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Hamiltonian Analysis for the Scalar Electrodynamics as 3*BF* Theory

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Abstract: The higher category theory can be employed to generalize the *BF* action to the so-called 3*BF* action, by passing from the notion of a gauge group to the notion of a gauge 3-group. The theory of scalar electrodynamics coupled to Einstein–Cartan gravity can be formulated as a constrained 3*BF* theory for a specific choice of the gauge 3-group. The complete Hamiltonian analysis of the 3*BF* action for the choice of a Lie 3-group corresponding to scalar electrodynamics is performed. This analysis is the first step towards a canonical quantization of a 3*BF* theory, an important stepping stone for the quantization of the complete scalar electrodynamics coupled to Einstein–Cartan gravity formulated as a 3*BF* action with suitable simplicity constraints. It is shown that the resulting dynamic constraints eliminate all propagating degrees of freedom, i.e., the 3*BF* theory for this choice of a 3-group is a topological field theory, as expected.

Keywords: Hamiltonian analysis; higher gauge theory; BF theory; topological theory; scalar electrodynamics

1. Introduction

The vast majority of physics community agrees that the quantum theory of gravity is necessary, even if they disagree on the quantization approach. The theory of loop quantum gravity is one of the well-formulated possible candidates for the desired theory of quantum gravity [1–3]. There are two approaches within the theory—the canonical and the covariant quantization method. The covariant quantization method is focused on obtaining a generating functional, by considering a triangulated spacetime manifold and defining the functional as a state sum over all configurations of a field living on simplices of the triangulation [2].

One of the key tools in the covariant quantization approach is the so-called *BF* theory. Given a Lie group G and its corresponding Lie algebra $\mathfrak{g}$, one considers a $\mathfrak{g}$-valued connection 1-form A, and its corresponding field strength 2-form $F \equiv \mathrm{d}A + A \wedge A$. Multiplying F with a $\mathfrak{g}$-valued Lagrange multiplier 2-form B and integrating over a four-dimensional spacetime manifold $\mathcal{M}$, one obtains the action of the *BF* theory,

$$S_{BF}[A,B] = \int_{\mathcal{M}} \langle B \wedge F \rangle_{\mathfrak{g}} \,,$$

where $\langle \_ , \_ \rangle_{\mathfrak{g}}$ is a G-invariant non-degenerate symmetric bilinear form. The *BF* theory derives its name from the symbols B and F for the Lagrange multiplier and the field strength present in the action. As it is defined, the *BF* theory is topological, containing no local propagating degrees of freedom. Therefore, for the purpose of building physically relevant actions, attention usually focuses not on the pure *BF* theory, but rather on the theory with constraints. The constrained *BF* models are based on deformations of the *BF* theory [4], by adding constraints to the topological *BF* action that promote some of the gauge degrees of freedom into physical ones. The well known example is the Plebanski model for general relativity [5]. Constrained BF models represent a starting point in the spinfoam approach to the construction of quantum gravity models [2].

The main shortcoming of building a quantum gravity model using a BF theory is the fact that it is very hard, if not impossible, to write the action for matter fields (specifically scalar and fermion fields) in the form of a constrained BF theory. Thus, the spinfoam quantization method is limited to pure gravity, and the problem of consistently coupling matter fields to gravity in this framework becomes highly nontrivial. One of the proposed ways to circumvent this issue is to generalize the notion of a BF theory using the mathematical apparatus of higher category theory.

The higher category theory [6] can be employed to generalize the BF action to the so-called nBF action, by passing from the notion of a gauge group to the notion of a gauge n-group (for a comprehensive review of n-groups see for example [7], and also Appendix C). Specifically, the notion of a 3-group in the framework of higher category theory is introduced as a 3-category with only one object where all the morphisms, 2-morphisms and 3-morphisms are invertible. Based on this generalization, recently a constrained $3BF$ action has been introduced, which describes the full Standard Model coupled to Einstein–Cartan gravity [8].

As a first step to the study of the Hamiltonian structure of such theories, in this work, we discuss the simplest nontrivial toy example, namely the theory of scalar electrodynamics coupled to gravity. The standard way to define scalar electrodynamics coupled to gravity is by the action:

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[-\frac{1}{16\pi l_p^2} R - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + g^{\mu\nu} \nabla_\mu \phi^* \nabla_\nu \phi - m^2 \phi^* \phi \right] . \tag{1}$$

Here, $g_{\mu\nu}$ is the spacetime metric, $g \equiv \det(g_{\mu\nu})$ is its determinant, R is the corresponding curvature scalar, and l_p is the Planck length, its square being equal to the Newton's gravitational constant, $l_p^2 = G$, in the natural system of units $\hbar = c = 1$. The total covariant derivative ∇_μ of the complex scalar field ϕ is defined as $\nabla_\mu \phi = (\partial_\mu + iqA_\mu)\phi$, and thus coupled to the electromagnetic potential A_μ via the coupling constant q (the electric charge of the field ϕ). See Appendix A for more detailed notation. In the next section, we will reformulate this model as a classically equivalent constrained $3BF$ theory for a specific choice of the gauge 3-group. Moreover, for reasons of simplicity, in the Hamiltonian analysis, we will focus only on the topological sector, disregarding the simplicity constraints. The Hamiltonian structure of the theory is important for various reasons, primarily for the canonical quantization program.

The layout of the paper is as follows. In Section 2, we introduce the 3-group structure corresponding to the theory of scalar electrodynamics coupled to Einstein–Cartan gravity and the corresponding constrained $3BF$ action. Section 3 contains the Hamiltonian analysis for the topological, $3BF$ sector of the action, with the resulting first-class and second-class constraints present in the theory, and their mutual Poisson brackets. In Section 4, we analyze the Bianchi identities that the first-class constraints satisfy, which enforce restrictions in the sense of Hamiltonian analysis, and reduce the number of independent first-class constraints present in the theory. Section 5 focuses on the counting of the dynamical degrees of freedom present in the theory, based on the results from Sections 3 and 4. Encouraged by these results, in Section 6, we construct the generator of the gauge symmetries for the topological theory and we find the form variations of all variables and their canonical momenta. Finally, Section 7 is devoted to the discussion of the results and the possible future lines of research. The Appendices contain various technical details.

The notation and conventions are as follows. The local Lorentz indices are denoted by the Latin letters $a, b, c, \dots$, take values $0, 1, 2, 3$, and are raised and lowered using the Minkowski metric η_{ab} with signature $(-, +, +, +)$. Spacetime indices are denoted by the Greek letters $\mu, \nu, \dots$, and are raised and lowered by the spacetime metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ are the tetrad fields. The inverse tetrad is denoted as $e^\mu{}_a$, so that the standard orthogonality conditions hold: $e^a{}_\mu e^\mu{}_b = \delta^a_b$ and $e^a{}_\mu e^\nu{}_a = \delta^\nu_\mu$. When needed, spacetime indices will be split into time and space indices, denoted with a 0 and lowcase Latin indices $i, j, \ldots$, respectively. All other indices that appear in the paper are dependent on the context, and their usage is explicitly defined in the text where they appear. The antisymmetrization over two indices is introduced with the factor one half that is $A_{[a_1|a_2\ldots a_{n-1}|a_n]} = \frac{1}{2}\left(A_{a_1a_2\ldots a_{n-1}a_n} - A_{a_na_2\ldots a_{n-1}a_1}\right)$, and the total antisymmetrization is introduced as $A_{[a_1\ldots a_n]} = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{\mathrm{sign}(\sigma)}A_{a_{\sigma(1)}\ldots a_{\sigma(n)}}$.

2. Scalar Electrodynamics as a Constrained 3*BF* Action

Let us begin by providing a short introduction into the construction and structure of a 3*BF* theory, after which we will impose appropriate simplicity constraints, in order to obtain the equations of motion for scalar electrodynamics coupled to gravity.

As was discussed in detail in [8], one formulates a topological 3*BF* action by specifying a particular gauge Lie 3-group. It has been proved that any strict 3-group is equivalent to a 2-crossed module [9,10]. A gauge theory for the manifold $\mathcal{M}_4$ and 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,,\triangleright\,,\{\_\,,\_\})$ can be constructed for the following choice of the three Lie groups as:

$$G = SO(3,1)\times U(1)\,,\quad H = \mathbb{R}^4\,,\quad L = \mathbb{R}^2\,.$$

The maps ∂ and δ are chosen to be trivial. The action of the algebra $\mathfrak{g}$ on $\mathfrak{h}$ and $\mathfrak{l}$ is chosen as:

$$\begin{array}{ll} M_{ab}\triangleright P_c = \triangleright_{ab,c}{}^{d}\,P_d = \delta_{[a|}{}^{d}\eta_{|b]c}\,P_d = \eta_{[b|c}\,P_{|a]}\,, & T\triangleright P_a = 0\,,\\ M_{ab}\triangleright P_A = 0\,, & T\triangleright P_A = \triangleright_A{}^{B}\,P_B \end{array} \tag{2}$$

where M_{ab} denote the six generators of $\mathfrak{so}(3,1)$, T is the sole generator of $\mathfrak{u}(1)$, P_a are the four generators of $\mathbb{R}^4$ and P_A are the two generators of $\mathbb{R}^2$. In the previous expression, the action of the algebra $\mathfrak{u}(1)$ on the algebra $\mathbb{R}^2$ is defined via

$$\triangleright_A{}^{B} = iq\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

The action of the algebra $\mathfrak{g}$ on itself is by definition given via the adjoint representation and, for the choice $\mathfrak{g} = \mathfrak{so}(3,1)\times\mathfrak{u}(1)$, one obtains

$$\begin{gathered} M_{ab}\triangleright M_{cd} = \triangleright_{ab\,,cd}{}^{ef}\,M_{ef} = f_{ab\,,cd}{}^{ef}\,M_{ef} = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}\,,\\ M_{ab}\triangleright T = 0\,,\quad T\triangleright M_{ab} = 0\,,\quad T\triangleright T = 0\,, \end{gathered} \tag{3}$$

as the consequence of the direct product structure and the Abelian nature of the subgroup $U(1)$. The Peiffer lifting

$$\{\_\,,\_\}: H\times H \to L$$

is also trivial, i.e., all the coefficients $X_{ab}{}^{A}$ are equal to zero:

$$\{P_a\,,P_b\} \equiv X_{ab}{}^{A}T_A = 0\,. \tag{4}$$

Given Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$, one can introduce a 3-connection (α,β,γ) given by the algebra-valued differential forms $\alpha\in\mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$, $\beta\in\mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$ and $\gamma\in\mathcal{A}^3(\mathcal{M}_4\,,\mathfrak{l})$. The corresponding fake 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is then defined as:

$$\mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,,\qquad \mathcal{G} = \mathrm{d}\beta + \alpha\wedge^{\triangleright}\beta - \delta\gamma\,,\qquad \mathcal{H} = \mathrm{d}\gamma + \alpha\wedge^{\triangleright}\gamma + \{\beta\wedge\beta\}\,, \tag{5}$$

see [9,10] for details. For this specific choice of a 3-group, where $\alpha = \omega + A$, given by the algebra-valued differential forms $\omega \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{so}(3,1))$, $A \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{u}(1))$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathbb{R}^4)$ and $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,,\mathbb{R}^2)$, the corresponding 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is defined as

$$\begin{aligned} \mathcal{F} &= R^{ab}M_{ab} + FT &&= \left(\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\right)M_{ab} + \mathrm{d}A\; T\,, \\ \mathcal{G} &= \mathcal{G}^a P_a &&= \left(\mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b\right)P_a\,, \\ \mathcal{H} &= \mathcal{H}^A P_A &&= \left(\mathrm{d}\gamma^A + \rhd_B{}^A A \wedge \gamma^B\right)P_A\,. \end{aligned} \tag{6}$$

Note that the connection ω^{ab} is not present in the last expression, as follows from the definition of the action $\rhd$ and the Peiffer lifting $\{\_\,,\_\}$, see Equations (2) and (4):

$$\begin{aligned} \mathcal{H} &= \mathrm{d}\gamma + \alpha \wedge^{\rhd} \gamma + \{\beta \wedge \beta\} \\ &= \mathrm{d}\gamma^A P_A + (\omega^{ab}M_{ab} + AT) \wedge^{\rhd} (\gamma^A P_A) \\ &= \mathrm{d}\gamma^A P_A + \omega^{ab} \wedge \gamma^A M_{ab} \rhd P_A + A \wedge \gamma^A T \rhd P_A \\ &= \mathrm{d}\gamma^A P_A + A \wedge \gamma^A \rhd_A{}^B P_B \\ &= (\mathrm{d}\gamma^A + \rhd_B{}^A A \wedge \gamma^B) P_A\,. \end{aligned} \tag{7}$$

The coefficients of the differential 2-forms F and R^{ab}, 3-form $\mathcal{G}$, and 4-form $\mathcal{H}$ are:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\,, \\ R^{ab}{}_{\mu\nu} &= \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^a{}_{c\mu}\omega^{cb}{}_\nu - \omega^a{}_{c\nu}\omega^{cb}{}_\mu\,, \\ \mathcal{G}^a{}_{\mu\nu\rho} &= \partial_\mu \beta^a{}_{\nu\rho} + \partial_\nu \beta^a{}_{\rho\mu} + \partial_\rho \beta^a{}_{\mu\nu} + \omega^a{}_{b\mu}\,\beta^b{}_{\nu\rho} + \omega^a{}_{b\nu}\,\beta^b{}_{\rho\mu} + \omega^a{}_{b\rho}\,\beta^b{}_{\mu\nu}\,, \\ \mathcal{H}^A{}_{\mu\nu\rho\sigma} &= \partial_\mu \gamma^A{}_{\nu\rho\sigma} - \partial_\nu \gamma^A{}_{\rho\sigma\mu} + \partial_\rho \gamma^A{}_{\sigma\mu\nu} - \partial_\sigma \gamma^A{}_{\mu\nu\rho} \\ &\quad + \rhd_B{}^A A_\mu \gamma^B{}_{\nu\rho\sigma} - \rhd_B{}^A A_\nu \gamma^B{}_{\rho\sigma\mu} + \rhd_B{}^A A_\rho \gamma^B{}_{\sigma\mu\nu} - \rhd_B{}^A A_\sigma \gamma^B{}_{\mu\nu\rho}\,. \end{aligned} \tag{8}$$

Now, one can define a gauge invariant 3BF action as:

$$S_{3BF} = \int_{\mathcal{M}_4} \Big(\langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \Big)\,, \tag{9}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{so}(3,1))$, $C \in \mathcal{A}^1(\mathcal{M}_4\,,\mathbb{R}^4)$ and $D \in \mathcal{A}^0(\mathcal{M}_4\,,\mathbb{R}^2)$ are Lagrange multipliers. The forms $\langle\_\,,\_\rangle_{\mathfrak{g}}$, $\langle\_\,,\_\rangle_{\mathfrak{h}}$ and $\langle\_\,,\_\rangle_{\mathfrak{l}}$ are G-invariant bilinear symmetric nondegenerate forms on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, respectively, defined as

$$\langle M_{ab}\,, M_{cd}\rangle_{\mathfrak{g}} = g_{ab,cd}\,, \quad \langle T\,,T\rangle_{\mathfrak{g}} = 1\,, \quad \langle M_{ab}\,,T\rangle_{\mathfrak{g}} = 0\,, \quad \langle P_a\,,P_b\rangle_{\mathfrak{h}} = g_{ab}\,, \quad \langle P_A\,,P_B\rangle_{\mathfrak{l}} = g_{AB}\,,$$

where

$$g_{ab,cd} = \eta_{a[c|}\eta_{b|d]}\,, \qquad g_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad g_{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Identifying the Lagrange multiplier C^a as the tetrad field e^a, and the Lagrange multiplier D^A as the doublet of scalar fields ϕ^A,

$$\phi = \phi^A P_A = \phi P_1 + \phi^* P_2\,,$$

based on their transformation properties as discussed in [8,11], the Lagrangian of the action (9) obtains the form:

$$S_{3BF} = \int_{\mathcal{M}_4} \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} \big(\frac{1}{4}\, B^{ab}{}_{\mu\nu}\, R^{cd}{}_{\rho\sigma}\, g_{ab,cd} + \frac{1}{4}\, B_{\mu\nu} F_{\rho\sigma} + \frac{1}{3!}\, e^a{}_\mu\, \mathcal{G}^b{}_{\nu\rho\sigma}\, g_{ab} + \frac{1}{4!}\, \phi^A \mathcal{H}^B{}_{\mu\nu\rho\sigma}\, g_{AB} \big) \,. \quad (10)$$

Varying the action with respect to all the variables, one obtains the equations of motion:

varied variable	equation of motion	varied variable	equation of motion
δB^{ab}	$R_{ab} = 0$	δB	$F = 0$
$\delta\omega^{ab}$	$\nabla B_{ab} - e_{[a\vert} \wedge \beta_{\vert b]} = 0$	δA	$\mathrm{d}B + \phi_A \rhd_B{}^A \gamma^B = 0$
δe^a	$\mathcal{G}_a = 0$	$\delta\beta^a$	$\nabla e_a = 0$
$\delta\phi^A$	$\nabla\gamma_A = 0$	$\delta\gamma^A$	$\nabla\phi_A = 0$

(11)

Since one is interested in the doublet of scalar fields ϕ^A of mass m and charge q minimally coupled to gravity and electromagnetic field, we impose additional simplicity constraint terms to the topological action (9), in order to obtain the appropriate equations of motion equivalent to the equations of motion for the action (1):

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B \wedge F + e_a \wedge \nabla\beta^a + \phi_A\, \nabla\gamma^A \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda^A \wedge \left(\gamma_A - \frac{1}{2} H_{abcA} e^a \wedge e^b \wedge e^c \right) + \Lambda^{abA} \wedge \left(H_{abcA} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \nabla\phi_A \wedge e_a \wedge e_b \right) \\ & + \lambda \wedge \left(B - \frac{12}{q} M_{ab} e^a \wedge e^b \right) + \zeta^{ab} \left(M_{ab} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b \right) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi_A\, \phi^A \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,. \end{aligned} \quad (12)$$

For the notation used here and the equations of motion obtained by varying the action (12), see Appendix A.

The dynamical degrees of freedom are the tetrad fields e^a, the scalar doublet ϕ^A, and the electromagnetic potential A, while the remaining variables are algebraically determined in terms of them, as shown in Appendix A. The equation of motion for the field ϕ^A reduces to the covariant Klein-Gordon equation for the scalar field,

$$\left(\nabla_\mu \nabla^\mu - m^2 \right) \phi_A = 0 \,. \quad (13)$$

The differential equation of motion for the field A is:

$$\nabla_\mu F^{\mu\nu} = j^\nu \,, \qquad j^\mu \equiv \frac{1}{2} \left(\nabla^\nu \phi^A \rhd^B{}_A \phi_B - \phi_A \rhd_B{}^A \nabla^\nu \phi^B \right) = iq \left(\nabla\phi^*\, \phi - \phi^* \nabla\phi \right) . \quad (14)$$

Finally, the equation of motion for e^a becomes:

$$\begin{aligned} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R &= 8\pi l_p^2\; T^{\mu\nu} \,, \\ T^{\mu\nu} \equiv \nabla^\mu \phi_A\, \nabla^\nu \phi^A - \frac{1}{2} g^{\mu\nu} & \left(\nabla_\rho \phi_A\, \nabla^\rho \phi^A + m^2 \phi_A\, \phi^A \right) - \frac{1}{4q} \left(F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} + 4 F^{\mu\rho} F_\rho{}^\nu \right) . \end{aligned} \quad (15)$$

3. The Hamiltonian Analysis

The Hamiltonian analysis of the constrained $3BF$ action (12) for scalar electrodynamics is exceedingly complicated to study. A testament to this is the level of complexity of the constrained $2BF$ formulation of general relativity [12], which is merely one sector in the action (12). Therefore, in this paper, we will limit ourselves to the topological sector of the theory, namely the unconstrained $3BF$ theory (9), which consists of the terms in the first row of Equation (12), and is written in full detail in Equation (10). One should be aware that this restriction changes various properties of the theory. Namely, the simplicity constraints (everything but the first row in Equation (12)) substantially modify the dynamics of the theory—they increase the number of local propagating degrees of freedom of the theory, a property that was known since the original Plebanski model [5]. On the other hand, the unconstrained $3BF$ theory (9) is important even in its own right, and the Hamiltonian analysis may give important insight into the structure of both the unconstrained and the constrained theory.

In what follows, the complete Hamiltonian analysis for the action (9) is presented, see [13] for an overview and a comprehensive introduction of the Hamiltonian analysis. The Hamiltonian analysis for a $2BF$ action is performed in [12,14–16].

Under the standard assumption that the spacetime manifold is globally hyperbolic, $\mathcal{M}_4 = \mathbb{R} \times \Sigma_3$, the Lagrangian of the action (9) has the form:

$$L_{3BF} = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \epsilon^{\mu\nu\rho\sigma} \Big(\frac{1}{4} B^{ab}{}_{\mu\nu}\, R^{cd}{}_{\rho\sigma}\, g_{ab,cd} + \frac{1}{4} B_{\mu\nu} F_{\rho\sigma} + \frac{1}{3!} e^a{}_\mu\, \mathcal{G}^b{}_{\nu\rho\sigma}\, g_{ab} + \frac{1}{4!} \phi^A \mathcal{H}^B{}_{\mu\nu\rho\sigma}\, g_{AB} \Big) . \tag{16}$$

The canonical momentum $\pi(q)$ corresponding for the canonical coordinate q from the set of all variables in the theory, $q \in \{B^{ab}{}_{\mu\nu}, \omega^{ab}{}_\mu, B_{\mu\nu}, A_\mu, e^a{}_\mu, \beta^a{}_{\mu\nu}, \phi^A, \gamma^A{}_{\mu\nu\rho}\}$, is obtained as a derivative of the Lagrangian with respect to the appropriate velocity,

$$\pi(q) \equiv \frac{\delta L}{\delta \partial_0 q} ,$$

giving:

$$\begin{array}{rclrcl}
\pi(B)_{ab}{}^{\mu\nu} & = & 0 , & \pi(\omega)_{ab}{}^{\mu} & = & \epsilon^{0\mu\nu\rho} B_{ab\nu\rho} , \\
\pi(B)^{\mu\nu} & = & 0 , & \pi(A)^{\mu} & = & \frac{1}{2}\epsilon^{0\mu\nu\rho} B_{\nu\rho} , \\
\pi(e)_a{}^{\mu} & = & 0 , & \pi(\beta)_a{}^{\mu\nu} & = & -\epsilon^{0\mu\nu\rho} e_{a\rho} , \\
\pi(\phi)_A & = & 0 , & \pi(\gamma)_A{}^{\mu\nu\rho} & = & \epsilon^{0\mu\nu\rho} \phi_A .
\end{array} \tag{17}$$

Since these momenta cannot be inverted for the time derivatives of the variables, they all give rise to primary constraints:

$$\begin{array}{rclrcl}
P(B)_{ab}{}^{\mu\nu} & \equiv & \pi(B)_{ab}{}^{\mu\nu} \approx 0 , & P(\omega)_{ab}{}^{\mu} & \equiv & \pi(\omega)_{ab}{}^{\mu} - \epsilon^{0\mu\nu\rho} B_{ab\nu\rho} \approx 0 , \\
P(B)^{\mu\nu} & \equiv & \pi(B)^{\mu\nu} \approx 0 , & P(A)^{\mu} & \equiv & \pi(A)^{\mu} - \frac{1}{2}\epsilon^{0\mu\nu\rho} B_{\nu\rho} \approx 0 , \\
P(e)_a{}^{\mu} & \equiv & \pi(e)_a{}^{\mu} \approx 0 , & P(\beta)_a{}^{\mu\nu} & \equiv & \pi(\beta)_a{}^{\mu\nu} + \epsilon^{0\mu\nu\rho} e_{a\rho} \approx 0 , \\
P(\phi)_A & \equiv & \pi(\phi)_A \approx 0 , & P(\gamma)_A{}^{\mu\nu\rho} & \equiv & \pi(\gamma)_A{}^{\mu\nu\rho} - \epsilon^{0\mu\nu\rho} \phi_A \approx 0 .
\end{array} \tag{18}$$

Here, the symbol "$\approx$" denotes the so-called "weak" equality, i.e., the equality that holds on a subspace of the phase space determined by the constraints, while the equality that holds for any point of the phase space is referred to as the "strong" equality and it is denoted by the symbol "$=$". The expressions "on-shell" and "off-shell" are used for weak and strong equalities, respectively, and henceforth will be used in this paper.

The fundamental Poisson brackets are defined as:

$$
\begin{aligned}
\{ B^{ab}{}_{\mu\nu}(x)\,,\, \pi(B)_{cd}{}^{\rho\sigma}(y) \} &= 4\delta^{a}{}_{[c}\delta^{b}{}_{d]}\delta^{\rho}{}_{[\mu}\delta^{\sigma}{}_{\nu]}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ \omega^{ab}{}_{\mu}(x)\,,\, \pi(\omega)_{cd}{}^{\nu}(y) \} &= 2\delta^{a}{}_{[c}\delta^{b}{}_{d]}\delta^{\nu}{}_{\mu}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ B_{\mu\nu}(x)\,,\, \pi(B)^{\rho\sigma}(y) \} &= 2\delta^{\rho}{}_{[\mu}\delta^{\sigma}{}_{\nu]}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ A_{\mu}(x)\,,\, \pi(A)^{\nu}(y) \} &= \delta^{\nu}{}_{\mu}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ e^{a}{}_{\mu}(x)\,,\, \pi(e)_{b}{}^{\nu}(y) \} &= \delta^{a}{}_{b}\delta^{\nu}{}_{\mu}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ \beta^{a}{}_{\mu\nu}(x)\,,\, \pi(\beta)_{b}{}^{\rho\sigma}(y) \} &= 2\delta^{a}{}_{b}\, \delta^{\rho}{}_{[\mu}\delta^{\sigma}{}_{\nu]}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ \phi^{A}(x)\,,\, \pi(\phi)_{B}(y) \} &= \delta^{A}{}_{B}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ \gamma^{A}{}_{\mu\nu\rho}(x)\,,\, \pi(\gamma)_{B}{}^{\alpha\beta\gamma}(y) \} &= 3!\delta^{A}{}_{B}\, \delta^{\alpha}{}_{[\mu}\delta^{\beta}{}_{\nu}\delta^{\gamma}{}_{\rho]}\, \delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{19}
$$

Using these relations, one can calculate the algebra between the primary constraints,

$$
\begin{aligned}
\{ P(B)^{abjk}(x)\,,\, P(\omega)_{cd}{}^{i}(y) \} &= 4\epsilon^{0ijk}\, \delta^{a}{}_{[c}\delta^{b}{}_{d]}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ P(B)^{jk}(x)\,,\, P(A)^{i}(y) \} &= \epsilon^{0ijk}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ P(e)^{ak}\,,\, P(\beta)_{b}{}^{ij}(y) \} &= -\epsilon^{0ijk}\, \delta^{a}{}_{b}(x)\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ P(\phi)^{A}(x)\,,\, P(\gamma)_{B}{}^{ijk}(y) \} &= \epsilon^{0ijk}\, \delta^{A}{}_{B}\, \delta^{(3)}(\vec{x}-\vec{y})\,,
\end{aligned}
\tag{20}
$$

while all other Poisson brackets vanish. The canonical on-shell Hamiltonian is defined by

$$
\begin{aligned}
H_c = \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big[& \frac{1}{4}\pi(B)_{ab}{}^{\mu\nu}\,\partial_0 B^{ab}{}_{\mu\nu} + \frac{1}{2}\pi(\omega)_{ab}{}^{\mu}\,\partial_0\omega^{ab}{}_{\mu} + \frac{1}{2}\pi(B)^{\mu\nu}\,\partial_0 B_{\mu\nu} + \pi(A)^{\mu}\,\partial_0 A_{\mu} \\
& + \pi(e)_{a}{}^{\mu}\,\partial_0 e^{a}{}_{\mu} + \frac{1}{2}\pi(\beta)_{a}{}^{\mu\nu}\,\partial_0\beta^{a}{}_{\mu\nu} + \pi(\phi)_{A}\,\partial_0 D^{A} + \frac{1}{3!}\pi(\gamma)_{A}{}^{\mu\nu\rho}\,\partial_0\gamma^{A}{}_{\mu\nu\rho} \Big] - L\,.
\end{aligned}
\tag{21}
$$

Rewriting the Hamiltonian (21) such that all the velocities are multiplied by the first class constraints and therefore in an on-shell quantity they drop out, one obtains:

$$
\begin{aligned}
H_c = -\int_{\Sigma_3} \mathrm{d}^3\vec{x}\,\epsilon^{0ijk} \Big[& \frac{1}{2}B_{ab0i}\,R^{ab}{}_{jk} + \frac{1}{2}B_{0i}F_{jk} + \frac{1}{6}e_{a0}\,\mathcal{G}^{a}{}_{ijk} + \beta^{a}{}_{0i}\nabla_j e_{ak} \\
& + \frac{1}{2}\omega^{ab}{}_{0}\Big(\nabla_i B_{ab\,jk} - e_{[a|i}\,\beta_{|b]jk}\Big) + \frac{1}{2}A_0\Big(\partial_i B_{jk} + \frac{1}{3}\,\phi_A \,\rhd_B{}^{A}\,\gamma^{B}{}_{ijk}\Big) + \frac{1}{2}\gamma^{A}{}_{0ij}\nabla_k\phi_A \Big]\,.
\end{aligned}
\tag{22}
$$

This expression does not depend on any of the canonical momenta and it contains only the fields and their spatial derivatives. By adding a Lagrange multiplier λ for each of the primary constraints we can build the off-shell Hamiltonian, which is given by:

$$
\begin{aligned}
H_T = H_c + \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big[& \frac{1}{4}\lambda(B)^{ab}{}_{\mu\nu}P(B)_{ab}{}^{\mu\nu} + \frac{1}{2}\lambda(\omega)^{ab}{}_{\mu}P(\omega)_{ab}{}^{\mu} + \frac{1}{2}\lambda(B)_{\mu\nu}P(B)^{\mu\nu} + \lambda(A)_{\mu}P(A)^{\mu} \\
& + \lambda(e)^{a}{}_{\mu}P(e)_{a}{}^{\mu} + \frac{1}{2}\lambda(\beta)^{a}{}_{\mu\nu}P(\beta)_{a}{}^{\mu\nu} + \lambda(\phi)^{A}P(\phi)_{A} + \frac{1}{3!}\lambda(\gamma)^{A}{}_{\mu\nu\rho}P(\gamma)_{A}{}^{\mu\nu\rho} \Big]\,.
\end{aligned}
\tag{23}
$$

Since the primary constraints must be preserved in time, one must impose the following requirement:

$$
\dot{P} \equiv \{ P\,,\, H_T \} \approx 0\,,
\tag{24}
$$

for each primary constraint P. By using the consistency condition (24) for the primary constraints $P(B)_{ab}{}^{0i}$, $P(\omega)_{ab}{}^{0}$, $P(B)^{0i}$, $P(A)^{0}$, $P(e)_{a}{}^{0}$, $P(\beta)_{a}{}^{0i}$, and $P(\gamma)_{A}{}^{0ij}$,

$$\begin{aligned} &\dot{P}(B)_{ab}{}^{0i} \approx 0\,, \quad \dot{P}(\omega)_{ab}{}^{0} \approx 0\,, \quad \dot{P}(B)^{0i} \approx 0\,, \quad \dot{P}(A)^{0} \approx 0\,, \\ &\dot{P}(e)_{a}{}^{0} \approx 0\,, \quad \dot{P}(\beta)_{a}{}^{0i} \approx 0\,, \quad \dot{P}(\gamma)_{A}{}^{0ij} \approx 0\,, \end{aligned} \tag{25}$$

one obtains the secondary constraints $\mathcal{S}$,

$$\begin{array}{lcllcl} \mathcal{S}(R)_{ab}{}^{i} & \equiv & \epsilon^{0ijk} R_{abjk} \approx 0\,, & \mathcal{S}(\nabla B)_{ab} & \equiv & \epsilon^{0ijk}\big(\nabla_i B_{ab\,jk} - e_{[a|i}\,\beta_{|b]\,jk}\big) \approx 0\,, \\ \mathcal{S}(F)^{i} & \equiv & \frac{1}{2}\epsilon^{0ijk} F_{jk} \approx 0\,, & \mathcal{S}(\nabla B) & \equiv & \frac{1}{2}\epsilon^{0ijk}\big(\partial_i B_{jk} + \frac{1}{3}\,\phi_A \rhd_B{}^{A}\,\gamma^{B}{}_{ijk}\big) \approx 0\,, \\ \mathcal{S}(\mathcal{G})_{a} & \equiv & \frac{1}{6}\epsilon^{0ijk}\mathcal{G}_{aijk} \approx 0\,, & \mathcal{S}(\nabla e)_{a}{}^{i} & \equiv & \epsilon^{0ijk}\nabla_j e_{ak} \approx 0\,, \\ \mathcal{S}(\nabla\phi)_{A}{}^{ij} & \equiv & \epsilon^{0ijk}\nabla_k \phi_A \approx 0\,, & & & \end{array} \tag{26}$$

while in the case of $P(B)_{ab}{}^{jk}$, $P(\omega)_{ab}{}^{k}$, $P(B)^{jk}$, $P(A)^{k}$, $P(e)_{a}{}^{k}$, $P(\beta)_{a}{}^{jk}$, $P(\phi)_A$ and $P(\gamma)_{A}{}^{ijk}$ the consistency conditions

$$\begin{aligned} &\dot{P}(B)_{ab}{}^{jk} \approx 0\,, \quad \dot{P}(\omega)_{ab}{}^{k} \approx 0\,, \quad \dot{P}(B)^{jk} \approx 0\,, \quad \dot{P}(A)^{k} \approx 0\,, \\ &\dot{P}(e)_{a}{}^{k} \approx 0\,, \quad \dot{P}(\beta)_{a}{}^{jk} \approx 0\,, \quad \dot{P}(\phi)_{A} \approx 0\,, \quad \dot{P}(\gamma)_{A}{}^{ijk} \approx 0\,, \end{aligned} \tag{27}$$

determine the following Lagrange multipliers:

$$\begin{array}{lcllcl} \lambda(\omega)_{ab}{}^{i} & \approx & \nabla^{i}\,\omega_{ab\,0}\,, & \lambda(B)^{ij} & \approx & 2\partial^{[i|}\,B^{0|j]} + \gamma_{A}{}^{0ij} \rhd_B{}^{A}\,\phi^{B}\,, \\ \lambda(A)^{i} & \approx & \partial^{i}\,A_0\,, & \lambda(\beta)_{a}{}^{ij} & \approx & 2\nabla^{[i|}\,\beta_{a}{}^{0|j]} - \omega_{ab}{}^{0}\,\beta^{b\,ij}\,, \\ \lambda(\phi)^{A} & \approx & A^{0} \rhd^{A}{}_{B}\,\phi^{B}\,, & \lambda(e)_{a}{}^{i} & \approx & \nabla^{i}\,e_{a}{}^{0} - \omega_{a}{}^{b\,0}\,e_{b}{}^{i}\,, \\ \lambda(B)_{ab}{}^{ij} & \approx & \multicolumn{4}{l}{2\nabla^{[i|}B_{ab}{}^{0|j]} + e_{[a|\,0}\beta_{|b]}{}^{ij} - 2e_{[a|}{}^{[i|}\beta_{|b]}{}^{0|j]} + 2\omega_{[a|}{}^{c}B_{|b]}{}^{c\,ij}\,,} \\ \lambda(\gamma)_{A}{}^{ijk} & \approx & \multicolumn{4}{l}{-A^{0} \rhd_{A}{}^{B}\,\gamma_{B}{}^{ijk} + \nabla^{i}\gamma_{A}{}^{0jk} - \nabla^{j}\gamma_{A}{}^{0ik} + \nabla^{k}\gamma_{A}{}^{0ij}\,.} \end{array} \tag{28}$$

Note that the consistency conditions leave the Lagrange multipliers

$$\lambda(B)^{ab}{}_{0i}\,, \qquad \lambda(\omega)^{ab}{}_{0}\,, \qquad \lambda(B)_{0i}\,, \qquad \lambda(A)_{0}\,, \qquad \lambda(e)^{a}{}_{0}\,, \qquad \lambda(\beta)^{a}{}_{0i}\,, \qquad \lambda(\gamma)^{A}{}_{0ij} \tag{29}$$

undetermined. The consistency conditions of the secondary constraints do not produce new constraints, since one can show that

$$\begin{array}{lclcl} \dot{\mathcal{S}}(R)^{abi} & = & \{\mathcal{S}(R)^{abi}\,,\,H_T\} & = & \omega^{[a|}{}_{c0}\,\mathcal{S}(R)^{c|b]i}\,, \\ \dot{\mathcal{S}}(\nabla B) & = & \{\mathcal{S}(\nabla B),\,H_T\} & = & -\rhd_B{}^{A}\,\gamma^{B}{}_{0ij}\,\mathcal{S}(\nabla\phi)_{A}{}^{ij}\,, \\ \dot{\mathcal{S}}(\mathcal{G})^{a} & = & \{\mathcal{S}(\mathcal{G})^{a}\,,\,H_T\} & = & \beta_{b0k}\,\mathcal{S}(R)^{abk} - \omega^{ab}{}_{0}\,\mathcal{S}(\mathcal{G})_{b}\,, \\ \dot{\mathcal{S}}(\nabla e)_{a}{}^{i} & = & \{\mathcal{S}(\nabla e)_{a}{}^{i}\,,\,H_T\} & = & e^{b}{}_{0}\,\mathcal{S}(R)_{ab}{}^{i} - \omega_{a}{}^{b}{}_{0}\,\mathcal{S}(\nabla e)_{b}{}^{i}\,, \\ \dot{\mathcal{S}}(\nabla\phi)_{A}{}^{ij} & = & \{\mathcal{S}(\nabla\phi)_{A}{}^{ij}\,,\,H_T\} & = & A_0 \rhd_{A}{}^{B}\mathcal{S}(\nabla\phi)_{B}{}^{ij}\,, \\ \dot{\mathcal{S}}(F)^{i} & = & \{\mathcal{S}(F)^{i}\,,\,H_T\} & = & 0\,, \\ \dot{\mathcal{S}}(\nabla B)_{ab} & = & \{\mathcal{S}(\nabla B)_{ab}\,,\,H_T\} & = & \mathcal{S}(R)_{[a|c}{}^{k}\,B^{c}{}_{|b]0k} + \omega_{[a|}{}^{c}{}_{0}\mathcal{S}(\nabla B)_{|b]c} \\ & & & & -\beta_{[a|0k}\,\mathcal{S}(\nabla e)_{|b]}{}^{k} + e_{[a|0}\,\mathcal{S}(\mathcal{G})_{|b]}\,. \end{array} \tag{30}$$

Then, the total Hamiltonian can be written as

$$
\begin{aligned}
H_T \;=\; \int_{\Sigma_3} \mathrm{d}^3\vec{x} \;\; \Big[& \frac{1}{2}\lambda(B)_{ab}{}^{0i}\,\Phi(B)^{ab}{}_i + \frac{1}{2}\lambda(\omega)_{ab}{}^{0}\,\Phi(\omega)^{ab} + \lambda(B)^{0i}\,\Phi(B)_i + \lambda(A)^0\,\Phi(A) \\
& + \lambda(e)_a{}^0\,\Phi(e)^a + \lambda(\beta)_a{}^{0i}\,\Phi(\beta)^a{}_i + \frac{1}{2}\lambda(\gamma)_A{}^{0ij}\Phi(\gamma)^A{}_{ij} \\
& - \frac{1}{2}B_{ab0i}\,\Phi(R)^{abi} - \frac{1}{2}\omega_{ab0}\,\Phi(\nabla B)^{ab} - B_{0i}\,\Phi(F)^i - A_0\,\Phi(\nabla B) \\
& - e_{a0}\,\Phi(\mathcal{G})^a - \beta_{a0i}\,\Phi(\nabla e)^{ai} - \frac{1}{2}\gamma_{A0ij}\,\Phi(\nabla\phi)^{Aij} \Big]\,,
\end{aligned} \tag{31}
$$

where

$$
\begin{array}{llllll}
\Phi(B)^{ab}{}_i & = & P(B)^{ab}{}_{0i}\,, & \Phi(\gamma)^A{}_{ij} & = & P(\gamma)^A{}_{0ij}\,, \\
\Phi(\omega)^{ab} & = & P(\omega)^{ab}{}_0\,, & \Phi(F)^i & = & \mathcal{S}(F)^i - \partial_j P(B)^{ij}\,, \\
\Phi(B)_i & = & P(B)_{0i}\,, & \Phi(R)^{abi} & = & \mathcal{S}(R)^{abi} - \nabla_j P(B)^{ab\,ij}\,, \\
\Phi(A) & = & P(A)_0\,, & \Phi(\mathcal{G})^a & = & \mathcal{S}(\mathcal{G})^a + \nabla_i P(e)^{a\,i} - \frac{1}{4}\,\beta_{b\,ij}\,P(B)^{ab\,ij}\,, \\
\Phi(e)^a & = & P(e)^a{}_0\,, & \Phi(\nabla e)^{a\,i} & = & \mathcal{S}(\nabla e)^{a\,i} - \nabla_j P(\beta)^{a\,ij} + \frac{1}{2}\,e_{b\,j}\,P(B)^{ab\,ij}\,, \\
\Phi(\beta)^a{}_i & = & P(\beta)^a{}_{0i}\,, & \Phi(\nabla\phi)^{A\,ij} & = & \mathcal{S}(\nabla\phi)^{A\,ij} + \nabla_k P(\gamma)^{A\,ijk} - \triangleright_B{}^A\,\phi^B\,P(B)^{ij}\,, \\
\Phi(\nabla B) & = & \multicolumn{4}{l}{\mathcal{S}(\nabla B) + \partial_i P(A)^i + \frac{1}{3!}\,\gamma^A{}_{ijk}\,\triangleright_A{}^B\,P(\gamma)_B{}^{ijk} - \phi_A\,\triangleright_B{}^A\,P(\phi)^B\,,} \\
\Phi(\nabla B)^{ab} & = & \multicolumn{4}{l}{\mathcal{S}(\nabla B)^{ab} + \nabla_i P(\omega)^{abi} + B^{[a|}{}_{c\,ij}\,P(B)^{c|b]\,ij} - 2e^{[a|}{}_i\,P(e)^{|b]\,i} - \beta^{[a|}{}_{ij}\,P(\beta)^{|b]\,ij}\,,}
\end{array} \tag{32}
$$

are the first-class constraints, while

$$
\begin{array}{llll}
\chi(B)_{ab}{}^{jk} = P(B)_{ab}{}^{jk}\,, & \chi(B)^{jk} = P(B)^{jk}\,, & \chi(e)_a{}^i = P(e)_a{}^i\,, & \chi(\phi)_A = P(\phi)_A\,, \\
\chi(\omega)_{ab}{}^i = P(\omega)_{ab}{}^i\,, & \chi(A)^i = P(A)^i\,, & \chi(\beta)_a{}^{ij} = P(\beta)_a{}^{ij}\,, & \chi(\gamma)_A{}^{ijk} = P(\gamma)_A{}^{ijk}\,,
\end{array} \tag{33}
$$

are the second-class constraints.

The PB algebra of the first-class constraints is given by:

$$
\begin{array}{lll}
\{\,\Phi(\mathcal{G})^a(x)\,,\,\Phi(\nabla e)_b{}^i(y)\,\} & = & -\,\Phi(R)^a{}_b{}^i(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\mathcal{G})^a(x)\,,\,\Phi(\nabla B)_{bc}(y)\,\} & = & 2\delta^a{}_{[b|}\,\Phi(\mathcal{G})_{|c]}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla e)^a{}_i(x)\,,\,\Phi(\nabla B)_{bc}(y)\,\} & = & 2\delta^a{}_{[b|}\Phi(\nabla e)_{|c]i}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(R)^{abi}(x)\,,\,\Phi(\nabla B)_{cd}(y)\,\} & = & -4\delta^{[a|}{}_{[c}\,\Phi(R)^{|b]}{}_{d]}{}^i(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\Phi(\nabla B)_{cd}(y)\,\} & = & -4\delta^{[a|}{}_{[c|}\,\Phi(\nabla B)^{|b]}{}_{|d]}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla B)(x)\,,\,\Phi(\nabla\phi)_A{}^{ij}(y)\,\} & = & -2\,\triangleright^B{}_A\,\Phi(\nabla\phi)_B{}^{ij}(x)\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{array} \tag{34}
$$

The PB algebra between the first and the second-class constraints is given by:

$$
\begin{aligned}
\{\,\Phi(R)^{abi}(x)\,,\,\chi(\omega)_{cd}{}^{j}(y)\,\} &= 4\,\delta^{[a|}{}_{[c|}\,\chi(B)^{|b]}{}_{|d]}{}^{ij}(x)\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\mathcal{G})^{a}(x)\,,\,\chi(\omega)_{cd}{}^{i}(y)\,\} &= 2\,\delta^{a}{}_{[c|}\,\chi(e)_{|d]}{}^{i}(x)\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\mathcal{G})^{a}(x)\,,\,\chi(\beta)_{c}{}^{jk}(y)\,\} &= -\frac{1}{2}\,\chi(B)^{a}{}_{c}{}^{jk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla e)^{ai}(x)\,,\,\chi(\omega)_{cd}{}^{j}(y)\,\} &= -2\,\delta^{a}{}_{[c|}\,\chi(\beta)_{|d]}{}^{ij}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla e)^{ai}(x)\,,\,\chi(e)_{b}{}^{j}(y)\,\} &= \frac{1}{2}\,\chi(B)^{a}{}_{b}{}^{ij}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\chi(\omega)_{cd}{}^{i}(y)\,\} &= 4\,\delta^{[a|}{}_{[c|}\,\chi(\omega)_{|d]}{}^{|b]i}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)(x)\,,\,\chi(A)^{i}(y)\,\} &= 2\,\chi(A)^{i}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\chi(\beta)_{c}{}^{jk}(y)\,\} &= -2\delta^{[a|}{}_{c}\,\chi(\beta)^{|b]jk}\,\delta^{(3)}(x-y)\,,\\
\{\,\Phi(\nabla B)(x)\,,\,\chi(\gamma)_{A}{}^{ijk}(y)\,\} &= \rhd_{A}{}^{B}\,\chi(\gamma)_{B}{}^{ijk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\chi(B)_{cd}{}^{jk}(y)\,\} &= 4\,\delta^{[a|}{}_{[c}\,\chi(B)_{d]}{}^{|b]jk}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\chi(e)_{a}{}^{i}(y)\,\} &= -2\delta^{[a|}{}_{c}\,\chi(e)^{|b]i}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)(x)\,,\,\chi(\phi)_{A}(y)\,\} &= -\rhd^{B}{}_{A}\,\chi(\phi)_{B}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla\phi)^{Aij}(x)\,,\,\chi(A)^{k}(y)\,\} &= -\rhd_{B}{}^{A}\,\chi(\gamma)^{Bijk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla\phi)^{Aij}(x)\,,\,\chi(\phi)_{B}(y)\,\} &= -\rhd_{B}{}^{A}\,\chi(B)^{ij}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{35}
$$

The PB algebra between the second-class constraints has already been calculated, and is given in Equations (20).

4. The Bianchi Identities

In order to calculate the number of degrees of freedom in the theory, one needs to make use of the *Bianchi identities* (BI), as well as additional, *generalized Bianchi identities* (GBI) that are an analogue of the ordinary BI for the additional fields present in the theory.

One uses BI associated with the 1-form fields ω^{ab} and e^{a}, as well as the GBI for the 1-form A. Namely, the corresponding 2-form curvatures

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^{a}{}_{c}\wedge\omega^{cb}\,, \qquad T^{a} = \mathrm{d}e^{a} + \omega^{a}{}_{b}\wedge e^{b}\,, \qquad F = \mathrm{d}A\,, \tag{36}$$

satisfy the following identities:

$$\epsilon^{\lambda\mu\nu\rho}\,\nabla_{\mu}R^{ab}{}_{\nu\rho} = 0\,, \tag{37}$$

$$\epsilon^{\lambda\mu\nu\rho}\left(\nabla_{\mu}T^{a}{}_{\nu\rho} - R^{ab}{}_{\mu\nu}\,e_{b\rho}\right) = 0\,, \tag{38}$$

$$\epsilon^{\lambda\mu\nu\rho}\,\nabla_{\mu}F_{\nu\rho} = 0\,. \tag{39}$$

Choosing the free index to be time coordinate $\lambda = 0$, these indentities, as the time-independent parts of the Bianchi identities, become the off-shell restrictions in the sense of the Hamiltonian analysis. On the other hand, choosing the free index to be a spatial coordinate, one obtains time-dependent pieces of the Bianchi identities, which do not enforce any restrictions, but can instead be derived as a consequence of the Hamiltonian equations of motion.

There are also GBI associated with the 2-form fields B^{ab}, B and β^a. The corresponding 3-form curvatures are given by

$$S^{ab} = \mathrm{d}B^{ab} + 2\omega^{[a|}{}_c \wedge B^{c\,|b]}\,, \qquad P = \mathrm{d}B\,, \qquad G^a = \mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b\,. \tag{40}$$

Differentiating these expressions, one obtains the following GBI:

$$\epsilon^{\lambda\mu\nu\rho}\left(\frac{1}{3}\nabla_\lambda\, S^{ab}{}_{\mu\nu\rho} - R^{[a|\,c}{}_{\lambda\mu}\, B_c{}^{|b]}{}_{\nu\rho}\right) = 0\,, \tag{41}$$

$$\epsilon^{\lambda\mu\nu\rho}\partial_\lambda\, P_{\mu\nu\rho} = 0\,, \tag{42}$$

$$\epsilon^{\lambda\mu\nu\rho}\left(\frac{2}{3}\nabla_\lambda\, G^a{}_{\mu\nu\rho} - R^{ab}{}_{\lambda\mu}\,\beta_{b\,\nu\rho}\right) = 0\,. \tag{43}$$

However, in four-dimensional spacetime, these identities will be single-component equations, with no free spacetime indices, and therefore necessarily feature time derivatives of the fields. Thus, they do not impose any off-shell restictions on the canonical variables.

Finally, there is also GBI associated with the 0-form ϕ. The corresponding 1-form curvature is:

$$Q^A = \mathrm{d}\phi^A + \rhd_B{}^A\, A \wedge \phi^B\,, \tag{44}$$

so that the GBI associated with this curvature is:

$$\epsilon^{\lambda\mu\nu\rho}\left(\nabla_\nu Q^A{}_\rho - \frac{1}{2}\rhd_B{}^A\, F_{\nu\rho}\phi^B\right) = 0. \tag{45}$$

This GBI consists of 12 component equations, corresponding to six possible choices of the free antisymmetrized spacetime indices $\lambda\mu$, and the 2 possible choices of the free group index A. However, not all of these 12 identities are independent. This can be seen by taking the derivative of the Equation (45) and obtaining eight identities of the form

$$\rhd_B{}^A\,\epsilon^{\lambda\mu\nu\rho}\,\partial_\mu\, F_{\nu\rho}\,\phi^B = 0\,, \tag{46}$$

which are automatically satisfied because of the GBI (39). One concludes there are only four independent identities (45). Now, fixing the value $\lambda = 0$, one obtains the time-independent components of both Equations (45) and (46),

$$\epsilon^{0ijk}\left(\nabla_j Q^A{}_k - \frac{1}{2}\rhd_B{}^A\, F_{jk}\phi^B\right) = 0\,, \tag{47}$$

and

$$\rhd_B{}^A\,\epsilon^{0ijk}\,\partial_i\, F_{jk}\,\phi^B = 0\,. \tag{48}$$

Of these, there are six components in Equation (47), but, because of the two components of Equation (48), there are overall only four independent GBI relevant for the Hamiltonian analysis.

5. Number of Degrees of Freedom

Let us now show that the structure of the constraints implies that there are no local degrees of freedom (DoF) in a $3BF$ theory. In the general case, if there are N initial fields in the theory and there are F independent first-class constraints per space point and S independent second-class constraints per space point, then the number of local DoF, i.e., the number of independent field components, is given by

$$n = N - F - \frac{S}{2}\,. \tag{49}$$

Equation (49) is a consequence of the fact that S second-class constraints are equivalent to vanishing of $S/2$ canonical coordinates and $S/2$ of their momenta. The F first-class constraints are equivalent to vanishing of F canonical coordinates, and since the first-class constraints generate the gauge symmetries, we can impose F gauge-fixing conditions for the corresponding F canonical momenta. Consequently, there are $2N - 2F - S$ independent canonical coordinates and momenta and therefore $2n = 2N - 2F - S$, giving rise to Equation (49).

In our case, N can be determined from the Table 1, giving rise to a total of $N = 120$ canonical coordinates. Similarly, the number of independent components for the second class constraints is determined by the Table 2, so that $S = 70$.

Table 1. The number of components for all fields present in the theory.

$\omega^{ab}{}_{\mu}$	A_{μ}	$\beta^{a}{}_{\mu\nu}$	$\gamma^{A}{}_{\mu\nu\rho}$	$B^{ab}{}_{\mu\nu}$	$B_{\mu\nu}$	$e^{a}{}_{\mu}$	ϕ^{A}
24	4	24	8	36	6	16	2

Table 2. The number of components for the second class constraints present in the theory.

$\chi(B)_{ab}{}^{jk}$	$\chi(B)^{jk}$	$\chi(e)_{a}{}^{i}$	$\chi(\phi)_{A}$	$\chi(\omega)_{ab}{}^{i}$	$\chi(A)^{i}$	$\chi(\beta)_{a}{}^{ij}$	$\chi(\gamma)_{A}{}^{ijk}$
18	3	12	2	18	3	12	2

The first-class constraints are not all independent because of BI and GBI. To see that, take the derivative of $\Phi(R)^{abi}$ to obtain

$$\nabla_i \Phi(R)^{abi} = \varepsilon^{0ijk} \nabla_i R^{ab}{}_{jk} + \frac{1}{2} R^{c[a|}{}_{ij} P(B)_c{}^{|b]ij} . \tag{50}$$

The first term on the right-hand side is zero off-shell because $\epsilon^{ijk} \nabla_i R^{ab}{}_{jk} = 0$, which is a $\lambda = 0$ component of the BI (37). The second term on the right-hand side is also zero off-shell, since it is a product of two constraints,

$$R^{c[a|}{}_{ij} P(B)_c{}^{|b]ij} \equiv \frac{1}{2} \epsilon_{0ijk} \mathcal{S}(R)^{c[a|k} P(B)_c{}^{|b]ij} = 0 . \tag{51}$$

Therefore, we have the off-shell identity

$$\nabla_i \Phi(R)^{abi} = 0 , \tag{52}$$

which means that six components of $\Phi(R)^{abi}$ are not independent of the others. In an analogous fashion, taking the derivative of $\Phi(F)^{i}$, one obtains

$$\partial_i \Phi(F)^i = \varepsilon^{0ijk} \partial_i F_{jk} + \frac{1}{2} F_{ij} P(B)^{ij} . \tag{53}$$

The first term on the right-hand side is zero off-shell because $\epsilon^{ijk} \partial_i F_{jk} = 0$, which is a $\lambda = 0$ component of the GBI (37). The second term on the right-hand side is also zero off-shell, since it is a product of two constraints,

$$F_{ij} P(B)^{ij} \equiv \frac{1}{2} \epsilon_{0ijk} \mathcal{S}(F)^k P(B)^{ij} = 0 . \tag{54}$$

Therefore, we have the off-shell identity

$$\partial_i \Phi(F)^i = 0 , \tag{55}$$

which means that one component of $\Phi(F)^i$ is not independent of the others. Similarly, one can demonstrate that

$$\nabla_i \Phi(\nabla e)_a{}^i - \frac{1}{2}\Phi(R)_{ab}{}^i e^b{}_i + \frac{1}{4}\epsilon^{0ijk}\mathcal{S}(R)_{abk}\, P(\beta)^b{}_{ij} = \frac{1}{2}\epsilon^{0ijk}\left(\nabla_i T_{ajk} - R_{ab\,ij}\, e^b{}_k\right) . \tag{56}$$

The right-hand side of the Equation (56) is the $\lambda = 0$ component of the BI (38), so that Equation (56) gives the relation:

$$\nabla_i \Phi(\nabla e)_a{}^i - \frac{1}{2}\Phi(R)_{ab}{}^i e^b{}_i = 0\,, \tag{57}$$

where we have omitted the term that is the product of two constraints. This relation means that four components of the constraints $\Phi(\nabla e)_a{}^i$ and $\Phi(R)_{ab}{}^i$ can be expressed in terms of the rest. Finally, one can also demonstrate that

$$\begin{aligned}\nabla_i \Phi(\nabla\phi)_A{}^{ij} &- \frac{1}{2}\epsilon_{0ikl} \rhd_A \mathcal{S}(F)^l \chi(\gamma)_B{}^{ijk} + \rhd^B{}_A \phi_B \Phi(F)^j \\ &+ \frac{1}{2}\epsilon_{0ilm} \rhd^B{}_A P(B)^{ij}\mathcal{S}(\nabla\phi)_B{}^{lm} = \epsilon^{0ijk}\left(\nabla_i Q_{Ak} + \frac{1}{2}\rhd^B{}_A F_{ik}\phi_B\right) ,\end{aligned} \tag{58}$$

which gives

$$\nabla_i \Phi(\nabla\phi)_A{}^{ij} + \frac{1}{2}\rhd^B{}_A \phi_B \Phi(F)^j = 0\,, \tag{59}$$

for $\lambda = 0$ component of the GBI (45), where we have again used that the product of two contraints is zero off-shell. This relation suggests that six components of two first-class constraints, $\Phi(\nabla\phi)_A{}^{ij}$ and $\Phi(F)^j$, are not independent of the others. However, in the previous section, we have discussed that only four of these six identities are mutually independent, which means that we have only four independent identities (59). A rigorous proof of this statement entails the evaluation of the corresponding Wronskian, and is left for future work.

Taking into account all of the above indentites (52), (55), (57), and (59), we can finally evaluate the total number of independent first-class constraints. From the Table 3, one can see that the total number of components of the first-class constraints is given by $F^* = 100$. However, the number of independent components of the first-class constraints is $F = 85$, obtained by subtracting the six relations (52), one relation (55), four relations (57) and four relations (59).

Table 3. The number of components for the first class constraints present in the theory. The identities (52), (55), (57), and (59) reduce the number of components which are independent. This reduction is explicitly denoted in the table.

$\Phi(B)_{ab}{}^i$	$\Phi(B)^i$	$\Phi(e)_a$	$\Phi(\omega)_{ab}$	$\Phi(A)$	$\Phi(\beta)_a{}^i$	$\Phi(\gamma)_A{}^{ij}$	$\Phi(R)_{ab}{}^i$	$\Phi(F)^i$	$\Phi(\mathcal{G})_a$	$\Phi(\nabla e)_a{}^i$	$\Phi(\nabla B)_{ab}$	$\Phi(\nabla B)$	$\Phi(\nabla\phi)_A{}^{ij}$
18	3	4	6	1	12	6	$18-6$	$3-1$	4	$12-4$	6	1	$6-4$

Therefore, substituting all the obtained results into Equation (49), one gets

$$n = 120 - 85 - \frac{70}{2} = 0, \tag{60}$$

which means that there are no propagating DoF in a $3BF$ theory described by the action (10).

6. Generator of the Gauge Symmetry

Based on the results of the Hamiltonian analysis of the action (10), it can also be interesting to calculate the generator of the complete gauge symmetry of the action. The gauge generator of the theory is obtained by using the Castellani's procedure (see Chapter V in [13] for details of the procedure), and one gets the following result (see Appendix B for details of the calculation):

$$
\begin{aligned}
G \;=\; & \int_{\Sigma_3} \mathrm{d}^3\vec{x}\bigg(\frac{1}{2}(\nabla_0\epsilon^{ab}{}_i)\Phi(B)_{ab}{}^i - \frac{1}{2}\epsilon^{ab}{}_i\Phi(R)_{ab}{}^i + \frac{1}{2}(\nabla_0\epsilon^{ab})\Phi(\omega)_{ab} - \frac{1}{2}\epsilon^{ab}\Phi(\nabla B)_{ab} \\
& + (\partial_0\epsilon_i)\Phi(B)^i - \epsilon_i\Phi(F)^i + (\partial_0\epsilon)\Phi(A) - \epsilon\Phi(\nabla B) \\
& + (\nabla_0\epsilon^a)\Phi(e)_a - \epsilon^a\Phi(\mathcal{G})_a + (\nabla_0\epsilon^a{}_i)\Phi(\beta)_a{}^i - \epsilon^a{}_i\Phi(\nabla e)_a{}^i \\
& + \frac{1}{2}(\nabla_0\epsilon^A{}_{ij})\Phi(\gamma)_A{}^{ij} - \frac{1}{2}\epsilon^A{}_{ij}\Phi(\nabla\phi)_A{}^{ij} \\
& + \epsilon^{ab}\left(\beta_{[a|0i}P(\beta)_{|b]}{}^i + e_{[a|0}P(e)_{|b]} + B_{[a|c0i}P(B)^c{}_{|b]}{}^i\right) - \epsilon\,\gamma_{A0ij} \rhd_B{}^A P(\gamma)^{Bij} \\
& + \epsilon^a\beta_{b0i}P(B)^{abi} + \epsilon^a{}_i\, e_{b0}P(B)_a{}^{bi}\bigg)\,.
\end{aligned}
\tag{61}
$$

Here, $\epsilon^{ab}{}_i$, ϵ^{ab}, ϵ_i, ϵ, ϵ^a, $\epsilon^a{}_i$ and $\epsilon^A{}_{ij}$ are the independent parameters of the gauge transformations.

Furthermore, one can employ the gauge generator to calculate the form-variations for all canonical coordinates and their corresponding momenta, by computing the Poisson bracket of the chosen variable $A(t,\vec{x})$ and the generator (61):

$$
\delta_0 A(t,\vec{x}) = \{A(t,\vec{x})\,,G\}\,. \tag{62}
$$

The results are given as follows:

$$
\begin{aligned}
\delta_0\omega^{ab}{}_0 &= \nabla_0\epsilon^{ab}\,, & \delta_0\pi(\omega)_{ab}{}^0 &= -2\epsilon_{[a|}{}^c{}_i\pi(B)_{c|b]}{}^{0i} - 2\epsilon_{[a|}{}^c\pi(\omega)_{c|b]}{}^0\,, \\
& & & \quad +2\epsilon_{[a|}\pi(e)_{|b]}{}^0 + 2\epsilon_{[a|i}\pi(\beta)_{|b]}{}^{0i}\,, \\
\delta_0\omega^{ab}{}_i &= \nabla_i\epsilon^{ab}\,, & \delta_0\pi(\omega)_{ab}{}^i &= -2\epsilon_{[a|}{}^c{}_j\,\pi(B)_{c|b]}{}^{ij} - 2\epsilon_{[a|}{}^c{}_i\,\pi(\omega)_{|b]c}{}^i \\
& & & \quad +2\epsilon_{[a|}\,\pi(e)_{|b]i} + 2\epsilon_{[a|j}\pi(\beta)_{|b]}{}^{ij} \\
& & & \quad +2\epsilon^{0ijk}\,\nabla_{[j|}\epsilon_{ab|k]} + \epsilon^{0ijk}\epsilon_{[a|}\beta_{|b]jk}\,, \\
\delta_0 B^{ab}{}_{0i} &= \nabla_0\epsilon^{ab}{}_i + \epsilon^{[a|}{}_i e^{|b]}{}_0 & \delta_0\pi(B)_{ab}{}^{0i} &= 2\epsilon_{[a|c}\,\pi(B)_{|b]}{}^{ci}\,, \\
& \quad +2\epsilon^{[a|c}B^{|b]}{}_{c0i} + \epsilon^{[a|}\beta^{|b]}{}_{0i}\,, & & \\
\delta_0 B^{ab}{}_{ij} &= 2\nabla_{[i|}\epsilon^{ab}{}_{|j]} + 2\epsilon^{[a|c}B^{|b]}{}_{cij} & \delta_0\pi(B)_{ab}{}^{ij} &= 2\epsilon_{[a|c}\,\pi(B)_{|b]}{}^{cij}\,, \\
& \quad +2\epsilon^{[a|}{}_{[i}e^{|b]}{}_{j]} + \epsilon^{[a|}\beta^{|b]}{}_{ij}\,, & & \\
\delta_0 A_0 &= \partial_0\epsilon\,, & \delta_0\pi(A)^0 &= -\tfrac{1}{2}\epsilon^A{}_{ij} \rhd^B{}_A\,\pi(\gamma)_B{}^{0ij}\,, \\
\delta_0 A_i &= \partial_i\epsilon\,, & \delta_0\pi(A)^i &= \epsilon^{0ijk}\partial_j\epsilon_k - \tfrac{1}{2}\epsilon^A{}_{jk} \rhd^B{}_A\,\pi(\gamma)_B{}^{ijk}\,, \\
\delta_0 B_{0i} &= \partial_0\epsilon_i\,, & \delta_0\pi(B)^{0i} &= 0\,, \\
\delta_0 B_{ij} &= 2\,\partial_{[i|}\epsilon_{|j]} + \epsilon^A{}_{ij} \rhd^B{}_A\,\phi_B\,, & \delta_0\pi(B)^{ij} &= -\epsilon^{0ijk}\partial_k\epsilon\,, \\
\delta_0\beta^a{}_{0i} &= \nabla_0\epsilon^a{}_i - \epsilon^{ab}\beta_{b0i}\,, & \delta_0\pi(\beta)_a{}^{0i} &= -\epsilon_{ab}\pi(\beta)^{b0i} + \tfrac{1}{2}\epsilon^b\pi(B)_{ab}{}^{0i}\,, \\
\delta_0\beta^a{}_{ij} &= 2\nabla_{[i|}\epsilon^a{}_{|j]} - \epsilon^{ab}\,\beta_{bij}\,, & \delta_0\pi(\beta)_a{}^{ij} &= -\epsilon_{ab}\,\pi(\beta)^{bij} + \tfrac{1}{2}\epsilon^b\,\pi(B)_{ab}{}^{ij} \\
& & & \quad -\epsilon^{0ijk}\,\nabla_k\epsilon^a\,, \\
\delta_0 e^a{}_0 &= \nabla_0\epsilon^a - \epsilon^{ab}\,e_{b0}\,, & \delta_0\pi(e)_a{}^0 &= -\epsilon_{ab}\,\pi(e)^{b0} + \tfrac{1}{2}\epsilon^b{}_i\,\pi(B)_{ab}{}^{0i}\,, \\
\delta_0 e^a{}_i &= \nabla_i\epsilon^a - \epsilon^{ab}\,e_{bi}\,, & \delta_0\pi(e)_a{}^i &= -\epsilon_{ab}\,\pi(e)^{bi} + \epsilon^{0ijk}\Big(\nabla_{[j|}\epsilon_{a|k]} + \epsilon_{ab}\beta^{bjk}\Big) \\
& & & \quad +\tfrac{1}{2}\epsilon^b{}_j\,\pi(B)_{ab}{}^{ij}\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta_0\gamma^A{}_{0ij} &= \nabla_0\epsilon^A{}_{ij} - \epsilon\,\gamma^B{}_{0ij} \rhd^A{}_B\,, & \delta_0\pi(\gamma)_A{}^{0ij} &= \epsilon \rhd^B{}_A\,\pi(\gamma)_B{}^{0ij}\,,\\
\delta_0\gamma^A{}_{ijk} &= -\,\epsilon\,\gamma^B{}_{ijk} \rhd_B{}^A + \nabla_i\epsilon^A{}_{jk} & \delta_0\pi(\gamma)_A{}^{ijk} &= \epsilon \rhd_A{}^B\left(\pi(\gamma)_B{}^{ijk} + \epsilon^{0ijk}\,\phi_B\right),\\
&\quad -\nabla_j\epsilon^A{}_{ik} + \nabla_k\epsilon^A{}_{ij}\,, & & \qquad (63)\\
\delta_0\phi^A &= \epsilon\,\phi^B \rhd^A{}_B\,, & \delta_0\pi(\phi)_A &= -\epsilon \rhd^B{}_A\,\pi(\phi)_B + \frac{1}{3!}\,\epsilon\,\epsilon^{0ijk} \rhd^B{}_A\,\gamma_{Bijk}\\
& & &\quad -\frac{1}{2}\rhd_{A\,B}\,\epsilon^B{}_{ij}\,\pi(B)^{ij} - \frac{1}{2}\epsilon^{0ijk}\,\nabla_i\epsilon^A{}_{jk}\,,
\end{aligned}
$$

These transformations are an extension of the form-variations in the case of the Poincaré 2-group obtained in [17].

7. Conclusions

Let us summarize the results of the paper. In Section 2, we have demonstated in detail how to use the idea of a categorical ladder to introduce the 3-group structure corresponding to the theory of scalar electrodynamics coupled to Einstein–Cartan gravity. We have introduced the topological $3BF$ action corresponding to this choice of a 3-group, as well as the constrained $3BF$ action which gives rise to the standard equations of motion for the scalar electrodynamics. In order to perform the canonical quantization of this theory, the complete Hamiltonian analysis of the full theory with constraints has to be performed, but the important step towards this goal is the Hamiltonian analysis of the topological $3BF$ action. This has been done in Section 3. Here, the first-class and second-class constraints of the theory, as well as their Poisson brackets, have been obtained. In Section 4, we have discussed the Bianchi identities and also the generalized Bianchi identities, since they enforce restrictions in the sense of Hamiltonian analysis, and reduce the number of independent first-class constraints present in the theory. With this background material in hand, in Section 5, the counting of the dynamical degrees of freedom present in the theory has been performed and it was established that the considered $3BF$ action is a topological theory, i.e., the diffeomorphism invariant theory without any propagating degrees of freedom. In Section 6, we have constructed the generator of the gauge symmetries for the theory, and we found the form-variations for all the variables and their canonical momenta.

