial_+\bar z_\alpha\Psi^\alpha_\mu\Theta^{\mu\nu}_-\partial_-\gamma_\nu + \frac{1}{2}\partial_+\bar z_\alpha{}^bF^{\alpha\beta}(V^{(0)})\partial_-z_\beta \\ &-\frac{1}{2}\partial_-\bar z_\alpha(\alpha)^{\alpha\beta}\partial_+z_\beta\Big]. \end{aligned} \tag{3.38}$$

From here, we can read background fields of T-dual theory

$$\begin{aligned} {}^{bf}\bar\Pi^{\mu\nu}_+ &= \frac{1}{4}\bar\Theta^{\mu\nu}_- - \frac{1}{2}{}^b\bar\Psi^{\mu\alpha}\left({}^bF^{-1}(V^{(0)})\right)_{\alpha\beta}{}^b\Psi^{\nu\beta} = \Theta^{\mu\nu}_-, \\ {}^{bf}\left(F^{-1}(x)\right)^{\alpha\beta} &= {}^bF^{\alpha\beta}(x) = F^{\alpha\beta}(x) + \frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu, \\ {}^{bf}\bar\Psi^\mu_\alpha\,{}^{bf}\left(F^{-1}(x)\right)^{\alpha\beta} &= {}^b\bar\Psi^{\mu\beta} = \frac{1}{2}\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu \\ &\to \quad {}^{bf}\bar\Psi^\mu_\alpha = \frac{1}{2}\bar\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu\left(F^{-1}(x)\right)_{\beta\alpha}, \\ {}^{bf}\left(F^{-1}(x)\right)^{\alpha\beta}\,{}^{bf}\Psi^\nu_\beta &= {}^b\Psi^{\nu\alpha} = -\frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_- \\ &\to \quad {}^{bf}\Psi^\nu_\beta = -\frac{1}{2}\left(F^{-1}(x)\right)_{\beta\alpha}\Psi^\alpha_\mu\bar\Theta^{\mu\nu}_-. \end{aligned} \tag{3.39}$$

Comparing background fields in different stages of T-dualization we notice that both fermionic T-duality and bosonic T-duality affect all field, where all T-dual theories now have coordinate dependent fields. It should also be noted that non-commutative relations in theory emerge only after performing bosonic T-duality. Fermionic T-dual coordinates are always only proportional to fermionic momenta therefore Poisson brackets between fermionic coordiantes always remain zero.

3.3. Bosonic T-Duality of Fermionic T-Dual Theory

For completion sake, we will also T-dualize fermionic T-dual action (3.20) along x^μ coordinates. In this specific case, where only RR field depends on bosonic coordinate, we expect that bosonic and ferionic T-dualities commute. Therefore, this section can be thought of as a check for calculations from previous section.

Bosonic T-duality is mostly the same as fermionic one,[10–13] where only difference is the lack of introduction of Fadeev-Popov and gauge fixing actions. We again start by localizing translational symmetry, inserting Lagrange multipliers and fixing gauge fields. This produces following auxiliary action

$$ {}^{f}S_{aux} = \kappa \int d^2\xi \Big[v_+^\mu \Pi_{+\mu\nu} v_-^\nu + \frac{1}{2}\partial_+ \bar{z}_\alpha f^{\alpha\beta}\partial_- z_\beta + \frac{1}{2}\partial_+\bar{z}_\alpha C_\mu^{\alpha\beta}\partial_- z_\beta \Delta V^\mu + \frac{1}{2}\partial_+\bar{z}_\alpha \Psi_\mu^\alpha v_-^\mu - \frac{1}{2} v_+^\mu \bar{\Psi}_\mu^\alpha \partial_- z_\alpha - \frac{1}{2}\partial_-\bar{z}_\alpha \alpha^{\alpha\beta}\partial_+ z_\beta + \frac{1}{2}y_\mu(\partial_+ v_-^\mu - \partial_- v_+^\mu)\Big]. \tag{3.40} $$

Introducing the variables

$$ Y_{+\mu} = \partial_+ y_\mu - \partial_+ \bar{z}_\alpha \Psi_\mu^\alpha, \quad Y_{-\mu} = \partial_- y_\mu - \bar{\Psi}_\mu^\alpha \partial_- z_\alpha, \tag{3.41} $$

the action (3.40) gets much simpler form

$$ {}^{f}S_{aux} = \kappa \int d^2\xi \Big[v_+^\mu \Pi_{+\mu\nu} v_-^\nu + \frac{1}{2}\partial_+ \bar{z}_\alpha f^{\alpha\beta}\partial_- z_\beta + \frac{1}{2}\partial_+\bar{z}_\alpha C_\mu^{\alpha\beta}\partial_- z_\beta \Delta V^\mu - \frac{1}{2} Y_{+\mu} v_-^\mu + \frac{1}{2} v_+^\mu Y_{-\mu}\Big]. \tag{3.42} $$

Varying the above action with respect to gauge fields v_+^μ and v_-^μ, we get, respectively,

$$ \Pi_{+\mu\nu} v_-^\nu = -\Big(\frac{1}{2}Y_{-\mu} + \beta_{+\mu}(V)\Big), \tag{3.43} $$

$$ v_+^\nu \Pi_{+\nu\mu} = \frac{1}{2}Y_{+\mu} - \beta_{-\mu}(V), \tag{3.44} $$

where $\beta_{\pm\mu}$ are the beta functions obtained from coordinate dependent term in the action

$$ \beta_{\pm\mu} = \mp\frac{1}{8}\Big(\bar{z}_\alpha C_\mu^{\alpha\beta}\partial_\mp z_\beta - \partial_\mp \bar{z}_\alpha C_\mu^{\alpha\beta} z_\beta\Big). \tag{3.45} $$

Inserting (3.43) and (3.44) into the auxiliary action (3.42), keeping the terms linear in $C_\mu^{\alpha\beta}$, we obtain fully T-dualized action (first fermionic, then bosonic T-dualization)

$$ {}^{fb}S = \kappa \int d^2\xi \Big[\frac{1}{2}\partial_+\bar{z}_\alpha F^{\alpha\beta}(\Delta V)\partial_- z_\beta + \frac{1}{4}Y_{+\mu}(\Pi_+^{-1})^{\mu\nu}Y_{-\nu}\Big]. \tag{3.46} $$

Expanding above action we prove that it is identical to one given in (3.38).

4. Few Notes on Non-commutativity

In paper[13] it has been shown that bosonic T-duality produces non-commutative relations between bosonic T-dual coordinates. With this in mind, following question naturally arises: can we expect emergence of same behavior for fermionic coordinates after fermionic T-dualization? To get the answer for this question we have to express fermionic T-dual coordinates as some combination of starting coordinates and their momenta and connect T-dual Poisson brackets with Poisson brackets of original theory. Original theory is geometric theory with regular Poisson structure

$$ \{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta_\nu^\mu \delta(\sigma-\bar\sigma), $$

$$ \{\theta^\alpha(\sigma), \pi_\beta(\bar\sigma)\} = \{\bar\theta^\alpha(\sigma), \bar\pi_\beta(\bar\sigma)\} = \delta_\beta^\alpha\delta(\sigma-\bar\sigma), \tag{4.1} $$

where all other Poisson brackets vanish.

We start with case that has only been T-dualized along fermionic coordinates. To find how T-dual coordinates depend on starting ones and their momenta we can begin by finding fermionic momenta of starting theory. It is useful to remember that starting theory did not posses terms that are proportional to $\partial_+\theta^\alpha$ and $\partial_-\bar\theta^\alpha$ and that this symmetry was fixed with BRST formalism. Addition of gauge fixing term introduced modification to momenta of starting theory and to obtain correct non-commutative relations we should be working with theories that have gauge fixing term in them. With this in mind, it is easy to find fermionic momentum of original theory (3.13)

$$ \pi_\beta = -\frac{k}{2}\Big[(\partial_+\bar\theta^\alpha + \partial_+ x^\mu \bar\Psi_\mu^\alpha)\big(F^{-1}(x)\big)_{\alpha\beta} - \partial_-\bar\theta^\alpha(\alpha^{-1})_{\alpha\beta}\Big], \tag{4.2} $$

$$ \bar\pi_\alpha = \frac{k}{2}\Big[\big(F^{-1}(x)\big)_{\alpha\beta}(\partial_-\theta^\beta + \Psi_\nu^\beta\partial_- x^\nu) - (\alpha^{-1})_{\alpha\beta}\partial_+\theta^\beta\Big]. \tag{4.3} $$

Since we want to obtain Poisson brackets for equal τ we want to find σ partial derivatives of dual coordinate

$$ \partial_\sigma z_\alpha = \partial_+ z_\alpha - \partial_- z_\alpha = \frac{2}{k}\bar\pi_\alpha, \tag{4.4} $$

$$ \partial_\sigma \bar z_\alpha = \partial_+ \bar z_\alpha - \partial_- \bar z_\alpha = -\frac{2}{k}\pi_\alpha. \tag{4.5} $$

Momenta of original theory commute with each other and with x^μ coordinates, therefore we deduce that there has been no change to geometric structure of this theory.

For fully dualized theory, transformation laws (3.33) (3.34) (3.35) all depend on dual bosonic coordinate however, when we insert transformation laws that connect original bosonic coordinates with T-dual ones (more details in[13])

$$ \partial_+ y_\mu = 2\Big[\partial_+ x^\nu \bar\Pi_{+\nu\mu} + \beta_\mu^-(x)\Big] + \partial_+\bar\theta^\alpha\big(F^{-1}(x)\big)_{\alpha\beta}\Psi_\mu^\beta, \tag{4.6} $$

$$ \partial_- y_\nu = -2\Big[\bar\Pi_{+\mu\nu}\partial_- x^\nu + \beta_\mu^+(x)\Big] - \bar\Psi_\nu^\alpha\big(F^{-1}(x)\big)_{\alpha\beta}\partial_-\theta^\beta, \tag{4.7} $$

into transformation laws for fermionic coordinates (3.33), (3.34) and (3.35) we again obtain relations (4.4) and (4.5).

On a first glance it would seem that fermionic T-duality has not produced any new Poisson brackets, however this is not the case. While it is true that there are no modifications to Poisson brackets between fermions, we have new Poisson bracket structure between fermions and bosons. This can be seen from σ derivative of bosonic T-dual coordinate

$$ y'_\mu \cong \frac{\pi_\mu}{\kappa} + \beta_\mu^0(x), \tag{4.8} $$

where $\beta_\mu^0(x)$ is combination $\beta_\mu^+(x) + \beta_\mu^-(x)$ given as

$$ \beta_\mu^0(x) = \frac{1}{2}\partial_\sigma\Big[\bar\theta^\alpha + x^{\nu_1}\bar\Psi_{\nu_1}^\alpha\Big](f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\Big[\theta^\beta + \Psi_{\nu_2}^\beta x^{\nu_2}\Big] - \frac{1}{2}\Big[\bar\theta^\alpha + x^{\nu_1}\bar\Psi_{\nu_1}^\alpha\Big](f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\partial_\sigma\Big[\theta^\beta + \Psi_{\nu_2}^\beta x^{\nu_2}\Big]. \tag{4.9} $$

Finding Poisson brackets between σ derivatives of coordinates and integrating twice we obtain following relations

$$\{y_\mu(\sigma), \bar{z}_\beta(\bar{\sigma})\} = \frac{1}{k}\Big[\bar{\theta}^\alpha(\sigma) + x^{\nu_1}(\sigma)\bar{\Psi}^\alpha_{\nu_1} - 2\Big(\bar{\theta}^\alpha(\bar{\sigma}) + x^{\nu_1}(\bar{\sigma})\bar{\Psi}^\alpha_{\nu_1}\Big)\Big] \times (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta} H(\sigma - \bar{\sigma}), \tag{4.10}$$

$$\{y_\mu(\sigma), z_\alpha(\bar{\sigma})\} = \frac{1}{k}(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta} \times \Big[\theta^\beta(\sigma) + \Psi^\beta_{\nu_2} x^{\nu_2}(\sigma) - 2\Big(\theta^\beta(\bar{\sigma}) + \Psi^\beta_{\nu_2} x^{\nu_2}(\bar{\sigma})\Big)\Big] \times H(\sigma - \bar{\sigma}). \tag{4.11}$$

5. Conclusions

In this article we examined effects of fermionic T-duality performed on action of type II superstring in pure spinor formalism. We carried out our investigation in two cases, one where we performed fermionic T-duality on previously non dualized action and in second case where we had action that was already dualized along bosonic coordinates. Starting (non dualized) action that we worked with described closed string that propagates in presence of Ramond-Ramond field with linear coordinate dependence. We made a decision to only consider dependence on bosonic coordinates, furthermore this dependence was tied to infinitesimal antisymmetric term $C^{\alpha\beta}_\mu$. Rest of the background fields were held constant. Terms in action that were non-linearly dependent on fermionic coordinates were neglected. These choices were in accordance with consistency conditions for background fields and were made in order to keep calculations manageable.

On the other hand, bosonic T-duality of starting action provided us with theory that was non-local. Unlike starting theory that only dependent on coordinates through RR field, this theory manifested coordinate dependence on all background fields. Furthermore, bosonic T-dual coordinates now exhibit non-commutative properties.

Before we could start with T-dualization we noticed that both cases posses additional local symmetry which removed terms proportional to $\partial_+\theta^\alpha$ and $\partial_-\bar{\theta}^\alpha$. In order to obtain correct T-dual theory this symmetry was fixed through BRST formalism. In both cases procedure for obtaining fermionic T-duality was the same, we employed Buscher T-dualizing procedure. Procedure is based on localization of translational symmetry where we replace partial derivatives with covariant ones. Introduction of covariant derivatives carries with itself new degrees of freedom in shape of gauge fields. By demanding that starting and T-dual theory give description of same physical system we inevitably demand for both theories to posses same degrees of freedom. Thus, all additional degrees of freedom must be removed with Lagrange multipliers. By utilizing gauge freedom we can also remove all instances of fermionic coordinates in action obtaining action that is only a function of gaguge fields and Lagrange multipliers. Finding equations of motion for gauge fields of this gauge fixed action and inserting their solutions into the action we obtain T-dual theory.

Carrying Buscher procedure for fermionic coordinates of non dualized action we obtain local theory where all fields depend on bosonic coordinates. This theory is commutative, its Poisson brackets are identical to Poisson brackets of starting theory.

Buscher procedure in case of theory that has been dualizied along bosonic coordinates does not change coordinate dependence of the background fields. All fields are still dependent on both bosonic and fermionic coordinates and theory is still non-local. However, this theory posseses two additional non-trivial Poisson brackets. We have emergence of non-commutativity between bosonic and fermionic coordinates, where non-commutativity is proportional to infinitesimal constant $C^{\alpha\beta}_\mu$.

Same result is obtained even in case where we first perform fermionic and then bosonic T-duality. Commutativity between different dualities was expected since fully T-dual theory must be unique. Only distinction between different paths of T-dualization procedures can be noticed in intermediate theories, where most important change is transition of theory from being local to non-local.

We suspect that it is possible to obtain T-dual theory that is fully non-commutative, theory that has non-commutativity even between fermionic coordinates, but we would need starting theory that has background fields that depend on both bosonic and fermionic coordinates.

Acknowledgements

Work supported in part by the Serbian Ministry of Education,Science and Technological Development.

Conflict of Interest

The authors have declared no conflict of interest.

Keywords

T-duality, Buscher procedure, non-commutativity, fermionic T-duality, string theory

Received: September 30, 2022

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Noncommutativity and Nonassociativity of Type II Superstring with Coordinate Dependent RR Field

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In this paper we will consider noncommutativity that arises from bosonic T-dualization of type II superstring in presence of Ramond-Ramond (RR) field, which linearly depends on the bosonic coordinates x^μ. The derivative of the RR field $C_\mu^{\alpha\beta}$ is infinitesimal. We will employ generalized Buscher procedure that can be applied to cases that have coordinate dependent background fields. Bosonic part of newly obtained T-dual theory is non-local. It is defined in non-geometric space spanned by Lagrange multipliers y_μ. We will apply generalized Buscher procedure once more on T-dual theory and prove that original theory can be salvaged. Finally, we will use T-dual transformation laws along with Poisson brackets of original theory to derive Poisson bracket structure of T-dual theory and nonassociativity relation. Noncommutativity parameter depends on the supercoordinates x^μ, θ^α and $\bar{\theta}^\alpha$, while nonassociativity parameter is a constant tensor containing infinitesimal $C_\mu^{\alpha\beta}$.

1. Introduction

In 1982 emerged a model[1] that would offer the possibility of obtaining bosonic coordinates of space-time as emergent properties of more fundamental fermionic coordinates. While this model worked with supersymmetric particle, this approach suggested that maybe we can express bosonic coordinates as a Poisson bracket of fermionic coordinates. In addition to these Poisson brackets, Poisson brackets between bosonic coordinates as well as between bosonic and fermionic coordinates would remain zero. Later, in paper,[2] it has been suggested that the same result can be obtained in context of string theory in case of coordinate dependent RR field. It is also suggested that unlike supersymmetric particle model, coordinate dependent RR field would produce full spectrum of non-commutative relations, bosonic coordinates themselves would become non-commutative. In this paper our goal is to determine if coordinate dependent RR field, while remaining background fields are as simple as consistency relations allow, can produce suggested non-commutative relations.

Superstring theory, as a theory of extended objects propagating in space-time, is defined in 10 dimensions.[3,4] In order to establish link between this mathematical model and real world observations, surplus space-like dimensions are compactified on circles of radius R. From this kind of geometry arises new kind of symmetry, T-duality, that links theories that have radii of compactification R with ones that have radii of compactification α'/R.[5,6] Existence of T-duality between different theories implies that those theories are physically equivalent and it gives us a way to explore how geometry and topology of one theory is connected to other. This connection between different geometries makes T-duality a useful tool in examining emergence of non-commutativity in context of closed strings.[7]

While in string theory both open and closed strings, under certain conditions, exhibit emergence of non-commutativity, mechanisms that enable this emergence are different. In case of open string, we have that endpoints of string that propagates in presence of constant metric and Kalb-Ramond field become non-commutative.[8] Basic idea of open string non-commutativity is that initial coordinates can be expressed as linear combination of effective coordinates and momenta by employing boundary conditions. In case of closed string, we do not have string endpoints therefore we don't have emergence of non-commutativity when string propagates in presence of constant background fields. In order to achieve same effect as in case of open string, we have to use coordinate dependent background fields. By finding T-dual of theories with this kind of geometry we obtain T-dual theory in non-geometric background, where T-dual coordinates are expressed as linear combinations of original coordinates and their conjugated momenta.

Mathematical framework for obtaining T-dual theories is standard Buscher procedure.[9,10] Procedure is based on existence of shift symmetry in relevant action and its implementation can be summarized in few steps. First step is localization of translational symmetry by introduction of covariant derivatives and introduction of Lagrange multipliers that make newly introduced gauge fields nonphysical. By gauge fixing and finding equations of motion of both gauge fields and Lagrange multipliers we obtain T-dual transformation laws. These transformation laws inserted into gauge fixed action produce T-dual action. For cases where we have coordinate dependent background fields there exist generalized Buscher procedure,[11–15] this extension has one additional step, replacement of all initial coordinates with invariant coordinates. Further extension of generalized Buscher procedure is possible[16] and it is applicable to theories that do not posses shift symmetry.

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DOI: 10.1002/prop.202200048

In this article we will deal with closed superstring propagating in presence of linearly coordinate dependent Ramond-Ramond (RR) field using type II superstring model in pure spinor formulation. All calculations we will do in approximation of diluted flux, which means that in all calculations we keep constant and linear terms in infinitesimal derivative of the RR field strength. Rest of the fields, metric, Kalb-Ramond and gravitino fields are constant. Furthermore all dependence of background field on fermionic terms will be neglected for mathematical simplicity. This choice of field configuration is in full accordance with consistency equations for background fields.[17]

Because we are currently only interested in non-commutative relations between bosonic coordinates, T-dualization procedure will be applied only on bosonic part of action. To find T-dual action and T-dual transformation laws we will employ extension of generalized Buscher procedure that works with coordinate dependent background fields.[11] After finding T-dual theory, we will apply Buscher procedure once more to see if we can obtain original theory.

Transformation laws that connect variables from initial with variables from T-dual theory will be written in canonical form, where initial momenta are expressed in terms of the T-dual coordinates. By inverting these transformation laws we obtain how sigma derivatives of T-dual theory depend on linear combinations of coordinates and momenta of original theory. Taking into account that original theory is geometrical, both locally and globally, we have that its coordinate and conjugated momenta satisfy standard Poisson brackets. By using this fact we are able to find Poisson structure of sigma derivatives of T-dual coordinates and by doing integration, Poisson structure of T-dual coordinates is obtained.

The form of obtained non-commutativity is such that non-commutativity exists when arguments are different, $\sigma \neq \bar{\sigma}$. Imposing trivial winding conditions, we obtain string winding numbers from Poisson brackets.

In the end, we give conclusions and in appendix we present some technical details regarding derivation of $\beta^{\pm}_{\mu}$ functions.

2. General Type II Superstring Action and Choice of Background Fields

Starting point of this investigation will be action of type II superstring theory in pure spinor formulation.[18–21] We will present and explain assumed approximations in order to obtain type II pure spinor action with non-constant RR field-strength. It turns out that ghost fields are neglected and only quadratic terms are considered. Final form of this kind of action will be used in subsequent sections.

2.1. General Form of the Pure Spinor Type II Superstring Action

Sigma model of type IIB superstring has the following form[17]

$$S = S_0 + V_{SG}. \tag{2.1}$$

This general form of action is expressed as a sum of the part that describes the motion of string in flat background

$$S_0 = \int_\Sigma d^2\xi \Big(\frac{\kappa}{2}\eta_{\mu\nu}\partial_m x^\mu \partial_n x^\nu \eta^{mn} - \pi_\alpha \partial_- \theta^\alpha + \partial_+ \bar{\theta}^\alpha \bar{\pi}_\alpha \Big) + S_\lambda + S_{\bar{\lambda}}\,, \tag{2.2}$$

and part that governs the modifications to the background fields

$$V_{SG} = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar{X}^N. \tag{2.3}$$

Modifications to the flat background are introduced by integrated form of massless type II supergravity vertex operator V_{SG}. The terms S_λ and $S_{\bar{\lambda}}$ in (2.2) are free-field actions for pure spinors

$$S_\lambda = \int_\Sigma d^2\xi \omega_\alpha \partial_- \lambda^\alpha, \quad S_{\bar{\lambda}} = \int_\Sigma d^2\xi \bar{\omega}_\alpha \partial_+ \bar{\lambda}^\alpha\,. \tag{2.4}$$

Here, λ^α and $\bar{\lambda}^\alpha$ are pure spinors whose canonically conjugated momenta are ω_α and $\bar{\omega}_\alpha$, respectively. Pure spinors satisfy pure spinor constraints

$$\lambda^\alpha (\Gamma^\mu)_{\alpha\beta} \lambda^\beta = \bar{\lambda}(\Gamma^\mu)_{\alpha\beta}\bar{\lambda}^\beta = 0\,. \tag{2.5}$$

In general case, vectors X^M and X^N as well as a supermatrix A_{MN} are given by

$$X^M = \begin{pmatrix} \partial_+\theta^\alpha \\ \Pi_+^\mu \\ d_\alpha \\ \frac{1}{2}N_+^{\mu\nu} \end{pmatrix}, \quad \bar{X}^M = \begin{pmatrix} \partial_-\bar{\theta}^\lambda \\ \Pi_-^\mu \\ \bar{d}_\lambda \\ \frac{1}{2}\bar{N}_-^{\mu\nu} \end{pmatrix},$$

$$A_{MN} = \begin{bmatrix} A_{\alpha\beta} & A_{\alpha\nu} & E_\alpha{}^\beta & \Omega_{\alpha,\mu\nu} \\ A_{\mu\beta} & A_{\mu\nu} & \bar{E}_\mu^\beta & \Omega_{\mu,\nu\rho} \\ E^\alpha{}_\beta & E^\alpha_\nu & P^{\alpha\beta} & C^\alpha{}_{\mu\nu} \\ \Omega_{\mu\nu,\beta} & \Omega_{\mu\nu,\rho} & \bar{C}^\beta{}_{\mu\nu} & S_{\mu\nu,\rho\sigma} \end{bmatrix}, \tag{2.6}$$

where notation is in accordance with Ref [17]. The components of matrix A_{MN} are generally functions of x^μ, θ^α and $\bar{\theta}^\alpha$. Components themselves are derived as expansions in powers of θ^α and $\bar{\theta}^\alpha$ (for details consult[17]). The superfields $A_{\mu\nu}$, $\bar{E}^\alpha_\mu$, E^α_μ and $P^{\alpha\beta}$ are known as physical superfields, while superfields that are in the first row and the first column are known as auxiliary because they can be expressed in terms of physical ones.[17] Remaining superfields $\Omega_{\mu,\nu\rho}$ ($\Omega_{\mu\nu,\rho}$), $C^\alpha{}_{\mu\nu}$ ($\bar{C}^\beta{}_{\mu\nu}$) and $S_{\mu\nu,\rho\sigma}$, are curvatures (field strengths) for physical fields. Components of vectors X^M and $\bar{X}^N$ are defined as

$$\Pi_+^\mu = \partial_+ x^\mu + \frac{1}{2}\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\partial_+\theta^\beta, \quad \Pi_-^\mu = \partial_- x^\mu + \frac{1}{2}\bar{\theta}^\alpha (\Gamma^\mu)_{\alpha\beta}\partial_-\bar{\theta}^\beta, \tag{2.7}$$

$$d_\alpha = \pi_\alpha - \frac{1}{2}(\Gamma_\mu \theta)_\alpha \Big[\partial_+ x^\mu + \frac{1}{4}(\theta\Gamma^\mu \partial_+\theta)\Big],$$

$$\bar{d}_\alpha = \bar{\pi}_\alpha - \frac{1}{2}(\Gamma_\mu \bar{\theta})_\alpha \Big[\partial_- x^\mu + \frac{1}{4}(\bar{\theta}\Gamma^\mu \partial_-\bar{\theta})\Big], \tag{2.8}$$

$$N_+^{\mu\nu} = \frac{1}{2}\omega_\alpha(\Gamma^{[\mu\nu]})^\alpha{}_\beta\lambda^\beta, \quad \bar{N}_-^{\mu\nu} = \frac{1}{2}\bar{\omega}_\alpha(\Gamma^{[\mu\nu]})^\alpha{}_\beta\bar{\lambda}^\beta. \tag{2.9}$$

The world sheet Σ is parameterized by $\xi^m = (\xi^0 = \tau, \xi^1 = \sigma)$ and world sheet light-cone partial derivatives are defines as $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Superspace in which string propagates is spanned both by bosonic x^μ ($\mu = 0, 1, \dots, 9$) and fermionic θ^α, $\bar{\theta}^\alpha$ ($\alpha = 1, 2, \dots, 16$) coordinates. Variables π_α and $\bar{\pi}_\alpha$ represent canonically conjugated momenta of fermionic coordinates θ^α and $\bar{\theta}^\alpha$, respectively. Fermionic coordinates and their canonically conjugated momenta are Majorana-Weyl spinors. It means that each of these spinors has 16 independent real valued components.

2.2. Choice of the Background Fields

In this particular case we will work with the supermatrix A_{MN} where all background fields, except RR field strength $P^{\alpha\beta}$, are constants. RR field strength will have linear coordinate dependence on bosonic coordinate x^μ. With these restrictions in mind, supermatrix A_{MN} has the following form

$$A_{MN} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \kappa\left(\frac{1}{2}g_{\mu\nu} + B_{\mu\nu}\right) & \bar{\Psi}_\mu^\beta & 0 \\ 0 & -\Psi_\nu^\alpha & \frac{2}{\kappa}(f^{\alpha\beta} + C_\rho^{\alpha\beta}x^\rho) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.10}$$

Here $g_{\mu\nu}$ is symmetric tensor, $B_{\mu\nu}$ is Kalb-Ramond antisymmetric field, Ψ_μ^α and $\bar{\Psi}_\mu^\alpha$ are Mayorana-Weyl gravitino fields, and finally, $f^{\alpha\beta}$ and $C_\rho^{\alpha\beta}$ are constants. Let us stress that dilaton field Φ is assumed to be constant, so, the factor e^Φ is included in $f^{\alpha\beta}$ and $C_\rho^{\alpha\beta}$. This will be a classical analysis and we will not calculate the dilaton shift under T-duality transformation. Based on the chirality of spinors, there are type IIA superstring theory for opposite chirality and type IIB superstring theory for same chirality.

This particular choice of supermatrix imposes following restriction on background fields

$$\gamma^\mu_{\alpha\beta}C_\mu^{\beta\gamma} = 0, \quad \gamma^\mu_{\alpha\beta}C_\mu^{\gamma\beta} = 0. \tag{2.11}$$

Remaining constraints[17] are trivial and applied only to nonphysical fields.

In addition to choice of supermatrix, in order to simplify calculation of bosonic T-duality, because all background fields are expanded in powers of θ^α and $\bar{\theta}^\alpha$, all θ^α and $\bar{\theta}^\alpha$ non-linear terms in X^M and $\bar{X}^N$ will be neglected. With this in mind, components of these two vectors reduce into the following form

$$\Pi_\pm^\mu \to \partial_\pm x^\mu, \quad d_\alpha \to \pi_\alpha, \quad \bar{d}_\alpha \to \bar{\pi}_\alpha. \tag{2.12}$$

Taking into account all these assumptions, the action (2.1) takes the form

$$S = \int_\Sigma d^2\xi \left[\frac{\kappa}{2}\Pi_{+\mu\nu}\partial_+x^\mu\partial_-x^\nu - \pi_\alpha(\partial_-\theta^\alpha + \Psi_\nu^\alpha\partial_-x^\nu) + (\partial_+\bar{\theta}^\alpha + \partial_+x^\mu\bar{\Psi}_\mu^\alpha)\bar{\pi}_\alpha + \frac{2}{\kappa}\pi_\alpha(f^{\alpha\beta} + C_\rho^{\alpha\beta}x^\rho)\bar{\pi}_\beta\right]. \tag{2.13}$$

Here, new tensor $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}$ is introduced, where $G_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu}$ is metric tensor. Terms for S_λ and $S_{\bar{\lambda}}$ are fully decoupled from action and they will not be considered from now on.

Before considering T-duality, we can notice that fermionic momenta act as auxiliary fields in full actions. These fields can be integrated out and final action will be function of only coordinates and their derivatives. Finding equations for motion for both π_α and $\bar{\pi}_\alpha$ we get following two equations

$$\bar{\pi}_\beta = \frac{\kappa}{2}\left(F^{-1}(x)\right)_{\beta\alpha}\left(\partial_-\theta^\alpha + \Psi_\nu^\alpha\partial_-x^\nu\right), \tag{2.14}$$

$$\pi_\alpha = -\frac{\kappa}{2}\left(\partial_+\bar{\theta}^\beta + \partial_+x^\mu\bar{\Psi}_\mu^\beta\right)\left(F^{-1}(x)\right)_{\beta\alpha}, \tag{2.15}$$

where $F^{\alpha\beta}(x)$ and $(F^{-1}(x))_{\alpha\beta}$ are of the form

$$F^{\alpha\beta}(x) = f^{\alpha\beta} + C_\mu^{\alpha\beta}x^\mu,$$
$$(F^{-1}(x))_{\alpha\beta} = (f^{-1})_{\alpha\beta} - (f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\,x^\mu. \tag{2.16}$$

In order to invert previous equations and T-dual transformation laws, as well as to simplify calculations, we take two additional assumptions. First assumption is that $C_\mu^{\alpha\beta}$ is infinitesimal. Second assumption is that $(f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}$ is antisymmetric under exchange of first and last index. In other words, tensor $(F^{-1}(x))_{\alpha\beta}$ has only antysimetric part that depends on x^μ and it is infinitesimal. These assumptions are in full accordance with constraints.[17]

Substituting equations (2.14) and (2.15) into (2.13) the final form of action is

$$S = \kappa\int_\Sigma d^2\xi \left[\Pi_{+\mu\nu}\partial_+x^\mu\partial_-x^\nu + \frac{1}{2}(\partial_+\bar{\theta}^\alpha + \partial_+x^\mu\bar{\Psi}_\mu^\alpha)\left(F^{-1}(x)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi_\nu^\beta\partial_-x^\nu)\right]. \tag{2.17}$$

In the following sections, this form of action will be used for investigation of bosonic T-duality and for obtaining transformation laws between starting and T-dual coordinates.

3. T-dualization

In this section T-duality will be performed along all bosonic coordinates in order to find relations that connect T-dual coordinates with coordinates and momenta of original theory. These transformation laws will then be used in subsequent chapters to find non-commutativity relations between coordinates of T-dual theory.

Starting point for considering T-duality will be generalized Buscher T-dualization procedure.[11] Standard Buscher procedure[9,10] is designed to be applied along isometry directions on which background fields do not depend. Generalized Buscher procedure can be applied to theories with coordinate dependent background fields. The shift symmetry in the generalized procedure is localized by introduction of covariant derivatives, invariant coordinates and additional gauge fields. These newly introduced gauge fields produce additional degrees of freedom. Since we expect that starting and T-dual theory have exactly the same number of degrees of freedom we need to eliminate all excessive degrees of freedom. This is accomplished by demanding that field strength of gauge fields ($F_{+-} = \partial_+ v_- - \partial_- v_+$) vanishes by addition of Lagrange multipliers. Next step in procedure is fixing the gauge symmetry such that starting coordinates are constant and action is only left with gauge fields and its derivatives. From this gauge fixed action, finding equations of motion for gauge fields, expressing gauge fields as function of Lagrange multipliers and inserting those equations into action we can obtain T-dual action, were Lagrange multipliers of original theory now play the role of T-dual coordinates.

In cases where shift symmetry is absent, T-duality can still be performed by extending generalized Buscher procedure.[16] This extension is based on replacing original action with translation invariant auxiliary action. Form of this auxiliary action is exactly the same as the form of action where translation symmetry was localized and gauged fixed, that is, derivatives have been replaced with gauge fields and coordinates with integrals of gauge fields. Auxiliary action gives correct T-dual theory only if original action can be salvaged from it. In cases where this is possible, original theory is obtained by finding equations of motion with respect to Lagrange multipliers and inserting their solutions into auxiliary action.

Action (2.17) is invariant to translation symmetry, by the virtue of antisymmetric part of $F^{-1}_{\alpha\beta}$, tensor $(f^{-1}C_\mu f^{-1})_{\alpha\beta}$. Following antisymmetricity of this tensor, we can rewrite the action (2.17) in the following way

$$S = \kappa \int_\Sigma d^2\xi \Big[\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu + \frac{1}{2}\epsilon^{mn}\partial_m(\bar\theta^\alpha + x^\mu \bar\Psi^\alpha_\mu)\big(F^{-1}(x)\big)_{\alpha\beta}\partial_n(\theta^\beta + \Psi^\beta_\nu x^\nu)\Big]. \tag{3.1}$$

Let us now consider the global shift symmetry $\delta x^\mu = \lambda^\mu$ and vary the action (3.1)

$$\delta S = -\frac{\kappa}{2}(f^{-1}C_\mu f^{-1})_{\alpha\beta}\lambda^\mu \int_\Sigma d^2\xi \epsilon^{mn}\partial_m(\bar\theta^\alpha + \bar\Psi^\alpha_\nu x^\nu)\partial_n(\theta^\beta + \Psi^\beta_\rho x^\rho)\,, \tag{3.2}$$

where m, n are indices of the twodimensional worldsheet. After one partial integration, we obtain one surface term and one term which is identically zero because it is summation of symmetric, $\partial_m\partial_n$, and antisymmetric, ϵ^{mn}, tensor. The surface term is zero for trivial topology. So, the shift isometry exists.

In order to find T-dual action we have to implement following substitutions

$$\partial_\pm x^\mu \to D_\pm x^\mu = \partial_\pm x^\mu + v^\mu_\pm, \tag{3.3}$$

$$x^\rho \to x^\mu_{inv} = \int_P d\xi^m D_m x^\mu = x^\mu(\xi) - x^\mu(\xi_0) + \Delta V^\mu\,,$$
$$\Delta V^\mu = \int_P d\xi^m v^\rho_m(\xi), \tag{3.4}$$

$$S \to S + \frac{\kappa}{2}\int_\Sigma d^2\xi\big[v^\mu_+\partial_- y_\mu - v^\mu_-\partial_+ y_\mu\big]. \tag{3.5}$$

Because of the shift symmetry we fix the gauge, $x^\mu(\xi) = x^\mu(\xi_0)$ and, inserting these substitutions into action (2.17), we obtain auxiliary action suitable for T-dualization

$$S_{aux} = \kappa \int_\Sigma d^2\xi \Big[\Pi_{+\mu\nu}v^\mu_+ v^\nu_- + \frac{1}{2}(\partial_+\bar\theta^\alpha + v^\mu_+\bar\Psi^\alpha_\mu)\big(F^{-1}(\Delta V)\big)_{\alpha\beta} \times (\partial_-\theta^\beta + \Psi^\beta_\nu v^\nu_-) + \frac{1}{2}(v^\mu_+\partial_- y_\mu - v^\mu_-\partial_+ y_\mu)\Big]. \tag{3.6}$$

It should be noted that, path P that is taken in expression for ΔV^ρ goes from some starting point ξ_0 to end point ξ. Introduction of this element makes this action non-local, however, this is a necessary step in order to find T-dual theory of coordinate dependent background fields.[11]

In order to check if substitutions we had introduced are valid and that they will lead to correct T-dual theory of starting action, we need to be able to obtain original action by finding solutions to equations of motion for Lagrange multipliers. Equations of motion for Lagrange multipliers give us

$$\partial_- v^\mu_+ - \partial_+ v^\mu_- = 0 \quad \Rightarrow \quad v^\mu_\pm = \partial_\pm x^\mu. \tag{3.7}$$

Inserting this result into (3.4) we get the following

$$\Delta V^\rho = \int_P d\xi'^m \partial_m x^\rho(\xi') = x^\rho(\xi) - x^\rho(\xi_0) = \Delta x^\rho. \tag{3.8}$$

Since, we had shift symmetry in original action, we can let $x^\rho(\xi_0)$ be any arbitrary constant. Taking all this into account and inserting (3.7), (3.8) into (3.6) we obtain our starting action (2.17).