The results obtained in this paper represent the straightforward generalization of Hamiltonian analysis done in [15] for the Poincaré 2-group, and a first example of the Hamiltonian analysis of a $3BF$ action. The fact that the theory was found to be topological is nontrivial, since it relies on the existence of the generalized Bianchi identities, which have been identified for the first time. In addition to that, it was demonstrated that the algebra of constraint closes, which is an important consistency check for the theory. There is another very interesting aspect of the constraint algebra. Namely, one can recognize, looking at the structure of Equations (34) that the subalgebra generated by the first-class constraint $\Phi(\nabla\phi)_A{}^{ij}$ is in fact an *ideal* of the constraint algebra because the Poisson bracket between this constraint and all other constraints is again proportional to that constraint. It is curious that precisely the constraint $\Phi(\nabla\phi)_A{}^{ij}$ is the only one related to the Lie group L from the 3-group, according to its index structure, and also that the structure constant of the ideal is determined by the action $\rhd$ of the group G on L. Let us also note that the action $\rhd$ appears as well in the structure constants of the algebra between the first-class and second-class constraints.

The results of this work open several avenues for future research. From the point of view of mathematics, the relationship between the algebraic structures mentioned above should be understood in more detail. More generally, one should understand the correspondence between the gauge group generated by the generator (61) and the 3-group structure used to define the theory. This is not viable in the special case of the 3-group discussed in this work, but instead needs to be done in the case of a generic 3-group, where homomorphisms δ and ∂ and the Peiffer lifting $\{\_,\_\}$ are nontrivial. From the point of view of physics, the obtained results represent the fundamental building blocks for the construction of the quantum theory of scalar electrodynamics coupled to gravity, as well as a convenient model to discuss before proceeding to the Hamiltonian analysis and canonical quantization of the full Standard Model coupled to gravity, formulated as a $3BF$ action with suitable constraints [8]. Both the Hamiltonian analysis of constrained 3BF models and the corresponding canonical quantization programme need to be further developed in order to achieve these goals. Our work is a first step in this direction.

Finally, let us note in the end that the above list of topics for future research is by no means complete, and there are potentially many other interesting topics that can be studied in this context.

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Abbreviations

The following abbreviations are used in this manuscript:

LQG	Loop Quantum Gravity
BI	Bianchi Identities
GBI	Generalized Bianchi Identities
DoF	Degrees of Freedom
PB	Poisson Bracket

Appendix A. The Equations of Motion for the Scalar Electrodynamics

The action of scalar electrodynamics coupled to Einstein–Cartan gravity is given in the form (12):

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B \wedge F + e_a \wedge \nabla \beta^a + \phi_A \nabla \gamma^A \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda^A \wedge \left(\gamma_A - \frac{1}{2} H_{abcA} e^a \wedge e^b \wedge e^c \right) + \Lambda^{abA} \wedge \left(H_{abcA} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \nabla \phi_A \wedge e_a \wedge e_b \right) \\ & + \lambda \wedge \left(B - \frac{12}{q} M_{ab} e^a \wedge e^b \right) + \zeta^{ab} \left(M_{ab} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b \right) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi_A \phi^A \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,. \end{aligned} \tag{A1}$$

Varying the total action (12) with respect to the variables B_{ab}, B, ω_{ab}, β_a, λ_{ab}, Λ^{abA}, γ^A, λ^A, H_{abcA}, ζ^{ab}, M_{ab}, λ, A, ϕ^A and e^a, one obtains the equations of motion:

$$R^{ab} - \lambda^{ab} = 0 \,, \tag{A2}$$

$$F + \lambda = 0 \,, \tag{A3}$$

$$\nabla B^{ab} - e^{[a|} \wedge \beta^{|b]} = 0 \,, \tag{A4}$$

$$\nabla e^a = 0 \,, \tag{A5}$$

$$B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d = 0 \,, \tag{A6}$$

$$H_{abcA}\varepsilon^{cdef}e_d \wedge e_e \wedge e_f - \nabla\phi_A \wedge e_a \wedge e_b = 0\,, \tag{A7}$$

$$\nabla\phi_A - \lambda_A = 0\,, \tag{A8}$$

$$\gamma_A - \frac{1}{2}H_{abcA}e^a \wedge e^b \wedge e^c = 0\,, \tag{A9}$$

$$-\frac{1}{2}\lambda^A \wedge e^a \wedge e^b \wedge e^c + \varepsilon^{cdef}\Lambda^{abA} \wedge e_d \wedge e_e \wedge e_f = 0\,, \tag{A10}$$

$$M_{ab}\varepsilon_{cdef}e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b = 0\,, \tag{A11}$$

$$-\frac{12}{q}\lambda \wedge e^a \wedge e^b + \zeta^{ab}\varepsilon_{cdef}e^c \wedge e^d \wedge e^e \wedge e^f = 0\,, \tag{A12}$$

$$B - \frac{12}{g}M_{ab}e^a \wedge e^b = 0\,, \tag{A13}$$

$$-\,\mathrm{d}B + \mathrm{d}(\zeta^{ab}e_a \wedge e_b) - \phi_A \rhd_B{}^A\gamma^B - \Lambda^{abA} \rhd^B{}_A\,\phi_B \wedge e_a \wedge e_b = 0\,, \tag{A14}$$

$$\nabla\gamma_A - \nabla(\Lambda^{ab}{}_A \wedge e_a \wedge e_b) - \frac{1}{4!}m^2\,\phi_A\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d = 0\,, \tag{A15}$$

$$\begin{aligned}&\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{3}{2}H_{abcA}\lambda^A \wedge e^b \wedge e^c + 3H^{defA}\varepsilon_{abcd}\Lambda_{efA} \wedge e^b \wedge e^c\\ &-2\Lambda_{abA} \wedge \nabla\phi^A \wedge e^b - 2\frac{1}{4!}m^2\phi_A\;\phi^A\varepsilon_{abcd}e^b \wedge e^c \wedge e^d\\ &-\frac{24}{q}M_{ab}\lambda \wedge e^b + 4\zeta^{ef}M_{ef}\varepsilon_{abcd}e^b \wedge e^c \wedge e^d - 2\zeta_{ab}F \wedge e^b = 0\,.\end{aligned} \tag{A16}$$

The dynamical degrees of freedom are the tetrad fields e^a, the scalar field ϕ^A, and the electromagnetic potential A, while the remaining variables are algebraically determined in terms of them. Specifically, Equations (A2)–(A13) give

$$\begin{gathered}\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,, \qquad \gamma^A{}_{\mu\nu\rho} = -\frac{1}{2e}\varepsilon_{\mu\nu\rho\sigma}\,\nabla^\sigma\phi^A\,,\\ \Lambda^{abA}{}_\mu = \frac{1}{12e}g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\nabla_\nu\phi^A\,e^a{}_\rho e^b{}_\sigma\,, \qquad \beta^a{}_{\mu\nu} = 0\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,,\\ H^{abcA} = \frac{1}{6e}\varepsilon^{\mu\nu\rho\sigma}\,\nabla_\mu\phi^A\,e^a{}_\nu e^b{}_\rho e^c{}_\sigma\,, \qquad \lambda^A{}_\mu = \nabla_\mu\phi^A\,,\\ \lambda_{\mu\nu} = F_{\mu\nu}\,, \quad B_{\mu\nu} = -\frac{1}{2eq}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,,\\ M^{ab} = -\frac{1}{4e}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,e^a{}_\rho e^b{}_\sigma\,, \quad \zeta^{ab} = \frac{1}{4eq}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,e^a{}_\rho e^b{}_\sigma\,.\end{gathered} \tag{A17}$$

Note that from the Equations (A4)–(A6) it follows that $\beta^a = 0$, as in the pure gravity case. The equation of motion (A15) reduces to the covariant Klein–Gordon equation for the scalar field coupled to the electromagnetic potential A,

$$\left(\nabla_\mu\nabla^\mu - m^2\right)\phi_A = 0\,. \tag{A18}$$

From Equation (A14), we obtain the differential equation of motion for the field A:

$$\nabla_\mu F^{\mu\nu} = j^\nu\,, \qquad j^\mu \equiv \frac{1}{2}\left(\nabla^\nu\phi^A \rhd^B{}_A\phi_B - \phi_A \rhd_B{}^A\nabla^\nu\phi^B\right) = iq\left(\nabla\phi^*\,\phi - \phi^*\nabla\phi\right). \tag{A19}$$

Finally, the equation of motion (A16) for e^a becomes:

$$
\begin{aligned}
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2 \, T^{\mu\nu} \,, \\
T^{\mu\nu} \equiv \nabla^\mu \phi_A \, \nabla^\nu \phi^A - \frac{1}{2} g^{\mu\nu} \left(\nabla_\rho \phi_A \, \nabla^\rho \phi^A + m^2 \phi_A \, \phi^A \right) - \frac{1}{4q} \left(F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} + 4 F^{\mu\rho} F_\rho{}^\nu \right) .
\end{aligned}
\tag{A20}
$$

The system of Equations (A2)–(A16) is equivalent to the system of Equations (A17)–(A20).

Appendix B. The Calculation of the Gauge Generator

The gauge generator of the theory is obtained by the standard Castellani procedure (see [13] for an introduction). One starts from the generic form for the generator,

$$
\begin{aligned}
G \;=\; & \int_{\Sigma_3} \partial^3 \vec{x} \Big(\frac{1}{2} (\partial_0 \epsilon^{ab}{}_i) G_{1ab}{}^i + \frac{1}{2} \epsilon^{ab}{}_i G_{0ab}{}^i + \frac{1}{2} (\partial_0 \epsilon^{ab}) G_{1ab} + \frac{1}{2} \epsilon^{ab} G_{0ab} \\
& + (\partial_0 \epsilon_i) G_1{}^i + \epsilon_i G_0{}^i + (\partial_0 \epsilon) G_1 + \epsilon G_0 \\
& + (\partial_0 \epsilon^a) G_{1a} + \epsilon^a G_{0a} + (\partial_0 \epsilon^a{}_i) G_{1a}{}^i + \epsilon^a{}_i G_{0a}{}^i \\
& + \frac{1}{2} (\partial_0 \epsilon^A{}_{ij}) G_{1A}{}^{ij} + \frac{1}{2} \epsilon^A{}_{ij} G_{0A}{}^{ij} \Big) \,,
\end{aligned}
\tag{A21}
$$

where the generators G_0 and G_1 are obtained by the standard prescription [13]:

$$
\begin{aligned}
G_1 &= C_{PFC} \,, \\
G_0 + \{ \, G_1 \,, \, H_T \, \} &= C_{PFC} \,, \\
\{ \, G_0 \,, \, H_T \, \} &= C_{PFC} \,,
\end{aligned}
\tag{A22}
$$

where C_{PFC} is a primary first-class constraint. For example, one choses $G_{1ab}{}^i = \Phi(B)_{ab}{}^i$. From the conditions

$$
\begin{aligned}
G_{0ab}{}^i + \{ \, \Phi(B)_{ab}{}^i \,, \, H_T \, \} = G_{0ab}{}^i + \Phi(R)_{ab}{}^i = C_{PFC} \,, \\
\{ \, G_{0ab}{}^i \,, \, H_T \, \} = C_{PFC}{}^* = \{ \, C_{PFC} - \Phi(R)_{ab}{}^i \,, \, H_T \, \} \,,
\end{aligned}
\tag{A23}
$$

we solve for $G_{0ab}{}^i$ by determining C_{PFC} from the second equation. Evaluating one PB, one can reexpress the second equation in the form:

$$
\{ \, C_{PFC} \,, \, H_T \, \} = C_{PFC}{}^* + 2\omega_{[a|}{}^d{}_0 \Phi(R)_{|b]d}{}^i = \{ \, 2\omega_{[a|}{}^d{}_0 P(B)_{|b]d}{}^i \,, \, H_T \, \} \,. \tag{A24}
$$

From the second equality, we recognize that

$$
C_{PFC} = 2\omega_{[a|}{}^d{}_0 P(B)_{|b]d}{}^i \,, \tag{A25}
$$

which can then be substituted into the first condition above, giving

$$
G_{0ab}{}^i = 2\omega_{[a|}{}^d{}_0 \Phi(B)_{|b]d}{}^i - \Phi(R)_{ab}{}^i \,. \tag{A26}
$$

One thus obtains

$$
\frac{1}{2} (\partial_0 \epsilon^{ab}{}_i)(G_1)_{ab}{}^i + \frac{1}{2} \epsilon^{ab}{}_i G_{0ab}{}^i = \frac{1}{2} \nabla_0 \epsilon^{ab}{}_i \Phi(B)_{ab}{}^i - \frac{1}{2} \epsilon^{ab}{}_i \Phi(R)_{ab}{}^i \,.
$$

The other G_0 and G_1 terms are obtained in a similar way, and the generator (61) is derived.

Appendix C. Introduction to 3-Groups

The notion of a 3-*group* is usually introduced in the framework of higher category theory [6]. In category theory, every *group* can be understood as a *category* which has only one element, and morphisms which are all invertible. The group elements are then individual morphisms that map the category element to itself, while the group operation is the categorical composition of the morphisms. In such a case, the axioms of the category guarantee the validity of all axioms of a group. This kind of construction can be generalized to 2-*groups,* 3-*groups* and, in general, *n-groups*. Namely, a 2-group is by definition a 2-category which has only one element, and whose morphisms and 2-morhisms (i.e., morphisms between morphisms) are invertible. Similarly, a 3-group is by definition a 3-category which has only one element, while its morphisms, 2-morphisms, and 3-morphisms are invertible.

The above definition of a 3-group is very abstract, and while theoretically very important, in itself not very useful for practical calculations and applications in physics. Fortunately, there is a theorem of equivalence between 3-groups and the so-called 2-*crossed modules*, which are algebraic structures with more familiar properties [9,10]. For the applications in physics, attention focuses on the so-called *strict Lie* 3-*groups*, and their corresponding differential (Lie algebra) structure, which corresponds to the *differential Lie* 2-*crossed module*. Let us therefore give a brief overview of the latter.

A *differential Lie* 2-*crossed module* $(\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}, \triangleright, \{\_\,,\_\})$ is given by three Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, maps $\delta : \mathfrak{l} \to \mathfrak{h}$ and ∂: $\mathfrak{h} \to \mathfrak{g}$, together with a map called the Peiffer lifting,

$$\{\_\,,\_\} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{l}\,, \tag{A27}$$

and an action $\triangleright$ of the algebra $\mathfrak{g}$ on all three algebras.

Let us introduce the bases in the three algebras, $\tau_\alpha \in \mathfrak{g}$, $t_a \in \mathfrak{h}$ and $T_A \in \mathfrak{l}$, and structure constants in those bases, as follows:

$$[\tau_\alpha\,, \tau_\beta] = f_{\alpha\beta}{}^\gamma \tau_\gamma\,, \quad [t_a\,, t_b] = f_{ab}{}^c t_c\,, \quad [T_A\, T_B] = f_{AB}{}^C T_C\,. \tag{A28}$$

Now, the maps ∂ and δ can be written as

$$\partial(t_a) = \partial_a{}^\alpha\, \tau_\alpha\,, \qquad \delta(T_A) = \delta_A{}^a\, t_a\,, \tag{A29}$$

and the action of the algebra $\mathfrak{g}$ on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ as:

$$\tau_\alpha \triangleright \tau_\beta = \triangleright_{\alpha\beta}{}^\gamma\, \tau_\gamma\,, \quad \tau_\alpha \triangleright t_a = \triangleright_{\alpha a}{}^b\, t_b\,, \quad \tau_\alpha \triangleright T_A = \triangleright_{\alpha A}{}^B\, T_B\,. \tag{A30}$$

Finally, the Peiffer lifting can be encoded into coefficients $X_{ab}{}^A$ as:

$$\{t_a,\, t_b\} = X_{ab}{}^A\, T_A\,. \tag{A31}$$

A differential Lie 2-crossed module has the following properties (we write all equations in the abstract and their corresponding component forms, side by side):

1. The action of the algebra $\mathfrak{g}$ on itself is via the adjoint representation, i.e., $\forall g, g_1 \in \mathfrak{g}$:

$$g \triangleright g_1 = [g, g_1]\,, \qquad \triangleright_{\alpha\beta}{}^\gamma = f_{\alpha\beta}{}^\gamma\,. \tag{A32}$$

2. The action of the algebra $\mathfrak{g}$ on algebras $\mathfrak{h}$ and $\mathfrak{l}$ is $\mathfrak{g}$-equivariant, i.e., $\forall g \in \mathfrak{g}$, $h \in \mathfrak{h}$, $l \in \mathfrak{l}$:

$$\partial(g \triangleright h) = g \triangleright \partial(h)\,, \qquad \partial_a{}^\beta\, f_{\alpha\beta}{}^\gamma = \triangleright_{\alpha a}{}^b\, \partial_b{}^\gamma\,, \tag{A33}$$

$$\delta(g \triangleright l) = g \triangleright \delta(l)\,, \qquad \delta_A{}^a\, \triangleright_{\alpha a}{}^b = \triangleright_{\alpha A}{}^B\, \delta_B{}^b\,. \tag{A34}$$

3. The Peiffer lifting is a $\mathfrak{g}$-equivariant map, i.e., for every $g \in \mathfrak{g}$ and $h_1, h_2 \in \mathfrak{h}$:

$$g \rhd \{h_1, h_2\} = \{g \rhd h_1, h_2\} + \{h_1, g \rhd h_2\}\,, \qquad X_{ab}{}^B \rhd_{\alpha B}{}^A = \rhd_{\alpha a}{}^c X_{cb}{}^A + \rhd_{\alpha b}{}^c X_{ac}{}^A\,. \quad \text{(A35)}$$

4. For every h_1, $h_2 \in \mathfrak{h}$, the following identity holds:

$$\delta(\{h_1, h_2\}) = [h_1, h_2] - \partial(h_1) \rhd h_2\,, \qquad X_{ab}{}^A\, \delta_A{}^c = f_{ab}{}^c - \partial_a{}^\alpha \rhd_{\alpha b}{}^c\,. \quad \text{(A36)}$$

5. For all l_1, $l_2 \in \mathfrak{l}$, the following identity holds:

$$[l_1, l_2] = \{\delta(l_1), \delta(l_2)\}\,, \qquad f_{AB}{}^C = \delta_A{}^a\, \delta_B{}^b\, X_{ab}{}^C\,. \quad \text{(A37)}$$

6. For all $h_1, h_2, h_3 \in \mathfrak{h}$:

$$\begin{aligned} \{[h_1, h_2], h_3\} &= \partial(h_1) \rhd \{h_2, h_3\} + \{h_1, [h_2, h_3]\} - \partial(h_2) \rhd \{h_1, h_3\} - \{h_2, [h_1, h_3]\}\,, \\ f_{ab}{}^d X_{dc}{}^B &= \partial_a{}^\alpha X_{bc}{}^A \rhd_{\alpha A}{}^B + X_{ad}{}^B f_{bc}{}^d - \partial_b{}^\alpha \rhd_{\alpha A}{}^B X_{ac}{}^A - X_{bd}{}^B f_{ac}{}^d\,. \end{aligned} \quad \text{(A38)}$$

7. For all $h_1, h_2, h_3 \in \mathfrak{h}$:

$$\begin{aligned} \{h_1, [h_2, h_3]\} &= \{\delta\{h_1, h_2\}, h_3\} - \{\delta\{h_1, h_3\}, h_2\}\,, \\ X_{ad}{}^A f_{bc}{}^d &= X_{ab}{}^B\, \delta_B{}^d\, X_{dc}{}^A - X_{ac}{}^B\, \delta_B{}^d X_{db}{}^A\,. \end{aligned} \quad \text{(A39)}$$

8. For all $l \in \mathfrak{l}$ and $\forall h \in \mathfrak{h}$:

$$\{\delta(l), h\} + \{h, \delta(l)\} = -\partial(h) \rhd l\,, \qquad 2\,\delta_A{}^a\, X_{\{ab\}}{}^B = -\partial_b{}^\alpha \rhd_{\alpha A}{}^B\,. \quad \text{(A40)}$$

Finally, when dealing with various algebra valued differential forms, one multiplies them as differential forms using the ordinary wedge product $\wedge$, and simultaneously as algebra elements using one of maps defined above. For example, the product with an action $\wedge^{\rhd}$ of the $\mathfrak{g}$-valued n-form ρ on the $\mathfrak{h}$-valued m-form η is defined as:

$$\begin{aligned} \rho \wedge^{\rhd} \eta &= \frac{1}{n!m!}\, \rho^\alpha{}_{\mu_1\dots\mu_m}\, \eta^a{}_{\nu_1\dots\nu_n}\, \tau_\alpha \rhd t_a\, \mathrm{d}x^{\mu_1} \wedge \dots \mathrm{d}x^{\mu_m} \wedge \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_n} \\ &= \frac{1}{n!m!}\, \rho^\alpha{}_{\mu_1\dots\mu_m}\, \eta^a{}_{\nu_1\dots\nu_n} \rhd_{\alpha a}{}^b t_b\, \mathrm{d}x^{\mu_1} \wedge \dots \mathrm{d}x^{\mu_m} \wedge \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_n}\,. \end{aligned} \quad \text{(A41)}$$

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Topological invariant of 4-manifolds based on a 3-group

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Abstract: We study a generalization of 4-dimensional BF-theory in the context of higher gauge theory. We construct a triangulation independent topological state sum Z, based on the classical $3BF$ action for a general 3-group and a 4-dimensional spacetime manifold $\mathcal{M}_4$. This state sum coincides with Porter's TQFT for $d = 4$ and $n = 3$. In order to verify that the constructed state sum is a topological invariant of the underlying 4-dimensional manifold, its behavior under Pachner moves is analyzed, and it is obtained that the state sum Z remains the same. This paper is a generalization of the work done by Girelli, Pfeiffer, and Popescu for the case of state sum based on the classical $2BF$ action with the underlying 2-group structure.

Keywords: Differential and Algebraic Geometry, Models of Quantum Gravity, Topological Field Theories, Gauge Symmetry

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Contents

1 Introduction

Within the Loop Quantum Gravity framework, one studies the nonperturbative quantization of gravity, both canonically and covariantly, see [1–4] for an overview and a comprehensive introduction. The covariant approach focuses on defining the path integral for the gravitational field by considering a triangulation of a spacetime manifold and specifying the path integral as a discrete state sum of the gravitational field configurations living on the simplices in the triangulation. This quantization technique is usually referred to as the *spinfoam quantization method*, and it can be divided into three major steps:

1. first, one writes the classical action $S[g]$ as a topological BF-like action plus simplicity constraints,
2. then one uses the algebraic structure underlying the topological sector of the action to define a topological state sum Z,

3. and finally, one deforms the topological state sum by imposing simplicity constraints, thus promoting it into a path integral for a physical theory.

Spinfoam models for gravity are usually constructed by constraining the topological gauge theory known as BF theory, obtaining the Plebanski formulation of general relativity [5]. For example, in 3 dimensions, the prototype spinfoam model is known as the Ponzano-Regge model [6]. In 4 dimensions there are multiple models, such as the Barrett-Crane model [7, 8], the Ooguri model [9], and the most sophisticated EPRL/FK model [10, 11] (see also [12–14]). All these models aim to define a viable theory of a quantum gravitational field alone, without matter fields. The attempts to include matter fields have had limited success [15], mainly because the mass terms cannot be expressed in the theory due to the absence of the tetrad fields from the topological BF sector of the theory.

In order to overcome this problem, a new approach has been developed within the framework of *higher gauge theory* (for a review of higher gauge theory, see [16, 17], and for its applications in physics see [18–29]). Within higher gauge theory formalism, one generalizes the BF action, based on some Lie group, to an $2BF$ action based on the 2-group structure. Within this approach [30], one rewrites the action for general relativity as a constrained $2BF$ action, such that the tetrad fields are present in the topological sector. This result opened up the possibility to couple all matter fields to gravity in a straightforward way. Nevertheless, the matter fields could not be naturally expressed using the underlying algebraic structure of a 2-group, rendering the spinfoam quantization method only half-implementable, since the matter sector of the classical action could not be expressed as a topological term plus a simplicity constraint, which means that the steps 2 and 3 above could not be performed for the matter sector of the action.

This final issue has recently been resolved in [31], where one more step in the categorical ladder is performed in order to generalize the underlying algebraic structure from a 2-group to a 3-group (see also [32] for the 4-group formulation). This generalization then naturally gives rise to the so-called $3BF$ action, which proves to be suitable for a unified description of both gravity and matter fields. The first step of the spinfoam quantization program is carried out in [31] where the suitable gauge 3-groups have been specified, and the corresponding constrained $3BF$ actions constructed so that the desired classical dynamics of the gravitational and matter fields are obtained. A reader interested in the construction of the constrained $2BF$ actions describing the Yang-Mills field and Einstein-Cartan gravity, and $3BF$ actions describing the Klein-Gordon, Dirac, Weyl, and Majorana fields, each coupled to gravity in the standard way, is referred to [30, 31].

In this paper, we focus our attention on the second step of the spinfoam quantization program: we will construct a triangulation independent topological state sum Z, based on the classical $3BF$ action for a general 3-group and a 4-dimensional spacetime manifold $\mathcal{M}_4$. This state sum coincides with Porter's TQFT [33, 34] for $d = 4$ and $n = 3$. In order to verify that the constructed state sum is topological, we analyze its behavior under Pachner moves [35]. Pachner moves are local changes of a triangulation that preserve topology, such that any two triangulations of the same manifold are connected by a finite number of Pachner moves. In 4 dimensions, there are five different Pachner moves: the $3-3$ move,

$4-2$ move, and $5-1$ move, and their inverses. After defining the state sum, we calculate its behavior under these Pachner moves. We obtain that the state sum Z remains the same, proving that it is a topological invariant of the underlying 4-dimensional manifold. This construction thus completes the second step of the quantization procedure. Our result paves the way for the third step of the covariant quantization procedure and a formulation of a quantum theory of gravity and matter by imposing the simplicity constraints on the state sum. We leave the third step for future work.

The layout of the paper is as follows. In section 2 we review the pure and the constrained nBF theories describing some of the physically relevant models — the constrained $2BF$ actions describing the Yang-Mills field and Einstein-Cartan gravity, and constrained $3BF$ actions describing the Klein-Gordon and Dirac fields coupled to Yang-Mills fields and gravity in the standard way. In section 3, we review the relevant algebraic tools involved in the description of higher gauge theory, 2-crossed modules, and 3-gauge theory. Starting from the notion of Lie 3-groups, we generalize the integral picture of gauge theory to a 3-gauge theory that involves curves, surfaces, and volumes labeled with elements of non-Abelian groups. In section 4, we define the discrete state sum model of topological higher gauge theory in dimension $d=4$. The model is defined for any closed and oriented combinatorial 4-dimensional manifold $\mathcal{M}_4$. The proof that the state sum is invariant under the Pachner moves and thus independent of the chosen triangulation is presented in appendix B.

Notations and conventions throughout the paper are as follows. The local Lorentz indices are denoted by the Latin letters $a, b, c, \dots$, that take values $0,1,2,3$, and are raised and lowered using the Minkowski metric η_{ab} with signature $(-,+,+,+)$. The spacetime indices are denoted by the Greek letters $\mu, \nu, \dots$, and are raised and lowered by the spacetime metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ denotes the tetrad fields. If G is a finite group, $\int_G dg = 1/|G| \sum_{g\in G}$ denotes the normalized sum over all group elements, while δ_G denotes the corresponding δ-distribution on G. The δ-distribution is defined for every element $g \in G$ such that $\delta_G(g) = |G|$ if g is the unit element of the group, i.e., $g = e$, and $\delta_G(g) = 0$ if it is not, i.e., $g \neq e$. If G is a Lie group, $\int_G dg$ and δ_G denote the Haar measure and the δ-distribution on G, respectively. The set of all k-simplices, $0 \leq k \leq d$, is denoted by Λ_k. The set of vertices Λ_0 is finite and ordered, and every k-simplex is labeled by $(k+1)$-tuples of vertices $(i_0 \dots i_k)$, where $i_0, \dots, i_k \in \Lambda_0$ such that $i_0 < \dots < i_k$.

2 Review of the classical theory

2.1 Topological nBF theories

For a given Lie group G whose Lie algebra $\mathfrak{g}$ is equipped with the G-invariant symmetric nondegenerate bilinear form $\langle \_\,, \_ \rangle_{\mathfrak{g}}$, and for a given 4-dimensional spacetime manifold $\mathcal{M}_4$, one can introduce the BF action as

$$S_{BF} = \int_{\mathcal{M}_4} \langle B \wedge F \rangle_{\mathfrak{g}} \,, \tag{2.1}$$

where 2-form $F \equiv d\alpha + \alpha \wedge \alpha$ is the curvature for the $\mathfrak{g}$-valued connection 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$ and 2-form $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ is an $\mathfrak{g}$-valued Lagrange multiplier. Varying the

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action (2.1) with respect to the Lagrange multiplier B and the connection α, one obtains the equations of motion of the theory,

$$F = 0\,, \qquad \nabla B \equiv \mathrm{d}B + \alpha \wedge B = 0\,. \tag{2.2}$$

From the first equation of motion, one sees that α is a flat connection, which then, together with the second equation of motion, implies that B is constant. Therefore, the theory given by the BF action has no local propagating degrees of freedom, i.e., the theory is topological. For more details about the BF theory see [5, 36, 37].

Within the framework of Higher Gauge Theory, by passing from the notion of a gauge group to the notion of a gauge 2-group, one defines the categorical generalization of the BF action, called the $2BF$ action. A 2-group has a naturally associated notion of a 2-connection $(\alpha\,,\beta)$, described by the usual $\mathfrak{g}$-valued 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$ and an $\mathfrak{h}$-valued 2-form $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$, where $\mathfrak{h}$ is a Lie algebra of the Lie group H. The 2-connection gives rise to the so-called fake 2-curvature $(\mathcal{F},\mathcal{G})$, where $\mathcal{F}$ is a $\mathfrak{g}$-valued fake curvature 2-form $\mathcal{F} \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{g})$ and $\mathcal{G}$ is an $\mathfrak{h}$-valued curvature 3-form $\mathcal{G} \in \mathcal{A}^3(\mathcal{M}_4\,,\mathfrak{h})$, defined as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta\,. \tag{2.3}$$

Representing the 2-group as a crossed-module $(H \overset{\partial}{\to} G\,,\triangleright)$, and seeing the next section for the definition and notation, one introduces a $2BF$ action using the fake 2-curvature (2.3) as

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F}\rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G}\rangle_{\mathfrak{h}}\,, \tag{2.4}$$

where the 2-form $B \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{g})$ and the 1-form $C \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{h})$ are Lagrange multipliers, and $\langle \_\,,\_\rangle_{\mathfrak{g}}$ and $\langle \_\,,\_\rangle_{\mathfrak{h}}$ denote the G-invariant symmetric nondegenerate bilinear forms for the algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Similarly as in the case of the BF theory, varying the $2BF$ action (2.4) with respect to the Lagrange multipliers B and C one obtains the equations of motion,

$$\mathcal{F} = 0\,, \qquad \mathcal{G} = 0\,, \tag{2.5}$$

i.e., the conditions that the curvature 2-form $\mathcal{F}$ and the curvature 3-form $\mathcal{G}$ vanish, while varying with respect to the connections α and β one obtains

$$\nabla B + C \wedge^{\mathcal{T}} \beta = 0\,, \qquad \nabla C - \partial(B) = 0\,. \tag{2.6}$$

Similar to the case of the BF action, the $2BF$ action defines a topological theory, i.e., a theory with no propagating degrees of freedom, see [38–41] for review and references.

Continuing the categorical ladder one step further, one can generalize the $2BF$ action to the $3BF$ action, by passing from the notion of a 2-group to the notion of a 3-group. Representing the 3-group with a 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,,\triangleright\,,\{\_\,,\_\}_{\mathrm{p}})$, and seeing next section for definition and notation, one can define a 3-connection as an ordered triple (α,β,γ), where α, β, and γ are appropriate algebra-valued differential forms, $\alpha \in \mathcal{A}^1(\mathcal{M}_4,\mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4,\mathfrak{h})$, and $\gamma \in \mathcal{A}^3(\mathcal{M}_4,\mathfrak{l})$. The corresponding fake 3-curvature

$(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is defined as:

$$\begin{aligned} \mathcal{F} &= \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \\ \mathcal{H} &= \mathrm{d}\gamma + \alpha \wedge^{\triangleright} \gamma + \{\beta \wedge \beta\}_{\mathrm{p}}\,. \end{aligned} \tag{2.7}$$

Then, similar to the construction of BF and $2BF$ actions, one defines the $3BF$ action as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}\,, \tag{2.8}$$

where $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ denote the Lie algebras corresponding to the Lie groups G, H, and L and the forms $\langle \_, \_ \rangle_{\mathfrak{g}}$, $\langle \_, \_ \rangle_{\mathfrak{h}}$, and $\langle \_, \_ \rangle_{\mathfrak{l}}$ are G-invariant symmetric nondegenerate bilinear forms on $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$, respectively. The variables $B \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$, $C \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$, and $D \in \mathcal{A}^0(\mathcal{M}_4, \mathfrak{l})$ are Lagrange multipliers, and their associated equations of motion are the conditions that the 3-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ vanishes,

$$\mathcal{F} = 0\,, \qquad \mathcal{G} = 0\,, \qquad \mathcal{H} = 0\,. \tag{2.9}$$

Additionally, varying with respect to the 3-connection variables α, β, and γ one gets:

$$\nabla B + C \wedge^{\mathcal{T}} \beta - D \wedge^{\mathcal{S}} \gamma = 0\,, \tag{2.10}$$

$$\nabla C - \partial(B) - D \wedge^{(\chi_1 + \chi_2)} \beta = 0\,, \tag{2.11}$$

$$\nabla D + \delta(C) = 0\,. \tag{2.12}$$

For further details see [22, 42, 43] for the definition of the 3-group, and [31] for the definition of the pure $3BF$ action.

All the above actions are topological, in the sense that they do not contain any local propagating degrees of freedom [44, 45]. In this sense, they are not very interesting for the description of realistic physics, which should feature nontrivial dynamics. Nevertheless, by choosing the convenient underlying 2-crossed module structure and imposing the appropriate simplicity constraints onto the degrees of freedom present in the $3BF$ action, one can obtain the nontrivial classical dynamics of the gravitational and matter fields, as we will see in the following subsection.

2.2 Models with relevant dynamics

Let us review how one can employ the n-group structure to introduce the topological nBF actions corresponding to gravity and matter fields, as well as the form of the appropriate simplicity constraints to be imposed on these fields to obtain the classical dynamics.

First we review the most important constrained $2BF$ actions. We begin by rewriting general relativity as a constrained $2BF$ action based on the underlying Poincaré 2-group. The Poincaré 2-group is equivalent to a crossed module $(H \xrightarrow{\partial} G, \triangleright)$, where the groups are choosen as $G = \mathrm{SO}(3,1)$ and $H = \mathbb{R}^4$, and the map ∂ is trivial. The action $\triangleright$ is a natural action of $\mathrm{SO}(3,1)$ on $\mathbb{R}^4$, defined as

$$M_{ab} \triangleright P_c = \eta_{[bc} P_{a]}\,, \tag{2.13}$$

where M_{ab} and P_a are the generators of groups $\mathrm{SO}(3,1)$ and $\mathbb{R}^4$, respectively. The action $\triangleright$ of $\mathrm{SO}(3,1)$ on itself is given via conjugation, by definition of a crossed module. Then, Poincaré 2-group gives rise to the 2-connection (α, β), given by the algebra-valued differential forms

$$\alpha = \omega^{ab} M_{ab} \,, \qquad \beta = \beta^a P_a \,, \tag{2.14}$$

where we have interpreted the connection 1-form α^{ab} as the ordinary spin connection ω^{ab}. Also, the corresponding 2-curvature $(\mathcal{F}, \mathcal{G})$ is given as

$$\begin{aligned} \mathcal{F} &= (\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}) M_{ab} \equiv R^{ab} M_{ab} \,, \\ \mathcal{G} &= (\mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b) P_a \qquad \equiv \nabla \beta^a P_a \equiv G^a P_a \,, \end{aligned} \tag{2.15}$$

where we can recognize the standard Riemann curvature 2-form R^{ab} in $\mathcal{F}$. Having these variables in hand, one defines $2BF$ action (2.4) for the Poincaré 2-group as

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a \,. \tag{2.16}$$

Here, the crucial insight is that the Lagrange multiplier fields C^a can be identified with the tetrads [30], since one can show that 1-forms C^a transform in the same way as the tetrad 1-forms e^a under the Lorentz transformations and diffeomorphisms. One can now construct the action for general relativity by simply adding the additional simplicity constraint term to the action (2.16):

$$S = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) . \tag{2.17}$$

Here λ_{ab} is a Lagrange multiplier 2-form associated to the simplicity constraint term, and l_p is the Planck length. It is straightforward to show that the corresponding equations of motion reduce to vacuum Einstein field equations. Thus the action (2.17) is classically equivalent to general relativity. The construction of the action (2.17) is analogous to the Plebanski model, where general relativity is constructed by adding a simplicity constraint to the BF theory based on the Lorentz group. However, one clear advantage of this model over the Plebanski model is that the tetrads are explicitly present in the topological sector of the action. Upon the covariant quantization, tetrads are therefore fundamental, off-shell quantities, in contrast to the Plebanski model where they appear only on-shell, as solutions of the classical equations of motion. The off-shell presence of the tetrads facilitates the straightforward coupling of the matter fields to gravity, and thus overcomes the problems present in the spinfoam models [15].

The Poincaré 2-group can be easily extended to include the coupling of the $\mathrm{SU}(N)$ Yang-Mills fields to gravity [31]. To achieve this, one constructs the crossed module $(H \stackrel{\partial}{\to} G, \triangleright)$, where the groups are chosen as $G = \mathrm{SO}(3,1) \times \mathrm{SU}(N)$ and $H = \mathbb{R}^4$, while the map ∂ remains trivial, as before. The action $\triangleright$ of the group G on H is such that the $\mathrm{SO}(3,1)$ subgroup acts on $\mathbb{R}^4$ via the vector representation (2.13), while the action of the $\mathrm{SU}(N)$ subgroup is trivial,

$$\tau_I \triangleright P_a = 0 \,, \tag{2.18}$$

where τ_I are the SU(N) generators. This crossed module yields the 2-connection (α,β), where algebra-valued 1-form α and algebra valued 2-form β are defined as follows,

$$\alpha = \omega^{ab} M_{ab} + A^I \tau_I \,, \qquad \beta = \beta^a P_a \,, \tag{2.19}$$

where we can identify the gauge connection 1-form A^I. This connection gives rise to the 2-curvature $(\mathcal{F},\mathcal{G})$, where $\mathcal{F}$ as defined as

$$\mathcal{F} = R^{ab} M_{ab} + F^I \tau_I \,, \qquad F^I \equiv \mathrm{d}A^I + f_{JK}{}^I A^J \wedge A^K \,, \tag{2.20}$$

while the curvature $\mathcal{G}$ for β remains the same as before. Given these variables, the Lagrange multiplier B in the first term of the topological action (2.4) also splits into two pieces corresponding to the direct product of the group G, giving

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a \,, \tag{2.21}$$

where 2-form $B^I \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{su}(N))$ is the second piece of the Lagrange multiplier. To obtain the non-trivial classical dynamics for gravity and the Yang-Mills field, we add the appropriate simplicity constraint terms to the action (2.21), and construct the constrained $2BF$ action:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda^I \wedge \left(B_I - \frac{12}{g} M_{abI} e^a \wedge e^b \right) + \zeta^{abI} \left(M_{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - g_{IJ} F^J \wedge e_a \wedge e_b \right) . \end{aligned} \tag{2.22}$$

Here, the first row is the topological sector and the familiar simplicity constraint for gravity from (2.17), while the second row contains the appropriate simplicity constraints for Yang Mills field, featuring the Lagrange multipliers λ^I and ζ^{abI}. The action (2.22) provides two dynamical equations — the equation for A^I,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + \Gamma^\rho{}_{\lambda\rho} F^{I\lambda\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0 \,, \tag{2.23}$$

where $\Gamma^\lambda{}_{\mu\nu}$ is the standard Levi-Civita connection, and an equation for e^a which is the Einstein field equation with the SU(N) gauge field source term,

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2 \, T^{\mu\nu} \,, \quad T^{\mu\nu} \equiv -\frac{1}{4g} \left(F_{\rho\sigma}{}^I F^{\rho\sigma}{}_I g^{\mu\nu} + 4 F^{\mu\rho}{}_I F_\rho{}^{\nu I} \right) . \tag{2.24}$$

In this way, we see that both gravity and gauge fields can be successfully represented within a unified framework of higher gauge theory, based on a 2-group structure. A generalization from SU(N) Yang-Mills case to the more complicated cases, such as SU$(3) \times$ SU$(2) \times$ U(1), is straightforward.

Let us now review how one can use the 3-group structure and the corresponding constrained $3BF$ theory to describe general relativity coupled to Klein-Gordon and Dirac fields. To describe a single real Klein-Gordon field coupled to gravity, one begins by specifying a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G \,, \triangleright \,, \{\_\,,\_\}_{\mathrm{p}})$, as follows. The groups are given as

$G = \mathrm{SO}(3,1)$, $H = \mathbb{R}^4$, and $L = \mathbb{R}$. The group G acts on H via the vector representation, and on L via the trivial representation. The maps ∂ and δ are chosen to be trivial, as well as the Peiffer lifting. Given this choice of a 2-crossed module, the 3-connection $(\alpha\,,\beta\,,\gamma)$ takes the form

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma \mathbb{I}\,, \tag{2.25}$$

where $\mathbb{I}$ is the sole generator of the Lie group L. This 3-connection gives rise to the fake 3-curvature $(\mathcal{F},\mathcal{G},\mathcal{H})$,

$$\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,, \qquad \mathcal{H} = \mathrm{d}\gamma\,. \tag{2.26}$$

The importance of the $3BF$ theory for this choice of the 2-crossed module lies in the fact that the Lagrange multiplier D can transform as a scalar with respect to Lorentz symmetry, $M_{ab} \triangleright \mathbb{I} = 0$, and it transforms as a scalar with respect to diffeomorphisms since D is also a 0-form. In other words, one can interpret the Lagrange multiplier D to be a real scalar field, $D \equiv \phi$, and write the topological $3BF$ action (2.8) as:

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \phi\, \mathrm{d}\gamma\,. \tag{2.27}$$

In order to obtain the Klein-Gordon field ϕ of mass m coupled to gravity in the standard way, the appropriate simplicity constraints are imposed, and the constrained $3BF$ action takes the form:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \phi\,\mathrm{d}\gamma - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c \wedge e_d\right) \\ & + \lambda \wedge \left(\gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c\right) + \Lambda^{ab} \wedge \left(H_{abc}\varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b\right) \\ & - \frac{1}{2\cdot 4!} m^2 \phi^2 \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d\,. \end{aligned} \tag{2.28}$$

The first row is the topological sector (2.27) and the simplicity constraint for gravity from the action (2.17), the second row contains two new simplicity constraints featuring the Lagrange multiplier 1-forms λ and Λ^{ab} and the 0-form H_{abc}, and the third row features the mass term for the scalar field. The action (2.28) has two dynamical equations of motion — the equation for the scalar field ϕ is the covariant Klein-Gordon equation,

$$\left(\nabla_\mu \nabla^\mu - m^2\right)\phi = 0\,, \tag{2.29}$$

while the equation for the tetrads e^a is the Einstein field equation with the scalar field source term,

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2\, T^{\mu\nu}\,, \quad T^{\mu\nu} \equiv \partial^\mu\phi\partial^\nu\phi - \frac{1}{2} g^{\mu\nu}\left(\partial_\rho\phi\partial^\rho\phi + m^2\phi^2\right)\,. \tag{2.30}$$

We see that the obtained theory is classically equivalent to general relativity coupled to a scalar field. Most importantly, one sees that the choice of the group L dictates the matter

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content of the theory, while the action $\triangleright$ of G on L specifies the transformation properties of the matter fields.

Finally, in order to describe the Dirac field coupled to Einstein-Cartan gravity, the 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\}_{\rm p})$ has to be chosen as follows. The groups are $G = {\rm SO}(3,1)$. $H = \mathbb{R}^4$, and $L = \mathbb{R}^8(\mathbb{G})$, where $\mathbb{G}$ is the algebra of complex Grassmann numbers. The maps ∂, δ, and the Peiffer lifting are trivial, as before. The action of the group G on H is via vector representation, and on L via spinor representation, in the following way. Denoting the eight generators of the Lie group $\mathbb{R}^8(\mathbb{G})$ as P_α and P^α, where the bispinor index α takes the values $1,\dots,4$, the action $\triangleright$ of G on L is given explicitly as

$$M_{ab} \triangleright P_\alpha = \frac{1}{2}(\sigma_{ab})^\beta{}_\alpha P_\beta\,, \qquad M_{ab} \triangleright P^\alpha = -\frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta\,, \tag{2.31}$$

where $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, and γ_a are the usual Dirac matrices. This choice of the 2-crossed module gives rise to the 3-connection $(\alpha\,,\beta\,,\gamma)$, defined as

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma^\alpha P_\alpha + \bar{\gamma}_\alpha P^\alpha\,, \tag{2.32}$$

where the 3-connection 3-forms γ^α and $\bar{\gamma}_\alpha$ should not be confused with the Dirac matrices γ_a due to different types of indices. The 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is given as:

$$\begin{aligned}
&\mathcal{F} = R^{ab} M_{ab}\,, \qquad\qquad \mathcal{G} = \nabla \beta^a P_a\,,\\
&\mathcal{H} = \left({\rm d}\gamma^\alpha + \frac{1}{2}\omega^{ab}(\sigma_{ab})^\alpha{}_\beta \gamma^\beta\right) P_\alpha + \left({\rm d}\bar{\gamma}_\alpha - \frac{1}{2}\omega^{ab}\bar{\gamma}_\beta(\sigma_{ab})^\beta{}_\alpha\right) P^\alpha \equiv (\overset{\rightarrow}{\nabla}\gamma)^\alpha P_\alpha + (\bar{\gamma}\overset{\leftarrow}{\nabla})_\alpha P^\alpha\,.
\end{aligned} \tag{2.33}$$

As in the case of the scalar field, the choice of the group L and action $\triangleright$ of G on L dictates the matter content of the theory and its transformation properties. The group L prescribes that D contains eight independent real anticommuting matter fields as its components. Then, since D is a 0-form and it transforms according to the spinorial representation of ${\rm SO}(3,1)$, these eight real Grassmann-valued fields can be identified with the four complex Dirac bispinor fields, and one can write the corresponding topological $3BF$ action as:

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + (\bar{\gamma}\overset{\leftarrow}{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overset{\rightarrow}{\nabla}\gamma)^\alpha\,. \tag{2.34}$$

In order to obtain the action that gives us the dynamics of Einstein-Cartan theory of gravity coupled to a Dirac field, we add the following simplicity constraints:

$$\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + (\bar{\gamma}\overset{\leftarrow}{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overset{\rightarrow}{\nabla}\gamma)^\alpha - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c \wedge e_d\right)\\
& - \lambda^\alpha \wedge \left(\bar{\gamma}_\alpha - \frac{i}{6}\varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar{\psi}\gamma^d)_\alpha\right) + \bar{\lambda}_\alpha \wedge \left(\gamma^\alpha + \frac{i}{6}\varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^\alpha\right)\\
& - \frac{1}{12} m\,\bar{\psi}\psi\,\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d + 2\pi i l_p^2\,\bar{\psi}\gamma_5\gamma^a\psi\,\varepsilon_{abcd} e^b \wedge e^c \wedge \beta^d.
\end{aligned} \tag{2.35}$$

The topological sector is in the first row, as well as the gravitational simplicity constraint, the second row contains the new simplicity constraints for the Dirac field, while the third row contains the mass term for the Dirac field and a term that ensures the correct coupling between the torsion and the spin of the Dirac field. Varying the action (2.35), one obtains the following dynamical equations of motion — the equations for ψ and $\bar{\psi}$ which are the standard covariant Dirac equation and its conjugate,

$$(i\gamma^a e^\mu{}_a \overrightarrow{\nabla}_\mu - m)\psi = 0\,, \quad \bar{\psi}(i\overleftarrow{\nabla}_\mu e^\mu{}_a \gamma^a + m) = 0\,, \tag{2.36}$$

and the differential equation of motion for e^a which is the Einstein field equation with a Dirac field source term,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \quad T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\gamma^a \overleftrightarrow{\nabla}^\nu e^\mu{}_a \psi - \frac{1}{2}g^{\mu\nu}\bar{\psi}\Big(i\gamma^a \overleftrightarrow{\nabla}_\rho e^\rho{}_a - 2m\Big)\psi\,, \tag{2.37}$$

where $\overleftrightarrow{\nabla} = \overrightarrow{\nabla} - \overleftarrow{\nabla}$. Moreover, one obtains the desired equation of motion for the torsion,

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a\,, \quad s_a = i\varepsilon_{abcd}e^b \wedge e^c \bar{\psi}\gamma_5\gamma^d\psi\,, \tag{2.38}$$

where s_a is the Dirac spin 2-form. The equations of motion (2.36), (2.37), and (2.38) are precisely the equations of motion of the Einstein-Cartan-Dirac theory.

The natural presence of a scalar and Dirac field in the $3BF$ action is an essential property of the specific choices of the 3-group structures in a 4-dimensional spacetime, just like the existence of the tetrad field e^a in the topological $2BF$ action is an essential property of the $2BF$ action and the Poincaré 2-group. In this way, both the scalar field and the Dirac field appear in the topological sector of the action, making the quantization procedure feasible. Similarly, one can introduce Weyl and Majorana fields as well, see [31].

3 A review of 2-groups and 3-groups

As we have seen in the previous section, the gauge symmetry of 3-gauge theory is described by an algebraic structure known as a 3-*group*. In this section, we present the relevant definition of the 3-group, and we briefly explain how this structure is used to equip curves, surfaces, and volumes with holonomies. The results obtained in this section are necessary for the construction of the topological invariant, which will be studied in section IV.

3.1 3-Groups

In the category theory, a 2-group is defined as a 2-category consisting of only one object, where all the morphisms and 2-morphisms are invertible. It has been shown that every strict 2-group is equivalent to a crossed module $(H \stackrel{\partial}{\to} G\,, \triangleright)$.

A *pre-crossed module* $(H \stackrel{\partial}{\to} G\,, \triangleright)$ of groups G and H, is given by a group map $\partial : H \to G$, together with a left action $\triangleright$ of G on both groups, by automorphisms, such that the group G acts on itself via conjugation, i.e., for each $g_1, g_2 \in G$,

$$g_1 \triangleright g_2 = g_1 g_2 g_1^{-1}\,,$$

and for each $h_1\,, h_2 \in H$ and $g \in G$ the following identity holds:

$$g\partial h g^{-1} = \partial(g \triangleright h)\,.$$

In a pre-crossed module the *Peiffer commutator* is defined as:

$$\langle h_1\,, h_2\rangle_{\rm p} = h_1 h_2 h_1^{-1} \partial(h_1) \triangleright h_2^{-1}\,. \tag{3.1}$$

A pre-crossed module is said to be a *crossed module* if all of its Peiffer commutators are trivial, which is to say that the *Peiffer identity* is satisfied:

$$(\partial h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}\,. \tag{3.2}$$

Continuing the categorical generalization one step further, one can generalize the notion of a 2-group to the notion of a 3-group. Similar to the definition of a group and a 2-group within the category theory formalism, a 3-group is defined as a 3-category with only one object, where all morphisms, 2-morphisms, and 3-morphisms are invertible. Moreover, in analogy with how a crossed module encodes a strict 2-group, it has been proved that a semistrict 3-group — Gray group is equivalent to a 2-crossed module [42, 46].

A 2-*crossed module* $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G,\, \triangleright,\, \{\_\,,\,\_\}_{\rm p})$ is a chain complex of groups, given by three groups G, H, and L, together with maps ∂ and δ,

$$L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,,$$

such that $\partial\delta = 1_G$, an action $\triangleright$ of the group G on all three groups, and a map $\{\_\,,\_\}_{\rm p}$ called the *Peiffer lifting*:

$$\{\_\,,\_\}_{\rm p} : H \times H \to L\,.$$

The maps ∂ and δ, and the Peiffer lifting are G-equivariant, i.e., for each $g \in G$ and $h \in H$

$$g \triangleright \partial(h) = \partial(g \triangleright h)\,, \qquad g \triangleright \delta(l) = \delta(g \triangleright l)\,,$$

and for each h_1, $h_2 \in H$ and $g \in G$:

$$g \triangleright \{h_1\,, h_2\}_{\rm p} = \{g \triangleright h_1,\, g \triangleright h_2\}_{\rm p}\,.$$

The action of the group G on the groups H and L is a smooth left action by automorphisms, i.e., for each $g, g_1, g_2 \in G$, $h_1, h_2 \in H$, $l_1, l_2 \in L$ and $k \in H, L$,

$$g_1 \triangleright (g_2 \triangleright k) = (g_1 g_2) \triangleright k\,, \qquad g \triangleright (h_1 h_2) = (g \triangleright h_1)(g \triangleright h_2)\,, \qquad g \triangleright (l_1 l_2) = (g \triangleright l_1)(g \triangleright l_2)\,.$$

The action of the group G on itself is again via conjugation. Further, the following identities are satisfied:

$$\delta(\{h_1, h_2\}_{\rm p}) = \langle h_1\,, h_2\rangle_{\rm p}\,, \qquad \forall h_1, h_2 \in H\,; \tag{3.3a}$$

$$[l_1, l_2] = \{\delta(l_1)\,, \delta(l_2)\}_{\rm p}\,, \qquad \forall l_1\,, l_2 \in L\,, \quad \text{where } [l,k] = lkl^{-1}k^{-1}; \tag{3.3b}$$

$$\{h_1 h_2, h_3\}_{\rm p} = \{h_1, h_2 h_3 h_2^{-1}\}_{\rm p} \partial(h_1) \triangleright \{h_2, h_3\}_{\rm p}\,, \qquad \forall h_1, h_2, h_3 \in H\,; \tag{3.3c}$$

$$\{h_1, h_2 h_3\}_{\rm p} = \{h_1, h_2\}_{\rm p} \{h_1, h_3\}_{\rm p} \{\langle h_1, h_3\rangle_{\rm p}^{-1}\,, \partial(h_1) \triangleright h_2\}_{\rm p}\,, \qquad \forall h_1, h_2, h_3 \in H\,; \tag{3.3d}$$

$$\{\delta(l), h\}_{\rm p} \{h, \delta(l)\}_{\rm p} = l(\partial(h) \triangleright l^{-1})\,, \qquad \forall h \in H\,, \qquad \forall l \in L\,. \tag{3.3e}$$

In a 2-crossed module the structure $(L \xrightarrow{\delta} H, \triangleright')$ is a crossed module, with action of the group H on the group L defined for each $h \in H$ and $l \in L$ as:

$$h \triangleright' l = l\, \{\delta(l)^{-1},\, h\}_{\mathrm{p}}\,,$$

and it follows that the Peiffer identity is satisfied for each $l_1, l_2 \in L$:

$$\delta(l_1) \triangleright' l_2 = l_1\, l_2\, l_1^{-1}\,.$$

However, the structure $(H \xrightarrow{\partial} G\,, \triangleright)$ in the general case does not form a crossed module, but a pre-crossed module, and for each $h, h' \in H$ the Peiffer commutator does not necessarily vanish.

The following identities hold, for each $h_1, h_2, h_3 \in H$ [42]:

$$\{h_1h_2, h_3\}_{\mathrm{p}} = (h_1 \triangleright' \{h_2, h_3\}_{\mathrm{p}})\{h_1, \partial(h_2) \triangleright h_3\}_{\mathrm{p}}\,, \tag{3.4}$$

$$\{h_1, h_2h_3\}_{\mathrm{p}} = \{h_1, h_2\}_{\mathrm{p}}(\partial(h_1) \triangleright h_2) \triangleright' \{h_1, h_3\}_{\mathrm{p}}\,, \tag{3.5}$$

and are of prime importance for the proof of the Pachner moves invariance. By using the condition (3.3e) of the definition of a 2-crossed module, it follows that for each $h \in H$ and $l \in L$ the following identity holds:

$$\{h, \delta(l)^{-1}\}_{\mathrm{p}} = (h \triangleright' l^{-1})(\partial(h) \triangleright l)\,. \tag{3.6}$$

Moreover, for each $h_1, h_2 \in H$,

$$\{h_1, h_2\}_{\mathrm{p}}^{-1} = h_1 \triangleright' \{h_1^{-1}, \partial(h_1) \triangleright h_2\}_{\mathrm{p}}\,, \tag{3.7}$$

$$\{h_1, h_2\}_{\mathrm{p}}^{-1} = \partial(h_1) \triangleright \{h_1^{-1}, h_1h_2h_1^{-1}\}_{\mathrm{p}}\,, \tag{3.8}$$

$$\{h_1, h_2\}_{\mathrm{p}}^{-1} = (h_1h_2h_1^{-1}) \triangleright' \{h_1, h_2^{-1}\}_{\mathrm{p}}\,, \tag{3.9}$$

$$\{h_1, h_2\}_{\mathrm{p}}^{-1} = (\partial(h_1) \triangleright h_2) \triangleright' \{h_1, h_2^{-1}\}_{\mathrm{p}}\,. \tag{3.10}$$

A reader interested in more details about 3-groups is referred to [43].

3.2 3-gauge theory

In this subsection, we will describe how the language of 3-gauge theory can be used in order to define compositions of labeled paths, surfaces, and volumes. In a 3-gauge theory, one labels geometric objects at three levels. Curves are labeled by elements of G. Their composition and orientation reversal is defined as in conventional gauge theory. In addition, surfaces are labeled with elements of H, and volumes are labeled with the elements of L. The reader interested in the formulation of a 2-gauge theory is referred to [47].

Curves are labeled with the elements of G, and the elements are composed as in the ordinary gauge theory, i.e. , for each $g_1, g_2 \in G$,

$$\bullet \xleftarrow{g_1} \bullet \xleftarrow{g_2} \bullet \;=\; \bullet \xleftarrow{g_1 g_2} \bullet\,,$$

the composition of the elements results in the element $g_1g_2 \in G$. The orientation of a curve can be reversed if it is labeled by the inverse element g^{-1} instead.

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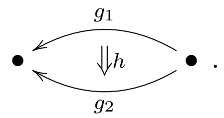
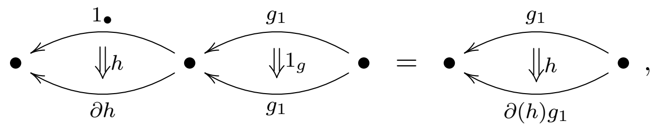
Surfaces are labeled with the elements $h \in H$. For each surface, we choose two reference points on the boundary, and split the boundary into two curves, the source curve labeled with $g_1 \in G$, and the target curve labeled with $g_2 \in G$, as demonstrated in the diagram



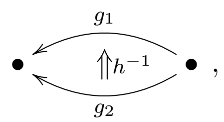
The 2-arrow $h \in H$ maps the curve $g_1 \in G$ to the curve $\partial(h)g_1 \in G$,



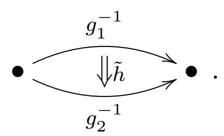
so that the label $h \in H$ of the surface is required to satisfy the following condition:

$$\partial(h) = g_2 g_1^{-1} \,. \tag{3.11}$$

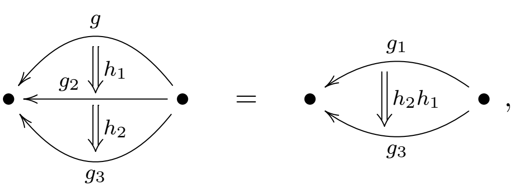
The orientation of the surface can be reversed and labeled with the inverse element instead,



while the orientation reversal of the curves leads to the surface element labeled with $\tilde{h} = g_1^{-1} \triangleright h^{-1}$:



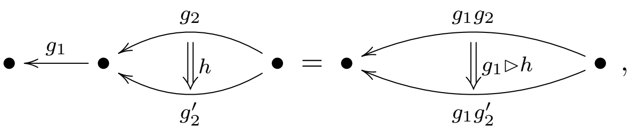
One can now compose 2-morphisms vertically. Let us denote the source and the target of the k-arrow ($k = 1, 2$) of the 2-morphism h as $\partial_k^-(h)$ and $\partial_k^+(h)$, respectively. Then, the vertical composition of 2-morphisms (g_1, h_1) and (g_2, h_2), when they are compatible, i.e., when $\partial_2^+(h_1) = \partial_2^-(h_2)$,



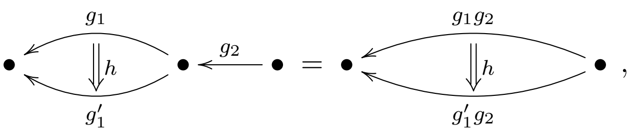
results in a 2-morphism $(g_1, h_2 h_1)$,

$$(g_2, h_2) \#_2 (g_1, h_1) = (g_1, h_2 h_1) \,. \tag{3.12}$$

An important operation is known as whiskering. One can whisker a 2-morphism h with a morphism g_1 by attaching the whisker g_1 to the surface h from the left, i.e., such that $\partial_1^-(g_1) = \partial_1^+(h)$,

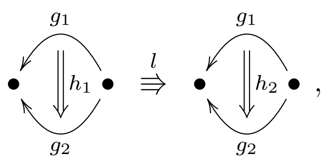


which results in the 2-morphism with the source curve $g_1 g_2$ and target curve $g_1 g_2'$, carrying the label $g_1 \triangleright h$. Similarly, by attaching whisker g_2 to a surface h from the right, i.e., such that $\partial_1^-(h) = \partial_1^+(g_2)$,



one obtains the 2-morphism with the source curve $g_1 g_2$ and target curve $g_1' g_2$, carrying the label h.

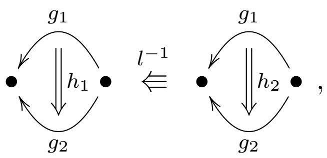
The volumes are labeled with the elements $l \in L$. Let us denote the source and the target of the k-arrow ($k = 1, 2, 3$) of the 3-morphism l as $\partial_k^-(l)$ and $\partial_k^+(l)$, respectively. For each volume, we split the boundary into two surfaces, the source surface labeled with $\partial_3^-(l) = h_1$ and the target surface labeled with $\partial_3^+(l) = h_2$. On the common boundary of the source and target surface, we choose two reference points, and split the boundary into two curves, the source curve labeled with $\partial_2^-(l) = g_1$ and the target curve labeled with $\partial_2^+(l) = g_2$, as demonstrated in the diagram below



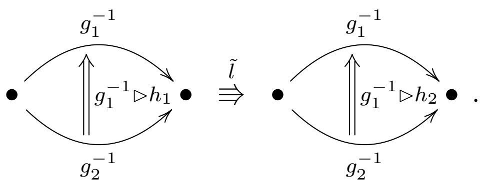
so that the volume label $l \in L$ is required to satisfy the following condition:

$$\delta(l) = h_2 h_1^{-1} . \tag{3.13}$$

The orientation of the volume can be reversed if one labels it with the inverse element l^{-1}:



while the orientation reversal of the curves and surfaces leads to the surface element labeled with $\tilde{l} = g_1^{-1} \triangleright l$,



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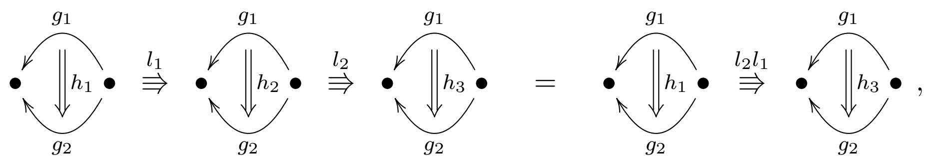
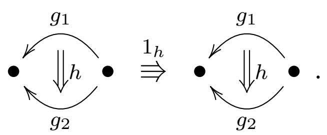
One can compose two 3-morphisms via the *upward composition* (visualizing a third axis, orthogonal to the plane of the paper, as the direction up). The upward composition of 3-morphisms (g_1, h_1, l_1) and (g_1, h_2, l_2), when they are compatible, i.e., when $\partial_3^+(l_1) = \partial_3^-(l_2)$,



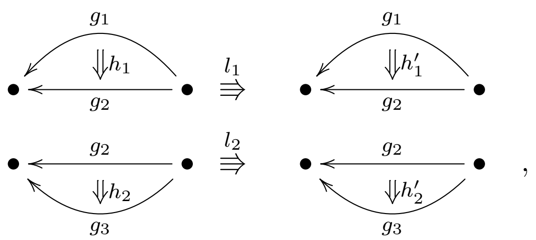
results in a 3-morphism $(g_1, h_1, l_2 l_1)$:

$$(g_1, h_2, l_2)\#_3(g_1, h_1, l_1) = (g_1, h_1, l_2 l_1)\,. \tag{3.14}$$

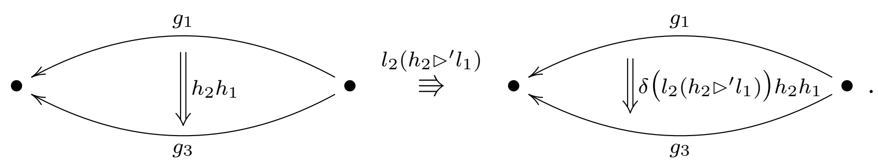
The upward composition of 3-morphisms is associative, and for every $h \in H$ there is a 3-morphism that is an identity for the upward composition of 3-morphisms



The vertical composition of two 3-morphisms (g_1, h_1, l_1) and (g_2, h_2, l_2), when they are compatible, i.e., when $\partial_2^+(l_1) = \partial_2^-(l_2)$,



results in a 3-morphism $(g_1, h_2 h_1, l_2(h_2 \triangleright' l_1))$,



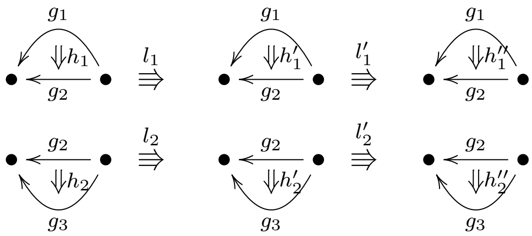
One can write, for (g_1, h_1, l_1) and (g_2, h_2, l_2),

$$(g_2, h_2, l_2)\#_2(g_1, h_1, l_1) = (g_1, h_2 h_1, l_2(h_2 \triangleright' l_1))\,. \tag{3.15}$$

The vertical composition of 3-morphisms is an associative operation. Composition of 3-morphisms is invariant under the change of order of upward composition and vertical composition of 3-morphisms, i.e.,

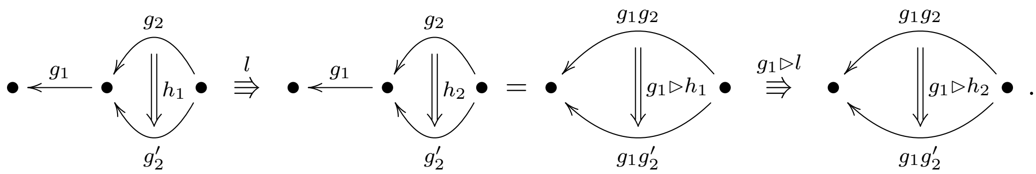
$$\begin{aligned} &((g_2, h_2', l_2')\#_3(g_2, h_2, l_2))\#_2\big((g_1, h_1', l_1')\#_3(g_1, h_1, l_1)\big) \\ =&((g_2, h_2', l_2')\#_2(g_1, h_1', l_1'))\#_3\big((g_2, h_2, l_2)\#_2(g_1, h_1, l_1)\big)\,, \end{aligned} \tag{3.16}$$

which is demonstrated in the diagram notation, where the diagram



uniquely determines the 3-morphism. The proof of the equation (3.16) is given in the appendix A.

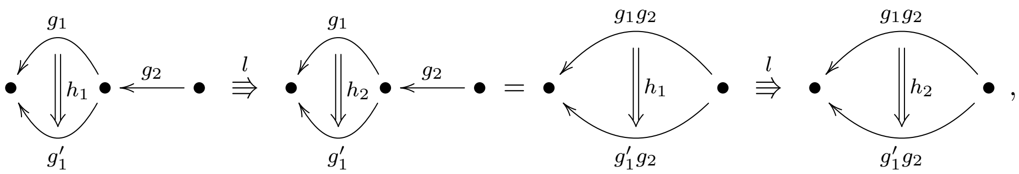
One can whisker the 3-morphisms with morphisms and 2-morphisms. Whiskering of a 3-morphism by a morphism from the left is the composition of a volume $l \in L$ and curve $g_1 \in G$ from the left, when they are compatible, i.e. , when $\partial_1^+(l) = \partial_1^-(g_1)$,



The composition results in a 3-morphism:

$$g_1 \#_1 (g_2, h_1, l) = (g_1 g_2, g_1 \triangleright h_1, g_1 \triangleright l) \,. \tag{3.17}$$

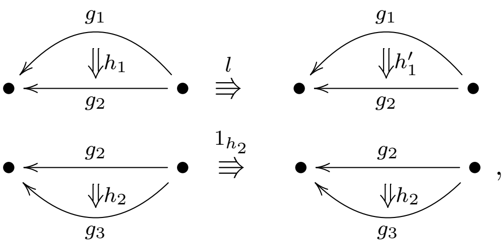
Similarly, one can whisker a 3-morphism by a morphism from the right, when they are compatible, i.e. , $\partial_1^-(l) = \partial_1^+(g_2)$,



which results in the 3-morphism:

$$(g_1, h_1, l) \#_1 g_2 = (g_1 g_2, h_1, l) \,. \tag{3.18}$$

Whiskering of a 3-morphism with a 2-morphisms from below, when they are compatible, i.e. , $\partial_2^+(l) = \partial_2^-(h_2)$, is formed as a vertical composition of 3-morphisms (g_1, h_1, l) and $(g_2, h_2, 1_{h_2})$,



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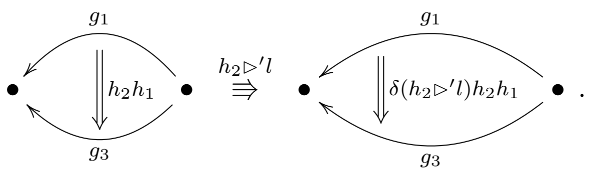
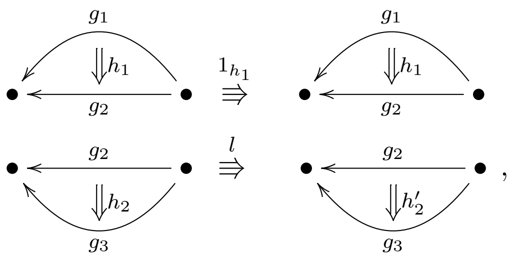
which results in a 3-morphism



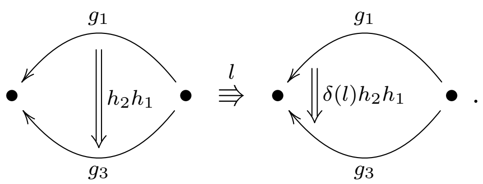
One writes,

$$(g_2, h_2)\#_2(g_1, h_1, l) = (g_1, h_2h_1, h_2 \triangleright' l)\,. \tag{3.19}$$

Whiskering a 3-morphism by 2-morphism from above, when they are compatible, i.e., when $\partial_2^-(l) = \partial_2^+(h_1)$, is formed as a vertical composition of 3-morphisms $(g_1, h_1, 1_{h_1})$ and (g_2, h_2, l),



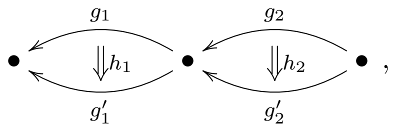
which results in a 3-morphism,



One obtains

$$(g_2, h_2, l)\#_2(g_1, h_1) = (g_1, h_2h_1, l)\,. \tag{3.20}$$

The interchanging 3-arrow is the horizontal composition of two 2-morphisms h_1 and h_2, when they are compatible, i.e., when $\partial_1^-(h_1) = \partial_1^+(h_2)$,

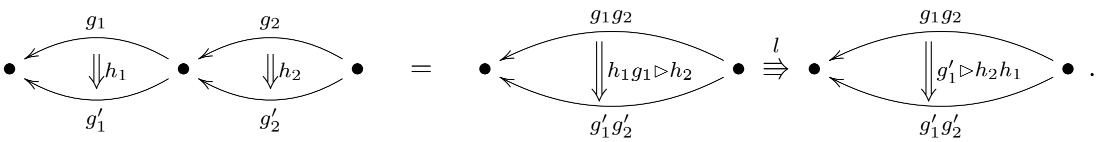


that results in a 3-morphism l, with source surface

$$\partial_3^-(l) = ((g_1, h_1)\#_1 g_2')\#_2(g_1\#_1(g_2, h_2))\,,$$

and target surface

$$\partial_3^+(l) = (g_1'\#_1(g_2, h_2))\#_2((g_1, h_1)\#_1 g_2)\,,$$



One obtains,

$$(g_1,h_1)\#_1(g_2,h_2) = (g_1g_2, h_1g_1 \triangleright h_2, l)\,, \tag{3.21}$$

where the 3-morphism l is Peiffer lifting $\{h_1, g_1 \triangleright h_2\}_{\mathrm{p}}^{-1}$. Using the condition (3.13), one obtains

$$((\partial(h_1)g_1) \triangleright h_2)h_1 = \delta(l)h_1(g_1 \triangleright h_2)\,, \tag{3.22}$$

and from the definition of the Peiffer commutator, the identity (3.1), and the property (3.3a) of the 2-crossed module, i.e. , $\delta(\{h_1,h_2\}_{\mathrm{p}}) = \langle h_1\,,h_2\rangle_{\mathrm{p}}$, one obtains

$$\delta(l)^{-1} = h_1g_1 \triangleright h_2h_1^{-1}(\partial(h_1)g_1) \triangleright h_2{}^{-1} = \langle h_1, g_1 \triangleright h_2\rangle_{\mathrm{p}} = \delta(\{h_1, g_1 \triangleright h_2\}_{\mathrm{p}})\,. \tag{3.23}$$

Given any collection of curves, surfaces, and volumes, a configuration of 3-gauge theory is an assignment of elements of G to the curves, elements of H to the surfaces, and elements of L to volumes so that the following conditions hold:

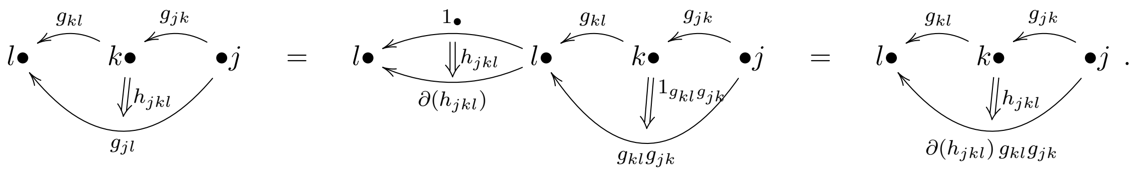
1. For each surface labeled by $h \in H$, one has that $\partial(h) = g_2g_1^{-1}$ where g_1 and g_2 are the source and target curve, respectively;
2. For each volume labeled by $l \in L$, one has that $\delta(l) = h_2h_1^{-1}$, where h_1 and h_2 are the source and target surface, respectively;
3. For each 4-simplex labeled by $(jk\ell mn) \in \Lambda_4$, the volume holonomy around it is trivial.

The defined configurations can be viewed as the classical configurations of 3-gauge theory or, in a path integral quantum theory, these are the configurations over which one integrates in the path integral.

3.3 Gauge invariant quantities

In subsection 3.2, we have introduced a number of operations by which we can combine labeled paths, surfaces, and volumes, in order to calculate the composition of elementary paths, surfaces, and volumes, to arbitrarily large ones. In this subsection, we will make use of these compositions in order to construct gauge invariant quantities that are associated with closed paths, surfaces, and volumes. In Lemmas 3.1, 3.2, and 3.3, this procedure is used for the boundary path of a triangle, the boundary surface of a tetrahedron, and the boundary volume of the 4-simplex. The result of Lemma 3.1 is already derived for the case of 2-groups and remains unchanged in the 3-gauge theory, see [38]. The higher flatness condition for the boundary surface of a tetrahedron derived in [38], is generalized for the case of 3-groups is Lemma 3.2. One of the main results of the paper is Lemma 3.3 where we derived the higher flatness condition for the boundary volume of the 4-simplex.

Lemma 3.1. Let us consider a triangle, $(jk\ell)$. The edges $(jk)\,, j<k$, are labeled by group elements $g_{jk} \in G$ and the triangle $(jk\ell)\,, j<k<\ell$, by element $h_{jk\ell} \in H$. Consider the diagram (3.24).



(3.24)

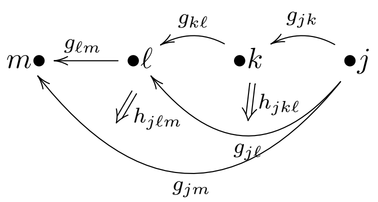
The curve $\gamma_1 = g_{k\ell}g_{jk}$ is the source and the curve $\gamma_2 = g_{j\ell}$ is the target of the surface morphism $\Sigma : \gamma_1 \to \gamma_2$, labeled by the group element $h_{jk\ell}$, i.e.,

$$g_{j\ell} = \partial(h_{jk\ell})g_{k\ell}g_{jk}\,. \tag{3.25}$$

Lemma 3.2. Let us consider a tetrahedron, $(jk\ell m)$. The edges $(jk)\,, j < k$, are labeled by group elements $g_{jk} \in G$ and the triangles $(jk\ell)\,, j < k < \ell$, by elements $h_{jk\ell} \in H$, and the tetrahedron $(jk\ell m)\,, j < k < \ell < m$ by the group element $l_{jk\ell m} \in L$. We have oriented the triangles $(jk\ell)$ so that they have the source is $g_{k\ell}g_{jk}$ and the target is $g_{j\ell}$, i.e. $g_{j\ell} = \partial(h_{jk\ell})g_{k\ell}g_{jk}\,.$

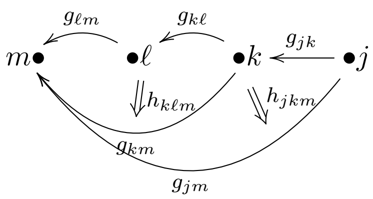
Let us first cut the tetrahedron surface along the edge (jm). This determines the ordering of the vertical composition of the constituent surfaces. One just has to make sure that all surfaces are composable, i.e., they have the suitable reference points and the correct orientation in order to compose them vertically.

Consider the diagram (3.26). We first move the curve from $g_{k\ell}g_{jk}$ to the curve $g_{j\ell}$. At this stage, one cannot compose the result with the triangle $(j\ell m)$, and one first has to whisker it from the left by $g_{\ell m}$. Now the two morphisms are vertically composable, and this moves the curve to g_{jm}. The following 2-morphism is obtained



$$\begin{aligned} &= (g_{\ell m}g_{j\ell}, h_{j\ell m})\#_2(g_{\ell m}\#_1(g_{k\ell}g_{jk}, h_{jk\ell})) \\ &= (g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell m}(g_{\ell m} \rhd h_{jk\ell}))\,. \end{aligned} \tag{3.26}$$

Let us then consider the diagram (3.27). We first move the curve from $g_{\ell m}g_{k\ell}$ to the curve g_{km}. At this stage, one cannot compose the result with the triangle (jkm), and one first has to whisker it from the right by g_{jk}. Now the two morphisms are vertically composable, and this moves the curve to g_{jm}. The following 2-morphism is obtained



$$\begin{aligned} &= (g_{km}g_{jk}, h_{jkm})\#_2((g_{\ell m}g_{k\ell}, h_{k\ell m})\#_1 g_{jk}) \\ &= (g_{\ell m}g_{k\ell}g_{jk}, h_{jkm}h_{k\ell m})\,. \end{aligned} \tag{3.27}$$

The two surfaces have the same source and target, $\Sigma_1 : g_{\ell m}g_{k\ell}g_{jk} \to g_{jm}$ and $\Sigma_2 : g_{\ell m}g_{k\ell}g_{jk} \to g_{jm}$. Now, transition from the surface shown on the diagram (3.26) to the surface shown on the diagram (3.27) is given by the volume morphism $\mathcal{V} : \Sigma_1 \to \Sigma_2$

determined by the group element $l_{jk\ell m}$, i.e. ,

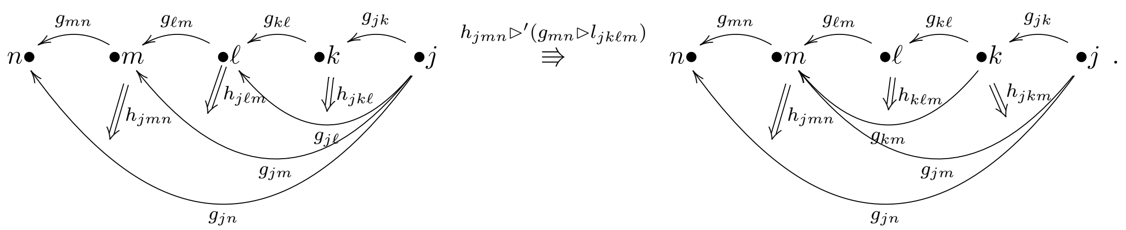
$$(g_{\ell m}g_{k\ell}g_{jk}, h_{jkm}h_{k\ell m}) = (g_{\ell m}g_{k\ell}g_{jk}, \delta(l_{jk\ell m})h_{j\ell m}(g_{\ell m} \rhd h_{jk\ell}))\,, \tag{3.28}$$

that gives the relation,

$$h_{jkm}h_{k\ell m} = \delta(l_{jk\ell m})h_{j\ell m}(g_{\ell m} \rhd h_{jk\ell})\,. \tag{3.29}$$

Lemma 3.3. Let us consider a 4-simplex, $(jk\ell mn)$. The edges $(jk)\,, j < k$, are labeled by group elements $g_{jk} \in G$, the triangles $(jk\ell)\,, j < k < \ell$, by elements $h_{jk\ell} \in H$, and the tetrahedrons $(jk\ell m)\,, j < k < \ell < m$, by the group element $l_{jk\ell m} \in L$. We have oriented the triangles $(jk\ell)$ so that the source curve is $g_{k\ell}g_{jk}$ and the target curve is $g_{j\ell}$, i.e. , $g_{j\ell} = \partial(h_{jk\ell})g_{k\ell}g_{jk}$, and the tetrahedrons $(jk\ell m)$ so that the source surface is $h_{j\ell m}(g_{\ell m} \rhd h_{jk\ell})$ and the target surface is $h_{jkm}h_{k\ell m}$, i.e. , $h_{jkm}h_{k\ell m} = \delta(l_{jk\ell m})h_{j\ell m}(g_{\ell m} \rhd h_{jk\ell})$.

Let us first cut the 4-simplex volume along the surface $h_{jmn}g_{mn} \rhd (h_{j\ell m}g_{\ell m} \rhd h_{jk\ell})$. This surface determines the ordering of the vertical composition of the constituent volumes. We have to make sure that all volumes are composable, i.e. , they have the suitable reference points and the correct orientation in order to compose them vertically. First, let us consider the diagram (3.30). We first move the surface from $h_{j\ell m}g_{\ell m} \rhd h_{jk\ell}$ to surface $h_{jkm}h_{k\ell m}$ with the 3-arrow $l_{jk\ell m}$. To compose the resulting 3-morphism with the surface h_{jmn} one must first whisker it from the left with g_{mn}. The obtained 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{mn} \rhd (h_{j\ell m}g_{\ell m} \rhd h_{jk\ell}), g_{mn} \rhd l_{jk\ell m})$ can be whiskered from below with the 2-morphism $(g_{mn}g_{jm}, h_{jmn})$, and the resulting 3-morphism is $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \rhd (h_{j\ell m}g_{\ell m} \rhd h_{jk\ell}), h_{jmn} \rhd' (g_{mn} \rhd l_{jk\ell m}))$, with the source surface $h_{jmn}g_{mn} \rhd (h_{j\ell m}g_{\ell m} \rhd h_{jk\ell})$ and the target surface $h_{jmn}g_{mn} \rhd (h_{jkm}h_{k\ell m})$,



(3.30)

Let us move the surface to $h_{jkn}h_{kmn}g_{m\ell} \rhd h_{k\ell m}$, see diagram (3.31). To do that, we consider the 3-morphism $(g_{mn}g_{km}g_{jk}, h_{jmn}g_{mn} \rhd h_{jkm}, l_{jkmn})$ with the source surface $h_{jmn}g_{mn} \rhd h_{jkm}$ and target surface $h_{jkn}h_{kmn}$. This 3-morphism can be whiskered from above with the 2-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{mn} \rhd h_{k\ell m})$, and the obtained 3-morphism is $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}\, g_{mn} \rhd (h_{jkm}h_{k\ell m}), l_{jkmn})$, with the source surface $h_{jmn}g_{mn} \rhd (h_{jkm}h_{k\ell m})$ and target surface

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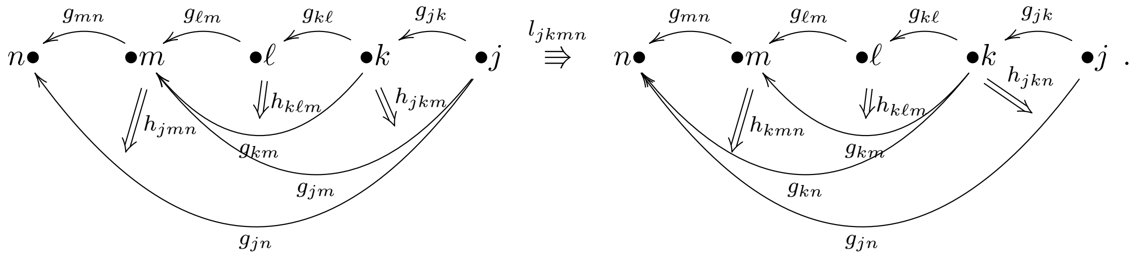
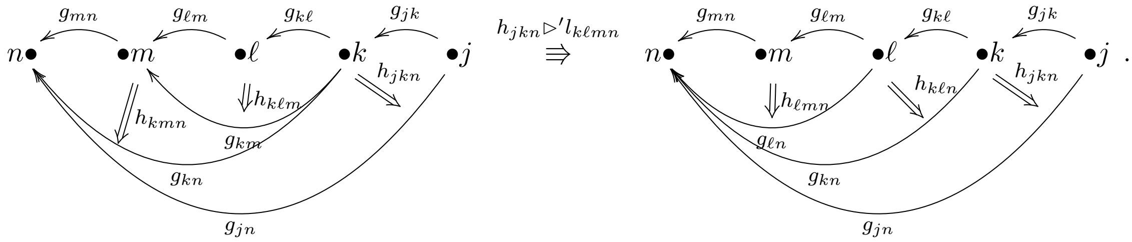
$h_{jkn}h_{kmn}g_{mn} \triangleright h_{k\ell m}$,



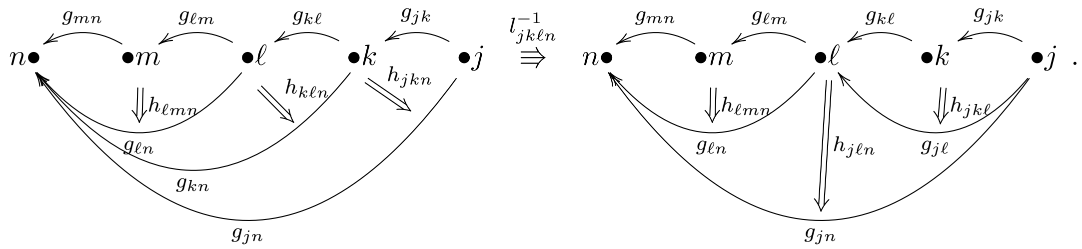
(3.31)

Next, we want to move the surface $h_{jkn}h_{kmn}g_{mn} \triangleright h_{k\ell m}$ to surface $h_{jkn}h_{k\ell n}h_{\ell mn}$, as shown on the diagram (3.32). We whisker the 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}, h_{kmn}g_{mn} \triangleright h_{k\ell m}, l_{k\ell mn})$, with the source surface $h_{kmn}g_{mn} \triangleright h_{k\ell m}$ and target surface $h_{k\ell n}h_{\ell mn}$, with the morphism g_{jk} from the right, obtaining the 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{kmn}g_{mn} \triangleright h_{k\ell m}, l_{k\ell mn})$. Now, we whisker this 3-morphism with the 2-morphism $(g_{kn}g_{jk}, h_{jkn})$ from below, and we obtain the 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{kmn}g_{mn} \triangleright h_{k\ell m}, h_{jkn} \triangleright' l_{k\ell mn})$,



(3.32)

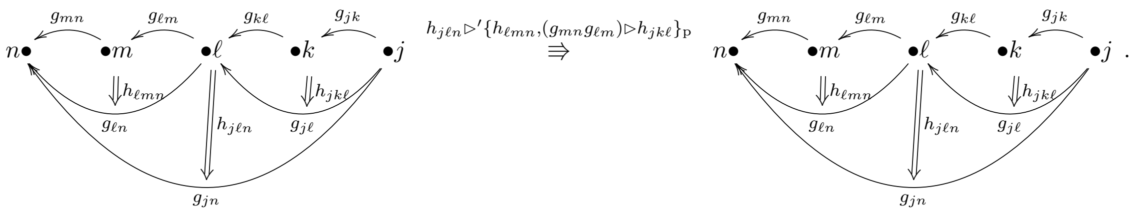
The mapping of the surface $h_{jkn}h_{k\ell n}h_{\ell mn}$ to the surface $h_{j\ell n}g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}$ in shown on the diagram (3.33). The 3-morphism with the appropriate source and target is constructed by whiskering the 3-morphism $(g_{\ell n}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}, l^{-1}_{jk\ell n})$ with 2-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{\ell mn})$ from above. The obtained 3-morphism is $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}h_{\ell mn}, l^{-1}_{jk\ell n})$,



(3.33)

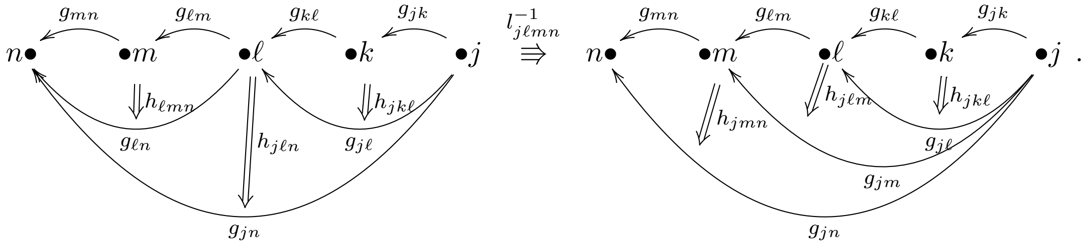
Next we map the surface $h_{j\ell n}g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}$ to the surface $h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \triangleright h_{jk\ell}$, see the diagram (3.34). We use the inverse interchanging 2-arrow composition to map the surface $g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}$ to the surface $h_{\ell mn}(g_{mn}g_{\ell m}) \triangleright h_{jk\ell}$, resulting in the 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}, \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\mathrm{p}})$. Next, we whisker the obtained 3-morphism with the 2-morphism $(g_{\ell n}g_{j\ell}, h_{j\ell n})$ from below. The obtained 3-morphism with the appropriate source and target surfaces is $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell n}g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}, h_{j\ell n} \triangleright'$

$\{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\rm p})$,



(3.34)

Finally, we construct the 3-morphism that maps the surface $h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \triangleright h_{jk\ell}$ to the starting surface $h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$. To obtain the 3-morphism with the appropriate source and target surfaces we first move the surface $h_{j\ell n}h_{\ell mn}$ to the surface $h_{jmn}g_{mn} \triangleright h_{j\ell m}$ with the 3-arrow $(g_{mn}g_{\ell m}g_{j\ell}, h_{j\ell n}h_{\ell mn}, l^{-1}_{j\ell mn})$. Next, we whisker the 3-morphism $(g_{mn}g_{\ell m}g_{j\ell}, h_{j\ell n}h_{\ell mn}, l^{-1}_{j\ell mn})$ with the 2-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell})$ from above. The obtained 3-morphism $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \triangleright h_{jk\ell}, l^{-1}_{j\ell mn})$ moves the surface to the starting surface, as shown on the diagram (3.35),



(3.35)

After the upward composition of the 3-morphisms given by the diagrams (3.30)–(3.35), the obtained 3-morphism is:

$$
\begin{aligned}
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \triangleright h_{jk\ell}, l^{-1}_{j\ell mn})\#_3\\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{\ell n} \triangleright h_{jk\ell}h_{\ell mn}, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\rm p})\#_3\\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}h_{\ell mn}, l^{-1}_{jk\ell n})\#_3\\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{kmn}g_{m\ell} \triangleright h_{k\ell m}, h_{jkn} \triangleright' l_{jkmn})\#_3\\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{jkm}h_{k\ell m}), l_{jkmn})\#_3\\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}), h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}))\\
=&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell}), l^{-1}_{j\ell mn}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\rm p}\\
&l^{-1}_{jk\ell n}(h_{jkn} \triangleright' l_{k\ell mn})l_{jkmn}h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}))\,.
\end{aligned}
\tag{3.36}
$$

The obtained 3-morphism is the identity morphism with source and target surface $\mathcal{V}_1 = \mathcal{V}_2 = h_{jmn}g_{mn} \triangleright (h_{j\ell m}g_{\ell m} \triangleright h_{jk\ell})$, i.e.,

$$
l^{-1}_{j\ell mn}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\rm p}\, l^{-1}_{jk\ell n}(h_{jkn} \triangleright' l_{k\ell mn})l_{jkmn}h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m}) = e\,.
\tag{3.37}
$$

4 Quantization of the topological $3BF$ theory

In conventional BF theory, one chooses the action in such a way that the theory does not depend on any background field, but only the spacetime manifold. The classical field equations of the theory require the gauge connection to be flat, i.e., in terms of the holonomy variables, that any null-homotopic closed curve corresponds to the identity of the gauge group. In the framework of higher gauge theory, specifically 2-gauge theory, one generalizes this idea by imposing the *higher flatness condition* requiring that the surface holonomy around the boundary 2-sphere of any 3-ball be trivial instead. One can continue further categorical generalization by choosing a 3-group structure to describe the gauge symmetry of the theory, and formulate a $3BF$ theory whose equations of motion impose a higher flatness condition for a 3-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$. In this section, a combinatorial description of such model for any triangulation of any smooth manifold of dimension $d = 4$ is presented. This model coincides with Porter's abstract definition of a TQFT [33] for $d = 4$ and $n = 3$, which is itself a generalization of Yetter's work [48, 49].

Let us show how to construct a state sum model from the classical action (2.8) by the usual heuristic spinfoam quantization procedure. We consider the path integral for the action S_{3BF},

$$Z = \int \mathcal{D}\alpha\, \mathcal{D}\beta\, \mathcal{D}\gamma\, \mathcal{D}B\, \mathcal{D}C\, \mathcal{D}D\ \exp\left(i \int_{M_4} \langle B \wedge \mathcal{F}\rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G}\rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H}\rangle_{\mathfrak{l}} \right) . \tag{4.1}$$

The formal integration over the Lagrange multipliers B, C, and D leads to:

$$Z = \mathcal{N} \int \mathcal{D}\alpha\, \mathcal{D}\beta\, \mathcal{D}\gamma\ \delta(\mathcal{F})\delta(\mathcal{G})\delta(\mathcal{H}) . \tag{4.2}$$

Similarly to conventional gauge theory, the connection 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$ is discretized by colouring the edges $\epsilon = (jk) \in \Lambda_1$ of the triangulation with group elements $g_\epsilon \in G$. The connection 2-form $\beta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{h})$ is represented by group elements $h_\Delta \in H$ coloring the triangles $\Delta = (jk\ell) \in \Lambda_2$. The connection 3-form $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,, \mathfrak{l})$ is represented by group elements $l_\tau \in L$ coloring the tetrahedrons $\tau = (jk\ell m) \in \Lambda_3$.

The path integral measures of (4.1) are discretized by replacing

$$\int \mathcal{D}\alpha \quad \mapsto \quad \prod_{(jk)\in\Lambda_1} \int_G dg_{jk}\,, \tag{4.3}$$

$$\int \mathcal{D}\beta \quad \mapsto \quad \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell}, \tag{4.4}$$

$$\int \mathcal{D}\gamma \quad \mapsto \quad \prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m}\,, \tag{4.5}$$

where dg_{jk}, $dh_{jk\ell}$, and $dl_{jk\ell m}$ denote integration with respect to the Haar measures of G, H, and L, respectively. The vanishing fake curvature condition is discretized on each triangle $(jkl) \in \Lambda_2$ by discretizing $\delta(\mathcal{F})$. When passing from a smooth manifold to its triangulation, the δ distribution is defined over the appropriate set of simplices as follows,

$$\delta(\mathcal{F}) = \prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell})\,, \tag{4.6}$$

where for each $(jkl) \in \Lambda_2$ the δ-function $\delta_G(g_{jkl})$ is given by:

$$\delta_G(g_{jk\ell}) = \delta_G\big(\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk}\, g_{j\ell}^{-1}\big)\,. \tag{4.7}$$

Similarly, on the triangulated manifold the condition $\delta(\mathcal{G})$ on the fake curvature 3-form reads

$$\delta(\mathcal{G}) = \prod_{(jk\ell m)\in\Lambda_3} \delta_H(h_{jk\ell m})\,, \tag{4.8}$$

where for every tetrahedron $(jk\ell m) \in \Lambda_3$ one has:

$$\delta_H(h_{jk\ell m}) = \delta_H\big(\delta(l_{jk\ell m})h_{j\ell m}\,(g_{\ell m} \triangleright h_{jk\ell})\, h_{k\ell m}^{-1}\, h_{jkm}^{-1}\big)\,. \tag{4.9}$$

Finally, the condition $\delta(\mathcal{H})$ is discretized as

$$\delta(\mathcal{H}) = \prod_{(jk\ell mn)\in\Lambda_4} \delta_L(l_{jk\ell mn})\,, \tag{4.10}$$

where for each 4-simplex $(jk\ell mn) \in \Lambda_4$ one has:

$$\delta_L(l_{jk\ell mn}) = \delta_L\big(l_{j\ell mn}^{-1}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\rm p}\, l_{jk\ell n}^{-1} (h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m})\big)\,. \tag{4.11}$$

The identities (4.7), (4.9), and (4.11) are the results of Lemmas 3.1, 3.2, and 3.3, respectively.

After substituting the expressions for discretized measures (4.3)–(4.5) and δ-functions (4.6), (4.8), and (4.10) into the equation (4.2) one obtains:

$$Z = \mathcal{N} \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell} \prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m} \Big(\prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell})\Big) \Big(\prod_{(jk\ell m)\in\Lambda_3} \delta_H(h_{jk\ell m})\Big) \Big(\prod_{(jk\ell mn)\in\Lambda_4} \delta_L(l_{jk\ell mn})\Big). \tag{4.12}$$

By inserting (4.7), (4.9), and (4.11) into (4.12), we obtain an explicit expression for the state sum over a given triangulation of the manifold $\mathcal{M}_4$. This expression can be made independent of the triangulation if one appropriately chooses the constant factor $\mathcal{N}$, obtained after the integration over the Lagrange multipliers B, C, and D. This is done by requiring that the state sum is invariant under the Pachner moves, which leads us to the appropriate form of the constant factor $\mathcal{N}$, as given by the definition 4.1.

Definition 4.1. Let $\mathcal{M}_4$ be a compact and oriented combinatorial d-manifold, $d = 4$, and $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\}_{\rm pf})$ be a 2-crossed module. The state sum of *topological higher gauge theory* is defined by

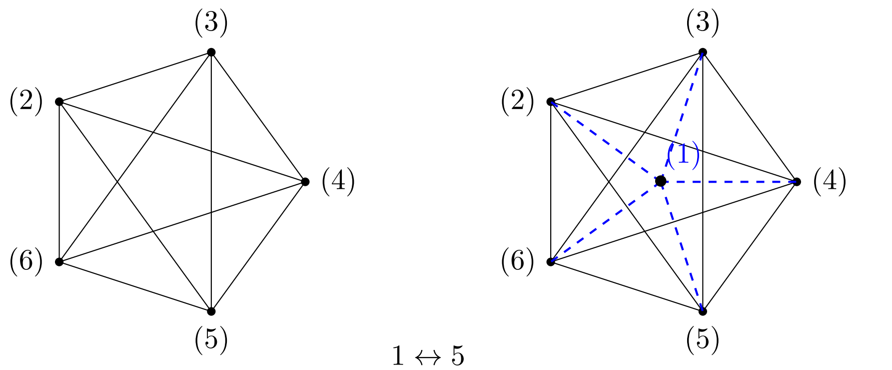
$$\begin{aligned} Z = \; & |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|} |H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}\, |L|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|+|\Lambda_3|-|\Lambda_4|} \\ & \times \Big(\prod_{(jk)\in\Lambda_1} \int_G dg_{jk}\Big) \Big(\prod_{(jk\ell)\in\Lambda_2} \int_H dh_{jk\ell}\Big) \Big(\prod_{(jk\ell m)\in\Lambda_3} \int_L dl_{jk\ell m}\Big) \\ & \times \Big(\prod_{(jk\ell)\in\Lambda_2} \delta_G\big(\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk}\, g_{j\ell}^{-1}\big)\Big) \Big(\prod_{(jk\ell m)\in\Lambda_3} \delta_H\big(\delta(l_{jk\ell m})h_{j\ell m}\,(g_{\ell m} \triangleright h_{jk\ell})\, h_{k\ell m}^{-1}\, h_{jkm}^{-1}\big)\Big) \\ & \times \Big(\prod_{(jk\ell mn)\in\Lambda_4} \delta_L\big(l_{j\ell mn}^{-1}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \triangleright h_{jk\ell}\}_{\rm p}\, l_{jk\ell n}^{-1} (h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m})\big)\Big)\,. \end{aligned} \tag{4.13}$$

Here we integrate over $g_{jk} \in G$ for every edge $(jk) \in \Lambda_1$, over $h_{jk\ell} \in H$ for every triangle $(jk\ell) \in \Lambda_2$ and over l_{jklm} for every tetrahedron $(jk\ell m) \in \Lambda_3$. The δ-distributions under the integral impose the following conditions. First, the condition that $\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk} = g_{j\ell}$ for each triangle $(jk\ell) \in \Lambda_2$, i.e., that each surface label $h_{jk\ell}$ has got the appropriate source and target, see Lemma 3.1. Second, the condition that $h_{jkm}\, h_{k\ell m} = \delta(l_{jk\ell m}) h_{j\ell m}\, (g_{\ell m} \triangleright h_{jk\ell})$ for each tetrahedron $(jk\ell m) \in \Lambda_3$, i.e., that each volume label $l_{jk\ell m}$ has got the appropriate source and target, see Lemma 3.2. Finally, the condition that the volume holonomy around every 4-simplex $(jk\ell mn) \in \Lambda_4$ is trivial, i.e., that $l^{-1}_{j\ell mn}\, h_{j\ell n} \triangleright' \{h_{\ell mn}, (g_{mn} g_{\ell m}) \triangleright h_{jk\ell}\}_{\text{p}}\, l^{-1}_{jk\ell n} (h_{jkn} \triangleright' l_{k\ell mn}) l_{jkmn} h_{jmn} \triangleright' (g_{mn} \triangleright l_{jk\ell m})$ is equal to the neutral element of the group L for each 4-simplex $(jk\ell mn) \in \Lambda_4$, see Lemma 3.3.

Theorem 4.2. Let $\mathcal{M}_4$ be a closed and oriented combinatorial 4-manifold and $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\}_{\text{pf}})$ be a 2-crossed module. The state sum (4.13) is invariant under Pachner moves.

The statements of Pachner move invariance are formulated in the following subsections, while corresponding proofs are given in the appendix B.

4.1 Pachner move $1 \leftrightarrow 5$



$1 \leftrightarrow 5$

Let us verify that the state sum (4.13) is invariant under $1-5$ Pachner move. Since the partition function is independent of the total order of vertices, let us fix the ordering and verify the move in only one case. Let us denote the vertices of the 4-simplex on the left hand side of the $1-5$ Pachner move as (23456). Then, adding a vertex 1 on the right hand side of the Pachner move one obtains five 4-simplices $M_4 = \{(13456), (12456), (12356), (12346), (12345)\}$. On the r.h.s. there are tetrahedrons $M_3 = \{(1234), (1235), (1236), (1245), (1246), (1256), (1345), (1346), (1356), (1456)\}$, triangles $(jk\ell) \in M_2 = \{(123), (124), (125), (126), (134), (135), (136), (145), (146), (156)\}$, edges $(jk) \in M_1 = \{(12), (13), (14), (15), (16)\}$ and vertices $(j) \in M_0 = \{(1)\}$. All other simplices are present on both sides of the move.

	$\lvert\Lambda_0\rvert$	$\lvert\Lambda_1\rvert$	$\lvert\Lambda_2\rvert$	$\lvert\Lambda_3\rvert$	$\lvert\Lambda_4\rvert$
l.h.s.	5	10	10	5	1
r.h.s.	6	15	20	15	5

Table 1. Number of vertices $|\Lambda_0|$, edges $|\Lambda_1|$, triangles $|\Lambda_2|$, tetrahedrons $|\Lambda_3|$, and 4-simplices $|\Lambda_4|$ on both sides of the $1 \leftrightarrow 5$ move.

If the $1-5$ Pachner move does not change the state sum (4.13), then the state sum of the right hand side,

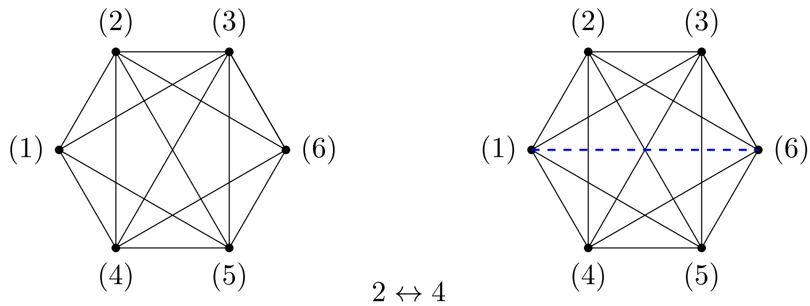
$$Z^{1\leftrightarrow 5}_{\text{right}} = |G|^{-11}|H|^{-4}|L|^{-1} \int_{G^5} \prod_{(jk)\in M_1} dg_{jk} \int_{H^{10}} \prod_{(jk\ell)\in M_2} dh_{jk\ell} \int_{L^{10}} \prod_{(jklm)\in M_3} dl_{jklm} \cdot \left(\prod_{(jk\ell)\in M_2} \delta_G(g_{jk\ell}) \right) \left(\prod_{(jk\ell m)\in M_3} \delta_H(h_{jk\ell m}) \right) \left(\prod_{(jk\ell mn)\in M_4} \delta_L(l_{jk\ell mn}) \right) Z_{\text{remainder}} \,, \tag{4.14}$$

should be equal to the state sum of the left hand side,

$$Z^{1\leftrightarrow 5}_{\text{left}} = |G|^{-5}|H|^{0}|L|^{-1}\delta_L(l_{23456}) Z_{\text{remainder}} \,. \tag{4.15}$$

Here, the prefactors $|G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}$, $|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}$, and $|L|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|+|\Lambda_3|-|\Lambda_4|}$ are $|G|^{-11}|H|^{-4}|L|^{-1}$ on the r.h.s. and $|G|^{-5}|H|^{0}|L|^{-1}$ on the l.h.s., as obtained by counting the numbers of the k-simplices on both sides of the $1-5$ move, shown in the table 1. The $Z_{\text{remainder}}$ denotes the part of the state sum that is the same on both sides of the move, and thus irrelevant for the proof of invariance. The proof that $Z_{\text{left}} = Z_{\text{right}}$ is given in the appendix B.

4.2 Pachner move $2 \leftrightarrow 4$



$2 \leftrightarrow 4$

In order to verify the state sum (4.13) invariance under $2-4$ Pachner move, we order the vertices in such a way that on the l.h.s. of the move we have two 4-simplices $M_4^{\text{left}} = \{(23456), (12345)\}$, while on the r.h.s. we have four 4-simplices $M_4^{\text{right}} = \{(12346), (12356), (12456), (13456)\}$. On the l.h.s. we have one tetrahedron $M_3^{\text{left}} = \{(2345)\}$, whereas on the r.h.s. there are six tetrahedrons $M_3^{\text{right}} = \{(1236), (1246), (1256), (1346), (1356), (1456)\}$. All other tetrahedrons appear on both sides of the move. On the r.h.s. there are triangles $M_2^{\text{right}} = \{(126), (136), (146), (156)\}$, and one edge $M_1^{\text{right}} = \{(16)\}$, while the rest of the triangles and edges appear on both sides of the move.

	$\lvert\Lambda_0\rvert$	$\lvert\Lambda_1\rvert$	$\lvert\Lambda_2\rvert$	$\lvert\Lambda_3\rvert$	$\lvert\Lambda_4\rvert$
l.h.s.	6	14	16	9	2
r.h.s.	6	15	20	14	4

Table 2. Number of vertices $|\Lambda_0|$, edges $|\Lambda_1|$, triangles $|\Lambda_2|$, tetrahedrons $|\Lambda_3|$, and 4-simplices $|\Lambda_4|$ on both sides of the $2 \leftrightarrow 4$ move.

On the l.h.s. there is the state sum,

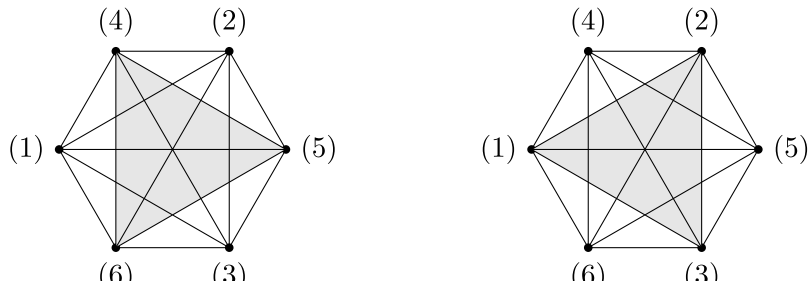
$$Z^{2\leftrightarrow 4}_{\text{left}} = |G|^{-8}|H|^{-1}|L|^{-1} \int_L dl_{2345}\delta_H(h_{2345}) \Bigg(\prod_{(jk\ell mn)\in M_4^{\text{left}}} \delta_L(l_{jk\ell mn}) \Bigg) Z_{\text{remainder}} \,, \qquad (4.16)$$

whereas on the r.h.s. the state sum reads:

$$Z^{2\leftrightarrow 4}_{\text{right}} = |G|^{-11}|H|^{-3}|L|^{-1} \int_G dg_{16} \int_{H^4} dh_{126}dh_{136}dh_{146}dh_{156} \int_L dl_{1236}dl_{1246}dl_{1256}dl_{1346}dl_{1356}dl_{1456}$$
$$\Bigg(\prod_{(jk\ell)\in M_2^{\text{right}}} \delta_G(g_{jk\ell}) \Bigg) \Bigg(\prod_{(jk\ell m)\in M_3^{\text{right}}} \delta_H(h_{jk\ell m}) \Bigg) \Bigg(\prod_{(jk\ell mn)\in M_4^{\text{right}}} \delta_L(l_{jk\ell mn}) \Bigg) Z_{\text{remainder}} . \qquad (4.17)$$

Here the prefactors $|G|^{-8}|H|^{-1}|L|^{-1}$ on the l.h.s. and $|G|^{-11}|H|^{-3}|L|^{-1}$ on the r.h.s. are obtained by counting the numbers of k-simplices on both sides of the $2-4$ move, as shown in the table 2. The term $Z_{\text{remainder}}$ denotes the part of the state sum that is identical on both sides of the move, as before. The proof that $Z_{\text{left}} = Z_{\text{right}}$ is given in the appendix B.

4.3 Pachner move $3 \leftrightarrow 3$



$3 \leftrightarrow 3$

In order to verify the state sum invariance under $3-3$ Pachner move, we order the vertices in such a way that on the l.h.s. of the $3-3$ move, we have three 4-simplices $M_4^{\text{left}} = \{(23456), (13456), (12456)\}$, whereas on the r.h.s. we have the 4-simplices $M_4^{\text{right}} = \{(12356), (12346), (12345)\}$. On the l.h.s. there are tetrahedrons $M_3^{\text{left}} = \{(1456), (2456), (3456)\}$, and on the r.h.s. $M_3^{\text{right}} = \{(1234), (1235), (1236)\}$. One notices that the six tetrahedrons form the common boundary of both sides of the move, whereas on each side there are three tetrahedrons shared by two 4-simplices. On the l.h.s. one has the triangle $M_2^{\text{left}} = \{(456)\}$ and on the r.h.s. the triangle $M_3^{\text{right}} = \{(123)\}$. All other triangles appear on both sides of the move.

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Therefore on the l.h.s. there is the state sum,

$$Z^{3\leftrightarrow 3}_{\text{left}} = \int_H dh_{456} \int_{L^3} dl_{1456} dl_{2456} dl_{3456} \delta_G(g_{456})\, \delta_H(h_{3456}) \delta_H(h_{2456}) \delta_H(h_{1456}) \\ \delta_L(l_{23456}) \delta_L(l_{13456}) \delta_L(l_{12456}) Z_{\text{remainder}}\,, \tag{4.18}$$

whereas on the r.h.s. the state sum reads

$$Z^{3\leftrightarrow 3}_{\text{right}} = \int_H dh_{123} \int_{L^3} dl_{1234} dl_{1235} dl_{1236} \delta_G(g_{123})\, \delta_H(h_{1234}) \delta_H(h_{1235}) \delta_H(h_{1236}) \\ \delta_L(l_{12356}) \delta_L(l_{12346}) \delta_L(l_{12345}) Z_{\text{remainder}}\,. \tag{4.19}$$

The numbers of k-simplices agree on both sides of the $3-3$ move for all k, and the prefactors play no role in this case, therefore they are part of the $Z_{\text{remainder}}$. The proof that $Z_{\text{left}} = Z_{\text{right}}$ is given in the appendix B.

We obtain that the state sum given by the definition 4.1 is invariant under all three Pachner moves, and thus independent of triangulation of the underlying 4-dimensional manifold (see appendix B for the proof).

5 Conclusions

Let us summarize the results of the paper. In section 2 we reviewed the pure the constrained $2BF$ actions describing the Yang-Mills field and Einstein-Cartan gravity, and constrained $3BF$ actions describing the Klein-Gordon and Dirac fields coupled to Yang-Mills fields and gravity in the standard way. In section 3, we reviewed the relevant algebraic tools involved in the description of higher gauge theory, 2-crossed modules, and 3-gauge theory and generalized the integral picture of an ordinary gauge theory to a 3-gauge theory that involves curves, surfaces, and volumes labeled with elements of non-Abelian groups. We have also proved three key results, stated in Lemmas 3.1, 3.2, and 3.3, which are crucial for the construction of the invariant state sum. In section 4, we have presented the two main results of the paper. First, we constructed a triangulation independent state sum Z of a topological higher gauge theory for a general 3-group and a 4-dimensional spacetime manifold $\mathcal{M}_4$. Second, we proved the theorem that the constructed state sum is indeed independent of the choice of triangulation, i.e., that it is a genuine topological invariant.

The constructed state sum coincides with Porter's TQFT [33, 34] for $d = 4$ and $n = 3$. The proof that the state sum is invariant under the local changes of triangulation called the Pachner moves and thus independent of the chosen triangulation is presented in appendix B. It is obtained that the state sum is invariant under all five different Pachner moves: the $3-3$ move, $4-2$ move, and $5-1$ move, and their inverses. The state sum constructed this way can be thought of as a combinatorial construction of a topological quantum field theory (TQFT) in the sense of Atiyah's axioms, a topic that is beyond the scope of this paper and will be studied in a future work.

In order to finish the second step of the spinfoam quantization procedure, however, the generalizations of the Peter-Weyl and Plancharel theorems to 2-groups and 3-groups are required, which so far represent open problems. Namely, these theorems should provide a decomposition of a function on a 3-group into a sum over the corresponding irreducible representations of a 3-group. In this way, the spectrum of labels for the simplices, i.e. , the domain of values of the fields living on the simplices of the triangulation, would be specified. Nonetheless, one can still try to guess the irreducible representations of 3-groups, as was done for example in the case of 2-groups in the spincube model of quantum gravity [30], or obtain the state sum using other techniques, see for example [50–52]).

However, if one wants to describe a real physical theory, i.e. , the theory which contains local propagating degrees of freedom, one needs to construct the nontopological state sum, with the non-trivial dynamics. To do so, once the topological state sum is constructed, the final third step of the spinfoam quantization procedure is to impose the constraints that deform the topological theory into a realistic theory of gravity coupled to matter fields (as defined in [31]) at the quantum level. We leave the construction of the constrained state sum model for future work.

In addition to the above topics, there are also many other possible applications of the invariant state sum, both in physics and mathematics.

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A Proof of the invariance identity

Let us prove the identity (3.16). Using the definitions of the upward composition (3.14) and the vertical composition (3.15) of the 3-morphisms, one obtains that the left-hand side of the equation (3.16) is equal to:

$$\begin{aligned}((g_2,h'_2,l'_2)\#_3(g_2,h_2,l_2))\#_2((g_1,h'_1,l'_1)\#_3(g_1,h_1,l_1)) &= (g_2,h_2,l'_2l_2)\#_2(g_1,h_1,l'_1l_1) \\ &= (g_1,h_2h_1,l'_2l_2\,h_2\rhd'(l'_1l_1))\,.\end{aligned} \tag{A.1}$$

The right-hand side of the equation (3.16) is equal to:

$$\begin{aligned}((g_2,h'_2,l'_2)\#_2(g_1,h'_1,l'_1))\#_3((g_2,h_2,l_2)\#_2(g_1,h_1,l_1)) &= (g_1,h'_2h'_1,l'_2h'_2\rhd' l'_1)\#_3(g_1,h_2h_1,l_2h_2\rhd' l_1) \\ &= (g_1,h_2h_1,l'_2\,h'_2\rhd' l'_1\,l_2\,h_2\rhd' l_1) \quad (h'_2=\delta(l_2)h_2) \\ &= (g_1,h_2h_1,l'_2\,(\delta(l_2)h_2)\rhd' l'_1\,l_2\,h_2\rhd' l_1) \quad \text{eq. (A.3)} \\ &= (g_1,h_2h_1,l'_2\,\delta(l_2)\rhd'(h_2\rhd' l'_1)\,l_2\,h_2\rhd' l_1) \quad \text{(Peiffer identity)} \\ &= (g_1,h_2h_1,l'_2\,l_2(h_2\rhd' l'_1)l_2^{-1}\,l_2\,h_2\rhd' l_1) \quad (l_2^{-1}l_2=e) \\ &= (g_1,h_2h_1,l'_2\,l_2h_2\rhd' l'_1\,h_2\rhd' l_1) \quad \text{eq. (A.4)} \\ &= (g_1,h_2h_1,l'_2\,l_2h_2\rhd'(l'_1l_1))\,,\end{aligned} \tag{A.2}$$

where in the third and sixth line we use the identities

$$(h_1 h_2) \rhd' l = h_1 \rhd' (h_2 \rhd' l), \qquad \forall h_1, h_2 \in H, \qquad \forall l \in L\,, \tag{A.3}$$
$$h \rhd' (l_1 l_2) = h \rhd' l_1 \; h \rhd' l_2, \qquad \forall h \in H, \qquad \forall l_1, l_2 \in L\,. \tag{A.4}$$

This proves the equation (3.16).

B Proof of Pachner move invariance

In this section, a self contained proof in terms of Pachner moves that the partition function (4.13) is independent of the chosen triangulation is presented.

B.1 Pachner move $1 \leftrightarrow 5$

On the *left hand side of the move* there is the integrand $\delta_L(l_{23456})$:

$$\delta_L(l_{23456}) = \delta_L(l_{2346}{}^{-1}(h_{236} \rhd' l_{3456})l_{2356}h_{256} \rhd' (g_{56} \rhd l_{2345})l_{2456}{}^{-1}h_{246} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{234}\}_{\mathrm{p}}). \tag{B.1}$$

Let us examine the *right hand side of the move*, given by the equation (4.14). First, one integrates out g_{12} using $\delta_G(g_{123})$, g_{13} using $\delta_G(g_{134})$, g_{14} using $\delta_G(g_{145})$, and g_{15} using $\delta_G(g_{156})$, and obtains:

$$\begin{aligned} g_{12} &= g_{23}^{-1}\,\partial(h_{123})^{-1}\,g_{13}\,, \\ g_{13} &= g_{34}^{-1}\,\partial(h_{134})^{-1}\,g_{14}\,, \\ g_{14} &= g_{45}^{-1}\,\partial(h_{145})^{-1}\,g_{15}\,, \\ g_{15} &= g_{56}^{-1}\,\partial(h_{156})^{-1}\,g_{16}\,. \end{aligned} \tag{B.2}$$

One integrates out h_{123} using $\delta_H(h_{1234})$, h_{124} using $\delta_H(h_{1245})$, h_{125} using $\delta_H(h_{1256})$, h_{134} using $\delta_H(h_{1345})$, h_{135} using $\delta_H(h_{1356})$, and h_{145} using $\delta_H(h_{1456})$, and obtains:

$$\begin{aligned} h_{123} &= g_{34}^{-1} \rhd h_{134}^{-1}\, g_{34}^{-1} \rhd \delta(l_{1234})^{-1}\, g_{34}^{-1} \rhd h_{124}\, g_{34}^{-1} \rhd h_{234}\,, \\ h_{124} &= g_{45}^{-1} \rhd h_{145}^{-1}\, g_{45}^{-1} \rhd \delta(l_{1245})^{-1}\, g_{45}^{-1} \rhd h_{125}\, g_{45}^{-1} \rhd h_{245}\,, \\ h_{125} &= g_{56}^{-1} \rhd h_{156}^{-1}\, g_{56}^{-1} \rhd \delta(l_{1256})^{-1}\, g_{56}^{-1} \rhd h_{126}\, g_{56}^{-1} \rhd h_{256}\,, \\ h_{134} &= g_{45}^{-1} \rhd h_{145}^{-1}\, g_{45}^{-1} \rhd \delta(l_{1345})^{-1}\, g_{45}^{-1} \rhd h_{135}\, g_{45}^{-1} \rhd h_{345}\,, \\ h_{135} &= g_{56}^{-1} \rhd h_{156}^{-1}\, g_{56}^{-1} \rhd \delta(l_{1356})^{-1}\, g_{56}^{-1} \rhd h_{136}\, g_{56}^{-1} \rhd h_{356}\,, \\ h_{145} &= g_{56}^{-1} \rhd h_{156}^{-1}\, g_{56}^{-1} \rhd \delta(l_{1456})^{-1}\, g_{56}^{-1} \rhd h_{146}\, g_{56}^{-1} \rhd h_{456}\,. \end{aligned} \tag{B.3}$$

The δ-functions on the group G now read $\delta_G(e)^6$. Let us show this. First, for $\delta_G(g_{124})$ one obtains

$$\begin{aligned} \delta_G(g_{124}) &= \delta_G\Big(\partial(h_{124})\,g_{24}\,g_{12}\,g_{14}^{-1}\Big) \\ &= \delta_G\Big(\partial(h_{124})\,g_{24}\,g_{23}^{-1}\,\partial(h_{123})^{-1}\,g_{13}\,g_{14}^{-1}\Big) \\ &= \delta_G\Big(\partial(h_{124})\,g_{24}\,g_{23}^{-1}\,g_{34}^{-1}\,\partial(h_{234})^{-1}\partial(h_{124})^{-1}\partial(h_{134})\,g_{34}\,g_{13}\,g_{14}^{-1}\Big) \\ &= \delta_G\Big(\partial(h_{124})\,g_{24}\,g_{23}^{-1}\,g_{34}^{-1}\,(g_{34}\,g_{23}^{-1}\,g_{24}^{-1})\,\partial(h_{124})^{-1}\,e\Big) \\ &= \delta_G(e)\,, \end{aligned} \tag{B.4}$$

Next, for δ-function $\delta_G(g_{125})$ one obtains,

$$\begin{aligned}
\delta_G(g_{125}) &= \delta_G\Big(\partial(h_{125})\, g_{25}\, g_{12}\, g_{15}^{-1}\Big)\,,\\
&= \delta_G\Big(\partial(h_{125})\, g_{25}\, g_{23}^{-1}\, \partial(h_{123})^{-1}\, g_{13}\, g_{15}^{-1}\Big)\\
&= \delta_G\Big(\partial(h_{125})\, g_{25}\, g_{23}^{-1}\, g_{34}^{-1}\, \partial(h_{234})^{-1}\partial(h_{124})^{-1}\partial(h_{134})\, g_{34}\, g_{13}\, g_{15}^{-1}\Big)\\
&= \delta_G\Big(\partial(h_{125})\, g_{25}\, g_{23}^{-1}\, g_{34}^{-1}\, \partial(h_{234})^{-1} g_{45}^{-1}(\partial(h_{245})^{-1}\partial(h_{125})^{-1}\partial(h_{145}))\, g_{45} g_{14}\, g_{15}^{-1}\Big)\\
&= \delta_G\Big(\partial(h_{125})\, g_{25}\, g_{23}^{-1}\, g_{34}^{-1}\, (g_{34}\, g_{23}^{-1}\, g_{24}^{-1}) g_{45}^{-1}(g_{45}\, g_{24}^{-1}\, g_{25}^{-1})\partial(h_{125})^{-1} e\Big)\\
&= \delta_G(e)\,.
\end{aligned} \tag{B.5}$$

Similarly, $\delta_G(g_{126})$ becomes

$$\begin{aligned}
\delta_G(g_{126}) &= \delta_G(\partial(h_{126}) g_{26} g_{12} g_{16}^{-1})\\
&= \delta_G(\partial(h_{126}) g_{26} g_{23}^{-1} \partial(h_{123})^{-1} g_{13} g_{16}^{-1})\\
&= \delta_G(\partial(h_{126}) g_{26} g_{23}^{-1} g_{34}^{-1} \partial(h_{234})^{-1} \partial(h_{124})^{-1} \partial(h_{134}) g_{34} g_{13} g_{16}^{-1})\\
&= \delta_G(\partial(h_{126}) g_{26} g_{23}^{-1} g_{34}^{-1} \partial(h_{234})^{-1} g_{45}^{-1} (\partial(h_{245})^{-1} \partial(h_{125})^{-1} \partial(h_{145})) g_{45} \partial(h_{134}) g_{34} g_{13} g_{16}^{-1})\\
&= \delta_G(\partial(h_{126}) g_{26} g_{23}^{-1} g_{34}^{-1} \partial(h_{234})^{-1} g_{45}^{-1} (\partial(h_{245})^{-1} g_{56}^{-1} \partial(h_{256})^{-1} \partial(h_{126})^{-1} \partial(h_{156}) g_{56}\\
&\quad \partial(h_{145})) g_{45} g_{14} g_{16}^{-1})\\
&= \delta_G(\partial(h_{126}) g_{26} g_{23}^{-1} g_{34}^{-1} (g_{34} g_{23}^{-1} g_{24}^{-1}) g_{45}^{-1} (g_{45} g_{24}^{-1} g_{25}^{-1}) g_{56}^{-1} (g_{56} g_{25}^{-1} g_{26}^{-1}) \partial(h_{126})^{-1}\\
&\quad (g_{16} g_{15}^{-1} g_{56}^{-1}) g_{56} g_{15} g_{16}^{-1})\\
&= \delta_G(e),
\end{aligned} \tag{B.6}$$

and $\delta_G(g_{135})$ now reads,

$$\begin{aligned}
\delta_G(g_{135}) &= \delta_G\Big(\partial(h_{135})\, g_{35}\, g_{13}\, g_{15}^{-1}\Big)\,,\\
&= \delta_G\Big(\partial(h_{135})\, g_{35}\, g_{34}^{-1}\, \partial(h_{134})^{-1}\, g_{14}\, g_{15}^{-1}\Big)\\
&= \delta_G\Big(\partial(h_{135})\, g_{35}\, g_{34}^{-1}\, g_{45}^{-1}\, \partial(h_{345})^{-1}\partial(h_{135})^{-1}\partial(h_{145})\, g_{45}\, g_{14}\, g_{15}^{-1}\Big)\\
&= \delta_G\Big(\partial(h_{135})\, g_{35}\, g_{34}^{-1}\, g_{45}^{-1}\, \partial(h_{345})^{-1}\partial(h_{135})^{-1}\, \partial(h_{145})\, g_{45}\, g_{45}^{-1}\, \partial(h_{145})^{-1}\, g_{15}\, g_{15}^{-1}\Big)\\
&= \delta_G\Big(\partial(h_{135})\, g_{35}\, g_{34}^{-1}\, g_{45}^{-1}\, (g_{45}\, g_{34}^{-1}\, g_{35}^{-1})\partial(h_{135})^{-1}\Big)\\
&= \delta_G(e)\,,
\end{aligned} \tag{B.7}$$

while $\delta_G(g_{136})$ reads:

$$\begin{aligned}\delta_G(g_{136}) &= \delta_G(\partial(h_{136})\, g_{36}\, g_{13}\, g_{16}^{-1}) \\ &= \delta_G(\partial(h_{136})\, g_{36}\, g_{34}^{-1}\, \partial(h_{134})^{-1}\, g_{14}\, g_{16}^{-1}) \\ &= \delta_G(\partial(h_{136})\, g_{36}\, g_{34}^{-1}\, g_{45}^{-1}\, \partial(h_{345})^{-1}\partial(h_{135})^{-1}\partial(h_{145})\, g_{45}\, g_{14}\, g_{16}^{-1}) \\ &= \delta_G(\partial(h_{136})\, g_{36}\, g_{34}^{-1}\, g_{45}^{-1}\, \partial(h_{345})^{-1} g_{56}^{-1}(\partial(h_{356})^{-1}\partial(h_{136})^{-1}\partial(h_{156}))\, g_{56}\partial(h_{145})\, g_{45}\, g_{14}\, g_{16}^{-1}) \\ &= \delta_G(\partial(h_{136})\, g_{36}\, g_{34}^{-1}\, g_{45}^{-1}\, (g_{45}\, g_{34}^{-1}\, g_{35}^{-1}) g_{56}^{-1}(g_{56}\, g_{35}^{-1}\, g_{36}^{-1})\partial(h_{136})^{-1} e) \\ &= \delta_G(e)\,.\end{aligned} \tag{B.8}$$

Finally, the δ-function $\delta_G(g_{146})$ reads:

$$\begin{aligned}\delta_G(g_{146}) &= \delta_G\Big(\partial(h_{146})\, g_{46}\, g_{14}\, g_{16}^{-1}\Big) \\ &= \delta_G\Big(\partial(h_{146})\, g_{46}\, (g_{45}^{-1}\, \partial(h_{145})^{-1}\, g_{15})\, g_{16}^{-1}\Big) \\ &= \delta_G\Big(\partial(h_{146})\, g_{46}\, g_{45}^{-1}\, \partial(h_{145})^{-1}\, (g_{56}^{-1}\, \partial(h_{156})^{-1}\, g_{16})\, g_{16}^{-1}\Big) \\ &= \delta_G\Big(\partial(h_{146})\, g_{46}\, g_{45}^{-1}\, g_{56}^{-1}\partial(h_{456})^{-1}\partial(h_{146})^{-1}\partial(h_{156}) g_{56}\, (g_{56}^{-1}\, \partial(h_{156})^{-1}\, g_{16})\, g_{16}^{-1}\Big) \\ &= \delta_G(e)\,.\end{aligned} \tag{B.9}$$

Next, one integrates out l_{1235} using $\delta_L(l_{12345})$, l_{1236} using $\delta_L(l_{12346})$, l_{1246} using $\delta_L(l_{12456})$, and l_{1346} using $\delta_L(l_{13456})$, and obtains

$$l_{1235} = (h_{125} \rhd' l_{2345}) l_{1245} h_{145} \rhd' (g_{45} \rhd l_{1234}) l_{1345}^{-1}\, h_{135} \rhd' \{h_{345}, (g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{p}}\,, \tag{B.10}$$

$$l_{1236} = (h_{126} \rhd' l_{2346}) l_{1246} h_{146} \rhd' (g_{46} \rhd l_{1234}) l_{1346}^{-1}\, h_{136} \rhd' \{h_{346}, (g_{46} g_{34}) \rhd h_{123}\}_{\mathrm{p}}\,, \tag{B.11}$$

$$l_{1246} = (h_{126} \rhd' l_{2456}) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1245}) {l_{1456}}^{-1}\, h_{146} \rhd' \{h_{456}, (g_{56} g_{45}) \rhd h_{124}\}_{\mathrm{p}}\,, \tag{B.12}$$

$$l_{1346} = (h_{136} \rhd' l_{3456}) l_{1356} h_{156} \rhd' (g_{56} \rhd l_{1345}) {l_{1456}}^{-1}\, h_{146} \rhd' \{h_{456}, (g_{56} g_{45}) \rhd h_{134}\}_{\mathrm{p}}\,. \tag{B.13}$$

Let us now show that the remaining δ-functions on the group H equal $\delta_H(e)^4$. First, $\delta_H(h_{1235})$ becomes:

$$\begin{aligned}\delta_H(h_{1235}) =& \delta_H(\delta(l_{1235}) h_{135} (g_{35} \rhd h_{123}) h_{235}^{-1} h_{125}^{-1}) \\ =& \delta_H\Big(\delta((h_{125} \rhd' l_{2345}) l_{1245} h_{145} \rhd' (g_{45} \rhd l_{1234}) l_{1345}^{-1} h_{135} \rhd' \{h_{345}, (g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{p}}) h_{135} (g_{35} \rhd h_{123}) h_{235}^{-1} h_{125}^{-1}\Big) \\ =& \delta_H\Big((h_{125} \delta(l_{2345}) h_{125}^{-1} \delta(l_{1245}) h_{145} (g_{45} \rhd \delta(l_{1234})) h_{145}^{-1} \delta(l_{1345})^{-1} h_{135} \delta(\{h_{345}, (g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{p}}) h_{135}^{-1}) \\ & h_{135} (g_{35} \rhd h_{123}) h_{235}^{-1} h_{125}^{-1}\Big) \\ =& \delta_H\Big(h_{235} h_{345} (g_{45} \rhd h_{234}^{-1}) h_{245}^{-1} h_{125}^{-1} h_{125} h_{245} (g_{45} \rhd h_{124}^{-1}) h_{145}^{-1} h_{145} (g_{45} \rhd (h_{124} h_{234} (g_{34} \rhd h_{123}^{-1}) h_{134}^{-1})) \\ & h_{145}^{-1} (h_{145} (g_{45} \rhd h_{134}) h_{345}^{-1} h_{135}^{-1}) h_{135} \delta(\{h_{345}, (g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{p}}) h_{135}^{-1} h_{135} (g_{35} \rhd h_{123}) h_{235}^{-1}\Big) \\ =& \delta_H(h_{345} ((g_{45} g_{34}) \rhd h_{123}^{-1}) h_{345}^{-1} \delta(\{h_{345}, (g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{p}}) (g_{35} \rhd h_{123}).\end{aligned} \tag{B.14}$$

Here, one uses the following identity

$$\delta\{h_1\,, h_2\}_{\mathrm{p}} (\partial(h_1) \rhd h_2) h_1 h_2^{-1} h_1^{-1} = e\,. \tag{B.15}$$

Substituting $g_{35} = \partial(h_{345})g_{45}g_{34}$, and applying the (B.15) identity for $h_1 = h_{345}$ and $h_2 = (g_{45}g_{34}) \triangleright h_{123}$, one obtains

$$\delta_H(h_{1235}) = \delta_H(e). \tag{B.16}$$

Similarly, one obtains for $\delta_H(h_{1236})$:

$$\begin{aligned}\delta_H(h_{1236}) =& \delta_H(\delta(l_{1236})h_{136}(g_{36}\triangleright h_{123})h_{236}^{-1}h_{126}^{-1}) \\ =& \delta_H\Big(\delta((h_{126}\triangleright' l_{2346})l_{1246}h_{146}\triangleright'(g_{46}\triangleright l_{1236}l_{1346}^{-1}h_{136}\triangleright'\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{p}})h_{136}(g_{36}\triangleright h_{123})h_{236}^{-1}h_{126}^{-1}\Big) \\ =& \delta_H\Big((h_{126}\delta(l_{2346})h_{126}^{-1}\delta(l_{1246})h_{146}(g_{46}\triangleright\delta(l_{1234}))h_{146}^{-1}\delta(l_{1346})^{-1}h_{136}\delta(\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{p}})h_{136}^{-1}) \\ & h_{136}(g_{36}\triangleright h_{123})h_{236}^{-1}h_{126}^{-1}\Big) \\ =& \delta_H\Big(h_{236}h_{346}(g_{46}\triangleright h_{234}^{-1})h_{246}^{-1}h_{126}^{-1}h_{126}h_{246}(g_{46}\triangleright h_{124}^{-1})h_{146}^{-1}h_{146}(g_{46}\triangleright(h_{124}h_{234}(g_{34}\triangleright h_{123}^{-1})h_{134}^{-1})) \\ & h_{146}^{-1}(h_{146}(g_{46}\triangleright h_{134})h_{346}^{-1}h_{136}^{-1})h_{136}\delta(\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{p}})h_{136}^{-1}h_{136}(g_{36}\triangleright h_{123})h_{236}^{-1}\Big) \\ =& \delta_H(h_{346}((g_{46}g_{34})\triangleright h_{123}^{-1})h_{346}^{-1}\delta(\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{p}})(g_{36}\triangleright h_{123}).\end{aligned} \tag{B.17}$$

Substituting $g_{36} = \partial(h_{346})g_{46}g_{34}$, and applying the (B.15) identity for $h_1 = h_{346}$ and $h_2 = (g_{46}g_{34}) \triangleright h_{123}$, one obtains

$$\delta_H(h_{1236}) = \delta_H(e)\,. \tag{B.18}$$

Similarly, one obtains that $\delta_H(h_{1246}) = \delta_H(h_{1346}) = \delta_H(e)$. The remaining δ-function on the group L $\delta_L(l_{12356})$ reads:

$$\delta_L(l_{12356}) = \delta_L\big({l_{1236}}^{-1}(h_{126}\triangleright' l_{2356})l_{1256}h_{156}\triangleright'(g_{56}\triangleright l_{1235}){l_{1356}}^{-1}h_{136}\triangleright'\{h_{356},(g_{56}g_{35})\triangleright h_{123}\}_{\mathrm{p}}\big)\,. \tag{B.19}$$