Before we obtain equations for motion for gauge fields, we would like to make following substitution in action

$$Y_{+\mu} = \partial_+ y_\mu - \partial_+\bar\theta^\alpha\big(F^{-1}(\Delta V)\big)_{\alpha\beta}\Psi^\beta_\mu,$$
$$Y_{-\mu} = \partial_- y_\mu + \bar\Psi^\alpha_\mu\big(F^{-1}(\Delta V)\big)_{\alpha\beta}\partial_-\theta^\beta, \tag{3.9}$$

$$\bar\Pi_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar\Psi^\alpha_\mu\big(F^{-1}(\Delta V)\big)_{\alpha\beta}\Psi^\beta_\nu = \breve\Pi_{+\mu\nu} - \frac{1}{2}\bar\Psi^\alpha_\mu(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_\nu\Delta V^\rho, \tag{3.10}$$

$$\breve\Pi_{+\mu\nu} \equiv \Pi_{+\mu\nu} + \frac{1}{2}\bar\Psi^\alpha_\mu(f^{-1})_{\alpha\beta}\Psi^\beta_\nu. \tag{3.11}$$

With these substitutions in mind we have that auxiliary action takes the following form

$$S_{aux} = \kappa \int_\Sigma d^2\xi \Big[\bar\Pi_{+\mu\nu}v^\mu_+ v^\nu_- + \frac{1}{2}v^\mu_+ Y_{-\mu} - \frac{1}{2}v^\mu_- Y_{+\mu} + \frac{1}{2}\partial_+\bar\theta^\alpha\big(F^{-1}(\Delta V)\big)_{\alpha\beta}\partial_-\theta^\beta\Big]. \tag{3.12}$$

This action produces following equations of motion for gauge fields

$$\bar\Pi_{+\mu\nu}v^\nu_- = -(\frac{1}{2}Y_{-\mu} + \beta^+_\mu(V)), \qquad \bar\Pi_{+\mu\nu}v^\mu_+ = \frac{1}{2}Y_{+\nu} - \beta^-_\nu(V). \tag{3.13}$$

Here, function $\beta^\pm(V)$ is obtained from variation of term containing ΔV^ρ in expression for $F^{-1}(\Delta V)$ (details are presented in

Appendix A)

$$\beta^-_\mu(V) = \frac{1}{4}\partial_+\left[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right]$$
$$- \frac{1}{4}\left[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_+\left[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right], \tag{3.14}$$

$$\beta^+_\mu(V) = \frac{1}{4}\left[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_-\left[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right]$$
$$- \frac{1}{4}\partial_-\left[\bar\theta^\alpha + V^{\nu_1}\bar\Psi^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left[\theta^\beta + \Psi^\beta_{\nu_2}V^{\nu_2}\right]. \tag{3.15}$$

Here we have took advantage of the fact that $\partial_\pm V^\mu = v^\mu_\pm$ (more details in Appendix A). Let us note that V^μ in the expressions for beta functions is actually $V^{(0)\mu}$ because it stands besides $C^{\alpha\beta}_\mu$. We omit index (0) just in order to simplify the form of the expressions.

In order to find how gauge fields depend on Lagrange multipliers, we need to invert equations of motion (3.13). Since $C^{\alpha\beta}_\mu$ is an infinitesimal constant, these equations can be inverted iteratively.[22] We separate variables into two parts, one finite and one proportional to $C^{\alpha\beta}_\mu$. After doing this we have

$$v^\nu_- = -\bar\Theta^{\nu\mu}_-\left[\frac{1}{2}Y_{-\mu} + \beta^+_\mu(V^{(0)})\right], \qquad v^\mu_+ = \left[\frac{1}{2}Y_{+\nu} - \beta^-_\nu(V^{(0)})\right]\bar\Theta^{\nu\mu}_-. \tag{3.16}$$

Functions $\beta_{\pm\mu}(V^{(0)})$ are obtained by substituting first order of expression for $v_\pm$ into $\beta_{\pm\mu}(V)$, where $V^{(0)}$ is given by

$$\Delta V^{(0)\rho} = \int_P d\xi^m v^{(0)\rho}_m$$
$$= \frac{1}{2}\int_P d\xi^+\check\Theta^{\rho_1\rho}_-\left[\partial_+y_{\rho_1} - \partial_+\bar\theta^\alpha(f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right]$$
$$- \frac{1}{2}\int_P d\xi^-\check\Theta^{\rho\rho_1}_-\left[\partial_-y_{\rho_1} + \bar\Psi^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right]. \tag{3.17}$$

Where $\bar\Theta^{\mu\nu}_-$ is inverse tensor of $\bar\Pi_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar\Psi^\alpha_\mu(F^{-1}(\Delta V))_{\alpha\beta}\Psi^\beta_\nu$, defined as

$$\bar\Theta^{\mu\nu}_-\bar\Pi_{+\nu\rho} = \delta^\mu_\rho, \tag{3.18}$$

where

$$\bar\Theta^{\mu\nu}_- = \check\Theta^{\mu\nu}_- + \frac{1}{2}\check\Theta^{\mu\mu_1}_-\bar\Psi^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho V^{(0)\rho}(f^{-1})_{\beta_1\beta}\Psi^{\beta_1}_{\nu_1}\check\Theta^{\nu_1\nu}_-, \tag{3.19}$$

$$\check\Theta^{\mu\nu}_-\Pi_{\nu\rho} = \delta^\mu_\rho, \qquad \check\Theta^{\mu\nu}_- = \Theta^{\mu\nu}_- - \frac{1}{2}\Theta^{\mu\mu_1}_-\bar\Psi^\alpha_{\mu_1}(\bar f^{-1})_{\alpha\beta}\Psi^\beta_{\nu_1}\Theta^{\nu_1\nu}_- \tag{3.20}$$

$$\bar f^{\alpha\beta} = f^{\alpha\beta} + \frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu, \tag{3.21}$$

$$\Theta^{\mu\nu}_-\Pi_{+\mu\rho} = \delta^\mu_\rho, \qquad \Theta_- = -4(G_E^{-1}\Pi_-G^{-1})^{\mu\nu}. \tag{3.22}$$

Tensor $G_{E\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$ is known in the literature as the effective metric.

Inserting equations (3.16) into (3.6), keeping only terms that are linear in $C^{\alpha\beta}_\mu$ we obtain T-dual action

$$S^\star = \frac{\kappa}{2}\int_\Sigma\left[\frac{1}{2}\bar\Theta^{\mu\nu}_-Y_{+\mu}Y_{-\nu} + \partial_+\bar\theta^\alpha\left(F^{-1}(\Delta V)\right)_{\alpha\beta}\partial_-\theta^\beta\right]. \tag{3.23}$$

Comparing starting action (2.17) with T-dual action, were we note that $\partial_\pm x^\mu$ transforms into $\partial_\pm y_\mu$ and x^μ transforms into $V^{(0)}$, we can deduce that T-dual action has following arguments.

$${}^\star\bar\Pi^{\mu\nu}_+ = \frac{1}{4}\bar\Theta^{\mu\nu}_-, \tag{3.24}$$

$$\left({}^\star F^{-1}(V^{(0)})\right)_{\alpha\beta} = \left(F^{-1}(V^{(0)})\right)_{\alpha\beta}$$
$$- \frac{1}{2}\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta^{\mu\nu}_-\bar\Psi^{\beta_1}_\nu\left(F^{-1}(V^{(0)})\right)_{\beta_1\beta}, \tag{3.25}$$

$${}^\star\bar\Psi^{\mu\alpha}\left({}^\star F^{-1}(V^{(0)})\right)_{\alpha\beta} = \frac{1}{2}\bar\Theta^{\mu\nu}_-\bar\Psi^\alpha_\nu\left(F^{-1}(V^{(0)})\right)_{\alpha\beta}, \tag{3.26}$$

$$\left({}^\star F^{-1}(V^{(0)})\right)_{\alpha\beta}{}^\star\Psi^{\nu\beta} = -\frac{1}{2}\left(F^{-1}(V^{(0)})\right)_{\alpha\beta}\Psi^\beta_\mu\bar\Theta^{\mu\nu}_-. \tag{3.27}$$

In order to express T-dual gravitino background fields in terms of its components, it is useful to calculate inverse of field ${}^\star F^{-1}_{\alpha\beta}$

$${}^\star F^{\alpha\beta}(V^{(0)}) = F^{\alpha\beta}(V^{(0)}) + \frac{1}{2}\Psi^\alpha_\mu\Theta^{\mu\nu}_-\bar\Psi^\beta_\nu. \tag{3.28}$$

With this equation at hand it is straightforward to obtain T-dual gravitino fields. Here we present T-dual gravitino fields expanded in terms of their components

$${}^\star\bar\Psi^{\mu\alpha} = \frac{1}{2}\bar\Theta^{\mu\nu}\bar\Psi^\alpha_\nu + \frac{1}{4}\bar\Theta^{\mu\mu_1}_-\bar\Psi^\beta_{\mu_1}\left(F^{-1}(V^{(0)})\right)_{\beta\beta_1}\Psi^{\beta_1}_\nu\Theta^{\nu\nu_1}_-\bar\Psi^\alpha_{\nu_1}, \tag{3.29}$$

$${}^\star\Psi^{\nu\beta} = -\frac{1}{2}\Psi^\beta_\mu\bar\Theta^{\mu\nu}_- - \frac{1}{2}\Psi^\beta_\mu\Theta^{\mu\mu_1}_-\bar\Psi^\alpha_{\mu_1}\left(F^{-1}(V^{(0)})\right)_{\alpha\alpha_1}\Psi^{\alpha_1}_{\nu_1}\bar\Theta^{\nu_1\nu}_-. \tag{3.30}$$

The general conclusion is that all background fields get the linear corrections in $C^{\alpha\beta}_\mu$ comparing with the results of the case with constant background fields.[23] Also the coordinate dependence is present in all T-dual background fields.

From the above equations we see how background fields of original theory transform under T-duality. It should be noted that these actions are of the same form taking into account that initial coordinates x^μ are replaced by y_μ after T-dualization.

4. T-dualization of T-dual Theory

From requirement that original theory and T-dual theory be physically equivalent, it should be possible to obtain original theory from T-dual one by applying T-duality procedure a second time. Since original action possessed translation symmetry, we have that this symmetry is inherited by T-dual action. T-dual theory is invariant to translations of T-dual coordinate. However, even in cases where starting action is not invariant to translation symmetry we can expect emergence of this symmetry in T-dual theory. This is a natural consequence of introducing $\Delta V^{(0)}$ and of the fact that T-dual theory is intrinsically a non-local one. T-dualization of T-dual theory is obtained with generalized Buscher procedure and steps are identical as before.

$$\partial_\pm y_\mu \to D_\pm y_\mu = \partial_\pm y_\mu + u_{\pm\mu} \to D_\pm y_\mu = u_{\pm\mu}, \tag{4.1}$$

$$\Delta V^{(0)\rho} \to \Delta U^{(0)\rho}, \tag{4.2}$$

$$\Delta U^{(0)\rho} = \frac{1}{2}\int_P d\xi^+ \check{\Theta}_-^{\rho_1\rho}\left[u_{+\rho_1} - \partial_+\bar{\theta}^\alpha (f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] - \frac{1}{2}\int_P d\xi^- \check{\Theta}_-^{\rho\rho_1}\left[u_{-\rho_1} + \bar{\Psi}^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right], \tag{4.3}$$

$$Y_{+\mu} \to U_{+\mu} = u_{+\mu} - \partial_+\bar{\theta}^\alpha\left(F^{-1}(\Delta U^{(0)})\right)_{\alpha\beta}\Psi^\beta_\mu \tag{4.4}$$

$$Y_{-\mu} \to U_{-\mu} = u_{-\mu} + \bar{\Psi}^\alpha_\mu\left(F^{-1}(\Delta U^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta \tag{4.5}$$

$$^\star S \to {}^\star S + \frac{\kappa}{2}\int_\Sigma d^2\xi (u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu). \tag{4.6}$$

In first line we immediately fixed gauge by choosing $y(\xi) = const$. Inserting these substitutions into (3.23) we get

$$^\star S_{gfix} = \frac{k}{2}\int_\Sigma d^2\xi\left[\frac{1}{2}\bar{\Theta}_-^{\mu\nu}U_{+\mu}U_{-\nu} + \partial_+\bar{\theta}^\alpha\left(F^{-1}(\Delta U^{(0)})\right)_{\alpha\beta}\partial_-\theta^\beta + (u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu)\right]. \tag{4.7}$$

Finding equations of motion for Lagrange multipliers and inserting solution to those equations into gauge fixed action we return to the starting point of this chapter, T-dual action. On the other hand, finding equations of motion for gauge fields

$$u_{+\mu} = 2\left[\partial_+ x^\nu \bar{\Pi}_{+\nu\mu} + \beta^-_\mu(x)\right] + \partial_+\bar{\theta}^\alpha\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\mu, \tag{4.8}$$

$$u_{-\nu} = -2\left[\bar{\Pi}_{+\mu\nu}\partial_- x^\nu + \beta^+_\mu(x)\right] - \bar{\Psi}^\alpha_\nu\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta, \tag{4.9}$$

and inserting these equations into the gauge fixed action, keeping all terms linear with respect to $C^{\mu\nu}_\rho$, we obtain our original action (2.17). Here we use the freedom to choose $\Delta x^\mu = x(\xi) - x(\xi_0)$, with $x(\xi_0) = 0$.

5. Non-commutative Relations

Having found T-dual action and equations that link T-dual coordinate with original coordinates in previous chapters, in this chapter we will focus on establishing a relationship between Poisson brackets of original and T-dual theory. Furthermore, we will mainly focus on Poisson brackets between bosonic variables and their momenta. Original theory is a geometric one with variables $x^\mu(\xi)$ and $\pi_\mu(\xi)$. Therefore, it is natural to impose standard Poisson structure on original theory

$$\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu_\nu\delta(\sigma - \bar{\sigma}), \quad \{x^\mu(\sigma), x^\nu(\bar{\sigma})\} = 0, \quad \{\pi_\mu(\sigma), \pi_\nu(\bar{\sigma})\} = 0. \tag{5.1}$$

In order to find Poisson brackets of T-dual theory, we need to find T-dual transformation laws which connect the initial and T-dual coordinates. Starting with relations (4.8) and (4.9) and using equations of motion for Lagrange multipliers x^μ, $u_{\pm\mu} = \partial_\pm y_\mu$, we obtain T-dual transformation laws

$$\partial_+ y_\mu \cong 2\left[\partial_+ x^\nu \bar{\Pi}_{+\nu\mu} + \beta^-_\mu(x)\right] + \partial_+\bar{\theta}^\alpha\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\mu, \tag{5.2}$$

$$\partial_- y_\mu \cong -2\left[\bar{\Pi}_{+\mu\nu}\partial_- x^\nu + \beta^+_\mu(x)\right] - \bar{\Psi}^\alpha_\nu\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta, \tag{5.3}$$

where symbol $\cong$ denotes T-dual transformation. Subtracting these two equations, we get

$$y'_\mu \cong \bar{\Pi}_{+\mu\nu}\partial_- x^\nu + \partial_+ x^\nu \bar{\Pi}_{+\nu\mu} + \beta^+_\mu + \beta^-_\mu + \frac{1}{2}\partial_+\bar{\theta}^\alpha\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\mu + \frac{1}{2}\bar{\Psi}^\alpha_\nu\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta. \tag{5.4}$$

Taking into account that bosonic momenta, π_μ of original theory are of the form

$$\pi_\mu = \kappa\left[\bar{\Pi}_{+\mu\nu}\partial_- x^\nu + \partial_+ x^\nu \bar{\Pi}_{+\nu\mu} + \frac{1}{2}\bar{\Psi}^\alpha_\mu\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta + \frac{1}{2}\partial_+\bar{\theta}^\alpha\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_\nu\right], \tag{5.5}$$

and $\beta^0_\mu = \beta^+_\mu + \beta^-_\mu$, we obtain

$$y'_\mu \cong \frac{\pi_\mu}{\kappa} + \beta^0_\mu(x). \tag{5.6}$$

Here $\beta^0_\mu(x)$ is given by

$$\beta^0_\mu(x) = \frac{1}{2}\partial_\sigma\left[\bar{\theta}^\alpha + x^{\nu_1}\bar{\Psi}^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left[\theta^\beta + \Psi^\beta_{\nu_2}x^{\nu_2}\right] - \frac{1}{2}\left[\bar{\theta}^\alpha + x^{\nu_1}\bar{\Psi}^\alpha_{\nu_1}\right](f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\partial_\sigma\left[\theta^\beta + \Psi^\beta_{\nu_2}x^{\nu_2}\right]. \tag{5.7}$$

To find Poisson bracket between T-dual coordinates, we can start by finding Poisson bracket of sigma derivatives of T-dual coordinates and then integrating twice (see [13, 24, 25], B). Implementing this procedure we have that Poisson bracket is given as

$$\{y_{\nu_1}(\sigma), y_{\nu_2}(\bar{\sigma})\} \cong \frac{1}{2k}[2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2} - \delta^{\mu_2}_{\nu_1}\delta^{\mu_1}_{\nu_2}]\left[K_{\mu_1\mu_2}(\bar{\sigma}) + K_{\mu_2\mu_1}(\sigma)\right]H(\sigma - \bar{\sigma}), \tag{5.8}$$

where, for the sake of simplicity, we introduced

$$K_{\mu\nu}(\sigma) = \left(\bar{\theta}^\alpha(\sigma) + x^{\mu_1}(\sigma)\bar{\Psi}^\alpha_{\mu_1}\right)(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\Psi^\beta_\nu - \bar{\Psi}^\alpha_\nu(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}\left(\theta^\beta(\sigma) + \Psi^\beta_{\nu_1}x^{\nu_1}(\sigma)\right). \tag{5.9}$$

Here, $H(\sigma - \bar{\sigma})$ is same step function defined in Appendix B. It should be noted that these Poisson brackets are zero when $\sigma = \bar{\sigma}$. However, in cases where string in curled around compactified dimension, that is cases where $\sigma - \bar{\sigma} = 2\pi$, we have following situation

$$\{y_{\nu_1}(\sigma + 2\pi), y_{\nu_2}(\sigma)\}$$
$$\cong \frac{1}{2k}[2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2} - \delta^{\mu_2}_{\nu_1}\delta^{\mu_1}_{\nu_2}]\left[K_{\mu_1\mu_2}(\sigma) + K_{\mu_2\mu_1}(\sigma)\right]$$
$$+ \frac{\pi}{k}N^\mu \bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_{\mu_2}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_3}[\delta^{\mu_1}_{\mu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} - \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\mu}$$
$$+ \delta^{\mu_1}_{\mu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\mu}]. \tag{5.10}$$

Here we used fact that $H(2\pi) = 1$, while N^ρ is winding number around compactified coordinate defined as

$$x^\mu(\sigma + 2\pi) - x^\mu(\sigma) = 2\pi N^\mu. \tag{5.11}$$

From this relation we can see that if we choose $x^\mu(\sigma) = 0$ than Poisson bracket has linear dependence on winding number. In cases where we do not have any winding number, we still have non-commutativity that is proportional to background fields.

Using the expression for sigma derivative of y_ν (5.6) and expression for Poisson bracket of T-dual coordinates (5.8), we can find non-associative relations. Procedure is the same as for finding Poisson brackets of T-dual theory, we find Poisson bracket of sigma derivative and integrate with respect to sigma coordinate, this time integration is done once. Going along with this procedure we have the final result

$$\{y_\nu(\sigma), \{y_{\nu_1}(\sigma_1), y_{\nu_2}(\sigma_2)\}\}$$
$$\cong \frac{1}{2k}H(\sigma_1 - \sigma_2)\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_{\mu_2}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_3}$$
$$\times\Bigg[H(\sigma_1 - \sigma)[2\delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} - 2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu} - \delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} + \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu}]$$
$$+H(\sigma_2 - \sigma)[2\delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} - 2\delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu} - \delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} + \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu}]\Bigg]. \tag{5.12}$$

Since Jacobi identity is non-zero for T-dual theory we have that coordinate dependent RR field produces non-associative theory. However putting $\sigma = \sigma_2 = \bar{\sigma}$ and $\sigma_1 = \bar{\sigma} + 2\pi$ we have following Jacobi identity

$$\{y_\nu(\bar{\sigma}), \{y_{\nu_1}(\bar{\sigma} + 2\pi), y_{\nu_2}(\bar{\sigma})\}\}$$
$$\cong \bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_{\mu_2}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_3}[2\delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu_1} - 2\delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\delta^{\mu_3}_{\nu}$$
$$- \delta^{\mu_1}_{\nu}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu_2} + \delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1}\delta^{\mu_3}_{\nu}]. \tag{5.13}$$

Examining equation (5.6), we notice that $\partial_\sigma y_\mu$ is not only a linear combination of initial coordinate and its momenta but also has terms that are proportional to fermionic coordinates. This might lead us to believe that T-dual theory would have nontrivial Poisson bracket between T-dual coordinate and fermionic coordinates. However, this is not the case, and it can be directly calculated by finding Poisson bracket between sigma derivative of T-dual coordinate and fermion coordinates (more details in B).

$$\{\theta^\alpha(\sigma), y_\mu(\bar{\sigma})\} \cong 0, \qquad \{\bar{\theta}^\alpha(\sigma), y_\mu(\bar{\sigma})\} \cong 0. \tag{5.14}$$

6. Conclusion

In this article we examined type II superstring propagating in presence of coordinate dependent RR field. This choice of background was in accordance with consistency conditions for background field and all calculations were made in approximation that are linear with respect to the space-time derivative of the RR field, $C^{\alpha\beta}_\mu$, which is infinitesimal one. We have also excluded parts that were non-linear in fermionic coordinates and neglected pure spinor actions. Using equations of motion for fermionic momenta we obtained action that was expressed in terms of bosonic coordinates, their derivatives and derivatives of fermionic coordinates.

Action with our choice of background fields possessed translation symmetry, therefore we use generalized Buscher procedure that was developed for such cases. By substituting starting action with auxiliary action we gave up on locality in order to be able to find T-dual theory. Finding equations of motion of newly introduced Lagrange multipliers we were able to salvage starting action giving us assurance that auxiliary action we selected would produce correct T-dual theory. After this we found equations of motion for gauge fields and by inserting them into action, we found T-dual theory.

Having found T-dual theory, we applied T-dual procedure once again as a more thorough way of checking if action we obtained was in fact correct T-dual of starting action. Unlike starting action, T-dual action was non-local from the start by virtue of containing $V^{(0)}$ term. Applying steps of generalized Buscher procedure we obtained starting action, again confirming that our choice of auxiliary action was correct.

We obtained non-commutativity relations in context of T-dual theory, where we used T-dual transformation laws as a bridge between Poisson brackets of starting theory and T-dual theory. T-dual transformation laws were expressed in terms of coordinates and momenta of original theory, which produced non-commutativity in T-dual theory. From expression for Poisson brackets (5.8) we can see that non-commutativity is proportional to infinitesimal part of RR field. Non-commutativity relations are zero in case when $\sigma = \bar{\sigma}$, while in case where $\sigma = \bar{\sigma} + 2\pi$ we see the emergence of winding numbers. Noncommutativity parameters are linearly dependent on bosonic coordinates x^μ as well as on fermionic ones, θ^α and $\bar{\theta}^\alpha$.

Taking into account Poisson brackets of T-dual coordinates and expression for sigma derivative of T-dual coordinate we were able to find non-associative relation for T-dual theory. In general case this relation was non-zero and it was proportional to infinitesimal constant, which is proportional to $C^{\alpha\beta}_\mu$. In special case when we put $\sigma_1 = \sigma_2 = \bar{\sigma}$ and $\sigma_3 = \bar{\sigma} + 2\pi$ we noticed that non-associativity relation remains constant.

It should be noted that since we did not preform T-dualization along fermionic coordinates their Poisson structure would remain the same as in original theory. However, unlike original theory, T-dual coordinates depend on sigma derivatives of fermionic coordinates. This dependence does not affect the Poisson brackets of the T-dual coordinates and fermionic coordinates (5.14). So, T-dual SUSY algebra has non zero Poisson bracket of the bosonic coordinates, while the rest ones are zero. In further investigation we will study fermionic T-dualization and we expect the effect on the algebra of the fermionic coordinates.

Appendix A: Obtaining $\beta^{\pm}_{\mu}$ Terms

In this paper function $\beta^{\pm}_{\mu}(V)$ emerged in T-dual transformation laws as a consequence of variation of term that was proportional to ΔV. Here we will present derivation of this function.

Here we will use substitutions $\partial_+\bar{\Theta}^\alpha = \partial_+\bar{\theta}^\alpha + v_+^{\nu_1}\bar{\Psi}^\alpha_{\nu_1}$, $\partial_-\Theta^\beta = \partial_-\theta^\beta + \Psi^\beta_{\nu_2} v_-^{\nu_2}$, also we will use $F_{\alpha\beta\rho}$ to represent term containing infinitesimal constant

$$\int_\Sigma d^2\xi \partial_+\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_-\Theta^\beta = \int_\Sigma d^2\xi \epsilon^{mn}\partial_m\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_n\Theta^\beta$$
$$= \int_\Sigma d^2\xi \Big[\frac{1}{2}\epsilon^{mn}\partial_m\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_n\Theta^\beta - \frac{1}{2}\epsilon^{mn}\partial_n\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Delta V^{(0)\rho}\partial_m\Theta^\beta\Big]$$
$$= -\frac{1}{2}\int_\Sigma d^2\xi \Big[\epsilon^{mn}\bar{\Theta}^\alpha F_{\alpha\beta\rho}\partial_m\Delta V^{(0)\rho}\partial_n\Theta^\beta + \frac{1}{2}\epsilon^{mn}\partial_n\bar{\Theta}^\alpha F_{\alpha\beta\rho}\partial_m\Delta V^{(0)\rho}\Theta^\beta\Big]$$
$$= -\frac{1}{2}\int_\Sigma d^2\xi \epsilon^{mn}\partial_m\Delta V^{(0)\rho}\big[\bar{\Theta}^\alpha F_{\alpha\beta\rho}\partial_n\Theta^\beta - \partial_n\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Theta^\beta\big]$$
$$= -\frac{1}{2}\int_\Sigma d^2\xi \epsilon^{mn}v^\rho_m\big[\bar{\Theta}^\alpha F_{\alpha\beta\rho}\partial_n\Theta^\beta - \partial_n\bar{\Theta}^\alpha F_{\alpha\beta\rho}\Theta^\beta\big]$$
$$= \int_\Sigma d^2\xi v^\rho_m \beta^m_\rho. \tag{A.1}$$

Variation with respect to gauge field $v^\rho_\pm$, and setting $F_{\alpha\beta\rho} = -(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}$ produces desired $\beta^\pm_\rho$ functions (3.14), (3.15) in equations of motion (3.13). Here we have used the property that $(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\mu(f^{-1})_{\beta_1\beta}$, ie. $F_{\alpha\beta\rho}$, is antisymmetric under exchange of α and β, this, in combination with the fact that we can express $\partial_+\bar{\Theta}\partial_-\Theta$ as $\epsilon^{nm}\partial_n\bar{\Theta}\partial_m\Theta$, removes all terms proportional to $\partial_+\partial_-$, using identity $\epsilon^{mn}\bar{\Theta}\partial_m\partial_n\Theta = 0$.

It should be noted that $\beta_{\pm\mu}(V)$ functions are not unique, we could have obtained different function simply by not using using symmetrization in (A.1). In case of non-symmetric $\beta_{\pm\mu}(V)$, all results that have been obtained would take a simpler form. We have chosen to work with symmetric function because results that are deduced from this case can be easily reduced, by neglecting terms, to simpler case.

Appendix B: Poisson Bracket Between Sigma Derivatives of T-dual Coordinate

In this article, in order to find Poisson brackets of the T-dual coordinates we had to find first Poisson brackets of the sigma derivative of T-dual coordinates, $y'_\mu \equiv \partial_\sigma y_\mu(\sigma)$. In this section we will demonstrate how to obtain Poisson brackets from Poisson brackets that contain sigma derivatives. We will use canonical form of the T-dual transformation law (5.6) and standard Poisson algebra, because the initial theory is geometric one. First, we have to calculate the following Poisson bracket

$$\{\partial_{\sigma_1}y_{\nu_1}(\sigma_1), \partial_{\sigma_2}y_{\nu_2}(\sigma_2)\}$$
$$= \frac{1}{2k}\Big[K_{\nu_2\nu_1}(\sigma_2)\partial_{\sigma_2}\delta(\sigma_2-\sigma_1) - K_{\nu_1\nu_2}(\sigma_1)\partial_{\sigma_1}\delta(\sigma_1-\sigma_2) + \partial_{\sigma_1}K_{\nu_1\nu_2}(\sigma_1)\delta(\sigma_1-\sigma_2) - \partial_{\sigma_2}K_{\nu_2\nu_1}(\sigma_2)\delta(\sigma_2-\sigma_1)\Big], \tag{B.1}$$

where $K_{\mu\nu}(\sigma)$ is given by (5.9).

On the other side we have

$$\{\Delta y_{\nu_1}(\sigma_0,\sigma), \Delta y_{\nu_2}(\bar{\sigma}_0,\bar{\sigma})\} = \int_{\sigma_0}^{\sigma} d\sigma_1 \int_{\bar{\sigma}_0}^{\bar{\sigma}} d\sigma_2 \{\partial_{\sigma_1}y_{\nu_1}(\sigma_1), \partial_{\sigma_2}y_{\nu_2}(\sigma_2)\}$$
$$= \{y_{\nu_1}(\sigma), y_{\nu_2}(\bar{\sigma})\} - \{y_{\nu_1}(\sigma), y_{\nu_2}(\bar{\sigma}_0)\} - \{y_{\nu_1}(\sigma_0), y_{\nu_2}(\bar{\sigma})\} + \{y_{\nu_1}(\sigma_0), y_{\nu_2}(\bar{\sigma}_0)\}, \tag{B.2}$$

where

$$\Delta y_\mu(\sigma_0,\sigma) \equiv \int_{\sigma_0}^{\sigma} d\sigma_1 \partial_{\sigma_1}y_\mu(\sigma_1) = y_\mu(\sigma) - y_\mu(\sigma_0). \tag{B.3}$$

Combining the equations (B.1) and (B.2) we have

$$\{\Delta y_{\nu_1}(\sigma_0,\sigma), \Delta y_{\nu_2}(\bar{\sigma}_0,\bar{\sigma})\} = \frac{1}{2k}\int_{\sigma_0}^{\sigma} d\sigma_1 \int_{\bar{\sigma}_0}^{\bar{\sigma}} d\sigma_2 \times \Big[K_{\nu_2\nu_1}(\sigma_2)\partial_{\sigma_2}\delta(\sigma_2-\sigma_1) - K_{\nu_1\nu_2}(\sigma_1)\partial_{\sigma_1}\delta(\sigma_1-\sigma_2) + \partial_{\sigma_1}K_{\nu_1\nu_2}(\sigma_1)\delta(\sigma_1-\sigma_2) - \partial_{\sigma_2}K_{\nu_2\nu_1}(\sigma_2)\delta(\sigma_2-\sigma_1)\Big]. \tag{B.4}$$

By applying partial integration, it is straightforward to extract the Poisson bracket of T-dual coordinates given by (5.8).

In paper[13] it has been shown that Poisson brackets between σ derivatives of coordinates have following form

$$\{\partial_{\sigma_1}X_\mu(\sigma_1), \partial_{\sigma_2}Y_\nu(\sigma_2)\} \cong \partial_{\sigma_1}K_{\mu\nu}(\sigma_1)\delta(\sigma_1-\sigma_2) + L_{\mu\nu}(\sigma_1)\partial_{\sigma_1}\delta(\sigma_1-\sigma_2). \tag{B.5}$$

Applying integrating twice and using partial integration this equation reduces to

$$\{X_\mu(\sigma_1), Y_\nu(\sigma_2)\} \cong -\big[K_{\mu\nu}(\sigma_1) - K_{\mu\nu}(\sigma_2) + L_{\mu\nu}(\sigma_2)\big]\delta(\sigma_1-\sigma_2). \tag{B.6}$$

In our case, we can bring equation (B.1) to the form of equation (B.5) by making following substitutions

$$\begin{aligned} &\partial_{\sigma_1} K_{\nu_1\nu_2}(\sigma_1) = \partial_{\sigma_1} K_{\nu_1\nu_2}(\sigma_1), \qquad \partial_{\sigma_2} K_{\nu_2\nu_1}(\sigma_2) = \partial_{\sigma_2} K_{\nu_2\nu_1}(\sigma_2), \\ &K_{\nu_1\nu_2}(\sigma_1) = -L_{\nu_1\nu_2}(\sigma_1), \qquad K_{\nu_2\nu_1}(\sigma_2) = -L_{\nu_2\nu_1}(\sigma_2). \end{aligned} \tag{B.7}$$

Because we chose to work with symmetric $\beta^{\pm}_{\mu}$ function we obtain duplicated terms in (B.5).

Same procedure can be applied to find Poisson bracket between T-dual coordinate and fermionic momenta. That is, we start from Poisson bracket for sigma derivative of T-dual coordinate and fermionic momenta, then integrate once and compare left and right hand sides.

The step function $H(x)$ is defined as

$$H(x) = \int_0^x ds\delta(s) = \frac{1}{2\pi}\Sigma_{n\in Z}\int_0^x e^{ins} = \begin{cases} 0 & \text{if} \quad x = 0 \\ 1/2 & \text{if} \quad 0 < x < 2\pi \\ 1 & \text{if} \quad x = 2\pi \end{cases}. \tag{B.8}$$

Acknowledgements

Work supported in part by the Serbian Ministry of Education, Science and Technological Development.

Keywords

string theory, noncommutativity, nonassociativity, T-duality

Received: March 27, 2022
Revised: March 27, 2022
Published online: April 12, 2022

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Published for SISSA by Springer

Received: *January 9, 2019*
Revised: *March 1, 2019*
Accepted: *March 11, 2019*
Published: *March 22, 2019*

Directly from H-flux to the family of three nonlocal R-flux theories

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Abstract: In this article we consider T-dualization of the 3D closed bosonic string in the weakly curved background — constant metric and Kalb-Ramond field with one non-zero component, $B_{xy} = Hz$, where field strength H is infinitesimal. We use standard and generalized Buscher T-dualization procedure and make T-dualization starting from coordinate z, via y and finally along x coordinate. All three theories are *nonlocal*, because variable ΔV, defined as line integral, appears as an argument of background fields. After the first T-dualization we obtain commutative and associative theory, while after we T-dualize along y, we get, κ-Minkowski-like, noncommutative and associative theory. At the end of this T-dualization chain we come to the theory which is both noncommutative and nonassociative. The form of the final T-dual action does not depend on the order of T-dualization while noncommutativity and nonassociativity relations could be obtained from those in the $x \to y \to z$ case by replacing $H \to -H$.

Keywords: Bosonic Strings, String Duality

ArXiv ePrint: 1901.01040

https://doi.org/10.1007/JHEP03(2019)136

JHEP03(2019)136

Contents

1 Introduction

Noncommutativity of coordinates has come into focus of physics about hundred years ago when the problem with infinite value of physical quantities occurred. The solution was proposed by Heisenberg in the form of noncommutative coordinates. But after developing of renormalization procedure coordinate noncommutativity was forgotten as a tool for cancelling of infinities.

Commuting of coordinates means that there is no minimal possible length in Nature i.e. that we can measure the position of particle with infinite precision. The return of noncommutativity into physics starts with the article of Hartland Snyder [1]. Usually we treat space-time as continuum but Snyder showed that there is Lorentz invariant discrete space-time. Consequently, this means that commutator of coordinates is nonzero, and noncommutativity parameter dictates the scale at which noncommutativity exists.

In the paper [2] existence of noncommutative manifold was shown using propagators in open bosonic string theory with constant metric and constant Kalb-Ramond field. This result is proven in many articles [3–12] after that but using different mathematical methods. Obtained noncommutativity with constant noncommutativity parameter is known

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in literature as canonical noncommutativity. Consequently, canonical noncommutativity implies that theory is still associative one.

One of the first application of canonical noncommutativity was in Yang-Mills (YM) theories [13–16]. Noncommutative YM theories are constructed and their renormalisability properties are analyzed. It turned out that some processes forbidden in commutative YM are allowed in noncommutative YM theories. Consequently, cross sections for those decays and processes are calculated [17, 18]. Such predictions offer the possibility of indirect check of idea of noncommutativity.

The next type of noncommuatativity which is considered in literature is Lie-algebraic one, which means that commutator of two coordinates is proportional to the coordinate. The κ-Minkowski space-time is an example of this kind of noncommutativity and it is considered in various contexts [19–24]. The κ-Minkowski space is noncommutative but it is easy to check that is associative one. But, in general, if the commutator of the coordinates is proportional to the some linear combination of coordinates, then the space is nonassociative because jacobiator and associator are nonzero. For example, such spaces are closely related to the L_∞ algebra [25].

The mathematical framework for T-dualization is standard Buscher procedure [26, 27]. It consists of the localization of the shift symmetry and adding a term with Lagrange multiplier in order to make gauge fields unphysical degrees of freedom. Also there is an improvement of standard Buscher procedure developed and applied in refs. [28–31], generalized Buscher procedure. In the application of the generalized procedure of T-dualization there is one additional step with respect to the standard one. We introduce invariant coordinate in order to localize shift symmetry in the coordinate dependent backgrounds.

The first articles addressing the subject of coordinate dependent backgrounds appear in the last ten years [32–43]. A 3-torus with constant metric and Kalb-Ramond field with just one nonzero component, $B_{xy} = Hz$, was considered within standard Buscher procedure [33]. Authors made two successive T-dualzation along isometry directions x and y, and, using nontrivial winding conditions, obtained noncommutativity with parameter proportional to field strength H and winding number N_3.

Using generalized T-duality procedure [30, 44] we obtained coordinate dependent noncommutativity and, consequently, nonassociativity. Also it is shown that final theory is nonlocal. In ref. [30] the bosonic string is considered in the weakly curved background — constant metric and linearly coordinate dependent Kalb-Ramond field with infinitesimal field strength, while in [44] we consider the same model as in [33], but T-dualizing along all three directions and imposing trivial winding conditions. Obtained nonlocality comes from the coordinate dependent background, or more precisely, from invariant coordinates. At the end of T-dualization procedure background fields depend on ΔV, defined as line integral. Nonlocality has been become very important issue in the quantum mechanical considerations [45].

In this article we will deal with closed bosonic string propagating in the constant metric and linear dependent Kalb-Ramond field with $B_{xy} = Hz$, the same background as in [33, 44]. But our goal here is to examine the influence of order of T-dualizations. In ref. [44] we T-dualize first along isometry directions, first along x and then along y, and at

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the end, along direction z. The first T-dualization produces configuration known as twisted torus which is commutative, and it is globally and locally well defined. After second T-dualization we obtained nongeometric theory with Q flux which is still locally well defined and it is commutative. The final T-dualization along z direction produces nonlocal theory which is noncommutative and nonassociative one. This line of T-dualizations we will call xyz one.

But what it will happen, if we change the order of T-dualizations, regrading (non)locality issue as well as (non)commutativity and (non)associativity? It is quite obvious that nothing will be changed if we T-dualize along line yxz, because the first two directions, which are T-dualized, are isometry ones. Some nontrivial issues could be expected if we T-dualize first along z direction. In this article we will present T-dualization of the model from [33, 44] along the T-dualization line zyx. After every step of T-dualization we will rewrite the T-dual transformation law in canonical form using the expressions for canonical momenta of the initial theory. Also we will check whether the obtained theory is commutative or not and, consequently, we will see whether it is associative or not.

The fact which is quite sure is that all three theories which we will obtain from the T-dualization line zyx are *nonlocal*. The explanation comes from the fact that background field $B_{\mu\nu}$ is z dependent and according to the generalized T-dualization procedure, after T-dualization along z, we obtain quantity ΔV which is defined as line integral. Consequently, the theory is nonlocal. But because y and x T-dualizations do not affect ΔV, all three theories obtained in zyx T-dualization line are nonlocal. That is a difference with respect to the xyz T-dualization line considered in [44].