After substituting the equations (B.10), (B.11), (B.12), and (B.13), one obtains:

$$\begin{aligned}\delta_L(l_{12356}) =& \delta_L\Big(h_{136}\triangleright'\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{p}}^{-1}(h_{136}\triangleright' l_{3456})l_{1356}h_{156}\triangleright'(g_{56}\triangleright l_{1345}){l_{1456}}^{-1} \\ & h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{134}\}_{\mathrm{p}}h_{146}\triangleright'(g_{46}\triangleright l_{1234})^{-1}h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{124}\}_{\mathrm{p}}^{-1}l_{1456} \\ & h_{156}\triangleright'(g_{56}\triangleright l_{1245})^{-1}l_{1256}^{-1}(h_{126}\triangleright' l_{2456})^{-1}(h_{126}\triangleright' {l_{2346}}^{-1})(h_{126}\triangleright' l_{2356})l_{1256} \\ & h_{156}\triangleright'(g_{56}\triangleright((h_{125}\triangleright' l_{2345})l_{1245}h_{145}\triangleright'(g_{45}\triangleright l_{1234})l_{1345}^{-1}h_{135}\triangleright'\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{p}})) \\ & {l_{1356}}^{-1}h_{136}\triangleright'\{h_{356},(g_{56}g_{35})\triangleright h_{123}\}_{\mathrm{p}}\Big)\,.\end{aligned} \tag{B.20}$$

Using the identity (3.4) the delta function $\delta_L(l_{12356})$ becomes:

$$\begin{aligned}\delta_L(l_{12356}) =& \delta_L\Big((h_{136}\triangleright' l_{3456})l_{1356}h_{156}\triangleright'(g_{56}\triangleright l_{1345}){l_{1456}}^{-1} \\ & h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{134}\}_{\mathrm{p}}h_{146}\triangleright'(g_{46}\triangleright l_{1234})^{-1}h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{124}\}_{\mathrm{p}}^{-1}l_{1456} \\ & \delta(h_{156}\triangleright'(g_{56}\triangleright l_{1245})^{-1})\triangleright'\Big((\delta(l_{1256})^{-1}h_{126})\triangleright'(l_{2456}^{-1}l_{2346}^{-1}l_{2356})h_{156}\triangleright'(g_{56}\triangleright(h_{125}\triangleright' l_{2345}))\Big) \\ & h_{156}\triangleright'(g_{56}\triangleright(h_{145}\triangleright'(g_{45}\triangleright l_{1234})l_{1345}^{-1})){l_{1356}}^{-1}(h_{136}h_{346})\triangleright'\{h_{346}^{-1}h_{356}g_{56}\triangleright h_{345}, \\ & (g_{56}g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{p}}\Big)\,.\end{aligned} \tag{B.21}$$

Commuting the elements, one obtains

$$\delta_L(l_{12356}) = \delta_L\Big((h_{156}\rhd'(g_{56}\rhd\delta(l_{1245})^{-1})\delta(l_{1256})^{-1}h_{126})\rhd'\big(l_{2456}^{-1}l_{2346}^{-1}l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})\big)$$
$$h_{156}\rhd'\big(g_{56}\rhd(h_{145}\rhd'(g_{45}\rhd l_{1234})l_{1345}^{-1})\big)l_{1356}{}^{-1}(h_{136}h_{346})\rhd'\{h_{346}^{-1}h_{356}g_{56}\rhd h_{345},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}}$$
$$h_{136}\rhd' l_{3456}l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1345})(\delta(l_{1456})^{-1}h_{146})\rhd'\big(\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{p}}\big)$$
$$(\delta(l_{1456})^{-1}h_{146})\rhd'\big((g_{46}\rhd l_{1234})^{-1}\big)(\delta(l_{1456})^{-1}h_{146})\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\mathrm{p}}^{-1}\Big). \tag{B.22}$$

The tetrahedron (3456) is part of the integrand on both sides of the move, so using the condition (4.9) for $\delta_H(h_{3456})$ one can write $h_{346}^{-1}h_{356}g_{56}\rhd h_{345} = h_{346}^{-1}\rhd'\delta(l_{3456})^{-1}h_{456}$. Then, using the identity (3.4) one obtains that

$$\begin{aligned}\{h_{346}^{-1}h_{356}g_{56}\rhd h_{345},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}} &= \{h_{346}^{-1}\rhd'\delta(l_{3456})^{-1}h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}}\\ &= \big(h_{346}^{-1}\rhd'\delta(l_{3456})^{-1}\big)\rhd'\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}}\\ &\quad\{h_{346}^{-1}\rhd'\delta(l_{3456})^{-1},(g_{46}g_{34})\rhd h_{123}\}_{\mathrm{p}}\\ &= h_{346}^{-1}\rhd' l_{3456}^{-1}\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}}\\ &\quad\big((g_{46}g_{34})\rhd h_{123}h_{346}^{-1}\big)\rhd' l_{3456}\,,\end{aligned} \tag{B.23}$$

where in the last row the definition of the action $\rhd'$ is used. Substituting the equation (B.23) in the equation (B.22) one obtains

$$\delta_L(l_{12356})=\delta_L\Big((h_{156}\rhd'(g_{56}\rhd\delta(l_{1245})^{-1})\delta(l_{1256})^{-1}h_{126}\delta(l_{2456})^{-1})\rhd'\Big(l_{2346}^{-1}l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})l_{2456}^{-1}\Big)$$
$$h_{156}\rhd'(g_{56}\rhd(h_{145}\rhd'(g_{45}\rhd l_{1234})))(h_{156}\rhd'(g_{56}\rhd\delta(l_{1345})^{-1})\delta(l_{1356})^{-1}h_{136}\delta(l_{3456})^{-1}h_{346})\rhd'$$
$$(\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}}((g_{46}g_{34})\rhd h_{123})\rhd' l_{3456})(\delta(l_{1456})^{-1}h_{146})\rhd'(\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{p}})$$
$$(\delta(l_{1456})^{-1}h_{146})\rhd'((g_{46}\rhd l_{1234})^{-1})(\delta(l_{1456})^{-1}h_{146})\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\mathrm{p}}^{-1}\Big). \tag{B.24}$$

Commuting the element l_{3456} to the end of the expression, one obtains

$$\delta_L(l_{12356})=\delta_L((h_{156}\rhd'(g_{56}\rhd\delta(l_{1245})^{-1})\delta(l_{1256})^{-1}h_{126}\delta(l_{2456})^{-1})\rhd'(l_{2346}^{-1}l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})l_{2456}^{-1})$$
$$h_{156}\rhd'(g_{56}\rhd(h_{145}\rhd'(g_{45}\rhd l_{1234})))(h_{156}\rhd'(g_{56}\rhd\delta(l_{1345})^{-1})\delta(l_{1356})^{-1}h_{136}\delta(l_{3456})^{-1}h_{346})\rhd'$$
$$(\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}})(\delta(l_{1456})^{-1}h_{146})\rhd'(\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{p}})$$
$$(\delta(l_{1456})^{-1}h_{146})\rhd'((g_{46}\rhd l_{1234})^{-1})(\delta(l_{1456})^{-1}h_{146})\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\mathrm{p}}^{-1}$$
$$(h_{156}g_{56}\rhd h_{145}h_{246}g_{46}\rhd h_{234}h_{346}^{-1})\rhd' l_{3456})). \tag{B.25}$$

Acting to the whole expression with $(h_{156}\rhd'(g_{56}\rhd\delta(l_{1245})^{-1})\delta(l_{1256})^{-1}h_{126}\delta(l_{2456})^{-1})^{-1}\rhd'$, one obtains,

$$\delta_L(l_{12356})=\delta_L(l_{2346}^{-1}l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})l_{2456}^{-1}(h_{246}h_{456}(g_{56}g_{45})\rhd h_{124}^{-1})\rhd'$$
$$\Big((g_{56}g_{45})\rhd l_{1234}((g_{56}g_{45})\rhd h_{134}h_{456}^{-1})\rhd'\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}}$$
$$h_{456}^{-1}\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{p}}h_{456}^{-1}\rhd g_{46}\rhd l_{1234}^{-1}(h_{456}^{-1}g_{46}\rhd h_{124})\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}^{-1}\}_{\mathrm{p}}\Big)$$
$$(h_{246}g_{46}\rhd h_{234}h_{346}^{-1})\rhd' l_{3456}. \tag{B.26}$$

Using the identity (3.5) for $\{h_{456},(g_{56}g_{45})\rhd(h_{134}g_{34}\rhd h_{123})\}_{\mathrm{p}}$,

$$\{h_{456},(g_{56}g_{45})\rhd(h_{134}g_{34}\rhd h_{123})\}_{\mathrm{p}} = \{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{p}}(g_{46}\rhd h_{134})\rhd'\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{p}}\,, \tag{B.27}$$

one obtains:

$$\delta_L(l_{12356}) = \delta_L\big(l_{2346}^{-1} l_{2356} h_{256} \rhd' (g_{56} \rhd l_{2345}) l_{2456}^{-1} h_{246} \rhd' \Big((h_{456}(g_{56}g_{45}) \rhd h_{124}^{-1}) \rhd' \\ \Big((g_{56}g_{45}) \rhd l_{1234} h_{456}^{-1} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd (h_{134} g_{34} \rhd h_{123})\}_{\mathrm{p}} \\ h_{456}^{-1} \rhd g_{46} \rhd l_{1234}^{-1} \Big) \{h_{456}, (g_{56}g_{45}) \rhd h_{124}^{-1}\}_{\mathrm{p}} \Big) (h_{246} g_{46} \rhd h_{234} h_{346}^{-1}) \rhd' l_{3456}\big) . \tag{B.28}$$

Using the identity (3.5) for $\{h_{456}, (g_{56}g_{45}) \rhd (h_{124}^{-1}\delta(l_{1234}) h_{134} g_{34} \rhd h_{123})\}_{\mathrm{p}}$ one obtains the terms featuring l_{1234} cancel, i.e. ,

$$\begin{aligned}\delta_L(l_{12356}) =& \delta_L\big(l_{2346}^{-1} l_{2356} h_{256} \rhd' (g_{56} \rhd l_{2345}) l_{2456}^{-1} \\ & h_{246} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd (h_{124}^{-1}\delta(l_{1234}) h_{134} g_{34} \rhd h_{123})\}_{\mathrm{p}} (h_{246} g_{46} \rhd h_{234} h_{346}^{-1}) \rhd' l_{3456} \\ =& \delta_L\big({l_{2346}}^{-1} l_{2356} h_{256} \rhd' (g_{56} \rhd l_{2345}) {l_{2456}}^{-1} h_{246} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{234}\}_{\mathrm{p}} (\delta(l_{2346})^{-1} h_{236}) \rhd' l_{3456})\big) \\ =& \delta_L(l_{23456}),\end{aligned} \tag{B.29}$$

the delta function $\delta_L(l_{12356})$ on the r.h.s. reduces to the delta function $\delta_L(l_{23456})$ of the l.h.s. The integrations over l_{1234}, l_{1245}, l_{1256}, l_{1345}, l_{1356}, and l_{1456} are trivial, and finally one obtains,

$$r.h.s. = \delta_G(e)^6 \delta_H(e)^4 \delta_L(l_{23456}) = |G|^6 |H|^4 \delta_L(l_{23456}) . \tag{B.30}$$

The prefactors $|G|^{-11}|H|^{-4}|L|^{-1}$ on the r.h.s. and $|G|^{-5}|H|^{0}|L|^{-1}$ on the l.h.s., compensate for left-over factors.

B.2 Pachner move $2 \leftrightarrow 4$

On the left hand side of the move one has the following integrals and the integrand,

$$\int_L dl_{2345} \delta_H(h_{2345}) \delta_L(l_{23456}) \delta_L(l_{12345}). \tag{B.31}$$

Integrating out l_{2345} using $\delta_L(l_{12345})$, one obtains

$$l_{2345} = {h_{125}}^{-1} \rhd' (l_{1235} h_{135} \rhd' \{h_{345}, (g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{p}}^{-1} l_{1345} h_{145} \rhd' (g_{45} \rhd l_{1234})^{-1} l_{1245}^{-1}) . \tag{B.32}$$

The δ-function $\delta_H(h_{2345})$ now reads,

$$\begin{aligned}\delta_H(h_{2345}) &= \delta_H\big(\delta(l_{2345}) h_{245}\, (g_{45} \rhd h_{234})\, h_{345}^{-1}\, h_{235}^{-1}\big) \\ &= \delta_H\Big({h_{125}}^{-1} \delta(l_{1235}) h_{135} \delta(\{h_{345}, (g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{p}}^{-1}) h_{135}^{-1} \delta(l_{1345}) h_{145} (g_{45} \rhd \delta(l_{1234}))^{-1} h_{145}^{-1} \\ &\quad \delta(l_{1245})^{-1} h_{125} h_{245}\, (g_{45} \rhd h_{234})\, h_{345}^{-1}\, h_{235}^{-1}\Big) .\end{aligned} \tag{B.33}$$

Using the identity (4.9) for the tetrahedrons (1235), (1345), (1234), and (1245), the equation (B.33) reduces to:

$$\begin{aligned}\delta_H(h_{2345}) = \delta_H\Big(& {h_{125}}^{-1} h_{125}\, h_{235}\, (g_{35} \rhd h_{123}^{-1})\, h_{135}^{-1} h_{135} \delta(\{h_{345}, (g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{p}}^{-1}) h_{135}^{-1} h_{135}\, h_{345}\, (g_{45} \rhd h_{134}^{-1}) \\ & h_{145}^{-1} h_{145} g_{45} \rhd (h_{134}(g_{34} \rhd h_{123}) h_{234}^{-1} h_{124}^{-1}) h_{145}^{-1} h_{145} (g_{45} \rhd h_{124}) h_{245}^{-1} h_{125}^{-1} h_{125} h_{245}\, (g_{45} \rhd h_{234})\, h_{345}^{-1}\, h_{235}^{-1}\Big) \\ = \delta_H\Big(& (g_{35} \rhd h_{123}^{-1})\, \delta(\{h_{345}, (g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{p}}^{-1})\, h_{345}\, (g_{45}g_{34}) \rhd h_{123})\ h_{345}^{-1}\Big) .\end{aligned} \tag{B.34}$$

Here, one uses the following identity

$$\delta\{h_1\,,h_2\}_{\rm p}(\partial(h_1)\rhd h_2)h_1h_2^{-1}h_1^{-1}=e\,, \tag{B.35}$$

for $h_1 = h_{345}$ and $h_2 = (g_{45}g_{34}) \rhd h_{123}$, and the identity $g_{35} = \partial(h_{345})g_{45}g_{34}$, and obtains

$$\delta_H(h_{2345})=\delta_H(e)\,. \tag{B.36}$$

The remaining δ-function $\delta_L(l_{23456})$, reads

$$\delta_L(l_{23456})=\delta_L(l_{2346}{}^{-1}(h_{236}\rhd' l_{3456})l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})l_{2456}{}^{-1}h_{246}\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{234}\}_{\rm p})\,. \tag{B.37}$$

Substituting the equation (B.33), one obtains

$$\begin{aligned}\delta_L(l_{23456})=\delta_L\Big(&l_{2346}{}^{-1}(h_{236}\rhd' l_{3456})l_{2356}h_{256}\rhd'\Big(g_{56}\rhd(h_{125}{}^{-1}\rhd'(l_{1235}h_{135}\rhd'\{h_{345},(g_{45}g_{34})\rhd h_{123}\}_{\rm p}^{-1}\\&l_{1345}h_{145}\rhd'(g_{45}\rhd l_{1234})^{-1}l_{1245}^{-1}))\Big)l_{2456}{}^{-1}h_{246}\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{234}\}_{\rm p}\Big)\,.\end{aligned} \tag{B.38}$$

Commuting the elements one obtains

$$\begin{aligned}\delta_L(l_{23456})=\delta_L\Big(&l_{2456}{}^{-1}l_{2346}{}^{-1}l_{2356}(h_{256}g_{56}\rhd h_{125}{}^{-1})\rhd' g_{56}\rhd l_{1235}(h_{256}g_{56}\rhd h_{125}{}^{-1}g_{56}\rhd h_{135})\rhd'\\&\Big((g_{35}\rhd h_{123}h_{356}^{-1})\rhd' l_{3456})\{g_{56}\rhd h_{345},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\rm p}^{-1}\\&(g_{56}\rhd h_{345}(g_{56}g_{45})\rhd(h_{123}h_{234}^{-1})h_{456}^{-1})\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{234}\}_{\rm p}\Big)\\&(h_{256}g_{56}\rhd h_{125}{}^{-1})\rhd' g_{56}\rhd l_{1345}(h_{256}g_{56}\rhd h_{125}{}^{-1}g_{56}\rhd h_{145})\rhd'((g_{56}g_{45})\rhd l_{1234})^{-1}\\&(h_{256}g_{56}\rhd h_{125}{}^{-1})\rhd' g_{56}\rhd l_{1245}^{-1}\Big)\,.\end{aligned} \tag{B.39}$$

Finally, the l.h.s. reads:

$$\text{l.h.s.}=\delta_H(e)\delta_L(l_{23456})=|H|\delta_L(l_{23456})\,. \tag{B.40}$$

Let us now examine the right hand side of the move, i.e., the integral (4.17). First, one integrates out g_{16} using $\delta_G(g_{126})$, and obtains

$$g_{16}=\partial(h_{126})\,g_{26}\,g_{12}\,. \tag{B.41}$$

Next, one integrates out h_{126} using $\delta_H(h_{1236})$, h_{136} using $\delta_H(h_{1346})$, and h_{146} using $\delta_H(h_{1456})$, and obtains

$$\begin{aligned}h_{126}&=\delta(l_{1236})h_{136}\,(g_{36}\rhd h_{123})\,h_{236}^{-1}\,,\\h_{136}&=\delta(l_{1346})h_{146}\,(g_{46}\rhd h_{134})\,h_{346}^{-1}\,,\\h_{146}&=\delta(l_{1456})h_{156}\,(g_{56}\rhd h_{145})\,h_{456}^{-1}\,.\end{aligned} \tag{B.42}$$

The remaining δ-functions on the group G reduces to $\delta_G(e)^3$. The δ-function $\delta_G(g_{136})$

$$\delta_G(g_{136})=\delta_G\big(\partial(h_{136})\,g_{36}\,g_{13}\,g_{16}^{-1}\big)\,, \tag{B.43}$$

after substituting the equation (B.41) reads:

$$\delta_G(g_{136}) = \delta_G\big(\partial(h_{136})\, g_{36}\, g_{13}\, g_{12}^{-1} g_{26}^{-1} \partial(h_{126})^{-1}\big)\,. \tag{B.44}$$

Using the equations (B.42) for h_{126}, and h_{136}, and h_{146}, and the identity $\partial(\delta l) = 0$ for every element $l \in L$, the δ-function $\delta_G(g_{136})$ reduces to $\delta_G(e)$ after implementing the identity (4.7) for the triangles (156), (145), (456) (134), (346), (236), and (123). Similarly, one obtains $\delta_G(g_{146}) = \delta_G(g_{156}) = \delta_G(e)$.

One integrates out l_{1236} using $\delta_L(l_{12346})$ and obtains

$$l_{1236} = (h_{126} \rhd' l_{2346}) l_{1246} h_{146} \rhd' (g_{46} \rhd l_{1234}) l_{1346}^{-1}\, h_{136} \rhd' \{h_{346}, (g_{46} g_{34}) \rhd h_{123}\}_{\mathrm{p}}, \tag{B.45}$$

l_{1246} using $\delta_L(l_{12456})$ and obtains

$$l_{1246} = (h_{126} \rhd' l_{2456}) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1245}) {l_{1456}}^{-1}\, h_{146} \rhd' \{h_{456}, (g_{56} g_{45}) \rhd h_{124}\}_{\mathrm{p}}, \tag{B.46}$$

and l_{1346} using $\delta_L(l_{13456})$ and obtains

$$l_{1346} = (h_{136} \rhd' l_{3456}) l_{1356} h_{156} \rhd' (g_{56} \rhd l_{1345}) {l_{1456}}^{-1}\, h_{146} \rhd' \{h_{456}, (g_{56} g_{45}) \rhd h_{134}\}_{\mathrm{p}}. \tag{B.47}$$

The remaining δ-functions on H reduce on $\delta_H(e)^3$, similarly as in the case of $1-5$ Pachner move, i.e. , one obtains $\delta_H(h_{1256}) = \delta_H(h_{1356}) = \delta_H(h_{1456}) = \delta_H(e)$. For the remaining δ-function $\delta_L(l_{12356})$,

$$\delta_L(l_{12356}) = \delta_L\Big({l_{1236}}^{-1} (h_{126} \rhd' l_{2356}) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1235}) {l_{1356}}^{-1} h_{136} \rhd' \{h_{356}, (g_{56} g_{35}) \rhd h_{123}\}_{\mathrm{p}}\Big)\,, \tag{B.48}$$

one obtains, after substituting the equations (B.45), (B.46), and (B.47), the following

$$\begin{aligned}
\delta_L(l_{12356}) = \delta_L\Big(& h_{136} \rhd' \{h_{346}, (g_{46} g_{34}) \rhd h_{123}\}_{\mathrm{p}}{}^{-1} l_{1346} h_{146} \rhd' (g_{46} \rhd l_{1234})^{-1} l_{1246}^{-1} (h_{126} \rhd' l_{2346})^{-1} \\
& (h_{126} \rhd' l_{2356}) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1235}) {l_{1356}}^{-1} h_{136} \rhd' \{h_{356}, (g_{56} g_{35}) \rhd h_{123}\}_{\mathrm{p}}\Big) \\
= \delta_L\Big(& (h_{126} \rhd' l_{2456})^{-1} (h_{126} \rhd' l_{2346})^{-1} (h_{126} \rhd' l_{2356}) (h_{256} g_{56} \rhd {h_{125}}^{-1}) \rhd' l_{1235} \\
& \delta(l_{1256}) \rhd' \Big(\delta(l_{1356})^{-1} \rhd' (h_{136} \rhd' \{h_{356}, (g_{56} g_{35}) \rhd h_{123}\}_{\mathrm{p}} (h_{136} h_{346}) \rhd' \{h_{346}^{-1}, g_{36} \rhd h_{123}\}_{\mathrm{p}} \\
& (h_{136} \rhd' l_{3456})) h_{156} \rhd' (g_{56} \rhd l_{1345}) {l_{1456}}^{-1}\, h_{146} \rhd' \{h_{456}, (g_{56} g_{45}) \rhd h_{134}\}_{\mathrm{p}} h_{146} \rhd' (g_{46} \rhd l_{1234})^{-1} \\
& h_{146} \rhd' \{h_{456}, (g_{56} g_{45}) \rhd h_{124}\}_{\mathrm{p}}^{-1} l_{1456} h_{156} \rhd' (g_{56} \rhd l_{1245})^{-1}\Big)\Big)\,.
\end{aligned} \tag{B.49}$$

Commuting the elements in order to match the l.h.s. of the move, i.e. , the δ-function given by the equation (B.39), and using the identity (3.4), i.e. ,

$$\{h_{346}^{-1} h_{356}, (g_{56} g_{35}) \rhd h_{123}\}_{\mathrm{p}} = h_{346}^{-1} \rhd' \{h_{356}, (g_{56} g_{35}) \rhd h_{123}\}_{\mathrm{p}} \{h_{346}^{-1}, g_{36} \rhd h_{123}\}_{\mathrm{p}}\,, \tag{B.50}$$

one obtains

$$\begin{aligned}
\delta_L(l_{12356}) = \delta_L\Big(& (h_{126} \rhd' l_{2456})^{-1} (h_{126} \rhd' l_{2346})^{-1} (h_{126} \rhd' l_{2356}) (h_{126} h_{256} g_{56} \rhd {h_{125}}^{-1}) \rhd' l_{1235} \\
& \delta(l_{1256}) \rhd' \Big(\delta(l_{1356})^{-1} \rhd' ((h_{136} h_{346}) \rhd' \{h_{346}^{-1} h_{356}, (g_{56} g_{35}) \rhd h_{123}\}_{\mathrm{p}} (h_{136} \rhd' l_{3456})) \\
& h_{156} \rhd' (g_{56} \rhd l_{1345}) (\delta(l_{1456})^{-1}\, h_{146}) \rhd' (\{h_{456}, (g_{56} g_{45}) \rhd h_{134}\}_{\mathrm{p}} (g_{46} \rhd l_{1234})^{-1} \\
& \{h_{456}, (g_{56} g_{45}) \rhd h_{124}\}_{\mathrm{p}}^{-1}) h_{156} \rhd' (g_{56} \rhd l_{1245})^{-1}\Big)\Big)\,.
\end{aligned} \tag{B.51}$$

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Using the identity (3.4) again one rewrites the following term as

$$\begin{aligned}(h_{136}h_{346}) \rhd' \{h_{346}^{-1}h_{356}, (g_{56}g_{35}) \rhd h_{123}\}_{\mathrm{p}}(h_{136} \rhd' l_{3456}) =\\ (h_{136}h_{346}) \rhd' \{h_{346}^{-1} \rhd' \delta(l_{3456})^{-1}h_{456}g_{56} \rhd h_{345}^{-1}, (g_{56}g_{35}) \rhd h_{123}\}_{\mathrm{p}}(h_{136} \rhd' l_{3456}) =\\ (h_{136} \rhd' \delta(l_{3456})^{-1}h_{136}h_{346}) \rhd' (\{h_{456}g_{56} \rhd h_{345}^{-1}, (g_{56}g_{35}) \rhd h_{123}\}_{\mathrm{p}}((g_{46}g_{34}) \rhd h_{123}h_{346}^{-1}) \rhd' l_{3456}^{-1})\,,\end{aligned} \tag{B.52}$$

and substituting it in the equation (B.51) the δ-function becomes:

$$\begin{aligned}\delta_L(l_{12356}) = \delta_L\Big(&(h_{126} \rhd' l_{2456})^{-1}(h_{126} \rhd' l_{2346})^{-1}(h_{126} \rhd' l_{2356})(h_{126}h_{256}g_{56} \rhd h_{125}{}^{-1}) \rhd' l_{1235}\\ &\delta(l_{1256}) \rhd' \Big((\delta(l_{1356})^{-1}h_{136} \rhd' \delta(l_{3456})^{-1}h_{136}h_{346})\rhd'\\ &(\{h_{456}g_{56} \rhd h_{345}^{-1}, (g_{56}g_{35}) \rhd h_{123}\}_{\mathrm{p}}((g_{46}g_{34}) \rhd h_{123}h_{346}^{-1}) \rhd' l_{3456}))\\ &(h_{156}g_{56} \rhd h_{135}g_{56} \rhd (h_{345}g_{45} \rhd h_{134}^{-1})h_{456}^{-1}) \rhd' (\{h_{456}, (g_{56}g_{45}) \rhd h_{134}\}_{\mathrm{p}}(g_{46} \rhd l_{1234})^{-1}\\ &\{h_{456}, (g_{56}g_{45}) \rhd h_{124}\}_{\mathrm{p}}^{-1})\Big)(h_{126}h_{256}g_{56} \rhd h_{125}{}^{-1}) \rhd' (h_{156} \rhd' (g_{56} \rhd l_{1345})(g_{56} \rhd l_{1245})^{-1})\Big)\,.\end{aligned} \tag{B.53}$$

Commuting the elements l_{3456} and $\{h_{456}g_{56} \rhd h_{345}, (g_{56}g_{35}) \rhd h_{123}\}_{\mathrm{p}}$, and using the identity (3.4) to rewrite this Peiffer lifting, one obtains

$$\begin{aligned}\delta_L(l_{12356}) = \delta_L\Big(&(h_{126} \rhd' l_{2456})^{-1}(h_{126} \rhd' l_{2346})^{-1}(h_{126} \rhd' l_{2356})(h_{126}h_{256}g_{56} \rhd h_{125}{}^{-1}) \rhd' l_{1235}\\ &(h_{126}h_{256}g_{56} \rhd h_{125}{}^{-1}h_{135}(g_{56}g_{35}) \rhd h_{123}g_{56} \rhd h_{356}^{-1}) \rhd' g_{56} \rhd l_{3456}\\ &(h_{126}h_{256}g_{56} \rhd h_{125}^{-1}g_{56} \rhd h_{135}g_{56} \rhd h_{345}) \rhd' \Big(\{g_{56} \rhd h_{345}^{-1}, (g_{56}g_{35}) \rhd h_{123}\}_{\mathrm{p}}\\ &h_{456}^{-1} \rhd' \{h_{456}, (g_{56}g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{p}}((g_{56}g_{45}) \rhd h_{134}^{-1}h_{456}^{-1}) \rhd' (\{h_{456}, (g_{56}g_{45}) \rhd h_{134}\}_{\mathrm{p}}(g_{46} \rhd l_{1234})^{-1}\\ &\{h_{456}, (g_{56}g_{45}) \rhd h_{124}\}_{\mathrm{p}}^{-1})\Big)(h_{126}h_{256}g_{56} \rhd h_{125}{}^{-1}) \rhd' (h_{156} \rhd' (g_{56} \rhd l_{1345})(g_{56} \rhd l_{1245})^{-1})\Big)\,.\end{aligned} \tag{B.54}$$

After the similar transformations as in the case of $1-5$ move, commuting the element l_{1234} so that the order of the elements matches the order in the expression (B.39), and acting to the whole expression with h_{126}^{-1} one obtains

$$\begin{aligned}\delta_L(l_{12356}) = \delta_L\Big(&l_{2456}{}^{-1}l_{2346}{}^{-1}l_{2356}(h_{256}g_{56} \rhd h_{125}{}^{-1}) \rhd' g_{56} \rhd l_{1235}(h_{256}g_{56} \rhd h_{125}{}^{-1}g_{56} \rhd h_{135})\rhd'\\ &\Big((g_{35} \rhd h_{123}h_{356}^{-1}) \rhd' l_{3456})\{g_{56} \rhd h_{345}, (g_{56}g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{p}}^{-1}(g_{56} \rhd h_{345}(g_{56}g_{45}) \rhd (h_{123}h_{234}^{-1})h_{456}^{-1})\rhd'\\ &\{h_{456}, (g_{56}g_{45}) \rhd h_{234}\}_{\mathrm{p}}\Big)(h_{256}g_{56} \rhd h_{125}{}^{-1}) \rhd' g_{56} \rhd l_{1345}\\ &(h_{256}g_{56} \rhd h_{125}{}^{-1}g_{56} \rhd h_{145}) \rhd' ((g_{56}g_{45}) \rhd l_{1234})^{-1}(h_{256}g_{56} \rhd h_{125}{}^{-1}) \rhd' g_{56} \rhd l_{1245}^{-1}\Big)\,.\end{aligned} \tag{B.55}$$

which is precisely the equation (B.39). The remaining integration over the element h_{156} of the group H and remaining integration over the three elements of the group L, l_{1246}, l_{1256}, and l_{1356}, are trivial, yielding the result on the r.h.s. to:

$$\text{r.h.s.} = \delta_G(e)^3\,\delta_H(e)^3\,\delta_L(l_{12356}) = |G|^3\,|H|^3\,\delta_L(l_{12356})\,. \tag{B.56}$$

The prefactors are $|G|^{-8}|H|^{-1}|L|^{-1}$ on the l.h.s., and $|G|^{-11}|H|^{-3}|L|^{-1}$ on the r.h.s. compensate for the left-over factors.

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B.3 Pachner move $3 \leftrightarrow 3$

Let us first investigate the r.h.s. of the move. First, one integrates out the l_{1235}, exploiting $\delta_L(l_{12345})$ and obtains

$$l_{1235} = (h_{125} \triangleright' l_{2345}) l_{1245} h_{145} \triangleright' (g_{45} \triangleright l_{1234}) l_{1345}{}^{-1}\, h_{135} \triangleright' \{h_{345}, (g_{45} g_{34}) \triangleright h_{123}\}_{\rm p}, \qquad \text{(B.57)}$$

and one integrates out l_{1236}, exploiting $\delta_L(l_{12356})$ and obtains

$$l_{1236} = (h_{126} \triangleright' l_{2356}) l_{1256} h_{156} \triangleright' (g_{56} \triangleright l_{1235}) l_{1356}{}^{-1} h'_{136} \triangleright \{h_{356}, (g_{56} g_{35}) \triangleright h_{123}\}_{\rm p}. \qquad \text{(B.58)}$$

Next, one integrates out h_{123}, exploiting $\delta_H(l_{1234})$ and obtains:

$$h_{123} = g_{34}^{-1} \triangleright h_{134}^{-1}\, g_{34}^{-1} \triangleright \delta(l_{1234})^{-1}\, g_{34}^{-1} \triangleright h_{124}\, g_{34}^{-1} \triangleright h_{234}. \qquad \text{(B.59)}$$

The δ-function $\delta_G(g_{123})$, when using the equation (B.59) reads

$$\delta_G(g_{123}) = \delta_G\big(g_{34}^{-1} \triangleright \partial(h_{134})^{-1}\, g_{34}^{-1} \triangleright \partial(\delta(l_{1234}))^{-1}\, g_{34}^{-1} \triangleright \partial(h_{124})\, g_{34}^{-1} \triangleright \partial(h_{234})\, g_{23}\, g_{12}\, g_{13}^{-1}\big)\,, \qquad \text{(B.60)}$$

which then using the condition $\partial\delta = 0$, reduces to

$$\delta_G(g_{123}) = \delta_G\big(\partial(h_{134})^{-1}\, \partial(h_{124})\, \partial(h_{234})\, g_{34}^{-1} g_{23}\, g_{12}\, g_{13}^{-1} g_{34}\big)\,. \qquad \text{(B.61)}$$

Using the condition (4.7) for the triangles (134), (124), and (234), it finally reduces to

$$\delta_G(g_{123}) = \delta_G(e)\,. \qquad \text{(B.62)}$$

For the δ-function $\delta_H(h_{1235})$, one obtains, after using the equation (B.57):

$$\begin{aligned}\delta_H(h_{1235}) = \delta_H\Big(&(h_{125}\delta(l_{2345})h_{125}^{-1})\delta(l_{1245})(h_{145}(g_{45} \triangleright \delta(l_{1234}))h_{145}^{-1})\delta(l_{1345})^{-1}\\ &h_{135} \triangleright' \{h_{345}, g_{35} \triangleright h_{123}\}_{\rm p}\, h_{135}((g_{35}g_{34}{}^{-1}) \triangleright (h_{134}^{-1}\, \delta(l_{1234})^{-1}\, h_{124}\, h_{234}))\, h_{235}^{-1}\, h_{125}^{-1}\Big)\,.\end{aligned} \qquad \text{(B.63)}$$

Using the δ-functions $\delta_L(h_{2345})$, $\delta_L(h_{1245})$, and $\delta_L(h_{1345})$, that appear on both sides of the move, and are thus part of the integrand,

$$\begin{aligned}\delta(l_{2345}) &= h_{235}\, h_{345}\, (g_{45} \triangleright h_{234}^{-1})\, h_{245}^{-1}\,,\\ \delta(l_{1245}) &= h_{125}\, h_{245}\, (g_{45} \triangleright h_{124}^{-1})\, h_{145}^{-1}\,,\\ \delta(l_{1345})^{-1} &= h_{145}\, (g_{45} \triangleright h_{134})\, h_{345}^{-1}\, h_{135}^{-1}\,,\end{aligned} \qquad \text{(B.64)}$$

one obtains:

$$\begin{aligned}\delta_H(h_{1235}) = \delta_H\Big(&h_{125} h_{235}\, h_{345}\, (g_{45} \triangleright h_{234}^{-1})\, h_{245}^{-1} h_{125}^{-1} h_{125}\, h_{245}\, (g_{45} \triangleright h_{124}^{-1})\, h_{145}^{-1} h_{145} (g_{45} \triangleright \delta(l_{1234})) h_{145}^{-1}\\ &h_{145}\, (g_{45} \triangleright h_{134})\, h_{345}^{-1}\, h_{135}^{-1} h_{135} \triangleright \delta(\{h_{345}, (g_{45}g_{34}) \triangleright h_{123}\}_{\rm p})\\ &h_{135}\, ((g_{35}g_{34}{}^{-1}) \triangleright (h_{134}^{-1} \delta(l_{1234})^{-1}\, h_{124}\, h_{234}))\, h_{235}^{-1}\, h_{125}^{-1}\Big)\\ = \delta_H\Big(&h_{345}(g_{45}g_{34}) \triangleright h_{123}^{-1}\, h_{345}^{-1} \delta(\{h_{345}, (g_{45}g_{34}) \triangleright h_{123}\}_{\rm p})\, (g_{35} \triangleright h_{123})\Big)\,.\end{aligned} \qquad \text{(B.65)}$$

Substituting $g_{35}=\partial(h_{345})g_{45}g_{34}$, and applying the identity

$$\delta\{h_1\,,h_2\}_{\rm p}(\partial(h_1)\rhd h_2)h_1h_2^{-1}h_1^{-1}=e\,, \tag{B.66}$$

for $h_1=h_{345}$ and $h_2=(g_{45}g_{34})\rhd h_{123}$, one obtains

$$\delta_H(h_{1235})=\delta_H(e). \tag{B.67}$$

Similarly, one obtains that $\delta_H(h_{1236})=\delta_H(e)$. The remaining δ-function $\delta_H(l_{12346})$ reads

$$\delta_L(l_{12346})=\delta_L(l_{1236}{}^{-1}(h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234})l_{1346}{}^{-1}\,h_{136}\rhd'\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\rm p})\,. \tag{B.68}$$

After substituting the equation (B.58), and then the equation (B.57), one obtains:

$$\begin{aligned}\delta_L(l_{12346})=\;&\delta_L(h_{136}\rhd'\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\rm p}^{-1}l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1235})^{-1}l_{1256}^{-1}h_{126}\rhd' l_{2356}^{-1}\\&(h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234})l_{1346}{}^{-1}h_{136}\rhd'\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\rm p})\\=\;&\delta_L(h_{136}\rhd'\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\rm p}^{-1}l_{1356}\\&h_{156}\rhd'(g_{56}\rhd((h_{125}\rhd' l_{2345})l_{1245}h_{145}\rhd'(g_{45}\rhd l_{1234})l_{1345}{}^{-1}\,h_{135}\rhd'\{h_{345},(g_{45}g_{34})\rhd h_{123}\}_{\rm p}))^{-1}\\&l_{1256}^{-1}h_{126}\rhd' l_{2356}^{-1}(h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234})l_{1346}{}^{-1}h_{136}\rhd'\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\rm p})\,.\end{aligned} \tag{B.69}$$

After commuting the elements, i.e. , using the Peiffer identity for the crossed module $(L\xrightarrow{\delta}H,\rhd')$, one obtains

$$\begin{aligned}\delta_L(l_{12346})=\;&\delta_L(h_{136}\rhd'\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\rm p}^{-1}\\&(\delta(l_{1356})h_{156}g_{56}\rhd h_{135})\rhd' g_{56}\rhd\{h_{345},(g_{45}g_{34})\rhd h_{123}\}_{\rm p}^{-1}l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1345})\\&(h_{156}g_{56}\rhd h_{145})\rhd'((g_{56}g_{45})\rhd l_{1234})^{-1}h_{156}\rhd'(g_{56}\rhd l_{1245})^{-1}(h_{156}g_{56}\rhd h_{125})\rhd'(g_{56}\rhd l_{2345}^{-1})l_{1256}^{-1}\\&h_{126}\rhd' l_{2356}^{-1}(h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234})l_{1346}{}^{-1}h_{136}\rhd'\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\rm p})\\=\;&\delta_L(((\delta(l_{1346})^{-1}h_{136})\rhd'\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\rm p}(\delta(l_{1346})^{-1}h_{136})\rhd'\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\rm p}^{-1}\\&((\delta(l_{1346})^{-1}\delta(l_{1356})h_{156}g_{56}\rhd h_{135})\rhd' g_{56}\rhd\{h_{345},(g_{45}g_{34})\rhd h_{123}\}_{\rm p}^{-1}\\&(\delta(l_{1346})^{-1}\delta(l_{1356})h_{156}\rhd'(g_{56}\rhd\delta(l_{1345}))h_{156}g_{56}\rhd h_{145})\rhd'((g_{56}g_{45})\rhd l_{1234})^{-1}l_{1346}^{-1}\\&l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1345})h_{156}\rhd'(g_{56}\rhd l_{1245})^{-1}(h_{156}g_{56}\rhd h_{125})\rhd'(g_{56}\rhd l_{2345}^{-1})\\&l_{1256}^{-1}h_{126}\rhd' l_{2356}^{-1}(h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234}))\,.\end{aligned} \tag{B.70}$$

Using the identity (3.7) one obtains that

$$\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\rm p}=h_{346}\rhd'\{h_{346}^{-1},g_{36}\rhd h_{123}\}_{\rm p}^{-1}\,. \tag{B.71}$$

Using a variant of the identity (3.4), i.e. , that

$$\{h_1h_2h_3,h_4\}_{\rm p}^{-1}=\{h_1,\partial(h_2h_3)\rhd h_4\}_{\rm p}^{-1}h_1\rhd'\{h_2,\partial(h_2)\rhd h_4\}_{\rm p}^{-1}(h_1h_2)\rhd'\{h_3,h_4\}_{\rm p}^{-1}\,, \tag{B.72}$$

one obtains that

$$\begin{aligned}\{h_{346}^{-1}\,h_{356}\,(g_{56}\rhd h_{345}),(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\rm p}^{-1}=\;&\{h_{346}^{-1},(g_{46}g_{34})\rhd h_{123}\}_{\rm p}^{-1}h_{346}^{-1}\rhd'\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\rm p}^{-1}\\&(h_{346}^{-1}h_{356})\rhd'\{g_{56}\rhd h_{345},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\rm p}^{-1}\,,\end{aligned} \tag{B.73}$$

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rendering the expression (B.70) to

$$
\begin{aligned}
\delta_L(l_{12346}) = \delta_L\big(&(h_{146}g_{46} \triangleright h_{134}) \triangleright' \{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}g_{34}) \triangleright h_{123}\}_{\mathrm{p}}^{-1} \\
&(\delta(l_{1346})^{-1}\delta(l_{1356})h_{156} \triangleright' (g_{56} \triangleright \delta(l_{1345}))h_{156}g_{56} \triangleright h_{145}) \triangleright' ((g_{56}g_{45}) \triangleright l_{1234})^{-1} \\
&l_{1346}^{-1}l_{1356}h_{156} \triangleright' (g_{56} \triangleright l_{1345})h_{156} \triangleright' (g_{56} \triangleright l_{1245})^{-1}(h_{156}g_{56} \triangleright h_{125}) \triangleright' (g_{56} \triangleright l_{2345}^{-1})l_{1256}^{-1} \\
&h_{126} \triangleright' l_{2356}^{-1}(h_{126} \triangleright' l_{2346})l_{1246}h_{146} \triangleright' (g_{46} \triangleright l_{1234})\big)\,.
\end{aligned}
\tag{B.74}
$$

Substituting the equation (B.59), and using the identity (3.5), one obtains that the expression,

$$
\begin{aligned}
\{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}g_{34}) \triangleright h_{123}\}_{\mathrm{p}}^{-1} &= \{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}) \triangleright ((h_{134}^{-1} \triangleright' \delta(l_{1234})^{-1})h_{134}^{-1}h_{124}h_{234}\}_{\mathrm{p}}^{-1} \\
&= (g_{46} \triangleright (h_{134}^{-1} \triangleright' \delta(l_{1234})^{-1})) \triangleright' \{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}) \triangleright \\
&\quad (h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{p}}^{-1}\{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}) \triangleright (h_{134}^{-1} \triangleright' \delta(l_{1234})^{-1})\}_{\mathrm{p}}^{-1}\,,
\end{aligned}
\tag{B.75}
$$

using the identity (3.9), i.e. , that

$$
\begin{aligned}
\{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}) \triangleright (h_{134}^{-1} \triangleright' \delta(l_{1234})^{-1})\}_{\mathrm{p}}^{-1} = g_{46} \triangleright (h_{134}^{-1} \triangleright' l_{1234}^{-1})(h_{346}^{-1}\, h_{356} \\
(g_{56} \triangleright h_{345})) \triangleright' ((g_{56}g_{45}) \triangleright (h_{134}^{-1} \triangleright' l_{1234}))\,,
\end{aligned}
\tag{B.76}
$$

reduces to

$$
\begin{aligned}
\{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}g_{34}) \triangleright h_{123}\}_{\mathrm{p}}^{-1} = g_{46} \triangleright (h_{134}^{-1} \triangleright' \delta(l_{1234})^{-1}) \\
\{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}) \triangleright (h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{p}}^{-1} \\
(h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345})) \triangleright' ((g_{56}g_{45}) \triangleright (h_{134}^{-1} \triangleright' l_{1234}))\,.
\end{aligned}
\tag{B.77}
$$

Substituting this result in the expression (B.74) the terms featuring l_{1234} cancel, and finally the delta function $\delta_L(l_{12346})$ reads:

$$
\begin{aligned}
\delta_L(l_{12346}) = \delta_L\big(&(h_{146}g_{46} \triangleright h_{134}) \triangleright' \{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}) \triangleright (h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{p}}^{-1}l_{1346}^{-1}l_{1356} \\
&h_{156} \triangleright' (g_{56} \triangleright l_{1345})h_{156} \triangleright' (g_{56} \triangleright l_{1245})^{-1}(h_{156}g_{56} \triangleright h_{125}) \triangleright' (g_{56} \triangleright l_{2345}^{-1}) \\
&l_{1256}^{-1}h_{126} \triangleright' l_{2356}^{-1}(h_{126} \triangleright' l_{2346})l_{1246}\big)\,.
\end{aligned}
\tag{B.78}
$$

One obtains that the integration over l_{1234} is trivial, and the r.h.s. of the move finally reads

$$
\begin{aligned}
\text{r.h.s.} = \delta_G(e)\delta_H(e)^2\delta_L\big(&h_{156} \triangleright' (g_{56} \triangleright l_{1245})^{-1}\, h_{156} \triangleright' (g_{56} \triangleright (h_{125} \triangleright' l_{2345}))^{-1}\, l_{1256}^{-1} \\
&h_{126} \triangleright' l_{2356}^{-1}(h_{126} \triangleright' l_{2346})l_{1246}(h_{146}g_{46} \triangleright h_{134}) \triangleright' \\
&\{h_{346}^{-1}\, h_{356}\, (g_{56} \triangleright h_{345}), (g_{56}g_{45}) \triangleright (h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{p}}^{-1}l_{1346}^{-1}l_{1356}h_{156} \triangleright' (g_{56} \triangleright l_{1345})\,.
\end{aligned}
\tag{B.79}
$$

The integral of the l.h.s. reads

$$
\int_H dh_{456} \int_{L^3} dl_{1456}dl_{2456}dl_{3456}\delta_G(g_{456})\,\delta_H(h_{3456})\delta_H(h_{2456})\delta_H(h_{1456})\delta_L(l_{23456})\delta_L(l_{13456})\delta_L(l_{12456})\,. \tag{B.80}
$$

First, one integrates out the l_{1456}, exploiting $\delta_L(l_{13456})$ and obtains

$$
l_{1456} = h_{146} \triangleright \{h_{456}, (g_{56}g_{45}) \triangleright h_{134}\}{l_{1346}}^{-1}(h_{136} \triangleright' l_{3456})l_{1356}h_{156} \triangleright' (g_{56} \triangleright l_{1345}). \tag{B.81}
$$

Next, one integrates out the l_{2456}, exploiting $\delta_L(l_{23456})$ and obtains

$$l_{2456} = h_{246} \rhd \{h_{456}, (g_{56}g_{45}) \rhd h_{234}\} {l_{2346}}^{-1}(h_{236} \rhd' l_{3456}) l_{2356} h_{256} \rhd' (g_{56} \rhd l_{2345})\,. \quad \text{(B.82)}$$

Next, one integrates out h_{456}, exploiting $\delta_H(h_{3456})$ and obtains

$$h_{456} = h_{346}^{-1}\,\delta(l_{3456}) h_{356}\,(g_{56} \rhd h_{345})\,. \quad \text{(B.83)}$$

Using the equation (B.83), one obtains that

$$\delta_G(g_{456}) = \delta_G\big(\partial(h_{346})^{-1}\,\partial(h_{356})\,g_{56} \rhd \partial(h_{345})\,g_{56}\,g_{45}\,g_{46}^{-1}\big)\,, \quad \text{(B.84)}$$

which, using the identity (4.7) for triangles (346), (356), and (345), reduces to:

$$\delta_G(g_{456}) = \delta_G(e)\,. \quad \text{(B.85)}$$

Similarly as done for the right-hand side of the move, one shows that $\delta_H(h_{1456})$, when using the equation (B.81), and $\delta_H(h_{2456})$, when using the equation (B.82), reduce to $\delta_H(e)^2$. The remaining $\delta_L(l_{12456})$ now reads

$$\delta_L(l_{12456}) = \delta_L({l_{1246}}^{-1}(h_{126} \rhd' l_{2456}) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1245}) {l_{1456}}^{-1}\, h_{146} \rhd \{h_{456}, (g_{56}g_{45}) \rhd h_{124}\}_{\mathrm{p}})\,. \quad \text{(B.86)}$$

Substituting the equations (B.81) and (B.82), one obtains

$$\begin{aligned}\delta_L(l_{12456}) = \delta_L({l_{1246}}^{-1}(h_{126} \rhd' (h_{246} \rhd \{h_{456},(g_{56}g_{45}) \rhd h_{234}\}_{\mathrm{p}} {l_{2346}}^{-1}(h_{236} \rhd' l_{3456}) l_{2356} \\ h_{256} \rhd' (g_{56} \rhd l_{2345}))) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1245}) h_{156} \rhd' (g_{56} \rhd l_{1345})^{-1} l_{1356}^{-1}(h_{136} \rhd' l_{3456})^{-1} \\ l_{1346} h_{146} \rhd \{h_{456},(g_{56}g_{45}) \rhd h_{134}\}_{\mathrm{p}}^{-1} h_{146} \rhd \{h_{456},(g_{56}g_{45}) \rhd h_{124}\}_{\mathrm{p}}).\end{aligned} \quad \text{(B.87)}$$

After commuting the elements, i.e., using the Peiffer identity for the crossed module $(L \xrightarrow{\delta} H, \rhd')$, one obtains

$$\begin{aligned}\delta_L(l_{12456}) = \delta_L\big((\delta(l_{1246})^{-1} h_{126} h_{246}) \rhd \{h_{456}, (g_{56}g_{45}) \rhd h_{234}\}_{\mathrm{p}} (\delta(l_{1246})^{-1} h_{126} \rhd \delta(l_{2346})^{-1} h_{126} h_{236}) \rhd' l_{3456} \\ l_{1246}^{-1} h_{126} \rhd' {l_{2346}}^{-1} h_{126} \rhd' l_{2356} (h_{126} h_{256}) \rhd' (g_{56} \rhd l_{2345})\,) \\ l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1245}) h_{156} \rhd' (g_{56} \rhd l_{1345})^{-1} l_{1356}^{-1} l_{1346} (\delta(l_{1346})^{-1} h_{136}) \rhd' {l_{3456}}^{-1} \\ h_{146} \rhd \{h_{456}, (g_{56}g_{45}) \rhd h_{134}\}_{\mathrm{p}}^{-1}\, h_{146} \rhd \{h_{456}, (g_{56}g_{45}) \rhd h_{124}\}_{\mathrm{p}})\,.\end{aligned} \quad \text{(B.88)}$$

Using the identity (3.10) for the inverse of the element $\{h_{456}, (g_{56}g_{45}) \rhd h_{134}\}_{\mathrm{p}}^{-1}$, and then the variant of the identity (3.5), i.e., that is,

$$\{h_1, h_2h_3h_4\}_{\mathrm{p}} = \{h_1, h_2\}_{\mathrm{p}}(\partial(h_1) \rhd h_2) \rhd' \{h_1, h_3\}_{\mathrm{p}}(\partial(h_1) \rhd (h_2h_3)) \rhd' \{h_1, h_4\}_{\mathrm{p}}\,, \quad \text{(B.89)}$$

one obtains

$$\begin{aligned}\{h_{456}, (g_{56}g_{45}) \rhd (h_{134}^{-1} h_{124} h_{234})\}_{\mathrm{p}} = \{h_{456}, (g_{56}g_{45}) \rhd h_{134}^{-1}\}_{\mathrm{p}}(g_{46} \rhd h_{134}^{-1}) \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{124}\}_{\mathrm{p}} \\ (g_{46} \rhd (h_{134}^{-1} h_{124})) \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{124}\}_{\mathrm{p}}\,,\end{aligned} \quad \text{(B.90)}$$

rendering the equation (B.88) to

$$\delta_L(l_{12456}) = \delta_L\big((\delta(l_{1246})^{-1}h_{126} \rhd \delta(l_{2346})^{-1}h_{126}h_{236}) \rhd' l_{3456}$$
$$l_{1246}^{-1}h_{126} \rhd' {l_{2346}}^{-1}h_{126} \rhd' l_{2356}(h_{126}h_{256}) \rhd' (g_{56} \rhd l_{2345})\,)$$
$$l_{1256}h_{156} \rhd' (g_{56} \rhd l_{1245})h_{156} \rhd' (g_{56} \rhd l_{1345})^{-1}l_{1356}^{-1}l_{1346}(\delta(l_{1346})^{-1}h_{136}) \rhd' {l_{3456}}^{-1}$$
$$(h_{146}g_{46} \rhd h_{134}) \rhd' \{h_{456}, (g_{56}g_{45}) \rhd (h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{p}})\,. \tag{B.91}$$

Using the equation (B.83), and the identities (3.4) and (3.6), similarly as for the r.h.s. of the move, one obtains that the terms featuring l_{3456} cancel, i.e. , the delta function $\delta_L(l_{12456})$ reads

$$\delta_L(l_{12456}) = \delta_L(l_{1246}^{-1}h_{126} \rhd' {l_{2346}}^{-1}h_{126} \rhd' l_{2356}(h_{126}h_{256}) \rhd' (g_{56} \rhd l_{2345}))l_{1256}h_{156} \rhd' (g_{56} \rhd l_{1245})$$
$$h_{156} \rhd' (g_{56} \rhd l_{1345})^{-1}l_{1356}^{-1}l_{1346}(h_{146}g_{46} \rhd h_{134}) \rhd' \{h_{456}, (g_{56}g_{45}) \rhd (h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{p}})\,. \tag{B.92}$$

It follows that the integral over l_{3456} is now trivial and l.h.s. of the move finally reduces to:

$$\text{l.h.s.} = \delta_G(e)\delta_H(e)^2\delta_L\big(h_{126} \rhd' l_{2346}l_{1246}(h_{146}g_{46} \rhd h_{134}) \rhd' \{h_{456}, (g_{56}g_{45}) \rhd (h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{p}}^{-1}$$
$$l_{1346}^{-1}\, l_{1356}\, h_{156} \rhd' (g_{56} \rhd l_{1345})\, h_{156} \rhd' (g_{56} \rhd l_{1245})^{-1}(h_{156}g_{56} \rhd h_{125}) \rhd' (g_{56} \rhd l_{2345})^{-1}$$
$$l_{1256}^{-1}\, h_{126} \rhd' l_{2356}^{-1}\,)\,. \tag{B.93}$$

The expressions (B.79) and (B.86) are the same, which proves the invariance of the state sum (4.1) under the Pachner move $3-3$. The numbers of k-simplices agree on both sides of the $3-3$ move for all k, and the prefactors play no role in this case.

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Higher gauge theories based on 3-groups

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Abstract: We study the categorical generalizations of a BF theory to $2BF$ and $3BF$ theories, corresponding to 2-groups and 3-groups, in the framework of higher gauge theory. In particular, we construct the constrained $3BF$ actions describing the correct dynamics of Yang-Mills, Klein-Gordon, Dirac, Weyl, and Majorana fields coupled to Einstein-Cartan gravity. The action is naturally split into a topological sector and a sector with simplicity constraints, adapted to the spinfoam quantization programme. In addition, the structure of the 3-group gives rise to a novel gauge group which specifies the spectrum of matter fields present in the theory, just like the ordinary gauge group specifies the spectrum of gauge bosons in the Yang-Mills theory. This allows us to rewrite the whole Standard Model coupled to gravity as a constrained $3BF$ action, facilitating the nonperturbative quantization of both gravity and matter fields. Moreover, the presence and the properties of this new gauge group open up a possibility of a nontrivial unification of all fields and a possible explanation of fermion families and all other structure in the matter spectrum of the theory.

Keywords: Models of Quantum Gravity, Topological Field Theories, Gauge Symmetry, Beyond Standard Model

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1 Introduction

The quantization of the gravitational field is one of the most prominent open problems in modern theoretical physics. Within the Loop Quantum Gravity framework, one can study the nonperturbative quantization of gravity, both canonically and covariantly, see [1–3] for an overview and a comprehensive introduction. The covariant approach focuses on the definition of the path integral for the gravitational field,

$$Z = \int \mathcal{D}g \, e^{iS[g]} \,, \tag{1.1}$$

by considering a triangulation of a spacetime manifold, and defining the path integral as a discrete state sum of the gravitational field configurations living on the simplices in the triangulation. This quantization technique is known as the *spinfoam* quantization method, and roughly goes along the following lines:

1. first, one writes the classical action $S[g]$ as a topological BF action plus a simplicity constraint,

2. then one uses the algebraic structure (a Lie group) underlying the topological sector of the action to define a triangulation-independent state sum Z,

3. and finally, one imposes the simplicity constraints on the state sum, promoting it into a path integral for a physical theory.

This quantization prescription has been implemented for various choices of the action, the Lie group, and the spacetime dimension. For example, in 3 dimensions, the prototype spinfoam model is known as the Ponzano-Regge model [4]. In 4 dimensions there are multiple models, such as the Barrett-Crane model [5, 6], the Ooguri model [7], and the most sophisticated EPRL/FK model [8, 9]. All these models aim to define a viable theory of quantum gravity, with variable success. However, virtually all of them are focused on pure gravity, without matter fields. The attempts to include matter fields have had limited success [10], mainly because the mass terms could not be expressed in the theory due to the absence of the tetrad fields from the BF sector of the theory.

In order to resolve this issue, a new approach has been developed, using the categorical generalization of the BF action, within the framework of *higher gauge theory* (see [11] for a review). In particular, one uses the idea of a categorical ladder to promote the BF action, which is based on some Lie group, into a $2BF$ action, which is based on the so-called 2-group structure. If chosen in a suitable way, the 2-group structure should hopefully introduce the tetrad fields into the action. This approach has been successfully implemented [12], rewriting the action for general relativity as a constrained $2BF$ action, such that the tetrad fields are present in the topological sector. This result opened up a possibility to couple all matter fields to gravity in a straightforward way. Nevertheless, the matter fields could not be naturally expressed using the underlying algebraic structure of a 2-group, rendering the spinfoam quantization method only half-implementable, since the matter sector of the classical action could not be expressed as a topological term plus a simplicity constraint, which means that the steps 2 and 3 above could not be performed for the matter sector of the action.

We address this problem in this paper. As we will show, it turns out that it is necessary to perform one more step in the categorical ladder, generalizing the underlying algebraic structure from a 2-group to a 3-group. This generalization then naturally gives rise to the so-called $3BF$ action, which proves to be suitable for a unified description of both gravity and matter fields. The steps of the categorical ladder can be conveniently summarized in the following table:

categorical structure	algebraic structure	linear structure	topological action	degrees of freedom
Lie group	Lie group	Lie algebra	BF theory	gauge fields
Lie 2-group	Lie crossed module	differential Lie crossed module	$2BF$ theory	tetrad fields
Lie 3-group	Lie 2-crossed module	differential Lie 2-crossed module	$3BF$ theory	scalar and fermion fields

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Once the suitable gauge 3-group has been specified and the corresponding $3BF$ action constructed, the most important thing that remains, in order to complete the step 1 of the spinfoam quantization programme, is to impose appropriate simplicity constraints onto the degrees of freedom present in the $3BF$ action, so that we obtain the desired classical dynamics of the gravitational and matter fields. Then one can proceed with steps 2 and 3 of the spinfoam quantization, hopefully ending up with a viable model of quantum gravity and matter.

In this paper, we restrict our attention to the first of the above steps: we will construct a constrained $3BF$ action for the cases of Klein-Gordon, Dirac, Weyl and Majorana fields, as well as Yang-Mills and Proca vector fields, all coupled to the Einstein-Cartan gravity in the standard way. This construction will lead us to an unexpected novel result. As we shall see, the scalar and fermion fields will be *naturally associated to a new gauge group*, generalizing the notion of a gauge group in the Yang-Mills theory, which describes vector bosons. This new group opens up a possibility to use it as an algebraic way of classifying matter fields, describing the structures such as quark and lepton families, and so on. The insight into the existence of this new gauge group is the consequence of the categorical ladder and is one of the main results of the paper. However, given the complexity of the algebraic properties of 3-groups, we will restrict ourselves only to the reconstruction of the already known theories, such as the Standard Model (SM), in the new framework. In this sense, any potential explanation of the spectrum of matter fields in the SM will be left for future work.

The layout of the paper is as follows. In subsection 2.1 we will give a short overview of the constrained BF actions, including the well-known example of the Plebanski action for general relativity, and a completely new example of the Yang-Mills theory rewritten as a constrained BF model. In the subsection 2.2 we also introduce the formalism of the constrained $2BF$ actions, reviewing the example of general relativity as a constrained $2BF$ action, first introduced in [12]. In addition, we will demonstrate how to couple gravity in a natural way within the formalism of 2-groups. Section 3 contains the main results of the paper and is split into 4 subsections. The subsection 3.1 introduces the formalism of 3-groups, and the definition and properties of a $3BF$ action, including the three types of gauge transformations. The subsection 3.2 focuses on the construction of a constrained $3BF$ action which describes a single real scalar field coupled to gravity. It provides the most elementary example of the insight that matter fields correspond to a gauge group. Encouraged by these results, in the subsection 3.3 we construct the constrained $3BF$ action for the Dirac field coupled to gravity and specify its gauge group. Finally, the subsection 3.4 deals with the construction of the constrained $3BF$ action for the Weyl and Majorana fields coupled to gravity, thereby covering all types of fields potentially relevant for the Standard Model and beyond. After the construction of all building blocks, in section 4 we apply the results of sections 2 and 3 to construct the constrained $3BF$ action corresponding to the full Standard Model coupled to Einstein-Cartan gravity. Finally, section 5 is devoted to the discussion of the results and the possible future lines of research. The appendices contain some mathematical reminders and technical details.

The notation and conventions are as follows. The local Lorentz indices are denoted by the Latin letters $a, b, c, \dots$, take values $0, 1, 2, 3$, and are raised and lowered using the

Minkowski metric η_{ab} with signature $(-,+,+,+)$. Spacetime indices are denoted by the Greek letters $\mu, \nu, \dots$, and are raised and lowered by the spacetime metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ are the tetrad fields. The inverse tetrad is denoted as $e^\mu{}_a$. All other indices that appear in the paper are dependent on the context, and their usage is explicitly defined in the text where they appear. A lot of additional notation is defined in appendix A. We work in the natural system of units where $c = \hbar = 1$, and $G = l_p^2$, where l_p is the Planck length.

2 BF and $2BF$ models, ordinary gauge fields and gravity

Let us begin by giving a short review of BF and $2BF$ theories in general. For additional information on these topics, see for example [11, 13–18].

2.1 BF theory

Given a Lie group G and its corresponding Lie algebra $\mathfrak{g}$, one can introduce the so-called BF action as

$$S_{BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} \,. \tag{2.1}$$

Here, $\mathcal{F} \equiv \mathrm{d}\alpha + \alpha \wedge \alpha$ is the curvature 2-form for the algebra-valued connection 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$ on some 4-dimensional spacetime manifold $\mathcal{M}_4$. In addition, $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ is a Lagrange multiplier 2-form, while $\langle \_\,, \_ \rangle_{\mathfrak{g}}$ denotes the G-invariant bilinear symmetric nondegenerate form.

From the structure of (2.1), one can see that the action is diffeomorphism invariant, and it is usually understood to be gauge invariant with respect to G. In addition to these properties, the BF action is topological, in the following sense. Varying the action (2.1) with respect to B^β and α^β, where the index β counts the generators of $\mathfrak{g}$ (see appendix A for notation and conventions), one obtains the equations of motion of the theory,

$$\mathcal{F} = 0\,, \qquad \nabla B \equiv \mathrm{d}B + \alpha \wedge B = 0\,. \tag{2.2}$$

From the first equation of motion, one immediately sees that α is a flat connection, which then together with the second equation of motion implies that B is constant. Therefore, there are no local propagating degrees of freedom in the theory, and one then says that the theory is topological.

Usually, in physics one is interested in theories which are nontopological, i.e., which have local propagating degrees of freedom. In order to transform the BF action into such a theory, one adds an additional term to the action, commonly called the *simplicity constraint*. A very nice example is the Yang-Mills theory for the $\mathrm{SU}(N)$ group, which can be rewritten as a constrained BF theory in the following way:

$$S = \int B_I \wedge F^I + \lambda^I \wedge \left(B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b \right) + \zeta^{abI} \left(M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - g_{IJ} F^J \wedge \delta_a \wedge \delta_b \right). \tag{2.3}$$

Here $F \equiv \mathrm{d}A + A \wedge A$ is again the curvature 2-form for the connection $A \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{su}(N))$, and $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{su}(N))$ is the Lagrange multiplier 2-form. The Killing form $g_{IJ} \equiv$

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$\langle \tau_I, \tau_J \rangle_{\mathfrak{su}(N)} \propto f_{IK}{}^L f_{JL}{}^K$ is used to raise and lower the indices $I, J, \dots$ which count the generators of $\mathrm{SU}(N)$, where $f_{IJ}{}^K$ are the structure constants for the $\mathfrak{su}(N)$ algebra. In addition to the topological $B \wedge F$ term, we also have two simplicity constraint terms, featuring the Lagrange multiplier 2-form λ^I and the Lagrange multiplier 0-form ζ^{abI}. The 0-form M_{abI} is also a Lagrange multiplier, while g is the coupling constant for the Yang-Mills theory.

Finally, δ^a is a nondynamical 1-form, such that there exists a global coordinate frame in which its components are equal to the Kronecker symbol $\delta^a{}_\mu$ (hence the notation δ^a). The 1-form δ^a plays the role of a background field, and defines the global spacetime metric, via the equation

$$\eta_{\mu\nu} = \eta_{ab} \delta^a{}_\mu \delta^b{}_\nu \,, \tag{2.4}$$

where $\eta_{ab} \equiv \mathrm{diag}(-1, +1, +1, +1)$ is the Minkowski metric. Since the coordinate system is global, the spacetime manifold $\mathcal{M}_4$ is understood to be flat. The indices $a, b, \dots$ are local Lorentz indices, taking values $0, \dots, 3$. Note that the field δ^a has all the properties of the tetrad 1-form e^a in the flat Minkowski spacetime. Also note that the action (2.3) is manifestly diffeomorphism invariant and gauge invariant with respect to $\mathrm{SU}(N)$, but not background independent, due to the presence of δ^a.

The equations of motion are obtained by varying the action (2.3) with respect to the variables ζ^{abI}, M_{abI}, A^I, B_I, and λ^I, respectively (note that we do not take the variation of the action with respect to the background field δ^a):

$$M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - F_I \wedge \delta_a \wedge \delta_b = 0 \,, \tag{2.5}$$

$$-\frac{12}{g} \lambda^I \wedge \delta^a \wedge \delta^b + \zeta^{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f = 0 \,, \tag{2.6}$$

$$-\mathrm{d}B_I + f_{JI}{}^K B_K \wedge A^J + \mathrm{d}(\zeta^{ab}{}_I \delta_a \wedge \delta_b) - f_{JI}{}^K \zeta^{ab}{}_K \delta_a \wedge \delta_b \wedge A^J = 0 \,, \tag{2.7}$$

$$F_I + \lambda_I = 0 \,, \tag{2.8}$$

$$B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b = 0 \,, \tag{2.9}$$

From the algebraic equations (2.5), (2.6), (2.8) and (2.9) one obtains the multipliers as functions of the dynamical field A^I:

$$M_{abI} = \frac{1}{48} \varepsilon_{abcd} F_I{}^{cd} \,, \quad \zeta^{abI} = \frac{1}{4g} \varepsilon^{abcd} F^I{}_{cd} \,, \quad \lambda_{Iab} = F_{Iab} \,, \quad B_{Iab} = \frac{1}{2g} \varepsilon_{abcd} F_I{}^{cd} \,. \tag{2.10}$$

Here we used the notation $F_{Iab} = F_{I\mu\nu} \delta_a{}^\mu \delta_b{}^\nu$, where we used the fact that $\delta^a{}_\mu$ is invertible, and similarly for other variables. Using these equations and the differential equation (2.7) one obtains the equation of motion for gauge field A^I,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0 \,. \tag{2.11}$$

This is precisely the classical equation of motion for the free Yang-Mills theory. Note that in addition to the Yang-Mills theory, one can easily extend the action (2.3) in order to describe the massive vector field and obtain the Proca equation of motion. This is done by adding a mass term

$$-\frac{1}{4!} m^2 A_{I\mu} A^I{}_\nu \eta^{\mu\nu} \varepsilon_{abcd} \delta^a \wedge \delta^b \wedge \delta^c \wedge \delta^d \tag{2.12}$$

to the action (2.3). Of course, this term explicitly breaks the $SU(N)$ gauge symmetry of the action.

Another example of the constrained BF theory is the Plebanski action for general relativity [15], see also [13] for a recent review. Starting from a gauge group $SO(3,1)$, one constructs a constrained BF action as

$$S = \int_{\mathcal{M}_4} B_{ab} \wedge R^{ab} + \phi_{abcd} B^{ab} \wedge B^{cd} \,. \tag{2.13}$$

Here R^{ab} is the curvature 2-form for the spin connection ω^{ab}, B_{ab} is the usual Lagrange multiplier 2-form, while ϕ_{abcd} is the Lagrange multiplier 0-form corresponding to the simplicity constraint term $B^{ab} \wedge B^{cd}$. It can be shown that the variation of this action with respect to B_{ab}, ω^{ab} and ϕ_{abcd} gives rise to equations of motion which are equivalent to vacuum general relativity. However, the tetrad fields appear in the model as a solution to the simplicity constraint equation of motion $B^{ab} \wedge B^{cd} = 0$. Thus, being intrinsically on-shell objects, they are not present in the action and cannot be quantized. This renders the Plebanski model unsuitable for coupling of matter fields to gravity [10, 12, 19]. Nevertheless, as a model for pure gravity, the Plebanski model has been successfully quantized in the context of spinfoam models, see [1, 2, 8, 9] for details and references.

2.2 $2BF$ theory

In order to circumvent the issue of coupling of matter fields, a recent promising approach has been developed [12, 19–23] in the context of higher category theory [11]. In particular, one employs the higher category theory construction to generalize the BF action to the so-called $2BF$ action, by passing from the notion of a gauge group to the notion of a gauge 2-group. In order to introduce it, let us first give a short review of the 2-group formalism.

In the framework of category theory, the group as an algebraic structure can be understood as a specific type of category, namely a category with only one object and invertible morphisms [11]. The notion of a category can be generalized to the so-called *higher categories*, which have not only objects and morphisms, but also 2-morphisms (morphisms between morphisms), and so on. This process of generalization is called the *categorical ladder*. Similarly to the notion of a group, one can introduce a 2-group as a 2-category consisting of only one object, where all the morphisms and 2-morphisms are invertible. It has been shown that every strict 2-group is equivalent to a crossed module $(H \stackrel{\partial}{\to} G \,, \triangleright)$, see appendix A for definition. Here G and H are groups, δ is a homomorphism from H to G, while $\triangleright : G \times H \to H$ is an action of G on H.

An important example of this structure is a vector space V equipped with an isometry group O. Namely, V can be regarded as an Abelian Lie group with addition as a group operation, so that a representation of O on V is an action $\triangleright$ of O on the group V, giving rise to the crossed module $(V \stackrel{\partial}{\to} O \,, \triangleright)$, where the homomorphism ∂ is chosen to be trivial, i.e., it maps every element of V into a unit of O. We will make use of this example below to introduce the Poincaré 2-group.

Similarly to the case of an ordinary Lie group G which has a naturally associated notion of a connection α, giving rise to a BF theory, the 2-group structure has a naturally associated notion of a 2-connection $(\alpha\,,\beta)$, described by the usual $\mathfrak{g}$-valued 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$ and an $\mathfrak{h}$-valued 2-form $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$, where $\mathfrak{h}$ is a Lie algebra of the Lie group H. The 2-connection gives rise to the so-called *fake 2-curvature* $(\mathcal{F},\mathcal{G})$, given as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,, \qquad \mathcal{G} = d\beta + \alpha\wedge^{\triangleright}\beta\,. \tag{2.14}$$

Here $\alpha\wedge^{\triangleright}\beta$ means that α and β are multiplied as forms using $\wedge$, and simultaneously multiplied as algebra elements using $\triangleright$, see appendix A. The curvature pair $(\mathcal{F},\mathcal{G})$ is called fake because of the presence of the $\partial\beta$ term in the definition of $\mathcal{F}$, see [11] for details.

Using these variables, one can introduce a new action as a generalization of the BF action, such that it is gauge invariant with respect to both G and H groups. It is called the $2BF$ action and is defined in the following way [16, 17]:

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}} + \langle C\wedge\mathcal{G}\rangle_{\mathfrak{h}}\,, \tag{2.15}$$

where the 2-form $B\in\mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{g})$ and the 1-form $C\in\mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{h})$ are Lagrange multipliers. Also, $\langle\_\,,\_\rangle_{\mathfrak{g}}$ and $\langle\_\,,\_\rangle_{\mathfrak{h}}$ denote the G-invariant bilinear symmetric nondegenerate forms for the algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. As a consequence of the axiomatic structure of a crossed module (see appendix A), the bilinear form $\langle\_\,,\_\rangle_{\mathfrak{h}}$ is H-invariant as well. See [16, 17] for review and references.

Similarly to the BF action, the $2BF$ action is also topological, which can be seen from equations of motion. Varying with respect to B and C one obtains

$$\mathcal{F} = 0\,, \qquad \mathcal{G} = 0\,, \tag{2.16}$$

while varying with respect to α and β one obtains the equations for the multipliers,

$$\mathrm{d}B_\alpha - g_{\alpha\beta}{}^{\gamma}B_\gamma\wedge\alpha^\beta - \triangleright_{\alpha a}{}^{b}C_b\wedge\beta^a = 0\,, \tag{2.17}$$

$$\mathrm{d}C_a - \partial_a{}^{\alpha}B_\alpha + \triangleright_{\alpha a}{}^{b}C_b\wedge\alpha^\alpha = 0\,. \tag{2.18}$$

One can either show that these equations have only trivial solutions, or one can use the Hamiltonian analysis to show that there are no local propagating degrees of freedom (see for example [21, 22]), demostrating the topological nature of the theory.

An example of a 2-group relevant for physics is the Poincaré 2-group, which is constructed using the aforementioned example of a vector space equipped with an isometry group. One constructs a crossed module by choosing

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \tag{2.19}$$

while $\triangleright$ is a natural action of SO(3, 1) on $\mathbb{R}^4$, and the map ∂ is trivial. The 2-connection (α,β) is given by the algebra-valued differential forms

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \tag{2.20}$$

where ω^{ab} is the spin connection, while M_{ab} and P_a are the generators of groups SO(3, 1) and $\mathbb{R}^4$, respectively. The corresponding 2-curvature in this case is given by

$$\mathcal{F} = (\mathrm{d}\omega^{ab}+\omega^a{}_c\wedge\omega^{cb})M_{ab} \equiv R^{ab}M_{ab}\,, \quad \mathcal{G} = (\mathrm{d}\beta^a+\omega^a{}_b\wedge\beta^b)P_a \equiv \nabla\beta^a P_a \equiv G^a P_a\,, \tag{2.21}$$

where we have evaluated $\wedge^{\triangleright}$ using the equation $M_{ab} \triangleright P_c = \eta_{[bc}P_{a]}$. Note that, since ∂ is trivial, the fake curvature is the same as ordinary curvature. Using the bilinear forms

$$\langle M_{ab}, M_{cd}\rangle_{\mathfrak{g}} = \eta_{a[c}\eta_{bd]}\,, \qquad \langle P_a, P_b\rangle_{\mathfrak{h}} = \eta_{ab}\,, \tag{2.22}$$

one can show that 1-forms C^a transform in the same way as the tetrad 1-forms e^a under the Lorentz transformations and diffeomorphisms, so the fields C^a can be identified with the tetrads. Then one can rewrite the $2BF$ action (2.15) for the Poincaré 2-group as

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + e_a \wedge \nabla\beta^a\,. \tag{2.23}$$

In order to obtain general relativity, the topological action (2.23) can be modified by adding a convenient simplicity constraint, like it is done in the BF case:

$$S = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + e_a \wedge \nabla\beta^a - \lambda_{ab}\wedge\left(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd}e_c\wedge e_d\right). \tag{2.24}$$

Here λ_{ab} is a Lagrange multiplier 2-form associated to the simplicity constraint term, and l_p is the Planck length. Varying the action (2.24) with respect to B_{ab}, e_a, ω_{ab}, β_a and λ_{ab}, one obtains the following equations of motion:

$$R_{ab} - \lambda_{ab} = 0\,, \tag{2.25}$$

$$\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc}\wedge e^d = 0\,, \tag{2.26}$$

$$\nabla B_{ab} - e_{[a}\wedge\beta_{b]} = 0\,, \tag{2.27}$$

$$\nabla e_a = 0\,, \tag{2.28}$$

$$B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd}e_c\wedge e_d = 0\,. \tag{2.29}$$

The only dynamical fields are the tetrads e^a, while all other fields can be algebraically determined, as follows. From the equations (2.28) and (2.29) we obtain that $\nabla B^{ab} = 0$, from which it follows, using the equation (2.27), that $e_{[a}\wedge\beta_{b]} = 0$. Assuming that the tetrads are nondegenerate, $e \equiv \det(e^a{}_\mu) \neq 0$, it can be shown that this is equivalent to the condition $\beta^a = 0$ (for the proof see appendix in [12]). Therefore, from the equations (2.25), (2.27), (2.28) and (2.29) we obtain

$$\lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu}\,, \quad \beta^a{}_{\mu\nu} = 0\,, \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \quad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,. \tag{2.30}$$

Here the Ricci rotation coefficients are defined as

$$\triangle^{ab}{}_\mu \equiv \frac{1}{2}(c^{abc} - c^{cab} + c^{bca})e_{c\mu}\,, \tag{2.31}$$

where

$$c^{abc} = e^\mu{}_b e^\nu{}_c\left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu\right)\,. \tag{2.32}$$

Finally, the remaining equation (2.26) reduces to

$$\varepsilon_{abcd}R^{bc} \wedge e^d = 0\,, \tag{2.33}$$

which is nothing but the vacuum Einstein field equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$. Therefore, the action (2.24) is classically equivalent to general relativity.

The main advantage of the action (2.24) over the Plebanski model and similar approaches lies in the fact that the tetrad fields are explicitly present in the topological sector of the theory. This allows one to couple matter fields in a straightforward way, as demonstrated in [12]. However, one can do even better, and couple gauge fields to gravity within a unified framework of 2-group formalism.

Let us demonstrate this on the example of the SU(N) Yang-Mills theory. Begin by modifying the Poincaré 2-group structure to include the SU(N) gauge group, as follows. We choose the two Lie groups as

$$G = \mathrm{SO}(3,1) \times \mathrm{SU}(N)\,, \qquad H = \mathbb{R}^4\,, \tag{2.34}$$

and we define the action $\triangleright$ of the group G in the following way. As in the case of the Poincaré 2-group, it acts on itself via conjugation. Next, it acts on H such that the SO(3,1) subgroup acts on $\mathbb{R}^4$ via the vector representation, while the action of SU(N) subgroup is trivial. The map ∂ also remains trivial, as before. The 2-connection (α, β) now obtains the form which reflects the structure of the group G,

$$\alpha = \omega^{ab}M_{ab} + A^I\tau_I\,, \qquad \beta = \beta^a P_a\,, \tag{2.35}$$

where A^I is the gauge connection 1-form, while τ_I are the SU(N) generators. The curvature for α is thus

$$\mathcal{F} = R^{ab}M_{ab} + F^I\tau_I\,, \qquad F^I \equiv \mathrm{d}A^I + f_{JK}{}^I A^J \wedge A^K\,. \tag{2.36}$$

The curvature for β remains the same as before, since the action $\triangleright$ of SU(N) on $\mathbb{R}^4$ is trivial, i.e., $\tau_I \triangleright P_a = 0$. Finally, the product structure of the group G implies that its Killing form $\langle \_ , \_ \rangle_{\mathfrak{g}}$ reduces to the Killing forms for the SO(3,1) and SU(N), along with the identity $\langle M_{ab}, \tau_I\rangle_{\mathfrak{g}} = 0$.

Given a crossed module defined in this way, its corresponding topological $2BF$ action (2.15) becomes

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla\beta^a\,, \tag{2.37}$$

where $B^I \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{su}(N))$ is the new Lagrange multiplier. In order to transform this topological action into action with nontrivial dynamics, we again introduce the appropriate simplicity constraints. The constraint giving rise to gravity is the same as in (2.24), while the constraint for the gauge fields is given as in the action (2.3) with the substitution $\delta^a \to e^a$:

$$S = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla\beta^a - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd}e_c \wedge e_d\right) \tag{2.38}$$
$$+ \lambda^I \wedge \left(B_I - \frac{12}{g}M_{abI}e^a \wedge e^b\right) + \zeta^{abI}\Big(M_{abI}\varepsilon_{cdef}e^c \wedge e^d \wedge e^e \wedge e^f - g_{IJ}F^J \wedge e_a \wedge e_b\Big)\,.$$

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It is crucial to note that the action (2.38) is a combination of the pure gravity action (2.24) and the Yang-Mills action (2.3), such that the nondynamical background field δ^a from (2.3) gets promoted to a dynamical field e^a. The relationship between these fields has already been hinted at in the equation (2.4), which describes the connection between δ^a and the flat spacetime metric $\eta_{\mu\nu}$. Once promoted to e^a, this field becomes dynamical, while the equation (2.4) becomes the usual relation between the tetrad and the metric,

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \,, \tag{2.39}$$

further confirming that the Lagrange multiplier C^a should be identified with the tetrad. Moreover, the total action (2.38) now becomes background independent, as expected in general relativity. All this is a consequence of the fact that the tetrad field is explicitly present in the topological sector of the action (2.24), establishing an improvement over the Plebanski model.

By varying the action (2.38) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, ζ^{abI}, M_{abI}, B_I, λ^I, A^I, and e^a, we obtain the following equations of motion, respectively:

$$R^{ab} - \lambda^{ab} = 0\,, \tag{2.40}$$

$$\nabla B^{ab} - e^{[a} \wedge \beta^{b]} = 0\,, \tag{2.41}$$

$$\nabla e^a = 0\,, \tag{2.42}$$

$$B_{ab} - \frac{1}{16\pi l_p^2} \varepsilon_{abcd} e^c \wedge e^d = 0\,, \tag{2.43}$$

$$M_{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F_I \wedge e_a \wedge e_b = 0\,, \tag{2.44}$$

$$-\frac{12}{g} \lambda^I \wedge e^a \wedge e^b + \zeta^{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f = 0\,, \tag{2.45}$$

$$F_I + \lambda_I = 0\,, \tag{2.46}$$

$$B_I - \frac{12}{g} M_{abI} e^a \wedge e^b = 0\,, \tag{2.47}$$

$$-\mathrm{d}B_I + B_K \wedge g_{JI}{}^K A^J + \mathrm{d}(\zeta_I^{ab} e_a \wedge e_b) - \zeta_K^{ab} e_a \wedge e_b \wedge g_{JI}{}^K A^J = 0\,, \tag{2.48}$$

$$\begin{aligned} &\nabla \beta_a + \frac{1}{8\pi l_p^2} \varepsilon_{abcd} \lambda^{bc} \wedge e^d - \frac{24}{g} M_{abI} \lambda^I \wedge e^b \\ &+ 4\zeta^{ef}{}^I M_{ef\,I} \varepsilon_{abcd} e^b \wedge e^c \wedge e^d - 2\zeta_{ab}{}^I F_I \wedge e^b = 0\,. \end{aligned} \tag{2.49}$$

In the above system of equations, we have two dynamical equations for e^a and A^I, while all other variables are algebraically determined from these. In particular, from equations (2.40)–(2.47), we have:

$$\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \quad \beta_{a\mu\nu} = 0\,, \quad \omega_{ab\mu} = \triangle_{ab\mu}\,, \quad \lambda_{abI} = F_{abI}\,, \quad B_{\mu\nu I} = -\frac{e}{2g} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}{}_I\,, \tag{2.50}$$

$$B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu\,, \quad M_{abI} = -\frac{1}{4eg} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma\,, \quad \zeta^{abI} = \frac{1}{4eg} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma\,.$$

Then, substituting all these into (2.48) and (2.49) we obtain the differential equation of motion for A^I,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + \Gamma^\rho{}_{\lambda\rho} F^{I\lambda\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0\,, \tag{2.51}$$

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where $\Gamma^\lambda{}_{\mu\nu}$ is the standard Levi-Civita connection, and a differential equation of motion for e^a,

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2\, T^{\mu\nu}\,, \qquad T^{\mu\nu} \equiv -\frac{1}{4g}\left(F_{\rho\sigma}{}^I F^{\rho\sigma}{}_I g^{\mu\nu} + 4 F^{\mu\rho}{}_I F_\rho{}^{\nu I} \right)\,. \tag{2.52}$$

The system of equations (2.50)–(2.52) is equivalent to the system (2.40)–(2.49). Note that we have again obtained that $\beta^a = 0$, as in the pure gravity case.

In this way, we see that both gravity and gauge fields can be represented within a unified framework of higher gauge theory based on a 2-group structure.

3 3*BF* models, scalar and fermion matter fields

While the structure of a 2-group can successfully accommodate both gravitational and gauge fields, unfortunately it cannot include other matter fields, such as scalars or fermions. In order to construct a unified description of all matter fields within the framework of higher gauge theory, we are led to make a further generalization, passing from the notion of a 2-group to the notion of a 3-group. As it turns out, the 3-group structure is a perfect fit for the description of all fields that are present in the Standard Model, coupled to gravity. Moreover, this structure gives rise to a new gauge group, which corresponds to the choice of the scalar and fermion fields present in the theory. This is a novel and unexpected result, which has the potential to open up a new avenue of research with the aim of explaining the structure of the matter sector of the Standard Model and beyond.

In order to demonstrate this in more detail, we first need to introduce the notion of a 3-group, which we will afterward use to construct constrained 3*BF* actions describing scalar and fermion fields on an equal footing with gravity and gauge fields.

3.1 3-groups and topological 3*BF* action

Similarly to the concepts of a group and a 2-group, one can introduce the notion of a 3-group in the framework of higher category theory, as a 3-category with only one object where all the morphisms, 2-morphisms and 3-morphisms are invertible. It has been proved that a strict 3-group is equivalent to a 2-*crossed module* [24], in the same way as a 2-group is equivalent to a crossed module.

A Lie 2-crossed module, denoted as $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$, is a algebraic structure specified by three Lie groups G, H and L, together with the homomorphisms δ and ∂, an action $\triangleright$ of the group G on all three groups, and a G-equivariant map

$$\{\_\,,\_\} : H \times H \to L\,.$$

called the Peiffer lifting. See appendix A for more details.

In complete analogy to the construction of *BF* and 2*BF* topological actions, one can define a gauge invariant topological 3*BF* action for the manifold $\mathcal{M}_4$ and 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$. Given $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ as Lie algebras corresponding to the groups G, H and L, one can introduce a 3-connection (α, β, γ) given by the algebra-valued differential forms $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$ and $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,,\mathfrak{l})$. The corresponding fake 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is then defined as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha\wedge^{\triangleright}\beta - \delta\gamma\,, \qquad \mathcal{H} = \mathrm{d}\gamma + \alpha\wedge^{\triangleright}\gamma + \{\beta\wedge\beta\}\,. \quad (3.1)$$

see [24, 25] for details. Then, a $3BF$ action is defined as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}} + \langle C\wedge\mathcal{G}\rangle_{\mathfrak{h}} + \langle D\wedge\mathcal{H}\rangle_{\mathfrak{l}}\,, \quad (3.2)$$

where $B\in\mathcal{A}^2(\mathcal{M}_4,\mathfrak{g})$, $C\in\mathcal{A}^1(\mathcal{M}_4,\mathfrak{h})$ and $D\in\mathcal{A}^0(\mathcal{M}_4,\mathfrak{l})$ are Lagrange multipliers. The forms $\langle\_,\_\rangle_{\mathfrak{g}}$, $\langle\_,\_\rangle_{\mathfrak{h}}$ and $\langle\_,\_\rangle_{\mathfrak{l}}$ are G-invariant bilinear symmetric nondegenerate forms on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, respectively. Under certain conditions, the forms $\langle\_,\_\rangle_{\mathfrak{h}}$ and $\langle\_,\_\rangle_{\mathfrak{l}}$ are also H-invariant and L-invariant, see appendix B for details.

One can see that varying the action with respect to the variables B, C and D, one obtains the equations of motion

$$\mathcal{F}=0\,, \qquad \mathcal{G}=0\,, \qquad \mathcal{H}=0\,, \quad (3.3)$$

while varying with respect to α, β, γ one obtains

$$\mathrm{d}B_\alpha - g_{\alpha\beta}{}^{\gamma}B_\gamma\wedge\alpha^\beta - \triangleright_{\alpha a}{}^{b}C_b\wedge\beta^a + \triangleright_{\alpha B}{}^{A}D_A\wedge\gamma^B = 0\,, \quad (3.4)$$

$$\mathrm{d}C_a - \partial_a{}^{\alpha}B_\alpha + \triangleright_{\alpha a}{}^{b}C_b\wedge\alpha^\alpha + 2X_{\{ab\}}{}^{A}D_A\wedge\beta^b = 0\,, \quad (3.5)$$

$$\mathrm{d}D_A - \triangleright_{\alpha A}{}^{B}D_B\wedge\alpha^\alpha + \delta_A{}^{a}C_a = 0\,. \quad (3.6)$$

Regarding the gauge transformations, the $3BF$ action is invariant with respect to three different types of transformations, generated by the groups G, H and L, respectively. Under the G-gauge transformations, the 3-connection transforms as

$$\alpha' = g^{-1}\alpha g + g^{-1}\mathrm{d}g\,, \qquad \beta' = g^{-1}\triangleright\beta\,, \qquad \gamma' = g^{-1}\triangleright\gamma\,, \quad (3.7)$$

where $g:\mathcal{M}_4\to G$ is an element of the G-principal bundle over $\mathcal{M}_4$. Next, under the H-gauge transformations, generated by $\eta\in\mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{h})$, the 3-connection transforms as

$$\alpha' = \alpha + \partial\eta\,, \qquad \beta' = \beta + \mathrm{d}\eta + \alpha'\wedge^{\triangleright}\eta - \eta\wedge\eta\,, \qquad \gamma' = \gamma - \{\beta'\wedge\eta\} - \{\eta\wedge\beta\}\,. \quad (3.8)$$

Finally, under the L-gauge transformations, generated by $\theta\in\mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{l})$, the 3-connection transforms as

$$\alpha' = \alpha\,, \qquad \beta' = \beta - \delta\theta\,, \qquad \gamma' = \gamma - \mathrm{d}\theta - \alpha\wedge\theta\,. \quad (3.9)$$

As a consequence of the definition (3.1) and the above transformation rules, the curvatures transform under the G-gauge transformations as

$$\mathcal{F}\to g^{-1}\mathcal{F}g\,, \qquad \mathcal{G}\to g^{-1}\triangleright\mathcal{G}\,, \qquad \mathcal{H}\to g^{-1}\triangleright\mathcal{H}\,, \quad (3.10)$$

under the H-gauge transformations as

$$\mathcal{F}\to\mathcal{F}\,, \qquad \mathcal{G}\to\mathcal{G} + \mathcal{F}\wedge^{\triangleright}\eta\,, \qquad \mathcal{H}\to\mathcal{H} - \{\mathcal{G}'\wedge\eta\} + \{\eta\wedge\mathcal{G}\}\,, \quad (3.11)$$

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and under the L-gauge transformations as

$$\mathcal{F} \to \mathcal{F}\,, \qquad \mathcal{G} \to \mathcal{G}\,, \qquad \mathcal{H} \to \mathcal{H} - \mathcal{F} \wedge^{\triangleright} \theta\,. \tag{3.12}$$

For more details, the reader is referred to [25].

In order to make the action (3.2) gauge invariant with respect to the transformations (3.7), (3.8) and (3.9), the Lagrange multipliers B, C and D must transform under the G-gauge transformations as

$$B \to g^{-1} B g\,, \qquad C \to g^{-1} \triangleright C\,, \qquad D \to g^{-1} \triangleright D\,, \tag{3.13}$$

under the H-gauge transformations as

$$B \to B + C' \wedge^{\mathcal{T}} \eta - \eta \wedge^{\mathcal{D}} \eta \wedge^{\mathcal{D}} D\,, \qquad C \to C + D \wedge^{\mathcal{X}_1} \eta + D \wedge^{\mathcal{X}_2} \eta\,, \qquad D \to D\,, \tag{3.14}$$

while under the L-gauge transformations they transform as

$$B \to B - D \wedge^{\mathcal{S}} \theta\,, \qquad C \to C\,, \qquad D \to D\,. \tag{3.15}$$

See appendix B for details, for the definition of the maps $\mathcal{T}$, $\mathcal{D}$, $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{S}$, and for the notation of the $\wedge^{\mathcal{T}}$, $\wedge^{\mathcal{D}}$, $\wedge^{\mathcal{X}_1}$, $\wedge^{\mathcal{X}_2}$, and $\wedge^{\mathcal{S}}$ products.

3.2 Constrained 3*BF* action for a real Klein-Gordon field

Once the topological $3BF$ action is specified, we can proceed with the construction of the constrained $3BF$ action, describing a realistic case of a scalar field coupled to gravity. In order to perform this construction, we have to define a specific 2-crossed module which gives rise to the topological sector of the action, and then we have to impose convenient simplicity constraints.

We begin by defining a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\})$, as follows. The groups are given as

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}\,. \tag{3.16}$$

The group G acts on itself via conjugation, on H via the vector representation, and on L via the trivial representation. This specifies the definition of the action $\triangleright$. The map ∂ is chosen to be trivial, as before. The map δ is also trivial, that is, every element of L is mapped to the identity element of H. Finally, the Peiffer lifting is trivial as well, mapping every ordered pair of elements in H to an identity element in L. This specifies one concrete 2-crossed module.

Given this choice of a 2-crossed module, the 3-connection $(\alpha\,, \beta\,, \gamma)$ takes the form

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma \mathbb{I}\,, \tag{3.17}$$

where $\mathbb{I}$ is the sole generator of the Lie group $\mathbb{R}$. From (3.1), the fake 3-curvature $(\mathcal{F}\,, \mathcal{G}\,, \mathcal{H})$ reduces to the ordinary 3-curvature,

$$\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla \beta^a P_a\,, \qquad \mathcal{H} = \mathrm{d}\gamma\,, \tag{3.18}$$

where we used the fact that G acts trivially on L, that is, $M_{ab} \triangleright \mathbb{I} = 0$. The topological $3BF$ action (3.2) now becomes

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma \,, \tag{3.19}$$

where the bilinear form for L is $\langle \mathbb{I}, \mathbb{I} \rangle_{\mathfrak{l}} = 1$.

It is important to note that the Lagrange multiplier D in (3.2) is a 0-form and transforms trivially with respect to G, H and L gauge transformations for our choice of the 2-crossed module, as can be seen from (3.13), (3.14) and (3.15). Thus, D has all the *hallmark properties of a real scalar field*, allowing us to make identification between them, and conveniently relabel D into ϕ in (3.19). This is a crucial property of the 3-group structure in a 4-dimensional spacetime and is one of the main results of the paper. It follows the line of reasoning used in recognizing the Lagrange multiplier C^a in the $2BF$ action for the Poincaré 2-group as a tetrad field e^a. It is also important to stress that the choice of the third gauge group, L, dictates the number and the structure of the matter fields present in the action. In this case, $L = \mathbb{R}$ implies that we have only one real scalar field, corresponding to a single generator $\mathbb{I}$ of $\mathbb{R}$. The trivial nature of the action $\triangleright$ of $\mathrm{SO}(3,1)$ on $\mathbb{R}$ also implies that ϕ transforms as a scalar field. Finally, the scalar field appears as a degree of freedom in the topological sector of the action, making the quantization procedure feasible.

As in the case of BF and $2BF$ theories, in order to obtain nontrivial dynamics, we need to impose convenient simplicity constraints on the variables in the action (3.19). Since we are interested in obtaining the scalar field ϕ of mass m coupled to gravity in the standard way, we choose the action in the form:

$$\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma \\
& - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\
& + \lambda \wedge \left(\gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c \right) + \Lambda^{ab} \wedge \left(H_{abc} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b \right) \\
& - \frac{1}{2 \cdot 4!} m^2 \phi^2 \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,.
\end{aligned} \tag{3.20}$$

Note that the first row is the topological sector (3.19), the second row is the familiar simplicity constraint for gravity from the action (2.24), the third row contains the new simplicity constraints corresponding to the Lagrange multiplier 1-forms λ and Λ^{ab} and featuring the Lagrange multiplier 0-form H_{abc}, while the fourth row is the mass term for the scalar field.

Varying the total action (3.20) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, Λ_{ab}, γ, λ, H_{abc}, ϕ and e^a one obtains the equations of motion:

$$R^{ab} - \lambda^{ab} = 0 \,, \tag{3.21}$$

$$\nabla B^{ab} - e^{[a} \wedge \beta^{b]} = 0 \,, \tag{3.22}$$

$$\nabla e^a = 0 \,, \tag{3.23}$$

$$B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd}e^c \wedge e^d = 0\,, \qquad (3.24)$$

$$H_{abc}\varepsilon^{cdef}e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b = 0\,, \qquad (3.25)$$

$$\mathrm{d}\phi - \lambda = 0\,, \qquad (3.26)$$

$$\gamma - \frac{1}{2}H_{abc}e^a \wedge e^b \wedge e^c = 0\,, \qquad (3.27)$$

$$-\frac{1}{2}\lambda \wedge e^a \wedge e^b \wedge e^c + \varepsilon^{cdef}\Lambda^{ab} \wedge e_d \wedge e_e \wedge e_f = 0\,, \qquad (3.28)$$

$$\mathrm{d}\gamma - \mathrm{d}(\Lambda^{ab} \wedge e_a \wedge e_b) - \frac{1}{4!}m^2\phi\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d = 0\,, \qquad (3.29)$$

$$\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{3}{2}H_{abc}\lambda \wedge e^b \wedge e^c + 3H^{def}\varepsilon_{abcd}\Lambda_{ef} \wedge e^b \wedge e^c$$
$$-2\Lambda_{ab} \wedge \mathrm{d}\phi \wedge e^b - 2\frac{1}{4!}m^2\phi\varepsilon_{abcd}e^b \wedge e^c \wedge e^d = 0\,. \qquad (3.30)$$

The dynamical degrees of freedom are e^a and ϕ, while the remaining variables are algebraically determined in terms of them. Specifically, the equations (3.21)–(3.28) give

$$\begin{aligned}
&\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, && \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,, \quad \gamma_{\mu\nu\rho} = -\frac{e}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\sigma\phi\,, \\
&\Lambda^{ab}{}_\mu = \frac{1}{12e}g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\partial_\nu\phi e^a{}_\rho e^b{}_\sigma\,, && \beta^a{}_{\mu\nu} = 0\,, \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \\
&H^{abc} = \frac{1}{6e}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\phi e^a{}_\nu e^b{}_\rho e^c{}_\sigma\,, && \lambda_\mu = \partial_\mu\phi\,.
\end{aligned} \qquad (3.31)$$

Note that from the equations (3.22), (3.23) and (3.24) it follows that $\beta^a = 0$, as in the pure gravity case. The equation of motion (3.29) reduces to the covariant Klein-Gordon equation for the scalar field,

$$\left(\nabla_\mu\nabla^\mu - m^2\right)\phi = 0\,. \qquad (3.32)$$

Finally, the equation of motion (3.30) for e^a becomes:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \qquad T^{\mu\nu} \equiv \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\left(\partial_\rho\phi\partial^\rho\phi + m^2\phi^2\right)\,. \qquad (3.33)$$

The system of equations (3.21)–(3.30) is equivalent to the system of equations (3.31)–(3.33). Note that in addition to the correct covariant form of the Klein-Gordon equation, we have also obtained the correct form of the stress-energy tensor for the scalar field.

3.3 Constrained 3BF action for the Dirac field

Now we pass to the more complicated case of the Dirac field. We first define a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\})$ as follows. The groups are:

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^8(\mathbb{G})\,, \qquad (3.34)$$

where $\mathbb{G}$ is the algebra of complex Grassmann numbers. The maps ∂, δ and the Peiffer lifting are trivial. The action of the group G on itself is given via conjugation, on H via vector representation, and on L via spinor representation, as follows. Denoting the

8 generators of the Lie group $\mathbb{R}^8(\mathbb{G})$ as P_α and P^α, where the index α takes the values $1,\dots,4$, the action of G on L is thus given explicitly as

$$M_{ab} \rhd P_\alpha = \frac{1}{2}(\sigma_{ab})^\beta{}_\alpha P_\beta\,, \qquad M_{ab} \rhd P^\alpha = -\frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta\,, \tag{3.35}$$

where $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, and γ_a are the usual Dirac matrices, satisfying the anticommutation rule $\{\gamma_a\,,\,\gamma_b\} = -2\eta_{ab}$.

As in the case of the scalar field, the choice of the group L dictates the matter content of the theory, while the action $\rhd$ of G on L specifies its transformation properties. To see this explicitly, let us construct the corresponding $3BF$ action. The 3-connection $(\alpha\,,\beta\,,\gamma)$ now takes the form

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma^\alpha P_\alpha + \bar{\gamma}_\alpha P^\alpha\,, \tag{3.36}$$

while the 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$, defined in (3.1), is given as

$$\mathcal{F} = R^{ab}M_{ab}\,, \mathcal{G} = \nabla\beta^a P_a\,, \tag{3.37}$$
$$\mathcal{H} = \left(\mathrm{d}\gamma^\alpha + \frac{1}{2}\omega^{ab}(\sigma_{ab})^\alpha{}_\beta\gamma^\beta\right)P_\alpha + \left(\mathrm{d}\bar{\gamma}_\alpha - \frac{1}{2}\omega^{ab}\bar{\gamma}_\beta(\sigma_{ab})^\beta{}_\alpha\right)P^\alpha \equiv (\overset{\rightarrow}{\nabla}\gamma)^\alpha P_\alpha + (\bar{\gamma}\overset{\leftarrow}{\nabla})_\alpha P^\alpha\,,$$

where we have used (3.35). The bilinear form $\langle\_,\_\rangle_\mathfrak{l}$ is defined as

$$\langle P_\alpha, P_\beta\rangle_\mathfrak{l} = 0\,, \qquad \langle P^\alpha, P^\beta\rangle_\mathfrak{l} = 0\,, \qquad \langle P_\alpha, P^\beta\rangle_\mathfrak{l} = -\delta_\alpha^\beta\,, \qquad \langle P^\alpha, P_\beta\rangle_\mathfrak{l} = \delta^\alpha_\beta\,. \tag{3.38}$$

Note that, for general $A, B \in \mathfrak{l}$, we can write

$$\langle A, B\rangle_\mathfrak{l} = A^I B^J g_{IJ}\,, \qquad \langle B, A\rangle_\mathfrak{l} = B^J A^I g_{JI}\,. \tag{3.39}$$

Since we require the bilinear form to be symmetric, the two expressions must be equal. However, since the coefficients in $\mathfrak{l}$ are Grassmann numbers, we have $A^I B^J = -B^J A^I$, so it follows that $g_{IJ} = -g_{JI}$. Hence the antisymmetry of (3.38).

Now we use the properties of the group L and the action $\rhd$ of G on L to recognize the physical nature of the Lagrange multiplier D in (3.2). Indeed, the choice of the group L dictates that D contains 8 independent complex Grassmannian matter fields as its components. Moreover, due to the fact that D is a 0-form and that it transforms according to the spinorial representation of $\mathrm{SO}(3,1)$, we can identify its components with the Dirac bispinor fields, and write

$$D = \psi^\alpha P_\alpha + \bar{\psi}_\alpha P^\alpha\,, \tag{3.40}$$

where it is assumed that ψ and $\bar{\psi}$ are independent fields, as usual. This is again an illustration of the fact that information about the structure of the matter sector in the theory is specified by the choice of the group L in the 2-crossed module, and another main result of the paper.

Given all of the above, now we can finally write the $3BF$ action (3.2) corresponding to this choice of the 2-crossed module as

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + e_a \wedge \nabla\beta^a + (\bar{\gamma}\overset{\leftarrow}{\nabla})_\alpha\psi^\alpha + \bar{\psi}_\alpha(\overset{\rightarrow}{\nabla}\gamma)^\alpha\,. \tag{3.41}$$

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In order to promote this action into a full theory of gravity coupled to Dirac fermions, we add the convenient constraint terms to the action, as follows:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + (\bar{\gamma} \overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla} \gamma)^\alpha \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & - \lambda^\alpha \wedge \left(\bar{\gamma}_\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar{\psi} \gamma^d)_\alpha \right) + \bar{\lambda}_\alpha \wedge \left(\gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^\alpha \right) \\ & - \frac{1}{12} m \, \bar{\psi} \psi \, \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d + 2\pi i l_p^2 \, \bar{\psi} \gamma_5 \gamma^a \psi \, \varepsilon_{abcd} e^b \wedge e^c \wedge \beta^d \, . \end{aligned} \tag{3.42}$$

Here the first row is the topological sector, the second row is the gravitational simplicity constraint term from (2.24), while the third row contains the new simplicity constraints for the Dirac field corresponding to the Lagrange multiplier 1-forms λ^α and $\bar{\lambda}_\alpha$. The fourth row contains the mass term for the Dirac field, and a term which ensures the correct coupling between the torsion and the spin of the Dirac field, as specified by the Einstein-Cartan theory. Namely, we want to ensure that the torsion has the form

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a \, , \tag{3.43}$$

where

$$s_a = i \varepsilon_{abcd} e^b \wedge e^c \bar{\psi} \gamma_5 \gamma^d \psi \tag{3.44}$$

is the spin 2-form. Of course, other couplings should also be straightforward to implement, but we choose this particular coupling because we are interested in reproducing the standard Einstein-Cartan gravity coupled to the Dirac field.

Varying the action (3.42) with respect to B_{ab}, λ^{ab}, $\bar{\gamma}_\alpha$, γ^α, λ^α, $\bar{\lambda}_\alpha$, $\bar{\psi}_\alpha$, ψ^α, e^a, β^a and ω^{ab} one obtains the equations of motion:

$$R^{ab} - \lambda^{ab} = 0 \, , \tag{3.45}$$

$$B_{ab} - \frac{1}{16\pi l_p^2} \varepsilon_{abcd} e^c \wedge e^d = 0 \, , \tag{3.46}$$

$$(\overrightarrow{\nabla} \psi)^\alpha - \lambda^\alpha = 0 \, , \tag{3.47}$$

$$(\bar{\psi} \overleftarrow{\nabla})_\alpha - \bar{\lambda}_\alpha = 0 \, , \tag{3.48}$$

$$\bar{\gamma}_\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar{\psi} \gamma^d)_\alpha = 0 \, , \tag{3.49}$$

$$\gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^\alpha = 0 \, , \tag{3.50}$$

$$\begin{aligned} \mathrm{d}\gamma^\alpha + \omega^\alpha{}_\beta \wedge \gamma^\beta + \frac{i}{6} \lambda^\beta \wedge \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \gamma^{d\alpha}{}_\beta + \frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \psi^\alpha & \\ + i 2\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\gamma_5 \gamma^d \psi)^\alpha = 0 \, , & \end{aligned} \tag{3.51}$$

$$\begin{aligned} \mathrm{d}\bar{\gamma}_\alpha - \bar{\gamma}_\beta \wedge \omega^\beta{}_\alpha + \frac{i}{6} \bar{\lambda}_\beta \wedge \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \gamma^{d\beta}{}_\alpha - \frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \bar{\psi}_\alpha & \\ - i 2\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\bar{\psi} \gamma_5 \gamma^d)_\alpha = 0 \, , & \end{aligned} \tag{3.52}$$

$$\nabla\beta_a + 2\varepsilon_{abcd}\lambda^{bc}\wedge e^d - \frac{i}{2}\varepsilon_{abcd}\lambda^\alpha\wedge e^b\wedge e^c(\bar\psi\gamma^d)_\alpha + \frac{i}{2}\varepsilon_{abcd}\bar\lambda_\alpha\wedge e^b\wedge e^c(\gamma^d\psi)^\alpha$$
$$-\frac{1}{3}\varepsilon_{abcd}e^b\wedge e^c\wedge e^d m\bar\psi\psi - 4\pi l_p^2 i\varepsilon_{abcd}e^b\wedge\beta^c\bar\psi\gamma_5\gamma^d\psi = 0\,, \quad (3.53)$$
$$\nabla e_a - i2\pi l_p^2\varepsilon_{abcd}e^b\wedge e^c\bar\psi\gamma_5\gamma^d\psi = 0\,, \quad (3.54)$$
$$\nabla B_{ab} - e_{[a}\wedge\beta_{b]} + \bar\gamma\frac{1}{8}[\gamma_a,\gamma_b]\psi + \bar\psi\frac{1}{8}[\gamma_a,\gamma_b]\gamma = 0\,. \quad (3.55)$$

The dynamical degrees of freedom are e^a, ψ^α and $\bar\psi_\alpha$, while the remaining variables are determined in terms of the dynamical variables, and are given as:

$$\begin{aligned}
&B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, && \lambda^\alpha{}_\mu = (\vec\nabla_\mu\psi)^\alpha\,, && \bar\lambda_{\alpha\mu} = (\bar\psi\overleftarrow\nabla_\mu)_\alpha\,,\\
&\bar\gamma_{\alpha\mu\nu\rho} = i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\bar\psi\gamma^d)_\alpha\,, && \gamma^\alpha{}_{\mu\nu\rho} = -i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\gamma^d\psi)^\alpha\,, && \\
&\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, && \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu + K^{ab}{}_\mu\,. &&
\end{aligned} \quad (3.56)$$

Here $K^{ab}{}_\mu$ is the contorsion tensor, constructed in the standard way from the torsion tensor, whereas from (3.54) we have

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a\,, \quad (3.57)$$

which is precisely the desired equation (3.43). Further, from the equation (3.46) one obtains

$$\nabla B^{ab} = -\frac{1}{8\pi l_p^2}\varepsilon^{abcd}\left(e_c\wedge\nabla e_d\right)\,. \quad (3.58)$$

Substituting this expression in the equation (3.55) it follows that

$$2\varepsilon_{abcd}e^c\wedge\left(-\frac{1}{16\pi l_p^2}\nabla e^d + \frac{1}{8}s^d\right) - e_{[a}\wedge\beta_{b]} = 0\,. \quad (3.59)$$

The expression in the parentheses is equal to zero, according to the equation (3.54). From the remaining term $e_{[a}\wedge\beta_{b]} = 0$ it again follows that

$$\beta = 0\,. \quad (3.60)$$

Using this result, the equation of motion (3.51) for fermions becomes

$$\frac{i}{6}\varepsilon_{abcd}e^a\wedge e^b\wedge\left(2e^c\wedge\gamma^d\vec\nabla + \frac{im}{2}e^c\wedge e^d - 3(\nabla e^c)\gamma^d\right)\psi = 0\,. \quad (3.61)$$

Using equation (3.54), the last term in the parentheses vanishes, and the equation reduces to the covariant Dirac equation,

$$(i\gamma^a e^\mu{}_a\vec\nabla_\mu - m)\psi = 0\,, \quad (3.62)$$

where $e^\mu{}_a$ is the inverse tetrad. Similarly, the equation (3.52) gives the conjugated Dirac equation:

$$\bar\psi(i\overleftarrow\nabla_\mu e^\mu{}_a\gamma^a + m) = 0\,. \quad (3.63)$$

Finally, the equation of motion (3.53) for tetrad field reduces to

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \qquad T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\gamma^{\nu}\overset{\leftrightarrow}{\nabla}{}^{a}e^{\mu}{}_{a}\psi - \frac{1}{2}g^{\mu\nu}\bar{\psi}\Big(i\gamma^{a}\overset{\leftrightarrow}{\nabla}_{\rho}e^{\rho}{}_{a} - 2m\Big)\psi\,, \quad (3.64)$$

Here, we used the notation $\overset{\leftrightarrow}{\nabla} = \overset{\rightarrow}{\nabla} - \overset{\leftarrow}{\nabla}$. The system of equations (3.45)–(3.55) is equivalent to the system of equations (3.56), (3.60), (3.62)–(3.64). As we expected, the equations of motion (3.57), (3.62), (3.63) and (3.64) are precisely the equations of motion of the Einstein-Cartan theory coupled to a Dirac field.

3.4 Constrained $3BF$ action for the Weyl and Majorana fields

A general solution of the Dirac equation is not an irreducible representation of the Lorentz group, and one can rewrite Dirac fermions as left-chiral and right-chiral fermion fields that both retain their chirality under Lorentz transformations, implying their irreducibility. Hence, it is useful to rewrite the action for left and right Weyl spinors as a constrained $3BF$ action. For simplicity, we will discuss only left-chiral spinor field, while the right-chiral field can be treated analogously. Both Weyl and Majorana fermions can be treated in the same way, the only difference being the presence of an additional mass term in the Majorana action.

We being by defining a 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$, as follows. The groups are:

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^4(\mathbb{G})\,. \tag{3.65}$$

The maps ∂, δ and the Peiffer lifting are trivial. The action $\triangleright$ of the group G on G, H and L is given in the same way as for the Dirac case, whereas the spinorial representation reduces to

$$M_{ab} \triangleright P^{\alpha} = \frac{1}{2}(\sigma_{ab})^{\alpha}{}_{\beta}P^{\beta}\,, \qquad M_{ab} \triangleright P_{\dot\alpha} = \frac{1}{2}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha}P_{\dot\beta}\,, \tag{3.66}$$

where $\sigma^{ab} = -\bar\sigma^{ab} = \frac{1}{4}(\sigma^a\bar\sigma^b - \sigma^b\bar\sigma^a)$, for $\sigma^a = (1,\vec\sigma)$ and $\bar\sigma^a = (1,-\vec\sigma)$, in which $\vec\sigma$ denotes the set of three Pauli matrices. The four generators of the group L are denoted as P^{α} and $P_{\dot\alpha}$, where the Weyl indices $\alpha, \dot\alpha$ take values $1, 2$.

The 3-connection $(\alpha\,, \beta\,, \gamma)$ now takes the form corresponding to this choice of Lie groups,

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma_{\alpha}P^{\alpha} + \bar\gamma^{\dot\alpha}P_{\dot\alpha}\,, \tag{3.67}$$

while the fake 3-curvature $(\mathcal{F}\,, \mathcal{G}\,, \mathcal{H})$ defined in (3.1) is

$$\mathcal{F} = R^{ab}M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,, \tag{3.68}$$

$$\mathcal{H} = \Big(\mathrm{d}\gamma_{\alpha} + \frac{1}{2}\omega^{ab}(\sigma^{ab})^{\beta}{}_{\alpha}\gamma_{\beta}\Big)P^{\alpha} + \Big(\mathrm{d}\bar\gamma^{\dot\alpha} + \frac{1}{2}\omega_{ab}(\bar\sigma^{ab})^{\dot\alpha}{}_{\dot\beta}\bar\gamma^{\dot\beta}\Big)P_{\dot\alpha} \equiv (\overset{\rightarrow}{\nabla}\gamma)_{\alpha}P^{\alpha} + (\bar\gamma\overset{\leftarrow}{\nabla})^{\dot\alpha}P_{\dot\alpha}\,.$$

Introducing the spinor fields ψ_{α} and $\bar\psi^{\dot\alpha}$ via the Lagrange multiplier D as

$$D = \psi_{\alpha}P^{\alpha} + \bar\psi^{\dot\alpha}P_{\dot\alpha}\,, \tag{3.69}$$

and using the bilinear form $\langle\_,\_\rangle_{\mathfrak{l}}$ for the group L,

$$\langle P^{\alpha}, P^{\beta}\rangle_{\mathfrak{l}} = \varepsilon^{\alpha\beta}\,, \qquad \langle P_{\dot\alpha}, P_{\dot\beta}\rangle_{\mathfrak{l}} = \varepsilon_{\dot\alpha\dot\beta}\,, \qquad \langle P^{\alpha}, P_{\dot\beta}\rangle_{\mathfrak{l}} = 0\,, \qquad \langle P_{\dot\alpha}, P^{\beta}\rangle_{\mathfrak{l}} = 0\,, \tag{3.70}$$

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where $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$ are the usual two-dimensional antisymmetric Levi-Civita symbols, the topological $3BF$ action (3.2) for spinors coupled to gravity becomes

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \psi^\alpha \wedge (\overrightarrow{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}} \wedge (\bar{\gamma}\overleftarrow{\nabla})^{\dot{\alpha}} . \tag{3.71}$$

In order to obtain the suitable equations of motion for the Weyl spinors, we again introduce appropriate simplicity constraints, so that the action becomes:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \psi^\alpha \wedge (\overrightarrow{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}} \wedge (\bar{\gamma}\overleftarrow{\nabla})^{\dot{\alpha}} \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & - \lambda^\alpha \wedge \left(\gamma_\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} \right) - \bar{\lambda}_{\dot{\alpha}} \wedge \left(\bar{\gamma}^{\dot{\alpha}} + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta \right) \\ & - 4\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\bar{\psi}_{\dot{\alpha}} \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta) . \end{aligned} \tag{3.72}$$

The new simplicity constraints are in the third row, featuring the Lagrange multiplier 1-forms λ_α and $\bar{\lambda}^{\dot{\alpha}}$. Also, using the coupling between the Dirac field and torsion from Einstein-Cartan theory as a model, the term in the fourth row is chosen to ensure that the coupling between the Weyl spin tensor

$$s_a \equiv i\varepsilon_{abcd} e^b \wedge e^c \; \psi^\alpha \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} , \tag{3.73}$$

and torsion is given as:

$$T_a = 4\pi l_p^2 s_a . \tag{3.74}$$

The case of the Majorana field is introduced in exactly the same way, albeit with an additional mass term in the action, of the form:

$$-\frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d (\psi^\alpha \psi_\alpha + \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}) . \tag{3.75}$$

Varying the action (3.72) with respect to the variables B_{ab}, λ^{ab}, γ_α, $\bar{\gamma}^{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, e^a, β^a and ω^{ab} one again obtains the complete set of equations of motion, displayed in the appendix C. The only dynamical degrees of freedom are ψ_α, $\bar{\psi}^{\dot{\alpha}}$ and e^a, while the remaining variables are algebraically determined in terms of these as:

$$\begin{aligned} & \lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu} , \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu , \quad \lambda_{\alpha\mu} = \nabla_\mu \psi_\alpha , \quad \bar{\lambda}^{\dot{\alpha}}{}_\mu = \nabla_\mu \bar{\psi}^{\dot{\alpha}} , \\ & \gamma_{\alpha\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} , \quad \bar{\gamma}^{\dot{\alpha}}{}_{\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta , \quad \omega_{ab\mu} = \triangle_{ab\mu} + K_{ab\mu} . \end{aligned} \tag{3.76}$$

In addition, one also maintains the result $\beta = 0$ as before. Finally, the equations of motion for the dynamical fields are

$$\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta = 0 , \qquad \sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} = 0 , \tag{3.77}$$

and

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2 \, T^{\mu\nu} , \tag{3.78}$$

where

$$T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} - g^{\mu\nu}\frac{1}{2}\left(i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi}\right). \qquad (3.79)$$

Here we have suppressed the spinor indices. In the case of the Majorana field, the equations of motion (3.76) remain the same, while the equations of motion for ψ_α and $\bar{\psi}^{\dot{\alpha}}$ take the form

$$i\sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} - m\psi_\alpha = 0\,, \qquad i\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta - m\bar{\psi}^{\dot{\alpha}} = 0\,, \qquad (3.80)$$

whereas the stress-energy tensor takes the form

$$\begin{aligned} T^{\mu\nu} \equiv\ & \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} \\ & - g^{\mu\nu}\frac{1}{2}\left[i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi} - \frac{1}{2}m\left(\psi\psi + \bar{\psi}\bar{\psi}\right)\right]. \end{aligned} \qquad (3.81)$$

4 The Standard Model

The Standard Model 3-group can be defined as:

$$G = \mathrm{SO}(3,1)\times\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\,, \quad H = \mathbb{R}^4\,, \quad L = \mathbb{R}^4(\mathbb{C})\times\mathbb{R}^{64}(\mathbb{G})\times\mathbb{R}^{64}(\mathbb{G})\times\mathbb{R}^{64}(\mathbb{G})\,, \qquad (4.1)$$

where $\mathbb{C}$ denotes the field of complex numbers. The motivation for this choice of the group L is given in the table below.

1. lepton generation	red color 1. quark generation	green color 1. quark generation	blue color 1. quark generation
$\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L$	$\begin{pmatrix}u_r \\ d_r\end{pmatrix}_L$	$\begin{pmatrix}u_g \\ d_g\end{pmatrix}_L$	$\begin{pmatrix}u_b \\ d_b\end{pmatrix}_L$
$(\nu_e)_R$	$(u_r)_R$	$(u_g)_R$	$(u_b)_R$
$(e^-)_R$	$(d_r)_R$	$(d_g)_R$	$(d_b)_R$

We see that in order to introduce one generation of matter one needs to provide 16 spinors, or equivalently the group L has to be chosen as $L = \mathbb{R}^{64}(\mathbb{G})$. As there are three generations of matter, the part of the group L that corresponds to the fermion fields in the theory is chosen to be $L = \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G})$. To define the Higgs sector one needs two complex scalar fields $\begin{pmatrix}\phi^+ \\ \phi_0\end{pmatrix}$, or equivalently the scalar sector of the group L is given as $L = \mathbb{R}^4(\mathbb{C})$.

The maps ∂, δ and the Peiffer lifting are trivial. The action of the group G on itself is given via conjugation. The action of the $\mathrm{SO}(3,1)$ subgroup of G on H is via vector representation and the action of $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ subgroup on H is via trivial representation. The action of the $\mathrm{SO}(3,1)$ on L is via trivial representation for the generators corresponding to the scalar fields, i.e. the $\mathbb{R}^4(\mathbb{C})$ subgroup of L, and via spinor representation for the every quadruple of generators corresponding to the fermion fields, given as in the section 3. The information how spinors transform under the $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ group is encoded in the action of that subgroup of G on L, as specified in the table above. For simplicity, in the following, only one family of the lepton sector and only electroweak part of the gauge sector of the Standard model is considered.

The groups are chosen as:

$$G=\mathrm{SO}(3,1)\times\mathrm{SU}(2)\times\mathrm{U}(1)\,,\quad H=\mathbb{R}^4\,,\quad L^{\text{leptons}}=\mathbb{R}^{16}(\mathbb{G})\times\mathbb{R}^4(\mathbb{C})\,. \tag{4.2}$$

The 3-connection then takes the form

$$\begin{aligned}&\alpha=\omega^{ab}M_{ab}+W^IT_I+AY\,, && \beta=\beta^aP_a\,,\\ &\gamma=\gamma_\alpha{}^{\tilde L}P^\alpha{}_{\tilde L}+\gamma^{\dot\alpha}{}_{\tilde L}P_{\dot\alpha}{}^{\tilde L}+\gamma_\alpha{}^{\tilde R}P^\alpha{}_{\tilde R}+\gamma^{\dot\alpha}{}_{\tilde R}P_{\dot\alpha}{}^{\tilde R}+\gamma^{\tilde a}P_{\tilde a}\,.\end{aligned} \tag{4.3}$$

Here the indices $I,J,\dots$ take the values $1,2,3$ and counts the Pauli matrices, generators of the group SU(2), the indices $\tilde L,\tilde L',\dots$ take the values $1,2$ and count the components of left doublet, $\tilde R$ denotes the right singlet $(e^-)_R$ and right singlet $(\nu_e)_R$, and indices $\tilde a,\tilde b,\dots$ take values $1,2$ and count the components of the scalar doublet. It is also useful to define $\tilde i=(\tilde L,\tilde R)$ which takes values $1,\dots,4$.

The action of the group G on L is defined as:

$$\begin{aligned}&M_{ab}\triangleright P^\alpha{}_i=\frac12(\sigma_{ab})^\alpha{}_\beta P^\beta{}_i\,, \qquad M_{ab}\triangleright P_{\dot\alpha i}=\frac12(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha}P_{\dot\beta i}\,, \qquad M_{ab}\triangleright P_{\tilde a}=0\,,\\ &T_I\triangleright P^\alpha{}_{\tilde L}=\frac12(\sigma_I)^{\tilde L'}{}_{\tilde L}P^\alpha{}_{\tilde L'}\,, \qquad T_I\triangleright P_{\dot\alpha\tilde L}=\frac12(\sigma_I)^{\tilde L'}{}_{\tilde L}P_{\dot\alpha\tilde L'}\,,\\ &T_I\triangleright P^\alpha{}_{\tilde R}=0\,, \qquad T_I\triangleright P_{\dot\alpha\tilde R}=0\,, \qquad T_I\triangleright P_{\tilde a}=\frac12(\sigma_I)^{\tilde b}{}_{\tilde a}P_{\tilde b}\,,\\ &Y\triangleright P^\alpha{}_{\tilde L}=-P^\alpha{}_{\tilde L}\,, \quad Y\triangleright P^\alpha{}_{e_R}=-2P^\alpha{}_{e_R}\,, \quad Y\triangleright P^\alpha{}_{\nu_R}=-2P^\alpha{}_{\nu_R}\,, \quad Y\triangleright P_{\tilde a}=P_{\tilde a}\,,\\ &Y\triangleright P_{\dot\alpha\tilde L}=-P_{\dot\alpha\tilde L}\,, \qquad Y\triangleright P_{\dot\alpha e_R}=-2P_{\dot\alpha e_R}\,, \qquad Y\triangleright P_{\dot\alpha\nu_R}=-2P_{\dot\alpha\nu_R}\,.\end{aligned} \tag{4.4}$$

The 3-curvatures are given as:

$$\begin{aligned}&\mathcal F=R^{ab}M_{ab}+F^IT_I+FY\,, \qquad \mathcal G=\nabla\beta^aP_a\,,\\ &\mathcal H=(\overrightarrow{\nabla}\gamma^{\tilde L})_\alpha P^\alpha{}_{\tilde L}+(\bar\gamma_{\tilde L}\overleftarrow{\nabla})^{\dot\alpha}P_{\dot\alpha}{}^{\tilde L}+(\overrightarrow{\nabla}\gamma^{\tilde R})_\alpha P^\alpha{}_{\tilde R}+(\bar\gamma_{\tilde R}\overleftarrow{\nabla})^{\dot\alpha}P_{\dot\alpha}{}^{\tilde R}+\mathrm d\gamma^{\tilde a}P_{\tilde a}\,.\end{aligned} \tag{4.5}$$

The topological $3BF$ action is defined as:

$$S=\int B_{ab}R^{ab}+B_IF^I+BF+e_a\nabla\beta^a+\psi^\alpha{}_{\tilde i}(\overrightarrow{\nabla}\gamma^{\tilde i})_\alpha+\bar\psi_{\dot\alpha}{}^{\tilde i}(\bar\gamma_{\tilde i}\overleftarrow{\nabla})^{\dot\alpha}+\phi^{\tilde a}\mathrm d\gamma_{\tilde a}\,. \tag{4.6}$$

At this point, it is useful to simplify the notation and denote all indices of the group G by $\hat\alpha$, of the group H by $\hat a$ and L by $\hat A$. In order to promote this action to a full theory of first lepton family coupled to electroweak gauge fields, Higgs field, and gravity, we again introduce the appropriate simplicity constraint, as follows

$$
\begin{aligned}
S = \int & B_{\hat{\alpha}} \wedge \mathcal{F}^{\hat{\alpha}} + e_{\hat{a}} \wedge \mathcal{G}^{\hat{a}} + D_{\hat{A}} \wedge \mathcal{H}^{\hat{A}} \\
& + \left(B_{\hat{\alpha}} - C_{\hat{\alpha}}{}^{\hat{\beta}} M_{cd\hat{\beta}} e^c \wedge e^d \right) \wedge \lambda^{\hat{\alpha}} - \left(\gamma_{\hat{A}} - e^a \wedge e^b \wedge e^c C_{\hat{A}}{}^{\hat{B}} M_{abc\hat{B}} \right) \wedge \lambda^{\hat{A}} \\
& + \zeta^{ab}{}_{\hat{\alpha}} \wedge \left(M_{ab}{}^{\hat{\alpha}} \varepsilon^{cdef} e_c \wedge e_d \wedge e_e \wedge e_f - F^{\hat{\alpha}} \wedge e_c \wedge e_d \right) \\
& + \zeta^{ab}{}_{\hat{A}} \wedge \left(M_{abc}{}^{\hat{A}} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - F^{\hat{A}} \wedge e_a \wedge e_b \right) \\
& - \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \; \left(Y_{\hat{A}\hat{B}\hat{C}} D^{\hat{A}} D^{\hat{B}} D^{\hat{C}} + M_{\hat{A}\hat{B}} D^{\hat{A}} D^{\hat{B}} + L_{\hat{A}\hat{B}\hat{C}\hat{D}} D^{\hat{A}} D^{\hat{B}} D^{\hat{C}} D^{\hat{D}} \right) \\
& - 4\pi i \, l_p^2 \, \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c D_{\hat{A}} T^{d\hat{A}}{}_{\hat{B}} D^{\hat{B}} \,,
\end{aligned} \tag{4.7}
$$

where:

$$
\begin{aligned}
& B_{\hat{\alpha}} = \left[B_{ab} \; B_I \; B \right], \qquad \mathcal{F}^{\hat{\alpha}} = \left[R_{ab} \; F_I \; F \right]^T, \qquad D_{\hat{A}} = \left[\; \psi^{\alpha}{}_{\tilde{L}} \; \bar{\psi}_{\dot{\alpha}\tilde{L}} \; \psi^{\alpha}{}_{R} \; \bar{\psi}_{\dot{\alpha}R} \; \phi_{\tilde{a}} \right], \\
& \mathcal{H}^{\hat{A}} = \left[\, (\overrightarrow{\nabla} \gamma_{\tilde{L}})_{\alpha} \;\; (\bar{\gamma}_{\tilde{L}} \overleftarrow{\nabla})^{\dot{\alpha}} \;\; (\overrightarrow{\nabla} \gamma_{\tilde{R}})_{\alpha} \;\; (\bar{\gamma}_{\tilde{R}} \overleftarrow{\nabla})^{\dot{\alpha}} \;\; \mathrm{d}\gamma_{\tilde{a}} \; \right]^T, \quad \gamma_{\hat{A}} = \left[\gamma^{\alpha}{}_{\tilde{L}} \; \bar{\gamma}_{\dot{\alpha}\tilde{L}} \; \gamma^{\alpha}{}_{\tilde{R}} \; \bar{\gamma}_{\dot{\alpha}\tilde{R}} \; \gamma_{\tilde{a}} \right], \\
& \lambda^{\hat{\alpha}} = \left[-\lambda^{ab} \; \lambda^I \; \lambda \right]^T, \qquad \zeta^{cd}{}_{\hat{\alpha}} = \left[0 \; \zeta^{cd}{}_I \; \zeta^{cd} \right], \qquad \zeta^{ab}{}_{\hat{A}} = \left[\zeta^{ab} \; 0 \; 0 \right], \\
& \lambda^{\hat{A}} = \left[\lambda_{\alpha L} \; \bar{\lambda}^{\dot{\alpha}}{}_L \; \lambda_{\alpha R} \; \bar{\lambda}^{\dot{\alpha}}{}_R \; \lambda^{\tilde{a}} \right]^T, \qquad M_{cd\hat{\alpha}} = \left[\varepsilon_{abcd} \; M_{cdI} \; M_{cd} \right], \\
& M_{abc\hat{A}} = \left[\; \varepsilon_{abcd} \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}}{}_L \;\; \varepsilon_{abcd} \bar{\sigma}^{d\dot{\alpha}\beta} \psi_{\beta L} \;\; \varepsilon_{abcd} \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}}{}_R \;\; \varepsilon_{abcd} \bar{\sigma}^{d\dot{\alpha}\beta} \psi_{\beta R} \;\; M_{abc\tilde{a}} \; \right].
\end{aligned}
$$

The matrices $C^{\hat{\alpha}}{}_{\hat{\beta}}$, $C^{\hat{A}}{}_{\hat{B}}$, $M_{\hat{A}\hat{B}}$, $Y_{\hat{A}\hat{B}\hat{C}}$, $L_{\hat{A}\hat{B}\hat{C}\hat{D}}$ and $T^{d\hat{A}}{}_{\hat{B}}$ are constant matrices, and carry the information about gauge coupling constants, mass of the Higgs field, Yukawa couplings and mixing angles, Higgs self-coupling constant and torsion coupling, respectively.

5 Conclusions

Let us summarize the results of the paper. In section 2 we have given a short reminder of the BF theory and described how one can use it to construct the action for general relativity (the well known Plebanski model), and the action for the Yang-Mills theory in flat spacetime, in a novel way. Passing on to higher gauge theory, we have reviewed the formalism of 2-groups and the corresponding $2BF$ theory, using it again to construct the action for general relativity (a model first described in [12]), and the unified action of general relativity and Yang-Mills theory, both naturally described using the 2-group formalism. With this background material in hand, in section 3 we have used the idea of a categorical ladder yet again, generalizing the $2BF$ theory to $3BF$ theory, with the underlying structure of a 3-group instead of a 2-group. This has led us to the main insight that the *scalar and fermion fields can be specified using a gauge group*, namely the third gauge group, denoted L, present in the 2-crossed module corresponding to a given 3-group. This has allowed us to single out specific gauge groups corresponding to the Klein-Gordon, Dirac, Weyl and Majorana fields, and to construct the relevant constrained $3BF$ actions that describe all these fields coupled to gravity in the standard way.

The obtained results represent the fundamental building blocks for the construction of the complete Standard Model of elementary particles coupled to Einstein-Cartan gravity as a $3BF$ action with suitable simplicity constraints, as demonstrated in section 4. In this way, we can complete the first step of the spinfoam quantization programme for the complete theory of gravity and all matter fields, as specified in the Introduction. This is a clear improvement over the ordinary spinfoam models based on an ordinary constrained BF theory.

In addition to this, the gauge group which determines the matter spectrum of the theory is a completely novel structure, not present in the Standard Model. This new gauge group stems from the 3-group structure of the theory, so it is not surprising that it is invisible in the ordinary formulation of the Standard Model, since the latter does not use any 3-group structure in an explicit way. In this paper, we have discussed the choices of this group which give rise to all relevant matter fields, and these can simply be directly multiplied to give the group corresponding to the full Standard Model, encoding the quark and lepton families and all other structure of the matter spectrum. However, the true potential of the matter gauge group lies in a possibility of nontrivial unification of matter fields, by choosing it to be something other than the ordinary product of its component groups. For example, instead of choosing $\mathbb{R}^8(\mathbb{G})$ for the Dirac field, one can try a noncommutative SU(3) group, which also contains 8 generators, but its noncommutativity requires that the maps δ and $\{\_\,,\_\}$ be nontrivial, in order to satisfy the axioms of a 2-crossed module. This, in turn, leads to a distinction between 3-curvature and fake 3-curvature, which can have consequences for the dynamics of the theory. In this way, by studying nontrivial choices of a 3-group, one can construct various different 3-group-unified models of gravity and matter fields, within the context of higher gauge theory. This idea resembles the ordinary grand unification programme within the framework of the standard gauge theory, where one constructs various different models of vector fields by making various choices for the Yang-Mills gauge group. The detailed discussion of these 3-group unified models is left for future work.

As far as the spinfoam quantization programme is concerned, having completed the step 1 (as outlined in the Introduction), there is a clear possibility to complete the steps 2 and 3 as well. First, the fact that the full action is written completely in terms of differential forms of various degrees, allows us to adapt it to a triangulated spacetime manifold, in the sense of Regge calculus. In particular, all fields and their field strengths present in the $3BF$ action can be naturally associated to the appropriate d-dimensional simplices of a 4-dimensional triangulation, by matching 0-forms to vertices, 1-forms to edges, etc. This leads us to the following table:

d	triangulation	dual triangulation	form	fields	field strengths
0	vertex	4-polytope	0-form	ϕ, $\psi_{\tilde{\alpha}}$, $\bar{\psi}^{\tilde{\alpha}}$	
1	edge	3-polyhedron	1-form	ω^{ab}, A^I, e^a	
2	triangle	face	2-form	β^a, B^{ab}	R^{ab}, F^I, T^a
3	tetrahedron	edge	3-form	γ, $\gamma_{\tilde{\alpha}}$, $\bar{\gamma}^{\tilde{\alpha}}$	$\mathcal{G}^a$
4	4-simplex	vertex	4-form		$\mathcal{H}$, $\mathcal{H}_{\tilde{\alpha}}$, $\bar{\mathcal{H}}^{\tilde{\alpha}}$

Once the classical Regge-discretized topological $3BF$ action is constructed, one can attempt to construct a state sum Z which defines the path integral for the theory. The topological nature of the pure $3BF$ action, together with the underlying structure of the 3-group, should ensure that such a state sum Z is a topological invariant, in the sense that it is triangulation independent. Unfortunately, in order to perform this step precisely, one needs a generalization of the Peter-Weyl and Plancharel theorems to 2-groups and 3-groups, a mathematical result that is presently still missing. The purpose of the Peter-Weyl theorem is to provide a decomposition of a function on a group into a sum over the corresponding irreducible representations, which ultimately specifies the appropriate spectrum of labels for the d-simplices in the triangulation, fixing the domain of values for the fields living on those d-simplices. In the case of 2-groups and especially 3-groups, the representation theory has not been developed well enough to allow for such a construction, with a consequence of the missing Peter-Weyl theorem for 2-groups and 3-groups. However, until the theorem is proved, we can still try to *guess* the appropriate structure of the irreducible representations of the 2- and 3-groups, as was done for example in [12], leading to the so-called *spincube model* of quantum gravity.

Finally, if we remember that for the purpose of physics we are not really interested in a topological theory, but instead in one which contains local propagating degrees of freedom, we are therefore not really engaged in constructing a topological invariant Z, but rather a state sum which describes nontrivial dynamics. In particular, we need to impose the simplicity constraints onto the state sum Z, which is the step 3 of the spinfoam quantization programme. In light of that, one of the main motivations and also main results of our paper was to rewrite the action for gravity and matter in a way that explicitly distinguishes the topological sector from the simplicity constraints. Imposing the constraints is therefore straightforward in the context of a 3-group gauge theory, and completing this step would ultimately lead us to a state sum corresponding to a tentative theory of quantum gravity with matter. This is also a topic for future work.