The interesting thing is that transformation laws can be obtained from the corresponding ones in [44] by replacing $H \to -H$, but because in this article we T-dualize in the opposite direction, that produces theories of the different commutative and associative features with respect to [44]. After first T-dualization we get commutative and associative theory which is the same as in xyz case from [44]. But the second T-dualization here produces *noncommutative* and associative theory of κ-Minkowski type. That is different with respect to the xyz case, where second theory in the line is both commutative and associative. At the end we obtain the same theory as in [44] which is nonassociative and noncommutative. The noncommutativity and nonassociativity parameters have one additional "$-$" sign comparing with the corresponding ones in [44]. In this article as well as in [44], we impose trivial winding conditions which means $x^\mu(\sigma+2\pi) = x^\mu(\sigma) + 2\pi N^\mu$, where N^μ is a winding number.

At the end we comment some quantum aspects of the problem and add two appendices. The first one contains conventions regarding light-cone coordinates, while the second one is related to the mathematical details concerning derivation of two kinds of Poisson brackets appearing in the article.

2 Bosonic string action and choice of background fields

In this section we will introduce the action for bosonic string propagating in 3D space with constant metric and Kalb-Ramond field which single component is different from zero,

$B_{xy} = Hz$. This model is well known in literature as torus with H-flux. Since we are working with the same model as in [33, 44], for completeness we will repeat most of the steps from introductory part in the [44].

The closed bosonic string which propagates in the presence of the space-time metric $G_{\mu\nu}(x)$, Kalb-Ramond field $B_{\mu\nu}(x)$, and dilaton field $\Phi(x)$ is described by action [46–48]

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\} , \tag{2.1}$$

where world-sheet surface Σ is parameterized by $\xi^\alpha = (\tau, \sigma)$ $[(\alpha = 0, 1),\ \sigma \in (0, \pi)]$, while x^μ $(\mu = 0, 1, 2, \ldots, D-1)$ are space-time coordinates. Intrinsic world sheet metric is denoted by $g_{\alpha\beta}$, and the corresponding scalar curvature with $R^{(2)}$.

Conformal symmetry on the quantum level is not preserved for any choice of background fields. If we want to keep conformal symmetry on the quantum level, background fields must obey the space-time field equations [49]

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2 D_\mu a_\nu = 0 \,, \tag{2.2}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2 a_\rho B^\rho{}_{\mu\nu} = 0 \,, \tag{2.3}$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - R - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = c \,, \tag{2.4}$$

where c is an arbitrary constant. From

$$D^\nu \beta^G_{\nu\mu} + \partial_\mu \beta^\Phi = 0 \,, \tag{2.5}$$

it follows that third beta function, β^Φ, is equal to an arbitrary constant. Here $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to the space-time metric $G_{\mu\nu}$. Field strength for Kalb-Ramond field $B_{\mu\nu}$ and dilaton gradient are defined as

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \,, \quad a_\mu = \partial_\mu \Phi \,. \tag{2.6}$$

One of the solutions of these equations which is important for us here is the solution where some background fields are coordinate dependent. Let us choose Kalb-Ramond field to be linearly coordinate dependent and dilaton field to be constant. The equation (2.2) turns into

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0 \,. \tag{2.7}$$

If we assume that field strength is infinitesimal, then we take $G_{\mu\nu}$ to be constant in approximation linear in $B_{\mu\nu\rho}$. Consequently, the third equation (2.4) is of the form

$$2\pi\kappa \frac{D-26}{6} = c \,. \tag{2.8}$$

The constant c is arbitrary, and fixing its value at $c = -\frac{23\pi\kappa}{3}$, we obtain $D = 3$, dimension of the space in which we will work further.

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The choice of background fields in the case we will consider is

$$G_{\mu\nu} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.9}$$

where $R_\mu(\mu = 1,2,3)$ are radii of the compact dimensions. In terms of radii, the imposed condition that H is infinitesimal, can be rewritten as

$$\left(\frac{H}{R_1 R_2 R_3} \right)^2 = 0\,. \tag{2.10}$$

Physically, infinitesimality of H means that we work with sufficiently large torus (diluted flux approximation). If we rescale the coordinates

$$x^\mu \longmapsto x'^\mu = R_\mu x^\mu\,, \tag{2.11}$$

where indices on the right hand-side of equation are not summed, the form of the metric simplifies

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

Taking all assumption into consideration, the action is of the form

$$\begin{aligned} S &= \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \\ &= \kappa \int_\Sigma d^2\xi \left[\frac{1}{2} \left(\partial_+ x \partial_- x + \partial_+ y \partial_- y + \partial_+ z \partial_- z \right) + \partial_+ x H z \partial_- y - \partial_+ y H z \partial_- x \right], \end{aligned} \tag{2.13}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ is world-sheet derivative with respect to the light-cone coordinates $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$ and

$$x^\mu = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2.14}$$

T-dualization of dilaton is done within quantum formalism and here it will not be presented.

3 Family of three R flux non-local theories

In this section we will perform T-dualization of closed bosonic string equipped by H-flux torus background fields, one direction at time. T-dualization procedure will go along zyx line. We will show that all three theories are nonlocal with R-flux. Also we will find expressions connecting initial and T-dual variables, so called T-dual transformation laws. Using transformation laws in canonical form, we will check after every step whether obtained theory is (non)commutative and/or (non)associative.

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3.1 T-dualization along z direction — shortcut to R-flux

Unlike the cases considered in [33, 44], where T-dualization drives along xyz line, let us do that in opposite direction and perform generalized T-dualization [28] of action (2.13) along z direction.

3.1.1 T-dualization procedure

It looks like that this direction is not isometry one. But we can show that it can be treated like isometry direction. Let us consider the global transformation

$$\delta x^\mu = \lambda^\mu \,, \tag{3.1}$$

and vary the action with respect to this transformation

$$\delta S = \frac{\kappa}{3} B_{\mu\nu\rho}\lambda^\rho \int_\Sigma d^2\xi \partial_+ x^\mu \partial_- x^\nu = \frac{2k}{3} B_{\mu\nu\rho}\lambda^\rho \epsilon^{\alpha\beta} \int_\Sigma d^2\xi [\partial_\alpha (x^\mu \partial_\beta x^\nu) - x^\mu (\partial_\alpha \partial_\beta x^\nu)] \,. \tag{3.2}$$

The second term vanishes as a consequence of contraction of antisymmetric ($\epsilon^{\alpha\beta}$) and symmetric ($\partial_\alpha \partial_\beta$) tensors, while the first one, surface term, survives, and it is, in general, different from zero. But, the expression δS is an topological invariant, so it vanishes if the map from the world-sheet to D-dimensional space-time is topologically trivial. Essentially, infinitesimal field strength H does not affect the vanishing of the surface term.

There is one more explanation of vanishing of this surface term. It is more technical and adjusted to the approximation we used in this article which essence is the explanation in paragraph above. Because we work in the approximation up to the linear terms in H, x^μ satisfies equation of motion for constant $G_{\mu\nu}$ and $B_{\mu\nu}$, $\partial_+\partial_- x^\mu = 0$, which solution is well known in literature. If the winding number is equal to zero, it holds $x^\mu(2\pi + \sigma) = x^\mu(\sigma)$, and since the configuration in the initial τ_i and final moment τ_f is fixed, the surface term vanishes.

So, in the weakly curved background case (H-flux torus background is such like that), z direction is an isometry one. Localization of the shift symmetry of the action (2.13) along z starts with introducing the covariant derivative

$$\partial_\pm z \longrightarrow D_\pm z = \partial_\pm z + v_\pm \,, \tag{3.3}$$

where $v_\pm$ is a gauge field. In order to make gauge fields unphysical ones, we introduce term with Lagrange multiplier

$$S_{add} = \frac{\kappa}{2} \int_\Sigma d^2\xi y_3 (\partial_+ v_- - \partial_- v_+) \,. \tag{3.4}$$

These two steps are the part of the standard Buscher procedure. Because of coordinate dependent background field $B_{\mu\nu}$, generalized T-dualization procedure has an additional step, introducing of an invariant coordinate

$$z^{inv} = \int_P d\xi^\alpha D_\alpha z = \int_P d\xi^+ D_+ z + \int_P d\xi^- D_- z = z(\xi) - z(\xi_0) + \Delta V \,, \tag{3.5}$$

where

$$\Delta V = \int_P d\xi^\alpha v_\alpha = \int_P (d\xi^+ v_+ + d\xi^- v_-)\,. \tag{3.6}$$

The form of the action is now

$$\bar{S} = \kappa \int_\Sigma d^2\xi \left[H z^{inv}(\partial_+ x \partial_- y - \partial_+ y \partial_- x) + \frac{1}{2}(\partial_+ x \partial_- x + \partial_+ y \partial_- y + D_+ z D_- z) \right. \\ \left. + \frac{1}{2}\, y_3(\partial_+ v_- - \partial_- v_+) \right]. \tag{3.7}$$

Fixing the gauge, $z(\xi) = z(\xi_0)$, we get gauged fixed action in the form

$$S_{fix} = \kappa \int_\Sigma d^2\xi \left[H\Delta V(\partial_+ x \partial_- y - \partial_+ y \partial_- x) + \frac{1}{2}(\partial_+ x \partial_- x + \partial_+ y \partial_- y + v_+ v_-) \right. \\ \left. + \frac{1}{2} y_3(\partial_+ v_- - \partial_- v_+) \right]. \tag{3.8}$$

The equation of motion for Lagrange multiplier y_3 obtained from above action (3.8) produces

$$\partial_+ v_- - \partial_- v_+ = 0 \Longrightarrow v_\pm = \partial_\pm z\,, \tag{3.9}$$

which drives us back to the initial action (2.13). On the other side, if we found equations of motion for gauge fields $v_\pm$, we get

$$v_\pm = \pm\partial_\pm y_3 - 2\beta^\mp\,, \tag{3.10}$$

where $\beta^\pm$ functions are defined as

$$\beta^\pm = \mp\frac{1}{2} H(x\partial_\mp y - y\partial_\mp x)\,. \tag{3.11}$$

The $\beta^\pm$ functions stem from the variation of the term containing ΔV. The derivation of beta functions $\beta^\pm$ is based on the relation $\partial_\pm \Delta V = v_\pm$. In the derivation of the beta functions there is one nontrivial technical point and that is vanishing of the surface term after one partial integration. That surface term is of the same form as in eq. (3.2), so the same reasons for surface term vanishing hold here. Mathematical details regarding derivation of $\beta^\pm$ functions can be found in refs. [28–31, 44].

Inserting the relations (3.10) into the gauge fixed action, keeping linear terms in H, we obtain the T-dual action

$${}_zS = \kappa \int_\Sigma d^2\xi \partial_+ {}_zX^\mu {}_z\Pi_{+\mu\nu} \partial_- {}_zX^\nu\,, \tag{3.12}$$

where

$${}_zX^\mu = \begin{pmatrix} x \\ y \\ y_3 \end{pmatrix}, \qquad {}_z\Pi_{+\mu\nu} = {}_zB_{\mu\nu} + \frac{1}{2} {}_zG_{\mu\nu}\,, \tag{3.13}$$

$${}_zB_{\mu\nu} = \begin{pmatrix} 0 & H\Delta V & 0 \\ -H\Delta V & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad {}_zG_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.14}$$

Let us note that presence of ΔV, defined as line integral, represents the source of nonlocality of the T-dual theory.

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3.1.2 T-dual transformation law

Combining the equations of motion for Lagrange multiplier (3.9) and for gauge fields (3.10), we obtain T-dual transformation laws

$$\partial_\pm z \cong \pm\partial_\pm y_3 \mp H(x\partial_\pm y - y\partial_\pm x)\,, \tag{3.15}$$

where $\cong$ is used here to mark T-dual relation. Momentum of the initial theory (2.13) canonically conjugated to the coordinate z is of the form

$$\pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \kappa \dot{z}\,, \tag{3.16}$$

where $\mathcal{L}$ is a Lagrangian density defined as $S = \int_\Sigma d^2\xi \mathcal{L}$. Calculating $\dot{z}$ using T-dual transformation law (3.15), we get the T-dual transformation law in canonical form

$$y_3' \cong \frac{1}{\kappa}\pi_z + H(xy' - yx')\,, \tag{3.17}$$

which is of the same form as in the xyz case.

In all further expressions we will keep the symbol ΔV, but we must have in mind that we used equations of motion for Lagrange multipliers (3.9) at the end of T-dulization procedure along z coordinate, so, having in mind (3.6) and (3.15), we get

$$\Delta V = \Delta z \cong \int d\xi^+ \partial_+ y_3 - \int d\xi^- \partial_- y_3 \equiv \tilde{y}_3\,. \tag{3.18}$$

The variable ΔV is multiplied by infinitesimal field strength H, so, in the above expression we used $\partial_\pm z \cong \pm\partial_\pm y_3$, as a consequence of diluted flux approximation.

3.1.3 (Non)commutativity and (non)associativity

The initial theory is geometric one and its variables satisfy the standard Poisson algebra

$$\{x^\mu(\sigma), x^\nu(\bar{\sigma})\} = \{\pi_\mu(\sigma), \pi_\nu(\bar{\sigma})\} = 0\,, \quad \{x^\mu, \pi_\nu(\bar{\sigma})\} = \delta^\mu{}_\nu \delta(\sigma - \bar{\sigma})\,, \tag{3.19}$$

where x^μ are the coordinates of the initial theory, while π_μ are their canonically conjugated momenta. Using expression (3.17) and standard Poisson algebra (3.19), we obtain that coordinates of the theory obtained after one T-dualization, ${}_zX^\mu$, are commutative. Consequently, Jacobiator is equal to zero, which means that theory is associative.

Summarizing this first step of T-dualization, obtained theory is *commutative and associative nonlocal R-flux theory*. Comparing with the results of the ref. [44] after first T-dualization, qualitatively we obtain the same result, but with the essential difference that here obtained theory is nonlocal R-flux theory unlike that in [44] which is geometrical one, locally and globally well defined.

3.2 Step 2 — T-dualization along y direction

Our starting point is the action given in eq. (3.12). The background fields are independent of y, so, we apply standard Buscher procedure. This means that, unlike the previous case, we perform just first two steps in T-dualization procedure and skip the third one — introducing of invariant coordinate. The T-dualization procedure is already presented, so, we will skip explaining procedure steps further.

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3.2.1 T-dualization procedure

The gauge fixed action is of the form

$$S_{fix} = \kappa \int_{\Sigma} d^2\xi \left[\frac{1}{2} (\partial_+ x \partial_- x + v_+ v_- + \partial_+ y_3 \partial_- y_3) + H\Delta V (v_- \partial_+ x - v_+ \partial_- x) \right]$$
$$+ \frac{\kappa}{2} \int_{\Sigma} d^2\xi y_2 (\partial_+ v_- - \partial_- v_+) \,. \tag{3.20}$$

Varying with respect to the Lagrange multiplier y_2 we get

$$v_\pm = \partial_\pm y \,, \tag{3.21}$$

while the equations of motion for gauge fields are

$$v_\pm = \pm \partial_\pm y_2 \mp 2H\Delta V \partial_\pm x \,. \tag{3.22}$$

Inserting the expression for gauge fields (3.22) into gauge fixed action (3.20), we obtain the T-dual action

$$_{zy}S = \kappa \int_{\Sigma} d^2\xi \; \partial_+ \, _{zy}X^\mu \; _{zy}\Pi_{+\mu\nu} \; \partial_- \, _{zy}X^\nu \,, \tag{3.23}$$

where

$$_{zy}X^\mu = \begin{pmatrix} x \\ y_2 \\ y_3 \end{pmatrix} , \qquad _{zy}\Pi_{+\mu\nu} = \, _{zy}B_{\mu\nu} + \frac{1}{2} \, _{zy}G_{\mu\nu} \,, \tag{3.24}$$

$$_{zy}B_{\mu\nu} = 0 \,, \qquad _{zy}G_{\mu\nu} = \begin{pmatrix} 1 & -2H\Delta V & 0 \\ -2H\Delta V & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.25}$$

Let us note that after two T-dualizations in the xyz case in [44] we also obtained that T-dual Kalb-Ramond field is zero.

3.2.2 T-dual transformation law

Combining equations of motion (3.21) and (3.22) we get the corresponding transformation law

$$\partial_\pm y \cong \pm \partial_\pm y_2 \mp 2H\Delta V \partial_\pm x \,. \tag{3.26}$$

Let us now prescribe the transformation law in canonical form. The momentum canonically conjugated to the initial coordinate y is obtained by variation of the initial action (2.13) with respect to the $\dot{y}$ and it is of the form

$$\pi_y = \kappa (\dot{y} + 2Hzx') \,, \tag{3.27}$$

while from transformation law (3.26) we have

$$\dot{y} \cong y_2' - 2H\Delta V x' \,. \tag{3.28}$$

Combining last two equations and using the fact that, in the approximation linear in H, ΔV and z are T-dual to each other, we get

$$y_2' \cong \frac{1}{\kappa} \pi_y \,. \tag{3.29}$$

As we see the transformation law is the same as in the xyz case.

3.2.3 (Non)commutativity and (non)associativity

In this paragraph we will calculate Poisson brackets of the coordinates ${}_{zy}X^\mu$ using transformation laws in canonical form given by eqs. (3.17) and (3.29).

With the help of the standard Poisson algebra (3.19) and instructions from appendix B, it is easy to see that

$$\{x(\sigma), x(\bar\sigma)\} = \{y_2(\sigma), y_2(\bar\sigma)\} = \{y_3(\sigma), y_3(\bar\sigma)\} = \{x(\sigma), y_2(\bar\sigma)\} = \{x(\sigma), y_3(\bar\sigma)\} = 0\,. \tag{3.30}$$

The only non-zero Poisson bracket is

$$\{y_2'(\sigma), y_3'(\bar\sigma)\} \cong \frac{H}{\kappa}\left[2x'(\sigma)\delta(\sigma-\bar\sigma) + x(\sigma)\delta'(\sigma-\bar\sigma)\right]\,, \tag{3.31}$$

where $\delta' \equiv \partial_\sigma \delta(\sigma-\bar\sigma)$. This result is obtained by straightforward calculation using T-dual transformation laws, (3.17) and (3.29), and standard Poisson algebra (3.19). The relation (3.31) is of the form (B.1), where $A'(\sigma) = y_2'(\sigma)$, $B'(\bar\sigma) = y_3'(\bar\sigma)$, $U'(\sigma) = \frac{H}{\kappa}2x'(\sigma)$ and $V(\sigma) = \frac{H}{\kappa}x(\sigma)$. With these substitutions in mind, we have that final expression is of the form (B.8)

$$\{y_2(\sigma), y_3(\bar\sigma)\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar\sigma)\right]\theta(\sigma-\bar\sigma)\,. \tag{3.32}$$

For $\sigma \to \sigma + 2\pi$ and $\bar\sigma \to \sigma$ we have

$$\{y_2(\sigma+2\pi), y_3(\sigma)\} \cong -\frac{H}{\kappa}\left[x(\sigma) + 4\pi N_x\right]\,, \tag{3.33}$$

because $\theta(2\pi) = 1$ (B.6), while N_x is winding number for x coordinate

$$x(\sigma+2\pi) - x(\sigma) = 2\pi N_x\,. \tag{3.34}$$

As we can see the noncommutativity relation (3.32) is of κ-Minkowski type. It is straightforward to see that

$$\{x(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\} + \{y_2(\sigma_2), \{y_3(\sigma_3), x(\sigma_1)\}\} + \{y_3(\sigma_3), \{x(\sigma_1), y_2(\sigma_2)\}\} \cong 0\,. \tag{3.35}$$

Because the Jacobiator is zero, we conclude that this R-flux theory is **noncommutative** and **associative** one.

3.3 Step 3 — T-dualization along x direction

In this subsection we will finish T-dualization procedure not repeating the mathematical details, but giving just the important equations and results.

The gauge fixed action is given by the following equation

$$S_{fix} = \kappa\int_\Sigma d^2\xi\left[\frac{1}{2}(v_+v_- + \partial_+y_2\partial_-y_2 + \partial_+y_3\partial_-y_3) - H\Delta V(v_+\partial_-y_2 + \partial_+y_2\, v_-)\right] + \frac{\kappa}{2}\int_\Sigma d^2\xi y_1(\partial_+v_- - \partial_-v_+)\,. \tag{3.36}$$

The equations of motion for Lagrange multiplier produces

$$v_\pm = \partial_\pm x \,, \tag{3.37}$$

while the equations of motion for gauge fields $v_\pm$ give

$$v_\pm = \pm\partial_\pm y_1 + 2H\Delta V\, \partial_\pm y_2 \,. \tag{3.38}$$

Inserting expressions for $v_\pm$ into gauge fixed action we get the T-dual action

$${}_{zyx}S = \kappa \int_\Sigma d^2\xi \partial_+\, {}_{zyx}X^\mu\, {}_{zyx}\Pi_{+\mu\nu}\, {}_{zyx}\, X^\nu \,, \tag{3.39}$$

where

$${}_{zyx}X^\mu = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad {}_{zyx}\Pi_{+\mu\nu} = {}_{zyx}B_{\mu\nu} + \frac{1}{2}\, {}_{zyx}G_{\mu\nu} \tag{3.40}$$

$${}_{zyx}B_{\mu\nu} = \begin{pmatrix} 0 & -H\Delta V & 0 \\ H\Delta V & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad {}_{zyx}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.41}$$

Combining the equations of motion (3.37) and (3.38) we obtain the T-dual transformation law

$$\partial_\pm x \cong \pm\partial_\pm y_1 + 2H\Delta V \partial_\pm y_2 \,. \tag{3.42}$$

It directly follows that

$$\dot{x} \cong y_1' + 2H\Delta V \dot{y}_2 \,. \tag{3.43}$$

From the initial action (2.13) it is obvious that momentum canonically conjugated to x is of the form

$$\pi_x = \kappa\dot{x} - 2\kappa H z y' \,. \tag{3.44}$$

The T-dual transformation law for y (3.26), in the approximation linear in H, produces that $y' \cong \dot{y}_2$. Taking into account the relation (3.43), we get the canonical form of the T-dual transformation law

$$y_1' \cong \frac{1}{\kappa}\pi_x \,. \tag{3.45}$$

As we see the full set of T-dual transformation laws, (3.17), (3.29) and (3.45), are the same as in the case where T-dualization was along xyz line [44] up to $H \to -H$. The full T-dualized theory is of the same form as in [44] with the expressions for **noncommutativity**

$$\{y_1(\sigma), y_3(\bar\sigma)\} \cong \frac{H}{\kappa}\left[2y(\sigma) - y(\bar\sigma)\right]\theta(\sigma - \bar\sigma)\,, \tag{3.46}$$

$$\{y_2(\sigma), y_3(\bar\sigma)\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar\sigma)\right]\theta(\sigma - \bar\sigma)\,, \tag{3.47}$$

and **nonassociativity**

$$\begin{aligned}&\{y_1(\sigma_1), y_2(\sigma_2), y_3(\sigma_3)\} \equiv \\ &\{y_1(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\} + \{y_2(\sigma_2), \{y_3(\sigma_3), y_1(\sigma_1)\}\} + \{y_3(\sigma_3), \{y_1(\sigma_1), y_2(\sigma_2)\}\} \cong \\ &\frac{2H}{\kappa^2}\left[\theta(\sigma_1-\sigma_2)\theta(\sigma_2-\sigma_3) + \theta(\sigma_2-\sigma_1)\theta(\sigma_1-\sigma_3) + \theta(\sigma_1-\sigma_3)\theta(\sigma_3-\sigma_2)\right],\end{aligned} \tag{3.48}$$

which can be obtained from the corresponding ones in xyz case [44] by replacing $H \to -H$.

4 Quantum aspects of T-dualization in the weakly curved background

In proving isometry and computing the $\beta^{\pm}$ functions we assumed the trivial topology and the surface term occurring there vanishes. Now we want to discuss some quantum aspects of the considered problems in nontrivial topologies. We will consider the action for bosonic string in the weakly curved background — constant metric and Kalb-Ramond field depending on all coordinates and with infinitesimal field strength. Torus with infinitesimal H-flux is special case of this model.

On th classical level there are a few problems in the theory. In order to perform the generalized T-dualization procedure the invariant coordinate x^{μ}_{inv} is introduced. But it is multivalued and the proof of equivalence of gauged and initial theories needs the part considering global characteristics. Moreover, in the quantum theory at higher genus, the holonomies of the world-sheet gauge fields complicate the situation a little bit. Fortunately, these problems can be resolved in Abelian case in the quantum theory [50–52].

First, we make Wick rotation $\tau \to -i\tau$, which makes the term which contains metric tensor $G_{\mu\nu}$ gets multiplier i, while the terms which contain Kalb-Ramond field $B_{\mu\nu}$ and Lagrange multiplier y_μ stay unchanged. Then the partition function is of the form

$$Z = \sum_{g=0}^{\infty} \int \mathcal{D}y\mathcal{D}v \, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\, {}^\star v + i\kappa \int_\Sigma vB[V]v + \frac{i\kappa}{2}\int_\Sigma v dy} \,. \tag{4.1}$$

We use differential forms and omit the space-time indices to simplify writing of equations. The Hodge duality operator is denoted by star. The index g denotes the genus of manifold.

The first step in the calculation process is separation the one form dy into the exact part dy_e (y_e is single valued) and the harmonic part y_h ($dy_h = 0 = d^\dagger y_h$)

$$dy = dy_e + y_h. \tag{4.2}$$

For the closed forms the co-exact term $d^\dagger y_{co}$ in the Hodge decomposition is missing.

The path integral (4.1) goes over all degrees of freedom including local degrees of freedom as well as the sum over different topologies. Consequently, according to the (4.2), we substitute $\mathcal{D}y$ with the path integral over y_e and the sum over all possible topologically nontrivial states contained in y_h (marked by H_y)

$$\mathcal{D}y \to \mathcal{D}y_e \sum_{H_y} . \tag{4.3}$$

The integration over y_e induces vanishing of the field strength

$$Z = \int \mathcal{D}v \, \delta(dv) \, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\, {}^\star v + i\kappa \int_\Sigma vB[V]v} \sum_{H_y} e^{\frac{i\kappa}{2}\int_\Sigma v y_h}. \tag{4.4}$$

The 1-form v can be expressed as sum of exact, co-exact and the harmonic parts

$$v = dx + d^\dagger v_{ce} + v_h, \tag{4.5}$$

which means that

$$\mathcal{D}v \to \mathcal{D}x\mathcal{D}d^\dagger v_{ce}\, dH_v. \tag{4.6}$$

The functional integration over harmonic part v_h drives to the ordinary integration over topologically nontrivial periods (marked by symbol H_v). After integration over $d^\dagger v_{ce}$ we get

$$Z = \int \mathcal{D}x dH_v\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G^\star v + i\kappa \int_\Sigma vB[V]v} \sum_{H_y} e^{\frac{i\kappa}{2}\int_\Sigma v y_h}. \tag{4.7}$$

The last term in the exponent is responsible for nontrivial holonomies. Eliminating v_{ce} part, the 1-form v becomes closed and the Riemann bilinear relation becomes usable

$$\int_\Sigma v y_h = \sum_{i=1}^{g} \Big[\oint_{a_i} v \oint_{b_i} y_h - \oint_{a_i} y_h \oint_{b_i} v \Big]. \tag{4.8}$$

The symbols $a_i, b_i\, (i = 1, 2, \ldots, g)$ represent the canonical homology basis for the world-sheet. Because of the periodicity of the Lagrange multiplier y, its periods are just the winding numbers around cycles a_i and b_i

$$N_{a_i} = \oint_{a_i} y_h, \quad N_{b_i} = \oint_{b_i} y_h\, . \tag{4.9}$$

Denoting the periods with

$$A_i = \oint_{a_i} v, \quad B_i = \oint_{b_i} v\, , \tag{4.10}$$

we get

$$\int_\Sigma v y_h = \sum_{i=1}^{g} (N_{b_i} A_i - N_{a_i} B_i). \tag{4.11}$$

Now the partition function (4.7) gets the form

$$Z = \int \mathcal{D}x\, dA_i dB_i e^{-\frac{\kappa}{2}\int_\Sigma v\, G^\star v + i\kappa\int_\Sigma vB[V]v} \sum_{N_{a_i}, N_{b_i} \in Z} e^{\frac{i\kappa}{2}\sum_{i=1}^{g}(N_{b_i}A_i - N_{a_i}B_i)}. \tag{4.12}$$

The periodic delta function is defined as $\delta(x) = \frac{1}{2\pi}\sum_{n\in Z} e^{inx}$, which produces

$$Z = \int \mathcal{D}x\, dA_i dB_i \delta\left(\frac{\kappa}{2}A_i\right)\delta\left(\frac{\kappa}{2}B_i\right)\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G^\star v + i\kappa\int_\Sigma vB[V]v}\, . \tag{4.13}$$

It is useful to examine the path dependence of the variable V^μ, which form is now

$$V^\mu(\xi) = x^\mu(\xi) - x^\mu(\xi_0) + \int_P v_h^\mu. \tag{4.14}$$

Let us consider two paths, P_1 and P_2, with the same initial ξ_0^α and the final points ξ^α. Now we will subtract from the value of V^μ along P_1 the value along path P_2 and obtain the integral over closed curve $P_1 P_2^{-1}$ of the harmonic form

$$V^\mu[P_1] - V^\mu[P_2] = \oint_{P_1 P_2^{-1}} v_h^\mu. \tag{4.15}$$

Establishing the homology between the closed curve $P_1P_2^{-1}$ and curve $\sum_i \left[n_i a_i + m_i b_i\right]$, $(n_i, m_i \in Z)$ we get finally

$$V^\mu[P_1] = V^\mu[P_2] + \sum_i (n_i A_i^\mu + m_i B_i^\mu). \tag{4.16}$$

The variable $V^\mu(\xi)$ in classical theory is path dependent if holonomies are nontrivial.

Integrating eq. (4.13) over A_i and B_i implies that periods A_i and B_i are zero. Consequently

$$v = dx. \tag{4.17}$$

The variable V^μ becomes single valued, and the initial theory is restored

$$Z = \int \mathcal{D}x e^{-\frac{\kappa}{2}\int_\Sigma dx\, G \star dx + i\kappa \int_\Sigma dx B[x] dx} = \int \mathcal{D}x e^{-\kappa \int_\Sigma d^2\xi \partial x \Pi_+[x] \bar{\partial} x}. \tag{4.18}$$

Consequently, starting with partition function of the gauged fixed action of bosonic string in the weakly curved background, within path integral formalism and in the presence of nontrivial topologies, we came to the partition function of the initial theory. That means that introducing coordinate dependent Kalb-Ramond field is consistent with path integral quantization process.

5 Conclusion

In this article we studied the 3D closed bosonic string propagating in the geometry known as torus with H-flux — constant metric and Kalb-Ramond field with just one nonzero component, $B_{xy} = -B_{yx} = Hz$. The choice of background fields is consistent with the consistency conditions if we work in the diluted flux approximation which assumes that in all calculations we keep just the constant terms and those linear in the infinitesimal field strength H. Our goal was to study the T-dualization line which goes in the opposite direction from the standard one. First, we T-dualize z direction, then y and at the end along x direction — so-called zyx T-dualization line. We analyzed in every step the (non)commutativity and (non)associativity of the obtained theory and made comparisons with the case of xyz T-dualization line considered in [33, 44].

The common fact for all three theories obtained in the process of T-dualization step by step is that all three ones are nonlocal R-flux theories. The nonlocality comes as a result of the first step in T-dualization procedure, T-dualization along z direction. Generalized T-dualization procedure has one additional step with respect to the standard Buscher procedure and that is introduction of invariant coordinate. In the process of T-dualization invariant coordinate turns into variable ΔV which is defined as line integral. Consequently, this means that obtained theory is nonlocal. Further T-dualizations does not affect ΔV and, all three theories are nonlocal ones. As we know, in the case of xyz T-dualization line [44], we obtained three different theories in geometrical sense — twisted torus, Q-flux theory (which is local) and nonlocal R-flux theory.

The dualization along z direction produces nonlocal R-flux theory unlike the xyz case [33, 44] where the theory obtained after first T-dualization is locally and globally well defined. Because initial theory is geometrical one, its variables satisfy standard Pois-

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son algebra (3.19). Using (3.19) and T-dual transformation law written in the canonical form (3.17), we showed that theory obtained after T-dualization along z coordinate (using generalized T-dualization procedure) is *commutative* and, consequently, *associative* one as in [44].

The second step in T-dualization is T-dualization along y direction. Using standard Buscher procedure, we obtained the form of the T-dual theory and the corresponding T-dual transformation law, which is rewritten in the canonical form (3.29) in terms of the coordinates and momenta of the initial theory. Using standard Poisson algebra (3.19) and T-dual transformation laws in canonical form, (3.17) and (3.29), we easily proved that theory after two T-dualizations is *noncommutative*, but it is still *associative* one. In this article we used trivial winding condition (3.34) and showed that T-dual coordinates $y_2(\sigma)$ and $y_3(\bar{\sigma})$ are commutative for equal arguments, $\sigma = \bar{\sigma}$, but they are noncommutative if $\sigma - \bar{\sigma} = 2\pi$. The result is qualitatively similar to the result of [33], where after two T-dualizations the obtained theory is noncommutative one. But, the difference is in the winding condition which is nontrivial in [33], mixing different coordinates. The different winding condition induces the noncommutativity for $\sigma = \bar{\sigma}$ (for more details see [33]). On the other hand in the analysis presented in [44] (xyz T-dualization line) the theory obtained after two T-dualizations is commutative under trivial winding condition.

The final step in T-dualization procedure is T-dualization along x direction. The theory after full T-dualization is the same as in xyz case [44] with the noncommutativity and nonassociativity parameters which can be obtained from those in xyz case [44] adding "$-$" sign. This is a consequence of the fact that the full set of T-dual transformation laws is the same as in [44] up to the replacing $H \to -H$. This difference up to the "$-$" sign stems from the initial actions. In this article we start from (2.13), while in [44] the starting action for z T-dualization is Q-flux action, formally the same as (2.13) up to the replacing $H \to -H$.

Finishing the discussion of the results obtained in this paper it is interesting to make comparison with some similar efforts. We studied the abelian isometries using both standard and generalized T-duality procedure, while in the paper [53] nonabelian isometries using standard Buscher procedure are considered. The authors of [53] showed that spaces with isometry maps to the nonisometry spaces, while in this paper there is isometry in every T- dualization step. One of their conclusions that T-dual transformations are more than continuous isometry can be added to the concluding remarks of this paper. In the ref. [43] generalized T-duality and nongeometric background are considered, but using low energy effective action, unlike here, where we used sigma model action. The paper [54] deals with T-dualizations along nonisometry directions like in [31], using extension of gauge symmetry, while the authors of [31] use the generalized T-dualization procedure introducing invariant coordinates (in [54] they call them "covariant" coordinates). In this paper we use this generalized T-dualization procedure but all directions considered here are isometry ones. It is useful to mention that in the paper [55] bosonic string in the presence of the weakly curved backgrounds is considered using double space formalism as well as the influence of the order of T-dualizations. The double space formalism gives the result which is in accordance with the result of the current paper.

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Consequently, we conclude that in the case of the full T-dualization the form of the T-dual theory do not depend on the order of T-dualization, while parameters of noncommutativity and nonassociativity change sign.

A Light-cone coordinates

In the paper we often use light-cone coordinates defined as

$$\xi^{\pm} = \frac{1}{2}(\tau \pm \sigma)\,. \tag{A.1}$$

The corresponding partial derivatives are

$$\partial_{\pm} \equiv \frac{\partial}{\partial \xi^{\pm}} = \partial_\tau \pm \partial_\sigma\,. \tag{A.2}$$

Two dimensional Levi-Civita $\varepsilon^{\alpha\beta}$ is chosen in (τ,σ) basis as $\varepsilon^{\tau\sigma} = -1$. Consequently, in the light-cone basis the form of tensor is

$$\varepsilon^{\alpha\beta}_{lc} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{A.3}$$

The flat world-sheet metric is of the form in (τ,σ) and light-cone basis, respectively

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad \eta^{lc}_{\alpha\beta} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{A.4}$$

Let us stress that in whole article we use standard notation for τ and σ derivatives — $\dot{A} \equiv \partial_\tau A$ and $A' \equiv \partial_\sigma A$, where A is an arbitrary variable.

B Two types of Poisson brackets used in the paper

In this paper, we have seen that T-dual transformation laws connect derivatives of T-dual coordinates with coordinates and momenta of initial theory. While initial theory satisfies standard Poisson brackets, in order to find Poisson brackets for T-dual theory, we first need to find Poisson brackets between σ derivatives of T-dual coordinates. This type of Poisson bracket will, in general case, be some function of initial coordinates, Dirac delta functions and their derivatives with respect to σ. Having this in mind, general case for our Poisson brackets will have following form

$$\{A'(\sigma), B'(\bar{\sigma})\} = U'(\sigma)\delta(\sigma - \bar{\sigma}) + V(\sigma)\delta'(\sigma - \bar{\sigma})\,, \tag{B.1}$$

where $\delta'(\sigma - \bar{\sigma}) \equiv \partial_\sigma \delta(\sigma - \bar{\sigma})$. For terms $A'(\sigma)$, $U'(\sigma)$ and $B'(\bar{\sigma})$, symbol $'$ stands for partial derivative with respect to σ and $\bar{\sigma}$, respectively. If we want to calculate the Poisson bracket

$$\{A(\sigma), B(\bar{\sigma})\}\,,$$

first we have to calculate the following one

$$\{\Delta A(\sigma, \sigma_0), \Delta B(\bar{\sigma}, \bar{\sigma}_0)\}\,, \tag{B.2}$$

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where

$$\Delta A(\sigma,\sigma_0)=\int_{\sigma_0}^{\sigma}dxA'(x)=A(\sigma)-A(\sigma_0)\,,\quad \Delta B(\bar\sigma,\bar\sigma_0)=\int_{\bar\sigma_0}^{\bar\sigma}dxB'(x)=B(\bar\sigma)-B(\bar\sigma_0)\,. \tag{B.3}$$

Substituting the expressions (B.3) into (B.2), we have

$$\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=\int_{\sigma_0}^{\sigma}dx\int_{\bar\sigma_0}^{\bar\sigma}dy\;\left[U'(x)\delta(x-y)+V(x)\delta'(x-y)\right]\,. \tag{B.4}$$

After integration over y we get

$$\begin{aligned}&\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=\\&=\int_{\sigma_0}^{\sigma}dx\{U'(x)\left[\theta(x-\bar\sigma_0)-\theta(x-\bar\sigma)\right]+V(x)\left[\delta(x-\bar\sigma_0)-\delta(x-\bar\sigma)\right]\},\end{aligned} \tag{B.5}$$

where $\theta(x)$ is defined as

$$\theta(x)=\int_0^x d\eta\delta(\eta)=\frac{1}{2\pi}\left[x+2\sum_{n\geq1}\frac{1}{n}\sin(nx)\right]=\begin{cases}0 & \text{if } x=0\\ 1/2 & \text{if } 0<x<2\pi\,,\\ 1 & \text{if } x=2\pi\end{cases} \tag{B.6}$$

where $\delta(x)=\frac{1}{2\pi}\sum_{n\in Z}e^{inx}$. Finally, integrating over x, we obtain

$$\begin{aligned}\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=&\,U(\sigma)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma-\bar\sigma)]-U(\sigma_0)[\theta(\sigma_0-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma)]\\&-U(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0)]+U(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)]\\&+V(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0]-V(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)].\end{aligned} \tag{B.7}$$

From the last expression, using (B.3), we extract the searched Poisson bracket

$$\{A(\sigma),B(\bar\sigma)\}=-[U(\sigma)-U(\bar\sigma)+V(\bar\sigma)]\theta(\sigma-\bar\sigma)\,. \tag{B.8}$$

In order to calculate Jacobiator we have to find Poisson brackets of type $\{y(\sigma),x(\bar\sigma)\}$, where $y(\sigma)$ is coordinate T-dual to initial one $x(\sigma)$. Having this in mind, we start with the following Poisson bracket

$$\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}=\left\{\int_{\sigma_0}^{\sigma}d\eta y'(\eta),x(\bar\sigma)\right\}\,, \tag{B.9}$$

and using T-dual transformation law in canonical form

$$\pi\cong\kappa y'\,, \tag{B.10}$$

we get

$$\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}\cong\frac{1}{\kappa}\left\{\int_{\sigma_0}^{\sigma}d\eta\pi(\eta),x(\bar\sigma)\right\}\,, \tag{B.11}$$

where $\pi(\sigma)$ is momentum canonically conjugated to the coordinate $x(\sigma)$. Initial theory is geometric one which variables satisfy standard Poisson algebra, so, the final result is of the form

$$\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}\cong-\frac{1}{\kappa}\left[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)\right]\quad\Longrightarrow\{y(\sigma),x(\bar\sigma)\}\cong-\frac{1}{\kappa}\theta(\sigma-\bar\sigma)\,. \tag{B.12}$$

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Acknowledgments

Work supported in part by the Serbian Ministry of Education, Science and Technological Development, under contract No. 171031. I also want to thank to Prof. Dr. Branislav Sazdović and Dr. Ljubica Davidović from Institute of Physics Belgrade for useful discussions.

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JHEP03(2019)136

Published for SISSA by Springer

Received: *November 5, 2022*
Accepted: *December 5, 2022*
Published: *December 14, 2022*

Noncommutativity and nonassociativity of type II superstring with coordinate dependent RR field — the general case

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Abstract: In this paper we consider non-commutativity that arises from T-duality of bosonic coordinates of type II superstring in presence of coordinate dependent Ramond-Ramond field. Action with such choice of the background fields is not translational invariant. Consequently, we will employ generalization of Buscher procedure that can be applied to cases that have coordinate dependent fields and that do not possess translational isometry. Bosonic part of newly obtained T-dual theory is non-local and defined in non-geometric double space spanned by Lagrange multipliers y_μ and double coordinate ΔV^μ. We will apply Buscher procedure once more on T-dual theory to check if original theory can be salvaged. Finally, we will use T-dual transformation laws along with Poisson brackets of original theory to derive Poisson bracket structure of T-dual theory.

Keywords: String Duality, Superstrings and Heterotic Strings

ArXiv ePrint: 2211.00455

https://doi.org/10.1007/JHEP12(2022)078

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Contents

1 Introduction

String theory as a possible candidate for unification of all known interactions offers a framework for description of both gauge interactions and gravity. Analyzing the relation between world-sheet diffeomorphisms and transformations of the background fields for open bosonic string [1–3] it is concluded that Kalb-Ramond field gets one additional term that is in fact a field strength of some gauge field. In sigma-model action it looks like that gauge fields are attached at the string endpoints moving along Dp-brane. Finiteness of the gauge theories (UV cutoff) demands existence of some minimal length. Consequently, noncommutativity naturally arose in open bosonic string theory in the presence of the constant background fields [4–7].

The fact that noncommutativity appears together with gauge theory produces a new line of investigation in quantum field theory — noncommutative gauge theories [8–10], but not noncommutative gravity.

The open bosonic string in the presence of the constant background fields gives constant noncommutativity [4–7], but, consequently, Jacobi identity is zero and associativity is not broken. Closed string in the presence of the constant background fields remains commutative.

Noncommutativity in open string theory comes from the boundary conditions (see [5–7]). Coordinates and their canonically conjugated momenta are mixed in the boundary conditions, and because they obey standard Poisson algebra, at the end we get noncommutativity of the initial coordinates. All these facts tell us about the way how we can reach noncommutativity in the bosonic closed string case. Kalb-Ramond field must be, at least, linearly coordinate dependent, and the generalized T-dualization procedure is a machinery [11–19]. The noncommutativity relations are coordinate dependent and produce nonzero Jacobi identity — nonassociativity appears in the closed string theory [13, 18, 20–22].

The noncommutativity and nonassociativity can be considered also within superstring theory [23, 24]. In ref. [25] we considered one special case of the type II superstring theory in pure spinor formulation [26–30] — all physical background fields are constant except Ramond-Ramond (RR) field strength. The RR field strength consists of the constant part and linearly coordinate dependent one, which is infinitesimal. In accordance with consistency conditions, we have chosen constant part of RR field strength to be symmetric and cooordinate dependent part to be antisymmetric tensor.

The motivation for this choice of background fields is the quest for the anticommutation relation between fermionic coordinates suggested in [30, 31]. Formally, this case is similar to the bosonic string case with coordinate dependent Kalb-Ramond field (weakly curved background). The difference is that in the superstring case noncommutativity parameter depends both on the bosonic and fermionic coordinates. Also we obtained that Jacobiator is nonzero. Both noncommutativity and nonassociativity parameters are proportional to the infinitesimal tensor from RR field strength.

In this article we consider the same action as in [25], but we will not imply the additional restrictions on the constant and coordinate dependent part of RR field strength as in [25]. The fundamental difference in relation to the choice of background field in [25] is in the fact that action with RR field strength without restrictions does not possess translational isometry. In that sense this case can be considered as general one comparing with [25].

We will use the generalized T-dualization procedure [16, 19] along bosonic directions. Because this general case cannot be deduced to the form of the bosonic string with linearly dependent Kalb-Ramond field, as it could in the case [25], we obtained more complicated form of T-dual transformation laws and, consequently, the generalization of β_μ functions in the form of $N(\xi)$ functions. Besides the complexity of the T-dual transformation laws we succeeded to find expressions for noncommutativity and nonassociativity as well as the form of the T-dual theory.

At the end we give some concluding remarks. In the appendices we present the derivation of $N(\xi)$ functions and show their properties.

2 General type II superstring action and choice of background fields

In this section we will shortly present how we derive the action of type II superstring in pure spinor formulation with all constant background fields except RR field strength from the general form of that action given in [26–30].

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2.1 General form of the pure spinor type II superstring action

The general form of the type II superstring action in pure spinor formalism is derived and given in [30]. It consists of two parts and can be represent as their sum

$$S = S_0 + V_{SG}\,, \tag{2.1}$$

where S_0 describes the motion of string in flat background

$$S_0 = \int_\Sigma d^2\xi \left(\frac{k}{2}\eta_{\mu\nu}\partial_m x^\mu \partial_n x^\nu \eta^{mn} - \pi_\alpha \partial_- \theta^\alpha + \partial_+ \bar\theta^\alpha \bar\pi_\alpha \right) + S_\lambda + S_{\bar\lambda}\,, \tag{2.2}$$

while the second one contains all possible interactions

$$V_{SG} = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar X^N. \tag{2.3}$$

The second part of the action is expressed in terms of the integrated form of massless type II supergravity vertex operator V_{SG}. The actions S_λ and $S_{\bar\lambda}$ in (2.2) are free-field actions for pure spinors

$$S_\lambda = \int d^2\xi \omega_\alpha \partial_- \lambda^\alpha, \quad S_{\bar\lambda} = \int d^2\xi \bar\omega_\alpha \partial_+ \bar\lambda^\alpha\,. \tag{2.4}$$

Here, λ^α and $\bar\lambda^\alpha$ are pure spinors whose canonically conjugated momenta are ω_α and $\bar\omega_\alpha$, respectively.

The vectors X^M and X^N and matrix A_{MN} are of the form

$$X^M = \begin{pmatrix} \partial_+\theta^\alpha \\ \Pi_+^\mu \\ d_\alpha \\ \frac{1}{2}N_+^{\mu\nu} \end{pmatrix}, \quad \bar X^M = \begin{pmatrix} \partial_-\bar\theta^\lambda \\ \Pi_-^\mu \\ \bar d_\lambda \\ \frac{1}{2}\bar N_-^{\mu\nu} \end{pmatrix}, \quad A_{MN} = \begin{bmatrix} A_{\alpha\beta} & A_{\alpha\nu} & E_\alpha{}^\beta & \Omega_{\alpha,\mu\nu} \\ A_{\mu\beta} & A_{\mu\nu} & \bar E_\mu^\beta & \Omega_{\mu,\nu\rho} \\ E^\alpha{}_\beta & E_\nu^\alpha & P^{\alpha\beta} & C^\alpha{}_{\mu\nu} \\ \Omega_{\mu\nu,\beta} & \Omega_{\mu\nu,\rho} & \bar C^\beta{}_{\mu\nu} & S_{\mu\nu,\rho\sigma} \end{bmatrix}, \tag{2.5}$$

where notation is taken from refs. [25, 30]. Every component of the matrix A_{MN} is function of bosonic, x^μ, and fermionic, θ^α and $\bar\theta^\alpha$, coordinates. For more details about derivation of the components consult [30]. The superfields $A_{\mu\nu}$, $\bar E_\mu^\alpha$, E_μ^α and $P^{\alpha\beta}$ are known as physical superfields, superfields that are in the first row and the first column are known as auxiliary because they can be expressed in terms of physical ones [30]. Remaining superfields $\Omega_{\mu,\nu\rho}$ ($\Omega_{\mu\nu,\rho}$), $C^\alpha{}_{\mu\nu}$ ($\bar C^\beta{}_{\mu\nu}$) and $S_{\mu\nu,\rho\sigma}$, are curvatures (field strengths) for physical fields. Components of X^M and $\bar X^N$ are of the form

$$\Pi_+^\mu = \partial_+ x^\mu + \frac{1}{2}\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\partial_+\theta^\beta, \qquad \Pi_-^\mu = \partial_- x^\mu + \frac{1}{2}\bar\theta^\alpha (\Gamma^\mu)_{\alpha\beta}\partial_-\bar\theta^\beta, \tag{2.6}$$

$$d_\alpha = \pi_\alpha - \frac{1}{2}(\Gamma_\mu\theta)_\alpha \left[\partial_+ x^\mu + \frac{1}{4}(\theta\Gamma^\mu\partial_+\theta)\right],$$

$$\bar d_\alpha = \bar\pi_\alpha - \frac{1}{2}(\Gamma_\mu\bar\theta)_\alpha \left[\partial_- x^\mu + \frac{1}{4}(\bar\theta\Gamma^\mu\partial_-\bar\theta)\right], \tag{2.7}$$

$$N_+^{\mu\nu} = \frac{1}{2}\omega_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \lambda^\beta, \qquad \bar N_-^{\mu\nu} = \frac{1}{2}\bar\omega_\alpha (\Gamma^{[\mu\nu]})^\alpha{}_\beta \bar\lambda^\beta. \tag{2.8}$$

The world-sheet is spanned by $\xi^m = (\xi^0 = \tau, \xi^1 = \sigma)$, while world-sheet light-cone partial derivatives are defined as $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Superspace contains bosonic x^μ ($\mu = 0, 1, \dots, 9$) and fermionic θ^α, $\bar\theta^\alpha$ ($\alpha = 1, 2, \dots, 16$) coordinates. Variables π_α and $\bar\pi_\alpha$ are canonically conjugated momenta to the fermionic coordinates θ^α and $\bar\theta^\alpha$, respectively.

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2.2 Choice of the background fields

In this particular case we will use the supermatrix A_{MN} where all physical background fields, except RR field strength $P^{\alpha\beta}$, are constant. RR fields strength will have linear coordinate dependence on bosonic coordinate x^μ. Consequently, supermatrix A_{MN} is of the following form

$$A_{MN} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k(\frac{1}{2}g_{\mu\nu} + B_{\mu\nu}) & \bar{\Psi}^\beta_\mu & 0 \\ 0 & -\Psi^\alpha_\nu & \frac{2}{k}(f^{\alpha\beta} + C^{\alpha\beta}_\rho x^\rho) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.9}$$

where $g_{\mu\nu}$ is symmetric tensor, $B_{\mu\nu}$ is Kalb-Ramond antisymmetric field, Ψ^α_μ and $\bar{\Psi}^\alpha_\mu$ are Mayorana-Weyl gravitino fields and $f^{\alpha\beta}$ and $C^{\alpha\beta}_\rho$ are constant tensors. Let us stress this will be a classical analysis and we will not calculate the dilaton shift under T-duality transformation.

From the consistency conditions given in ref. [30], following this choice of background fields, it follows

$$\gamma^\mu_{\alpha\beta} C^{\beta\gamma}_\mu = 0, \quad \gamma^\mu_{\alpha\beta} C^{\gamma\beta}_\mu = 0. \tag{2.10}$$

Because all background fields are expanded in powers of θ^α and $\bar{\theta}^\alpha$, θ^α and $\bar{\theta}^\alpha$ terms in X^M and $\bar{X}^N$ will be neglected. Taking into account all imposed assumptions and approximations, the full action S is getting the form

$$\begin{aligned} S = \int_\Sigma d^2\xi \Big[& \frac{k}{2}\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu - \pi_\alpha(\partial_-\theta^\alpha + \Psi^\alpha_\nu \partial_- x^\nu) + (\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu \bar{\Psi}^\alpha_\mu)\bar{\pi}_\alpha \\ & + \frac{2}{k}\pi_\alpha (f^{\alpha\beta} + C^{\alpha\beta}_\rho x^\rho)\bar{\pi}_\beta \Big], \end{aligned} \tag{2.11}$$

where $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}$, and $G_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu}$ is metric tensor. The actions S_λ and $S_{\bar{\lambda}}$ are fully decoupled from the rest and they will not be analyzed from now on.

It is easy to notice that fermionic momenta play the roles of the auxiliary fields in full action. They can be integrated out finding equations for motion for both π_α and $\bar{\pi}_\alpha$

$$\bar{\pi}_\beta = \frac{k}{2}\left(F^{-1}(x)\right)_{\beta\alpha} (\partial_-\theta^\alpha + \Psi^\alpha_\nu \partial_- x^\nu), \tag{2.12}$$

$$\pi_\alpha = -\frac{k}{2}\left(\partial_+\bar{\theta}^\beta + \partial_+ x^\mu \bar{\Psi}^\beta_\mu\right)\left(F^{-1}(x)\right)_{\beta\alpha}, \tag{2.13}$$

where $F^{\alpha\beta}(x)$ and $(F^{-1}(x))_{\alpha\beta}$ are of the form

$$F^{\alpha\beta}(x) = f^{\alpha\beta} + C^{\alpha\beta}_\mu x^\mu, \quad (F^{-1}(x))_{\alpha\beta} = (f^{-1})_{\alpha\beta} - (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\rho x^\rho (f^{-1})_{\beta_1\beta}. \tag{2.14}$$

For practical reasons, we assume that $C^{\alpha\beta}{}_\mu$ is infinitesimal. This assumption is in accordance with constraints (2.10). Substituting equations (2.12) and (2.13) into (2.11) the final form of action is

$$S = k\int_\Sigma d^2\xi \left[\Pi_{+\mu\nu}\partial_+ x^\mu \partial_- x^\nu + \frac{1}{2}(\partial_+\bar{\theta}^\alpha + \partial_+ x^\mu \bar{\Psi}^\alpha_\mu)\left(F^{-1}(x)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi^\beta_\nu \partial_- x^\nu)\right]. \tag{2.15}$$

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Let us note that we did not impose any conditions on tensors $f^{\alpha\beta}$ and $C_\mu^{\alpha\beta}$ as we did in ref. [25]. The case considered in this article is more general than the case studied in [25], because action does not possess translational symmetry.

3 T-dualization

Here we will make T-dualization of all bosonic directions aiming to find T-dual transformation laws — relations between T-dual coordinates and canonical variables of the original theory. The T-dual transformation laws will be used to calculate Poisson brackets of the T-dual coordinates.

3.1 Implementation of the generalized T-dualization procedure

In implementing of T-dualization procedure we will use *generalized Buscher T-dualization procedure* [16]. The standard Buscher procedure [14, 15] is made to be used along directions on which background fields do not depend (isometry directions), while generalized Buscher procedure can be applied to theories with coordinate dependent background fields along all directions. The generalized T-dualization procedure follows three steps — introduction of covariant derivatives, invariant coordinates and additional gauge fields, which produces additional degrees of freedom. The starting and T-dual theory must have the same number of degrees of freedom. In order to achieve that we eliminate all excessive degrees of freedom demanding that field strength of gauge fields ($F_{+-} = \partial_+ v_- - \partial_- v_+$) vanishes by addition of Lagrange multipliers. Then we fix the gauge symmetry (shift symmetry) and action is left with gauge fields and their derivatives. Finding equations of motion for gauge fields, expressing in terms of the Lagrange multipliers and inserting those equations into action we obtain T-dual action, where Lagrange multipliers have roles of T-dual coordinates.

T-duality can be performed also in the cases of the absence of shift symmetry [19]. Then we replace original action with translation invariant auxiliary action. Form of the auxiliary action is exactly the same as the form of action where translation symmetry was localized and gauge fixed. It produces correct T-dual theory only if original action can be salvaged from it.

Action (2.15) is not translational invariant. Consequently, we make the following substitutions

$$\partial_\pm x^\mu \to v_\pm^\mu, \tag{3.1}$$

$$x^\rho \to \Delta V^\rho = \int_P d\xi'^m v_m^\rho(\xi'), \tag{3.2}$$

$$S \to S + \frac{k}{2}\int_\Sigma d^2\xi \left[v_+^\mu \partial_- y_\mu - v_-^\mu \partial_+ y_\mu\right], \tag{3.3}$$

and insert them in action (2.15). The result is auxiliary action convenient for T-dualization procedure

$$\begin{aligned} S_{aux} = k\int_\Sigma d^2\xi \Big[&\Pi_{+\mu\nu} v_+^\mu v_-^\nu + \frac{1}{2}(\partial_+\bar\theta^\alpha + v_+^\mu \bar\Psi_\mu^\alpha)\left(F^{-1}(\Delta V)\right)_{\alpha\beta}(\partial_-\theta^\beta + \Psi_\nu^\beta v_-^\nu) \\ &+\frac{1}{2}(v_+^\mu \partial_- y_\mu - v_-^\mu \partial_+ y_\mu)\Big]. \end{aligned} \tag{3.4}$$

Let us note that path P starts from ξ_0 and ends in ξ. In this way action becomes non-local.

Finding equations of motion for Lagrange multipliers

$$\partial_- v_+^\mu - \partial_+ v_-^\mu = 0 \quad v_\pm^\mu = \partial_\pm x^\mu, \tag{3.5}$$

and inserting them into (3.2) we have

$$\Delta V^\rho = \int_P d\xi'^m \partial_m x^\rho(\xi') = x^\rho(\xi) - x^\rho(\xi_0) = \Delta x^\rho. \tag{3.6}$$

In absence of translational symmetry, in order to extract starting action from auxiliary one, we impose $x^\rho(\xi_0) = 0$ as a constraint. Taking all this into account, we get the starting action (2.15).

Euler-Lagrange equations of motion for gauge fields $v_\pm(\kappa)$ give the following ones

$$-\frac{1}{2}\partial_- y_\mu(\kappa) = \Pi_{+\mu\nu} v_-^\nu(\kappa) + \frac{1}{2}\bar{\Psi}_\mu^\alpha \left(F^{-1}(\Delta V)\right)_{\alpha\beta} (\partial_-\theta^\beta(\kappa) + \Psi_\nu^\beta v_-^\nu(\kappa)) \tag{3.7}$$
$$-\frac{1}{2}\int_\Sigma d^2\xi [\partial_+\bar{\theta}^\alpha(\xi) + v_+^{\nu_1}(\xi)\bar{\Psi}_{\nu_1}^\alpha](f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1} (f^{-1})_{\beta_1\beta} N(\kappa^+)[\partial_-\theta^\beta(\xi) + \Psi_{\nu_2}^\beta v_-^{\nu_2}(\xi)],$$
$$\frac{1}{2}\partial_+ y_\mu(\kappa) = \Pi_{+\nu\mu} v_+^\nu(\kappa) + \frac{1}{2}(\partial_+\bar{\theta}^\alpha(\kappa) + v_+^\nu(\kappa)\bar{\Psi}_\nu^\alpha) \left(F^{-1}(\Delta V)\right)_{\alpha\beta} \Psi_\mu^\beta \tag{3.8}$$
$$-\frac{1}{2}\int_\Sigma d^2\xi [\partial_+\bar{\theta}^\alpha(\xi) + v_+^{\nu_1}(\xi)\bar{\Psi}_{\nu_1}^\alpha](f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1} (f^{-1})_{\beta_1\beta} N(\kappa^-)[\partial_-\theta^\beta(\xi) + \Psi_{\nu_2}^\beta v_-^{\nu_2}(\xi)].$$

Here, function $N(\kappa^\pm)$ is obtained from variation of term containing ΔV^ρ in expression for $F^{-1}(\Delta V)$ (details are presented in appendix A). They represent the generalization of beta functions introduced in ref. [25]

$$N(\kappa^+) = \delta\Big(\xi'^-((\xi'^+)^{-1}(\kappa^+)) - \kappa^-\Big)\left[H(\xi^+ - \kappa^+) - H(\xi_0^+ - \kappa^+)\right], \tag{3.9}$$
$$N(\kappa^-) = \delta\Big(\xi'^+((\xi'^-)^{-1}(\kappa^-)) - \kappa^+\Big)\left[H(\xi^- - \kappa^-) - H(\xi_0^- - \kappa^-)\right], \tag{3.10}$$

where more details on Dirac delta function and step function are given in appendix A. As we see the expressions for derivatives of y_μ are more complex comparing with those in [25], where translational symmetry is present.

Assuming that $C_\mu^{\alpha\beta}$ is an infinitesimal, we can iteratively invert equations of motion (3.7) and (3.8) [20]. Separating variables into two parts, one finite and one infinitesimal proportional to $C_\mu^{\alpha\beta}$, we have

$$\begin{aligned}
v_-^\nu(\kappa) = &-\frac{1}{2}\bar{\Theta}_-^{\nu\nu_1}\Big\{\partial_- y_{\nu_1}(\kappa) + \bar{\Psi}_{\nu_1}^\alpha (F^{-1}(\Delta V))_{\alpha\beta}\partial_-\theta^\beta(\kappa) \\
&+\frac{1}{2}\Psi_{\nu_1}^\alpha (f^{-1})_{\alpha\alpha_1} C_\rho^{\alpha_1\alpha_2}\Delta V^\rho (f^{-1})_{\alpha_2\alpha_3}\Psi_{\nu_2}^{\alpha_3}\bar{\Theta}_-^{\nu_2\nu_3}\Big(\partial_- y_{\nu_3}(\kappa) + \bar{\Psi}_{\nu_3}^{\beta_1}(f^{-1})_{\beta_1\beta}\partial_-\theta^\beta(\kappa)\Big) \\
&-\int_\Sigma d^2\xi\Big[\partial_+\bar{\theta}^\alpha(\xi) + \frac{1}{2}\Big(\partial_+ y_{\mu_1}(\xi) - \partial_+\bar{\theta}^{\gamma_1}(\xi)(f^{-1})_{\gamma_1\gamma_2}\Psi_{\mu_1}^{\gamma_2}\Big)\bar{\Theta}_-^{\mu_1\nu_1}\bar{\Psi}_{\nu_1}^\alpha\Big](f^{-1})_{\alpha\alpha_1} C_{\nu_1}^{\alpha_1\beta_1} \\
&\times (f^{-1})_{\beta_1\beta} N(\kappa^+)\Big[\partial_-\theta^\beta(\xi) - \frac{1}{2}\Psi_{\nu_2}^\beta\bar{\Theta}_-^{\nu_2\mu_2}\Big(\partial_- y_{\mu_2}(\xi) + \bar{\Psi}_{\mu_2}^{\gamma_3}(f^{-1})_{\gamma_3\gamma_4}\partial_-\theta^{\gamma_4}(\xi)\Big)\Big]\Big\},
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
v_+^\mu(\kappa) = &\frac{1}{2}\bar{\Theta}_-^{\mu_1\mu}\Big\{\partial_+ y_{\mu_1}(\kappa) - \partial_+\bar{\theta}^\alpha(\kappa)(F^{-1}(\Delta V))_{\alpha\beta}\Psi_{\mu_1}^\beta \\
&+\frac{1}{2}\Big(\partial_+ y_{\mu_2}(\kappa) - \partial_+\bar{\theta}^\alpha(\kappa)(f^{-1})_{\alpha\alpha_1}\Psi_{\mu_2}^{\alpha_1}\Big)\bar{\Theta}_-^{\mu_2\mu_3}\bar{\Psi}_{\mu_3}^{\beta_3}(f^{-1})_{\beta_3\beta_2} C_\rho^{\beta_2\beta_1}\Delta V^\rho (f^{-1})_{\beta_1\beta}\Psi_{\mu_1}^\beta \\
&+\int_\Sigma d^2\xi\Big[\partial_+\bar{\theta}^\alpha(\xi) + \frac{1}{2}\Big(\partial_+ y_{\mu_2}(\xi) - \partial_+\bar{\theta}^{\gamma_1}(\xi)(f^{-1})_{\gamma_1\gamma_2}\Psi_{\mu_2}^{\gamma_2}\Big)\bar{\Theta}_-^{\mu_2\nu_1}\bar{\Psi}_{\nu_1}^\alpha\Big](f^{-1})_{\alpha\alpha_1} C_{\mu_1}^{\alpha_1\beta_1} \\
&\times (f^{-1})_{\beta_1\beta} N(\kappa^-)\Big[\partial_-\theta^\beta(\xi) - \frac{1}{2}\Psi_{\nu_2}^\beta\bar{\Theta}_-^{\nu_2\mu_3}\Big(\partial_- y_{\mu_3}(\xi) + \bar{\Psi}_{\mu_3}^{\gamma_3}(f^{-1})_{\gamma_3\gamma_4}\partial_-\theta^{\gamma_4}(\xi)\Big)\Big]\Big\}.
\end{aligned} \tag{3.12}$$

Tensor $\bar{\Theta}_-^{\mu\nu}$ is inverse tensor to $\bar{\Pi}_{+\mu\nu} = \Pi_{+\mu\nu} + \frac{1}{2}\bar{\Psi}_\mu^\alpha (f^{-1})_{\alpha\beta}\Psi_\nu^\beta$

$$\bar{\Theta}_-^{\mu\nu}\bar{\Pi}_{+\nu\rho} = \delta_\rho^\mu, \tag{3.13}$$

where

$$\bar{\Theta}_-^{\mu\nu} = \Theta_-^{\mu\nu} - \frac{1}{2}\Theta_-^{\mu\mu_1}\bar{\Psi}_{\mu_1}^\alpha (\bar{f}^{-1})_{\alpha\beta}\Psi_{\nu_1}^\beta\Theta_-^{\nu_1\nu}, \tag{3.14}$$

$$\bar{f}^{\alpha\beta} = f^{\alpha\beta} + \frac{1}{2}\Psi_\mu^\alpha\Theta_-^{\mu\nu}\bar{\Psi}_\nu^\beta, \tag{3.15}$$

$$\Theta_-^{\mu\nu}\Pi_{+\mu\rho} = \delta_\rho^\mu, \qquad \Theta_- = -4(G_E^{-1}\Pi_- G^{-1})^{\mu\nu}. \tag{3.16}$$

Effective metric tensor is defined as $G_{E\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$.

In above expressions ΔV is a quantity in the zeroth order in $C_\mu^{\alpha\beta}$

$$\begin{aligned}
\Delta V^\rho &= \int d\xi^+ v_+^\rho + \int d\xi^- v_-^\rho \\
&= \frac{1}{2}\int_P d\xi^+\bar{\Theta}_-^{\rho_1\rho}\Big[\partial_+ y_{\rho_1} - \partial_+\bar{\theta}^\alpha (f^{-1})_{\alpha\beta}\Psi_{\rho_1}^\beta\Big] - \frac{1}{2}\int_P d\xi^-\bar{\Theta}_-^{\rho\rho_1}\Big[\partial_- y_{\rho_1} + \bar{\Psi}_{\rho_1}^\alpha (f^{-1})_{\alpha\beta}\partial_-\theta^\beta\Big].
\end{aligned} \tag{3.17}$$

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Using (3.7) and (3.8) and inserting them into (3.4), we get T-dual action

$$\begin{aligned}
S_{T-dual} = k\int_P d^2\xi\Bigg[&\frac{1}{4}\bar\Theta_-^{\mu\nu}\partial_+y_\mu\partial_-y_\nu \qquad (3.18)\\
+&\frac{1}{8}\bar\Theta_-^{\mu\mu_1}\bar\Psi^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_1}\bar\Theta_-^{\nu_1\nu}\partial_+y_\mu\partial_-y_\nu\\
+&\frac{1}{2}\partial_+\bar\theta^\alpha\Big((F^{-1}(\Delta V))_{\alpha\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\alpha_2}\Delta V^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\\
&-\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_3}_\nu(f^{-1})_{\beta_3\beta_2}C_\rho^{\beta_2\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta}\\
&-\frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\mu_1}\bar\Psi^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_3}\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\bar\Theta_-^{\nu_1\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\Big)\partial_-\theta^\beta\\
+&\frac{1}{4}\partial_+y_\mu\bar\Theta_-^{\mu\mu_1}\bar\Psi^\alpha_{\mu_1}\Big((F^{-1}(\Delta V))_{\alpha\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_3}\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\bar\Theta_-^{\nu_1\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\Big)\\
&\times\partial_-\theta^\beta\\
-&\frac{1}{4}\partial_+\bar\theta^\alpha\Big((F^{-1}(\Delta V))_{\alpha\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\mu_1}\bar\Psi^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta}\Big)\Psi^\beta_{\nu_1}\bar\Theta_-^{\nu_1\nu}\\
&\times\partial_-y_\nu\Bigg].
\end{aligned}$$

Let us note that above we kept terms up to the first order in $C_\mu^{\alpha\beta}$.

T-dual action contains all terms as initial action (2.15) up to the change $x^\mu\to y_\mu$. Consequently, T-dual background fields are of the form

$$\begin{aligned}
{}^\star\Pi_+^{\mu\nu}=&\frac{1}{4}\bar\Theta_-^{\mu\nu}+\frac{1}{8}\bar\Theta_-^{\mu\mu_1}\bar\Psi^\alpha_{\mu_1}\Big[(F^{-1}(\Delta V))_{\alpha\beta}+(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta} \qquad (3.19)\\
&-\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_{\mu_2}\bar\Theta_-^{\mu_2\nu_2}\bar\Psi^{\beta_1}_{\nu_2}(f^{-1})_{\beta_1\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\alpha_2}\Delta V^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_{\mu_2}\bar\Theta_-^{\mu_2\nu_2}\bar\Psi^{\beta_1}_{\nu_2}(f^{-1})_{\beta_1\beta}\\
&+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_{\mu_2}\bar\Theta_-^{\mu_2\nu_2}\bar\Psi^{\beta_3}_{\nu_2}(f^{-1})_{\beta_3\beta_2}C_\rho^{\beta_2\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta}\\
&-\frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_{\mu_2}\bar\Theta_-^{\mu_2\mu_3}\bar\Psi^{\alpha_2}_{\mu_3}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_3}\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_3}\bar\Theta_-^{\nu_3\nu_2}\bar\Psi^{\beta_1}_{\nu_2}(f^{-1})_{\beta_1\beta}\Big]\Psi^\beta_{\nu_1}\bar\Theta_-^{\nu_1\nu},\\
{}^\star(F^{-1}(x))_{\alpha\beta}=&(F^{-1}(\Delta\bar y))_{\alpha\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\alpha_2}\Delta V^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta} \qquad (3.20)\\
&-\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}+\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\nu}\bar\Psi^{\beta_3}_\nu(f^{-1})_{\beta_3\beta_2}C_\rho^{\beta_2\beta_1}\Delta V^\rho(f^{-1})_{\beta_1\beta}\\
&-\frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar\Theta_-^{\mu\mu_1}\bar\Psi^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_3}\Delta V^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\bar\Theta_-^{\nu_1\nu}\bar\Psi^{\beta_1}_\nu(f^{-1})_{\beta_1\beta},\\
{}^\star\bar\Psi^{\mu\alpha}=&\frac{1}{2}\Theta_-^{\mu\nu}\bar\Psi^\alpha_\nu, \qquad {}^\star\Psi_{\nu\beta}=-\frac{1}{2}\Psi^\beta_\mu\Theta_-^{\mu\nu}. \qquad (3.21)
\end{aligned}$$

Comparing background field of T-dual theory with background fields from [25] we immediately notice that background fields have become more complex. However, this is just an illusion. In both cases background field are exactly the same only difference is that here we did not introduce tensor $\breve\Pi_{+\mu\nu}=\Pi_{+\mu\nu}+\frac{1}{2}\bar\Psi^\alpha_\mu F^{-1}(\Delta V)_{\alpha\beta}\Psi^\beta_\nu$ and its inverse, therefore we are missing ingredients to express our fields in more compactified format.

3.2 T-dualization of T-dual theory

Since the initial theory is not symmetric under translations, T-dual action that is obtained from auxiliary action (3.4) and it is now invariant to translations of T-dual coordinates. Consequently, we can dualize T-dual theory by generalized Buscher procedure. We start with the introduction of following substitutions

$$\partial_\pm y_\mu \to D_\pm y_\mu = \partial_\pm y_\mu + u_{\pm\mu} \to D_\pm y_\mu = u_{\pm\mu}, \tag{3.22}$$

$$\Delta\bar{y}^\rho \to \Delta\bar{u}^\rho, \tag{3.23}$$

$$\begin{aligned}\Delta\bar{u}^\rho = &\frac{1}{2}\int_P d\xi^+ \bar{\Theta}_-^{\rho_1\rho}\left[u_{+\rho_1} - \partial_+\bar{\theta}^\alpha (f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] \\ &-\frac{1}{2}\int_P d\xi^- \bar{\Theta}_-^{\rho\rho_1}\left[u_{-\rho_1} + \bar{\Psi}^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right],\end{aligned} \tag{3.24}$$

$$S \to S + \frac{1}{2}(u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu)\,. \tag{3.25}$$

From the first line we see that gauge is fixed choosing $y(\xi) = const.$ Inserting these substitutions into (3.18) we obtain

$$\begin{aligned}S_{\text{gauge fix}} = \kappa\int_P d^2\xi\Bigg[&\frac{1}{4}\bar{\Theta}_-^{\mu\nu}u_{+\mu}u_{-\nu} \\ &+\frac{1}{8}\bar{\Theta}_-^{\mu\mu_1}\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}\Delta\bar{u}^\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_1}\bar{\Theta}_-^{\nu_1\nu}u_{+\mu}u_{-\nu} \\ &+\frac{1}{2}\partial_+\bar{\theta}^\alpha\Big((F^{-1}(\Delta\bar{u}))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\alpha_2}\Delta\bar{u}^\rho(f^{-1})_{\alpha_2\alpha_3}\Psi^{\alpha_3}_\mu\bar{\Theta}_-^{\mu\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta} \\ &\quad-\frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar{\Theta}_-^{\mu\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar{\Theta}_-^{\mu\nu}\bar{\Psi}^{\beta_3}_\nu(f^{-1})_{\beta_3\beta_2}C_\rho^{\beta_2\beta_1}\Delta\bar{u}^\rho(f^{-1})_{\beta_1\beta} \\ &\quad-\frac{1}{4}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar{\Theta}_-^{\mu\mu_1}\bar{\Psi}^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_3}\Delta\bar{u}^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\bar{\Theta}_-^{\nu_1\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\Big)\partial_-\theta^\beta \\ &+\frac{1}{4}u_{+\mu}\bar{\Theta}_-^{\mu\mu_1}\bar{\Psi}^\alpha_{\mu_1}\Big((F^{-1}(\Delta\bar{u}))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_3}\Delta\bar{u}^\rho(f^{-1})_{\beta_3\beta_2}\Psi^{\beta_2}_{\nu_1}\bar{\Theta}_-^{\nu_1\nu}\bar{\Psi}^{\beta_1}_\nu(f^{-1})_{\beta_1\beta}\Big) \\ &\quad\times\partial_-\theta^\beta \\ &-\frac{1}{4}\partial_+\bar{\theta}^\alpha\Big((F^{-1}(\Delta\bar{u}))_{\alpha\beta} + \frac{1}{2}(f^{-1})_{\alpha\alpha_1}\Psi^{\alpha_1}_\mu\bar{\Theta}_-^{\mu\mu_1}\bar{\Psi}^{\alpha_2}_{\mu_1}(f^{-1})_{\alpha_2\alpha_3}C_\rho^{\alpha_3\beta_1}\Delta\bar{u}^\rho(f^{-1})_{\beta_1\beta}\Big)\Psi^\beta_{\nu_1}\bar{\Theta}_-^{\nu_1\nu} \\ &\quad\times u_{-\nu} \\ &+\frac{1}{2}(u_{+\mu}\partial_- x^\mu - u_{-\mu}\partial_+ x^\mu)\Bigg].\end{aligned} \tag{3.26}$$

Using equations of motion for Lagrange multipliers, we return to the T-dual action. Finding equations of motion for gauge fields, we have

$$\begin{aligned}u_{+\mu}(\kappa) = \;&2\bar{\Pi}_{+\nu\mu}\partial_+ x^\nu(\kappa) - \partial_+ x^\nu(\kappa)\bar{\Psi}^\alpha_\nu(f^{-1})_{\alpha\alpha_1}C_\rho^{\alpha_1\beta_1}\Delta x^\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_\mu \\ &+\partial_+\bar{\theta}^\alpha(\kappa)(F^{-1}(\Delta\bar{x}))_{\alpha\beta}\Psi^\beta_\mu \\ &-\int_\Sigma d^2\xi\Big(\partial_+\bar{\theta}^\alpha(\xi) + \partial_+ x^{\mu_1}(\xi)\bar{\Psi}^\alpha_{\mu_1}\Big)(f^{-1})_{\alpha\alpha_1}C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}N(\kappa^-) \\ &\times\Big(\partial_-\theta^\beta(\xi) + \Psi^\beta_\nu\partial_- x^\nu(\xi)\Big),\end{aligned} \tag{3.27}$$

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$$
\begin{aligned}
u_{-\nu}(\kappa) = & -2\bar{\Pi}_{+\nu\mu}\partial_- x^\mu(\kappa) + \bar{\Psi}^\alpha_\nu (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\rho \Delta x^\rho (f^{-1})_{\beta_1\beta}\Psi^\beta_\mu \partial_- x^\mu(\kappa) \\
& - \bar{\Psi}^\alpha_\nu (F^{-1}(\Delta\bar{x}))_{\alpha\beta}\partial_-\theta^\beta(\kappa) \\
& + \int_\Sigma d^2\xi \Big(\partial_+\bar{\theta}^\alpha(\xi) + \partial_+ x^\mu(\xi)\bar{\Psi}^\alpha_\mu\Big)(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\nu (f^{-1})_{\beta_1\beta} N(\kappa^-) \\
& \times \Big(\partial_-\theta^\beta(\xi) + \Psi^\beta_{\nu_1}\partial_- x^{\nu_1}(\xi)\Big)\,.
\end{aligned}
\tag{3.28}
$$

Here we have that $\Delta x^\mu = x(\xi) - x(\xi_0)$, and inserting these equations into the gauge fixed action, keeping all terms linear with respect to $C^{\mu\nu}_\rho$ and selecting ξ_0 such that $x(\xi_0) = 0$, we obtain our original action (2.15).