In the end, let us also mention that aside from the unification and quantization programmes, there is also a plethora of additional studies one can perform with the constrained $3BF$ action, such as the analysis of the Hamiltonian structure of the theory (suitable for a potential canonical quantization programme), the idea of imposing the simplicity constraints using a spontaneous symmetry breaking mechanism, and finally a detailed study of the mathematical structure and properties of the simplicity constraints. This list is of course not conclusive, and there may be many more interesting related topics to study in both physics and mathematics.

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A Category theory, 2-groups and 3-groups

Definition 1 (Pre-crossed module and crossed module) *A pre-crossed module* $(H \overset{\partial}{\to} G\,, \triangleright)$ *of groups* G *and* H*, is given by a group map* $\partial : H \to G$*, together with a left action* $\triangleright$ *of* G *on* H*, by automorphisms, such that for each* $h_1\,, h_2 \in H$ *and* $g \in G$ *the following identity hold:*

$$g \partial h g^{-1} = \partial(g \triangleright h)\,.$$

In a pre-crossed module the **Peiffer commutator** *is defined as:*

$$\langle h_1\,, h_2 \rangle_{\rm p} = h_1 h_2 h_1^{-1} \partial(h_1) \triangleright h_2^{-1}\,.$$

A pre-crossed module is said to be a **crossed module** *if all of its Peiffer commutators are trivial, which is to say that*

$$(\partial h) \triangleright h' = h h' h^{-1}\,,$$

i.e. the **Peiffer identity** *is satisfied.*

Definition 2 (2-crossed module) *A 2-crossed module* $(L \overset{\delta}{\to} H \overset{\partial}{\to} G, \triangleright, \{-,-\})$ *is given by three groups* G*,* H *and* L*, together with maps* ∂ *and* δ *such that:*

$$L \overset{\delta}{\to} H \overset{\partial}{\to} G\,,$$

where $\partial\delta = 1$*, an action* $\triangleright$ *of the group* G *on all three groups, and an* G*-equivariant map called the* **Peiffer lifting***:*

$$\{-\,, -\} : H \times H \to L\,.$$

The following identities are satisfied:

1. *The maps* ∂ *and* δ *are* G*-equivariant, i.e. for each* $g \in G$ *and* $h \in H$*:*

$$g \triangleright \partial(h) = \partial(g \triangleright h)\,, \qquad g \triangleright \delta(l) = \delta(g \triangleright l)\,,$$

   the action of the group G *on the groups* H *and* L *is a smooth left action by automorphisms, i.e. for each* $g, g_1, g_2 \in G$*,* $h_1, h_2 \in H$*,* $l_1, l_2 \in L$ *and* $e \in H, L$*:*

$$g_1 \triangleright (g_2 \triangleright e) = (g_1 g_2) \triangleright e\,, \quad g \triangleright (h_1 h_2) = (g \triangleright h_1)(g \triangleright h_2)\,, \quad g \triangleright (l_1 l_2) = (g \triangleright l_1)(g \triangleright l_2)\,,$$

   and the Peiffer lifting is G*-equivariant, i.e. for each* h_1*,* $h_2 \in H$ *and* $g \in G$*:*

$$g \triangleright \{h_1\,, h_2\} = \{g \triangleright h_1,\, g \triangleright h_2\}\,;$$

2. *the action of the group* G *on itself is via conjugation, i.e. for each* $g\,, g_0 \in G$*:*

$$g \triangleright g_0 = g\, g_0\, g^{-1}\,;$$

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3. *In a 2-crossed module the structure* $(L \overset{\delta}{\to} H, \triangleright')$ *is a crossed module, with action of the group* H *on the group* L *is defined for each* $h \in H$ *and* $l \in L$ *as:*

$$h \triangleright' l = l\,\{\delta(l)^{-1},\, h\}\,,$$

but $(H \overset{\partial}{\to} G\,, \triangleright)$ *may not be one, and the Peiffer identity does not necessary hold. However, when* ∂ *is chosen to be trivial and group* H *Abelian, the Peiffer identity is satisfied, i.e. for each* $h,\, h' \in H$*:*

$$\delta(h) \triangleright h' = h\, h'\, h^{-1}\,;$$

4. $\delta(\{h_1, h_2\}) = \langle h_1\,, h_2\rangle_{\mathrm{p}}, \qquad \forall h_1, h_2 \in H,$

5. $[l_1, l_2] = \{\delta(l_1)\,, \delta(l_2)\}, \qquad \forall l_1\,, l_2 \in L.$ *Here, the notation* $[l, k] = lkl^{-1}k^{-1}$ *is used;*

6. $\{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\}\partial(h_1) \triangleright \{h_2, h_3\}, \qquad \forall h_1, h_2, h_3 \in H;$

7. $\{h_1, h_2 h_3\} = \{h_1, h_2\}\{h_1, h_3\}\{\langle h_1, h_3\rangle_{\mathrm{p}}^{-1}, \partial(h_1) \triangleright h_2\}, \qquad \forall h_1, h_2, h_3 \in H;$

8. $\{\delta(l), h\}\{h, \delta(l)\} = l(\partial(h) \triangleright l^{-1}), \qquad \forall h \in H\,, \quad \forall l \in L.$

Definition 3 (Differential pre-crossed module, differential crossed module) *A differential pre-crossed module* $(\mathfrak{h} \overset{\partial}{\to} \mathfrak{g}\,, \triangleright)$ *of algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$ *is given by a Lie algebra map* $\partial : \mathfrak{h} \to \mathfrak{g}$ *together with an action* $\triangleright$ *of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *such that for each* $\underline{h} \in \mathfrak{h}$ *and* $\underline{g} \in \mathfrak{g}$*:*

$$\partial(\underline{g} \triangleright \underline{h}) = [\underline{g}, \partial(\underline{h})]\,.$$

The action $\triangleright$ *of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *is on left by derivations, i.e. for each* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ *and each* $\underline{g} \in \mathfrak{g}$*:*

$$\underline{g} \triangleright [\underline{h}_1,\, \underline{h}_2] = [\underline{g} \triangleright \underline{h}_1,\, \underline{h}_2] + [\underline{h}_1,\, \underline{g} \triangleright \underline{h}_2]\,.$$

In a differential pre-crossed module, the Peiffer commutators are defined for each $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ *as:*

$$\langle \underline{h}_1,\, \underline{h}_2\rangle_{\mathrm{p}} = [\underline{h}_1, \underline{h}_2] - \partial(\underline{h}_1) \triangleright \underline{h}_2\,.$$

The map $(\underline{h}_1,\, \underline{h}_2) \in \mathfrak{h} \times \mathfrak{h} \to \langle \underline{h}_1,\, \underline{h}_2\rangle_{\mathrm{p}} \in \mathfrak{h}$ *is bilinear* $\mathfrak{g}$*-equivariant map called the* **Peiffer paring***, i.e. all* $\underline{h}_1\,, \underline{h}_2 \in \mathfrak{h}$ *and* $\underline{g} \in \mathfrak{g}$ *satisfy the following identity:*

$$\underline{g} \triangleright \langle \underline{h}_1\,, \underline{h}_2\rangle_{\mathrm{p}} = \langle \underline{g} \triangleright \underline{h}_1\,, \underline{h}_2\rangle + \langle \underline{h}_1\,, \underline{g} \triangleright \underline{h}_2\rangle_{\mathrm{p}}\,.$$

A differential pre-crossed module is said to be a **differential crossed module** *if all of its Peiffer commutators vanish, which is to say that for each* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$*:*

$$\partial(\underline{h}_1) \triangleright \underline{h}_2 = [\underline{h}_1,\, \underline{h}_2]\,.$$

Definition 4 (Differential 2-crossed module) *A differential 2-crossed module is given by a complex of Lie algebras:*

$$\mathfrak{l} \overset{\delta}{\to} \mathfrak{h} \overset{\partial}{\to} \mathfrak{g}\,,$$

together with left action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{h}$, $\mathfrak{l}$, by derivations, and on itself via adjoint representation, and a $\mathfrak{g}$-equivariant bilinear map called the **Peiffer lifting**:

$$\{-\,,\,-\}:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{l}$$

Fixing the basis in algebra $T_A\in\mathfrak{l}$, $t_a\in\mathfrak{h}$ and $\tau_\alpha\in\mathfrak{g}$:

$$[T_A,T_B]=f_{AB}{}^C\,T_C\,,\qquad [t_a,t_b]=f_{ab}{}^c\,t_c\,,\qquad [\tau_\alpha,\tau_\beta]=f_{\alpha\beta}{}^\gamma\,\tau_\gamma\,,$$

one defines the maps ∂ and δ as:

$$\partial(t_a)=\partial_a{}^\alpha\,\tau_\alpha\,,\qquad \delta(T_A)=\delta_A{}^a\,t_a\,,$$

and action of $\mathfrak{g}$ on the generators of $\mathfrak{l}$, $\mathfrak{h}$ and $\mathfrak{g}$ is, respectively:

$$\tau_\alpha\triangleright T_A=\triangleright_{\alpha A}{}^B\,T_B\,,\qquad \tau_\alpha\triangleright t_a=\triangleright_{\alpha a}{}^b\,t_b\,,\qquad \tau_\alpha\triangleright\tau_\beta=\triangleright_{\alpha\beta}{}^\gamma\,\tau_\gamma\,.$$

Note that when η is $\mathfrak{g}$-valued differential form and ω is $\mathfrak{l}$, $\mathfrak{h}$ or $\mathfrak{g}$ valued differential form the previous action is defined as:

$$\eta\triangleright\omega=\eta^\alpha\wedge\omega^A\,\triangleright_{\alpha A}{}^B\,T_B\,,\qquad \eta\triangleright\omega=\eta^\alpha\wedge\omega^a\,\triangleright_{\alpha a}{}^b\,t_b\,,\qquad \eta\triangleright\omega=\eta^\alpha\wedge\omega^\beta f_{\alpha\beta}{}^\gamma\,\tau_\gamma\,.$$

The coefficients $X_{ab}{}^A$ are introduced as:

$$\{t_a,\,t_b\}=X_{ab}{}^A T_A\,.$$

The following identities are satisfied:

1. *In the differential crossed module $(L\overset{\delta}{\to}H\,,\triangleright')$ the action $\triangleright'$ of $\mathfrak{h}$ on $\mathfrak{l}$ is defined for each $\underline{h}\in\mathfrak{h}$ and $\underline{l}\in\mathfrak{l}$ as:*

$$\underline{h}\triangleright'\underline{l}=-\{\delta(\underline{l}),\,\underline{h}\}\,,$$

or written in the basis where $t_a\triangleright' T_A=\triangleright'{}_{aA}{}^B T_B$ the previous identity becomes:

$$\triangleright'{}_{aA}{}^B=-\delta_A{}^b X_{ba}{}^B\,;$$

2. *The action of $\mathfrak{g}$ on itself is via adjoint representation:*

$$\triangleright_{\alpha\beta}{}^\gamma=f_{\alpha\beta}{}^\gamma\,;$$

3. *The action of $\mathfrak{g}$ on $\mathfrak{h}$ and $\mathfrak{l}$ is equivariant, i.e. the following identities are satisfied:*

$$\partial_a{}^\beta f_{\alpha\beta}{}^\gamma=\triangleright_{\alpha a}{}^b\partial_b{}^\gamma\,,\qquad \delta_A{}^a\,\triangleright_{\alpha a}{}^b=\triangleright_{\alpha A}{}^B\delta_B{}^b\,;$$

4. *The Peiffer lifting is $\mathfrak{g}$-equivariant, i.e. for each $\underline{h}_1,\underline{h}_2\in\mathfrak{h}$ and $\underline{g}\in\mathfrak{g}$:*

$$\underline{g}\triangleright\{\underline{h}_1,\underline{h}_2\}=\{\underline{g}\triangleright\underline{h}_1,\underline{h}_2\}+\{\underline{h}_1,\,\underline{g}\triangleright\underline{h}_2\}\,,$$

or written in the basis:

$$X_{ab}{}^B\triangleright_{\alpha B}{}^A=\triangleright_{\alpha a}{}^c X_{cb}{}^A+\triangleright_{\alpha b}{}^c X_{ac}{}^A\,;$$

5. $\delta(\{\underline{h}_1, \underline{h}_2\}) = \langle \underline{h}_1, \underline{h}_2 \rangle_{\rm p}$, $\quad \forall \underline{h}_1, \underline{h}_2 \in \mathfrak{h}$, *i.e.*

$$X_{ab}{}^A \delta_A{}^c = f_{ab}{}^c - \partial_a{}^\alpha \rhd_{\alpha b}{}^c ;$$

6. $[\underline{l}_1, \underline{l}_2] = \{\delta(\underline{l}_1), \delta(\underline{l}_2)\}$, $\quad \forall \underline{l}_1, \underline{l}_2 \in \mathfrak{l}$, *i.e.*

$$f_{AB}{}^C = \delta_A{}^a \delta_B{}^b X_{ab}{}^C ;$$

7. $\{[\underline{h}_1, \underline{h}_2], \underline{h}_3\} = \partial(\underline{h}_1) \rhd \{\underline{h}_2, \underline{h}_3\} + \{\underline{h}_1, [\underline{h}_2, \underline{h}_3]\} - \partial(\underline{h}_2) \rhd \{\underline{h}_1, \underline{h}_3\} - \{\underline{h}_2, [\underline{h}_1, \underline{h}_3]\}$, $\forall \underline{h}_1, \underline{h}_2, \underline{h}_3 \in \mathfrak{h}$, *i.e.*

$$\{[\underline{h}_1,\underline{h}_2],\underline{h}_3\} = \{\partial(\underline{h}_1)\rhd\underline{h}_2,\underline{h}_3\} - \{\partial(\underline{h}_2)\rhd\underline{h}_1,\underline{h}_3\} - \{\underline{h}_1,\delta\{\underline{h}_2,\underline{h}_3\}\} + \{\underline{h}_2,\delta\{\underline{h}_1,q,\underline{h}_3\}\},$$

$$f_{ab}{}^d X_{dc}{}^B = \partial_a{}^\alpha X_{bc}{}^A \rhd_{\alpha A}{}^B + X_{ad}{}^B f_{bc}{}^d - \partial_b{}^\alpha \rhd_{\alpha A}{}^B X_{ac}{}^A - X_{bd}{}^B f_{ac}{}^d ;$$

8. $\{\underline{h}_1, [\underline{h}_2, \underline{h}_3]\} = \{\delta\{\underline{h}_1, \underline{h}_2\}, \underline{h}_3\} - \{\delta\{\underline{h}_1, \underline{h}_3\}, \underline{h}_2\}$, $\quad \forall \underline{h}_1, \underline{h}_2, \underline{h}_3 \in \mathfrak{h}$, *i.e.*

$$X_{ad}{}^A f_{bc}{}^d = X_{ab}{}^B \delta_B{}^d X_{dc}{}^A - X_{ac}{}^B \delta_B{}^d X_{db}{}^A ;$$

9. $\{\delta(\underline{l}), \underline{h}\} + \{\underline{h}, \delta(\underline{l})\} = -\partial(\underline{h}) \rhd \underline{l}$, $\quad \forall \underline{l} \in \mathfrak{l}$, $\quad \forall \underline{h} \in \mathfrak{h}$, *i.e.*

$$\delta_A{}^a X_{ab}{}^B + \delta_A{}^a X_{ba}{}^B = -\partial_b{}^\alpha \rhd_{\alpha A}{}^B .$$

Note that the property 6. implies that either trivial map δ or the trivial Peiffer lifting imply that L is an Abelian group. Conversely, if L is Abelian, property 6. implies that either the map δ or the Peiffer lifting is trivial, or both.

In the case of an Abelian group H and trivial map ∂, among the aforementioned properties the only non-trivial remaining are:

1. $\delta\{\underline{h}_1, \underline{h}_2\} = 0$, $\quad \forall \underline{h}_1 , \underline{h}_2 \in \mathfrak{h}$;
2. $[\underline{l}_1, \underline{l}_2] = \{\delta(\underline{l}_1), \delta(\underline{l}_2)\}$, $\quad \forall \underline{l}_1 , \underline{l}_2 \in \mathfrak{l}$;
3. $\{\delta(\underline{l}), \underline{h}\} = -\{\underline{h}, \delta(\underline{l})\}$, $\quad \forall \underline{h} \in \mathfrak{h}$, $\quad \forall \underline{l} \in \mathfrak{l}$.

A reader intrested in more details about 3-groups is referred to [25].

B The construction of gauge-invariant actions for 3BF theory

Symmetric bilinear invariant nondegenerate forms are defined as:

$$\langle T_A , T_B \rangle_{\mathfrak{l}} = g_{AB} , \qquad \langle t_a , t_b \rangle_{\mathfrak{h}} = g_{ab} , \qquad \langle \tau_\alpha , \tau_\beta \rangle_{\mathfrak{g}} = g_{\alpha\beta} .$$

They satisfy the following properties:

- $\langle \_ , \_ \rangle_{\mathfrak{g}}$ is G-invariant:

$$\langle g\tau_\alpha g^{-1} , g\tau_\beta g^{-1} \rangle_{\mathfrak{g}} = \langle \tau_\alpha , \tau_\beta \rangle_{\mathfrak{g}} , \quad \forall g \in G ;$$

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- $\langle \_ , \_ \rangle_{\mathfrak{h}}$ is G-invariant:

$$\langle g \rhd t_a \,, g \rhd t_b \rangle_{\mathfrak{h}} = \langle t_a \,, t_b \rangle_{\mathfrak{h}} \,, \quad \forall g \in G \,,$$

  and, when $(H \stackrel{\partial}{\to} G \,, \rhd)$ is a crossed module, consequently H-invariant:

$$\langle h t_a h^{-1} \,, h t_b h^{-1} \rangle_{\mathfrak{h}} = \langle \partial(h) \rhd t_a \,, \partial(h) \rhd t_b \rangle_{\mathfrak{h}} = \langle t_a \,, t_b \rangle_{\mathfrak{h}} \,, \quad \forall h \in H \,;$$

- $\langle \_ , \_ \rangle_{\mathfrak{l}}$ is G-invariant:

$$\langle g \rhd T_A \,, g \rhd T_B \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall g \in G \,,$$

  and in the case when the Peiffer lifting or the map δ is trivial consequently H-invariant:

$$\langle h \rhd' T_A \,, h \rhd' T_B \rangle_{\mathfrak{l}} = \langle T_A - \{\delta(T_A), h\} \,, T_B - \{\delta(T_B), h\} \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall h \in H \,.$$

  From the H-invariance of $\langle \_ , \_ \rangle_{\mathfrak{l}}$ and properties of a crossed module $(L \stackrel{\delta}{\to} H \,, \rhd')$ follows L-invariance:

$$\langle l T_A l^{-1} \,, l T_B l^{-1} \rangle_{\mathfrak{l}} = \langle \delta(l) \rhd' T_A \,, \delta(l) \rhd' T_B \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall l \in L \,.$$

From the invariance of the bilinear forms follows the existence of gauge-invariant topological $3BF$ action of the form:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle \mathcal{C} \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle \mathcal{D} \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,, \tag{B.1}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4 \,, \mathfrak{g})$, $C \in \mathcal{A}^1(\mathcal{M}_4 \,, \mathfrak{h})$ and $D \in \mathcal{A}^0(\mathcal{M}_4 \,, \mathfrak{l})$ are Lagrange multipliers, and $\mathcal{F} \in \mathcal{A}^2(\mathcal{M}_4 \,, \mathfrak{g})$, $\mathcal{G} \in \mathcal{A}^3(\mathcal{M}_4 \,, \mathfrak{h})$ and $\mathcal{H} \in \mathcal{A}^4(\mathcal{M}_4 \,, \mathfrak{l})$ are curvatures defined as in (3.1). Written in the basis:

$$\begin{aligned}
&\mathcal{F} = \frac{1}{2} \mathcal{F}^{\alpha}{}_{\mu\nu} \tau_{\alpha} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \,, \qquad \mathcal{G} = \frac{1}{3!} \mathcal{G}^{a}{}_{\mu\nu\rho} t_a \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\rho} \,, \\
&\mathcal{H} = \frac{1}{4!} \mathcal{H}^{A}{}_{\mu\nu\rho\sigma} T_A \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\rho} \wedge \mathrm{d}x^{\sigma} \,,
\end{aligned}$$

the coefficients are:

$$\begin{aligned}
\mathcal{F}^{\alpha}{}_{\mu\nu} &= \partial_{\mu} \alpha^{\alpha}{}_{\nu} - \partial_{\nu} \alpha^{\alpha}{}_{\mu} + f_{\beta\gamma}{}^{\alpha} \alpha^{\beta}{}_{\mu} \alpha^{\gamma}{}_{\nu} - \beta^{a}{}_{\mu\nu} \partial_{a}{}^{\alpha} \,, \\
\mathcal{G}^{a}{}_{\mu\nu\rho} &= \partial_{\mu} \beta^{a}{}_{\nu\rho} + \partial_{\nu} \beta^{a}{}_{\rho\mu} + \partial_{\rho} \beta^{a}{}_{\mu\nu} \\
&\quad + \alpha^{\alpha}{}_{\mu} \beta^{b}{}_{\nu\rho} \rhd_{\alpha b}{}^{a} + \alpha^{\alpha}{}_{\nu} \beta^{b}{}_{\rho\mu} \rhd_{\alpha b}{}^{a} + \alpha^{\alpha}{}_{\rho} \beta^{b}{}_{\mu\nu} \rhd_{\alpha b}{}^{a} - \gamma^{A}{}_{\mu\nu\rho} \delta_{A}{}^{a} \,, \\
\mathcal{H}^{A}{}_{\mu\nu\rho\sigma} &= \partial_{\mu} \gamma^{A}{}_{\nu\rho\sigma} - \partial_{\nu} \gamma^{A}{}_{\rho\sigma\mu} + \partial_{\rho} \gamma^{A}{}_{\sigma\mu\nu} - \partial_{\sigma} \gamma^{A}{}_{\mu\nu\rho} \\
&\quad + 2\beta^{a}{}_{\mu\nu} \beta^{b}{}_{\rho\sigma} X_{\{ab\}}{}^{A} - 2\beta^{a}{}_{\mu\rho} \beta^{b}{}_{\nu\sigma} X_{\{ab\}}{}^{A} + 2\beta^{a}{}_{\mu\sigma} \beta^{b}{}_{\nu\rho} X_{\{ab\}}{}^{A} \\
&\quad + \alpha^{\alpha}{}_{\mu} \gamma^{B}{}_{\nu\rho\sigma} \rhd_{\alpha B}{}^{A} - \alpha^{\alpha}{}_{\nu} \gamma^{B}{}_{\rho\sigma\mu} \rhd_{\alpha B}{}^{A} + \alpha^{\alpha}{}_{\rho} \gamma^{B}{}_{\sigma\mu\nu} \rhd_{\alpha B}{}^{A} - \alpha^{\alpha}{}_{\sigma} \gamma^{B}{}_{\mu\nu\rho} \rhd_{\alpha B}{}^{A} \,.
\end{aligned}$$

Note that the wedge product $A \wedge B$ when A is a 0-form and B is a p-form is defined as $A \wedge B = \frac{1}{p!} A B_{\mu_1 \dots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge x^{\mu_p}$.

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Given G-invariant symmetric non-degenerate bilinear forms in $\mathfrak{g}$ and $\mathfrak{h}$, one can define a bilinear antisymmetric map $\mathcal{T} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ by the rule:

$$\langle \mathcal{T}(\underline{h}_1, \underline{h}_2)\,, \underline{g}\rangle_{\mathfrak{g}} = -\langle \underline{h}_1, \underline{g} \rhd \underline{h}_2\rangle_{\mathfrak{h}}, \qquad \forall \underline{h}_1, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{g} \in \mathfrak{g}\,.$$

See [17] for more properties and the construction of $2BF$ invariant topological action using this map. To define $3BF$ invariant topological action one has to first define a bilinear antisymmetric map $\mathcal{S} : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{g}$ by the rule:

$$\langle \mathcal{S}(\underline{l}_1, \underline{l}_2), \underline{g}\rangle_{\mathfrak{g}} = -\langle \underline{l}_1, \underline{g} \rhd \underline{l}_2\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{l}_1, \forall \underline{l}_2 \in \mathfrak{l}\,, \quad \forall \underline{g} \in \mathfrak{g}\,.$$

Note that $\langle \_, \_\rangle_{\mathfrak{g}}$ is non-degenerate and

$$\langle \underline{l}_1, \underline{g} \rhd \underline{l}_2\rangle_{\mathfrak{l}} = -\langle \underline{g} \rhd \underline{l}_1, \underline{l}_2\rangle_{\mathfrak{l}} = -\langle \underline{l}_2, \underline{g} \rhd \underline{l}_1\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{g} \in \mathfrak{g}, \quad \forall \underline{l}_1, \underline{l}_2 \in \mathfrak{l}\,.$$

Morever, given $g \in G$ and $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$ one has:

$$\mathcal{S}(g \rhd \underline{l}_1, g \rhd \underline{l}_2) = g\, \mathcal{S}(\underline{l}_1, \underline{l}_2)\, g^{-1}\,,$$

since for each $\underline{g} \in \mathfrak{g}$ and $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$:

$$\begin{aligned}\langle \underline{g}, g^{-1}\mathcal{S}(g \rhd \underline{l}_1\,, g \rhd \underline{l}_2) g\rangle_{\mathfrak{g}} &= \langle g\underline{g}g^{-1}, \mathcal{S}(g \rhd \underline{l}_1, g \rhd \underline{l}_2)\rangle_{\mathfrak{g}} \\ &= -\langle (g\,\underline{g}\,g^{-1}) \rhd g \rhd \underline{l}_1, g \rhd \underline{l}_2\rangle_{\mathfrak{l}} \\ &= -\langle \underline{g} \rhd \underline{l}_1\,, \underline{l}_2\rangle_{\mathfrak{l}} = \langle \underline{g}\,, \mathcal{S}(\underline{l}_1, \underline{l}_2)\rangle_{\mathfrak{g}}\,,\end{aligned}$$

where the following mixed relation has been used:

$$g \rhd (\underline{g} \rhd \underline{l}) = (g\,\underline{g}\,g^{-1}) \rhd g \rhd \underline{l}\,. \tag{B.2}$$

We thus have the following identity:

$$\mathcal{S}(\underline{g} \rhd \underline{l}_1, \underline{l}_2) + \mathcal{S}(\underline{l}_1, \underline{g} \rhd \underline{l}_2) = [\underline{g}, \mathcal{S}(\underline{l}_1, \underline{l}_2)]\,.$$

As far as the bilinear antisymmetric map $\mathcal{S} : l \times l \to g$, one can write it in the basis:

$$\mathcal{S}(T_A, T_B) = \mathcal{S}_{AB}{}^{\alpha}\tau_{\alpha}\,,$$

so that the defining relation for $\mathcal{S}$ becomes the relation:

$$\mathcal{S}_{AB}{}^{\alpha} g_{\alpha\beta} = -\rhd_{\alpha[B}{}^{C} g_{A]C}\,.$$

Given two $\mathfrak{l}$-valued forms η and ω, one can define a $\mathfrak{g}$-valued form:

$$\omega \wedge^{\mathcal{S}} \eta = \omega^A \wedge \eta^B \mathcal{S}_{AB}{}^{\alpha}\tau_{\alpha}\,.$$

Now one can define the transformations of the Lagrange multipliers under L-gauge transformations (3.15).

Further, to define the transformations of the Lagrange multipliers under H-gauge transformations one needs to define the bilinear map $\mathcal{X}_1 : \mathfrak{l} \times \mathfrak{h} \to \mathfrak{h}$ by the rule:

$$\langle \mathcal{X}_1(\underline{l},\, \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}} = -\langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}\,,$$

and bilinear map $\mathcal{X}_2 : \mathfrak{l} \times \mathfrak{h} \to \mathfrak{h}$ by the rule:

$$\langle \mathcal{X}_2(\underline{l},\, \underline{h}_2),\, \underline{h}_1\rangle_{\mathfrak{h}} = -\langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}\,.$$

As far as the bilinear maps $\mathcal{X}_1$ and $\mathcal{X}_2$ one can define the coefficients in the basis as:

$$\mathcal{X}_1(T_A, t_a) = \mathcal{X}_{1Aa}{}^b\, t_b\,, \qquad \mathcal{X}_2(T_A, t_a) = \mathcal{X}_{2Aa}{}^b\, t_b\,.$$

When written in the basis the defining relations for the maps $\mathcal{X}_1$ and $\mathcal{X}_2$ become:

$$\mathcal{X}_{1Ab}{}^c g_{ac} = -X_{ba}{}^B g_{AB}\,, \qquad \mathcal{X}_{2Ab}{}^c g_{ac} = -X_{ab}{}^B g_{AB}\,.$$

Given $\mathfrak{l}$-valued differential form ω and $\mathfrak{h}$-valued differential form η, one defines a $\mathfrak{h}$-valued form as:

$$\omega \wedge^{\mathcal{X}_1} \eta = \omega^A \wedge \eta^a \mathcal{X}_{1Aa}{}^b t_b\,, \qquad \omega \wedge^{\mathcal{X}_2} \eta = \omega^A \wedge \eta^a \mathcal{X}_{2Aa}{}^b t_b\,.$$

Given any $g \in G$, $\underline{l} \in \mathfrak{l}$ and $\underline{h} \in \mathfrak{h}$ one has:

$$\mathcal{X}_1(g \rhd \underline{l},\, g^{-1} \rhd \underline{h}) = g \rhd \mathcal{X}_1(\underline{l},\, \underline{h})\,, \qquad \mathcal{X}_2(g \rhd \underline{l},\, g \rhd \underline{h}) = g^{-1} \rhd \mathcal{X}_2(\underline{l},\, \underline{h})\,,$$

since for each $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ and $\underline{l} \in \mathfrak{l}$:

$$\begin{aligned}\langle \underline{h}_2,\, g^{-1} \rhd \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{h}_1)\rangle_{\mathfrak{h}} &= \langle g \rhd \underline{h}_2,\, \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{h}_1)\rangle_{\mathfrak{h}} = \langle g \rhd \underline{l},\, \{g \rhd \underline{h}_1,\, g \rhd \underline{h}_2\}\rangle_{\mathfrak{l}} \\ \langle g \rhd \underline{l},\, g \rhd \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}} &= \langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}} = \langle \underline{h}_2,\, \mathcal{X}_1(l,\, \underline{h}_1)\rangle_{\mathfrak{h}}\,,\end{aligned}$$

and similarly for $\mathcal{X}_2$. Finaly, one needs to define a trilinear map $\mathcal{D} : \mathfrak{h} \times \mathfrak{h} \times \mathfrak{l} \to \mathfrak{g}$ by the rule:

$$\langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l}),\, \underline{g}\rangle_{\mathfrak{g}} = -\langle \underline{l},\, \{\underline{g} \rhd \underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}, \quad \forall \underline{g} \in \mathfrak{g}\,,$$

One can define the coefficients of the trilinear map as:

$$\mathcal{D}(t_a,\, t_b,\, T_A) = \mathcal{D}_{abA}{}^\alpha \tau_\alpha\,,$$

and the defining relation for the map $\mathcal{D}$ expressed in terms of coefficients becomes:

$$\mathcal{D}_{abA}{}^\beta g_{\alpha\beta} = -\rhd_{\alpha a}{}^c X_{cb}{}^B g_{AB}\,.$$

Given two $\mathfrak{h}$-valued forms ω and η, and $\mathfrak{l}$-valued form ξ, the g-valued form is given by the formula:

$$\omega \wedge^{\mathcal{D}} \eta \wedge^{\mathcal{D}} \xi = \omega^a \wedge \eta^b \wedge \xi^A \mathcal{D}_{abA}{}^\beta \tau_\beta\,.$$

The following compatibility relation between the maps $\mathcal{X}_1$ and $\mathcal{D}$ hold:

$$\langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l}),\, \underline{g}\rangle_{\mathfrak{g}} = \langle \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}, \quad \forall \underline{g} \in \mathfrak{g}\,, \tag{B.3}$$

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which one can prove valid from the defining relations in terms of the coefficients. One can demonstrate that for each $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$ and $g \in G$:

$$\mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l}) = g\,\mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})\, g^{-1}\,,$$

since for each $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$, $\underline{g} \in \mathfrak{g}$ and $g \in G$:

$$\begin{aligned}
\langle g^{-1}\mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l})g,\, \underline{g}\rangle_{\mathfrak{g}} &= \langle \mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l}),\, g\underline{g}g^{-1}\rangle_{\mathfrak{g}} \\
&= \langle \mathcal{X}_1(g \rhd \underline{l},\, g\underline{g}g^{-1} \rhd g \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{g} \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle g \rhd \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})\,,\underline{g}\rangle_{\mathfrak{g}}\,,
\end{aligned}$$

where the relation (B.2) and the compatibility relation (B.3) were used. We thus have for each $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$ and $\underline{g} \in \mathfrak{g}$ the following identity:

$$\mathcal{D}(\underline{g} \rhd \underline{h}_1,\, \underline{h}_2,\, \underline{l}) + \mathcal{D}(\underline{h}_1,\, \underline{g} \rhd \underline{h}_2,\, \underline{l}) + \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{g} \rhd \underline{l}) = [\underline{g},\, \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})]\,.$$

Now one can define the transformations of the Lagrange multipliers under H-gauge transformations as in (3.14).

C The equations of motion for the Weyl and Majorana fields

The action for the Weyl spinor field coupled to gravity is given by (3.72). The variation of this action with respect to the variables B_{ab}, λ^{ab}, γ_α, $\bar{\gamma}^{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, e^a, β^a and ω^{ab} one obtains the complete set of equations of motion, as follows:

$$\begin{aligned}
R^{ab} - \lambda^{ab} &= 0\,, \\
B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd}e^c \wedge e^d &= 0\,, \\
\nabla\psi_\alpha + \lambda_\alpha &= 0\,, \\
\nabla\bar{\psi}^{\dot{\alpha}} + \bar{\lambda}^{\dot{\alpha}} &= 0\,, \\
-\gamma_\alpha + \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}} &= 0\,, \\
-\bar{\gamma}^{\dot{\alpha}} + \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta &= 0\,, \\
\nabla\gamma_\alpha - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}} &= 0\,, \\
\nabla\bar{\gamma}^{\dot{\alpha}} - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta}\lambda_\beta &= 0\,, \\
\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{i}{2}\varepsilon_{abcd}e^b \wedge e^c \wedge (\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta + \lambda^\alpha\sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}) & \\
-8\pi i l_p^2\varepsilon_{abcd}e^b\beta^c\big(\psi^\alpha(\sigma^d)_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}\big) &= 0\,,
\end{aligned}$$

$$\nabla e_a - 4\pi l_p^2 \varepsilon_{abcd} e^b \wedge e^c \wedge (\bar{\psi}_{\dot{\alpha}} \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta) = 0\,,$$
$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} - \frac{1}{2}\gamma \sigma^{ab}{}_\alpha{}^\beta \psi_\beta - \frac{1}{2}\bar{\gamma}_{\dot{\alpha}} \bar{\sigma}^{ab\dot{\alpha}}{}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}} = 0\,.$$

In the case of the Majorana field, one adds the mass term (3.75) to the action (3.72). Then, the variation of the action with respect to B_{ab}, ψ^{ab}, γ^α, $\bar{\gamma}_{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}_I^{\dot{\alpha}}$, e^a, β^a and ω_{ab} gives the equations of motion for the Majorana case, as follows:

$$R^{ab} - \lambda^{ab} = 0\,,$$
$$B_{ab} - \frac{1}{16\pi l_p^2} \varepsilon_{abcd} e^c \wedge e^d = 0\,,$$
$$-\nabla \psi_\alpha + \lambda_\alpha = 0\,,$$
$$-\nabla \bar{\psi}^{\dot{\alpha}} + \lambda^{\dot{\alpha}} = 0\,,$$
$$\gamma^\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \bar{\psi}_{\dot{\beta}} (\bar{\sigma}^d)^{\dot{\beta}\alpha} = 0\,,$$
$$\bar{\gamma}_{\dot{\alpha}} - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \psi^\beta (\sigma^d)_{\beta\dot{\alpha}} = 0\,,$$
$$\nabla \gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} \lambda^{\dot{\beta}} \wedge e^a \wedge e^b \wedge e^c (\sigma^d)^\alpha{}_{\dot{\beta}} - \frac{1}{6}\, m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \psi^\alpha$$
$$-4i\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c \bar{\psi}_{\dot{\beta}} (\bar{\sigma}^d)^{\dot{\beta}\alpha} = 0\,,$$
$$\nabla \bar{\gamma}_{\dot{\alpha}} + \frac{i}{6} \varepsilon_{abcd} \lambda_\beta \wedge e^a \wedge e^b \wedge e^c (\bar{\sigma}^d)_{\dot{\alpha}}{}^\beta - \frac{1}{6} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \psi_{\dot{\alpha}}$$
$$-4i\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c \psi^\beta (\sigma^d)_{\beta\dot{\alpha}} = 0\,,$$
$$\nabla \beta^a + \frac{1}{8\pi l_p^2} \varepsilon_{abcd} \lambda^{bc} \wedge e^d + \frac{i}{2} \varepsilon_{abcd} \lambda_\alpha \wedge e^b \wedge e^c \bar{\psi}_{\dot{\beta}} (\bar{\sigma}^d)^{\dot{\beta}\alpha} + \frac{i}{2} \varepsilon_{abcd} \lambda^{\dot{\alpha}} \wedge e^b \wedge e^c \psi^\beta (\sigma^d)_{\beta\dot{\alpha}}$$
$$-\frac{1}{3} m \varepsilon_{abcd} e^b \wedge e^c \wedge e^d (\psi^\alpha \psi_\alpha + \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}) - 8\pi i l_p^2 \varepsilon_{abcd} e^b \beta^c \big(\psi^\alpha (\sigma^d)_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}}\big) = 0\,,$$
$$\nabla e_a - 4i\pi l_p^2 \varepsilon_{abcd} e^b \wedge e^c \big(\psi^\alpha (\sigma^d)_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}}\big) = 0\,,$$
$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} - \frac{1}{2} \psi^\alpha (\sigma^{ab})_\alpha{}^\beta \gamma_\beta - \frac{1}{2} \bar{\psi}_{\dot{\alpha}} (\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\gamma}^{\dot{\beta}} = 0\,.$$

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Henneaux–Teitelboim Gauge Symmetry and Its Applications to Higher Gauge Theories

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Abstract: When discussing the gauge symmetries of any theory, the Henneaux–Teitelboim transformations are often underappreciated or even completely ignored, due to their on-shell triviality. Nevertheless, these gauge transformations play an important role in understanding the structure of the full gauge symmetry group of any theory, especially regarding the subgroup of diffeomorphisms. We give a review of the Henneaux–Teitelboim transformations and the resulting gauge group in the general case and then discuss its role in the applications to the class of topological theories called *nBF* models, relevant for the constructions of higher gauge theories and quantum gravity.

Keywords: gauge symmetry; trivial gauge transformations; *nBF* theory; Chern–Simons theory; diffeomorphism symmetry

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1. Introduction

In modern theoretical physics, gauge symmetries play a very prominent role. The two most-fundamental theories we have, which describe almost all observed phenomena in nature—namely Einstein's theory of general relativity and the Standard Model of elementary particle physics—are gauge theories. From Maxwell's electrodynamics to various approaches to quantum gravity, gauge theories play a central role, and gauge symmetry represents one of their most-important aspects. In light of this, there is one class of gauge transformations that is often slightly neglected in the literature, due to their specific nature and properties.

In order to introduce this particular gauge symmetry in the most-elementary way possible, let us look at the following simple example. Every action $S[\phi_1, \phi_2]$, which depends on the fields $\phi_1(x)$ and $\phi_2(x)$, is invariant under the following gauge transformation:

$$\delta_0\phi_1(x) = \epsilon(x)\frac{\delta S}{\delta\phi_2(x)}\,, \qquad \delta_0\phi_2(x) = -\epsilon(x)\frac{\delta S}{\delta\phi_1(x)}\,, \tag{1}$$

as one can see by calculating the variation of the action:

$$\delta S[\phi_1,\phi_2] = \frac{\delta S}{\delta\phi_1}\delta_0\phi_1 + \frac{\delta S}{\delta\phi_2}\delta_0\phi_2 = 0\,. \tag{2}$$

This gauge symmetry exists for every action that is a functional of at least two fields, irrespective of any other gauge symmetry that the action may or may not have. In the literature, this symmetry is often called *trivial* gauge symmetry, since the form variations of the fields are identically zero on-shell. This is in contrast to all other gauge symmetries, which perform some nontrivial change of the fields on-shell.

It should be noted that, being trivial on-shell, the above transformations cannot play a role in obtaining any predictions about observables in a given theory, due to the intrinsic on-shell nature of the physical observables. For example, in practical situations of scattering experiments and measurements of cross-sections, this trivial symmetry is irrelevant. Nevertheless, when constructing a new theory, in general, the off-shell properties of the theory are important. As a typical example, path integral quantization prescription depends not only on the classical equations of motion, but on the whole action of the theory. In this sense, while these trivial transformations are not relevant for making predictions, they do have methodological relevance and value in theory construction, despite their on-shell triviality.

For example, these transformations in fact represent a very important part of the gauge symmetry for any theory and play a crucial role in various contexts, such as in the Batalin–Vilkovisky formalism (see [1] for a review and also the original papers [2–6]), or when discussing the diffeomorphism symmetry of the BF-like class of theories [7–11]. Furthermore, in general, a commutator of two ordinary gauge transformations will remain an ordinary gauge transformation only up to the above trivial transformations, meaning that the latter are important for the algebraic closure of all gauge transformations into a group.

To the best of our knowledge, the most-complete treatment and discussion of the above gauge transformations can be found in the book [12] by Marc Henneaux and Claudio Teitelboim. Therefore, in this paper, we opted to call them Henneaux–Teitelboim (HT) transformations. This naming can also be justified with the paper [7] by Gary Horowitz (published two years before the book [12]), where the author attributes these transformations to Henneaux and Teitelboim in a footnote and thanks them "for explaining this to me".

Regarding terminology, we should also note that we use the terms "gauge symmetry" and "gauge transformations" with a certain level of charity. Namely, one could argue that there are two distinct types of local symmetries—those that are obtained by a localization procedure from a corresponding global symmetry group (the procedure of "gauging" a global symmetry) and those that are intrinsically local, not obtained by any such localization procedure. It is not known whether HT symmetry belongs to the former or the latter class, since a global symmetry whose localization would give rise to HT transformations has not yet been shown to exist. Either way, in the literature, there is no established terminology that distinguishes the two classes of symmetries, and most often, both are called "gauge symmetries". Therefore, in what follows, for a lack of better terminology, we will adhere to this practice and describe HT transformations as a gauge symmetry.

In some of the modern approaches to the problem of quantum gravity based on the spinfoam formalism of loop quantum gravity [13,14], as well as in other applications of the so-called higher gauge theory (see [15] for a review and [16] for an application to quantum gravity), the description of gauge symmetry is being extended from the notion of a Lie group to different algebraic structures, called 2-groups, 3-groups, and in general, n-groups [17–27]. In this context, it is important to revisit and study the specific class of HT gauge symmetries, since they provide a nontrivial insight into the properties of these more general algebraic structures, as well as the physics behind the symmetries they describe.

The purpose of this paper is to provide a review of HT transformations in general and then discuss their properties and applications in two concrete models—the Chern–Simons theory and the $3BF$ theory. The Chern–Simons case is simple enough to serve as an illustrative toy example, while the $3BF$ theory represents a basis for the construction of a realistic theory of quantum gravity with matter within the context of the spinfoam formalism (see also [16,28–32]), discussing that its HT symmetry represents an important stepping stone towards the goal of a more realistic theory. The main result of this work represents a clarification of the structure of the gauge symmetry of a pure topological $3BF$ action, as well as the corresponding symmetry for the constrained $2BF$ action, which is classically equivalent to Einstein's general relativity. We also discuss in detail the relationship between diffeomorphism symmetry and the HT symmetry for the Chern–Simons and $3BF$ theories and offer some conceptual suggestions regarding the notion of gauge symmetry as it is being used in the literature.

The layout of the paper is as follows. In Section 2, we give a review of the general theory of HT transformations and their main properties. Section 3 is devoted to the example of HT symmetry in Chern–Simons theory, which is convenient due to its simplicity. In Section 4, we discuss the main case of HT symmetry in the $3BF$ and $2BF$ theories, which are important for applications in quantum gravity models. Finally, Section 5 contains an overview of the results, future research directions, and some concluding remarks.

The notation and conventions in the paper are as follows. When important, we assume the $(-,+,+,+)$ signature of the spacetime metric. The Greek indices from the middle of the alphabet, $\lambda,\mu,\nu,\dots$, represent spacetime indices and take values $0,1,\dots,D-1$, where D is the dimension of the spacetime manifold $\mathcal{M}_D$ under consideration. The Greek indices from the beginning of the alphabet, $\alpha,\beta,\gamma,\dots$, represent group indices, as well as Latin indices $a,b,c,\dots$ and uppercase Latin indices $A,B,C,\dots$ and $I,J,K,\dots$. All these indices will be assigned to various Lie groups under consideration. Lowercase Latin indices from the middle of the alphabet, $i,j,k,\dots$, are generic and will be used to count all fields in a given theory or for some other purpose depending on the context. Throughout the paper, we denote the space of algebra-valued differential p-forms as

$$\mathcal{A}^p(\mathcal{M},\mathfrak{a}) \equiv \Lambda^p(\mathcal{M})\otimes\mathfrak{a}\,,$$

where $\Lambda^p(\mathcal{M})$ is the ordinary space of differential p-forms over the manifold $\mathcal{M}$, while $\mathfrak{a}$ is some Lie algebra.

2. Review of HT Symmetry

We begin by studying some basic general properties of HT transformations. After the definition, we demonstrate that the group of HT transformations represents a normal subgroup of the *total* gauge group of a given theory, and we discuss the triviality of HT transformations and that they exhaust all possible trivial transformations. Finally, before moving on to concrete theories, we study the subtleties of the dependence of HT symmetry on the choice of the action.

2.1. Definition of HT Transformations

Given an action $S[\phi^i]$ as a functional of fields $\phi^i(x)$ ($i\in\{1,\dots,N\}$ where we assume $N\geqslant 2$), the infinitesimal HT transformation is defined as

$$\phi^i(x)\to\phi'^i(x)=\phi^i(x)+\delta_0\phi^i(x)\,, \tag{3}$$

where the form variations of the fields are defined as

$$\delta_0\phi^i(x)=\epsilon^{ij}(x)\frac{\delta S}{\delta\phi^j(x)}\,. \tag{4}$$

The variation of the action under HT transformations then gives

$$\delta S=\frac{\delta S}{\delta\phi^i}\delta_0\phi^i=\frac{\delta S}{\delta\phi^i}\frac{\delta S}{\delta\phi^j}\epsilon^{ij}\,. \tag{5}$$

If the HT parameters are chosen to be antisymmetric,

$$\epsilon^{ij}(x)=-\epsilon^{ji}(x)\,, \tag{6}$$

the variation of the action (5) is identically zero, and HT transformations (4) represent a gauge symmetry of the theory.

The most-striking thing in the above definition is the fact that we did not specify the action in any way. Aside from the assumption $N\geqslant 2$, which excludes only actions describing a single real scalar field, every action is invariant with respect to the HT transformations. In other words, *HT transformations are a gauge symmetry of essentially every theory*.

The second striking property of the definition is that the form variations of fields become zero on-shell, according to (4). In this sense, the HT symmetry is sometimes called *trivial symmetry*, in contrast to ordinary gauge symmetries that a theory may have, which transform the fields in a nontrivial way on-shell. Triviality is also the reason why HT gauge symmetry does not feature in any way in the Hamiltonian analysis of a theory, so only the presence of ordinary gauge symmetries can be deduced from the Hamiltonian formalism.

2.2. HT Symmetry Group and Its Properties

There are two general properties that can be formulated for HT transformations. The first is that HT transformations form a normal subgroup within the full group of gauge symmetries, while the second is that HT transformations exhaust the set of all possible trivial transformations. The consequence of these properties is that one can always write the total symmetry group of any theory as

$$\mathcal{G}_{\text{total}} = \mathcal{G}_{\text{nontrivial}} \ltimes \mathcal{G}_{HT}\,, \tag{7}$$

where $\mathcal{G}_{\text{nontrivial}}$ is the symmetry group of ordinary gauge transformations (if there are any), $\mathcal{G}_{HT}$ is the HT symmetry group, and the symbol $\ltimes$ stands for a semidirect product. One can also reformulate (7) as

$$\mathcal{G}_{\text{nontrivial}} = \mathcal{G}_{\text{total}} / \mathcal{G}_{HT}\,, \tag{8}$$

so that the group of ordinary gauge symmetries is represented as a quotient group.

The easiest way to demonstrate (7) is to prove that the Lie algebra corresponding to $\mathcal{G}_{HT}$ represents an ideal within the Lie algebra corresponding to $\mathcal{G}_{\text{total}}$. To that end, pick an arbitrary form variation of fields that represents a symmetry of the action and write it in the form

$$\hat{\delta}_0\phi^i(x) = F^i(x)\,, \qquad \text{such that} \qquad \hat{\delta}S = \frac{\delta S}{\delta\phi^i}F^i \equiv 0\,. \tag{9}$$

Then, using (4), we can take concatenated variations of this form variation and the HT form variation as

$$\delta_0\hat{\delta}_0\phi^i = \frac{\delta F^i}{\delta\phi^j}\frac{\delta S}{\delta\phi^k}\epsilon^{jk}\,,$$

and

$$\hat{\delta}_0\delta_o\phi^i = \frac{\delta}{\delta\phi^k}\left(\epsilon^{ij}\frac{\delta S}{\delta\phi^j}\right)F^k = \frac{\delta\epsilon^{ij}}{\delta\phi^k}\frac{\delta S}{\delta\phi^j}F^k + \epsilon^{ij}\frac{\delta}{\delta\phi^j}\left(\frac{\delta S}{\delta\phi^k}F^k\right) - \epsilon^{ij}\frac{\delta S}{\delta\phi^k}\frac{\delta F^k}{\delta\phi^j}\,.$$

The term in the second parentheses is zero by (9), so the commutator of two-form variations becomes

$$[\delta_0\,,\hat{\delta}_0]\phi^i = \left(\epsilon^{jk}\frac{\delta F^i}{\delta\phi^j} - \epsilon^{ji}\frac{\delta F^k}{\delta\phi^j} - \frac{\delta\epsilon^{ik}}{\delta\phi^j}F^j\right)\frac{\delta S}{\delta\phi^k}\,, \tag{10}$$

which is again an HT transformation, since the expression in the parentheses is antisymmetric with respect to indices i,k. Therefore, the commutator is always an element of HT algebra, which means that HT algebra itself is an ideal of the total symmetry algebra. At the Lie group level, this translates into (7).

The second general property is the statement that there are no other trivial transformations beside the HT transformations. Assuming that some transformation described by the form variation $\bar{\delta}_0\phi^i$ is a gauge symmetry of the action that vanishes on-shell, i.e., that it satisfies

$$\frac{\delta S}{\delta\phi^i}\bar{\delta}_0\phi^i = 0\,, \qquad \text{and} \qquad \bar{\delta}_0\phi^i \approx 0\,,$$

then one can prove that this transformation is an HT transformation, i.e., there exists a choice of antisymmetric HT parameters ϵ^{ij} such that the form variation $\bar{\delta}_0\phi^i$ is of type (4):

$$\bar{\delta}_0\phi^i = \epsilon^{ij}\frac{\delta S}{\delta\phi^j}\,. \tag{11}$$

Provided certain suitable regularity conditions for the action S, this statement can be rigorously formulated as a theorem. However, we omitted the proof since it is technical and off topic for the purposes of this paper. The interested reader can find the details of both the theorem and the proof in [12], Appendix 10.A.2.

To sum up, the first property (10) tells us that one can always factorize the total gauge symmetry group into the form (7), while the second property (11) guarantees that the quotient group (8) contains only nontrivial gauge transformations. This factorization of the total symmetry group is a key result that lays the groundwork for any subsequent analysis of HT transformations in particular and gauge symmetry in general.

2.3. Dependence of HT Symmetry on the Action

The final property of HT transformations that needs to be discussed is their dependence on the choice of the action. Suppose we are given some action $S_{\rm old}[\phi^i]$, where $i \in \{1, \dots, N\}$, which has the corresponding HT transformation described as in (4):

$$\delta_0^{\rm old}\phi^i = \epsilon^{ij}\frac{\delta S_{\rm old}}{\delta\phi^j}\,. \tag{12}$$

Now, suppose that we modify that action into another one, $S_{\rm new}[\phi^i, \chi^k]$, where $k \in \{N + 1, \dots, N + M\}$, by adding an extra term to the old action:

$$S_{\rm new}[\phi^i, \chi^k] = S_{\rm old}[\phi^i] + S_{\rm extra}[\phi^i, \chi^k]\,. \tag{13}$$

Here, χ^j are additional fields that may be introduced into the new action. The HT transformation corresponding to the new action can be written in the block-matrix form, made of blocks of sizes N and M, as follows:

$$\left(\begin{array}{c} \delta_0^{\rm new}\phi^i \\ \delta_0^{\rm new}\chi^k \end{array}\right) = \left(\begin{array}{c|c} \epsilon^{ij} & \zeta^{il} \\ \hline \theta^{kj} & \psi^{kl} \end{array}\right)\left(\begin{array}{c} \dfrac{\delta S_{\rm new}}{\delta\phi^j} \\ \dfrac{\delta S_{\rm new}}{\delta\chi^l} \end{array}\right), \qquad \begin{array}{l} i, j \in \{1, \dots, N\}\,, \\ k, l \in \{N+1, \dots, N+M\}\,. \end{array} \tag{14}$$

Here, $\epsilon = -\epsilon^T$ is an antisymmetric $N \times N$ block of parameters ϵ^{ij}, ζ is a rectangular $N \times M$ block of parameters ζ^{il}, θ is a rectangular $M \times N$ block such that $\theta = -\zeta^T$, and finally, $\psi = -\psi^T$ is an antisymmetric $M \times M$ block of parameters ψ^{kl}. Overall, the total parameter matrix is antisymmetric, as required by (6).

The question one can now study is what is the relation between the two HT gauge symmetry groups $\mathcal{G}_{HT}^{\rm old}$ and $\mathcal{G}_{HT}^{\rm new}$ that correspond to the two actions. In practice, this question is most often relevant in cases when one introduces the piece $S_{\rm extra}$ as a gauge-fixing term, whose purpose is to break the ordinary gauge symmetry down to its subgroup:

$$G_{\rm nontrivial}^{\rm new} \subset G_{\rm nontrivial}^{\rm old}\,.$$

Naively, one might expect a similar relationship between the HT symmetry groups, $\mathcal{G}_{HT}^{\rm new} \subset \mathcal{G}_{HT}^{\rm old}$. However, looking at (12) and (14), this is obviously wrong. Namely, if $M \geqslant 1$, the HT symmetry of the new action is *larger* than the HT symmetry of the old action. Counting the number of independent parameters of both, one easily sees that

$$\dim(\mathcal{G}_{HT}^{\rm old}) = \frac{N(N-1)}{2}\,, \qquad \dim(\mathcal{G}_{HT}^{\rm new}) = \frac{(N+M)(N+M-1)}{2}\,,$$

so that the only possible relationship would be the opposite, $\mathcal{G}_{HT}^{\rm old} \subset \mathcal{G}_{HT}^{\rm new}$. However, in fact, this can also be shown to be wrong. Namely, one can choose the extra parameters ζ, θ and ψ to be zero in (14), reducing it to the form that is formally similar to (12):

$$\delta_0^{\rm new}\phi^i = \epsilon^{ij}\frac{\delta S_{\rm new}}{\delta\phi^j}\,.$$

However, taking into account the relationship (13) between the two actions, the HT transformation takes the form

$$\delta_0^{\text{new}} \phi^i = \epsilon^{ij} \frac{\delta S_{\text{old}}}{\delta \phi^j} + \epsilon^{ij} \frac{\delta S_{\text{extra}}}{\delta \phi^j} \,,$$

which is explicitly different from (12), due to the presence of the term S_{extra} in the action. Therefore, the gauge group $\mathcal{G}_{HT}^{\text{old}}$ is not a subgroup of $\mathcal{G}_{HT}^{\text{new}}$ either.

The overall conclusion is that introducing additional terms to the action changes the total gauge symmetry in a nontrivial way. On the one hand, the ordinary gauge symmetry group typically becomes *smaller* due to explicit symmetry breaking by the extra term. On the other hand, the HT gauge symmetry group may become *larger* if the extra term contains additional fields, but either way becomes *different*, as a consequence of the very presence of the extra term. Given this, one can conclude that the *total* symmetry groups for the two actions will always be mutually different:

$$\mathcal{G}_{\text{total}}^{\text{new}} = \mathcal{G}_{\text{nontrivial}}^{\text{new}} \ltimes \mathcal{G}_{HT}^{\text{new}} \qquad \neq \qquad \mathcal{G}_{\text{total}}^{\text{old}} = \mathcal{G}_{\text{nontrivial}}^{\text{old}} \ltimes \mathcal{G}_{HT}^{\text{old}} \,.$$

Specifically, one cannot claim that the group $\mathcal{G}_{\text{total}}^{\text{old}}$ is being broken down into $\mathcal{G}_{\text{total}}^{\text{new}}$ as its subgroup; such a relationship may hold exclusively for the quotient groups of ordinary gauge transformations.

In the next two sections, we will turn to explicit examples of all general properties and features of the HT symmetry that have been discussed above. Moreover, we will also discuss some additional particular properties, such as the fact that some nontrivial gauge subgroups of $\mathcal{G}_{\text{total}}$ are not simultaneously subgroups of $\mathcal{G}_{\text{nontrivial}}$, which is a consequence of the semidirect product in (7). One such example will be the diffeomorphism symmetry in the Chern–Simons and $3BF$ actions.

Let us conclude this section with one conceptual comment. Throughout the literature, the typical practice is to always take the quotient between the total and HT symmetry groups as in (8), in order to isolate the nontrivial gauge transformations, and call the latter simply as the "gauge symmetry" of a theory. This approach is in fact advocated for in [12]. However, we believe that this practice can be misleading and that one should instead describe the group $\mathcal{G}_{\text{total}}$ as "the gauge symmetry" of a theory, explicitly including the HT subgroup as a legitimate gauge symmetry group. Namely, despite the fact that it is often called "trivial", the consequences of its presence in $\mathcal{G}_{\text{total}}$ are far from trivial. Granted, it may often be enough to discuss the gauge symmetry on-shell, and then, one can indeed calculate all symmetry transformations only "up to equations of motion", with no mention of the HT subgroup. However, whenever one needs to discuss the gauge transformations off-shell, the HT subgroup simply cannot be ignored anymore. Typical situations include the Batalin–Vilkovisky formalism [1], various generalizations of gauge symmetry in the context of higher gauge theories and quantum gravity [33], and even the traditional contexts such as the Coleman–Mandula theorem [34]. The situations in which HT transformations play a significant role may be rare, but nevertheless, they tend to be important. Thus, in our opinion, it would be prudent to always be aware that, for any given theory, its total gauge symmetry group is in fact bigger, and more feature-rich, than just the group of ordinary gauge transformations that are typically discussed in the literature.

3. HT Symmetry in Chern–Simons Theory

As an illustrative example of the general properties of HT symmetry from the previous section, let us discuss the HT transformations for the simple case of the Chern–Simons theory. The Chern–Simons theory represents an excellent toy example since it is well known in the literature and most readers should be familiar with it.

Given any Lie group G, its corresponding Lie algebra $\mathfrak{g}$, and a three-dimensional manifold $\mathcal{M}_3$, the Chern–Simons theory can be defined as a topological field theory over a trivial principal bundle $G \to \mathcal{M}_3$, given by the action:

$$S_{CS} = \int_{\mathcal{M}_3} \langle A \wedge \mathrm{d}A \rangle_{\mathfrak{g}} + \frac{1}{3} \langle A \wedge [A \wedge A] \rangle_{\mathfrak{g}} \,. \tag{15}$$

Here, $A \in \mathcal{A}^1(\mathcal{M}_3, \mathfrak{g})$ is a $\mathfrak{g}$-valued connection one-form over a manifold $\mathcal{M}_3$, and $\langle \_, \_ \rangle_{\mathfrak{g}}$ is a G-invariant symmetric nondegenerate bilinear form on $\mathfrak{g}$. One often rewrites the Chern–Simons action within the framework of the enveloping algebra of $\mathfrak{g}$, introducing the notion of a *trace* as

$$\mathrm{Tr}(XY) \equiv \langle X, Y \rangle_{\mathfrak{g}} \,,$$

for every $X, Y \in \mathfrak{g}$. Then, the Chern–Simons action can be rewritten as

$$S_{CS} = \int_{\mathcal{M}_3} \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right), \tag{16}$$

where, for the second term, one employs the identity $\mathrm{Tr}(X[Y,Z]) = \mathrm{Tr}(XYZ) - \mathrm{Tr}(XZY)$ for every $X, Y, Z \in \mathfrak{g}$.

The gauge symmetry of the Chern–Simons action consists of G-gauge transformations, determined with the parameters $\epsilon_{\mathfrak{g}}{}^I(x)$. Using the basis of generators T_I to expand the connection A into components as

$$A = A^I{}_\mu(x)\, \mathrm{d}x^\mu \otimes T_I \,,$$

the form variation of the connection components $A^I{}_\mu$ corresponding to gauge transformations can then be written as

$$\delta_0 A^I{}_\mu = \partial_\mu \epsilon_{\mathfrak{g}}{}^I - f_{JK}{}^I \epsilon_{\mathfrak{g}}{}^J A^K{}_\mu \,, \tag{17}$$

where $f_{JK}{}^I$ are the structure constants corresponding to the generators T_I. Therefore, the gauge symmetry of the Chern–Simons theory is usually quoted as the initially chosen Lie group G:

$$\mathcal{G}_{CS} = G \,. \tag{18}$$

However, as we have seen in the previous section, this is not the complete set of gauge transformations, and the *total* gauge group should in fact be

$$\mathcal{G}_{\text{total}} = \mathcal{G}_{CS} \ltimes \mathcal{G}_{HT} \,. \tag{19}$$

Let us define the HT transformations for the Chern–Simons action (15). If we denote the dimension of the Lie algebra $\mathfrak{g}$ as $\dim(\mathfrak{g}) = p$, the number of independent field components $A^I{}_\mu$ is $N = 3p$. The HT transformation is then defined with the HT parameters $\epsilon^{IJ}{}_{\mu\nu}(x)$ as

$$\delta_0 A^I{}_\mu = \epsilon^{IJ}{}_{\mu\nu} \frac{\delta S}{\delta A^J{}_\nu} \,. \tag{20}$$

The requirement that the variation of the action vanishes:

$$\delta S = \frac{\delta S}{\delta A^I{}_\mu} \frac{\delta S}{\delta A^J{}_\nu} \epsilon^{IJ}{}_{\mu\nu} = 0 \,,$$

enforces the antisymmetry restriction on the HT parameters:

$$\epsilon^{IJ}{}_{\mu\nu} = -\epsilon^{JI}{}_{\nu\mu} \,.$$

Note that this equation can be satisfied in two different ways—the parameters can be either antisymmetric with respect to group indices IJ and symmetric with respect to spacetime indices $\mu\nu$, or vice versa. We, therefore, have two possible choices for their symmetry properties. The first possibility is defined as

$$\epsilon^{IJ}{}_{\mu\nu} = \epsilon^{IJ}{}_{\nu\mu} = -\epsilon^{JI}{}_{\mu\nu} = -\epsilon^{JI}{}_{\nu\mu}\,, \tag{21}$$

while the second possibility is defined as

$$\epsilon^{IJ}{}_{\mu\nu} = \epsilon^{JI}{}_{\mu\nu} = -\epsilon^{IJ}{}_{\nu\mu} = -\epsilon^{JI}{}_{\nu\mu}\,. \tag{22}$$

Varying the action, one obtains an explicit form of the HT transformation:

$$\delta_0 A^I{}_\mu = \epsilon^{IJ}{}_{\mu\nu}\varepsilon^{\nu\rho\sigma}\left(\partial_\rho A_{J\sigma} - \partial_\sigma A_{J\rho} + f_{KLJ}\, A^K{}_\rho A^L{}_\sigma\right). \tag{23}$$

In order to demonstrate that HT transformations have highly nontrivial implications, despite being trivial on-shell, it is instructive to discuss diffeomorphisms. Namely, looking at the action (15), one expects that the theory has diffeomorphism symmetry, since it is formulated in a manifestly covariant way using differential forms. However, one can check that diffeomorphisms are not a subgroup of the ordinary gauge symmetry group $\mathcal{G}_{CS}$ given by (18), but nevertheless can be obtained as a subgroup of the total gauge group (19). In other words, one can demonstrate that

$$Diff(\mathcal{M}_3) \not\subset \mathcal{G}_{CS}\,, \qquad \text{but} \qquad Diff(\mathcal{M}_3) \subset \mathcal{G}_{\text{total}} = \mathcal{G}_{CS} \ltimes \mathcal{G}_{HT}\,.$$

Let us examine this in detail. The diffeomorphism transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)\,, \tag{24}$$

determined by the parameter $\xi^\mu(x)$ represents a subgroup $Diff(\mathcal{M})$ of the full gauge symmetry of some given action, if for every field $\phi(x)$ in the theory and every choice of diffeomorphism parameters $\xi^\mu(x)$, there exists a choice of the gauge parameters $\epsilon^{\text{gauge}}(x)$ and the HT parameters $\epsilon^{\text{HT}}(x)$, such that:

$$\delta_0{}^{\text{diff}}\,\phi = \delta_0{}^{\text{gauge}}\phi + \delta_0{}^{\text{HT}}\phi\,. \tag{25}$$

In other words, if a theory has diffeomorphism symmetry, the diffeomorphism form variations of all the fields in the theory should be expressible in terms of their ordinary gauge and HT form variations.

In the case of Chern–Simons theory, this can be demonstrated explicitly. If one chooses the gauge parameters $\epsilon_{\mathfrak{g}}{}^I$ and the HT parameters $\epsilon^{IJ}{}_{\mu\nu}$ as

$$\epsilon_{\mathfrak{g}}{}^I = -\xi^\lambda A^I{}_\lambda\,, \qquad \epsilon^{IJ}{}_{\mu\nu} = -\frac{1}{2}\xi^\lambda \varepsilon_{\lambda\mu\nu} g^{IJ}\,, \tag{26}$$

where g^{IJ} is the inverse of $g_{IJ} \equiv \langle T_I\,, T_J\rangle_{\mathfrak{g}}$, one can apply Equations (25) using (17) and (23) to reproduce precisely the well-known diffeomorphism form variation of the connection $A^I{}_\mu$:

$$\delta_0{}^{\text{diff}} A^I{}_\mu = -A^I{}_\lambda \partial_\mu \xi^\lambda - \xi^\lambda \partial_\lambda A^I{}_\mu\,. \tag{27}$$

Therefore, as expected, despite the fact that $Diff(\mathcal{M}_3) \not\subset \mathcal{G}_{CS}$, one obtains that $Diff(\mathcal{M}_3) \subset \mathcal{G}_{\text{total}} = \mathcal{G}_{CS} \ltimes G_{HT}$. Note that the choice of HT parameters in (26) is nontrivial, which emphasizes the role of HT transformations and the fact that the full group of gauge symmetries is $\mathcal{G}_{\text{total}}$ rather than $\mathcal{G}_{CS}$. As we shall see in the next section, this property is not specific only to the Chern–Simons theory.

4. HT Symmetry in 3*BF* Theory

After discussing the Chern–Simons theory as a toy example, we move to the more important case of the 3*BF* theory. This theory is relevant for building models of quantum gravity; see [8,20,21,33,35]. Therefore, it is important to study its gauge symmetry and, in particular, the role of HT transformations.

4.1. Review of the 3BF Theory

Analogous to the fact that Chern–Simons theory is a topological theory based on a Lie group and a 3-dimensional manifold, the $3BF$ theory is also a topological theory based on a notion of a three-group and a 4-dimensional manifold. The notion of a three-group represents a categorical generalization of the notion of a group, in the context of higher gauge theory (HGT); see [15] for a review and motivation. For the purpose of defining the $3BF$ theory, we are interested in particular in a strict Lie three-group, which is known to be isomorphic to a so-called Lie two-crossed module; see [17–19] for details.

A Lie two-crossed module, denoted as $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\}_{\mathrm{pf}})$, is an algebraic structure specified by three Lie groups G, H, and L, together with the homomorphisms $\delta : L \to H$ and $\partial : H \to G$, an action $\triangleright$ of the group G on all three groups, and a G-equivariant map, called the Peiffer lifting:

$$\{\_\,,\_\}_{\mathrm{pf}} : H \times H \to L\,.$$

In order for this structure to form a two-crossed module, the structure constants of algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ (the Lie algebras corresponding to the Lie groups G, H, and L, respectively), as well as the maps ∂ and δ, the action $\triangleright$, and the Peiffer lifting, must satisfy certain axioms; see [20] for details.