4 Non-commutative relations

In this section we will establish a relationship between Poisson brackets of original and T-dual theory using results from the previous one. Original theory is a geometric one, which means that canonical variables $x^\mu(\xi)$ and $\pi_\mu(\xi)$ satisfy standard Poisson algebra

$$
\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu_\nu \delta(\sigma - \bar{\sigma}), \quad \{x^\mu(\sigma), x^\nu(\bar{\sigma})\} = 0, \quad \{\pi_\mu(\sigma), \pi_\nu(\bar{\sigma})\} = 0. \tag{4.1}
$$

We will find Poisson structure of T-dual theory using relations (3.7) and (3.8) and expressing them in terms of the coordinates and momenta of the initial theory. Replacing gauge fields with solutions of equations of motion for Lagrange multipliers, we get T-dual transformation laws in Lagrangian form. Because we implement here canonical approach, the next step is removing of all terms that are proportional to $\partial_\tau x^\mu(\xi)$. The most of the terms of this type will get incorporated into expression for canonical momenta $\pi_\mu(\xi)$, but term that is non-local and which is dependent on function $N(\xi^\pm)$ remains. One way of removing this term is to first use equations of motion for coordinate $x^\mu(\xi)$, and then replace remaining $\partial_\tau x^\mu$ term with canonical momentum. By doing all the steps that were outlined, we have following relationship between T-dual coordinate and variables of starting theory

$$
\begin{aligned}
\partial_\sigma y_\nu(\sigma) = & \, 2B_{\nu\mu}\partial_\sigma x^\mu - G_{\nu\mu}(\bar{\Pi}_+ + \bar{\Pi}^T_+)^{-1\nu_1\mu}\Bigg[\frac{\pi_{\nu_1}}{k} - \frac{1}{2}\bar{\Psi}^\alpha_{\nu_1}\left(F^{-1}(x)\right)_{\alpha\beta}\partial_-\theta^\beta \\
& - \frac{1}{2}\partial_+\bar{\theta}^\alpha \left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_{\nu_1} - \left[\Pi_{+\mu_1\mu_2} + \frac{1}{2}\bar{\Psi}^\alpha_{\mu_1}\left(F^{-1}(x)\right)_{\alpha\beta}\Psi^\beta_{\mu_2}\right](\delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2})\partial_\sigma x^{\nu_2} \\
& + \frac{1}{2}\bar{\Psi}^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho x^\rho(\sigma)(f^{-1})_{\beta_1\beta}\Psi^\beta_{\mu_2}(\delta^{\mu_1}_{\nu_2}\delta^{\mu_2}_{\nu_1} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2})(\bar{\Pi}_+ + \bar{\Pi}^T_+)^{-1\rho\nu_2} \\
& \times \left[\frac{\pi_\rho}{k} - \frac{1}{2}\bar{\Psi}^\gamma_\rho(f^{-1})_{\gamma\gamma_1}\partial_-\theta^{\gamma_1} - \frac{1}{2}\partial_+\bar{\theta}^\gamma(f^{-1})_{\gamma\gamma_1}\Psi^{\gamma_1}_\rho + \bar{\Pi}_{+\rho\rho_1}\partial_\sigma x^{\rho_1} - \bar{\Pi}_{+\rho_1\rho}\partial_\sigma x^{\rho_1}\right]\Bigg].
\end{aligned}
\tag{4.2}
$$

To find Poisson bracket between T-dual coordinates, we can start by finding Poisson bracket of sigma derivatives of T-dual coordinates and then integrating twice (see [25]).

Implementing this procedure we have that Poisson bracket for sigma derivatives is given as

$$\begin{aligned}
&\{\partial_{\sigma_1} y_{\nu_1}(\sigma_1), \partial_{\sigma_2} y_{\nu_2}(\sigma_2)\} = \qquad (4.3)\\
&= \frac{2}{k}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_1\mu_2}\Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}\partial_{\sigma_2}\delta(\sigma_1-\sigma_2) - B_{\nu_1\mu_1}G_{\nu_2\mu_2}\partial_{\sigma_1}\delta(\sigma_1-\sigma_2)\Big]\\
&\quad + \frac{1}{k}\bar{\Psi}^\alpha_{\nu_3}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_3\mu_1}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_4\mu_2}\\
&\quad \times \Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}x^\rho(\sigma_1)\partial_{\sigma_2}\delta(\sigma_1-\sigma_2) - B_{\nu_1\mu_1}G_{\nu_2\mu_2}x^\rho(\sigma_2)\partial_{\sigma_1}\delta(\sigma_1-\sigma_2)\Big].
\end{aligned}$$

Then we integrate with respect to σ_1 (σ_2), where we set boundaries as σ_0 ($\bar{\sigma}_0$) and σ ($\bar{\sigma}$). Extracting only Poisson bracket terms that contain σ and $\bar{\sigma}$, we have

$$\begin{aligned}
&\{y_{\nu_1}(\sigma), y_{\nu_2}(\bar{\sigma})\} = \frac{2}{k}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_1\mu_2}\Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2} + B_{\nu_1\mu_1}G_{\nu_2\mu_2}\Big]H(\sigma-\bar{\sigma}) \qquad (4.4)\\
&+ \frac{1}{k}\bar{\Psi}^\alpha_{\nu_3}(f^{-1})_{\alpha\alpha_1}G^{\alpha_1\beta_1}_\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_3\mu_1}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_4\mu_2}\\
&\times \Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}x^\rho(\bar{\sigma}) + B_{\nu_1\mu_1}G_{\nu_2\mu_2}x^\rho(\sigma)\Big]H(\sigma-\bar{\sigma}).
\end{aligned}$$

Here, $H(\sigma-\bar{\sigma})$ is same step function defined in appendix A. It should be noted that these Poisson brackets are zero when $\sigma = \bar{\sigma}$. However, in cases where string in curled around compactified dimension, that is cases where $\sigma - \bar{\sigma} = 2\pi$, we have following situation

$$\begin{aligned}
&\{y_{\nu_1}(\sigma+2\pi), y_{\nu_2}(\sigma)\} = \frac{2}{k}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_1\mu_2}\Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2} + B_{\nu_1\mu_1}G_{\nu_2\mu_2}\Big] \qquad (4.5)\\
&+ \frac{1}{k}\bar{\Psi}^\alpha_{\nu_3}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_3\mu_1}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_4\mu_2}\\
&\times \Big[4\pi G_{\nu_1\mu_1}B_{\nu_2\mu_2}N^\rho + (G_{\nu_1\mu_1}B_{\nu_2\mu_2} + B_{\nu_1\mu_1}G_{\nu_2\mu_2})x^\rho(\sigma)\Big].
\end{aligned}$$

We used fact that $H(2\pi) = 1$. The symbol N^μ denotes winding number around compactified coordinate, if is defined as

$$x^\mu(\sigma+2\pi) - x^\mu(\sigma) = 2\pi N^\mu. \qquad (4.6)$$

Let us note that if we choose $x^\mu(\sigma) = 0$ than Poisson bracket has linear dependence on winding number. In cases where we don't have any winding number, we still have non-commutativity that is proportional to background fields.

Using the expression for sigma derivative of y_ν (4.2) and expression for Poisson bracket of sigma derivatives (4.3), we can find non-associative relations. Procedure is the same as for finding Poisson brackets of T-dual theory, we find Poisson bracket of sigma derivatives and integrate with respect to sigma coordinate, this time integral is done trice. Going along with this procedure we have following final result

$$\begin{aligned}
&\{y_\nu(\sigma), \{y_{\nu_1}(\sigma_1), y_{\nu_2}(\sigma_2)\}\} = \frac{G_{\nu\mu}}{k^2}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\rho\mu} \qquad (4.7)\\
&\times \bar{\Psi}^\alpha_{\nu_3}(f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_4}(\delta^{\nu_3}_{\mu_3}\delta^{\nu_4}_{\mu_4} + \delta^{\nu_4}_{\mu_3}\delta^{\nu_3}_{\mu_4})(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_3\mu_1}(\bar{\Pi}_+ + \bar{\Pi}_+^T)^{-1\mu_4\mu_2}\\
&\times \Big[G_{\nu_1\mu_1}B_{\nu_2\mu_2}H(\sigma-\sigma_2) + B_{\nu_1\mu_1}G_{\nu_2\mu_2}H(\sigma-\sigma_1)\Big]H(\sigma_1-\sigma_2).
\end{aligned}$$

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Since Jacobi identity is non-zero for T-dual theory we have that coordinate dependent RR field produces non-associative theory. However putting $\sigma_1 = \sigma_2 = \bar{\sigma}$ and $\sigma = \bar{\sigma} + 2\pi$ we have that Jacobi identity disappears

$$\{y_\nu(\bar{\sigma} + 2\pi), \{y_{\nu_1}(\bar{\sigma}), y_{\nu_2}(\bar{\sigma})\}\} = 0. \tag{4.8}$$

5 Conclusion

In this article we examined type II superstring propagating in presence of coordinate dependent RR field. This choice of background was in accordance with consistency conditions for background field and all calculations were made in approximation that are linear with respect to coordinate dependent part of RR field. We have also excluded parts that were non-linear in fermionic coordinates and neglected pure spinor actions. Using equations of motion for fermionic momenta we obtained action that was expressed in terms of bosonic coordinates, their derivatives and derivatives of fermionic coordinates. Unlike [25] we do not impose any conditions on the constant and coordinate dependent part of RR field strength, so, it is not possible to deduce this case to the form of the weakly curved background one.

Action with our choice of background fields did not possess translation symmetry, therefore we needed to use Buscher procedure that was extended to such cases. By substituting starting action with auxiliary action we gave up on locality in order to be able to find T-dual theory. Finding equations of motion of newly introduced Lagrange multipliers we were able to salvage starting action giving us assurance that auxiliary action we selected would produce correct T-dual theory. After this we found equations of motion for gauge fields and by inserting them into action, we found T-dual theory.

Having found T-dual theory, we applied T-dual procedure once again as a more thorough way of checking if action we obtained was in fact correct T-dual of starting action. Unlike starting action, T-dual action possessed translation symmetry and was non-local from the start by virtue of having dual coordinate ΔV^μ. Applying steps of generalized Buscher procedure [16, 19] we obtained starting action, again confirming that our choice of auxiliary action was correct.

We obtained non-commutativity relations in context of T-dual theory, where we used T-dual transformation laws as a bridge between Poisson brackets of starting theory and T-dual theory. T-dual transformation laws were expressed as functions of coordinates and momenta of original theory and using their standard Poisson algebra, we got non-commutativity in T-dual theory. From expression for Poisson brackets (4.4) we can see that non-commutativity is proportional to infinitesimal part of RR field as well as to symmetrised inverse of field $\bar{\Pi}$. Non-commutativity relations are zero in case when $\sigma = \bar{\sigma}$, while in case where $\sigma = \bar{\sigma} + 2\pi$ we see the emergence of winding numbers.

Taking into account Poisson brackets of sigma derivatives and expression for sigma derivative of T-dual coordinate we were able to find non-associative relation for T-dual theory. In general case this relation was non-zero and it was proportional to infinitesimal constant $C^{\mu\nu}_\rho$. In special case when we put $\sigma_1 = \sigma_2 = \bar{\sigma}$ and $\sigma = \bar{\sigma} + 2\pi$ we noticed that non-associativity relation disappears. During the implementation of the T-dualization

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procedure and calculations, we obtained generalization of β_μ functions in the form of the N-functions.

It should be noted that since we did not preform T-dualization along fermionic coordinates their Poisson structure would remain the same as in original theory. Furthermore, since background fields do not depend on fermionic coordinates it should be expected, as in the case of bosonic coordinates [18], that T-duality would leave Poisson brackets between fermionic fields the same. We expect that, if proposed non-commutative relations from [30, 31] are even possible, we would need at least RR field that depends both on fermionic and bosonic coordinates.

Acknowledgments

Work supported in part by the Serbian Ministry of Education, Science and Technological Development.

A Obtaining $N(\kappa^\pm)$ terms

In this paper function $N(\kappa^\pm)$ emerged in T-dual transformation laws as a consequence of variation of term that was proportional to ΔV. Here we will present derivation of this function.

$$
\begin{aligned}
&\frac{\delta\left(F^{-1}(\Delta V)\right)_{\alpha\beta}}{\delta v_+^\mu(\kappa)} = -(f^{-1})_{\alpha\alpha_1} C_l^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_P d\xi'^m \frac{\delta v_m^\rho(\xi')}{\delta v_+^\mu(\kappa)} = \qquad (A.1)\\
&= -(f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_P d\xi'^+\delta(\xi'^+ - \kappa^+)\delta(\xi'^- - \kappa^-)\\
&= -(f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_{t_i}^{t_f} dt\frac{d\xi'^+}{dt}\delta(\xi'(t)^+ - \kappa^+)\delta(\xi'^-(t) - \kappa^-)\\
&= -(f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\int_{\xi_0{}^+}^{\xi^+} du\delta(u - \kappa^+)\delta\Big(\xi'^-\left((\xi'^+)^{-1}(u)\right) - \kappa^-\Big)\\
&= -(f^{-1})_{\alpha\alpha_1} C_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}\delta\Big(\xi'^-\left((\xi'^+)^{-1}(\kappa^+)\right) - \kappa^-\Big)\Big[H(\xi^+ - \kappa^+) - H(\xi_0^+ - \kappa^+)\Big]\\
&= -(f^{-1})_{\alpha\alpha_1} G_\mu^{\alpha_1\beta_1}(f^{-1})_{\beta_1\beta}N(\kappa^+).
\end{aligned}
$$

In third line we have parametrized the path with parameter t where $\xi'^+(t_i) = \xi_0{}^+$ and $\xi'^+(t_f) = \xi^+$. In fourth line we introduced substitution $u = \xi'^+(t)$, in delta function this substitute is inverted. Fifth line is obtained by using following integration rule for Dirac delta function

$$
\int_{\sigma_0}^{\sigma} d\eta f(\eta)\delta(\eta - \bar\eta) = f(\bar\eta)\left[H(\sigma - \bar\eta) - H(\sigma_0 - \bar\eta)\right]. \qquad (A.2)
$$

Here, $H(x)$ is a step function defined as

$$H(x) = \int_0^x d\eta\delta(\eta) = \frac{1}{2\pi}\Big[x + \sum_{n\geq 2}\frac{1}{n}\sin(nx)\Big]$$
$$= \begin{cases} 0 & \text{if} \quad x=0 \\ 1/2 & \text{if} \quad 0<x<2\pi \\ 1 & \text{if} \quad x=2\pi \end{cases} . \tag{A.3}$$

Procedure for obtaining $N(\kappa^-)$ is similar.

B Properties of $N(\kappa^\pm)$ terms

Here ve will list some properties of $N(\kappa^\pm)$ function.

$$N(\kappa^+) + N(\kappa^-) = N(\kappa^0), \tag{B.1}$$
$$N(\kappa^+) - N(\kappa^-) = N(\kappa^1), \tag{B.2}$$

Where κ^0 and κ^1 represent τ and σ coordinates respectevly

$$\int_\Sigma d^2\xi\partial_+ N(\kappa^+) = 1, \qquad \int_\Sigma d^2\xi\partial_- N(\kappa^+) = 0, \tag{B.3}$$
$$\int_\Sigma d^2\xi\partial_- N(\kappa^-) = 1, \qquad \int_\Sigma d^2\xi\partial_+ N(\kappa^-) = 0. \tag{B.4}$$

These relationships can be checked directly by applying partial derivatives to expressions from A.

$$\int_\Sigma d^2\xi\partial_+ N(\kappa^+) = \int_\Sigma d^2\xi\delta\Big(\xi'^-((\xi'^+)^{-1}(\kappa^+)) - \kappa^-\Big)\partial_+\Big[H(\xi^+ - \kappa^+) - H(\xi_0^+ - \kappa^+)\Big]$$
$$= \int_\Sigma d^2\xi\delta\Big(\xi'^-((\xi'^+)^{-1}(\kappa^+)) - \kappa^-\Big)\delta(\xi^+ - \kappa^+) = \int d\xi^-\delta\Big(\xi'^-((\xi'^+)^{-1}(\xi^+)) - \kappa^-\Big). \tag{B.5}$$

In appendix A we had following parametrisation of path P: $\xi'^+(t_i) = \xi_0{}^+$ and $\xi'^+(t_f) = \xi^+$. Applying inverse parametrisation we have $(\xi'^+)^{-1}(\xi_0^+) = t_i$ and $(\xi'^+)^{-1}(\xi^+) = t_f$. With these we have

$$\int d\xi^-\delta\Big(\xi'^-((\xi'^+)^{-1}(\xi^+)) - \kappa^-\Big) = \int_\Sigma d\xi^-\delta\Big(\xi'^-(t_f) - \kappa^-\Big)$$
$$= \int_\Sigma d\xi^-\delta\Big(\xi^- - \kappa^-\Big) = 1. \tag{B.6}$$

Same rules apply for $N(\kappa^-)$, $N(\kappa^0)$ and $N(\kappa^1)$. In cases where $F^{-1}(x)_{\alpha\beta}$ is antisymmetric we can transfer partial derivatives from $\partial_\pm V^m u$ to $N(\kappa^\pm)$ and obtain standard $\beta^\pm$ functions.

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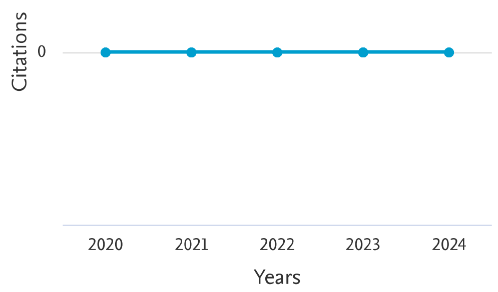

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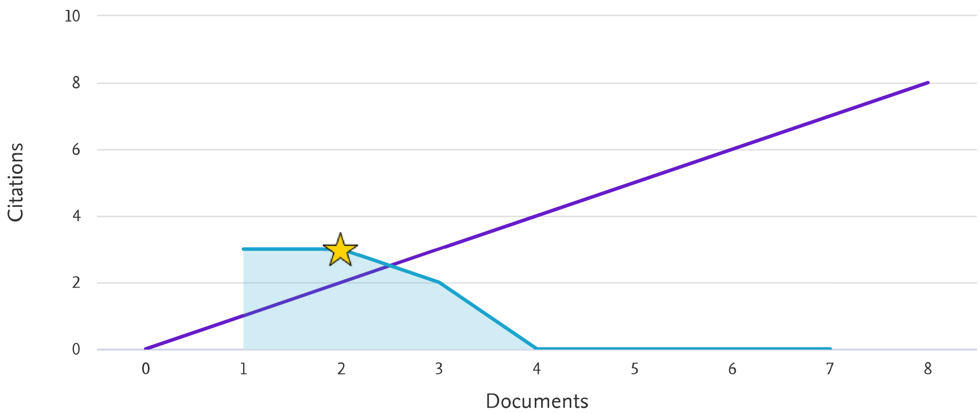

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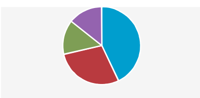

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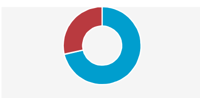

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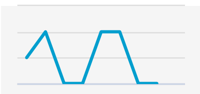

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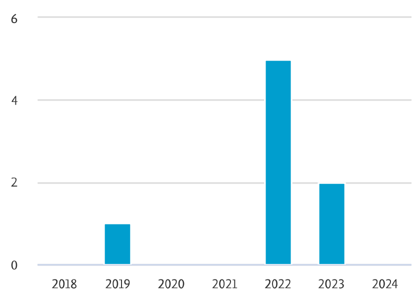

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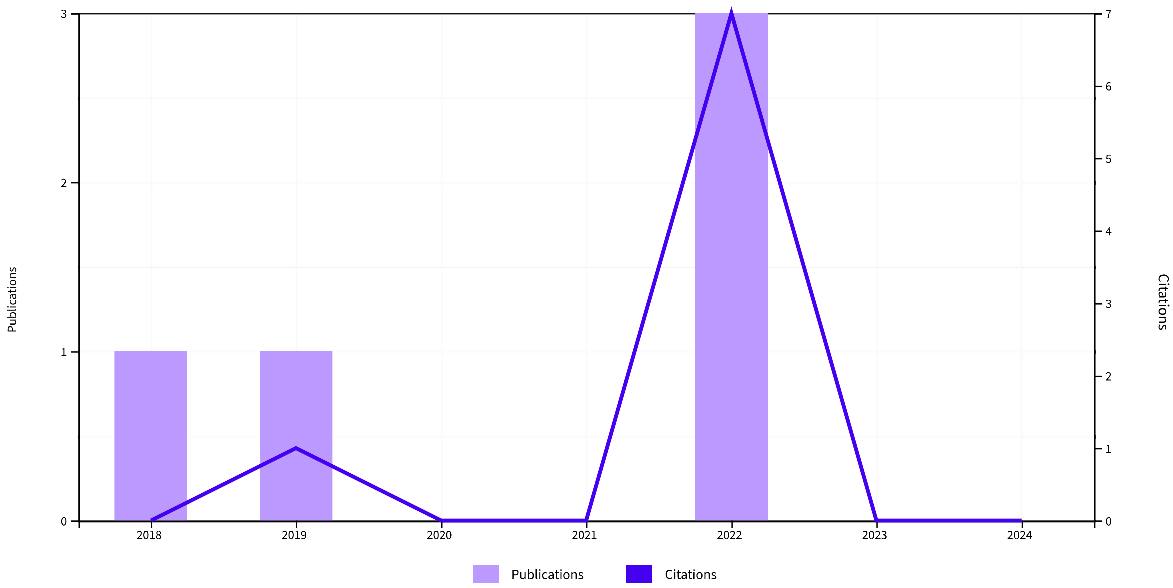

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Higher Quantum Geometry and Non-Geometric String Theory

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We present a concise overview of the physical and mathematical structures underpinning the appearence of nonassociative deformations of geometry in non-geometric string theory. Starting from a quick recap of the appearence of noncommutative product and commutator deformations of geometry in open string theory with B-fields, we argue on physical principles that closed strings should instead probe triproduct and tribracket deformations in backgrounds of locally non-geometric fluxes. After describing the toy model of electric charges moving in fields of smooth distributions of magnetic charge as a physical introduction to the notions of nonassociative geometry, we review the description of non-geometric fluxes in generalized geometry and double field theory, and the worldsheet calculations suggesting the appearence of nonassociative deformations, together with their caveats. We discuss how algebroids and their associated AKSZ sigma-models give a description of non-geometric backgrounds in terms of higher geometry, and consider the quantization of the membrane sigma-model which geometrizes closed strings with R-flux. From this we derive an explicit nonassociative star product for the quantum geometry of the closed string phase space, and apply it to derive the triproducts that appear in conformal field theory correlation functions, to describe a consistent treatment of nonassociative quantum mechanics, to demonstrate quantitatively the coarse-graining of spacetime due to R-flux, and to describe the quantization of Nambu brackets. We also briefly review how these constructions lead to a nonassociative theory of gravity, their uplifts to non-geometric M-theory, and the role played by L_∞-algebras in these developments.

Proceedings of the Corfu Summer Institute 2017 "School and Workshops on Elementary Particle Physics and Gravity"
2–28 September 2017
Corfu, Greece

*Speaker.

PoS(CORFU2017)151

Contents

1. Introduction: String theory and higher noncommutative geometry

The second Training School of the COST Action *Quantum Structure of Spacetime* was devoted to the general topic "Quantum Spacetime and Physics Models". While this is a broad topic with many potential directions (some covered by other lectures at this School), for the purposes of these lectures this refers to the problem that Einstein's general theory of relativity cannot be consistently quantized via quantum field theory, due to the ultraviolet divergences that plague perturbation theory around a flat background which require infinitely-many counterterms. To attempt to solve this problem one can consider physics models with a natural minimal length providing a suitable ultraviolet regularization. In this contribution we consider two such theories and how to reconcile them within the context of the School. Our treatment in this section and some other portions of this article have been influenced by various other reviews available, such as e.g. [85, 94, 91, 26].

One approach is based on noncommutative geometry. In attempting to reconcile quantum mechanics with gravity, which is a theory based on the geometry of spacetime, one is inevitably led to the notion of 'quantum geometry', which refers to the application of the principles of quantum mechanics to spacetime itself. One way to think of such a quantization is by promoting the spacetime coordinates x^i to 'operators' which do not commute:

$$[x^i, x^j] = \mathrm{i}\hbar\,\theta^{ij} \ , \tag{1.1}$$

for some bivector θ^{ij} which naturally incorporates a minimal area: Applying the standard uncertainty principle to the commutation relations (1.1) implies $\Delta x^i\,\Delta x^j \geqslant \frac{\hbar}{2}\,|\theta^{ij}|$, so that the minimal value of $|\theta^{ij}|$ may be thought of as the Planck length $\ell_{\rm P}$ of spacetime. Quantum field theory on such noncommutative spaces exhibits very interesting features of forbidden interactions, controlled Lorentz violation, and UV/IR mixing (see e.g. [113] for a review), and it can be extended to a noncommutative theory of gravity [14, 13] as discussed in the lectures of L. Castellani at this School. However, there are two major pitfalls to this approach. Firstly, although a minimal length scale is naturally introduced, there is no coarse-graining of spacetime that one would expect from a quantum theory of gravity: An underlying discrete structure such as a quantum of minimal volume does not appear in this framework. Secondly, most treatments assume that the brackets (1.1) satisfy the Jacobi identity; in particular, when the bivector $\theta = B^{-1}$ is invertible, it is not clear how to deal with the cases with non-vanishing flux $H = \mathrm{d}B \neq 0$. As we will discuss in the following, these two drawbacks are in fact related and are simultaneously dealt with by extending noncommutative geometry into the world of nonassociative geometry, which deals with deformations by higher structures in geometry.

Another approach is based on string theory, which certainly provides a concrete physical model of a quantum spacetime. Strings are extended one-dimensional degrees of freedom and so, unlike the point particle probes of quantum field theory, naturally come with intrinsic minimal length $\Delta x^i \geqslant \ell_s$, where ℓ_s is the string length. The interactions of strings thus violate locality, while the theory directly contains gravity and is on-shell ultraviolet finite. It is then natural to ask whether the spacetime approaches based on noncommutative geometry and string theory are related or are complementary to each other in some sense. The precise connection between *open strings* and noncommutative geometry was discovered near the end of the last millenium, see e.g. [52, 8, 9, 103, 45, 104, 101, 102], and is by now well-established. Open strings on D-branes in background B-fields (which provide a gauge flux on the D-brane worldvolume) probe a noncommutative worldvolume geometry. The massless bosonic field content of open string theory consists of gauge fields A_i on the worldvolume together with scalar fields ϕ^a governing the transverse fluctuations of the D-branes in spacetime. In the Seiberg-Witten scaling limit which decouples open and closed string modes, the effective low-energy dynamics is governed by a noncommutative gauge theory on the D-brane worldvolume.

Although this connection is precise, and has led to a flurry of investigation over the last 20 years, this does not explain the connection of noncommutative geometry with gravity, nor how noncommutative geometry can be used to formulate a consistent theory of quantum gravity. In the present context, one needs to look at *closed strings*. The massless bosonic field content of closed string theory consists of the spacetime metric g_{ij}, the Kalb-Ramond field B_{ij}, and the dilaton field Φ. Thus the closed string sector contains the data of background geometry and gravity, and it is here that one should seek analogs of a quantum geometry and suitable decoupling limits to make contact with the problem of quantizing gravity. Such a connection should certainly appear if noncommutative geometry indeed improves the ultraviolet behaviour of quantum gravity.

The connections between noncommutative geometry and closed string theory has been a topic of increasing interest over the last eight years, where it has been realised that closed strings are related to not only noncommutative but even *nonassociative* target space geometries. To understand how nonassociativity can arise in closed string theory, it is helpful to take a step back and look at in more detail why noncommutativity emerges in open string theory. An instructive pedagogical analogy from quantum mechanics is provided by the Landau problem [111]: The planar quantum dynamics of electrons of mass m and charge e propagating under the influence of a perpendicularly applied constant background magnetic field of magnitude B. The Lagrangian is

$$L=\frac{m}{2}\,\dot{\vec{x}}^{2}-e\,\dot{\vec{x}}\cdot\vec{A}\qquad\text{with}\quad A_i=-\frac{B}{2}\,\varepsilon_{ij}\,x^j\ . \tag{1.2}$$

The limit $B\gg m$ of strong magnetic field induces the projection onto the lowest Landau level, described by a first order Lagrangian $L\big|_{m=0}=\frac{eB}{2}\,\dot{x}^i\,\varepsilon_{ij}\,x^j$ with degenerate phase space whose canonical quantization gives the commutation relations of a noncommutative space:

$$[x^i,x^j]=\frac{\mathrm{i}\hbar}{eB}\,\varepsilon^{ij}=:\mathrm{i}\hbar\,\theta^{ij}\ . \tag{1.3}$$

This simple model is analogous to that of bosonic open strings in a B-field background, which at tree-level in string perturbation theory is described generally by the worldsheet action

$$S=\frac{1}{4\pi\,\ell_s^2}\int_{\Sigma_2}\big(g_{ij}(x)\,\mathrm{d}x^i\wedge *\,\mathrm{d}x^j-2\pi\,\ell_s^2\,B_{ij}(x)\,\mathrm{d}x^i\wedge\mathrm{d}x^j\big)\ , \tag{1.4}$$

where x^i are maps from the string worldsheet, which is a disk Σ_2, to the target spacetime M. The low-energy limit $\ell_s\to 0$ describes the decoupling of massive string states, while the Seiberg-Witten scaling limit $g_{ij}\sim\ell_s^4\to 0$ ensures that gravity is non-dynamical and that the bulk modes of the string decouple from the boundary degrees of freedom. In this limit the action (1.4) reduces to a simple topological action given by the pullback of the Kalb-Ramond two-form to the string worldsheet:

$$S\big|_{g,\ell_s=0}=-\frac{1}{2}\int_{\Sigma_2}B_{ij}(x)\,\mathrm{d}x^i\wedge\mathrm{d}x^j\ . \tag{1.5}$$

In the absence of NS–NS flux $H=\mathrm{d}B=0$, and for target space $M=\mathbb{R}^d$, using Stokes' theorem the action (1.5) gives a pure boundary interaction $S\big|_{g,\ell_s=0}=-\oint_{\partial\Sigma_2}\mathrm{d}t\ \dot{\vec{x}}\cdot\vec{A}$, where $B=\mathrm{d}A$ and $\dot{x}^i=\partial_t x^i$ denotes a tangential derivative of the string field along the worldsheet boundary circle $\partial\Sigma_2$. Again this action is of first order in worldsheet time derivatives, so that the open string endpoints have a degenerate phase space in the decoupling limit. For a constant B-field with the symmetric gauge choice $A_i=-\frac{1}{2}\,B_{ij}\,x^j$, the action becomes

$$S\big|_{g,\ell_s=0}=\frac{1}{2}\oint_{\partial\Sigma_2}\mathrm{d}t\ x^i\,B_{ij}\,\dot{x}^j\ , \tag{1.6}$$

and if the B-field is moreover non-degenerate then its canonical quantization produces the noncommutative coordinate algebra

$$[x^i, x^j] = \left(\frac{\mathrm{i}\hbar}{B}\right)^{ij} =: \mathrm{i}\hbar\,\theta^{ij}\ . \tag{1.7}$$

Thus the scaling limit of the open string sigma-model is formally analogous to projection to the lowest Landau level for charged particles in strong uniform magnetic fields.

The more precise dynamical mechanism behind this heuristic argument can be infered from studying the two-point disk correlators deformed by the non-zero two-form B_{ij}, which plays the role of a magnetic flux on the worldvolume and can be turned on by a left-right asymmetric rotation of the D-brane via T-duality. The B-field allows one to distinguish the insertions of string fields $x^i(t)$ and $x^j(t')$ on the boundary of the disk in the correlation function, which depends only on the ordering of the two boundary insertion points [102]:

$$\big\langle x^i(t)\,x^j(t')\big\rangle = -\ell_s^2\,G^{ij}\log(t-t')^2 + \frac{\mathrm{i}\hbar}{2}\,\theta^{ij}\,\mathrm{sgn}(t-t')\ , \tag{1.8}$$

where we used the open-closed string relations

$$\frac{1}{g+2\pi\,\ell_s^2\,B} = \frac{1}{G} + \frac{\theta}{2\pi\,\ell_s^2}\ , \tag{1.9}$$

with G the open string metric and the bivector θ is the source of noncommutativity since it is not symmetric under interchange of x^i with x^j. The transformation (1.9) is familiar from the Büscher rules for T-duality, with the precise connection suggested by the worldsheet approach of [55], and it explicitly determines the open string variables (G,θ) in terms of the closed string variables (g,B) by

$$G = g-(2\pi\,\ell_s^2)^2\,B\,g^{-1}\,B \qquad \text{and} \qquad \theta = -(2\pi\,\ell_s^2)^2\,G^{-1}\,B\,g^{-1}\ . \tag{1.10}$$

Note that $G=g$ and $\theta=0$ exactly when $B=0$. Using this correlation function, the operator product expansion of open string tachyon vertex operators on the boundary of the disk is computed to be

$$\mathrm{e}^{\,\mathrm{i}\,k\cdot x(t)}\cdot \mathrm{e}^{\,\mathrm{i}\,q\cdot x(t')} = (t-t')^{2\ell_s^2\,k_i\,G^{ij}\,q_j}\ \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}\,k_i\,\theta^{ij}\,q_j}\ \mathrm{e}^{\,\mathrm{i}\,(k+q)\cdot x(t')} + \cdots\ , \tag{1.11}$$

for $t>t'$. The second factor in (1.11) does not depend on the worldsheet coordinates and is purely a target space effect, and in the low-energy limit $\ell_s\to 0$ whereby $\theta = B^{-1}$, this phase factor is encoded in scattering amplitudes by the star product of fields f,g given by

$$(f\star g)(x) = \int \mathrm{d}k\,\int \mathrm{d}q\ \tilde f(k)\,\tilde g(q)\ \mathrm{e}^{-\frac{\mathrm{i}\hbar}{2}\,k_i\,\theta^{ij}\,q_j}\ \mathrm{e}^{\,\mathrm{i}\,(k+q)\cdot x} \tag{1.12}$$

in Fourier space, which is equivalent to the formal expansion in terms of a bidifferential operator:

$$f\star g = \cdot\ \exp\Big(\frac{\mathrm{i}\hbar}{2}\,\theta^{ij}\,\partial_i\otimes\partial_j\Big)(f\otimes g)\ . \tag{1.13}$$

This is simply the Moyal-Weyl star product which is a noncommutative deformation of the pointwise product $f\cdot g$ of functions on spacetime. Its characteristic features are that it quantizes the commutator $[x^i,x^j] = \mathrm{i}\hbar\,\theta^{ij}$ (by defining $[f,g]_\star := f\star g - g\star f$ and setting $f=x^i$, $g=x^j$), and it is associative: $f\star(g\star h)=(f\star g)\star h$. Since $f\star g$ differs from $f\cdot g$ by a total derivative, the star product deformation is consistent with the conformal $SL(2,\mathbb{R})$ symmetry of the worldsheet theory which leaves the cyclic ordering of boundary vertex operator insertions invariant, in the sense that the star product is *2-cyclic*:

$$\int \mathrm{d}x\ f\star g=\int \mathrm{d}x\ g\star f=\int \mathrm{d}x\ f\cdot g\ . \tag{1.14}$$

Moreover, consistency with associativity of the operator product expansion in conformal field theory only requires crossing symmetry of the worldsheet correlation functions, which leads to the weaker *3-cyclic* condition:

$$\int \mathrm{d}x\ f\star(g\star h)=\int \mathrm{d}x\ (f\star g)\star h\ . \tag{1.15}$$

It is in this way that one arrives at a noncommutative gauge theory for the massless bosonic open string modes A_i and ϕ^a; see e.g. [54, 110] for reviews and further details.

Now let us try to understand how an analogous scenario could be realised in a connection between noncommutative geometry and closed strings. Closed strings see geometry in a different way than open strings do, which from a target space perspective is due to T-duality. From the worldsheet perspective, the relevant tree-level amplitude involves correlation functions on the sphere S^2, but the situation must be different and one has to pass to higher correlators, as first pointed out by [28], since now the ordering on S^2 is ambiguous because two points can be interchanged by an $SL(2,\mathbb{R})$ transformation. However, the insertion of three string fields on the sphere depends only on the relative orientation of the three points, i.e. whether the insertion of a third point lies on the same or opposite hemisphere as the other two points. A trivector flux θ^{ijk} can be used to distinguish configurations, and in analogy with the Moyal-Weyl star product (1.13) it deforms the algebra of functions with the "triproduct"

$$f_1\vartriangle f_2\vartriangle f_3=\cdot\ \exp\Big(\frac{\mathrm{i}\hbar}{6}\,\theta^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k\Big)(f_1\otimes f_2\otimes f_3)\ , \tag{1.16}$$

which leads to a *nonassociative* tribracket defined by

$$[f_1,f_2,f_3]_\vartriangle:=\sum_{\tau\in S_3}\mathrm{sgn}(\tau)\,f_{\tau(1)}\vartriangle f_{\tau(2)}\vartriangle f_{\tau(3)}\ . \tag{1.17}$$

This quantizes the basic coordinate brackets

$$[x^i,x^j,x^k]_\vartriangle=\mathrm{i}\hbar\,\theta^{ijk}\ . \tag{1.18}$$

The purpose of these lectures is to discuss, and answer as far as possible, the following imminent questions at this stage:

(Q1) What is the trivector θ^{ijk}?

We will see that this trivector is a 'locally non-geometric flux', called the R-flux. To properly discuss this, we shall have to review some ingredients of non-geometric flux compactifications, generalized geometry, and double field theory, some aspects of which are discussed in the lectures by C. Hull at this School, and which we undertake in Section 3.

(Q2) What is the origin of the triproduct $\triangle$?

We will see that the nonassociativity encoded in off-shell closed string amplitudes is probed by suitable redefinitions of the coordinate fields x^i in linear flux backgrounds. We shall find in Sections 4 and 5 that the triproduct is not the fundamental algebraic entity, but arises as the result of a non-vanishing Jacobiator for a *nonassociative star product* on the closed string phase space.

(Q3) Is there a nonassociative version of the closed string effective action?

Recall that the closed string effective action for the massless bosonic modes g_{ij}, B_{ij} and Φ is given by

$$S_{\rm grav} = \frac{1}{16\pi G}\int_M \Big(*\,{\rm Ric} - \frac{1}{12}\,{\rm e}^{-\Phi/3}\,H\wedge *H - \frac{1}{6}\,{\rm d}\Phi\wedge *\,{\rm d}\Phi\Big)~. \tag{1.19}$$

Conformal invariance of the worldsheet theory at one-loop requires vanishing beta-functions, which are equivalent to the target space equations of motion resulting from (1.19). We shall discuss some aspects of this far-reaching future goal in Section 6, but will not provide a complete and decisive answer to the problem of the relevance of a nonassociative theory of gravity in closed string theory, which is currently a topic of ongoing investigation.

2. A first glimpse at nonassociative geometry: Magnetic monopoles

As we saw in the case of open strings, a simple yet instructive quantum mechanical analogue for the appearance of noncommutative geometry is provided by the motion of electric charges in background magnetic fields. A straightforward but far-reaching extension of this model likewise provides an instructive physical scenario in which to understand the appearence and implications of nonassociative geometry in the closed string sector. We shall see later on that this model has a precise analogue for closed strings propagating in locally non-geometric flux backgrounds, and it enables us to introduce some of the geometric ideas that will be used throughout this paper. The treatment of the quantum mechanical system of this section is originally due to [76, 77].

Consider the motion of a charged particle on $\mathbb{R}^3$ in a fixed magnetic field $\vec B$, possibly with sources. The kinematical momentum of the particle is $\vec p = m\dot{\vec x}$, which is the physical gauge-invariant quantity and is not to be confused with the (gauge-variant) canonical momentum. The Hamiltonian is taken to be the kinetic energy

$$H = \frac{\vec p^{\,2}}{2m}~. \tag{2.1}$$

In the quantum theory, the Lorentz-Heisenberg equations of motion

$$\dot{\vec p} = \frac{\rm i}{\hbar}\,[H,\vec p\,] = \frac{e}{2m}\,\big(\vec p\times\vec B - \vec B\times\vec p\,\big) \qquad \text{and} \qquad \dot{\vec x} = \frac{\rm i}{\hbar}\,[H,\vec x\,] = \frac{\vec p}{m} \tag{2.2}$$

require the *deformed* canonical commutation relations of a noncommutative momentum space:

$$[x^i,x^j]=0~, \qquad [x^i,p_j] = {\rm i}\,\hbar\,\delta^i{}_j~, \qquad [p_i,p_j] = {\rm i}\,\hbar\,e\,F_{ij}(\vec x) \qquad \text{with} \quad F_{ij} = \varepsilon_{ijk}\,B^k~. \tag{2.3}$$

This formulation depends only on the magnetic field $\vec{B}$, and in particular it allows for cases in which $\nabla\cdot\vec{B}\neq 0$.

Let us understand the geometric structure underlying these commutation relations. Writing phase space coordinates collectively as $x^I=(x^i,p_i)$, we can express the relations in the form

$$[x^I,x^J]=\mathrm{i}\,\hbar\,\Theta^{IJ}\qquad\text{with}\quad(\Theta^{IJ})=\begin{pmatrix}0 & 1_3\\ -1_3 & e\,F(\vec{x})\end{pmatrix}\ . \tag{2.4}$$

The phase space bivector $\Theta=\frac{1}{2}\,\Theta^{IJ}\,\partial_I\wedge\partial_J$ is *not* a Poisson bivector in general. The failure of the Jacobi identity for the quasi-Poisson brackets defined by Θ is controlled by the Schouten bracket

$$[\Theta,\Theta]_{\mathrm{S}}^{IJK}=\Theta^{[\underline{IL}}\,\partial_L\Theta^{\underline{JK}]}\ , \tag{2.5}$$

which is the natural extension of the usual Lie bracket of vector fields to multivector fields; here only underlined indices are antisymmetrized. Then $[\Theta,\Theta]_{\mathrm{S}}=0$ if and only if the Jacobiator

$$[x^I,x^J,x^K]:=[x^I,[x^J,x^K]]+[x^J,[x^K,x^I]]+[x^K,[x^I,x^J]] \tag{2.6}$$

vanishes. An easy calculation shows

$$[p_1,p_2,p_3]=3\,\hbar^2\,e\,\nabla\cdot\vec{B}=:3\,\hbar^2\,e\,\mu_0\,\rho_m\ . \tag{2.7}$$

Therefore the phase space algebra of the charged particle is *nonassociative* in the presence of magnetic sources $\rho_m\neq 0$.

For source-free magnetic fields $\vec{B}$, one has $\rho_m=0$ and $\nabla\cdot\vec{B}=0$, so that there exists a globally defined magnetic vector potential $\vec{A}$ on $\mathbb{R}^3$ such that $\vec{B}=\nabla\times\vec{A}$. Then the commutation relations can be transformed to canonical form with the canonical momentum $\vec{\pi}=\vec{p}+e\vec{A}$. However, since $\vec{B}=\nabla\times\vec{A}$ if and only if $\nabla\cdot\vec{B}=0$, we cannot work with canonical momenta and covariant derivatives in the presence of magnetic sources, i.e. for $\rho_m\neq 0$ we encounter nonassociativity and there is no linear operator $\vec{p}=\mathrm{i}\,\hbar\,\nabla-e\vec{A}$. Let us now explore how to understand magnetic sources and the ensuing violation of the Jacobi identity.

Since $[x^i,p_j]=\mathrm{i}\,\hbar\,\delta^i{}_j$, translations in the quantum theory are generated by the magnetic translation operators

$$T(\vec{a})=\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\,\vec{a}\cdot\vec{p}} \tag{2.8}$$

with $T^{-1}(\vec{a})\,\vec{x}\,T(\vec{a})=\vec{x}+\vec{a}$. These operators do not form a representation of the translation group on $\mathbb{R}^3$, as a simple calculation shows

$$T(\vec{a}_1)\,T(\vec{a}_2)=\mathrm{e}^{\frac{\mathrm{i}e}{\hbar}\,\Phi_2(\vec{x};\vec{a}_1,\vec{a}_2)}\,T(\vec{a}_1+\vec{a}_2)\ , \tag{2.9}$$

where

$$\Phi_2(\vec{x};\vec{a}_1,\vec{a}_2)=\int_{\langle\vec{a}_1,\vec{a}_2\rangle_{\vec{x}}}\vec{B}\cdot\mathrm{d}\vec{S} \tag{2.10}$$

is the magnetic flux through the oriented triangle $\langle\vec{a}_1,\vec{a}_2\rangle_{\vec{x}}$ based at $\vec{x}\in\mathbb{R}^3$ with sides $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_1+\vec{a}_2$. The Jacobi identity is the infinitesimal statement of associativity, and its failure results in the relations

$$\big(T(\vec{a}_1)\,T(\vec{a}_2)\big)\,T(\vec{a}_3)=\mathrm{e}^{\frac{\mathrm{i}e}{\hbar}\,\Phi_3(\vec{x};\vec{a}_1,\vec{a}_2,\vec{a}_3)}\,T(\vec{a}_1)\,\big(T(\vec{a}_2)\,T(\vec{a}_3)\big)\ , \tag{2.11}$$

where

$$\Phi_3(\vec{x};\vec{a}_1,\vec{a}_2,\vec{a}_3) = \int_{\partial\langle\vec{a}_1,\vec{a}_2,\vec{a}_3\rangle_{\vec{x}}} \vec{B}\cdot \mathrm{d}\vec{S} = \int_{\langle\vec{a}_1,\vec{a}_2,\vec{a}_3\rangle_{\vec{x}}} \nabla\cdot\vec{B}\ \mathrm{d}V \tag{2.12}$$

is the magnetic charge q_m enclosed by the oriented tetrahedron $\langle\vec{a}_1,\vec{a}_2,\vec{a}_3\rangle_{\vec{x}}$ based at $\vec{x}\in\mathbb{R}^3$ with sides $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$, $\vec{a}_1+\vec{a}_2$, $\vec{a}_2+\vec{a}_3$ and $\vec{a}_1+\vec{a}_2+\vec{a}_3$, and we have used the divergence theorem.

It follows that associativity of translations is ensured when [76]

$$\frac{\mu_0\, e\, q_m}{\hbar} \in 2\pi\,\mathbb{Z} \tag{2.13}$$

which is the celebrated Dirac charge quantization condition. Then the usual quantum mechanical formalism can be applied with linear operators on a separable Hilbert space. But this restricts the form of the magnetic field $\vec{B}$, which must be sourced by a point-like magnetic monopole (or a collection thereof), so that the phase Φ_3 does not lose its integrality when the translation vectors $\vec{a}_i$ are continuously varied. In this case the magnetic source must lie either inside or outside the tetrahedron $\langle\vec{a}_1,\vec{a}_2,\vec{a}_3\rangle_{\vec{x}}$, and is given by the Dirac monopole field

$$\vec{B} = \mu_0\, q_m\, \frac{\vec{x}}{|\vec{x}|^3} \qquad \text{with} \quad \nabla\cdot\vec{B} = 4\pi\,\mu_0\, q_m\, \delta(\vec{x}) \ . \tag{2.14}$$

What becomes of the Jacobi identity in this case? We note that it is violated precisely at the loci of the magnetic charges, which for Dirac monopoles occur at isolated points and so can be excised from $\mathbb{R}^3$, where the magnetic field $\vec{B}$ is singular. Such an excision is also natural from the point of view of angular momentum conservation, which imples that the electric charges never reach the monopoles and their wavefunction vanishes at the monopole locations [16]. This leads to a geometric description of Dirac monopoles in terms of connections on a non-trivial $U(1)$-bundle $P\to\mathbb{R}^3\setminus\{\vec{0}\}\simeq S^2$ of first Chern class $c_1(P)=\mu_0\, e\, q_m/2\pi\hbar$, and the wavefunctions of the particle live in the Hilbert space of square-integrable sections of P. In this case the map $\vec{a}\mapsto T(\vec{a})$ defines a projective representation of the translation group of $\mathbb{R}^3\setminus\{\vec{0}\}$ on this Hilbert space, and the projective phase $\mathrm{e}^{\frac{\mathrm{i}e}{\hbar}\Phi_2}$ is the group two-cocycle of the representation.

For our later considerations we are interested in situations corresponding to a constant homogeneous magnetic charge density background ρ_m, whose algebraic structure was first studied in [62]. The analogue of the rotationally symmetric field (2.14) in this case is given by

$$\vec{B} = \frac{\mu_0\,\rho_m}{3}\,\vec{x} \ . \tag{2.15}$$

The magnetic charge is now uniformly distributed over all space, so that the phase space coordinate algebra becomes everywhere nonassociative. In this case removing the magnetic sources from $\mathbb{R}^3$ would leave an empty space. In this sense the momentum space of an electric charge in a uniform magnetic charge distribution is 'locally non-geometric'. The constant magnetic charge density is not described by a connection on a $U(1)$-bundle over $\mathbb{R}^3\setminus\{\vec{0}\}$, but rather in terms of a connection on a (trivial) $U(1)$-gerbe on $\mathbb{R}^3$, i.e. by a "B-field"

$$F_{ij} = \varepsilon_{ijk}\, B^k = \frac{\mu_0\,\rho_m}{3}\,\varepsilon_{ijk}\, x^k \tag{2.16}$$

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with curvature $H = \mathrm{d}F = \mu_0\, \rho_m\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$; since $H \neq 0$ everywhere there is not even a local magnetic vector potential $\vec{A}$ in this instance. Moreover, now the phase factors Φ_2 yield two-cochains, rather than two-cocycles, whose coboundary is the phase $\mathrm{e}^{\frac{ie}{\hbar}\Phi_3}$ which hence defines a three-cocycle of the translation group of $\mathbb{R}^3$. Thus the quantum theory with this three-cocycle nessecitates *nonassociative quantum mechanics*. Geometrically, the bivector Θ in this case defines a 'twisted Poisson structure' on phase space. As we will see later on, such a quantum system makes perfect physical sense and possesses fascinating properties.

This simple quantum mechanical example illustrates the string theoretical considerations which will follow, in the topic of non-geometric flux compactifications that we turn to next. Our string theory considerations will lead naturally to an approach to nonassociative geometry via deformation quantization, analogously to the open string case, which will be the main tool of this paper. However, as phase space quantum mechanics comes with its own issues, as we discuss later, let us point out for completeness some alternative approaches to the nonassociative quantum mechanics alluded to above, of which there are currently three that each have their own deficiencies as well.

Firstly, and most straightforwardly, one may generalize the technique of symplectic realization from Poisson geometry to the twisted Poisson structure Θ and embed the nonassociative phase space into a symplectic manifold of twice the original dimension [82]. In this associative framework standard techniques of geometric or canonical quantization are available, and in particular a global magnetic vector potential exists on the doubled configuration space; its drawback is that, while the doubling is tantalizing reminescent of the framework of double field theory discussed below, it is not possible to eliminate the spurious auxiliary degrees of freedom that enable the reformulation in terms of associative geometry. Secondly, one can exploit the geometric structure of the gerbe associated to the distribution of magnetic charge to face the nonassociativity head on and define quantum states that live in the 2-Hilbert space of sections of this gerbe, in analogy to the ordinary Hilbert space of sections of a line bundle in the source-free case [114, 39]. This gives a geometric description of the magnetic translation operators (2.8) acting on this 2-Hilbert space with the three-cocycle above interpreted as a higher projective phase of a 2-representation; the drawback of this approach is that it is rather technically complicated and it is difficult to represent observables such as the Hamiltonian operator (2.1) on the 2-Hilbert space, while conceptually it is not clear what is the meaning of such higher quantum states. Thirdly, one can apply transgression techniques to map the gerbe to a line bundle over the loop space of the configuration manifold [98, 99, 40]. This approach is naturally suggested by the closed string origin of nonassociativity, and with it one can apply standard techniques of geometric quantization on loop space which successfully captures some predictions of string theory; the drawback of this approach is that it requires difficult infinite-dimensional analysis which makes computations of physical quantities, such as expectation values, seemingly intractable. Hence in all approaches one trades nonassociativity for some other sort of technical or conceptual complication.

3. Non-geometric fluxes and nonassociative geometry

The purpose of this section is to explain what the notion of "non-geometry" means in string theory, and in particular to answer Question (Q1) from Section 1. Recall that in closed string theory, in addition to the spacetime metric g_{ij}, there is a massless Kalb-Ramond two-form B in the

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NS–NS sector with curvature three-form $H = \mathrm{d}B$. The motion of the strings is described by the two-dimensional non-linear sigma-model action (1.4). The classical vacua are described by two-dimensional conformal field theories, and in this setting the target space geometry is emergent. On the other hand, there exist conformal field theories which cannot be identified with such simple large radius geometries, such as left-right asymmetric orbifolds, and in such instances we will advocate the point of view that the target space interpretation is related to noncommutative and nonassociative geometry. This is the case when a gauge flux is turned on in the worldvolume field theory of a D-brane, which corresponds to an asymmetric rotation in the boundary conformal field theory.

Another left-right asymmetric worldsheet operation is that of T-duality, which reflects right-moving strings while leaving the left-moving sector unchanged. From the perspective of a d-dimensional target space, a T-duality $\mathscr{T}_i$ along the i-th direction exchanges momentum modes p_i with winding modes w^i, and correspondingly for the canonically conjugate variables: position coordinates x^i are exchanged with their dual "winding coordinates" $\tilde{x}_i$. The collection of symmetry transformations form the split-signature orthogonal group $O(d,d)$ which is the continuous extension of the physical T-duality group $O(d,d;\mathbb{Z})$ of toroidally compactified closed string theory.

String theory with fluxes is of interest both for its geometric allure and because of its relevance to observable phenomenology and cosmology (see e.g. [60, 53, 32] for reviews): Flux compactifications can lead to generalized geometric structures, obtained for example by patching together with string symmetries, while at the same time they stabilize moduli on the string landscape. They are also of importance in the AdS/CFT correspondence. Starting from flat space with non-vanishing NS–NS H-flux, T-duality gives rise to a chain of geometric and non-geometric fluxes [71, 107]

$$H_{ijk} \xrightarrow{\mathscr{T}_i} f^i{}_{jk} \xrightarrow{\mathscr{T}_j} Q^{ij}{}_k \xrightarrow{\mathscr{T}_k} R^{ijk} \ . \tag{3.1}$$

Let us briefly describe the geometrical meaning of each member of this T-duality chain. The first term is of course the geometric NS–NS flux $H = \mathrm{d}B$, which represents the characteristic class of a $U(1)$-gerbe by generalized Dirac quantization of fluxes in string theory. The second term is a metric flux, which appears as torsion in the geometry: In a suitable basis of vielbeins e^i, with inverses e_i, it appears in the Maurer-Cartan equations $\mathrm{d}e^i = -\frac{1}{2} f^i{}_{jk}\, e^j \wedge e^k$, or equivalently as the structure constants of the non-trivial Lie bracket of vector fields $[e_i, e_j] = f^k{}_{ij}\, e_k$. The third member is the first example of a non-geometric frame, and $Q^{ij}{}_k$ is called a 'globally non-geometric' Q-flux. These are also called T-folds, which have a local description in terms of conventional Riemannian geometry, but globally the transition functions between local charts also involve T-duality transformations. The final member of the chain is the most mysterious frame and the one which shall occupy most of our attention: Here R^{ijk} is called a 'locally non-geometric' R-flux. This background cannot even be described locally by conventional geometry and has no clear target space interpretation. As we discuss below it is this frame that this gives rise to a *nonassociative geometry*.

Describing these fluxes and understanding this non-geometric regime of string theory requires generalized geometry and its extension to double field theory, which is an $O(d,d)$-symmetric theory treating the metric and B-field on equal footing, and which is covered in the lectures by C. Hull at this School.

3.1 Generalized geometry

The geometric and non-geometric fluxes appearing in the T-duality chain (3.1) can be regarded as structure constants of a generalized bracket

$$\begin{aligned} [e_i,e_j]_{\rm R} &= f^k{}_{ij}\,e_k + H_{ijk}\,e^k \ , \\ [e^i,e_j]_{\rm R} &= f^i{}_{jk}\,e^k - Q^{ik}{}_j\,e_k \ , \\ [e^i,e^j]_{\rm R} &= Q^{ij}{}_k\,e^k + R^{ijk}\,e_k \ , \end{aligned} \tag{3.2}$$

for a local vielbein basis of sections (e_i,e^i) of the vector bundle $E=TM\oplus T^*M$, which in generalized geometry is called the *generalized tangent bundle* [66, 61]. In this form the bracket is usually called the Roytenberg bracket [95, 65, 31], and its reductions for various choices of vanishing fluxes gives the usual Courant and Dorfman brackets of generalized geometry. It governs the worldsheet current algebras in flux compactifications of string theory, see e.g. [2, 64, 38].

The sections of the generalized tangent bundle $E=TM\oplus T^*M$ are denoted as $X+\xi$ with $X=X^i\,e_i$ a vector field and $\xi=\xi_i\,e^i$ a one-form on M. The bundle E carries a canonical $O(d,d)$-structure through the natural pairing $\langle e_i,e^j\rangle=\delta_i{}^j$ of the tangent bundle TM with the cotangent bundle T^*M of the target space M. The structure group $O(d,d)$ has two natural abelian subgroups acting on sections of the generalized tangent bundle in the following way:

- B-transforms: $\begin{pmatrix} 1_d & 0 \\ B & 1_d \end{pmatrix} \ : \ X+\xi \longmapsto X+\big(\xi+\iota_X B\big) \ ,$

- θ-transforms: $\begin{pmatrix} 1_d & \theta \\ 0 & 1_d \end{pmatrix} \ : \ X+\xi \longmapsto \big(X+\theta^\sharp\xi\big)+\xi \ ,$

where B is a two-form and θ is a bivector which induce the natural contraction maps $\iota:TM\to T^*M$, $(\iota_X B)_i=B_{ij}\,X^j$ and $\theta^\sharp:T^*M\to TM$, $(\theta^\sharp\xi)^i=\theta^{ij}\,\xi_j$. Any $O(d,d)$-transformation $\mathscr{O}\in O(d,d)$ can be written in the form

$$\mathscr{O}=\begin{pmatrix} 1_d & 0 \\ B & 1_d \end{pmatrix}\begin{pmatrix} N & 0 \\ 0 & N^{-\top} \end{pmatrix}\begin{pmatrix} 1_d & \theta \\ 0 & 1_d \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} N & 0 \\ 0 & N^{-\top} \end{pmatrix} \ : \ X+\xi \longmapsto \iota_X N+\big(N^{-\top}\big)^\sharp\xi \tag{3.3}$$

where $N\in GL(d)$ determines a general linear transformation of sections.

The generalization of the Lie bracket of vector fields X,Y on TM to sections $X+\xi,Y+\eta$ of $TM\oplus T^*M$ is provided by the Dorfman bracket

$$[X+\xi,Y+\eta]_{\rm D}=[X,Y]+\pounds_X\eta-\iota_Y\,{\rm d}\xi \ , \tag{3.4}$$

where $\pounds_X$ denotes the Lie derivative along X. Fluxes can be incorporated into this bracket structure by adding appropriate twisting terms, e.g. $\iota_X\,\iota_Y H$. Alternatively, they may be added by applying suitable $O(d,d)$-transformations of sections of the generalized tangent bundle, starting from the standard geometric frame with metric flux, basis (e_i,e^i) and the Dorfman brackets

$$[e_i,e_j]_{\rm D}=f^k{}_{ij}\,e_k \ , \qquad [e^i,e_j]_{\rm D}=f^i{}_{jk}\,e^k \qquad \text{and} \qquad [e^i,e^j]_{\rm D}=0 \ . \tag{3.5}$$

Then under a B-transform of the basis (e_i,e^i) these brackets map into

$$[e_i,e_j]_{\rm R}=f^k{}_{ij}\,e_k+H_{ijk}\,e^k \ , \qquad [e^i,e_j]_{\rm R}=f^i{}_{jk}\,e^k \qquad \text{and} \qquad [e^i,e^j]_{\rm R}=0 \ , \tag{3.6}$$

with the geometric NS–NS flux $H=\mathrm{d}B$, while under a θ-transform they map to

$$[e_i,e_j]_{\rm R}=f^k{}_{ij}\,e_k\ ,\quad [e^i,e_j]_{\rm R}=f^i{}_{jk}\,e^k-Q^{ik}{}_j\,e_k\quad\text{and}\quad [e^i,e^j]_{\rm R}=Q^{ij}{}_k\,e^k+R^{ijk}\,e_k\ , \tag{3.7}$$

with the globally and locally non-geometric fluxes

$$Q^{ij}{}_k=\partial_k\theta^{ij}+2\,f^{[i}{}_{kl}\,\theta^{l\underline{j}]}\qquad\text{and}\qquad R^{ijk}=3\,\big([\theta,\theta]_{\rm S}^{ijk}+f^{[\underline{i}}{}_{lm}\,\theta^{\underline{j}m}\,\theta^{\underline{k}m]}\big)\ . \tag{3.8}$$

In this way the local $O(d,d)$-transformations of the Dorfman bracket (3.5) reproduce the Roytenberg bracket (3.2). What is particularly noteworthy and relevant for us here is that the non-geometric fluxes are determined by a bivector θ.

3.2 Double field theory

Let us now briefly explain how double field theory [108, 109, 73] provides a *microscopic* description of Q-flux and R-flux, through the chain (3.1) of T-duality transformations, by a formal definition and unified description of non-geometric fluxes; see e.g. [1, 22, 69] for reviews. The idea of double field theory is to double the target space coordinates x^i to $x^I=(x^i,\tilde x_i)$, where $\tilde x_i$ are the T-dual “winding coordinates”. This gives a formalism with manifest $O(d,d)$-symmetry that allows one to perform such T-dualities to non-geometric frames.

Double field theory is a field theory for the massless modes of closed bosonic string theory that treats diffeomorphism symmetry and B-field gauge transformations on equal footing by assembling them into the *generalized metric*

$$\mathscr{H}=\begin{pmatrix} g^{-1} & -2\pi\,\ell_s^2\,g^{-1}\,B\\ 2\pi\,\ell_s^2\,B\,g^{-1} & g-(2\pi\,\ell_s^2)^2\,B\,g^{-1}\,B\end{pmatrix}\ . \tag{3.9}$$

This metric can be written in terms of a Schur decomposition

$$\mathscr{H}=\begin{pmatrix} 1_d & 0\\ 2\pi\,\ell_s^2\,B & 1_d\end{pmatrix}\begin{pmatrix} g^{-1} & 0\\ 0 & g\end{pmatrix}\begin{pmatrix} 1_d & -2\pi\,\ell_s^2\,B\\ 0 & 1_d\end{pmatrix}\ , \tag{3.10}$$

which identifies it as a B-transform of the doubled space metric when $B=0$. There is a global $O(d,d)$-symmetry that includes T-duality and acts as

$$x^I\longmapsto\mathscr{O}^I{}_J\,x^J\ ,\qquad \mathscr{H}\longmapsto\mathscr{O}^\top\,\mathscr{H}\,\mathscr{O}\qquad\text{with}\quad \mathscr{O}=(\mathscr{O}^I{}_J)\in O(d,d)\ . \tag{3.11}$$

One then constructs an $O(d,d)$-invariant action for $\mathscr{H}$ and halves the degrees of freedom by imposing the strong constraint $\partial_i\otimes\tilde\partial^i+\tilde\partial^i\otimes\partial_i=0$ on all products of fields of double field theory, which is a strong version of the level-matching condition $L_0-\overline{L}_0=0$ in the worldsheet conformal field theory, i.e. $p_i\,w^i=0$. Solving the strong constraint amounts to choosing a polarization on the doubled space [72]; for example, in the geometric supergravity frame one takes $\tilde\partial^i:=\frac{\partial}{\partial\tilde x_i}=0$, which reduces the fields of double field theory to fields of generalized geometry. Different polarisations define different T-duality frames, and any two frames are related by an $O(d,d)$-transformation.

Let us consider the example of flat space, $g^\circ=1_d$, with constant H-flux, and choose the Kalb-Ramond field in the symmetric gauge $B^\circ_{ij}=\frac13\,H_{ijk}\,x^k$; this is locally defined (for $x\in\mathbb{R}^d$), but not globally if the spacetime is e.g. a torus, where it is only defined up to large gauge transformations. A T-duality in the i-th direction interchanges x^i with $\tilde{x}_i$ and is implemented by the factorized $O(d,d)$-transformation matrix

$$\mathscr{T}_i = \begin{pmatrix} 1_d - E_i & E_i \\ E_i & 1_d - E_i \end{pmatrix} , \tag{3.12}$$

where E_i are the $d \times d$ matrix units $(E_i)_{kl} = \delta_{ki}\,\delta_{li}$. Consider now the effect of applying two successive T-duality transformations $\mathscr{T}_{(ij)} := \mathscr{T}_i\,\mathscr{T}_j$ to the corresponding generalized metric:

$$\begin{aligned} \mathscr{H}_{(ij)} &= \mathscr{T}_{(ij)}^{\top} \begin{pmatrix} 1_d & -2\pi\,\ell_s^2\,B^\circ \\ 2\pi\,\ell_s^2\,B^\circ & 1_d - (2\pi\,\ell_s^2)^2\,(B^\circ)^2 \end{pmatrix} \mathscr{T}_{(ij)} \\ &=: \begin{pmatrix} g^{-1} & -2\pi\,\ell_s^2\,g^{-1}\,B \\ 2\pi\,\ell_s^2\,B\,g^{-1} & g - (2\pi\,\ell_s^2)^2\,B\,g^{-1}\,B \end{pmatrix} . \end{aligned} \tag{3.13}$$

One easily computes that the new metric and Kalb-Ramond field (g,B) are not globally defined in the directions orthogonal to the (x^i,x^j)-plane, which is the earmark of the 'global non-geometry' of the T-fold in the geometric parameterization. However, a suitable field redefinition appropriate to the transformation from a geometric frame to a non-geometric frame yields a new parameterization of the generalized metric in double field theory as [6]

$$\mathscr{H}_{(ij)} = \begin{pmatrix} G^{-1} - \frac{1}{(2\pi\,\ell_s^2)^2}\,\theta\,G\,\theta & \frac{1}{2\pi\,\ell_s^2}\,\theta\,G \\ -\frac{1}{2\pi\,\ell_s^2}\,G\,\theta & G \end{pmatrix} , \tag{3.14}$$

where the new metric G and bivector $\theta = \frac{1}{2}\,\theta^{ij}\,\partial_i \wedge \partial_j$ are given precisely by the open-closed string relation (1.9), as anticipated from the Büscher rules. This metric can similarly be obtained from a θ-transform given by

$$\mathscr{H}_{(ij)} = \begin{pmatrix} 1_d & \frac{1}{2\pi\,\ell_s^2}\,\theta \\ 0 & 1_d \end{pmatrix} \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} 1_d & 0 \\ -\frac{1}{2\pi\,\ell_s^2}\,\theta & 1_d \end{pmatrix} . \tag{3.15}$$

In the present case one computes

$$G = 1_d \qquad \text{and} \qquad \theta^{ij} = Q^{ij}{}_k\,x^k , \tag{3.16}$$

which defines flat space with constant non-geometric Q-flux

$$Q^{ij}{}_k = \partial_k\,\theta^{ij} . \tag{3.17}$$

Finally, let us apply a T-duality transformation $\mathscr{T}_k \in O(d,d)$ along a remaining non-isometric direction, which exchanges the physical coordinate x^k with the winding coordinate $\tilde{x}_k$ and maps the geometric data (3.16) to

$$\tilde{G} = 1_d \qquad \text{and} \qquad \tilde{\theta}^{ij} = R^{ijk}\,\tilde{x}_k . \tag{3.18}$$

This defines flat space with constant non-geometric R-flux

$$R^{ijk} = \tilde{\partial}^{[i}\tilde{\theta}^{jk]} . \tag{3.19}$$

As the R-flux explicitly involves derivatives of the winding coordinates, it cannot be described in ordinary geometry and in this sense the R-flux frame is 'locally non-geometric'. In this simple example, one can alternatively regard $\tilde{\theta}$ as a two-form on the dual winding space of curvature $R = \mathrm{d}\tilde{\theta}$.

3.3 Overview of conformal field theory results

Let us now summarise the explicit worldsheet calculations which have suggested the appearance of noncommutative and nonassociative deformations of geometry in the non-geometric frames of closed string theory:

- The original suggestion of nonassociativity by [28] computes the cyclic equal time double commutator

$$[x^i,x^j,x^k] := \lim_{\sigma_i\to\sigma} [[x^i(\sigma_1,\tau),x^j(\sigma_2,\tau)],x^k(\sigma_3,\tau)] + \text{cyclic} \tag{3.20}$$

  in the $SU(2)$ Wess-Zumino-Witten model with H-flux, and finds a non-vanishing target space quantity.

- Explicit calculations of phase space commutators by canonical quantization of closed strings in flat space and in a linear B-field background were carried out in [84, 46, 5], by studying monodromy properties and the corresponding twisted closed string boundary conditions, which lead to a shifted closed string mode expansion analogous to the open string case and expansions in asymmetric orbifold string theories. These calculations reveal generally a doubled phase space nonassociative geometry in the different T-duality frames.

- Correlators of products of tachyon vertex operators in sigma-model perturbation theory about a flat geometry with small constant H-flux were computed in [33], and after conformal field theory T-duality shown to reproduce the triproducts discussed in Section 1.

Further evidence is provided in e.g. [42, 47, 24, 17, 93].

The resulting quantum geometry structures can be described as follows. Let us first consider the Q-flux frame. Naively, in analogy with the open string case, the bivector of (3.16) in the non-geometric parameterization would suggest noncommutativity, but in the closed string case one needs more: Only closed strings which wind in the Q-flux frame can probe a quantum deformation of the geometry, and the noncommutativity in this case is determined by a Wilson line of the Q-flux as [7]

$$[x^i,x^j] = \frac{\mathrm{i}\,\ell_s^4}{3\hbar}\oint \mathrm{d}\theta^{ij} = \frac{\mathrm{i}\,\ell_s^4}{3\hbar}\oint Q^{ij}{}_k\,\mathrm{d}x^k = \frac{\mathrm{i}\,\ell_s^4}{3\hbar}\,Q^{ij}{}_k\,w^k \ , \tag{3.21}$$

where w^k are the closed string winding numbers. Since the winding numbers are central elements in this algebra, i.e. $[w^i,w^j]=[x^i,w^j]=0$, these relations define a noncommutative but associative geometry.

To probe nonassociativity one needs to introduce local R-flux. The explicit computations above yield the tribracket

$$[x^i,x^j,x^k] = \ell_s^4\,R^{ijk} \ . \tag{3.22}$$

Let us quickly sketch how this bracket is derived from conformal perturbation theory, refering to the original work [33] and also to the nice review [26] for details of the calculation. Using complex coordinates on the Riemann sphere $S^2\simeq\mathbb{C}\cup\{\infty\}$, the worldsheet equations of motion to linear order in the H-flux for a flat target space read $\partial\overline{\partial}x^i = \frac12 H^i{}_{jk}\,\partial x^j\,\overline{\partial}x^k$, and thus the coordinate fields have to be modified in order to be consistent with the conformal field theory description, wherein the perturbation of the free worldsheet sigma-model by the H-flux should yield a marginal deformation. We therefore replace the usual conserved currents by

$$\mathscr{J}^{\,i} = \mathrm{i}\,\partial x^i - \frac{\mathrm{i}}{2}\,H^i{}_{jk}\,\partial x^j\,x^k_{\rm r} \qquad \text{and} \qquad \overline{\mathscr{J}}^{\,i} = \mathrm{i}\,\overline{\partial}x^i - \frac{\mathrm{i}}{2}\,H^i{}_{jk}\,x^j_{\rm l}\,\overline{\partial}x^k \tag{3.23}$$

where $x^i = x^i_{\rm l} + x^i_{\rm r}$ is the decomposition of the string fields into left and right moving modes; the dual winding fields are then given by $\tilde{x}^i = x^i_{\rm l} - x^i_{\rm r}$. The correlation functions of three insertions of the currents $\mathscr{J}^{\,i}$ are readily computed, and after writing $\mathscr{J}^{\,i} =: \mathrm{i}\,\partial\mathscr{X}^{\,i}$ and performing three worldsheet integrations, one arrives at the closed string generalization of the second term in the open string propagator (1.8) for the modified string fields $\mathscr{X}^{\,i}$ with the sign function, that arises from combinations of the complex logarithm function, replaced by certain combinations of the complex Rogers dilogarithm function, and the bivector θ^{ij} substituted by the trivector $\theta^{ijk} = \frac{\ell_s^4}{\hbar}\,H^{ijk}$.

A triple T-duality transformation $\mathscr{T}_{(ijk)} := \mathscr{T}_i\,\mathscr{T}_j\,\mathscr{T}_k$ is affected in the worldsheet conformal field theory as an asymmetric reflection of the right-moving string coordinates: $\mathscr{X}_{\rm l}^{\,i} \mapsto \mathscr{X}_{\rm l}^{\,i}$, $\mathscr{X}_{\rm r}^{\,i} \mapsto -\mathscr{X}_{\rm r}^{\,i}$, which maps winding modes w^i in the H-flux frame to momentum modes p_i in the R-flux frame. The three-point correlators of the corresponding tachyon vertex operators $\mathrm{e}^{\,\mathrm{i}\,k\cdot\mathscr{X}}$ thereby produce a phase which is trivial in the H-flux frame, but which is non-trivial in the R-flux frame and encoded in scattering amplitudes

$$(f\,\triangle\,g\,\triangle\,h)(x) = \int \mathrm{d}k\,\int \mathrm{d}q\,\int \mathrm{d}r\;\tilde{f}(k)\,\tilde{g}(q)\,\tilde{h}(r)\;\mathrm{e}^{\,-\frac{\mathrm{i}\,\ell_s^4}{6}\,R^{ijk}\,k_i\,q_j\,r_k}\;\mathrm{e}^{\,\mathrm{i}\,(k+q+r)\cdot x} \tag{3.24}$$

by the triproducts (1.16) with $\theta^{ijk} = \frac{\ell_s^4}{\hbar}\,R^{ijk}$. These phases are consistent with the crossing symmetry of conformal field theory correlation functions, or equivalently associativity of the operator product expansion, because by momentum conservation one has $p_i\,q_j\,r_k\,\theta^{ijk} = 0$ whenever $p+q+r=0$. This is equivalent to the cyclicity property of the triproduct

$$\int \mathrm{d}x\;f\,\triangle\,g\,\triangle\,h = \int \mathrm{d}x\;f\cdot g\cdot h\;, \tag{3.25}$$

which for constant R-flux follows from integration by parts since the two integrands differ by total derivatives. Hence the triproducts are consistent with the axioms of conformal field theory and nonassociativity is not probed by the on-shell theory. Note that it is the modified string coordinates $\mathscr{X}^{\,i}$ and not the original ones x^i which probe the nonassociative geometry, a feature which is also confirmed in other worldsheet approaches [17].

Alternatively, following what we did for the Q-flux frame, using the bivector from (3.18) we find noncommutativity probed by closed strings which propagate in the R-flux frame given by

$$[x^i,x^j] = \frac{\mathrm{i}\,\ell_s^4}{3\hbar}\,\oint \mathrm{d}\tilde{\theta}^{ij} = \frac{\mathrm{i}\,\ell_s^4}{3\hbar}\,\oint R^{ijk}\,\mathrm{d}\tilde{x}_k = \frac{\mathrm{i}\,\ell_s^4}{3\hbar}\,R^{ijk}\,p_k \tag{3.26}$$

where p_k are the closed string momentum modes. However, now the variables x^i and p_i are canonically conjugate, and so the tribracket (3.22) can be regarded as the Jacobiator of the precursor nonassociative phase space algebra

$$[x^i,x^j] = \frac{\mathrm{i}\,\ell_s^4}{3\hbar}\,R^{ijk}\,p_k\;, \qquad [x^i,p_j] = \mathrm{i}\,\hbar\,\delta^i{}_j \qquad \text{and} \qquad [p_i,p_j]=0\;. \tag{3.27}$$

Note that (3.27) for $d = 3$ is formally the same as the magnetic monopole phase space algebra (2.3) in the case of the linear magnetic field (2.15), after applying the magnetic duality transformation of order four given by

$$x^i \longmapsto -p_i \ , \qquad p_i \longmapsto x^i \qquad \text{and} \qquad \mu_0\,\rho_m\,\hbar\,e\,\varepsilon_{ijk} \longmapsto -\frac{\ell_s^4}{\hbar}\,R^{ijk} \ , \tag{3.28}$$

which in the absence of fluxes defines a canonical transformation. In particular, the locally non-geometric flux R now appears as the curvature three-form of a two-form connection on a gerbe over momentum space [89]. The quantization of the brackets (3.27) can be captured by an explicit nonassociative star product on the phase space T^*M of the original target space M [89, 16, 83], whose construction and physical implications will be discussed in detail later on.

3.4 Caveats

There are a few loopholes in the conformal field theory derivations mentioned above that should be pointed out:

1. The conformal field theory calculations are all performed in flat space with constant H-flux and constant dilaton. Such a background only satisfies the closed string equations of motion derived from the action (1.19) to linear order in H (in the critical dimension):

$$\begin{aligned} 0 &= \mathrm{Ric}_{ij} - \frac{1}{4}\,H_i{}^{jk}\,H_{jkl} + 2\,\nabla_i\nabla_j\Phi + O(\ell_s^4) \ , \\ 0 &= -\frac{1}{2}\,\nabla_k H^k{}_{ij} + \ell_s^2\,H_{ij}{}^k\,\nabla_k\Phi + O(\ell_s^4) \ , \\ 0 &= \ell_s^2\,\Big((\nabla\Phi)^2 - \frac{1}{2}\,\nabla^2\Phi - \frac{1}{24}\,H^2\Big) + O(\ell_s^4) \ . \end{aligned} \tag{3.29}$$

2. On a compact space with non-trivial topology such as the torus, the H-flux is quantized by virtue of generalized Dirac charge quantization in string theory. The true T-duality transformations connecting physically equivalent string theories are then valued in the discrete subgroup $O(d,d;\mathbb{Z}) \subset O(d,d)$. In this case the notion of linear order in H, as well as its extrapolation to non-geometric polarizations, is meaningless as H cannot be continuously varied.