Given a two-crossed module and a four-dimensional compact and orientable spacetime manifold $\mathcal{M}_4$, one can introduce the notion of a trivial principal three-bundle, in analogy with the notion of a trivial principal bundle constructed from an ordinary Lie group and a manifold; see [15]. Then, one can introduce the notion of a three-connection, an ordered triple (α, β, γ), where α, β, and γ are algebra-valued differential forms, $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{h})$, and $\gamma \in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{l})$; see [17–19]. The corresponding fake hree-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is defined as:

$$\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \\ \mathcal{H} = \mathrm{d}\gamma + \alpha \wedge^{\triangleright} \gamma + \{\beta \wedge \beta\}_{\mathrm{pf}}\,. \tag{28}$$

Then, for a four-dimensional manifold $\mathcal{M}_4$, one can define the gauge-invariant topological $3BF$ action, based on the structure of a two-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\}_{\mathrm{pf}})$, by the action

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}\,, \tag{29}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$, $C \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$, and $D \in \mathcal{A}^0(\mathcal{M}_4, \mathfrak{l})$ are Lagrange multipliers and $\mathcal{F} \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$, $\mathcal{G} \in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{h})$, and $\mathcal{H} \in \mathcal{A}^4(\mathcal{M}_4, \mathfrak{l})$ represent the fake three-curvature given by Equation (28). The forms $\langle \_\,,\_ \rangle_{\mathfrak{g}}$, $\langle \_\,,\_ \rangle_{\mathfrak{h}}$, and $\langle \_\,,\_ \rangle_{\mathfrak{l}}$ are G-invariant symmetric nondegenerate bilinear forms on $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$, respectively. The action (29) is an example of the so-called higher gauge theory.

By choosing the three bases of generators $\tau_\alpha \in \mathfrak{g}$, $t_a \in \mathfrak{h}$, and $T_A \in \mathfrak{l}$ of the three respective Lie algebras, one can expand all fields in the theory into components as

$$\begin{aligned} B &= \frac{1}{2} B^{\alpha}{}_{\mu\nu}(x)\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \otimes \tau_{\alpha}\,, & \alpha &= \alpha^{\alpha}{}_{\mu}(x)\, \mathrm{d}x^{\mu} \otimes \tau_{\alpha}\,, \\ C &= C^{a}{}_{\mu}(x)\, \mathrm{d}x^{\mu} \otimes t_a\,, & \beta &= \frac{1}{2} \beta^{a}{}_{\mu\nu}(x)\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \otimes t_a\,, \\ D &= D^{A}(x) T_A\,, & \gamma &= \frac{1}{3!} \gamma^{A}{}_{\mu\nu\rho}(x)\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\rho} \otimes T_A\,. \end{aligned}$$

One can also make use of the following notation for the components of all maps present in the theory, in the same three bases:

$$
\begin{array}{llll}
[\tau_\alpha\,,\tau_\beta] = f_{\alpha\beta}{}^{\gamma}\tau_\gamma\,, & g_{\alpha\beta} = \langle\tau_\alpha\,,\tau_\beta\rangle_{\mathfrak{g}}\,, & \tau_\alpha \triangleright \tau_\beta = \triangleright_{\alpha\beta}{}^{\gamma}\tau_\gamma\,, & \delta T_A = \delta_A{}^a t_a\,, \\
[t_a\,,t_b] = f_{ab}{}^{c}t_c\,, & g_{ab} = \langle t_a\,,t_b\rangle_{\mathfrak{h}}\,, & \tau_\alpha \triangleright t_a = \triangleright_{\alpha a}{}^{b}t_b\,, & \partial t_a = \partial_a{}^{\alpha}\tau_\alpha\,, \\
[T_A\,,T_B] = f_{AB}{}^{C}T_C\,, & g_{AB} = \langle T_A\,,T_B\rangle_{\mathfrak{l}}\,, & \tau_\alpha \triangleright T_A = \triangleright_{\alpha A}{}^{B}T_B\,, & \{t_a\,,t_b\}_{\rm pf} = X_{ab}{}^{A}T_A\,.
\end{array}
$$

The complete gauge symmetry of the $3BF$ action was studied in [8] using the techniques of Hamiltonian analysis. It consists of five types of gauge transformations, G-, H-, L-, M-, and N-gauge transformations, determined with the independent parameters $\epsilon_{\mathfrak{g}}{}^{\alpha}(x)$, $\epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}(x)$, $\epsilon_{\mathfrak{l}}{}^{A}{}_{\mu\nu}(x)$, $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu}(x)$, and $\epsilon_{\mathfrak{n}}{}^{a}(x)$, respectively. The form variations of the fields B, C, D, α, β, and γ, obtained in [8] are given as follows:

$$
\begin{array}{rcl}
\delta_0 B^{\alpha}{}_{\mu\nu} & = & f_{\beta\gamma}{}^{\alpha}\epsilon_{\mathfrak{g}}{}^{\beta}B^{\gamma}{}_{\mu\nu} + 2C_{a[\mu|}\epsilon_{\mathfrak{h}}{}^{b}{}_{|\nu]}\triangleright_{\beta b}{}^{a}g^{\alpha\beta} - D_A\triangleright_{\beta B}{}^{A}\epsilon_{\mathfrak{l}}{}^{B}{}_{\mu\nu}g^{\alpha\beta} - 2\nabla_{[\mu|}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{|\nu]} \\
 & & +\beta_{b\mu\nu}\triangleright_{\beta a}{}^{b}\epsilon_{\mathfrak{n}}{}^{a}g^{\alpha\beta}\,, \\
\delta_0 C^{a}{}_{\mu} & = & \triangleright_{\alpha b}{}^{a}\epsilon_{\mathfrak{g}}{}^{\alpha}C^{b}{}_{\mu} + 2D_A X_{(ab)}{}^{A}\epsilon_{\mathfrak{h}}{}^{b}{}_{\mu} - \partial^{a}{}_{\alpha}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu} - \nabla_{\mu}\epsilon_{\mathfrak{n}}{}^{a}\,, \\
\delta_0 D^{A} & = & \triangleright_{\alpha B}{}^{A}\epsilon_{\mathfrak{g}}{}^{\alpha}D^{B} + \delta^{A}{}_{a}\epsilon_{\mathfrak{n}}{}^{a}\,, \\
\delta_0 \alpha^{\alpha}{}_{\mu} & = & -\partial_{\mu}\epsilon_{\mathfrak{g}}{}^{\alpha} - f_{\beta\gamma}{}^{\alpha}\alpha^{\beta}{}_{\mu}\epsilon_{\mathfrak{g}}{}^{\gamma} - \partial_a{}^{\alpha}\epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}\,, \\
\delta_0 \beta^{a}{}_{\mu\nu} & = & \triangleright_{\alpha b}{}^{a}\epsilon_{\mathfrak{g}}{}^{\alpha}\beta^{b}{}_{\mu\nu} - 2\nabla_{[\mu|}\epsilon_{\mathfrak{h}}{}^{a}{}_{|\nu]} + \delta_A{}^{a}\epsilon_{\mathfrak{l}}{}^{A}{}_{\mu\nu}\,, \\
\delta_0 \gamma^{A}{}_{\mu\nu\rho} & = & \triangleright_{\alpha B}{}^{A}\epsilon_{\mathfrak{g}}{}^{\alpha}\gamma^{B}{}_{\mu\nu\rho} + 3!\beta^{a}{}_{[\mu\nu}\epsilon_{\mathfrak{h}}{}^{b}{}_{\rho]}X_{(ab)}{}^{A} + \nabla_{\mu}\epsilon_{\mathfrak{l}}{}^{A}{}_{\nu\rho} - \nabla_{\nu}\epsilon_{\mathfrak{l}}{}^{A}{}_{\mu\rho} + \nabla_{\rho}\epsilon_{\mathfrak{l}}{}^{A}{}_{\mu\nu}\,.
\end{array}
\tag{30}
$$

The gauge transformations (30) form a group $\mathcal{G}_{3BF}$:

$$
\mathcal{G}_{3BF} = \tilde{G} \ltimes (\tilde{H}_L \ltimes (\tilde{N} \times \tilde{M}))\,, \tag{31}
$$

where $\tilde{G}$ denotes the group of G-gauge transformations, the H-gauge transformations together with the L-gauge transformations form the group $\tilde{H}_L$, while $\tilde{M}$ and $\tilde{N}$ are the groups of M- and N-gauge transformations, respectively. All these groups are determined from the structure of the initial chosen two-crossed module that defines the theory; see [8] for details.

However, as we have seen in the general theory in Section 2 and in the example of the Chern–Simons theory in Section 3, the symmetry group $\mathcal{G}_{3BF}$ determined by the Hamiltonian analysis does not include HT transformations, and therefore, the *total* gauge group should in fact be

$$
\mathcal{G}_{\rm total} = \mathcal{G}_{3BF} \ltimes \mathcal{G}_{HT}\,. \tag{32}
$$

4.2. Explicit HT Transformations

Let us explicitly define the HT transformations for the $3BF$ action (29). If we denote the dimensions of the Lie algebras $\mathfrak{g}, \mathfrak{h}, \mathfrak{l}$ as

$$
\dim(\mathfrak{g}) = p\,, \qquad \dim(\mathfrak{h}) = q\,, \qquad \dim(\mathfrak{l}) = r\,,
$$

the number of independent field components in the theory can be counted according to the following table:

$B^{\alpha}{}_{\mu\nu}$	$C^{a}{}_{\mu}$	D^{A}	$\alpha^{\alpha}{}_{\mu}$	$\beta^{a}{}_{\mu\nu}$	$\gamma^{A}{}_{\mu\nu\rho}$
$6p$	$4q$	r	$4p$	$6q$	$4r$

The total number of independent field components is, therefore,

$$
N = 6p + 4q + r + 4p + 6q + 4r = 10p + 10q + 5r\,.
$$

Let ϕ^i denote all field components, where $i = 1, 2, \ldots, N$. We can write the fields schematically as a column-matrix with six blocks:

$$\phi^i = \begin{pmatrix} B^\alpha{}_{\mu\nu} \\ C^a{}_\mu \\ D^A \\ \alpha^\alpha{}_\mu \\ \beta^a{}_{\mu\nu} \\ \gamma^A{}_{\mu\nu\rho} \end{pmatrix} .$$

The HT transformation is then defined via the parameters $\epsilon^{ij}(x)$ as

$$\delta_0 \phi^i = \epsilon^{ij} \frac{\delta S}{\delta \phi^j} .$$

The requirement that the variation of the action vanishes enforces the antisymmetry restriction on the parameters, $\epsilon^{ij} = -\epsilon^{ji}$, for all $i, j \in \{1, \ldots, N\}$. These transformations can be represented more explicitly as a tensorial 6×6 block-matrix equation, in the following form:

$$\begin{pmatrix} \delta_0 B^\alpha{}_{\mu\nu} \\ \delta_0 C^a{}_\mu \\ \delta_0 D^A \\ \delta_0 \alpha^\alpha{}_\mu \\ \delta_0 \beta^a{}_{\mu\nu} \\ \delta_0 \gamma^A{}_{\mu\nu\rho} \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c} \epsilon^{\alpha\beta}{}_{\mu\nu\sigma\lambda} & \epsilon^{\alpha b}{}_{\mu\nu\sigma} & \epsilon^{\alpha B}{}_{\mu\nu} & \epsilon^{\alpha\beta}{}_{\mu\nu\sigma} & \epsilon^{\alpha b}{}_{\mu\nu\sigma\lambda} & \epsilon^{\alpha B}{}_{\mu\nu\sigma\lambda\xi} \\ \hline \mu^{a\beta}{}_{\mu\sigma\lambda} & \epsilon^{ab}{}_{\mu\sigma} & \epsilon^{aB}{}_{\mu} & \epsilon^{a\beta}{}_{\mu\sigma} & \epsilon^{ab}{}_{\mu\sigma\lambda} & \epsilon^{aB}{}_{\mu\sigma\lambda\xi} \\ \hline \mu^{A\beta}{}_{\sigma\lambda} & \mu^{Ab}{}_{\sigma} & \epsilon^{AB} & \epsilon^{A\beta}{}_{\sigma} & \epsilon^{Ab}{}_{\sigma\lambda} & \epsilon^{AB}{}_{\sigma\lambda\xi} \\ \hline \mu^{\alpha\beta}{}_{\mu\sigma\lambda} & \mu^{\alpha b}{}_{\mu\sigma} & \mu^{\alpha B}{}_{\mu} & \epsilon^{\alpha\beta}{}_{\mu\sigma} & \epsilon^{\alpha b}{}_{\mu\sigma\lambda} & \epsilon^{\alpha B}{}_{\mu\sigma\lambda\xi} \\ \hline \mu^{a\beta}{}_{\mu\nu\sigma\lambda} & \mu^{ab}{}_{\mu\nu\sigma} & \mu^{aB}{}_{\mu\nu} & \mu^{a\beta}{}_{\mu\nu\sigma} & \epsilon^{ab}{}_{\mu\nu\sigma\lambda} & \epsilon^{aB}{}_{\mu\nu\sigma\lambda\xi} \\ \hline \mu^{A\beta}{}_{\mu\nu\rho\sigma\lambda} & \mu^{Ab}{}_{\mu\nu\rho\sigma} & \mu^{AB}{}_{\mu\nu\rho} & \mu^{A\beta}{}_{\mu\nu\rho\sigma} & \mu^{Ab}{}_{\mu\nu\rho\sigma\lambda} & \epsilon^{AB}{}_{\mu\nu\rho\sigma\lambda\xi} \end{array}\right) \begin{pmatrix} \frac{1}{2} \frac{\delta S}{\delta B^\beta{}_{\sigma\lambda}} \\ \frac{\delta S}{\delta C^b{}_\sigma} \\ \frac{\delta S}{\delta D^B} \\ \frac{\delta S}{\delta \alpha^\beta{}_\sigma} \\ \frac{1}{2} \frac{\delta S}{\delta \beta^b{}_{\sigma\lambda}} \\ \frac{1}{3!} \frac{\delta S}{\delta \gamma^B{}_{\sigma\lambda\xi}} \end{pmatrix} . \quad (33)$$

The coefficients multiplying the variations of the action in the column on the right-hand side are there to compensate the overcounting of the independent field components. Due to the antisymmetry of HT parameters, all μ blocks (below the diagonal) are determined in terms of the ϵ blocks (above the diagonal), as follows. For the first column of the parameter matrix in (33), we have:

$$\begin{gathered} \mu^{b\alpha}{}_{\sigma\mu\nu} = -\epsilon^{\alpha b}{}_{\mu\nu\sigma} , \qquad \mu^{B\alpha}{}_{\mu\nu} = -\epsilon^{\alpha B}{}_{\mu\nu} , \qquad \mu^{\beta\alpha}{}_{\sigma\mu\nu} = -\epsilon^{\alpha\beta}{}_{\mu\nu\sigma} , \\ \mu^{b\alpha}{}_{\sigma\lambda\mu\nu} = -\epsilon^{\alpha b}{}_{\mu\nu\sigma\lambda} , \qquad \mu^{B\alpha}{}_{\sigma\lambda\xi\mu\nu} = -\epsilon^{\alpha B}{}_{\mu\nu\sigma\lambda\xi} . \end{gathered} \quad (34)$$

For the second column, we have:

$$\begin{gathered} \mu^{Ba}{}_{\mu} = -\epsilon^{aB}{}_{\mu} , \qquad \mu^{\beta a}{}_{\sigma\mu} = -\epsilon^{a\beta}{}_{\mu\sigma} , \\ \mu^{ba}{}_{\sigma\lambda\mu} = -\epsilon^{ab}{}_{\mu\sigma\lambda} , \qquad \mu^{Ba}{}_{\sigma\lambda\xi\mu} = -\epsilon^{aB}{}_{\mu\sigma\lambda\xi} . \end{gathered} \quad (35)$$

The μ parameters in the third column are determined via:

$$\mu^{\beta A}{}_{\sigma} = -\epsilon^{A\beta}{}_{\sigma} , \qquad \mu^{bA}{}_{\sigma\lambda} = -\epsilon^{Ab}{}_{\sigma\lambda} , \qquad \mu^{BA}{}_{\sigma\lambda\xi} = -\epsilon^{AB}{}_{\sigma\lambda\xi} , \quad (36)$$

while the remaining μ parameters in the fourth and fifth columns are determined as:

$$\mu^{b\alpha}{}_{\sigma\lambda\mu} = -\epsilon^{\alpha b}{}_{\mu\sigma\lambda} , \qquad \mu^{B\alpha}{}_{\sigma\lambda\xi\mu} = -\epsilon^{\alpha B}{}_{\mu\sigma\lambda\xi} , \qquad \mu^{Ba}{}_{\sigma\lambda\xi\mu\nu} = -\epsilon^{aB}{}_{\mu\nu\sigma\lambda\xi} . \quad (37)$$

Finally, in addition to all these, the parameters in the blocks on the diagonal also have to satisfy certain antisymmetry relations, specifically:

$$\begin{aligned} \epsilon^{\alpha\beta}{}_{\mu\nu\sigma\lambda} &= -\epsilon^{\beta\alpha}{}_{\sigma\lambda\mu\nu}\,, \qquad \epsilon^{ab}{}_{\mu\sigma} = -\epsilon^{ba}{}_{\sigma\mu}\,, \qquad \epsilon^{AB} = -\epsilon^{BA}\,, \\ \epsilon^{\alpha\beta}{}_{\mu\sigma} &= -\epsilon^{\beta\alpha}{}_{\sigma\mu}\,, \qquad \epsilon^{ab}{}_{\mu\nu\sigma\lambda} = -\epsilon^{ba}{}_{\sigma\lambda\mu\nu}\,, \qquad \epsilon^{AB}{}_{\mu\nu\rho\sigma\lambda\xi} = -\epsilon^{BA}{}_{\sigma\lambda\xi\mu\nu\rho}\,. \end{aligned} \tag{38}$$

Like in the example of the Chern–Simons theory from the previous section, these antisymmetry relations can be satisfied in various multiple ways. All those possibilities are allowed, as long as the identities (38) are satisfied. The final ingredient in (33) is the expressions for the variation of the action with respect to the fields, and these are given as follows:

$$\begin{aligned} \frac{\delta S}{\delta B^{\beta}{}_{\nu\rho}} &= \frac{1}{2}\varepsilon^{\nu\rho\sigma\tau}\mathcal{F}_{\beta\sigma\tau}\,, \\ \frac{\delta S}{\delta C^{b}{}_{\rho}} &= \frac{1}{3!}\varepsilon^{\rho\sigma\tau\lambda}\mathcal{G}_{b\sigma\tau\lambda}\,, \\ \frac{\delta S}{\delta D^{B}} &= \frac{1}{4!}\varepsilon^{\sigma\tau\lambda\xi}\mathcal{H}_{B\sigma\tau\lambda\xi}\,, \\ \frac{\delta S}{\delta \alpha^{\beta}{}_{\rho}} &= \frac{1}{2}\varepsilon^{\rho\tau\lambda\xi}\left(\nabla_\tau B_{\beta\lambda\xi} - \rhd_{\beta a}{}^{b} C_{b\tau}\beta^{a}{}_{\lambda\xi} + \frac{1}{3}\rhd_{\beta B}{}^{A} D_A \gamma^{B}{}_{\tau\lambda\xi}\right), \\ \frac{\delta S}{\delta \beta^{b}{}_{\nu\rho}} &= \varepsilon^{\nu\rho\sigma\tau}\left(\nabla_\sigma C_{b\tau} - \frac{1}{2}\partial_b{}^{\alpha} B_{\alpha\sigma\tau} + X_{(ab)}{}^{A} D_A \beta^{b}{}_{\sigma\tau}\right), \\ \frac{\delta S}{\delta \gamma^{B}{}_{\mu\nu\rho}} &= \varepsilon^{\mu\nu\rho\sigma}\left(\nabla_\sigma D_B + \delta_B{}^{a} C_{a\sigma}\right). \end{aligned} \tag{39}$$

4.3. Diffeomorphisms

As in the case of the Chern–Simons theory, it is instructive to discuss diffeomorphism symmetry. The $3BF$ action (29) obviously is diffeomorphism invariant, since it is formulated in a manifestly covariant way, using differential forms. However, one can check that the diffeomorphisms are not a subgroup of the gauge symmetry group $\mathcal{G}_{3BF}$ given by Equation (31), but nevertheless can be obtained as a subgroup of the total gauge group (32):

$$Diff(\mathcal{M}_4) \not\subset \mathcal{G}_{3BF}\,, \qquad \text{but} \qquad Diff(\mathcal{M}_4) \subset \mathcal{G}_{\text{total}} = \mathcal{G}_{3BF} \ltimes \mathcal{G}_{HT}\,. \tag{40}$$

Let us demonstrate this. Like in the Chern–Simons case, we want to demonstrate that the form variation of all fields corresponding to diffeomorphisms can be obtained as a suitable combination of the form variations for the ordinary gauge transformations (30) and the HT transformations (33). In other words, for an arbitrary choice of the diffeomorphism parameters $\xi^\mu(x)$ from (24), Equation (25) should hold in the case of the $3BF$ theory as well:

$$\delta_0{}^{\text{diff}}\phi = \delta_0{}^{\text{gauge}}\phi + \delta_0{}^{\text{HT}}\phi\,. \tag{41}$$

Indeed, this can be shown by a suitable choice of parameters. Regarding the parameters of the gauge transformations (30), the appropriate choice is given as:

$$\begin{aligned} \epsilon_{\mathfrak{g}}{}^{\alpha} = \xi^\lambda \alpha^{\alpha}{}_{\lambda}\,, \qquad \epsilon_{\mathfrak{h}}{}^{a}{}_{\mu} = -\xi^\lambda \beta^{a}{}_{\mu\lambda}\,, \qquad \epsilon_{\mathfrak{l}}{}^{A}{}_{\mu\nu} = -\xi^\lambda \gamma^{A}{}_{\mu\nu\lambda}\,, \\ \epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu} = -\xi^\lambda B^{\alpha}{}_{\mu\lambda}\,, \qquad \epsilon_{\mathfrak{n}}{}^{a} = \xi^\lambda C^{a}{}_{\lambda}\,. \end{aligned} \tag{42}$$

Regarding the parameters of the HT transformations (33), we chose the following special case, with the majority of the parameters equated to zero:

$$
\begin{pmatrix} \delta_0 B^{\alpha}{}_{\mu\nu} \\ \delta_0 C^{a}{}_{\mu} \\ \delta_0 D^{A} \\ \delta_0 \alpha^{\alpha}{}_{\mu} \\ \delta_0 \beta^{a}{}_{\mu\nu} \\ \delta_0 \gamma^{A}{}_{\mu\nu\rho} \end{pmatrix} = \left(\begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & \epsilon^{\alpha\beta}{}_{\mu\nu\sigma} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \epsilon^{ab}{}_{\mu\sigma\lambda} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \epsilon^{AB}{}_{\sigma\lambda\xi} \\ \hline \mu^{\alpha\beta}{}_{\mu\sigma\lambda} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & \mu^{ab}{}_{\mu\nu\sigma} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mu^{AB}{}_{\mu\nu\rho} & 0 & 0 & 0 \end{array}\right) \begin{pmatrix} \frac{1}{2}\frac{\delta S}{\delta B^{\beta}{}_{\sigma\lambda}} \\ \frac{\delta S}{\delta C^{b}{}_{\sigma}} \\ \frac{\delta S}{\delta D^{B}} \\ \frac{\delta S}{\delta \alpha^{\beta}{}_{\sigma}} \\ \frac{1}{2}\frac{\delta S}{\delta \beta^{b}{}_{\sigma\lambda}} \\ \frac{1}{3!}\frac{\delta S}{\delta \gamma^{B}{}_{\sigma\lambda\xi}} \end{pmatrix} . \tag{43}
$$

Of course, due to antisymmetry, the nonzero μ blocks take negative values of the corresponding ϵ blocks, in accordance with (34), (35), and (36). The three independent nonzero ϵ blocks are chosen as

$$
\epsilon^{\alpha\beta}{}_{\mu\nu\sigma} = \xi^{\rho} g^{\alpha\beta} \varepsilon_{\mu\nu\sigma\rho} \,, \qquad \epsilon^{ab}{}_{\mu\sigma\lambda} = \xi^{\rho} g^{ab} \varepsilon_{\rho\mu\sigma\lambda} \,, \qquad \epsilon^{AB}{}_{\sigma\lambda\xi} = \xi^{\rho} g^{AB} \varepsilon_{\sigma\lambda\xi\rho} \,. \tag{44}
$$

Finally, substituting (42) and (44) into (30) and (43), respectively, and then substituting all those results into (41), after a certain amount of work, one obtains precisely the standard form variations corresponding to diffeomorphisms:

$$
\begin{aligned}
\delta_0^{\text{diff}} B^{\alpha}{}_{\mu\nu} &= -B^{\alpha}{}_{\lambda\nu}\partial_\mu\xi^\lambda - B^{\alpha}{}_{\mu\lambda}\partial_\nu\xi^\lambda - \xi^\lambda\partial_\lambda B^{\alpha}{}_{\mu\nu} \,, \\
\delta_0^{\text{diff}} C^{a}{}_{\mu} &= -C^{a}{}_{\lambda}\partial_\mu\xi^\lambda - \xi^\lambda\partial_\lambda C^{a}{}_{\mu} \,, \\
\delta_0^{\text{diff}} D^{A} &= -\xi^\lambda\partial_\lambda D^{A} \,, \\
\delta_0^{\text{diff}} \alpha^{\alpha}{}_{\mu} &= -\alpha^{\alpha}{}_{\lambda}\partial_\mu\xi^\lambda - \xi^\lambda\partial_\lambda \alpha^{\alpha}{}_{\mu} \,, \\
\delta_0^{\text{diff}} \beta^{a}{}_{\mu\nu} &= -\beta^{a}{}_{\lambda\nu}\partial_\mu\xi^\lambda - \beta^{a}{}_{\mu\lambda}\partial_\nu\xi^\lambda - \xi^\lambda\partial_\lambda \beta^{a}{}_{\mu\nu} \,, \\
\delta_0^{\text{diff}} \gamma^{A}{}_{\mu\nu\rho} &= -\gamma^{A}{}_{\lambda\nu\rho}\partial_\mu\xi^\lambda - \gamma^{A}{}_{\mu\lambda\rho}\partial_\nu\xi^\lambda - \gamma^{A}{}_{\mu\nu\lambda}\partial_\rho\xi^\lambda - \xi^\lambda\partial_\lambda \gamma^{A}{}_{\mu\nu\rho} \,.
\end{aligned} \tag{45}
$$

This establishes both relations (40), as we set out to demonstrate. We note again that the HT transformations play a crucial role in obtaining the result, since we had to choose the parameters (44) in a nontrivial manner.

4.4. Symmetry Breaking in 2BF Theory

Let us now turn to the topic of symmetry breaking and the way it influences HT transformations. To that end, we studied the topological $2BF$ action, which is a special case of the $3BF$ action (29) without the last term:

$$
S_{2BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} \,. \tag{46}
$$

In order to be even more concrete, let us fix a two-crossed module structure with the following choice of groups:

$$
G = SO(3,1) \,, \qquad H = \mathbb{R}^4 \,, \qquad L = \{e\} \,.
$$

In other words, we interpret group G as the Lorentz group, group H as the spacetime translations group, while group L is trivial, for simplicity. This choice corresponds to the so-called Poincaré two-group; see [16] for details. Since the generators of the Lorentz group can be conveniently counted using the antisymmetric combinations of indices from the group of translations, instead of the G-group indices α, we shall systematically write $[ab] \in \{01, 02, 03, 12, 13, 23\}$, where $a, b \in \{0, 1, 2, 3\}$ are H-group indices, and the brackets denote antisymmetrization. With a further change in notation from the connection 1-form α to the spin-connection 1-form ω, the curvature 2-form $\mathcal{F}(\alpha)$ to $R(\omega)$, and interpreting the Lagrange multiplier 1-form C as the tetrad 1-form e, the $2BF$ action can be rewritten in new notation as

$$S_{2BF} = \int_{\mathcal{M}_4} B^{[ab]} \wedge R_{[ab]} + e^a \wedge \mathcal{G}_a \,. \tag{47}$$

The ordinary gauge symmetry group for this action has a form similar to (31):

$$\mathcal{G}_{2BF} = \tilde{G} \ltimes (\tilde{H} \ltimes (\tilde{N} \times \tilde{M}))\,, \tag{48}$$

while the total group of gauge symmetries is extended by the HT transformations, so that

$$\mathcal{G}_{\text{total}} = \mathcal{G}_{2BF} \ltimes \mathcal{G}_{HT}\,. \tag{49}$$

The explicit HT transformations are written as a tensorial 4×4 block-matrix equation, in the form

$$\begin{pmatrix} \delta_0 B^{[ab]}{}_{\mu\nu} \\ \delta_0 e^a{}_\mu \\ \delta_0 \omega^{[ab]}{}_\mu \\ \delta_0 \beta^a{}_{\mu\nu} \end{pmatrix} = \left(\begin{array}{c|c|c|c} \epsilon^{[ab][cd]}{}_{\mu\nu\sigma\lambda} & \epsilon^{[ab]c}{}_{\mu\nu\sigma} & \epsilon^{[ab][cd]}{}_{\mu\nu\sigma} & \epsilon^{[ab]c}{}_{\mu\nu\sigma\lambda} \\ \hline \mu^{a[cd]}{}_{\mu\sigma\lambda} & \epsilon^{ac}{}_{\mu\sigma} & \epsilon^{a[cd]}{}_{\mu\sigma} & \epsilon^{ac}{}_{\mu\sigma\lambda} \\ \hline \mu^{[ab][cd]}{}_{\mu\sigma\lambda} & \mu^{[ab]c}{}_{\mu\sigma} & \epsilon^{[ab][cd]}{}_{\mu\sigma} & \epsilon^{[ab]c}{}_{\mu\sigma\lambda} \\ \hline \mu^{a[cd]}{}_{\mu\nu\sigma\lambda} & \mu^{ac}{}_{\mu\nu\sigma} & \mu^{a[cd]}{}_{\mu\nu\sigma} & \epsilon^{ac}{}_{\mu\nu\sigma\lambda} \end{array}\right) \begin{pmatrix} \frac{1}{4}\frac{\delta S}{\delta B^{[cd]}{}_{\sigma\lambda}} \\ \frac{\delta S}{\delta e^c{}_\sigma} \\ \frac{1}{2}\frac{\delta S}{\delta \omega^{[cd]}{}_\sigma} \\ \frac{1}{2}\frac{\delta S}{\delta \beta^c{}_{\sigma\lambda}} \end{pmatrix}, \tag{50}$$

where the usual antisymmetry rules apply. Here, we have

$$\begin{aligned} \frac{\delta S}{\delta B^{[cd]}{}_{\sigma\lambda}} &= \varepsilon^{\mu\nu\sigma\lambda} R_{[cd]\mu\nu}\,, \\ \frac{\delta S}{\delta \omega^{[cd]}{}_\sigma} &= \varepsilon^{\sigma\mu\nu\rho} \left(\nabla_\mu B_{[cd]\nu\rho} - e_{[c|\mu} \beta_{|d]\nu\rho} \right), \\ \frac{\delta S}{\delta e^c{}_\sigma} &= \frac{1}{2} \varepsilon^{\sigma\mu\nu\rho} \nabla_\mu \beta_{c\nu\rho}\,, \\ \frac{\delta S}{\delta \beta^c{}_{\sigma\lambda}} &= \varepsilon^{\mu\nu\sigma\lambda} \nabla_\mu e_{c\nu}\,. \end{aligned} \tag{51}$$

The $2BF$ action (46) is topological, in the sense that it has no local propagating degrees of freedom. In this sense, it does not represent a theory of any realistic physics. In order to construct a more realistic theory, one proceeds by introducing the so-called *simplicity constraint* term into the action, which changes the equations of motion of the theory so that it does have nontrivial degrees of freedom. An example is the action

$$S_{GR} = \int_{\mathcal{M}_4} B^{[ab]} \wedge R_{[ab]} + e^a \wedge \nabla \beta_a - \lambda_{[ab]} \wedge \left(B^{[ab]} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right), \tag{52}$$

where the new constraint term features another Lagrange multiplier two-form $\lambda_{[ab]}$. By virtue of the simplicity constraint, the theory becomes equivalent to general relativity, in the sense that the corresponding equations of motion reduce to vacuum Einstein field equations (see [16] for the analysis and proof). In this sense, constraint terms of various types are important when building more realistic theories; see [20] for more examples.

However, adding the simplicity constraint term also changes the gauge symmetry of the theory. In particular, it breaks the gauge group $\mathcal{G}_{2BF}$ from (48) down to one of its subgroups, so that the symmetry group of the action S_{GR} is

$$\mathcal{G}_{GR} \subset \mathcal{G}_{2BF}\,. \tag{53}$$

This is expected and unsurprising. What is less obvious, however, is that the group of HT transformations $\tilde{\mathcal{G}}_{HT}$ of the action S_{GR} *is not a subgroup* of the HT group $\mathcal{G}_{HT}$ of the original action S_{2BF}:

$$\tilde{\mathcal{G}}_{HT} \not\subset \mathcal{G}_{HT}\,, \tag{54}$$

which implies that

$$\mathcal{G}^{GR}_{\text{total}} \not\subset \mathcal{G}^{2BF}_{\text{total}} \,, \tag{55}$$

despite (53).

Let us demonstrate this. Since the action (52) features an additional field $\lambda^{[ab]}{}_{\mu\nu}(x)$, the HT transformations (50) have to be modified to take this into account and obtain the following 5×5 block-matrix form:

$$\begin{pmatrix} \delta_0 B^{[ab]}{}_{\mu\nu} \\ \delta_0 e^a{}_\mu \\ \delta_0 \omega^{[ab]}{}_\mu \\ \delta_0 \beta^a{}_{\mu\nu} \\ \delta_0 \lambda^{[ab]}{}_{\mu\nu} \end{pmatrix} = \left(\begin{array}{c|c|c|c|c} \epsilon^{[ab][cd]}{}_{\mu\nu\sigma\lambda} & \epsilon^{[ab]c}{}_{\mu\nu\sigma} & \epsilon^{[ab][cd]}{}_{\mu\nu\sigma} & \epsilon^{[ab]c}{}_{\mu\nu\sigma\lambda} & \zeta^{[ab][cd]}{}_{\mu\nu\sigma\xi} \\ \hline \mu^{a[cd]}{}_{\mu\sigma\lambda} & \epsilon^{ac}{}_{\mu\sigma} & \epsilon^{a[cd]}{}_{\mu\sigma} & \epsilon^{ac}{}_{\mu\sigma\lambda} & \zeta^{a[cd]}{}_{\mu\sigma\xi} \\ \hline \mu^{[ab][cd]}{}_{\mu\sigma\lambda} & \mu^{[ab]c}{}_{\mu\sigma} & \epsilon^{[ab][cd]}{}_{\mu\sigma} & \epsilon^{[ab]c}{}_{\mu\sigma\lambda} & \zeta^{[ab][cd]}{}_{\mu\sigma\xi} \\ \hline \mu^{a[cd]}{}_{\mu\nu\sigma\lambda} & \mu^{ac}{}_{\mu\nu\sigma} & \mu^{a[cd]}{}_{\mu\nu\sigma} & \epsilon^{ac}{}_{\mu\nu\sigma\lambda} & \zeta^{a[cd]}{}_{\mu\nu\sigma\xi} \\ \hline \theta^{[ab][cd]}{}_{\mu\nu\sigma\lambda} & \theta^{[ab]c}{}_{\mu\nu\sigma} & \theta^{[ab][cd]}{}_{\mu\nu\sigma} & \theta^{[ab]c}{}_{\mu\nu\sigma\lambda} & \psi^{[ab][cd]}{}_{\mu\nu\sigma\xi} \end{array} \right) \begin{pmatrix} \frac{1}{4}\frac{\delta S_{GR}}{\delta B^{[cd]}{}_{\sigma\lambda}} \\ \frac{\delta S_{GR}}{\delta e^c{}_\sigma} \\ \frac{1}{2}\frac{\delta S_{GR}}{\delta \omega^{[cd]}{}_\sigma} \\ \frac{1}{2}\frac{\delta S_{GR}}{\delta \beta^c{}_{\sigma\lambda}} \\ \frac{1}{4}\frac{\delta S_{GR}}{\delta \lambda^{[cd]}{}_{\sigma\xi}} \end{pmatrix}, \tag{56}$$

where

$$\begin{aligned} \frac{\delta S_{GR}}{\delta B^{[cd]}{}_{\sigma\lambda}} &= \varepsilon^{\mu\nu\sigma\lambda}\left(R_{[cd]\mu\nu} - \lambda_{[cd]\mu\nu}\right), \\ \frac{\delta S_{GR}}{\delta \omega^{[cd]}{}_\sigma} &= \varepsilon^{\sigma\mu\nu\rho}\left(\nabla_\mu B_{[cd]\nu\rho} - e_{[c|\mu}\beta_{|d]\nu\rho}\right), \\ \frac{\delta S_{GR}}{\delta e^c{}_\sigma} &= \frac{1}{2}\varepsilon^{\sigma\mu\nu\rho}\left(\nabla_\mu \beta_{c\nu\rho} + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{[ab]}{}_{\mu\nu}e^d{}_\rho\right), \\ \frac{\delta S_{GR}}{\delta \beta^c{}_{\sigma\lambda}} &= \varepsilon^{\mu\nu\sigma\lambda}\nabla_\mu e_{c\nu}\,, \\ \frac{\delta S_{GR}}{\delta \lambda^{[cd]}{}_{\sigma\xi}} &= -\varepsilon^{\sigma\xi\mu\nu}\left(B_{[cd]\mu\nu} - \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu\right). \end{aligned} \tag{57}$$

We can now investigate the differences in the form of HT transformations for the topological and constrained theory. First, comparing (56) to (50), we see that the HT transformations in the constrained theory feature *more gauge parameters* than are present in the topological theory. Namely, compared to S_{2BF}, the action S_{GR} features an extra Lagrange multiplier two-form $\lambda^{[ab]}$, which extends the matrix of HT parameters from 4×4 blocks to 5×5 blocks, and, therefore, introduces the new parameters ζ and ψ (and θ, which are the negative of ζ due to antisymmetry). This means that the group $\tilde{\mathcal{G}}_{HT}$ for the constrained theory is *larger* than the group $\mathcal{G}_{HT}$ for the topological theory. On the one hand, this immediately proves (54) and, consequently, (55). On the other hand, one can ask the opposite question—given that $\tilde{\mathcal{G}}_{HT}$ is larger than $\mathcal{G}_{HT}$, is the latter maybe a subgroup of the former?

The answer to this question is negative:

$$\mathcal{G}_{HT} \not\subset \tilde{\mathcal{G}}_{HT}\,, \tag{58}$$

which together with (54) implies our final conclusion:

$$\mathcal{G}_{HT} \neq \tilde{\mathcal{G}}_{HT}\,. \tag{59}$$

In order to demonstrate (58), we can try to set all extra parameters ζ, ψ, and θ to zero in (56), reducing it to the same form as (50). This would naively suggest that $\mathcal{G}_{HT}$ indeed is a subgroup of $\tilde{\mathcal{G}}_{HT}$. However, upon closer inspection, we can observe that this is not true, since the functional derivatives (57) are different from (51). Namely, even taking into account that the choice $\zeta = \psi = \theta = 0$ eliminates the fifth equation from (57), the first four equations are still different from their counterparts (51) because of the presence of the Lagrange multiplier $\lambda^{[ab]}$ in the action. The Lagrange multiplier is a field in the theory, and generically, it is not zero, since it is determined by the equation of motion:

$$\lambda^{[ab]}{}_{\mu\nu} = R^{[ab]}{}_{\mu\nu}\,.$$

Therefore, the HT transformations (56) in fact cannot be reduced to the HT transformations (50) by setting the extra parameters equal to zero, which proves (58) and (59).

The overall consequences from the above analysis are as follows. The topological action S_{2BF} has a large ordinary gauge group $\mathcal{G}_{2BF}$ and a small HT symmetry group $\mathcal{G}_{HT}$. When one changes the action to S_{GR} by adding a simplicity constraint term, two things happen—the ordinary gauge group breaks down to its subgroup $\mathcal{G}_{GR}$, so that it becomes smaller, while the HT symmetry group *grows larger* to a completely different group $\tilde{\mathcal{G}}_{HT}$. In effect, the *total* gauge groups for the two actions are intrinsically different:

$$\mathcal{G}^{2BF}_{\text{total}} = \mathcal{G}_{2BF} \ltimes \mathcal{G}_{HT} \qquad \neq \qquad \mathcal{G}^{GR}_{\text{total}} = \mathcal{G}_{GR} \ltimes \tilde{\mathcal{G}}_{HT} \,,$$

in the sense that neither is a subgroup of the other. This conclusion is often overlooked in the literature, which mostly puts emphasis on the symmetry breaking of the ordinary gauge group down to its subgroup.

Let us state here, without proof, that the action (52) represents an example of a non-topological action, for which one can also demonstrate a property analogous to (40), that diffeomorphisms are not a subgroup of its ordinary gauge group, but are a subgroup of the total gauge group. Simply put, given that the simplicity constraint term in (52) breaks the ordinary gauge symmetry group $\mathcal{G}_{2BF}$ into its subgroup $\mathcal{G}_{GR}$ (see (53)), one can expect that diffeomorphisms are not a subgroup of $\mathcal{G}_{GR}$, since they are not a subgroup of the larger group $\mathcal{G}_{2BF}$ of the topological action (46). Nevertheless, since the action (52) is written in a manifestly covariant form, diffeomorphisms are certainly a symmetry of the action and, thus, must be a subgroup of the total gauge group $\mathcal{G}^{GR}_{\text{total}} = \mathcal{G}_{GR} \ltimes \tilde{\mathcal{G}}_{HT}$, in line with the statement analogous to (40). We leave the details of the proof as an exercise for the reader. The point of this analysis was to demonstrate that the interplay (40) between diffeomorphisms and the HT symmetry is a generic property of a large class of actions, including the physically relevant ones, and not limited to examples of topological theories such as the Chern–Simons or nBF models.

As the last comment, let us remark that, in fact, almost all conclusions discussed for the cases of the Chern–Simons, $3BF$, and $2BF$ theories are not really specific to these concrete cases. One can easily generalize our analysis to any other theory, and the conclusions should remain unchanged, except maybe in some corner cases.

5. Conclusions

Let us review the results. In Section 2, we gave a short overview of HT gauge symmetry and discussed its most-important general properties. First, the HT group is a normal subgroup of the total group of gauge symmetries of any given action. Second, HT transformations exhaust all "trivial" (i.e., vanishing on-shell) symmetries, in the sense that there are no trivial symmetries that are not of the HT type. Finally, adding additional terms into the action substantially changes the HT group, often enlarging it. This may be considered a counterintuitive result, since usually adding additional terms in the action serves the purpose of fixing the gauge and, thus, is meant to reduce the gauge symmetry, rather than to enlarge it.

After these general results, in Section 3, we discussed the HT symmetry of the Chern–Simons action, which is a convenient toy example that neatly displays the general features from Section 2. Special attention was given to the issue of diffeomorphisms, and it was shown that, while they are not a subgroup of the ordinary gauge group of the Chern–Simons action, they nevertheless do represent a proper subgroup of the total gauge symmetry, and the HT subgroup plays a nontrivial role in demonstrating this.

Section 4 was devoted to the study of HT symmetry in the $2BF$ and $3BF$ theories, which are relevant for the constructions of realistic quantum gravity models within the generalized spinfoam approach and higher gauge theory. After a brief review and introduction to the notion of three-groups and the $3BF$ theory, appropriate HT transformations were explicitly constructed, complementing the ordinary group of gauge symmetries of the $3BF$ action based on a given three-group. This gave us the total gauge symmetry group for this class of theories. We again discussed the issue of diffeomorphisms and demonstrated again that they are a subgroup of the total gauge group, without being a subgroup of the ordinary gauge group, just like in the case of the Chern–Simons theory. Finally, we introduced a completely concrete example of the $2BF$ theory based on the Poincaré two-group, which becomes classically equivalent to Einstein's general relativity when one introduces the additional term into the action, called the simplicity constraint. As argued in general in Section 2, the presence of this constraint breaks the ordinary gauge group down into its subgroup, while simultaneously enlarging the HT group, since it introduces an additional Lagrange multiplier field into the action. This represents an explicit example of the general statement from Section 2 that the total gauge symmetry group changes nontrivially, as opposed to simply breaking down to its subgroup.

It should be noted that the analysis and results discussed here do not cover everything that can be said about HT symmetry. Among the topics not covered, one can mention the question of an explicit form of finite HT transformations, as opposed to infinitesimal ones. Can one write down finite HT transformations in closed form, either for some conveniently chosen action or maybe even in general? A related topic is the explicit evaluation of the commutator of two HT transformations, or equivalently, the structure constants of the HT Lie algebra, or in yet other words, the multiplication rule in the group $\mathcal{G}_{HT}$. Is the group Abelian or not and for which choices of the action? Finally, one would also like to know the topological properties of the group $\mathcal{G}_{HT}$, i.e., its global structure. All these are potentially interesting topics for future research.

As a particularly interesting topic for future research, we should mention the nontrivial change of the HT symmetry group when additional terms are being added to the action. In Section 4.4, we briefly demonstrated that HT symmetry does change in a nontrivial way, on the example action (52). Nevertheless, the precise properties and the physical interpretation of this change are yet to be studied in full and for a general choice of the action. This topic is the subject of ongoing research.

Finally, we would like to reiterate the differences in two possible approaches to the notion of "the gauge symmetry" of a theory. The overwhelmingly common approach throughout the literature is to factor out the HT group and work only with the ordinary, nontrivial gauge group as the relevant symmetry. Admittedly, this approach does feature a certain level of appeal due to its simplicity and economy, since it does not have to deal with HT symmetry at all. Nevertheless, there are important situations where this is not enough, and one really needs to take into account the *total* gauge symmetry group, which includes HT transformations. As a rule, these situations always involve the gauge symmetry off-shell, either for the purpose of quantization or otherwise. A typical example is the Batalin–Vilkovisky formalism, where one needs to explicitly keep track of HT transformations throughout the whole analysis. Another situation, which was discussed here in more detail, is the question of diffeomorphism symmetry, where HT transformations are required in order to prove that diffeomorphisms are a symmetry of the theory even off-shell. This is especially relevant for building quantum gravity models. Finally, the third scenario would be the discussion of the Coleman–Mandula theorem. One of the main assumptions of the theorem is that the Poincaré group is a subgroup of the full symmetry group of the theory. Given this assumption, and a number of other assumptions, the theorem implies that the full symmetry group must be a direct product of the Poincaré subgroup and the internal symmetry subgroup. In certain cases of theories (such as the $3BF$ action), the full symmetry group is not explicitly expressed as such a direct product, and moreover, it is not obvious that the Poincaré group is a subgroup of the full symmetry group to begin with. Therefore, in order to verify whether the above assumption of the theorem is satisfied, one needs to inspect if the Poincaré group is or is not a subgroup of the full symmetry group. At this point, one may run into a scenario similar to diffeomorphisms: the Poincaré group may fail to be a subgroup of the ordinary gauge group, but still be a subgroup of the total gauge group, once the HT symmetry is taken into account. In this sense, HT symmetry may become relevant for the proper analysis and application of the Coleman–Mandula theorem in certain contexts. This topic is the subject of ongoing research [34].

All of the above arguments suggest that it may be prudent to abandon the common approach of factoring out the HT group and instead adopt the description of the symmetry with the total gauge group, which includes HT transformations on equal footing as the ordinary gauge transformations. In the long run, this may be a conceptually cleaner approach. However, either way, we believe that HT symmetry is relevant for the overall symmetry structure of a theory and that better understanding of its properties can add value to and benefit research.

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Gauge symmetry of the 3*BF* theory for a generic semistrict Lie three-group

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Abstract

The higher category theory can be employed to generalize the *BF* action to the so-called 3*BF* action, by passing from the notion of a gauge group to the notion of a gauge three-group. In this work we determine the full gauge symmetry of the 3*BF* action. To that end, the complete Hamiltonian analysis of the 3*BF* action for an arbitrary semistrict Lie three-group is performed, by using the Dirac procedure. The Hamiltonian analysis is the first step towards a canonical quantization of a 3*BF* theory. This is an important stepping-stone for the quantization of the complete standard model of elementary particles coupled to Einstein–Cartan gravity, formulated as a 3*BF* action with suitable simplicity constraints. We show that the resulting gauge symmetry group consists of the familiar *G*-, *H*-, and *L*-gauge transformations, as well as additional *M*- and *N*-gauge transformations, which have not been discussed in the existing literature.

Keywords: quantum gravity, higher gauge theory, higher category theory, three-group, *BF* action, 3*BF* action, gauge symmetry

Contents

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1. Introduction

Among the most important open problems in contemporary theoretical physics is the problem of quantization of the gravitational field. Within the framework of loop quantum gravity (LQG), one of the most prominent candidates for the quantum theory of gravity, the study of nonperturbative quantization has evolved in two directions: the canonical and the covariant approach. See [1–4] for an overview and a comprehensive introduction to the theory.

The *covariant quantization* approach focuses on defining the gravitational path integral of the theory:

$$Z_{\rm gr} = \int \mathcal{D}g\, e^{iS_{\rm gr}[g]}. \tag{1}$$

In order to give the rigorous definition of the path integral, the classical action of the theory $S_{\rm gr}$ is written as a sum of the topological BF action, i.e. the action with no propagating degrees of freedom, and the part featuring the simplicity constraints, i.e. sum of products of Langrange multipliers and the corresponding simplicity constraints imposed on the variables of the topological part of the action. Next, one defines the path-integral of the topological theory given by the BF action, using the topological quantum field theory (TQFT) formalism. Once a path-integral is defined for the topological sector, it is deformed into a non-topological theory, by imposing the simplicity constraints. This quantization technique is known as the *spinfoam quantization* method.

The spinfoam quantization procedure has been successfully employed in various theories, including the three-dimensional topological Ponzano–Regge model of quantum gravity [5], the four-dimensional topological Ooguri model [6], the Barrett–Crane model of gravity in four dimensions [7–9], and others. The most successful among these is the renowned EPRL/FK model [10, 11], which had been specifically formulated to correspond to the quantum theory of gravity obtained by the *canonical loop quantization*, where a state of the gravitational field is described by the so-called *spin network*.

However, note that all mentioned models, formulated as constrained *BF* actions, are theories of pure gravity, without matter fields. Recently, as an endeavor to formulate a theory that unifies all the known interactions, one interesting new avenue of research has been opened, based on a categorical generalization of the *BF* action in the context of higher gauge theory (HGT) formalism [12]. One novel candidate discussed in the literature [13], uses the three-group structure to formulate the 3*BF* action as a categorical generalization of the *BF* theory. Then, modifying the pure 3*BF* action by adding the appropriate simplicity constraints, one obtains the *constrained* 3*BF* action, describing the theory of all the fields present in the standard model coupled in a standard way to Einstein–Cartan gravity.

Once the appropriate classical theory has been constructed, one needs to quantize it by constructing a topological state sum Z using the algebraic structure underlying the topological sector of the constrained 3*BF* action, i.e. the underlying two-crossed module. This construction has been recently carried out in [14], where a triangulation independent state sum Z of a topological HGT for an arbitrary two-crossed module and a four-dimensional closed and orientable spacetime manifold $\mathcal{M}_4$ is defined. Once the topological state sum is formulated, one could proceed to modify the amplitudes of the state sum in order to impose the simplicity constraints and obtain the state sum describing the full theory. In this way one would finally arrive at the rigorous definition of a path integral given by the equation (1).

In addition to the covariant approach, one can also study the constrained 3*BF* action, using the *canonical quantization*. This approach focuses on defining the quantum theory via a triple $(\mathcal{H}, \mathcal{A}, W)$, i.e. the Hilbert space of states $\mathcal{H}$, the algebra of observables $\mathcal{A}$, and the dynamics W given by the transition amplitudes. Specifically, in canonical LQG, the algebra of fields that are promoted to the quantum operators is chosen to be the algebra based on the holonomies of the gravitational connection. However, in the case of the 3*BF* theory, the notion of connection is generalized to the notion of three-connection, which makes its canonical quantization approach an interesting avenue of research. The first step toward the canonical quantization of the theory is the Hamiltonian analysis, resulting in the algebra of first-class and second-class constraints. The first-class constraints become conditions on the physical states determining the Hilbert space, while the Hamiltonian constraint determines the dynamics.

The results presented in this paper are the natural continuation of the results presented in [13]. The main result is the calculation of the full symmetry group of the pure 3*BF* action. To that end, the complete Hamiltonian analysis of the 3*BF* action for a semistrict Lie three-group is performed by using the Dirac procedure (see [15] for an overview and a comprehensive introduction to the Hamiltonian analysis). It is a generalization of the Hamiltonian analysis of a 2*BF* action performed in [16–19], and of the Hamiltonian analysis for the special case of a two-crossed module corresponding to the theory of scalar electrodynamics, carried out in [20]. The analysis of the Hamiltonian structure of the theory gives us the algebra of first-class and second-class constraints present in the theory. As usual, the first-class constraints generate gauge transformations, which do not change the physical state of the system. Using the Castellani's procedure, one can find the generator of the gauge transformations in the theory on a spatial hypersurface. Then, the results obtained by this method are generalized to the whole spacetime. The complete gauge symmetry, consisting of five types of finite gauge transformations, along with the proofs that they are indeed the gauge symmetries of 3*BF* action, is presented. With these results in hand, the structure of the full gauge symmetry group is analyzed, and its corresponding Lie algebra is determined.

The obtained results give rise to a connection between the gauge symmetry group of the 3*BF* action, and its underlining three-group structure, establishing a *duality* between the two. This analysis is an important step towards the study of the gauge symmetry group of the theory of gravity with matter, formulated as the constrained 3*BF* action [13], as well as its canonical quantization. Furthermore, it is important for the overall understanding of the physical meaning of the three-group structure and its interpretation as the underlining symmetry of the pure 3*BF* action, which represents a basis for the constrained 3*BF* action describing the physical theory.

The layout of the paper is as follows. In section 2, we give a brief overview of *BF* and 2*BF* theories, and introduce the 3*BF* action. Section 3 contains the Hamiltonian analysis for the 3*BF* theory. In subsection 3.1, the canonical structure of the theory is obtained, while in subsection 3.2 the resulting first-class and second-class constraints present in the theory, as well as the algebra of constraints, are presented. In the subsection 3.3 we analyze the Bianchi identities (BI) that the first-class constraints satisfy, which enforce restrictions in the sense of Hamiltonian analysis, and reduce the number of independent first-class constraints present in the theory. We then proceed with the counting of the physical degrees of freedom. Finally, this section concludes with the subsection 3.4 where we construct the generator of the gauge symmetries for the topological theory, based on the calculations done in section 3.2.

Section 4 contains the main results of our paper and is devoted to the analysis of the symmetries of the 3*BF* action. Having results of the subsection 3.4 in hand, we find the form variations of all variables and their canonical momenta, and use that result to determine all gauge transformations of the theory. This is done in four steps. The subsection 4.1 deals with the gauge group G, and the corresponding G-gauge transformations. In subsection 4.2 we discuss the gauge group $\tilde{H}_L$ which consists of the H-gauge and L-gauge transformations (familiar from [21]), while the subsection 4.3 examines the novel M-gauge and N-gauge transformations which also arise in the theory. The results of the subsections 4.1–4.3 are summarized in subsection 4.4, where the complete structure of the symmetry group is presented, including its Lie algebra. Our concluding remarks are given in section 5, containing a summary and a discussion of the obtained results, as well as possible future lines of investigation. The appendices contain various technical details concerning three-groups, additional relations of the constraint algebra, the computation of the generator of gauge symmetries, form-variations of all fields and momenta, and some other technical details.

Our notation and conventions are as follows. Spacetime indices, denoted by the mid-alphabet Greek letters $\mu, \nu, \ldots$, are raised and lowered by the spacetime metric $g_{\mu\nu}$. The spatial part of these is denoted with lowercase mid-alphabet Latin indices $i, j, \ldots$, and the time component is denoted with 0. The indices that are counting the generators of groups G, H, and L are denoted with initial Greek letters $\alpha, \beta, \ldots$, lowercase initial Latin letters $a, b, c, \ldots$, and uppercase Latin indices $A, B, C, \ldots$, respectively. The antisymmetrization over two indices is denoted as $A_{[a_1|a_2\ldots a_{n-1}|a_n]} = \frac{1}{2}\left(A_{a_1a_2\ldots a_{n-1}a_n} - A_{a_na_2\ldots a_{n-1}a_1}\right)$, while the total antisymmetrization is denoted as $A_{[a_1\ldots a_n]} = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{\mathrm{sgn}(\sigma)}A_{a_{\sigma(1)}\ldots a_{\sigma(n)}}$. Likewise, the symmetrization over two indices is denoted as $A_{(a_1|a_2\ldots a_{n-1}|a_n)} = \frac{1}{2}\left(A_{a_1a_2\ldots a_{n-1}a_n} + A_{a_na_2\ldots a_{n-1}a_1}\right)$, while the total symmetrization is denoted as $A_{(a_1\ldots a_n)} = \frac{1}{n!}\sum_{\sigma\in S_n}A_{a_{\sigma(1)}\ldots a_{\sigma(n)}}$. We work in the natural system of units, defined by $c = \hbar = 1$ and $G = l_{\mathrm{p}}^2$, where l_{p} is the Planck length. All additional notation and conventions used throughout the paper are explicitly defined in the text where they appear.

2. The 3BF theory

Given a Lie group G and its corresponding Lie algebra $\mathfrak{g}$, one can introduce the so-called BF action as

$$S_{BF} = \int_{\mathcal{M}_4} \langle B \wedge F \rangle_{\mathfrak{g}}, \tag{2}$$

where $F \equiv d\alpha + \alpha \wedge \alpha$ is the curvature two-form for the algebra-valued connection one-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$ on a trivial principal G-bundle over a four-dimensional compact and orientable spacetime manifold $\mathcal{M}_4$, and $B \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$ is a Lagrange multiplier two-form. The $\langle \_, \_ \rangle_{\mathfrak{g}}$ denotes the G-invariant bilinear symmetric nondegenerate form on $\mathfrak{g}$. For more details see [22–24].

Varying the action (2) with respect to the Lagrange multiplier B and the connection α, one obtains the equations of motion,

$$F = 0, \qquad \nabla B \equiv dB + \alpha \wedge B = 0. \tag{3}$$

These equations of motion imply that α is a flat connection, while the Lagrange multiplier B is a constant field. Therefore, the theory given by the BF action has no local propagating degrees of freedom, i.e. the theory is topological.

Within the framework of HGT, one can define the categorical generalization of the BF action to the so-called 2BF action, by passing from the notion of a gauge group to the notion of a gauge two-group, see [25–27]. In the category theory, a two-group is defined as a two-category consisting of only one object, where all the morphisms and two-morphisms are invertible. It has been shown that every strict two-group is equivalent to a crossed module $(H \xrightarrow{\partial} G, \triangleright)$, where G and H are groups, δ is a homomorphism from H to G, while $\triangleright : G \times H \to H$ is an action of G on H. Given a crossed-module $(H \xrightarrow{\partial} G, \triangleright)$, one can introduce a generalization of the BF action, the so-called 2BF action [25, 26]:

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}}, \tag{4}$$

where the two-form $B \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$ and the one-form $C \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ are Lagrange multipliers, and $\mathfrak{h}$ is a Lie algebra of the Lie group H. The variables $\mathcal{F} \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$ and $\mathcal{G} \in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{h})$ define the *fake* two-*curvature* $(\mathcal{F}, \mathcal{G})$ for the two-connection (α, β) on a trivial principal two-bundle over a four-dimensional compact and oriented spacetime manifold $\mathcal{M}_4$. See [28] for a rigorous definition. Here the two-connection (α, β) is given by $\mathfrak{g}$-valued one-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$ and an $\mathfrak{h}$-valued two-form $\beta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{h})$:

$$\mathcal{F} = d\alpha + \alpha \wedge \alpha - \partial\beta, \qquad \mathcal{G} = d\beta + \alpha \wedge^{\triangleright} \beta. \tag{5}$$

The two-curvature $(\mathcal{F}, \mathcal{G})$ is called *fake*, because of the additional term $\partial\beta$, see [12]. Also, $\langle \_, \_ \rangle_{\mathfrak{g}}$ and $\langle \_, \_ \rangle_{\mathfrak{h}}$ denote the G-invariant bilinear symmetric nondegenerate forms for the algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. See [25, 26] for review and references. Varying the 2BF action (4) with respect to variables B and C one obtains the equations of motion

$$\mathcal{F} = 0, \qquad \mathcal{G} = 0, \tag{6}$$

while varying with respect to α and β one obtains

$$dB_\alpha - f_{\alpha\beta}{}^\gamma B_\gamma \wedge \alpha^\beta - \triangleright_{\alpha a}{}^b C_b \wedge \beta^a = 0, \tag{7}$$

$$\mathrm{d}C_a - \partial_a{}^\alpha B_\alpha + \triangleright_{\alpha a}{}^b C_b \wedge \alpha^\alpha = 0. \tag{8}$$

Here, the coefficients $f_{\alpha\beta}{}^\gamma$ are the structure constants of the algebra $\mathfrak{g}$, $\triangleright_{\alpha a}{}^b$ are the coefficients of the action $\triangleright$ of the algebra $\mathfrak{g}$ on $\mathfrak{h}$, while $\partial_a{}^\alpha$ are the coefficients of the map ∂, given in the bases of algebras $\mathfrak{g}$ and $\mathfrak{h}$ (see the equations (10)–(12) below). Similarly to the case of the *BF* action, the 2*BF* action defines a topological theory, i.e. a theory with no propagating degrees of freedom, see [16, 19].

Continuing the categorical generalization one step further, one can generalize the notion of a two-group to the notion of a three-group. Similarly to the definition of a group and a two-group within the category theory formalism, a three-group is defined as a three-category with only one object, where all morphisms, two-morphisms, and three-morphisms are invertible. Moreover, analogously as a strict two-group is equivalent to a crossed-module, it has been proved that a semistrict three-group is equivalent to a two-crossed module [29].

A Lie two-crossed module, denoted as $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{\_,\_\}_{\mathrm{pf}})$ (see appendix A for the precise definition), is an algebraic structure specified by three Lie groups G, H, and L, together with the homomorphisms $\delta : L \to H$ and $\partial : H \to G$, an action $\triangleright$ of the group G on all three groups, and a G-equivariant map, called the Peiffer lifting:

$$\{\_,\_\}_{\mathrm{pf}} : H \times H \to L.$$

In order for this structure to be a three-group, the structure constants of algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$, together with the maps ∂ and δ, the action $\triangleright$, and the Peiffer lifting, must satisfy certain axioms, see [13]. Here $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ denote the Lie algebras corresponding to the Lie groups G, H, and L.

Analogously to the definition of a two-connection given in [28], one can define a three-connection as follows. Given a two-crossed module and a four-dimensional compact and orientable spacetime manifold $\mathcal{M}_4$, one can introduce a trivial principal three-bundle using the two-crossed module as a fiber over the base manifold $\mathcal{M}_4$. See [21, 29] for the precise definition of a corresponding three-holonomy. This gives rise to a three-connection, which can be represented as an ordered triple (α, β, γ), where α, β, and γ are algebra-valued differential forms, $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{h})$, and $\gamma \in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{l})$. The corresponding fake three-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is defined as:

$$\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta, \quad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma, \quad \mathcal{H} = \mathrm{d}\gamma + \alpha \wedge^{\triangleright} \gamma + \{\beta \wedge \beta\}_{\mathrm{pf}}. \tag{9}$$

Similarly as in the case of the 2*BF* theory, the three-curvature $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ is called *fake*, because of the additional terms $\partial\beta$, $\delta\gamma$, and $\{\beta \wedge \beta\}_{\mathrm{pf}}$. Fixing the bases in algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ as $\tau_\alpha \in \mathfrak{g}$, $t_a \in \mathfrak{h}$, and $T_A \in \mathfrak{l}$, one defines the structure constants

$$[\tau_\alpha, \tau_\beta] = f_{\alpha\beta}{}^\gamma\, \tau_\gamma, \qquad [t_a, t_b] = f_{ab}{}^c\, t_c, \qquad [T_A, T_B] = f_{AB}{}^C\, T_C, \tag{10}$$

maps $\partial : H \to G$ and $\delta : L \to H$ as

$$\partial(t_a) = \partial_a{}^\alpha\, \tau_\alpha, \qquad \delta(T_A) = \delta_A{}^a\, t_a, \tag{11}$$

and an action of $\mathfrak{g}$ on the generators of $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ as

$$\tau_\alpha \triangleright \tau_\beta = f_{\alpha\beta}{}^\gamma\, \tau_\gamma, \qquad \tau_\alpha \triangleright t_a = \triangleright_{\alpha a}{}^b\, t_b, \qquad \tau_\alpha \triangleright T_A = \triangleright_{\alpha A}{}^B\, T_B, \tag{12}$$

respectively. To define the Peiffer lifting in a basis, one specifies the coefficients $X_{ab}{}^A$:

$$\{t_a,\, t_b\}_{\mathrm{pf}} = X_{ab}{}^A T_A. \tag{13}$$

Writing the curvature in the bases of the corresponding algebras and differential forms

$$\mathcal{F} = \frac{1}{2}\mathcal{F}^{\alpha}{}_{\mu\nu}\tau_{\alpha}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}, \quad \mathcal{G} = \frac{1}{3!}\mathcal{G}^{a}{}_{\mu\nu\rho}t_{a}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\rho},$$
$$\mathcal{H} = \frac{1}{4!}\mathcal{H}^{A}{}_{\mu\nu\rho\sigma}T_{A}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\rho}\wedge\mathrm{d}x^{\sigma},$$

one obtains the corresponding components:

$$\begin{aligned}
\mathcal{F}^{\alpha}{}_{\mu\nu} &= \partial_{\mu}\alpha^{\alpha}{}_{\nu} - \partial_{\nu}\alpha^{\alpha}{}_{\mu} + f_{\beta\gamma}{}^{\alpha}\alpha^{\beta}{}_{\mu}\alpha^{\gamma}{}_{\nu} - \beta^{a}{}_{\mu\nu}\partial_{a}{}^{\alpha},\\
\mathcal{G}^{a}{}_{\mu\nu\rho} &= \partial_{\mu}\beta^{a}{}_{\nu\rho} + \partial_{\nu}\beta^{a}{}_{\rho\mu} + \partial_{\rho}\beta^{a}{}_{\mu\nu}\\
&\quad + \alpha^{\alpha}{}_{\mu}\beta^{b}{}_{\nu\rho}\rhd_{\alpha b}{}^{a} + \alpha^{\alpha}{}_{\nu}\beta^{b}{}_{\rho\mu}\rhd_{\alpha b}{}^{a} + \alpha^{\alpha}{}_{\rho}\beta^{b}{}_{\mu\nu}\rhd_{\alpha b}{}^{a} - \gamma^{A}{}_{\mu\nu\rho}\delta_{A}{}^{a},\\
\mathcal{H}^{A}{}_{\mu\nu\rho\sigma} &= \partial_{\mu}\gamma^{A}{}_{\nu\rho\sigma} - \partial_{\nu}\gamma^{A}{}_{\rho\sigma\mu} + \partial_{\rho}\gamma^{A}{}_{\sigma\mu\nu} - \partial_{\sigma}\gamma^{A}{}_{\mu\nu\rho}\\
&\quad + 2\beta^{a}{}_{\mu\nu}\beta^{b}{}_{\rho\sigma}X_{(ab)}{}^{A} - 2\beta^{a}{}_{\mu\rho}\beta^{b}{}_{\nu\sigma}X_{(ab)}{}^{A} + 2\beta^{a}{}_{\mu\sigma}\beta^{b}{}_{\nu\rho}X_{(ab)}{}^{A}\\
&\quad + \alpha^{\alpha}{}_{\mu}\gamma^{B}{}_{\nu\rho\sigma}\rhd_{\alpha B}{}^{A} - \alpha^{\alpha}{}_{\nu}\gamma^{B}{}_{\rho\sigma\mu}\rhd_{\alpha B}{}^{A} + \alpha^{\alpha}{}_{\rho}\gamma^{B}{}_{\sigma\mu\nu}\rhd_{\alpha B}{}^{A}\\
&\quad - \alpha^{\alpha}{}_{\sigma}\gamma^{B}{}_{\mu\nu\rho}\rhd_{\alpha B}{}^{A}.
\end{aligned} \tag{14}$$

Then, similarly to the construction of BF and $2BF$ actions, one can define the gauge invariant topological $3BF$ action, with the underlying structure of a three-group. For the four-dimensional compact and orientable manifold $\mathcal{M}_4$ and the two-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_,\_\}_{\mathrm{pf}})$, that gives rise to three-curvature (9), one defines the $3BF$ action as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}} + \langle C\wedge\mathcal{G}\rangle_{\mathfrak{h}} + \langle D\wedge\mathcal{H}\rangle_{\mathfrak{l}}, \tag{15}$$

where $B\in\mathcal{A}^2(\mathcal{M}_4,\mathfrak{g})$, $C\in\mathcal{A}^1(\mathcal{M}_4,\mathfrak{h})$, and $D\in\mathcal{A}^0(\mathcal{M}_4,\mathfrak{l})$ are Lagrange multipliers. The forms $\langle\_,\_\rangle_{\mathfrak{g}}$, $\langle\_,\_\rangle_{\mathfrak{h}}$, and $\langle\_,\_\rangle_{\mathfrak{l}}$ are G-invariant bilinear symmetric nondegenerate forms on $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$, respectively. Note that in the case of a semisimple Lie algebra, a natural choice for this bilinear form is the Killing form. However, one can also choose it differently, and moreover for a solvable Lie algebra one can introduce a non-trivial bilinear form, despite the fact that the Killing form is degenerate in this case. Fixing the basis in algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$, as defined in (10), the forms $\langle\_,\_\rangle_{\mathfrak{g}}$, $\langle\_,\_\rangle_{\mathfrak{h}}$, and $\langle\_,\_\rangle_{\mathfrak{l}}$ map pairs of basis vectors of algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$, to the metrics on their vector spaces, $g_{\alpha\beta}$, g_{ab}, and g_{AB}:

$$\langle\tau_{\alpha},\tau_{\beta}\rangle_{\mathfrak{g}} = g_{\alpha\beta}, \qquad \langle t_a, t_b\rangle_{\mathfrak{h}} = g_{ab}, \qquad \langle T_A, T_B\rangle_{\mathfrak{l}} = g_{AB}. \tag{16}$$

As the symmetric maps are nondegenerate, the inverse metrics $g^{\alpha\beta}$, g^{ab}, and g^{AB} are well defined, and are used to raise and lower indices of the corresponding algebras.