3. The triproduct violates the strong constraint between the background R^{ijk} and the fluctuations around it [34]. This is not a serious problem if polarization can be achieved by a weaker constraint; this issue is currently under debate as the strong constraint indeed seems too stringent for some considerations. We elaborate on this point further in Section 5.

4. Nonassociativity does not appear in the algebra of conformal currents of the worldsheet theory, but rather through non-conformal fields such as $\mathscr{X}^i$ above.

5. Thus far no construction is available of H-deformed *graviton* vertex operators, whose correlation functions would clarify if and how a nonassociative deformation of the gravity theory defined by (1.19) could be of relevance in closed string theory.

Despite these caveats, the geometric structures unveiled and their novel physical implications are so rich and beautiful that work has plugged along in this direction to further explore the ramifications and relevance of nonassociative geometry in closed string theory. For example, recent discussions reveal that Yang-Baxter deformations (in general a class of non-abelian T-duality) may be viewed as the open-closed string map (1.9) at the generalized supergravity level, see [18] and references therein; in this setting the Jacobi identity yields the classical Yang-Baxter equation for the isometry group of the background. In the remainder of this contribution we shall discuss further ways of naturally understanding the origins of nonassociativity, and then proceed to unravel some of the physics of this structure.

4. Higher geometrization of non-geometry

In Section 3 we have discussed two geometric ways of making sense of the globally and locally non-geometric frames of flux compactifications: One through an extension of geometry into the realm of generalized geometry and double field theory, and the other through noncommutative and nonassociative deformations of the closed string phase space. To better understand the relationship between these two points of view from a dynamical perspective, we shall now show that the non-geometry of string backgrounds is geometrized through a membrane sigma-model, suggesting that the proper probes of these backgrounds should be open membranes whose boundary modes are the closed string degrees of freedom. Such an approach was first suggested by [65], and developed in detail in [89]; see [43, 23] for further developments. In Section 5 below we shall show that the quantization of this sigma-model produces the nonassociative phase space star product that we have been advertising.

To understand the idea behind this framework, let us recall how we treated the open string sigma-model (1.4). In the Seiberg-Witten scaling limit it reduces to the action (1.5), whose first order formalism describes a two-dimensional topological field theory called the 'Poisson sigma-model'. As we will discuss in Section 5, the quantization of this sigma-model reproduces the commutators (1.7) for a constant B-field, and more generally quantizes the noncommutative geometry defined by the Poisson bivector $\theta = B^{-1}$ on $M = \mathbb{R}^d$ for vanishing NS–NS flux $H = \mathrm{d}B = 0$.

At the other extreme, consider a constant non-vanishing NS–NS flux H on $M = \mathbb{R}^d$ and choose the symmetric gauge $B^\circ_{ij} = \frac{1}{3} H_{ijk}\, x^k$ for the Kalb-Ramond field. In this case the worldsheet action (1.5) can be written as

$$S\big|_{g,\ell_s=0} = -\frac{1}{2}\oint_{\Sigma_2} \frac{1}{3} H_{ijk}\, x^k\, \mathrm{d}x^i \wedge \mathrm{d}x^j = -\frac{1}{2}\int_{\Sigma_3} H_{ijk}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k\ , \tag{4.1}$$

where Σ_3 is a three-dimensional worldvolume with boundary $\partial\Sigma_3 = \Sigma_2$ and we used Stokes' theorem. This is just the well-known Wess-Zumino action, which is needed in particular for a global formulation of the non-linear sigma-model when the target space M is e.g. a torus. The corresponding first order formulation of this action is called the 'H-twisted Poisson sigma-model', which captures the topological dynamics of closed strings in a non-trivial H-flux background. It is a particular case of a three-dimensional topological field theory called a 'Courant sigma-model', which is defined on a *membrane* worldvolume.

The somewhat heuristic considerations above can be formalized within the setting of generalized geometry, which provides a higher geometric framework in which to study the geometric and non-geometric frames of closed string theory as we have seen in Section 3 together with some aspects of its extension to double field theory.

4.1 AKSZ theory

The Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) construction [3] is a geometric framework for constructing Schwarz-type action functionals in the Batalin-Vilkovisky formalism for $n+1$-dimensional topological sigma-models whose target space is a symplectic Lie n-algebroid $E \to M$. These theories fit into a geometric ladder describing $n-1$-dimensional degrees of freedom in background fields, see e.g. [75] for a review. The open string and membrane sigma-models alluded to above are the first two members of this hierarchy of theories, whose geometric structures we shall now discuss.

Consider first the case $n = 1$. A Lie 1-algebroid is simply called a *Lie algebroid* and consists of a vector bundle E over the target space M that sits in a diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \rho\ } & TM \\ & {\scriptstyle \pi}\searrow & \downarrow \\ & & M \end{array} \tag{4.2}$$

where π is the bundle projection and the anchor map ρ to the tangent bundle is compatible with a given Lie bracket $[s,s']_E$ on sections s,s' of E, in the sense that the following conditions hold:

- $[s, f\,s']_E = f\,[s,s']_E + \big(\rho(s)f\big)\,s'$,
- $\rho\big([s,s']_E\big) = \big[\rho(s),\rho(s')\big]$,

for any function $f \in C^\infty(M)$. The first axiom is the Leibniz rule, while the second axiom states that the anchor is a homomorphism between the Lie algebras of sections of E and TM. If M is a point, then a Lie algebroid is the same thing as a Lie algebra $\mathfrak{g}$ with zero anchor. On the other hand, the tangent bundle TM of any manifold M is trivially a Lie algebroid with the identity anchor map, and is called the *standard* Lie algebroid. The notion of a Lie algebroid thus generalizes these two simple examples simultaneously. For our purposes, the most relevant example of a Lie algebroid is the cotangent bundle $E = T^*M$ over a Poisson manifold (M,θ). In this case the anchor is $\rho = \theta^\sharp$ and the Lie bracket is the Koszul bracket

$$[\eta,\xi]_{\rm K} = \pounds_{\theta^\sharp\eta}\,\xi - \pounds_{\theta^\sharp\xi}\,\eta - {\rm d}\big(\theta(\eta,\xi)\big) \tag{4.3}$$

for one-forms η,ξ on M; in particular, for functions f,g one has $[{\rm d}f,{\rm d}g]_E = {\rm d}\{f,g\}$ where $\{f,g\} = \theta({\rm d}f\wedge{\rm d}g)$ is the Poisson bracket induced by θ.

The AKSZ sigma-model in this case is the Poisson sigma-model which is described in more detail below. It quantizes point particles, viewed as boundaries of open strings, in background magnetic fields, with underlying Hamiltonian dynamics governed by the Poisson bracket. This leads to a noncommutative geometry.

Consider next the case $n = 2$. A symplectic Lie 2-algebroid is the same thing as a *Courant algebroid*, which is a vector bundle over M sitting in a diagram like (4.2) with a (not necessarily antisymmetric) bracket $[s,s']_E$ on sections s,s' of E and a fibrewise metric $\langle s,s'\rangle_E$ satisfying the following conditions:

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- $[s,[s',s'']_E]_E = [[s,s']_E,s'']_E + [s',[s,s'']_E]_E$,
- $[s,f\,s']_E = f\,[s,s']_E + \big(\rho(s)f\big)\,s'$,
- $\rho\big([s,s']_E\big) = \big[\rho(s),\rho(s')\big]$,
- $\rho(s'')\langle s,s'\rangle_E = \langle [s'',s]_E,s'\rangle_E + \langle s,[s'',s']_E\rangle_E$.

The first two properties endow the vector bundle E with the structure of a 'Leibniz algebroid', and are a generic feature of all symplectic Lie n-algebroids; if the bracket is antisymmetric, as in the case $n=1$, then the first axiom is equivalent to the Jacobi identity. If M is a point, then a Courant algebroid is the same thing as a quadratic Lie algebra, i.e. a Lie algebra $\mathfrak{g}$ with an invariant inner product; a natural class of examples is provided by the Drinfeld double $\mathfrak{g}\oplus\mathfrak{g}^*$ of a Lie bialgebra $\mathfrak{g}$. For our purposes, the significance of this higher geometric structure is that if the generalized tangent bundle $E=TM\oplus T^*M$ is endowed with the Dorfman bracket (3.5), the natural pairing $\langle e_i,e^j\rangle=\delta_i{}^j$ between TM and T^*M, and anchor given by the projection $\rho(X+\xi)=X$, then E forms a Courant algebroid called the *standard* Courant algebroid.

The AKSZ sigma-model in this case is the Courant sigma-model which is studied in detail below. It quantizes closed strings, regarded as boundaries of open membranes, in flux compactifications, with underlying worldsheet Hamiltonian dynamics governed by the Dorfman bracket. This leads to a nonassociative geometry.

The list continues, but the cases with $n\geqslant 3$ are not as well understood (see Section 6 below). Let us now turn to the particular cases of AKSZ sigma-models of direct relevance to us and clarify the statements made above.

4.2 Poisson sigma-models

For $n=1$, the most general two-dimensional topological field theory that can be constructed from the AKSZ theory is based on the symplectic Lie algebroid $E=T^*M$, and gives rise to the *Poisson sigma-model* [74, 100] defined by the degree zero part of the AKSZ action

$$S^{(1)}_{\rm AKSZ} = \int_{\Sigma_2}\Big(\xi_i\wedge{\rm d}x^i + \frac12\,\theta^{ij}(x)\,\xi_i\wedge\xi_j\Big)\ , \tag{4.4}$$

where $x:\Sigma_2\to M$ are the string fields, ξ are auxiliary one-forms on Σ_2 valued in the cotangent bundle T^*M, and $\theta=\frac12\,\theta^{ij}(x)\,\partial_i\wedge\partial_j$ is a Poisson bivector on M. When $\theta=B^{-1}$ is non-degenerate, integrating out the auxiliary one-form fields ξ_i yields the topological B-field amplitude (1.5). With suitable Dirichlet boundary conditions on the fields, the perturbative expansion of the corresponding path integral leads to the Kontsevich formality maps [80, 41], which we will discuss in Section 5 below. The on-shell condition derived from this action is equivalent to $[\theta,\theta]_{\rm S}=0$, i.e. that the bivector θ defines a Poisson structure on M, but the sigma-model makes sense off-shell as well, for instance when θ is a twisted Poisson structure.

4.3 Courant sigma-models

For $n=2$, we take the symplectic Lie 2-algebroid to be the standard Courant algebroid $E=TM\oplus T^*M$, and define the three-dimensional topological field theory on a membrane worldvolume

Σ_3 by the degree zero part of the AKSZ action

$$S^{(2)}_{\rm AKSZ} = \int_{\Sigma_3} \Big(\phi_i \wedge {\rm d}x^i + \frac{1}{2}\, \eta_{IJ}\, \alpha^I \wedge {\rm d}\alpha^J - \rho_I{}^i\, \phi_i \wedge \alpha^I + \frac{1}{6}\, T_{IJK}(x)\, \alpha^I \wedge \alpha^J \wedge \alpha^K \Big) \ , \tag{4.5}$$

where $x:\Sigma_3\to M$ are the membrane fields, α are one-forms on Σ_3 valued in E, and ϕ are auxiliary two-forms on Σ_3 valued in the cotangent bundle T^*M. The fibre metric has components $\eta_{IJ} = \langle s_I, s_J\rangle_E$ in a local basis of sections s_I of E, the anchor map has components $\rho(s_I) = \rho_I{}^i\, e_i$, and the three-form $T_{IJK}(x) = \langle s_I, [s_J, s_K]_E\rangle_E$ can accomodate all four geometric and non-geometric fluxes in the T-duality orbit (3.1). This is called the *Courant sigma-model* [96].

Let us begin by describing the geometric H-flux frame. We write the one-forms as $\alpha = (\alpha^I) = (\alpha^i, \xi_i)$ corresponding to the splitting $E = TM\oplus T^*M$ and use the H-twisted Dorfman bracket from (3.6) with vanishing torsion to write the open membrane action

$$S^{(2)}_H = \int_{\Sigma_3} \Big(\phi_i \wedge {\rm d}x^i + \alpha^i \wedge {\rm d}\xi_i - \phi_i \wedge \alpha^i + \frac{1}{6}\, H_{ijk}(x)\, \alpha^i \wedge \alpha^j \wedge \alpha^k \Big) \ . \tag{4.6}$$

With suitable Dirichlet boundary conditions on the fields, by integrating out the auxiliary two-form fields ϕ_i we arrive at the action on the closed string worldsheet $\Sigma_2 = \partial\Sigma_3$ given by

$$S^{(2)}_H = \oint_{\Sigma_2} \xi_i \wedge {\rm d}x^i + \int_{\Sigma_3} \frac{1}{6}\, H_{ijk}(x)\, {\rm d}x^i \wedge {\rm d}x^j \wedge {\rm d}x^k \ . \tag{4.7}$$

This is just the Wess-Zumino action, which for constant H-flux falls entirely on the worldsheet Σ_2 as in (4.1). One can also add a boundary perturbation to the membrane sigma-model by an arbitrary bivector θ on M of the form given in (4.4). This defines the action of the *H-twisted Poisson sigma-model* [78], whose on-shell conditions imply $[\theta,\theta]_{\rm S} = \bigwedge^3 \theta^\sharp H$, so that θ defines an H-twisted Poisson structure on the target space M; in particular, the Jacobi identity for the corresponding bracket defined by θ is violated by the NS–NS three-form flux.

Now let us consider the locally non-geometric R-flux frame. Using the R-twisted Dorfman bracket from (3.7) with vanishing torsion and Q-flux, and with notation as above, after integrating out the auxiliary fields ϕ_i again we arrive at the open membrane sigma-model with (rescaled) R-flux given by the action

$$S^{(2)}_R = \int_{\Sigma_3} \Big({\rm d}\xi_i \wedge {\rm d}x^i + \frac{\ell_s^4}{18\hbar^2}\, R^{ijk}(x)\, \xi_i \wedge \xi_j \wedge \xi_k \Big) \ . \tag{4.8}$$

When the R-flux is constant, with suitable Dirichlet boundary conditions the equations of motion for the boundary string fields x^i imply $\xi_i = {\rm d}p_i$ for local fields p valued in the fibres of the cotangent bundle T^*M (up to harmonic forms on Σ_2), so that using Stokes' theorem and linearizing the resulting action with auxiliary one-form fields η_I valued in E leads to the boundary action [89]

$$S^{(2)}_R = \oint_{\Sigma_2} \Big(\eta_I \wedge {\rm d}x^I + \frac{1}{2}\, \Theta^{IJ}\, \eta_I \wedge \eta_J \Big) \ . \tag{4.9}$$

Here $x = (x^I) = (x^i, p_i) : \Sigma_2 \to T^*M$ are string fields valued in the cotangent bundle of M, so that the effective target space is now the *phase space*. This action defines a Poisson sigma-model for the bivector

$$\Theta = (\Theta^{IJ}) = \begin{pmatrix} R(p) & 1_d \\ -1_d & 0 \end{pmatrix} \qquad \text{with} \quad R(p)^{ij} = \frac{\ell_s^4}{3\hbar^2}\, R^{ijk}\, p_k \ . \tag{4.10}$$

The corresponding quantum phase space brackets

$$[x^I, x^J] = \mathrm{i}\hbar\,\Theta^{IJ}(x) \tag{4.11}$$

coincide precisely with the R-space commutation relations (3.27). Note that for $d = 3$ this twisted Poisson structure is formally identical to (2.4) with a linear magnetic field (2.15) under the magnetic duality transformation (3.28), and in particular the commutators (4.11) together with the corresponding Jacobiators

$$[x^I, x^J, x^K] = -\hbar^2\,[\Theta,\Theta]_{\mathrm{S}}^{IJK} = \begin{pmatrix} \ell_s^4\,R^{ijk} & 0 \\ 0 & 0 \end{pmatrix} \tag{4.12}$$

yield a noncommutative and nonassociative phase space geometry.

One important caveat with this derivation is that the setting of the Q-flux to zero in (3.7) does not define a Courant algebroid structure, as is evident from (3.8). This simply reflects the fact that the nonassociative geometry of the R-flux frame violates the strong constraint of double field theory, as mentioned previously, so that the corresponding membrane sigma-model does not define a Courant sigma-model. It can, however, be obtained in a precise way via projection from a proper Courant sigma-model defined on the doubled space of double field theory that incorporates all fluxes of (3.1) in a manifestly T-duality invariant way [44], which clarifies precisely the geometric algebroid structure underlying the non-geometric frames of closed string theory, and also how gauge invariance is restored in the non-geometric membrane sigma-models through the Bianchi identities among the fluxes of the T-duality chain (3.1). In [44] it is also shown how to treat the noncommutative geometry of the globally non-geometric Q-flux frame in an analogous manner, and how to generally treat the non-geometry through Courant sigma-models via the open-closed string reparameterization (1.9) of the algebroid structure maps, thereby clarifying the evident similarities we have seen between the phase space and double field theory frameworks for non-geometry (see also [58, 59]).

5. Quantization of non-geometric backgrounds

In this section we will describe the quantization of the twisted Poisson structure (4.10) describing the nonassociative geometry of the closed string phase space in the R-flux frame, and then present some of its far reaching applications to non-geometric string theory.

5.1 Quantization of topological string theory

Suitable functional integrals in the R-flux Poisson sigma-model (4.9) reproduce Kontsevich's formality maps for global deformation quantization of twisted Poisson manifolds [80]. The formality maps U_n take n multivector fields $\mathscr{X}_1, \dots, \mathscr{X}_n$ of degrees $k_1, \dots, k_n$ to multidifferential operators $D_\Gamma(\mathscr{X}_1, \dots, \mathscr{X}_n)$ of degree $2 - 2n + k_1 + \cdots + k_n$. They have a combinatorial expansion as a sum over graphs

$$U_n(\mathscr{X}_1, \dots, \mathscr{X}_n) = \sum_{\Gamma_n \in G_n} w_{\Gamma_n}\, D_{\Gamma_n}(\mathscr{X}_1, \dots, \mathscr{X}_n)\ , \tag{5.1}$$

where G_n is the set of admissible graphs Γ_n, with n vertices and edges $\{e_i^1,\dots,e_i^{k_i}\}_{i=1}^n$, which can be drawn in the configuration space $\mathbb{H}_n$ of n points on the hyperbolic plane with prescribed weights

$$w_{\Gamma_n} = \frac{1}{(2\pi)^{\sum_i k_i}} \int_{\mathbb{H}_n} \bigwedge_{i=1}^{n} \big(\mathrm{d}\phi_{e_i^1}\wedge\cdots\wedge\mathrm{d}\phi_{e_i^{k_i}}\big) \ . \tag{5.2}$$

In particular, inserting any bivector $\mathscr{X}_i = \Theta = \frac12\,\Theta^{IJ}\,\partial_I\wedge\partial_J$ into all slots gives a sum of bi-differential operators, which can be represented diagrammatically by graphs with edges emanating along the legs of Θ and operating on functions f,g sitting on the boundary of the hyperbolic plane. This defines the star product

$$f\star g = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\hbar)^n}{n!}\, U_n(\Theta,\dots,\Theta)(f,g) =: \mathscr{P}(\Theta)(f,g) \ , \tag{5.3}$$

which up to order $\hbar^2$ is given explicitly by

$$\begin{aligned} f\star g = f\cdot g + \frac{\mathrm{i}\hbar}{2}\,\Theta^{IJ}\,\partial_I f\cdot\partial_J g - \frac{\hbar^2}{4}\,\Theta^{IJ}\,\Theta^{KL}\,\partial_I\partial_K f\cdot\partial_J\partial_L g \\ - \frac{\hbar^2}{6}\,\Theta^{IJ}\,\partial_J\Theta^{KL}\,\big(\partial_I\partial_K f\cdot\partial_L g - \partial_K f\cdot\partial_I\partial_L g\big) + O(\hbar^3) \ . \end{aligned} \tag{5.4}$$

The order $\hbar$ term is the semiclassical contribution which is proportional to the classical bracket $\{f,g\} = \Theta(\mathrm{d}f\wedge\mathrm{d}g)$ defined by the bivector Θ.

As a simple example, let us consider a constant bivector θ in the Poisson sigma-model (4.4). The basic graph Γ_1 with a single vertex and two edges contributes the weight

$$w_{\Gamma_1} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\psi\int_0^{\psi}\mathrm{d}\phi = \frac{1}{(2\pi)^2}\Big[\frac12\,\psi^2\Big]_0^{2\pi} = \frac12 \ . \tag{5.5}$$

In this case all graphs and hence weight integrals (5.2) factorize in terms of this basic graph Γ_1, so that the sum (5.1) truncates to

$$U_n(\theta,\dots,\theta) = \Big(\frac12\Big)^n\,\theta^{i_1j_1}\cdots\theta^{i_nj_n}\,\big(\partial_{i_1}\cdots\partial_{i_n}\big)\otimes\big(\partial_{j_1}\cdots\partial_{j_n}\big) \ . \tag{5.6}$$

The star product (5.3) thus becomes

$$f\star g = \sum_{n=0}^{\infty}\frac{(\mathrm{i}\hbar)^n}{n!}\Big(\frac12\Big)^n\,\theta^{i_1j_1}\cdots\theta^{i_nj_n}\,\partial_{i_1}\cdots\partial_{i_n}f\cdot\partial_{j_1}\cdots\partial_{j_n}g \ , \tag{5.7}$$

which is just the Moyal-Weyl star product (1.13).

In general, if Θ is not constant but still satisfies $\Theta^{IJ}\,\partial_J\Theta^{KL}=0$, then the series (5.3) again exponentiates exactly as in the case of a constant bivector which led to the Moyal-Weyl star product. This is the case for the bivector (4.10) with constant R-flux, leading to the star product [89]

$$f\star g = \cdot\,\exp\Big(\frac{\mathrm{i}\hbar}{2}\Big[\frac{\ell_s^4}{3\hbar^2}\,R^{ijk}\,p_k\,\partial_i\otimes\partial_j + \partial_i\otimes\tilde\partial^i - \tilde\partial^i\otimes\partial_i\Big]\Big)(f\otimes g) \ , \tag{5.8}$$

where $\tilde\partial^i = \frac{\partial}{\partial p_i}$ denote momentum derivatives; in Fourier space it reads as

$$(f\star g)(x,p) = \int \mathrm{d}k\,\mathrm{d}\tilde k\int\mathrm{d}q\,\mathrm{d}\tilde q\ \tilde f(k,\tilde k)\,\tilde g(q,\tilde q)\ \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}\,(\tilde k^i\,q_i - k_i\,\tilde q^i)}\ \mathrm{e}^{-\frac{\mathrm{i}\,\ell_s^4}{6\hbar}\,R^{ijk}\,k_i\,q_j\,p_k}\ \mathrm{e}^{\,\mathrm{i}\,(k+q)_i\,x^i} \ . \tag{5.9}$$

This is a deformation by R-flux of the usual phase space Moyal product for deformation quantization in ordinary quantum mechanics, see e.g. [118] for a concise review. Other approaches to deriving this star product can be found in [89, 16, 90, 83]. Various formal properties of this and other classes of nonassociative star products in deformation quantization are discussed in [37, 116].

The formality maps U_n define quasi-isomorphisms between differential graded L_∞-algebras, relating Schouten brackets $[\ ,\ ]_{\rm S}$, i.e. the obvious extensions of the Lie bracket of vector fields to multivector fields, to Gerstenhaber brackets $[\ ,\ ]_{\rm G}$, i.e. the obvious extensions of the commutator of differential operators to multidifferential operators. As such they satisfy a set of 'formality conditions'. In particular, one of these conditions reads as

$$
\mathrm{i}\hbar\,\mathscr{P}([\Theta,\Theta]_{\rm S}) = [\mathscr{P}(\Theta),\star]_{\rm G} \ . \tag{5.10}
$$

This quantifies nonassociativity, since when applied to triples of functions f,g,h, it follows that $[\Theta,\Theta]_{\rm S}\neq 0$ if and only if

$$
(f\star g)\star h - f\star(g\star h)\neq 0 \ . \tag{5.11}
$$

That is, Θ is a Poisson bivector if and only if the star product is associative. For the R-space with constant locally non-geometric flux, the star commutators

$$
[x^I,x^J]_\star := x^I\star x^J - x^J\star x^I = \mathrm{i}\hbar\,\Theta^{IJ}(x) \tag{5.12}
$$

have the corresponding non-vanishing star Jacobiators

$$
[x^i,x^j,x^k]_\star = \ell_s^4\,R^{ijk} \ . \tag{5.13}
$$

For general phase space functions, the nonassociativity of the star product can be expressed in closed form as

$$
(f\star g)\star h = \varphi_{f,g,h}\big(f\star(g\star h)\big) := \star\exp\Big(\frac{\ell_s^4}{6}\,R^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k\Big)\big(f\otimes(g\otimes h)\big) \ , \tag{5.14}
$$

where the associators

$$
f\star(g\star h) \xrightarrow{\ \varphi_{f,g,h}\ } (f\star g)\star h \tag{5.15}
$$

satisfy pentagon relations implied by the (higher) formality conditions [89], which state that the diagram

$$
\begin{array}{ccc}
 & (f\star g)\star(h\star k) & \\
\swarrow{\scriptstyle \varphi_{f\star g,h,k}} & & \nwarrow{\scriptstyle \varphi_{f,g,h\star k}} \\
\big((f\star g)\star h\big)\star k & & f\star\big(g\star(h\star k)\big) \\
\uparrow{\scriptstyle \varphi_{f,g,h}\otimes 1} & & \downarrow{\scriptstyle 1\otimes\varphi_{g,h,k}} \\
\big(f\star(g\star h)\big)\star k & \xleftarrow[\ \varphi_{f,g\star h,k}\ ]{} & f\star\big((g\star h)\star k\big)
\end{array}
\tag{5.16}
$$

commutes for all functions f,g,h,k. In particular, the translation group three-cocycle discussed in Section 2 is realised here through a nonassociative version of the Baker-Campbell-Hausdorff formula which yields

$$\big(\mathrm{e}^{\,\mathrm{i}k_I x^I} \star \mathrm{e}^{\,\mathrm{i}q_I x^I}\big) \star \mathrm{e}^{\,\mathrm{i}r_I x^I} = \mathrm{e}^{\frac{\mathrm{i}\ell_s^4}{6} R^{ijk} k_i q_j r_k}\; \mathrm{e}^{\,\mathrm{i}k_I x^I} \star \big(\mathrm{e}^{\,\mathrm{i}q_I x^I} \star \mathrm{e}^{\,\mathrm{i}r_I x^I}\big) \ . \tag{5.17}$$

In the remainder of this section we consider various applications of this nonassociative phase space star product formalism.

5.2 Triproducts

We shall now address Question (Q2) from Section 1. Given n functions $f_1,\dots,f_n \in C^\infty(M)$ on the target space M, using the nonassociative phase space star product we define the *triproducts*

$$f_1 \vartriangle f_2 \vartriangle \cdots \vartriangle f_n := \lim_{p\to 0}\ f_1 \star \big(f_2 \star \big(\cdots (f_{n-1} \star f_n)\cdots\big)\big) \ , \tag{5.18}$$

where we have chosen a particular bracketing for the star products of functions. It was shown by [11] that the result of this operation can be expressed in terms of a tridifferential operator as

$$f_1 \vartriangle f_2 \vartriangle \cdots \vartriangle f_n = \cdot\, \exp\Big(-\frac{\ell_s^4}{6} \sum_{1\leqslant a<b<c\leqslant n} R^{ijk}\, \partial_i^a \otimes \partial_j^b \otimes \partial_k^c\Big)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) \ , \tag{5.19}$$

where ∂_i^a denotes the action of the derivative ∂_i in the a-th factor of the tensor product $f_1 \otimes f_2 \otimes \cdots \otimes f_n$. This remarkable algebraic structure has the following properties.

Firstly, it agrees with the triproducts of tachyon vertex operators after T-duality in worldsheet perturbation theory around flat space with constant H-flux, as we discussed earlier. This formula was derived to linear order in the R-flux in [33], where its all orders exponential form was conjectured, and subsequently extrapolated to non-constant fluxes appropriate to curved backgrounds M in [34]. Here we find an explicit calculational *derivation* of these conjectural expressions from the nonassociative phase space star product, which lends further credibility to the purported description of the locally non-geometric flux background.

Secondly, since the momentum is set to $p=0$, for $n=2$ there is no deformation of the usual pointwise product of fields: $f \vartriangle g = f \cdot g$, as anticipated from the closed string theory considerations of Section 1. For $n=3$ the triproduct was originally introduced in [115] as a proposal for the quantization of Nambu brackets [92], which we will discuss in more detail later on. It quantizes the tribracket defined by (1.17), and in particular one has

$$[x^i,x^j,x^k]_\vartriangle = \ell_s^4\, R^{ijk} \ , \tag{5.20}$$

which agrees with the Jacobiator (5.13).

Thirdly, the triproducts exhibit on-shell associativity:

$$\int \mathrm{d}x\ f_1 \vartriangle \cdots \vartriangle f_n = \int \mathrm{d}x\ f_1 \cdots f_n \ , \tag{5.21}$$

which follows from on-shell momentum conservation [11]. As we discussed earlier, this is expected and in fact *necessary* from the string theory perspective: As on-shell closed string theory is described a two-dimensional quantum field theory, it is described by an associative operator algebra. In particular, the on-shell associativity agrees with the crossing symmetry of n-point correlation functions on the sphere S^2.

Finally, as also mentioned before, the triproducts violate the strong constraint of double field theory. Recalling the microscopic origin of the R-flux in terms of a bivector from (3.19), the expansion of the triproduct for $n = 3$ reads as

$$f \triangle g \triangle h = f \cdot g \cdot h - \frac{\ell_s^4}{6}\, \tilde{\partial}^{[i}\tilde{\theta}^{jk]}\, \partial_i f \cdot \partial_j g \cdot \partial_k h + O(\ell_s^8) \ , \tag{5.22}$$

and the corrections to the pointwise product of fields vanishes by the strong constraint. Thus the nonassociative geometry of the closed string background is only probed if the strong constraint between the background and fluctuations is weakened, a point which is presently under debate.

5.3 Nonassociative quantum mechanics

Let us now investigate some of the physical consequences of the nonassociative deformation of quantum mechanics in the R-flux background. The standard operator-state formulation of quantum mechanics cannot handle nonassociative structures: Operators which act on a Hilbert space necessarily associate. However, we can generalize the phase space formulation of quantum mechanics [118] to provide a completely quantitative and physically viable formulation of nonassociative quantum mechanics [90]. Other approaches to nonassociative quantum mechanics based on the magnetic monopole algebra (2.3) are found in [36, 35].

The idea behind phase space quantum mechanics is to treat position and momentum variables on equal footing. In this setting generic 'operators' become complex-valued functions on phase space, with the operator product provided by the star product and traces given by integration of functions; an 'observable' is then a real-valued function on phase space. Dynamics of observables A are governed by Heisenberg-type time evolution equations

$$\dot{A} = \frac{\mathrm{i}}{\hbar}\, [H,A]_\star \ , \tag{5.23}$$

for a given classical Hamiltonian H.

The key properties of the nonassociative phase space star product (5.8) that are needed in this description are as follows. Firstly, since the star product $f \star g$ again differs from $f \cdot g$ by a total derivative, and likewise $f \star (g \star h)$ differs from $(f \star g) \star h$ by a total derivative, one again has the 2-cyclicity and 3-cyclicity properties (1.14) and (1.15) that we encountered for the Moyal-Weyl product. Recall that these features were crucial for compatibility of the star product formalism with the axioms of conformal field theory; in the present context, they refer to the fact that the star product leads to a traceless commutator and associator. However, there are in general inequivalent quartic expressions, for example

$$\int \mathrm{d}x \ f \star \big(g \star (h \star k)\big) \neq \int \mathrm{d}x \ f \star \big((g \star h) \star k\big) \ , \tag{5.24}$$

see [90, 88] for a detailed discussion. Using 2-cyclicity we deduce the crucial positivity property

$$\int \mathrm{d}x \ \overline{f} \star f \geqslant 0 \ . \tag{5.25}$$

Finally, the nonassociative star product is Hermitian, in the sense that $\overline{f\star g}=\overline{g}\star\overline{f}$, and unital, in the sense that the constant unit function 1 is still an identity element for the nonassociative star product algebra: $f\star 1=f=1\star f$. All of these properties are completely analogous to those of the usual Moyal product in canonical phase space quantum mechanics and they mimick the expected features of the operator product; indeed, the star product (5.8) for the constant R-flux background is the simplest example of a nonassociative star product and provides the natural extension of the Moyal-Weyl star product (1.13) to the non-geometric string background.

A 'state' in nonassociative quantum mechanics is determined by a set of square-integrable phase space wavefunctions ψ_a which are normalized:

$$\int \mathrm{d}x\, |\psi_a|^2=1\ , \tag{5.26}$$

and a collection of statistical probabilities $\mu_a\in[0,1]$ obeying

$$\sum_a \mu_a=1\ . \tag{5.27}$$

The 'expectation value' of an operator A is then defined by

$$\langle A\rangle_\psi=\sum_a \mu_a\int \mathrm{d}x\ \overline{\psi_a}\star(A\star\psi_a)=\int \mathrm{d}x\, W_\psi\cdot A\ , \tag{5.28}$$

where we introduced the state function

$$W_\psi=\sum_a \mu_a\, \psi_a\star\overline{\psi_a} \tag{5.29}$$

which is the analogue of the Wigner distribution function; in particular, it is real-valued and normalized: $\langle 1\rangle_\psi=\int \mathrm{d}x\, W_\psi=1$. However, even in canonical phase space quantum mechanics [118], one of the pitfalls of the formalism is that state functions are not necessarily non-negative and so only determine quasi-probability distribution functions in general.

As a simple example to familiarize ourselves with the phase space formulation of quantum mechanics, let us consider the simplest case of the free particle with Hamiltonian

$$H(x,p)=\sum_{i=1}^d \frac{p_i^2}{2m} \tag{5.30}$$

and vanishing R-flux. We denote the corresponding Moyal star product from (5.8) with $R^{ijk}=0$ by $\star_0$ and look for solutions of the time-independent Schrödinger equation

$$H\star_0 W_\psi=E\, W_\psi\ . \tag{5.31}$$

This can be expressed as a second order partial differential equation

$$\frac{1}{2m}\sum_{i=1}^d\Big(p_i^2+\mathrm{i}\,\hbar\,\partial_i-\frac{\hbar^2}{4}\,\partial_i^2\Big)W_\psi=E\,W_\psi \tag{5.32}$$

for the real-valued Wigner distribution function $W_\psi(x,p)$. It collapses to a pair of partial differential equations, its real and imaginary parts. The imaginary part

$$\sum_{i=1}^d p_i\,\partial_i W_\psi(x,p)=0 \tag{5.33}$$

restricts $W_\psi(x,p) = W_\psi(p)$ to be independent of x. The real part

$$\sum_{i=1}^{d} \Big(p_i^2 - \frac{\hbar^2}{4}\,\partial_i^2 - 2m\,E\Big)W_\psi(p) = 0 \tag{5.34}$$

is satisfied for arbitrary real functions $W_\psi(p)$ of momentum with the energy eigenvalues

$$E = E_p = \sum_{i=1}^{d} \frac{p_i^2}{2m} \ . \tag{5.35}$$

This calculation can be extended to include an anisotropic harmonic oscillator potential of frequencies ω_i, for which the real part of the second order partial differential equation can be separated into Laguerre equations in the phase space variables $z_i = \frac{2}{m\hbar}\,(m\,\omega_i^2\,x_i^2 + p_i^2)$ for $i = 1,\dots,d$ [118]. The corresponding solutions in terms of Laguerre polynomials reproduce the anticipated harmonic oscillator spectrum

$$E = E_n = \sum_{i=1}^{d} \hbar\,\omega_i\,\Big(n_i + \frac12\Big) \qquad \text{with} \quad n = (n_i) \in \mathbb{N}_0^d \tag{5.36}$$

in d dimensions. The extension of these considerations to a free particle moving in a non-vanishing R-flux background is discussed in [114], and from the alternative associative framework of symplectic realisation of the twisted Poisson structure (4.10) in [82].

The operator-state correspondence can be described in this framework as follows. The idea is that operators should still be combined together with an associative operation, since nonassociativity should not affect correlation functions. To this end we introduce two conjugate "operator algebras" by defining left and right compositions of observables A,B as

$$(A\circ B)\star f := A\star(B\star f) \qquad \text{and} \qquad f\star(A\,\bar\circ\,B) := (f\star A)\star B \ , \tag{5.37}$$

for a test function f. By definition the composition products are associative, since

$$(A_1\circ A_2\circ\cdots\circ A_n)\star f = A_1\star\big(A_2\star\cdots\star(A_n\star f)\cdots\big) \ . \tag{5.38}$$

They are furthermore unital: $A\circ 1 = A = 1\circ A$, and the composition products of basic coordinate monomials coincide with their star products, e.g. $x^i\circ x^i = x^i\star x^i = (x^i)^2$ and $p_i\circ p_i = p_i\star p_i = (p_i)^2$. However, in general the composition product $A\circ B$ of two functions A and B is not a function, but rather a differential operator; the composition products arise as the embedding of the nonassociative star product algebra of functions as a *subspace* (but not as a subalgebra) of an associative algebra of differential operators on phase space. In the conventional frameworks where $\star$ is associative, the composition product and the star product coincide. The physical and geometric meaning of the associative composition algebra of differential operators is elucidated by [82] from the perspective of symplectic realisation of the twisted Poisson structure (4.10).

We make the convention that the conjugate composition product $\bar\circ$ is always evaluated before $\circ$. Then a 'state' ρ_ψ is an expression of the form

$$\rho_\psi = \sum_a \mu_a\,\psi_a\,\bar\circ\,\overline{\psi_a} \ . \tag{5.39}$$

This quantity should be thought of as a special representation of a "density matrix", which in the associative setting would be the same as the Wigner distribution function; in particular, the expectation values (5.28) of operators are determined via traces with ρ_ψ through

$$\langle A\rangle_\psi = \int \mathrm{d}x\, A\star\rho_\psi\ . \tag{5.40}$$

The expectation values of compositions of operators are then given by

$$\begin{aligned}\langle A_1\circ A_2\circ\cdots\circ A_n\rangle_\psi &= \int \mathrm{d}x\ (A_1\circ A_2\circ\cdots\circ A_n)\star\rho_\psi \\ &= \sum_a \mu_a \int \mathrm{d}x\ \big(A_1\star(A_2\star\cdots\star(A_n\star\psi_a)\cdots)\big)\star\overline{\psi_a}\ .\end{aligned} \tag{5.41}$$

Let us make some simple consistency checks of this purported nonassociative version of quantum mechanics. First, let us check reality. A straightforward calculation using the above definitions and the properties of the nonassociative star product gives

$$\overline{\langle A\rangle_\psi} = \sum_a \mu_a \int \mathrm{d}x\ \overline{(A\star\psi_a)}\star\psi_a = \sum_a \mu_a \int \mathrm{d}x\ \overline{\psi_a}\star(\overline{A}\star\psi_a) = \langle\overline{A}\rangle_\psi\ . \tag{5.42}$$

It follows that the expectation values of observables, i.e. real functions, are real. Similarly, one checks positivity: $\langle\overline{A}\circ A\rangle_\psi\geqslant 0$. These derivations are carried out in complete analogy with the corresponding calculations in canonical phase space quantum mechanics; however, in the nonassociative case an extra line or two is always required in the computation.

Next we check that observables, i.e. real functions $A=\overline{A}$, have real eigenvalues. For this, consider the "star-genvalue equation" $A\star f=\lambda\, f$ for a complex number λ; complex conjugation of this equation using Hermiticity of the star product yields $\overline{f}\star\overline{A}=\overline{\lambda}\ \overline{f}$. From this we calculate

$$\overline{f}\star(A\star f)-(\overline{f}\star A)\star f=(\lambda-\overline{\lambda}\,)(\overline{f}\star f)\ . \tag{5.43}$$

In the associative case this would immediately imply that $\lambda=\overline{\lambda}$ is real, since the left-hand side would automatically vanish. In the nonassociative case this is not generally true, but using 3-cyclicity and 2-cyclicity we can integrate both sides this equation to get

$$0=(\lambda-\overline{\lambda}\,)\int \mathrm{d}x\ |f|^2\ . \tag{5.44}$$

Since the integral on the right-hand side is non-zero for $f\neq 0$, it follows that $\lambda=\overline{\lambda}$. A similar calculation establishes that eigenfunctions with different eigenvalues are orthogonal in the L^2-inner product.

5.4 Spacetime quantization

As a concrete application of the formalism, we will now show how nonassociative quantum mechanics in the R-flux background leads to a coarse-graining of spacetime. There are several ways in which to see this. In standard quantum mechanics it is a fundamental result that pairs of non-commuting operators cannot have simultaneous eigenvalues. Similarly, in the present case we can show that triples of nonassociating operators cannot have simultaneous eigenvalues. Suppose that

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W_ψ is a state function which simultaneously diagonalizes a triple of basic phase space coordinate operators: $x^I\star W_\psi=\lambda^I\,W_\psi$, $x^J\star W_\psi=\lambda^J\,W_\psi$ and $x^K\star W_\psi=\lambda^K\,W_\psi$ for eigenvalues $\lambda^I,\lambda^J,\lambda^K\in\mathbb{R}$. By repeatedly applying the cyclicity properties of the star product we compute

$$\begin{aligned}\int \mathrm{d}x\ \big((x^I\star x^J)\star x^K\big)\star W_\psi &= \int \mathrm{d}x\ (x^I\star x^J)\star(x^K\star W_\psi)\\ &=\lambda^K\int \mathrm{d}x\ (x^I\star x^J)\star W_\psi\\ &=\lambda^K\int \mathrm{d}x\ x^I\star(x^J\star W_\psi)\\ &=\lambda^K\,\lambda^J\,\lambda^I\ ,\end{aligned}\tag{5.45}$$

and similarly

$$\int \mathrm{d}x\ \big(x^I\star(x^J\star x^K)\big)\star W_\psi=\lambda^I\,\lambda^K\,\lambda^J\ .\tag{5.46}$$

Taking the difference of these two equations implies the vanishing Jacobiators $[x^I,x^J,x^K]_\star=0$, which contradicts the basic nonassociative deformation provided by the position coordinate operators in (5.13). This implies a coarse-graining of spacetime by the non-geometric R-flux background.

To compute the coarse-graining quantitatively, we introduce oriented area and volume uncertainty operators

$$\begin{aligned}\mathsf{A}^{IJ}&=\mathrm{Im}\big([\widetilde{x}^I,\widetilde{x}^J]_\star\big)=-\,\mathrm{i}\,\big(\widetilde{x}^I\star\widetilde{x}^J-\widetilde{x}^J\star\widetilde{x}^I\big)\ ,\\ \mathsf{V}^{IJK}&=\frac13\,\mathrm{Re}\big(\widetilde{x}^I\star[\widetilde{x}^J,\widetilde{x}^K]_\star+\widetilde{x}^K\star[\widetilde{x}^I,\widetilde{x}^J]_\star+\widetilde{x}^J\star[\widetilde{x}^K,\widetilde{x}^I]_\star\big)\ ,\end{aligned}\tag{5.47}$$

where we introduced the shifted coordinates $\widetilde{x}^I:=x^I-\langle x^I\rangle_\psi$ appropriate to uncertainties in measurements. These definitions mimick the vector product and triple scalar product of vectors in the respective coordinate directions of phase space, and their expectation values measure the minimal area and volume uncertainties between corresponding operators. A simple computation using (5.12), (5.13) and the integration properties above leads to the non-vanishing expectation values

$$\langle\mathsf{A}^{x^i,p_j}\rangle_\psi=\hbar\,\delta^i{}_j\ ,\qquad\langle\mathsf{A}^{ij}\rangle_\psi=\frac{\ell_s^4}{3\hbar}\,R^{ijk}\,\langle p_k\rangle_\psi\qquad\text{and}\qquad\langle\mathsf{V}^{ijk}\rangle_\psi=\frac12\,\ell_s^4\,R^{ijk}\ .\tag{5.48}$$

The first expectation value gives the usual Planck cells of canonical quantum phase space with the Planck quantum of minimal area $\hbar$, while the second expectation value is a new uncertainty measurement giving minimal spacetime areas in the directions transverse to the motion of closed strings (as originally conjectured by [84]). The third expectation value is the most interesting: It implies a quantized spacetime with a quantum of minimal volume $\frac12\,\ell_s^4\,R^{ijk}$.

This coarse-graining of spacetime has the following physical interpretation from the perspective of non-geometric string theory, which corroborates the argument that there can be no D0-branes in the locally non-geometric background [117]. In $d=3$ dimensions, applying a triple T-duality $\mathscr{T}_{(123)}$ maps the R-space to the three-torus with H-flux and a D0-brane to a D3-brane wrapping the torus. However, this latter configuration is not allowed as it suffers from the Freed-Witten anomaly [56], i.e. it violates the Bianchi identity $\mathrm{d}F=H$ for the gauge flux F on a D3-brane.

Generally, Freed-Witten anomaly cancellation would require that the degree three integer cohomology class of the NS–NS flux H be equal to the torsion characteristic class which measures the obstruction to a spin^c structure on the target space M. But if M is a torus, then its cohomology is torsion-free and it is a spin^c manifold, so that non-vanishing H-flux is not allowed. In the T-dual frame, this implies that placing a point-like object in the R-flux background is *not* allowed. This is yet another manifestation of the local non-geometry of the R-flux background, which we have reproduced here in a completely quantitative way through our formalism of nonassociative quantum mechanics.

5.5 Quantization of Nambu brackets

As another application of the nonassociative phase space deformation quantization, let us examine the problem of quantizing Nambu-Poisson structures, which have notably appeared in recent years in effective theories of M2-branes and M5-branes in M-theory, see e.g. [67] for a review. Nambu mechanics involves multi-Hamiltonian dynamics with generalized Poisson brackets $\{f,g,h\}$ of functions obeying the Leibniz rule and a "fundamental identity" which is a higher generalization of the Jacobi identity for Lie brackets [92]. Nambu used this algebraic structure to reformulate the Euler equations describing the dynamics of a rotating rigid body in $\mathbb{R}^3$ in the absence of applied torques, in the hope of obtaining new generalized integrals of motion. Recall that in a rotating reference frame parallel to the principal axes of inertia of the rigid body, they read in general form as

$$\dot{\vec{L}} + \vec{\omega} \times \vec{L} = \vec{0} \ , \tag{5.49}$$

where $\vec{L} = \boldsymbol{I}\,\vec{\omega}$ is the angular momentum, with $\boldsymbol{I}$ the matrix of moments of inertia and $\vec{\omega}$ the angular velocity about the principal axes. By setting $T := \frac{1}{2}\vec{L}\cdot\vec{\omega}$ we can write the components of this equation in the principal reference frame, wherein $\boldsymbol{I}$ is constant, as the bi-Hamiltonian equations

$$\dot{L}_i = \{L_i, \vec{L}^2, T\} \tag{5.50}$$

for the Nambu bracket $\{f,g,h\} = \varepsilon^{ijk}\,\partial_{L_i} f \cdot \partial_{L_j} g \cdot \partial_{L_k} h$ on $\mathbb{R}^3$.

In the case of the R-flux background, the pertinent classical tribracket is given by

$$\{f,g,h\}_R = -\frac{\ell_s^4}{\hbar^2}\,R^{ijk}\,\partial_i f \cdot \partial_j g \cdot \partial_k h \ . \tag{5.51}$$

Our nonassociative phase space star product quantizes these tribrackets [89, 90], in the sense of the non-vanishing Jacobiators (5.13); this extends the quantization provided by the configuration space triproducts from (5.20). In particular, in the semi-classical limit one has $[f,g,h]_\star = -\hbar^2\,\{f,g,h\}_R + \cdots$. The issue of quantizing Nambu brackets is a longstanding problem, see e.g. [105, 106, 48, 49] for some approaches. Our phase space approach may assist by exploiting the various properties infered by the origin of the tribracket from the star product. For example, one of the formality conditions implies the pentagon identity [89]

$$[f\star g,h,k]_\star - [f,g\star h,k]_\star + [f,g,h\star k]_\star = f\star[g,h,k]_\star + [f,g,h]_\star \star k \tag{5.52}$$

for the Jacobiator, analogously to (5.16). This can be regarded as a 'quantum Leibniz rule', in the sense that at semiclassical order in the R-flux it coincides with the Leibniz rule for the classical tribracket (5.51). At present it is not clear what should serve as the quantum version of the classical fundamental identity, nor if one is really necessary, see e.g. [48] for a discussion of this point. The viability of our phase space model for the quantization of Nambu-Poisson structures is still to be thoroughly investigated.

6. Further developments

We conclude in this final section by briefly describing some extensions of the story described in this paper so far, all of which have important open problems which should be subject to future investigations.

6.1 Nonassociative gravity

One important omission from our considerations thus far has been addressing Question (Q3) from Section 1. A noncommutative theory of gravity on Moyal-Weyl spacetimes was constructed in [14], and is discussed in the lectures by L. Castellani at this School. As a fundamental length is incompatible with diffeomorphism symmetries, general relativity on noncommutative spacetime requires a Drinfeld twist via a two-cocycle of the Hopf enveloping algebra of the Lie algebra of vector fields (see e.g. [112] for a review). This defines a twisted tensor calculus, and leads to deformed Einstein equations [14]. Two major problems with this approach are that the Einstein equations are not generically real and so have questionable physical (and geometric) significance, and that twisted diffeomorphisms do not appear to be symmetries of string theory [4].

An analogous treatment can be followed to formulate a nonassociative theory of gravity, by defining a quasi-Hopf algebra of twisted diffeomorphisms with a two-cochain twist whose coboundary is the three-cocycle that controls nonassociativity, i.e. the associator φ, see (5.14). Such an approach was originally suggested by [90], and the pentagon relations (5.16) were used to consistently build nonassociative field theories in [88, 21]. The cochain twisting method was extended by [19, 20] to develop a rigorous and very general theory of nonassociative differential geometry, extending and generalizing the considerations of [10] in the associative case, which in particular leads to a vielbein or first order formalism for nonassociative gravity. The metric aspects were considered in [27, 12], and currently nonassociative Riemannian geometry is under further construction.

The problems with noncommutative gravity alluded to above are of course still present in this theory (together with many additional issues). But recalling the development of our perspective, we need to explore how gravity on phase space induces gravity on configuration space. This problem was addressed by [11], where it is shown how the nonassociative geometry of phase space can lead to deformations of configuration space geometry. The idea is to remove momentum dependence up to an $O(d,d)$-transformation, which is the structure group symmetry determined by the natural phase space metric $\gamma = \mathrm{d}x^i \otimes \mathrm{d}p_i + \mathrm{d}p_i \otimes \mathrm{d}x^i$, analogously to double field theory. This is achieved by choosing a polarization, i.e. a maximally isotropic splitting $T(T^*M) \simeq L \oplus L^*$ of the tangent bundle of phase space with respect to the metric γ, and considering foliated tensor fields T:

$$\iota_Z T = 0 = \pounds_Z T \ , \tag{6.1}$$

for every section Z of the rank d vector bundle $L^* \to T^*M$; if the distribution L is integrable, then by Frobenius' theorem it can be (locally) identified with the tangent bundle of some d-dimensional submanifold of T^*M. For example, the foliation of phase space by leaves of constant momentum is determined by taking $L = TM$ and $Z_i = \frac{\partial}{\partial p_i}$ for $i = 1, \dots, d$, which formalizes the way in which we extracted the spacetime triproducts of fields occuring in closed string scattering amplitudes from the nonassociative phase space star product in (5.18).

The metric formulation of nonassociative gravity on phase space admits a Ricci tensor and a unique metric-compatible torsion-free connection, and in this way it yields a non-trivial deformation of the Ricci tensor of spacetime by locally non-geometric fluxes, which is given by [12]

$$\begin{aligned} \mathrm{Ric}^{\circ}_{ij} = \mathrm{Ric}_{ij} + \frac{\ell_s^4}{12}\, R^{abc} \Big(& \partial_k\big(\partial_a g^{kl}\,(\partial_b g_{lm})\,\partial_c\Gamma^m_{ij}\big) - \partial_j\big(\partial_a g^{kl}\,(\partial_b g_{lm})\,\partial_c\Gamma^m_{ik}\big) \\ & + \partial_c g_{mn}\big(\partial_a(g^{lm}\,\Gamma^k_{lj})\,\partial_b\Gamma^n_{ik} - \partial_a(g^{lm}\,\Gamma^k_{lk})\,\partial_b\Gamma^n_{ij} \\ & + (\Gamma^l_{ik}\,\partial_a g^{km} - \partial_a\Gamma^l_{ik}\,g^{km})\,\partial_b\Gamma^n_{lj} - (\Gamma^l_{ij}\,\partial_a g^{km} - \partial_a\Gamma^l_{ij}\,g^{km})\,\partial_b\Gamma^n_{lk}\big)\Big) \ , \end{aligned} \tag{6.2}$$

where Ric_{ij} is the usual Ricci tensor of the classical Levi-Civita connection Γ^k_{ij} of a metric tensor g_{ij} on spacetime. This expression is valid to linear order in the R-flux, which we recall is the order at which the corresponding conformal field theory calculations are reliable; this is also consistent with the second order R-flux corrections to the closed string equations (3.29) after T-duality, as the corrections (6.2) are indeed of second order $\ell_s^4\,\hbar$ in the double expansion in the parameters of the R-flux model. Notably, it is *real*, and thus represents the first non-trivial starting point for understanding how to define a nonassociative theory of gravity describing the low-energy effective dynamics of closed strings in non-geometric backgrounds. An action principle for nonassociative gravity is currently unknown, as some of the classical constructions of Riemannian geometry have yet to be generalized to the nonassociative setting.

The precise relation of this gravity theory with string theory and double field theory remains mysterious. Part of the issue is the role of twisted diffeomorphisms mentioned above. At any order in the string length scale ℓ_s^2, the closed string effective action should be invariant under classical diffeomorphisms, while the effective action of double field theory should be invariant under generalized diffeomorphisms. Whereas the twisted diffeomorphism symmetries of phase space remain elusive, upon polarization to configuration space they may compare naturally with the expectations from string theory and double field theory. This was partially analysed by [11], but it is currently an open problem to understand precisely and systematically the meaning of the symmetries of nonassociative gravity on spacetime. In particular, one can ask if the effective theory retains the $O(d,d)$-symmetry of double field theory, in analogy to the case of open strings. Recall [54, 110] that noncommutative Yang-Mills theory on a d-dimensional torus is invariant under Morita equivalence, which is the structure inherited from T-duality in the decoupling limit. In this case the group $O(d,d)$ acts on the bivector θ, the open string metric G, and the Yang-Mills coupling constant $g_{\rm YM}$ as

$$\begin{aligned} \theta &\longmapsto (A\,\theta + B)\,(C\,\theta + D)^{-1} \ , \\ G &\longmapsto (C\,\theta + D)\,G\,(C\,\theta + D)^{\top} \ , \\ g_{\rm YM} &\longmapsto g_{\rm YM}\,\big|\det(C\,\theta + D)\big|^{1/4} \ , \end{aligned} \tag{6.3}$$

where A,B,C,D are $d\times d$ matrices which parameterize an element of $O(d,d)$; this transformation is defined only on the subset of bivectors θ for which $C\theta+D$ is nondegenerate, which is dense for every element in the discrete subgroup $SO(d,d;\mathbb{Z})$ of $O(d,d)$. It would be interesting to see if a similar set of T-duality transformation rules are global symmetries of nonassociative gravity.

6.2 Higher structures in non-geometric M-theory

It is natural to ask what becomes of the higher quantum geometry when we lift IIA string theory to M-theory, which is generally defined on a circle bundle

$$\begin{array}{ccc} S^1 & \hookrightarrow & \widetilde{M} \\ & & \downarrow \\ & & M \end{array} \tag{6.4}$$

over the string target space M, with the string coupling g_s realized geometrically as the radius λ of the S^1 fibres. This has been discussed for lifts of the string theory R-flux model in three dimensions by [63], in the context of the $SL(5)$ exceptional field theory [25] which lifts the $O(3,3)$ double field theory. Taking the base space M to be a three-dimensional twisted torus (the T-dual of the three-torus with H-flux), using the T-duality chain (3.1) we generate the string theory R-flux via a double T-duality transformation

$$f^i{}_{jk} \xrightarrow{\ \mathscr{T}_{(jk)}\ } R^{ijk} \ . \tag{6.5}$$

Let us lift this to M-theory on the trivial circle bundle $\widetilde{M}=M\times S^1$, with coordinates $x^\mu=(x^i,x^4)$ where x^4 is the local coordinate on S^1. Closed strings lift to closed M2-branes, and T-duality becomes U-duality which sends membrane wrapping modes w^{ij} to momentum modes p_i. The Kalb-Ramond two-form field B lifts to the three-form C-field of M-theory, and U-duality takes it to

$$C_{\mu\nu\rho} \xrightarrow{\ \mathscr{U}_{(\mu\nu\rho)}\ } \Omega^{\mu\nu\rho} \ , \tag{6.6}$$

where the trivector $\Omega^{\mu\nu\rho}$ defines the M-theory R-flux via its wrapping derivatives

$$R^{\mu,\nu\rho\alpha\beta} = \partial^{\mu[\nu}\Omega^{\rho\alpha\beta]} \ , \tag{6.7}$$

which is the lift of the relation (3.19). The M-theory R-flux is a mixed symmetry tensor: It transforms as a vector in its first index and is antisymmetric in its last four indices. The particular choice $R^{4,\mu\nu\alpha\beta}=R\,\varepsilon^{\mu\nu\alpha\beta}$ breaks the $SL(5)$ symmetry to $SO(4)$.

The M2-brane phase space has a peculiar structure: The fact that there are no D0-branes on M lifts to the statement that there are no momentum modes along the M-theory direction, i.e. $p_4=0$. In [63] it is conjectured that the covariant form of this constraint is given by

$$R^{\mu,\nu\rho\alpha\beta}\,p_\mu = 0 \tag{6.8}$$

and that the resulting seven-dimensional phase space has bracket structure given by

$$\begin{aligned} {}[x^i,x^j] &= \frac{\mathrm{i}\,\ell_s^4}{3\hbar}\,R^{4,ijk4}\,p_k \qquad \text{and} \qquad [x^4,x^i] = \frac{\mathrm{i}\,\lambda\,\ell_s^4}{3\hbar}\,R^{4,1234}\,p^i \ , \\ [x^i,p_j] &= \mathrm{i}\hbar\,\delta^i{}_j\,x^4+\mathrm{i}\hbar\,\lambda\,\varepsilon^i{}_{jk}\,x^k \qquad \text{and} \qquad [x^4,p_i] = \mathrm{i}\hbar\,\lambda^2\,x_i \ , \\ [p_i,p_j] &= -\mathrm{i}\hbar\,\lambda\,\varepsilon_{ijk}\,p^k \ , \end{aligned} \tag{6.9}$$

with the Jacobiators

$$\begin{aligned}
&[x^i,x^j,x^k] = \frac{\mathrm{i}\,\ell_s^4}{3\hbar}\,R^{4,ijk4}\,x^4 \qquad \text{and} \qquad [x^i,x^j,x^4] = -\frac{\mathrm{i}\,\lambda^2\,\ell_s^4}{3\hbar}\,R^{4,ijk4}\,x_k \ , \\
&[p_i,x^j,x^k] = \frac{\mathrm{i}\,\lambda\,\ell_s^4}{3\hbar}\,R^{4,1234}\,\big(\delta_i{}^j\,p^k-\delta_i{}^k\,p^j\big) \qquad \text{and} \qquad [p^i,x^j,x^4] = \frac{\mathrm{i}\,\lambda^2\,\ell_s^4}{3\hbar}\,R^{4,ijk4}\,p_k \ , \\
&[p_i,p_j,x^k] = -\mathrm{i}\,\hbar\,\lambda^2\,\varepsilon_{ij}{}^k\,x^4 - \mathrm{i}\,\hbar\,\lambda\,\big(\delta_j{}^k\,x_i-\delta_i{}^k\,x_j\big) \qquad \text{and} \qquad [p_i,p_j,x^4] = \mathrm{i}\,\hbar\,\lambda^3\,\varepsilon_{ijk}\,x^k \ , \\
&[p_i,p_j,p_k] = 0 \ .
\end{aligned} \tag{6.10}$$

In the contraction limit $\lambda = 0$, which is precisely the limit of weak string coupling $g_s \to 0$ sending M-theory to IIA string theory, these brackets reduce to those of the closed string R-flux algebra (3.27); in this case the M-theory direction x^4 becomes a central element of the contracted algebra, so we may set it to $x^4 = 1$.

These brackets originate from the nonassociative alternative algebra of octonions $\mathbb{O}$ by a suitable rescaling of the seven imaginary unit octonions [63]. Deformation quantization of this quasi-Poisson structure was carried out in [81]. It proceeds via a choice of a G_2-structure, i.e. a cross product on the real inner product space $\mathbb{R}^7$ given by structure constants of $\mathbb{O}$ in a suitable oriented basis, which is preserved by rotations of $\mathbb{R}^7$ in the subgroup $G_2 \subset SO(7)$. Using alternativity one can define octonion exponentials, and the corresponding nonassociative Baker-Campbell-Hausdorf formula is captured by a 2-group addition law on the seven-dimensional Fourier space, which gives the deformation quantization of the M2-brane phase space in terms of an explicit, albeit complicated, nonassociative phase space star product $f \star_\lambda g$. This quantization has a variety of interesting features which are described in [81]. For example, the corresponding configuration space triproducts $\triangle_\lambda$ defined by $\star_\lambda$ analogously to (5.18) quantize the 3-Lie algebra A_4:

$$[x^\mu,x^\nu,x^\alpha]_{\triangle_1} = \ell_s^4\,R\,\varepsilon^{\mu\nu\alpha\beta}\,x^\beta \ , \tag{6.11}$$

which is known from studies of multiple membranes in M-theory, see e.g. [15] for a review, and whose quantization has been previously largely unknown, see e.g. [48] for an earlier analysis of this problem. Moreover, the noncommutative M2-brane momentum space is familiar from the noncommutative spacetimes arising in three-dimensional quantum gravity, see e.g. [57]. Using $x^4 \star_\lambda f = x^4\,f + O(\lambda)$ one finds that the M2-brane star product reduces non-trivially to the closed string star product (5.8) in the weak string coupling limit:

$$\lim_{\lambda\to 0}\,(f \star_\lambda g)(x^i,x^4) = (f\star g)(x^i) \ . \tag{6.12}$$

In [81] it was also suggested how to extend the nonassociative M-theory R-flux algebra to the full unconstrained eight-dimensional phase space by a choice of a $Spin(7)$-structure, i.e. a triple cross product on the real inner product space $\mathbb{R}^8$, which is parameterized by a four-form ϕ invariant under rotations of $\mathbb{R}^8$ in the subgroup $Spin(7) \subset SO(8)$. This extends the representation of the cross product to all real and imaginary octonions, and it defines the eight-dimensional "covariant" M2-

brane phase space 3-algebra with the $SO(4)\times SO(4)$-symmetric 3-brackets given by

$$
\begin{aligned}
{[x^i,x^j,x^k]}_\phi &= -\frac{\ell_s^4}{2}\,R^{4,ijk4}\,x^4 \qquad \text{and} \qquad [x^i,x^j,x^4]_\phi = \frac{\lambda^2\,\ell_s^4}{2}\,R^{4,ijk4}\,x_k\ ,\\
{[p^i,x^j,x^k]}_\phi &= -\frac{\lambda^2\,\ell_s^4}{2}\,R^{4,ijk4}\,p_4-\frac{\lambda\,\ell_s^4}{2}\,R^{4,ijkl}\,p_l\ ,\\
{[p^i,x^j,x^4]}_\phi &= -\frac{\lambda^2\,\ell_s^4}{2}\,R^{4,1234}\,\delta^{ij}\,p_4-\frac{\lambda^2\,\ell_s^4}{2}\,R^{4,ijk4}\,p_k\ ,\\
{[p_i,p_j,x^k]}_\phi &= \frac{\lambda^2}{2}\,\varepsilon_{ij}{}^k\,x^4+\frac{\hbar^2\,\lambda}{2}\,\big(\delta_j{}^k\,x_i-\delta_i{}^k\,x_j\big)\ ,\\
{[p_i,p_j,x^4]}_\phi &= -\frac{\hbar^2\,\lambda^3}{2}\,\varepsilon_{ijk}\,x^k \qquad \text{and} \qquad [p_i,p_j,p_k]_\phi \ = \ -2\hbar^2\,\lambda\,\varepsilon_{ijk}\,p_4\ ,\\
{[p_4,x^i,x^j]}_\phi &= \frac{\lambda\,\ell_s^4}{2}\,R^{4,ijk4}\,p_k \qquad \text{and} \qquad [p_4,x^i,x^4]_\phi = -\frac{\lambda^2\,\ell_s^4}{2}\,R^{4,1234}\,p^i\ ,\\
{[p_4,p_i,x^j]}_\phi &= -\frac{\hbar^2\,\lambda}{2}\,\delta_i{}^j\,x^4-\frac{\hbar^2\,\lambda^2}{2}\,\varepsilon_i{}^{jk}\,x_k\ ,\\
{[p_4,p_i,x^4]}_\phi &= -\frac{\hbar^2\,\lambda^3}{2}\,x_i \qquad \text{and} \qquad [p_4,p_i,p_j]_\phi = -\frac{\hbar^2\,\lambda^2}{2}\,\varepsilon_{ijk}\,p^k\ .
\end{aligned}
\tag{6.13}
$$

For any constraint $G=0$ on the eight-dimensional phase space, these 3-brackets induce a 2-bracket by defining $[f,g]_G:=[f,g,G]_\phi$. In particular, taking $G=\frac{2}{\lambda\,\hbar^2}\,p_4$ (or $G=R^{\mu,\nu\rho\alpha\beta}\,p_\mu$) gauge fixes the 3-brackets (6.13) to the brackets (6.9) together with their Jacobiators. The quantization of this 3-algebra is currently an open problem; see [81] for some preliminary steps towards the construction of a suitable ternary product for deformation quantization which naturally incorporates both the star product and the triproduct.

The magnetic monopole system of Section 2 with constant uniform magnetic charge can be embedded in string theory as D0-branes bound to a uniform distribution of D6-branes. The M-theory lift of this configuration is identified by [87] as a non-geometric variant of the Kaluza-Klein monopole solution of M-theory, whose phase space brackets map to (6.9) under magnetic duality and reduce to the magnetic monopole algebra (2.3) in the contraction limit $\lambda=0$.

A suitable check of these purported claims should come from the next rung $n=3$ on the AKSZ geometric ladder of Section 4, which has been investigated partially in [79]; the symplectic Lie 3-algebroid structure in this case is called a *Lie algebroid up to homotopy*, and the corresponding AKSZ sigma-model describes closed M2-branes viewed as boundaries of open threebranes. The standard Lie algebroid up to homotopy is the anticipated generalized tangent bundle $T\widetilde{M}\oplus\bigwedge^2T^*\widetilde{M}$ of exceptional field theory [22], and the bracket structure in this case is the expected higher 2-Courant bracket. However, beyond the simplest case of the four-form flux $G=\mathrm{d}C$ of the M-theory C-field, it is not clear how to twist these higher algebroid structures by geometric and non-geometric fluxes; see [79] for a discussion of this point. This may be related to the fact that, unlike T-duality which maps closed strings to closed strings, U-duality maps M2-branes to M-waves (the lifts of D0-branes), and more generally to M5-branes in higher dimensions.

Locally non-geometric fluxes in M-theory on higher dimensional spacetimes have been discussed in [86]. In each dimensionality one encounters the same qualitative structure, missing momentum modes in the M2-brane phase space, but now with a host of R-type fluxes with varying tensorial structures. The corresponding bracket structures are not currently understood, and may also involve higher algebras.

6.3 L_∞-algebras

As mentioned in Section 5, the Kontsevich formality maps are quasi-isomorphisms between particular examples of L_∞-algebras, and it is natural to wonder if such higher algebraic structures govern the infinitesimal symmetries of the higher quantum geometries we have discussed. In [89] it was observed that the nonassociative phase space deformation (3.27) underlying the closed string R-flux background can be realized and understood as a 2-term L_∞-algebra, which is similar to the characterization [97] of the bracket structure underlying Courant algebroids from Section 4. This result was extended, and put into a more general and systematic framework by [70], showing in particular how the M-theory R-flux algebra (6.9) is realized in the same way. It would be interesting to similarly understand the covariant M2-brane 3-algebra (6.13) in this way, which may also be the appropriate framework for the higher bracket structures underlying the phase spaces of non-geometric M-theory in dimensions $d > 4$ where they are governed by the E_d exceptional field theory [86].

These occurences nicely match the original appearence of L_∞-algebras in physics as higher gauge symmetry algebras of closed string field theory in [119], and more recently as the higher symmetries underlying double field theory [51, 50, 68] and two-dimensional conformal field theory [29]. It was shown by [30] that the symmetries and dynamics of open string nonassociative gauge theories on curved D-brane worldvolumes are also governed by an underlying L_∞-algebra. Extrapolating this feature to non-geometric backgrounds could lead to a similar characterization of the diffeomorphism symmetries in nonassociative gravity on phase space, and its fate under polarization to spacetime which may clarify the connections with closed string theory and double field theory discussed above.

Acknowledgments

We would like to thank the organisors of the School for the invitation to deliver these lectures, and all the participants for their persistent questions and comments which have helped shape the structure and content of these lecture notes. Several points of view discussed in this article have emerged from numerous discussions over the years with various collaborators and colleagues; we would like to thank in particular Paolo Aschieri, Ralph Blumenhagen, Chris Hull, Dieter Lüst, Vladislav Kupriyanov, Emanuel Malek, Alexander Schenkel and Peter Schupp. This work was supported in part by the Action MP1405 QSPACE from the European Cooperation in Science and Technology (COST) and by the Consolidated Grant ST/P000363/1 from the UK Science and Technology Facilities Council (STFC).

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УНИВЕРЗИТЕТ У БЕОГРАДУ
ФИЗИЧКИ ФАКУЛТЕТ
Бр. 283/10
29. 9. 2023.
БЕОГРАД СТУДЕНТСКИ ТРГ 12-16
П. ФАХ 98

На основу члана 29 Закона о општем управном поступку («Службени гласник РС» број 18/2016 и 95/2018), и члана 149 Статута Универзитета у Београду - Физичког факултета, по захтеву ДАНИЈЕЛА ОБРИЋА, мастер физичара, издаје се следеће

У В Е Р Е Њ Е

ДАНИЈЕЛ ОБРИЋ, мастер физичар, дана 18. септембра 2023. године, одбранио је докторску дисертацију под називом

"T-DUALIZATION OF BOSONIC STRING AND TYPE IIB SUPERSTRING IN PRESENCE OF COORDINATE DEPENDENT BACKGROUND FIELDS" (Т-дуализација бозонске струне и тип IIB суперструне у пристуству координатно зависних позадинских поља)

пред Комисијом Универзитета у Београду - Физичког факултета и тиме испунио све услове за промоцију у ДОКТОРА НАУКА – ФИЗИЧКЕ НАУКЕ.

Уверење се издаје на лични захтев, а служи ради регулисања права из радног односа и важи до промоције, односно добијања докторске дипломе.

Уверење је ослобођено плаћања таксе.

ДЕКАН ФИЗИЧКОГ ФАКУЛТЕТА

Проф. др Иван Белча

Република Србија · УНИВЕРЗИТЕТ У БЕОГРАДУ · ФИЗИЧКИ ФАКУЛТЕТ · БЕОГРАД

Универзитет у Крагујевцу
ПРИРОДНО-МАТЕМАТИЧКИ ФАКУЛТЕТ
Број: 9/9
29. 01. 2024. године
К р а г у ј е в а ц

На основу захтева Управника Института за физику доц. др Владимира Марковића, издаје се следећа

П О Т В Р Д А

Потврђује се да је **др Данијел Обрић,** истраживач-сарадник Института за физику Универзитета у Београду, одржао научно-популарна предавања на Институту за физику Природно-математичког факултета у Крагујевцу, и то:

- 22.11.2023. године: **ICTP: Преглед дешавања у модерној теоријској физици,**
- 14.12.2023. године: **Општа теорија релативности: Модерна теорија простор-времена,**
- 14.12.2023. године: **Квантна теорија поља: Модерна теорија фундаменталних сила.**

Ова потврда се издаје ради регулисања одговарајућих права поводом напредовања у више звање, те се у друге сврхе не може користити.

за **Д Е К А Н** а

Проф. др Марија Станић

КОЛАРЧЕВА ЗАДУЖБИНА

ЗАДУЖБИНА ИЛИЈЕ М. КОЛАРЦА
основана 1878.
ЦЕНТАР ЗА ПРЕДАВАЧКУ ДЕЛАТНОСТ

Циклус предавања

КВАНТНА ГРАВИТАЦИЈА – СВЕТИ ГРАЛ САВРЕМЕНЕ ФИЗИКЕ

1. Зашто квантна механика
Др Игор Салом
26. септембар 2023. у 18.00

2. Зашто општа релативност
Др Данијел Обрић
3. октобар 2023. у 18.00

3. Зашто теорија поља
Др Бојан Николић
10. октобар 2023. у 18.00

4. Зашто поља у кривом простору
Др Марко Војиновић
17. октобар 2023. у 18.00

5. Зашто квантна гравитација
Др Тијана Раденковић
24. октобар 2023. у 18.00

Мала сала Коларчеве задужбине

Циклус је реализован у сарадњи са пројектом „Квантна гравитација из виших гејџ теорија“ (QGHG-2021), број 7745968 програма ИДЕЈЕ Фонда за науку

тел. 2637-609, 2638-472; факс: 3031-711
www.kolarac.rs; e-mail: predavanja@kolarac.rs
Програме подржавају **Секретаријат за културу Скупштине града Београда и Министарство за науку Републике Србије**

Subject **Kragujevac Journal of Science - invitation**
From <kjs@kg.ac.rs>
Date 2023-12-01 05:42

- KJS_63_review.docx (~221 KB)

Dear Professor Danijel Obric,

Given your expertise in the field and, particularly, the link between the topics tackled in the submission ID63 and your research activities, I am inviting you to be a reviewer for the above mentioned article.

I would very much appreciate it if you could promptly find the time to give a quick look at this article and decide whether you can accept the invitation to review it. If you kindly accept the invitation, please return your report within 60 days after you accepted the invitation.

I would appreciate your prompt response to this invitation, even if you should decide to decline it. In this latter event, I would greatly appreciate your suggestions for alternative referees (names and email addresses) who you consider suited to this task and who would possibly be willing to accept the invitation.

Please answer using link at the end of this email.

I do hope that you will accept this invitation and thank you in advance for your prompt reaction.

Best regards,
Editorial Office,
Kragujevac Journal of Science, https://www.pmf.kg.ac.rs/KJS/

Your username is [redacted]. Please answer using following link:
https://www.pmf.kg.ac.rs/KJS/en/index.php?page=message&revisor=1&secret=02d6c773589d50638e059d1209ba4f71

Subject **Kragujevac Journal of Science - new message**
From <kjs@kg.ac.rs>
Date 2023-12-18 17:43

Dear Professor Danijel Obric,

Thank you for sending your suggestion and report on the manuscript submitted for publication in Kragujevac Journal of Science.

You can access your review comments and (eventually) view the editor's decision letter by logging onto our Editorial System\â€™s site as a reviewer:

https://www.pmf.kg.ac.rs/KJS/

We also appreciate you helping Editorial Board of Kragujevac Journal of Science to maintain quality and reliability of our Journal.

Best regards,
Editorial Office,
Kragujevac Journal of Science, https://www.pmf.kg.ac.rs/KJS/

GRAVITY and STRING THEORY: NEW IDEAS FOR UNSOLVED PROBLEMS III

SEPTEMBER 7-9 2018
ZLATIBOR, SERBIA

HOME ORGANIZATION PROGRAMME SPEAKERS PHOTOS

PARTICIPANTS

ACCOMMODATION

TRAVEL

PRACTICAL INFO

REGISTRATION FORM

CONTACT

ABOUT ZLATIBOR

Programme

The lectures will take place at the congress hall 2 (below ground floor) at the Romanija Congress Center, 50 meters across the road from the hotel.

	Thursday	**Friday**	**Saturday**	**Sunday**
Timetable	**September 6**	**September 7**	**September 8**	**September 9**
09:45 - 10:00	Arrival	Opening		
10:00 - 11:00		Dragovich	Sazdović	Miković
11:00 - 12:00		Cvetković	Haack	Vojinović
12:00 - 12:30		Radovanović	Nikolić	Radenković
12:30 - 13:00				Đorđević

13:00 - 15:00		Lunch break		
15:00 - 15:30		Gočanin	Davidović	Departure
15:30 - 16:00		Konjik		
16:00 - 16:30		Salom	Čubrović	
16:30 - 17:00				
17:00 - 17:30		Ivanišević		
17:30 - 18:00		Obrić		

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Participants

List of participants:

Milutin Blagojević (Serbia)
Branislav Cvetković (Serbia)
Mihailo Čubrović (Serbia)

Ljubica Davidović (Serbia)
Aleksandra Dimić (Serbia)
Branko Dragovich (Serbia)
Goran Đorđević (Serbia)
Dragoljub Gočanin (Serbia)
Michael Haack (Germany)
Ilija Ivanišević (Serbia)
Nikola Konjik (Serbia)
Aleksandar Miković (Portugal)
Biljana Nikolić (Serbia)
Bojan Nikolić (Serbia)
Danijel Obrić (Serbia)
Tijana Radenković (Serbia)

Voja Radovanović (Serbia)

Igor Salom (Serbia)

Branislav Sazdović (Serbia)

Dejan Simić (Serbia)

Marko Vojinović (Serbia)