Varying the action (15) with respect to Lagrange multipliers B^{α}, C^{a}, and D^{A} one obtains the equations of motion

$$\mathcal{F}^{\alpha} = 0, \qquad \mathcal{G}^{a} = 0, \qquad \mathcal{H}^{A} = 0, \tag{17}$$

while varying with respect to the three-connection variables α^α, β^a, and γ^A one gets:

$$\mathrm{d}B_\alpha - f_{\alpha\beta}{}^\gamma B_\gamma \wedge \alpha^\beta - \rhd_{\alpha a}{}^b C_b \wedge \beta^a + \rhd_{\alpha B}{}^A D_A \wedge \gamma^B = 0, \tag{18}$$

$$\mathrm{d}C_a - \partial_a{}^\alpha B_\alpha + \rhd_{\alpha a}{}^b C_b \wedge \alpha^\alpha + 2X_{(ab)}{}^A D_A \wedge \beta^b = 0, \tag{19}$$

$$\mathrm{d}D_A - \rhd_{\alpha A}{}^B D_B \wedge \alpha^\alpha + \delta_A{}^a C_a = 0. \tag{20}$$

For further details see [21, 29, 30] for the definition of the three-group, and [13] for the definition of the pure 3BF action.

Choosing the convenient underlying two-crossed module structure and imposing the appropriate simplicity constraints onto the degrees of freedom present in the 3BF action, one can obtain the non-trivial classical dynamics of the gravitational and matter fields. A reader interested in the construction of the constrained 2BF actions describing the Yang–Mills field and Einstein–Cartan gravity, and 3BF actions describing the Klein–Gordon, Dirac, Weyl and Majorana fields coupled to gravity in the standard way, is referred to [13, 27]. One can also introduce higher dimensional, nBF actions, see for example [31]. Various properties of these models have been studied in [32–34]. Naturally, if one is interested in theories defined on a four-dimensional spacetime manifold, there is an upper limit on the order of the differential forms one can use to construct a n-connection, and in four dimensions that is $n = 3$.

3. Hamiltonian analysis of the 3BF theory

In this section, the canonical structure of the theory is presented, with the resulting first-class and second-class constraints present in the theory. The algebra of Poisson brackets between all, the first-class and the second-class constraints, is obtained. We will use this result to calculate the total number of degrees of freedom in the theory, and in order to do that, we will have to analyse the BI that the first-class constraints satisfy, which enforce restrictions in the sense of Hamiltonian analysis. They reduce the number of independent first-class constraints present in the theory, thus increasing the number of degrees of freedom. We will obtain that the pure 3BF theory is topological, i.e. there are no local propagating degrees of freedom. Finally, we will finish this section with the construction of the generator of gauge symmetries of the 3BF action, which is used to calculate the form-variations of all the variables and their canonical momenta. This result will be crucial for finding the gauge symmetries of 3BF action, which will be a topic of section 4.

3.1. Canonical structure and Hamiltonian

Assuming that the spacetime manifold $\mathcal{M}_4$ is globally hyperbolic, the Lagrangian on a spatial foliation Σ_3 of spacetime $\mathcal{M}_4$ corresponding to the 3BF action (15) is given as:

$$L_{3BF} = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \epsilon^{\mu\nu\rho\sigma} \left(\frac{1}{4} B^\alpha{}_{\mu\nu}\, \mathcal{F}^\beta{}_{\rho\sigma}\, g_{\alpha\beta} + \frac{1}{3!}\, C^a{}_\mu\, \mathcal{G}^b{}_{\nu\rho\sigma}\, g_{ab} + \frac{1}{4!} D^A \mathcal{H}^B{}_{\mu\nu\rho\sigma} g_{AB} \right). \tag{21}$$

For the Lagrangian (21), the canonical momenta corresponding to all variables $B^{\alpha}{}_{\mu\nu}$, $\alpha^{\alpha}{}_{\mu}$, $C^{a}{}_{\mu}$, $\beta^{a}{}_{\mu\nu}$, D^{A}, and $\gamma^{A}{}_{\mu\nu\rho}$ are:

$$
\begin{aligned}
\pi(B)_{\alpha}{}^{\mu\nu} &= \frac{\delta L}{\delta \partial_0 B^{\alpha}{}_{\mu\nu}} = 0, \\
\pi(\alpha)_{\alpha}{}^{\mu} &= \frac{\delta L}{\delta \partial_0 \alpha^{\alpha}{}_{\mu}} = \frac{1}{2}\epsilon^{0\mu\nu\rho}B_{\alpha\nu\rho}, \\
\pi(C)_{a}{}^{\mu} &= \frac{\delta L}{\delta \partial_0 C^{a}{}_{\mu}} = 0, \\
\pi(\beta)_{a}{}^{\mu\nu} &= \frac{\delta L}{\delta \partial_0 \beta^{a}{}_{\mu\nu}} = -\epsilon^{0\mu\nu\rho}C_{a\rho}, \\
\pi(D)_{A} &= \frac{\delta L}{\delta \partial_0 D^{A}} = 0, \\
\pi(\gamma)_{A}{}^{\mu\nu\rho} &= \frac{\delta L}{\delta \partial_0 \gamma^{A}{}_{\mu\nu\rho}} = \epsilon^{0\mu\nu\rho}D_{A}.
\end{aligned}
\tag{22}
$$

These momenta give rise to the six primary constraints of the theory, since none of them can be inverted for the time derivatives of the variables,

$$
\begin{aligned}
P(B)_{\alpha}{}^{\mu\nu} &\equiv \pi(B)_{\alpha}{}^{\mu\nu} \approx 0, \\
P(\alpha)_{\alpha}{}^{\mu} &\equiv \pi(\alpha)_{\alpha}{}^{\mu} - \frac{1}{2}\epsilon^{0\mu\nu\rho}B_{\alpha\nu\rho} \approx 0, \\
P(C)_{a}{}^{\mu} &\equiv \pi(C)_{a}{}^{\mu} \approx 0, \\
P(\beta)_{a}{}^{\mu\nu} &\equiv \pi(\beta)_{a}{}^{\mu\nu} + \epsilon^{0\mu\nu\rho}C_{a\rho} \approx 0, \\
P(D)_{A} &\equiv \pi(D)_{A} \approx 0, \\
P(\gamma)_{A}{}^{\mu\nu\rho} &\equiv \pi(\gamma)_{A}{}^{\mu\nu\rho} - \epsilon^{0\mu\nu\rho}D_{A} \approx 0.
\end{aligned}
\tag{23}
$$

Employing the following fundamental Poisson brackets,

$$
\begin{aligned}
\{\, B^{\alpha}{}_{\mu\nu}(\vec{x})\,,\ \pi(B)_{\beta}{}^{\rho\sigma}(\vec{y})\,\} &= 2\delta^{\alpha}_{\beta}\delta^{\rho}_{[\mu|}\delta^{\sigma}_{|\nu]}\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\, \alpha^{\alpha}{}_{\mu}(\vec{x})\,,\ \pi(\alpha)_{\beta}{}^{\nu}(\vec{y})\,\} &= \delta^{\alpha}_{\beta}\delta^{\nu}_{\mu}\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\, C^{a}{}_{\mu}(\vec{x})\,,\ \pi(C)_{b}{}^{\nu}(\vec{y})\,\} &= \delta^{a}_{b}\delta^{\nu}_{\mu}\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\, \beta^{a}{}_{\mu\nu}(\vec{x})\,,\ \pi(\beta)_{b}{}^{\rho\sigma}(\vec{y})\,\} &= 2\delta^{a}_{b}\,\delta^{\rho}_{[\mu|}\delta^{\sigma}_{|\nu]}\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\, D^{A}(\vec{x})\,,\ \pi(D)_{B}(\vec{y})\,\} &= \delta^{A}_{B}\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\, \gamma^{A}{}_{\mu\nu\rho}(\vec{x})\,,\ \pi(\gamma)_{B}{}^{\sigma\tau\xi}(\vec{y})\,\} &= 3!\delta^{A}_{B}\,\delta^{\sigma}_{[\mu}\delta^{\tau}_{\nu}\delta^{\xi}_{\rho]}\,\delta^{(3)}(\vec{x}-\vec{y}),
\end{aligned}
\tag{24}
$$

one obtains the *algebra of primary constraints*:

$$
\begin{aligned}
\{ P(B)_\alpha{}^{jk}(\vec{x}) \,, P(\alpha)_\beta{}^{i}(\vec{y}) \} &= \epsilon^{0ijk}\, g_{\alpha\beta}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y}), \\
\{ P(C)_a{}^{k}(\vec{x}) \,, P(\beta)_b{}^{ij}(\vec{y}) \} &= -\epsilon^{0ijk}\, g_{ab}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y}), \\
\{ P(D)_A(\vec{x}) \,, P(\gamma)_B{}^{ijk}(\vec{y}) \} &= \epsilon^{0ijk}\, g_{AB}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y}).
\end{aligned}
\tag{25}
$$

Note that all other Poisson brackets vanish. The *canonical, on-shell Hamiltonian* is given by the following expression:

$$
\begin{aligned}
H_{\mathrm{c}} = \int_{\Sigma_3} \mathrm{d}^3\vec{x} \left[\frac{1}{2}\pi(B)_\alpha{}^{\mu\nu}\, \partial_0 B^\alpha{}_{\mu\nu} + \pi(\alpha)_\alpha{}^{\mu}\, \partial_0 \alpha^\alpha{}_\mu + \pi(C)_a{}^{\mu}\, \partial_0 C^a{}_\mu \right. \\
\left. + \frac{1}{2}\pi(\beta)_a{}^{\mu\nu}\, \partial_0 \beta^a{}_{\mu\nu} + \pi(D)_A\, \partial_0 D^A + \frac{1}{3!}\pi(\gamma)_A{}^{\mu\nu\rho}\, \partial_0 \gamma^A{}_{\mu\nu\rho} \right] - L.
\end{aligned}
\tag{26}
$$

Employing the definition of the curvature components (14), the Hamiltonian (26) can be written as the sum of terms that are equal to the product of the primary constraints and time derivatives of the variables, and the remainder. As the primary constraints are zero on-shell, the terms multiplying the time derivatives vanish, and the canonical Hamiltonian becomes:

$$
\begin{aligned}
H_{\mathrm{c}} = -\int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \epsilon^{0ijk} \left[\frac{1}{2} B_{\alpha 0i}\, \mathcal{F}^\alpha{}_{jk} + \frac{1}{6} C_{a0}\, \mathcal{G}^a{}_{ijk} + \beta^a{}_{0i} \left(\nabla_j C_{ak} - \frac{1}{2}\partial_a{}^\alpha B_{\alpha\, jk} + \beta^b{}_{jk}\, D_A\, X_{(ab)}{}^A \right) \right. \\
\left. + \frac{1}{2}\alpha^\alpha{}_0 \left(\nabla_i B_{\alpha jk} - C_{ai} \rhd_{\alpha b}{}^a\, \beta^b{}_{jk} + \frac{1}{3} D_A \rhd_{\alpha B}{}^A\, \gamma^B{}_{ijk} \right) + \frac{1}{2}\gamma^A{}_{0ij} \left(\nabla_k D_A + C_{ak}\, \delta_A{}^a \right) \right].
\end{aligned}
\tag{27}
$$

Adding to the canonical Hamiltonian the product of the Lagrange multipliers λ and the primary constraints, for every primary constraint, one gets the *total, off-shell Hamiltonian*:

$$
\begin{aligned}
H_{\mathrm{T}} = H_{\mathrm{c}} + \int_{\Sigma_3} \mathrm{d}^3\vec{x} \left[\frac{1}{2}\lambda(B)^\alpha{}_{\mu\nu} P(B)_\alpha{}^{\mu\nu} + \lambda(\alpha)^\alpha{}_\mu P(\alpha)_\alpha{}^\mu + \lambda(C)^a{}_\mu P(C)_a{}^\mu + \frac{1}{2}\lambda(\beta)^a{}_{\mu\nu} P(\beta)_a{}^{\mu\nu} \right. \\
\left. + \lambda(D)^A P(D)_A + \frac{1}{3!}\lambda(\gamma)^A{}_{\mu\nu\rho} P(\gamma)_A{}^{\mu\nu\rho} \right].
\end{aligned}
\tag{28}
$$

3.2. Consistency conditions and algebra of constraints

In order for primary constraints to be preserved during the evolution of the system, they must satisfy the consistency conditions,

$$
\dot{P} \equiv \{ P \,, H_{\mathrm{T}} \} \approx 0,
\tag{29}
$$

for every primary constraint P. Imposing this condition on primary constraints $P(B)_\alpha{}^{0i}$, $P(\alpha)_\alpha{}^0$, $P(C)_a{}^0$, $P(\beta)_a{}^{0i}$, and $P(\gamma)_A{}^{0ij}$, one obtains the secondary constraints $\mathcal{S}$,

$$
\begin{aligned}
\mathcal{S}(\mathcal{F})_\alpha{}^i &\equiv \frac{1}{2}\epsilon^{0ijk}\mathcal{F}_{\alpha jk} \approx 0,\\
\mathcal{S}(\nabla B)_\alpha &\equiv \frac{1}{2}\epsilon^{0ijk}\left(\nabla_{[i}B_{\alpha\, jk]} - C_{a[i}\rhd_{\alpha b}{}^a\,\beta^b{}_{jk]} + \frac{1}{3}D_A\rhd_{\alpha B}{}^A\,\gamma^B{}_{ijk}\right) \approx 0,\\
\mathcal{S}(\mathcal{G})_a &\equiv \frac{1}{6}\epsilon^{0ijk}\mathcal{G}_{aijk} \approx 0,\\
\mathcal{S}(\nabla C)_a{}^i &\equiv \epsilon^{0ijk}\left(\nabla_{[j|}C_{a|k]} - \frac{1}{2}\partial_a{}^\alpha B_{\alpha\, jk} + \beta^b{}_{jk}\,D_A\,X_{(ab)}{}^A\right) \approx 0,\\
\mathcal{S}(\nabla D)_A{}^{ij} &\equiv \epsilon^{0ijk}\left(\nabla_k D_A + C_{ak}\,\delta_A{}^a\right) \approx 0,
\end{aligned}
\tag{30}
$$

while in the case of the constraints $P(\alpha)_\alpha{}^k$, $P(B)_\alpha{}^{jk}$, $P(\beta)_a{}^{jk}$, $P(C)_a{}^k$, $P(\gamma)_A{}^{ijk}$, and $P(D)_A$ the corresponding consistency conditions determine the following Lagrange multipliers:

$$
\begin{aligned}
\lambda(B)_{\alpha ij} &\approx \nabla_i B_{\alpha 0j} - \nabla_j B_{\alpha 0i} + C_{a0}\beta^b{}_{ij}\rhd_{\alpha b}{}^a + C_{bi}\rhd_\alpha{}^b{}_a\beta^a{}_{0j}\\
&\quad - C_{bj}\rhd_\alpha{}^b{}_a\beta^a{}_{0i} + g_{\beta\gamma}{}^\alpha\alpha^\beta{}_0 B^\gamma{}_{ij} + D_B\gamma^A{}_{0ij}\rhd_\alpha{}^B{}_A,\\
\lambda(\alpha)^\alpha{}_i &\approx \nabla_i\alpha^\alpha{}_0 + \partial_a{}^\alpha\beta^a{}_{0i},\\
\lambda(C)^a{}_i &\approx \nabla_i C^a{}_0 + C^b{}_i\rhd_\alpha{}^a{}_b\alpha^\alpha{}_0 - 2\beta_{b0i}D_A X^{(ba)A} + B_{\alpha 0i}\partial^{a\alpha},\\
\lambda(\beta)^a{}_{ij} &\approx \nabla_i\beta^a{}_{0j} - \nabla_j\beta^a{}_{0i} - \beta^b{}_{ij}\rhd_{\alpha b}{}^a\alpha^\alpha{}_0 + \gamma^A{}_{0ij}\delta_A{}^a,\\
\lambda(D)_A &\approx \alpha^\alpha{}_0 D_B\rhd_{\alpha A}{}^B - C_{a0}\delta_A{}^a,\\
\lambda(\gamma)^A{}_{ijk} &\approx -2\beta^a{}_{0i}\beta^b{}_{jk}X_{(ab)}{}^A + 2\beta^a{}_{0j}\beta^b{}_{ik}X_{(ab)}{}^A - 2\beta^a{}_{0k}\beta^b{}_{ij}X_{(ab)}{}^A\\
&\quad - \alpha^\alpha{}_0\rhd_{\alpha B}{}^A\gamma^B{}_{ijk} + \nabla_i\gamma^A{}_{0jk} - \nabla_j\gamma^A{}_{0ik} + \nabla_k\gamma^A{}_{0ij}.
\end{aligned}
\tag{31}
$$

Note that the rest of the Lagrange multipliers

$$
\lambda(B)^\alpha{}_{0i}, \qquad \lambda(\alpha)^\alpha{}_0, \qquad \lambda(C)^a{}_0, \qquad \lambda(\beta)^a{}_{0i}, \qquad \lambda(\gamma)^A{}_{0ij}, \tag{32}
$$

remain undetermined.

Further, as the secondary constraints must also be preserved during the evolution of the system, the consistency conditions of secondary constraints must be enforced. However, no tertiary constraints arise from these conditions (see equation (B.1) in appendix B), leading the iterative procedure to an end. Finally, the total Hamiltonian can be written in the following form:

$$
\begin{aligned}
H_{\mathrm{T}} = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \Bigg[& \lambda(B)^{\alpha}{}_{0i}\, \Phi(B)_{\alpha}{}^{i} + \lambda(\alpha)^{\alpha}\, \Phi(\alpha)_{\alpha} + \lambda(C)^{a}{}_{0}\, \Phi(C)_{a} + \lambda(\beta)^{a}{}_{0i}\, \Phi(\beta)_{a}{}^{i} \\
& + \frac{1}{2}\lambda(\gamma)^{A}{}_{0ij}\Phi(\gamma)_{A}{}^{ij} - B_{\alpha 0i}\, \Phi(\mathcal{F})^{ai} - \alpha_{\alpha 0}\, \Phi(\nabla B)^{\alpha} - C_{a0}\, \Phi(\mathcal{G})^{a} \\
& - \beta_{a0i}\, \Phi(\nabla C)^{ai} - \frac{1}{2}\gamma_{A0ij}\, \Phi(\nabla D)^{Aij} \Bigg],
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
\Phi(B)_{\alpha}{}^{i} &= P(B)_{\alpha}{}^{0i}, \\
\Phi(\alpha)_{\alpha} &= P(\alpha)_{\alpha}{}^{0}, \\
\Phi(C)_{a} &= P(C)_{a}{}^{0}, \\
\Phi(\beta)_{a}{}^{i} &= P(\beta)_{a}{}^{0i}, \\
\Phi(\gamma)_{A}{}^{ij} &= P(\gamma)_{A}{}^{0ij}, \\
\Phi(\mathcal{F})^{\alpha i} &= \mathcal{S}(\mathcal{F})^{\alpha i} - \nabla_{j}P(B)^{\alpha ij} - P(C)_{a}{}^{i}\partial^{a\alpha}, \\
\Phi(\mathcal{G})_{a} &= \mathcal{S}(\mathcal{G})_{a} + \nabla_{i}P(C)_{a}{}^{i} - \frac{1}{2}\beta_{bij} \rhd_{\alpha}{}^{b}{}_{a} P(B)^{\alpha ij} + P(D)^{A}\delta_{Aa}, \\
\Phi(\nabla C)_{a}{}^{i} &= \mathcal{S}(\nabla C)_{a}{}^{i} - \nabla_{j}P(\beta)_{a}{}^{ij} + C_{bj} \rhd_{\alpha}{}^{b}{}_{a} P(B)^{\alpha ij} \\
&\quad - \partial_{a}{}^{\alpha}P(\alpha)_{\alpha}{}^{i} + 2D_{A}X_{(ab)}{}^{A}P(C)^{bi} + \beta^{b}{}_{jk}X_{(ab)}{}^{A}P(\gamma)_{A}{}^{ijk}, \\
\Phi(\nabla B)_{\alpha} &= \mathcal{S}(\nabla B)_{\alpha} + \nabla_{i}P(\alpha)_{\alpha}{}^{i} - \frac{1}{2}f_{\alpha\gamma}{}^{\beta}B_{\beta ij}P(B)^{\gamma ij} - C_{bi} \rhd_{\alpha a}{}^{b} P(C)^{ai} \\
&\quad - \frac{1}{2}\beta_{bij} \rhd_{\alpha a}{}^{b} P(\beta)^{aij} - P(D)^{A}D_{B} \rhd_{\alpha A}{}^{B} + \frac{1}{3!}P(\gamma)_{A}{}^{ijk}\gamma^{B}{}_{ijk} \rhd_{\alpha B}{}^{A}, \\
\Phi(\nabla D)_{A}{}^{ij} &= \mathcal{S}(\nabla D)_{A}{}^{ij} + \nabla_{k}P(\gamma)_{A}{}^{ijk} - P(\beta)_{a}{}^{ij}\delta_{A}{}^{a} - P(B)^{\alpha ij} \rhd_{\alpha}{}^{B}{}_{A} D_{B},
\end{aligned}
\tag{34}
$$

are the first-class constraints. The second-class constraints in the theory are:

$$
\begin{aligned}
&\chi(B)_{\alpha}{}^{jk} = P(B)_{\alpha}{}^{jk}, \quad \chi(C)_{a}{}^{i} = P(C)_{a}{}^{i}, \quad \chi(D)_{A} = P(D)_{A}, \\
&\chi(\alpha)_{\alpha}{}^{i} = P(\alpha)_{\alpha}{}^{i}, \quad \chi(\beta)_{a}{}^{ij} = P(\beta)_{a}{}^{ij}, \quad \chi(\gamma)_{A}{}^{ijk} = P(\gamma)_{A}{}^{ijk}.
\end{aligned}
\tag{35}
$$

The PB algebra of the first-class constraints is given by

$$\begin{aligned}
\{\, \Phi(\mathcal{F})^{\alpha i}(\vec{x})\,,\, \Phi(\nabla B)_{\beta}(\vec{y}) \,\} &= f_{\beta\gamma}{}^{\alpha}\, \Phi(\mathcal{F})^{\gamma i}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y}),\\
\{\, \Phi(\nabla B)_{\alpha}(\vec{x})\,,\, \Phi(\nabla B)_{\beta}(\vec{y}) \,\} &= f_{\alpha\beta}{}^{\gamma}\, \Phi(\nabla B)_{\gamma}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y}),\\
\{\, \Phi(\mathcal{G})^{a}(\vec{x})\,,\, \Phi(\nabla C)_{b}{}^{i}(\vec{y}) \,\} &= -\triangleright_{\alpha b}{}^{a}\, \Phi(\mathcal{F})^{\alpha i}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y}),\\
\{\, \Phi(\nabla C)_{a}{}^{i}(\vec{x})\,,\, \Phi(\nabla C)_{b}{}^{j}(\vec{y}) \,\} &= -2X_{(ab)}{}^{A}\, \Phi(\nabla D)_{A}{}^{ij}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y}),\\
\{\, \Phi(\mathcal{G})^{a}(\vec{x})\,,\, \Phi(\nabla B)_{\alpha}(\vec{y}) \,\} &= \triangleright_{\alpha b}{}^{a}\, \Phi(\mathcal{G})^{b}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y}),\\
\{\, \Phi(\nabla C)^{ai}(\vec{x})\,,\, \Phi(\nabla B)_{\alpha}(\vec{y}) \,\} &= \triangleright_{\alpha b}{}^{a}\, \Phi(\nabla C)^{bi}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y}),\\
\{\, \Phi(\nabla B)_{\alpha}(\vec{x})\,,\, \Phi(\nabla D)_{A}{}^{ij}(\vec{y}) \,\} &= \triangleright_{\alpha A}{}^{B}\Phi(\nabla D)_{B}{}^{ij}(\vec{x})\delta^{(3)}(\vec{x}-\vec{y}).
\end{aligned} \tag{36}$$

The algebra between the first and the second class constraints is given in the appendix B, equation (B.2).

With the algebra of the constraints in hand, one can proceed to calculate the generator of gauge symmetries of the action. The generator will be used to calculate the form-variations of all the variables and their canonical momenta, which will help us find the finite gauge symmetries of the action. Additionally, we can determine the number of independent parameters of gauge transformations, since usually all the first class constraints generate unphysical transformations of dynamical variables, i.e. that to each parameter of the gauge symmetry there corresponds one first-class constraint. However, before we embark on the construction of the symmetry generator, we will devote some attention to the number of local propagating degrees of freedom in the theory, in order to determine if the $3BF$ action is topological or not.

3.3. Number of degrees of freedom

In this subsection, we will show that the structure of the constraints implies that there are no local degrees of freedom in a $3BF$ theory. To that end, let us first specify all the BI present in the theory.

The two-form curvatures corresponding to one-forms α and C, given by

$$F^{\alpha} = \mathrm{d}\alpha^{\alpha} + f_{\beta\gamma}{}^{\alpha}\, \alpha^{\beta}\wedge\alpha^{\gamma}, \qquad T^{a} = \mathrm{d}C^{a} + \triangleright_{\alpha b}{}^{a}\, \alpha^{\alpha}\wedge C^{b}, \tag{37}$$

satisfy the BI:

$$\epsilon^{\lambda\mu\nu\rho}\, \nabla_{\mu}F^{\alpha}{}_{\nu\rho} = 0, \tag{38}$$

$$\epsilon^{\lambda\mu\nu\rho}\left(\nabla_{\mu}T^{a}{}_{\nu\rho} - \triangleright_{\alpha b}{}^{a}F^{\alpha}{}_{\mu\nu}C^{b}{}_{\rho}\right) = 0. \tag{39}$$

Similarly, the three-form curvatures corresponding to two-forms B and β, given by

$$S^{\alpha} = \mathrm{d}B^{\alpha} + f_{\beta\gamma}{}^{\alpha}\, \alpha^{\beta}\wedge B^{\gamma}, \qquad G^{a} = \mathrm{d}\beta^{a} + \triangleright_{\alpha b}{}^{a}\, \alpha^{\alpha}\wedge\beta^{\beta}, \tag{40}$$

Table 1. The fields present in the $3BF$ theory.

$\alpha^{\alpha}{}_{\mu}$	$\beta^{a}{}_{\mu\nu}$	$\gamma^{A}{}_{\mu\nu\rho}$	$B^{\alpha}{}_{\mu\nu}$	$C^{a}{}_{\mu}$	D^{A}
$4p$	$6q$	$4r$	$6p$	$4q$	r

Table 2. Second-class constraints in the $3BF$ theory.

$\chi(B)_{\alpha}{}^{jk}$	$\chi(C)_{a}{}^{i}$	$\chi(D)_{A}$	$\chi(\alpha)_{\alpha}{}^{i}$	$\chi(\beta)_{a}{}^{ij}$	$\chi(\gamma)_{A}{}^{ijk}$
$3p$	$3q$	r	$3p$	$3q$	r

satisfy the BI:

$$\epsilon^{\lambda\mu\nu\rho}\left(\frac{2}{3}\nabla_{\lambda}S^{\alpha}{}_{\mu\nu\rho}-f_{\beta\gamma}{}^{\alpha}F^{\beta}{}_{\lambda\mu}B^{\gamma}{}_{\nu\rho}\right)=0, \tag{41}$$

$$\epsilon^{\lambda\mu\nu\rho}\left(\frac{2}{3}\nabla_{\lambda}G^{a}{}_{\mu\nu\rho}-\triangleright_{\alpha b}{}^{a}F^{\alpha}{}_{\lambda\mu}\beta^{b}{}_{\nu\rho}\right)=0. \tag{42}$$

Finally, defining the one-form curvature for D,

$$Q^{A}=\mathrm{d}D^{A}+\triangleright_{\alpha B}{}^{A}\alpha^{\alpha}\wedge D^{B}, \tag{43}$$

one can write the corresponding BI for Q^{A}:

$$\epsilon^{\lambda\mu\nu\rho}\left(\nabla_{\nu}Q^{A}{}_{\rho}-\frac{1}{2}\triangleright_{\alpha B}{}^{A}F^{\alpha}{}_{\nu\rho}D^{B}\right)=0. \tag{44}$$

These BI play an important role in determining the number of degrees of freedom present in the theory.

As the general theory states, if there are N fields in the theory, F independent first-class constraints per space point, and S independent second-class constraints per space point, the number of independent field components, i.e. the number of the physical degrees of freedom present in the theory, is given by:

$$n=N-F-\frac{S}{2}. \tag{45}$$

Let p denote the dimensionality of the group G, q the dimensionality of the group H, and r the dimensionality of the group L. Determining the number of fields present in the $3BF$ theory, by counting the field components listed in table 1, one obtains $N=10(p+q)+5r$. Similarly, one determines the number of independent components of the second-class constraints by counting the components listed in table 2 and obtains $S=6(p+q)+2r$. However, when counting the number of the first-class constraints F one notes they are not all mutually independent. Namely, one can prove the following identities, as a consequence of the BI.

Taking the derivative of $\Phi(\mathcal{F})_{\alpha}{}^{i}$ one obtains

$$\nabla_{i}\Phi(\mathcal{F})^{\alpha i}+\partial_{a}{}^{\alpha}\Phi(\mathcal{G})^{a}=\frac{1}{2}\epsilon^{0ijk}\nabla_{i}F^{\alpha}{}_{jk}-\frac{1}{2}f_{\beta\gamma}{}^{\alpha}\mathcal{F}^{\beta}{}_{ij}P(B)^{ij}. \tag{46}$$

This relation gives

$$\nabla_{i}\Phi(\mathcal{F})^{\alpha i}+\partial_{a}{}^{\alpha}\Phi(\mathcal{G})^{a}=0, \tag{47}$$

since the first term on the right-hand side of (46) is zero off-shell because $\epsilon^{ijk}\,\nabla_i F^a{}_{jk} = 0$ are the $\lambda = 0$ components of BI (38), and the second term on the right-hand side is also zero off-shell, since it is a product of two constraints:

$$\frac{1}{2} f_{\beta\gamma}{}^{\alpha} \mathcal{F}^{\beta}{}_{ij} P(B)^{ij} = \frac{1}{2} f_{\beta\gamma}{}^{\alpha} \epsilon_{0ijk} \mathcal{S}(\mathcal{F})^{\beta k} P(B)^{ij} = 0. \tag{48}$$

The relation (47) means that p components of the first-class constraints $\Phi(\mathcal{F})^{\alpha i}$ and $\Phi(\mathcal{G})^a$ are not independent of the others. Furthermore, taking the derivative of $\Phi(\nabla C)_a{}^i$ one obtains

$$\begin{aligned}
&\nabla_i \Phi(\nabla C)_a{}^i + C_{bi} \triangleright_{\alpha a}{}^b \Phi(\mathcal{F})^{\alpha i} + \partial_a{}^{\alpha} \Phi(\nabla B)_{\alpha} - \beta^b{}_{ij} X_{(ab)}{}^A \Phi(\nabla D)_A{}^{ij} - 2D_A X_{(ab)}{}^A\, \Phi(\mathcal{G})^b \\
&\quad = \frac{1}{2}\epsilon^{0ijk}\left(\nabla_i T_{ajk} - \triangleright_{\alpha b}{}^a F^{\alpha}{}_{jk} C^b{}_i\right) - \frac{1}{2}\epsilon^{0ijk} \triangleright_{\alpha a}{}^b\, P(B)^{\alpha}{}_{ij}\, S(\nabla C)_{bk} \\
&\qquad + \epsilon^{0ijk} X_{(ab)}{}^A\, P(C)^b{}_i\, S(\nabla D)_{A\,jk} + \frac{1}{3}\epsilon^{0ijk} X_{(ab)}{}^A\, P(\gamma)^A{}_{ijk}\, S(\mathcal{G})^b + \frac{1}{2}\epsilon^{0ijk} \triangleright_{\alpha a}{}^b\, P(\beta)^b{}_{ij}\, S(\mathcal{F})^{\alpha}{}_k.
\end{aligned} \tag{49}$$

Noting that the right-hand side of (49) is zero off-shell as the $\lambda = 0$ components of the BI (39), and the remaining terms on the right-hand side are zero off-shell as products of two constraints, one obtains the following relation:

$$\nabla_i \Phi(\nabla C)_a{}^i + C_{bi} \triangleright_{\alpha a}{}^b \Phi(\mathcal{F})^{\alpha i} + \partial_a{}^{\alpha} \Phi(\nabla B)_{\alpha} - \beta^b{}_{ij} X_{(ab)}{}^A \Phi(\nabla D)_A{}^{ij} - 2D_A X_{(ab)}{}^A \Phi(\beta)^b = 0. \tag{50}$$

This relation means that q components of the constraints $\Phi(\nabla C)_a{}^i$, $\Phi(\mathcal{F})^{\alpha i}$, $\Phi(\nabla B)_{\alpha}$, $\Phi(\nabla D)_A{}^{ij}$, and $\Phi(\beta)^b$, are not independent of the others, further lowering the number of the independent first-class constraints. Finally, the following relation is satisfied

$$\begin{aligned}
&\nabla^j \Phi(\nabla D)^A{}_{ij} - \triangleright_{\alpha B}{}^A D^B \Phi(\mathcal{F})_{\alpha i} - \delta^A{}_a \Phi(\nabla C)^a{}_i \\
&\quad = \epsilon_{0ijk}\left(\nabla^j Q_A{}^k + \frac{1}{2} \triangleright_{\alpha A}{}^B F^{\alpha jk} D_B\right) + \frac{1}{2}\epsilon^{0jkl} \triangleright_{\alpha B}{}^A\, P(\gamma)^B{}_{ijk}\, S(\mathcal{F})_{\alpha l} \\
&\qquad - \frac{1}{2}\epsilon^{0jkl} \triangleright_{\alpha B}{}^A\, P(B)^{\alpha}{}_{ij}\, S(\nabla D)^B{}_{kl}.
\end{aligned} \tag{51}$$

Since the first term on the right-hand side is precisely the $\lambda = 0$ component of the BI (44), while the second and third terms are equal to zero as products of two constraints, this gives:

$$\nabla^j \Phi(\nabla D)^A{}_{ij} - \triangleright_{\alpha B}{}^A D^B \Phi(\mathcal{F})_{\alpha i} - \delta^A{}_a \Phi(\nabla C)^a{}_i = 0. \tag{52}$$

This relation suggests that $3r$ components of the primary constraints $\Phi(\nabla D)^A{}_{ij}$, $\Phi(\mathcal{F})_{\alpha i}$, and $\Phi(C)^a{}_i$ are not independent of the others. However, this is slightly misleading, since the covariant derivative of the BI (44) is automatically satisfied as a consequence of the BI (38),

$$\epsilon^{\lambda\mu\nu\rho} D^B \triangleright_{\alpha B}{}^A \nabla_{\mu} F^{\alpha}{}_{\nu\rho} = 0, \tag{53}$$

Table 3. First-class constraints in the 3*BF* theory.

$\Phi(B)_\alpha{}^i$	$\Phi(C)_a$	$\Phi(\alpha)_\alpha$	$\Phi(\beta)_a{}^i$	$\Phi(\gamma)_A{}^{ij}$	$\Phi(\mathcal{F})^{\alpha i}$	$\Phi(\mathcal{G})^a$	$\Phi(\nabla C)^{ai}$	$\Phi(\nabla B)^\alpha$	$\Phi(\nabla D)_A{}^{ij}$
$3p$	q	p	$3q$	$3r$	$3p-p$	q	$3q-q$	p	$3r-2r$

which means that there are in fact only $2r$ components of the constraint (52). A formal proof of this statement would involve evaluating the Wronskian of all first-class constraints, and is out of the scope of this paper.

The number of independent components of first-class constraints is determined by counting the components listed in table 3, and then subtracting the number of independent relations (47), (50) and (52).

Bearing the previous analysis in mind, one obtains the number of independent first-class constraints:

$$F = 8(p+q) + 6r - p - q - 2r = 7(p+q) + 4r.$$

Finally, using the definition (45), the number of degrees of freedom in the 3*BF* theory is:

$$n = 10(p+q) + 5r - 7(p+q) - 4r - \frac{6(p+q)+2r}{2} = 0. \tag{54}$$

Therefore, there are no local propagating degrees of freedom in a 3*BF* theory.

3.4. Symmetry generator

The unphysical transformations of dynamical variables are often referred to as gauge transformations. The gauge transformations are *local*, meaning that the parameters of the transformations are arbitrary functions of space and time. We shall now construct the generator of all gauge symmetries of the theory governed by the total Hamiltonian (33), using the Castellani's algorithm (see chapter 5 in [15] for a comprehensive overview of the procedure). The details of the construction are given in appendix C, and the following result is obtained

$$\begin{aligned} G = \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big(& (\nabla_0 \epsilon_{\mathfrak{g}}{}^\alpha)\,(\tilde{G}_1)_\alpha + \epsilon_{\mathfrak{g}}{}^\alpha\,(\tilde{G}_0)_\alpha + (\nabla_0 \epsilon_{\mathfrak{h}}{}^a{}_i)\,(\tilde{H}_1)_a{}^i + \epsilon_{\mathfrak{h}}{}^a{}_i\,(\tilde{H}_0)_a{}^i + \frac{1}{2}(\nabla_0 \epsilon_{\mathfrak{l}}{}^A{}_{ij})\,(\tilde{L}_1)_A{}^{ij} \\ & + \frac{1}{2}\epsilon_{\mathfrak{l}}{}^A{}_{ij}\,(\tilde{L}_0)_A{}^{ij} + (\nabla_0 \epsilon_{\mathfrak{m}}{}^\alpha{}_i)\,(\tilde{M}_1)_\alpha{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i\,(\tilde{M}_0)_\alpha{}^i + (\nabla_0 \epsilon_{\mathfrak{n}}{}^a)\,(\tilde{N}_1)_a + \epsilon_{\mathfrak{n}}{}^a (\tilde{N}_0)_a \Big), \end{aligned} \tag{55}$$

where

$$
\begin{aligned}
(\tilde{G}_1)_\alpha &= -\Phi(\alpha)_\alpha, \\
(\tilde{G}_0)_\alpha &= -\bigg(f_{\alpha\gamma}{}^\beta B_{\beta 0i}\Phi(B)^{\gamma i} + C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^{b0} + \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)^{b0i} \\
&\quad - \frac{1}{2}\gamma^A{}_{0ij} \rhd_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij} - \Phi(\nabla B)_\alpha \bigg), \\
(\tilde{H}_1)_a{}^i &= -\Phi(\beta)_a{}^i, \\
(\tilde{H}_0)_a{}^i &= C_{b0} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} - 2\beta^b{}_{0j} X_{(ab)}{}^A \Phi(\gamma)_A{}^{ij} + \Phi(\nabla C)_a{}^i, \\
(\tilde{L}_1)_a{}^{ij} &= \Phi(\gamma)_A{}^{ij}, \\
(\tilde{L}_0)_a{}^{ij} &= -\Phi(\nabla D)_A{}^{ij}, \\
(\tilde{M}_1)_\alpha{}^i &= -\Phi(B)_\alpha{}^i, \\
(\tilde{M}_0)_\alpha{}^i &= \Phi(\mathcal{F})_\alpha{}^i, \\
(\tilde{N}_1)_a &= -\Phi(C)_a, \\
(\tilde{N}_0)_a &= \beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a,
\end{aligned}
\tag{56}
$$

and $\epsilon_{\mathfrak{g}}{}^\alpha$, $\epsilon_{\mathfrak{h}}{}^a{}_i$, $\epsilon_{\mathfrak{l}}^A{}_{ij}$, $\epsilon_{\mathfrak{m}}{}^\alpha{}_i$, and $\epsilon_{\mathfrak{n}}{}^a$ are the independent parameters of the gauge transformations.

The obtained gauge generator (55) is then employed to calculate the form variations of variables and their corresponding canonical momenta, denoted as $A(t, \vec{x})$, using the following equation,

$$
\delta_0 A(t, \vec{x}) = \{A(t, \vec{x}), G\}. \tag{57}
$$

The form variations of all fields and canonical momenta are given in appendix E, equation (E.2), while the algebra of the generators (56) is obtained in the appendix B, equations (B.4)–(B.10). However, one must bear in mind that the gauge generator (55) is the generator of the symmetry transformations on a slice of spacetime, i.e. on a hypersurface Σ_3. Having in hand all these results, specifically the form variations of all variables and their canonical momenta (E.2), we can determine the full gauge symmetry of the theory, which will be done in the next section.

4. Symmetries of the 3*BF* action

In order to systematically describe all gauge transformations of the 3*BF* action, we will discuss in turn each set of gauge parameters $\epsilon_{\mathfrak{g}}{}^\alpha$, $\epsilon_{\mathfrak{h}}{}^a{}_i$, $\epsilon_{\mathfrak{l}}^A{}_{ij}$, $\epsilon_{\mathfrak{m}}{}^\alpha{}_i$, and $\epsilon_{\mathfrak{n}}{}^a$, appearing in (55). The subsection 4.1 deals with the gauge group G, and the G-gauge transformations, which are already familiar from the ordinary *BF* theory. In subsection 4.2 we discuss the gauge group $\tilde{H}_L$ which consists of the *H*-gauge and *L*-gauge transformations, familiar from the previous literature [21], while the subsection 4.3 examines the *M*-gauge and *N*-gauge transformations which are also present in the theory. Finally, the results of the subsections 4.1–4.3 will be summarized in the subsection 4.4, where we will present the complete structure of the gauge symmetry group.

4.1. Gauge group G

First, consider the infinitesimal transformation with the parameter ${\epsilon_{\mathfrak{g}}}^{\alpha}$, given by the form variations

$$
\begin{aligned}
&\delta_0 {\alpha^{\alpha}}_{\mu} = -\partial_{\mu}{\epsilon_{\mathfrak{g}}}^{\alpha} - {f_{\beta\gamma}}^{\alpha}{\alpha^{\beta}}_{\mu}{\epsilon_{\mathfrak{g}}}^{\gamma}, \quad \delta_0 {B^{\alpha}}_{\mu\nu} = {f_{\beta\gamma}}^{\alpha}{\epsilon_{\mathfrak{g}}}^{\beta}{B^{\gamma}}_{\mu\nu}, \\
&\delta_0 {\beta^{a}}_{\mu\nu} = {\rhd_{\alpha b}}^{a}{\epsilon_{\mathfrak{g}}}^{\alpha}{\beta^{b}}_{\mu\nu}, \quad \delta_0 {C^{a}}_{\mu} = {\rhd_{\alpha b}}^{a}{\epsilon_{\mathfrak{g}}}^{\alpha}{C^{b}}_{\mu}, \\
&\delta_0 {\gamma^{A}}_{\mu\nu\rho} = {\rhd_{\alpha B}}^{A}{\epsilon_{\mathfrak{g}}}^{\alpha}{\gamma^{B}}_{\mu\nu\rho}, \quad \delta_0 D^{A} = {\rhd_{\alpha B}}^{A}{\epsilon_{\mathfrak{g}}}^{\alpha}D^{B},
\end{aligned}
\tag{58}
$$

which is analogous to writing the transformation as:

$$
\begin{aligned}
&\alpha \to \alpha' = \alpha - \nabla\epsilon_{\mathfrak{g}}, \quad B \to B' = B - [B, \epsilon_{\mathfrak{g}}], \\
&\beta \to \beta' = \beta + \epsilon_{\mathfrak{g}} \rhd \beta, \quad C \to C' = C + \epsilon_{\mathfrak{g}} \rhd C, \\
&\gamma \to \gamma' = \gamma + \epsilon_{\mathfrak{g}} \rhd \gamma, \quad D \to D' = D + \epsilon_{\mathfrak{g}} \rhd D.
\end{aligned}
\tag{59}
$$

Based on these infinitesimal transformations, one can extrapolate the finite symmetry transformations, defined in the theorem 1.

Theorem 1 (G-gauge transformations). *In the* 3*BF theory for the two-crossed module* $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_,\_\}_{\mathrm{pf}})$*, the following transformation is a gauge symmetry,*

$$
\begin{aligned}
&\alpha \to \alpha' = \mathrm{Ad}_g \alpha + g\mathrm{d}g^{-1}, \quad B \to B' = gBg^{-1}, \\
&\beta \to \beta' = g \rhd \beta, \quad C \to C' = g \rhd C, \\
&\gamma \to \gamma' = g \rhd \gamma, \quad D \to D' = g \rhd D,
\end{aligned}
\tag{60}
$$

where $g = \exp(\epsilon_{\mathfrak{g}} \cdot \hat{G}) = \exp({\epsilon_{\mathfrak{g}}}_{\alpha}\hat{G}^{\alpha}) \in G$, *and* $\epsilon_{\mathfrak{g}} : \mathcal{M}_4 \to \mathfrak{g}$ *is the parameter of the transformation.*

Proof. Note that if one considers an element of the group, $g \in G$, the transformations of the theorem 1 give rise to the following three-curvature transformation

$$
\mathcal{F} \to \mathcal{F}' = g\mathcal{F}g^{-1}, \qquad \mathcal{G} \to \mathcal{G}' = g \rhd \mathcal{G}, \qquad \mathcal{H} \to \mathcal{H}' = g \rhd \mathcal{H},
\tag{61}
$$

and the invariance of the 3*BF* action under this transformation follows from the *G*-invariance of the symmetric bilinear forms on $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$. _

Let us consider two subsequent infinitesimal *G*-gauge transformations, determined by the small parameters $\epsilon_{\mathfrak{g}\,1}^{\alpha}$ and $\epsilon_{\mathfrak{g}\,2}^{\beta}$. To calculate the commutator between the generators of the *G*-gauge transformations, we will make use of the Baker–Campbell–Hausdorff (BCH) formula in the case when the parameters of the transformations are small

$$\mathrm{e}^{\epsilon_{\mathfrak{g}}{}^{\alpha}{}_{1}\hat{G}_{\alpha}}\mathrm{e}^{\epsilon_{\mathfrak{g}}{}^{\beta}{}_{2}\hat{G}_{\beta}} = \mathrm{e}^{\epsilon_{\mathfrak{g}}{}^{\alpha}{}_{1}\hat{G}_{\alpha}+\epsilon_{\mathfrak{g}}{}^{\beta}{}_{2}\hat{G}_{\beta}+\frac{1}{2}\epsilon_{\mathfrak{g}}{}^{\alpha}{}_{1}\,\epsilon_{\mathfrak{g}}{}^{\beta}{}_{2}[\hat{G}_{\alpha},\hat{G}_{\beta}]+O(\epsilon_{\mathfrak{g}}{}^{3})}, \tag{62}$$

from which it follows:

$$\mathrm{e}^{\epsilon_{\mathfrak{g}}{}^{\alpha}{}_{1}\hat{G}_{\alpha}}\mathrm{e}^{\epsilon_{\mathfrak{g}}{}^{\beta}{}_{2}\hat{G}_{\beta}} - \mathrm{e}^{\epsilon_{\mathfrak{g}}{}^{\beta}{}_{2}\hat{G}_{\beta}}\mathrm{e}^{\epsilon_{\mathfrak{g}}{}^{\alpha}{}_{1}\hat{G}_{\alpha}} = \epsilon_{\mathfrak{g}}{}^{\alpha}{}_{1}\,\epsilon_{\mathfrak{g}}{}^{\beta}{}_{2}\,[\hat{G}_{\alpha},\hat{G}_{\beta}] + O(\epsilon_{\mathfrak{g}}{}^{3}). \tag{63}$$

Using the equation (63), we obtain that the generators of the *G*-gauge transformations defined in the theorem 1 satisfy the following commutation relations:

$$[\hat{G}_{\alpha},\hat{G}_{\beta}] = f_{\alpha\beta}{}^{\gamma}\hat{G}_{\gamma}, \tag{64}$$

where $f_{\alpha\beta}{}^{\gamma}$ are the structure constants of the algebra $\mathfrak{g}$. By noting that there exists an isomorphism between generators $\hat{G}_{\alpha} \cong \tau_{\alpha}$, one establishes that the group of the *G*-gauge transformations from the theorem 1 is the same as the group *G* of the two-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{\_,\_\}_{\mathrm{pf}})$. This is an important result, which will not be true for the remaining symmetry transformations, as we shall see below.

4.2. The gauge group $\tilde{H}_L$

Let us now consider the form variations of the variables corresponding to the parameter $\epsilon_{\mathfrak{h}}{}^{a}{}_{i}$. For example, one can see from the equation (E.2) that the form-variation of the variables $\alpha^{\alpha}{}_{0}$ and $\alpha^{\alpha}{}_{i}$ are:

$$\delta_0\alpha^{\alpha}{}_{o} = 0, \qquad \delta_0\alpha^{\alpha}{}_{i} = -\partial_{a}{}^{\alpha}\epsilon_{\mathfrak{h}}{}^{a}{}_{i}. \tag{65}$$

Taking into account that the action of the generator (55) gives the symmetry transformations on one hypersurface Σ_3 with the time component of the parameter equal to zero, $\epsilon_{\mathfrak{h}}{}^{a}{}_{0} = 0$, one can extrapolate that for parameter of the spacetime gauge transformations $\epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}$, the form-variation of the variable $\alpha^{\alpha}{}_{\mu}$ is given as:

$$\delta_0\alpha^{\alpha}{}_{\mu} = -\partial_{a}{}^{\alpha}\epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}, \tag{66}$$

and similarly for the rest of the variables. Thus, the infinitesimal symmetry transformations in the whole spacetime corresponding to the parameter $\epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}$ are given by the form variations:

$$\begin{aligned}
&\delta_0\alpha^{\alpha}{}_{\mu} = -\partial_{a}{}^{\alpha}\epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}, \quad \delta_0 B^{\alpha}{}_{\mu\nu} = 2C_{a[\,\mu|}\epsilon_{\mathfrak{h}}{}^{b}{}_{|\nu]}\,\triangleright_{\beta b}{}^{a}g^{\alpha\beta},\\
&\delta_0\beta^{a}{}_{\mu\nu} = -2\nabla_{[\mu|}\epsilon_{\mathfrak{h}}{}^{a}{}_{|\nu]}, \quad \delta_0 C^{a}{}_{\mu} = 2D_A X_{(ab)}{}^{A}\epsilon_{\mathfrak{h}}{}^{b}{}_{\mu},\\
&\delta_0\gamma^{A}{}_{\mu\nu\rho} = 3!\beta^{a}{}_{[\,\mu\nu}\epsilon_{\mathfrak{h}}{}^{b}{}_{\rho]}X_{(ab)}{}^{A}, \quad \delta_0 D = 0.
\end{aligned} \tag{67}$$

For these infinitesimal transformations one obtains the finite symmetry transformations given in theorem 2.

Theorem 2 (H-gauge transformations). *In the* 3*BF theory for the two-crossed module* $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{\_,\_\}_{\mathrm{pf}})$*, the following transformation is a symmetry:*

$$
\begin{aligned}
&\alpha \to \alpha' = \alpha - \partial\epsilon_{\mathfrak{h}}, \quad \beta \to \beta' = \beta - \nabla'\epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge \epsilon_{\mathfrak{h}}, \\
&\gamma \to \gamma' = \gamma + \{\beta', \epsilon_{\mathfrak{h}}\}_{\mathrm{pf}} + \{\epsilon_{\mathfrak{h}}, \beta\}_{\mathrm{pf}}, \;\; B \to B' = B - C' \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} D, \\
&C \to C' = C - D \wedge^{\mathcal{X}_1} \epsilon_{\mathfrak{h}} - D \wedge^{\mathcal{X}_2} \epsilon_{\mathfrak{h}}, \quad D \to D' = D.
\end{aligned}
\tag{68}
$$

where $\epsilon_{\mathfrak{h}} \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ *is an arbitrary* $\mathfrak{h}$*-valued one-form, and* ∇' *denotes the covariant derivative with respect to the connection* α'*. The maps* $\mathcal{T}$*,* $\mathcal{D}$*,* $\mathcal{X}_1$*, and* $\mathcal{X}_2$ *are defined in appendix* D.

Proof. Note that the three-curvature transforms as

$$
\begin{aligned}
&\mathcal{F} \to \mathcal{F}' = \mathcal{F}, \\
&\mathcal{G} \to \mathcal{G}' = \mathcal{G} - \mathcal{F} \wedge^{\triangleright} \epsilon_{\mathfrak{h}}, \\
&\mathcal{H} \to \mathcal{H}' = \mathcal{H} + \{\mathcal{G}', \epsilon_{\mathfrak{h}}\}_{\mathrm{pf}} - \{\epsilon_{\mathfrak{h}}, \mathcal{G}\}_{\mathrm{pf}}.
\end{aligned}
\tag{69}
$$

Taking into account the transformations of the three-curvature (69) and the transformations of the Lagrange multipliers, the action S_{3BF} transforms as:

$$
\begin{aligned}
S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \big(&-\langle C' \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}}, \mathcal{F} \rangle_{\mathfrak{g}} - \langle \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} D, \mathcal{F} \rangle_{\mathfrak{g}} \\
&- \langle C', \mathcal{F} \wedge^{\triangleright} \epsilon_{\mathfrak{h}} \rangle_{\mathfrak{h}} - \langle D \wedge^{\mathcal{X}_1} \epsilon_{\mathfrak{h}}, \mathcal{G} \rangle_{\mathfrak{h}} - \langle D \wedge^{\mathcal{X}_2} \epsilon_{\mathfrak{h}}, \mathcal{G} \rangle_{\mathfrak{h}} \\
&+ \langle D, \{\mathcal{G}, \epsilon_{\mathfrak{h}}\}_{\mathrm{pf}} \rangle_{\mathfrak{l}} - \langle D, \{\mathcal{F} \wedge^{\triangleright} \epsilon_{\mathfrak{h}}, \epsilon_{\mathfrak{h}}\}_{\mathrm{pf}} \rangle_{\mathfrak{l}} - \langle D, \{\epsilon_{\mathfrak{h}}, \mathcal{G}\}_{\mathrm{pf}} \rangle_{\mathfrak{l}} \big) .
\end{aligned}
\tag{70}
$$

Using the definitions of the maps $\mathcal{T}, \mathcal{D}$, $\mathcal{X}_1$, and $\mathcal{X}_2$, given in appendix D, one sees that the terms in the parentheses cancel, specifically the first term with the third, second with seventh, fourth with eighth, and fifth with the sixth term. _

The H-gauge transformations do not form a group. Namely, one can check that the two consecutive H-gauge transformations do not give a transformation of the same kind, i.e. the closure axiom of the group is not satisfied. This is analogous to the well-known structure of Lorentz group, where boost transformations are not closed, and thus do not form a group. Indeed, one must consider both rotations and boosts to obtain the set of transformations that forms the Lorentz group. In the case of the H-gauge transformations, we will show that together with the H-gauge transformations one needs to consider the transformations corresponding to the parameter $\epsilon_{\mathfrak{l}}{}^{A}{}_{ij}$. From the equation (E.2) one reads the form-variations on a space hypersurface Σ_3 corresponding to this parameter. Similarly as it is done in the case of the H-gauge transformations, one extrapolates that the form-variations for all the variables corresponding to the parameter $\epsilon_{\mathfrak{l}}{}^{A}{}_{\mu\nu}$ are given as:

$$
\begin{aligned}
&\delta_0 \alpha^{\alpha}{}_{\mu} = 0, \\
&\delta_0 B^{\alpha}{}_{\mu\nu} = -D_A \triangleright_{\beta B}{}^{A} \epsilon_{\mathfrak{l}}{}^{B}{}_{\mu\nu} g^{\alpha\beta}, \\
&\delta_0 \beta^{a}{}_{\mu\nu} = \delta_A{}^{a} \epsilon_{\mathfrak{l}}^{A}{}_{\mu\nu}, \\
&\delta_0 C^{a}{}_{\mu} = 0, \quad \delta_0 \gamma^{A}{}_{\mu\nu\rho} = \nabla_{\mu} \epsilon_{\mathfrak{l}}^{A}{}_{\nu\rho} - \nabla_{\nu} \epsilon_{\mathfrak{l}}^{A}{}_{\mu\rho} + \nabla_{\rho} \epsilon_{\mathfrak{l}}^{A}{}_{\mu\nu}, \\
&\delta_0 D^{A} = 0.
\end{aligned}
\tag{71}
$$

These infinitesimal transformations correspond to the finite symmetry transformations defined in theorem 3.

Theorem 3 (*L*-gauge transformations). *In the 3BF theory for the two-crossed module* $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{\_,\_\}_{\mathrm{pf}})$*, the following transformation is a symmetry*

$$
\begin{array}{ll}
\alpha \to \alpha' = \alpha, & B \to B' = B + D \wedge^{\mathcal{S}} \epsilon_{\mathfrak{l}}, \\
\beta \to \beta' = \beta + \delta \epsilon_{\mathfrak{l}}, & C \to C' = C, \\
\gamma \to \gamma' = \gamma + \nabla \epsilon_{\mathfrak{l}}, & D \to D' = D,
\end{array}
\tag{72}
$$

where $\epsilon_{\mathfrak{l}} \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{l})$ *is an arbitrary* $\mathfrak{l}$*-valued two-form, and the map* $\mathcal{S}$ *is defined in appendix* D.

Proof. Note that the three-curvature transforms as

$$
\begin{array}{l}
\mathcal{F} \to \mathcal{F}' = \mathcal{F}, \\
\mathcal{G} \to \mathcal{G}' = \mathcal{G}, \\
\mathcal{H} \to \mathcal{H}' = \mathcal{H} + \mathcal{F} \wedge^{\triangleright} \epsilon_{\mathfrak{l}}.
\end{array}
\tag{73}
$$

Taking into account the transformations (73) and the transformations of the Lagrange multipliers, the action transforms as:

$$
S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \left(\langle D \wedge^{\mathcal{S}} \epsilon_{\mathfrak{l}}, \mathcal{F} \rangle_{\mathfrak{g}} + \langle D, \mathcal{F} \wedge^{\triangleright} \epsilon_{\mathfrak{l}} \rangle_{\mathfrak{l}} \right). \tag{74}
$$

According to the definition of the map $\mathcal{S}$, the terms in the parentheses cancel. ∎

Let us denote the generators of the H-gauge transformations given by the theorem 2 and the L-gauge transformations given by the theorem 3 as $\hat{H}_a{}^{\mu}$ and $\hat{L}_A{}^{\mu\nu}$, respectively. As we have commented above, one can now check that the transformations defined in the theorem 2, i.e. the H-gauge transformations, do not form a group. If one performs two consecutive H-gauge transformations, defined with parameters $\epsilon_{\mathfrak{h}1}$ and $\epsilon_{\mathfrak{h}2}$, one obtains

$$
\mathrm{e}^{\epsilon_{\mathfrak{h}1} \cdot \hat{H}} \mathrm{e}^{\epsilon_{\mathfrak{h}2} \cdot \hat{H}} - \mathrm{e}^{\epsilon_{\mathfrak{h}2} \cdot \hat{H}} \mathrm{e}^{\epsilon_{\mathfrak{h}1} \cdot \hat{H}} = 2 \left(\{\epsilon_{\mathfrak{h}1} \wedge \epsilon_{\mathfrak{h}2}\}_{\mathrm{pf}} - \{\epsilon_{\mathfrak{h}2} \wedge \epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}} \right) \cdot \hat{L}, \tag{75}
$$

where $\epsilon_{\mathfrak{h}} \cdot \hat{H} = \epsilon_{\mathfrak{h}}{}^{a}{}_{\mu} \hat{H}_a{}^{\mu}$ and $\epsilon_{\mathfrak{l}} \cdot \hat{L} = \frac{1}{2} \epsilon_{\mathfrak{l}}{}^{A}{}_{\mu\nu} \hat{L}_A{}^{\mu\nu}$. Using the equation analogous to BCH formula (63), one obtains that the commutator of the generators of two H-gauge transformations is the generator of an L-gauge transformation (see appendix F for the details of the calculation):

$$\left[\hat{H}_a{}^\mu, \hat{H}_b{}^\nu\right] = 2X_{(ab)}{}^A \hat{L}_A{}^{\mu\nu}. \tag{76}$$

Next, note that the transformations defined in theorem 3 are the linear transformations, and the two subsequent L-gauge transformations give one L-gauge transformation with the parameter $\epsilon_{\mathfrak{l}1} + \epsilon_{\mathfrak{l}2}$. Formally, one can write the previous statement as

$$\mathrm{e}^{\epsilon_{\mathfrak{l}1}\cdot\hat{L}}\mathrm{e}^{\epsilon_{\mathfrak{l}2}\cdot\hat{L}} = \mathrm{e}^{(\epsilon_{\mathfrak{l}1}+\epsilon_{\mathfrak{l}2})\cdot\hat{L}}, \tag{77}$$

which leads to the conclusion that the generators of the L-gauge transformations are mutually commuting:

$$\left[\hat{L}_A{}^{\mu\nu}, \hat{L}_B{}^{\rho\sigma}\right] = 0. \tag{78}$$

Thus, the L-gauge transformations form an abelian group, which will be denoted as $\tilde{L}$. According to the index structure of the parameters and generators, we can conclude that the group $\tilde{L}$ is isomorphic to $\mathbb{R}^{6r}$, where r is the dimension of the group L:

$$\tilde{L} \cong \mathbb{R}^{6r}. \tag{79}$$

Our analogy with the case of the Lorentz group can once again prove useful, since the closure of the L-gauge transformations resembles the fact that the composition of two rotations is a rotation. The abelian group $\tilde{L}$ should not be confused with the non-abelian group L of the two-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{\_,\_\}_{\mathrm{pf}})$.

Let us now examine the relationship between H-gauge transformations and L-gauge transformations. The following result,

$$\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot\hat{H}}\mathrm{e}^{\epsilon_{\mathfrak{l}}\cdot\hat{L}} = \mathrm{e}^{\epsilon_{\mathfrak{l}}\cdot\hat{L}}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot\hat{H}}, \tag{80}$$

leads to the conclusion that the commutator of generators of the H-gauge transformations and generators of the L-gauge transformations vanishes:

$$\left[\hat{H}_a{}^\mu, \hat{L}_A{}^{\nu\rho}\right] = 0. \tag{81}$$

From the closure of the algebra (76), (78) and (81), one can conclude that the H-gauge transformations together with the L-gauge transformations form a group, which will be denoted as $\tilde{H}_L$. Lastly, the action of the group G on the H-gauge and L-gauge transformations is examined by calculating the expressions:

$$[\epsilon_{\mathfrak{g}}\cdot\hat{G}, \epsilon_{\mathfrak{h}}\cdot\hat{H}] = (\epsilon_{\mathfrak{g}}\triangleright\epsilon_{\mathfrak{h}})\cdot\hat{H}, \qquad [\epsilon_{\mathfrak{g}}\cdot\hat{G}, \epsilon_{\mathfrak{l}}\cdot\hat{L}] = (\epsilon_{\mathfrak{g}}\triangleright\epsilon_{\mathfrak{l}})\cdot\hat{L}, \tag{82}$$

which lead to the following commutators:

$$\begin{aligned} \left[\hat{G}_\alpha, \hat{H}_a{}^\mu\right] &= \triangleright_{\alpha a}{}^b\, \hat{H}_b{}^\mu, \\ \left[\hat{G}_\alpha, \hat{L}_A{}^{\mu\nu}\right] &= \triangleright_{\alpha A}{}^B\, \hat{L}_B{}^{\mu\nu}. \end{aligned} \tag{83}$$

Theorems 1–3 represent the G-, H-, and L-gauge transformations, which are already familiar from the previous literature (see for example [21, 30]).

4.3. The gauge groups M and N

Next, consider the infinitesimal transformation with the parameter ${\epsilon_{\mathfrak{m}}}^{\alpha}{}_{i}$, given by the form variations in appendix E. In a similar manner as done in the previous subsection, one establishes that the form variations obtained as a result of the Hamiltonian analysis are transformations on one hypersurface Σ_3, from which one can guess the symmetry in the whole spacetime. Keeping in mind that the variations on the hypersurface have the time component of the parameter set to ${\epsilon_{\mathfrak{m}}}^{\alpha}{}_{0} = 0$, one extrapolates the form-variations of the whole spacetime for the parameter ${\epsilon_{\mathfrak{m}}}^{\alpha}{}_{\mu}$ to be:

$$
\begin{aligned}
&\delta_0 \alpha^{\alpha}{}_{\mu} = 0, \\
&\delta_0 B^{\alpha}{}_{\mu\nu} = -2\nabla_{[\mu|}{\epsilon_{\mathfrak{m}}}^{\alpha}{}_{|\nu]}, \\
&\delta_0 \beta^{a}{}_{\mu\nu} = 0, \\
&\delta_0 C^{a}{}_{\mu} = -\partial^{a}{}_{\alpha}{\epsilon_{\mathfrak{m}}}^{\alpha}{}_{\mu}, \\
&\delta_0 \gamma^{A}{}_{\mu\nu\rho} = 0, \\
&\delta_0 D^{A} = 0.
\end{aligned}
\tag{84}
$$

Based on this result, one obtains the finite symmetry transformations in the whole spacetime, as defined in theorem 4, which we will refer to as the M-gauge transformations.

Theorem 4 (M-gauge transformations). *In the* 3*BF theory for the two-crossed module* $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{\_,\_\}_{\mathrm{pf}})$*, the following transformation is a symmetry*

$$
\begin{aligned}
\alpha \to \alpha' &= \alpha, \\
B \to B' &= B - \nabla\epsilon_{\mathfrak{m}}, \\
\beta \to \beta' &= \beta, \\
C^{a} \to C'^{a} &= C^{a} - \partial^{a}{}_{\alpha}{\epsilon_{\mathfrak{m}}}^{\alpha}, \\
\gamma \to \gamma' &= \gamma, \\
D \to D' &= D,
\end{aligned}
\tag{85}
$$

where $\epsilon_{\mathfrak{m}} \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$ *is an arbitrary* $\mathfrak{g}$*-valued one-form.*

Proof. Consider the transformation of the 3BF action under the transformations of the variables defined in the theorem 4. One obtains:

$$
S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} \left(-\frac{1}{2}(\nabla_{\mu}{\epsilon_{\mathfrak{m}}}^{\alpha}{}_{\nu})\mathcal{F}_{\alpha\rho\sigma} - \frac{1}{3!}\partial^{a}{}_{\alpha}{\epsilon_{\mathfrak{m}}}^{\alpha}{}_{\mu}\mathcal{G}_{a\nu\rho\sigma} \right). \tag{86}
$$

Using the definition of three-curvature, given by the expressions (14), one obtains:

$$
\begin{aligned}
S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} \bigg(&-\frac{1}{2}(\nabla_{\mu}{\epsilon_{\mathfrak{m}}}^{\alpha}{}_{\nu}) \left(F_{\alpha\rho\sigma} - \partial^{a}{}_{\alpha}\beta_{a\rho\sigma}\right) \\
&- \frac{1}{3!}\partial^{a}{}_{\alpha}{\epsilon_{\mathfrak{m}}}^{\alpha}{}_{\mu}\left(3\nabla_{\nu}\beta_{a\rho\sigma} - \delta^{A}{}_{a}\gamma_{A\nu\rho\sigma}\right) \bigg).
\end{aligned}
\tag{87}
$$

Taking into account that the second and the third term cancel, while the last term is zero because of the identity (A.1), the expression reduces to:

$$S'_{3BF} = S_{3BF} - \frac{1}{2}\int_{\mathcal{M}_4} \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu} \nabla_\nu F_{\alpha\rho\sigma}. \tag{88}$$

Finally, the term $\epsilon^{\mu\nu\rho\sigma}\nabla_\nu F_{\alpha\rho\sigma} = 0$ is the BI (38). One concludes that the action S_{3BF} is invariant under the transformation defined in theorem 4. □

Note that the transformations defined in theorem 4 are linear transformations, and the two subsequent M-gauge transformations give one M-gauge transformation with the parameter $\epsilon_{\mathfrak{m}1} + \epsilon_{\mathfrak{m}2}$. Denoting the generators of the M-gauge transformations as $\hat{M}_\alpha{}^\mu$, one can now write the previous statement formally as:

$$\mathrm{e}^{\epsilon_{\mathfrak{m}1}\cdot\hat{M}}\mathrm{e}^{\epsilon_{\mathfrak{m}2}\cdot\hat{M}} = \mathrm{e}^{(\epsilon_{\mathfrak{m}1}+\epsilon_{\mathfrak{m}2})\cdot\hat{M}}, \tag{89}$$

where $\epsilon_{\mathfrak{m}} \cdot \hat{M} = \epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu}\hat{M}_\alpha{}^\mu$, leading to the conclusion that:

$$\left[\hat{M}_\alpha{}^\mu, \hat{M}_\beta{}^\nu\right] = 0. \tag{90}$$

Thus, the M-gauge transformations form an abelian group, which will be denoted as $\tilde{M}$. According to the index structure of its parameters and generators, we see that this group is isomorphic to $\mathbb{R}^{4p}$, where p is the dimension of the group G:

$$\tilde{M} \cong \mathbb{R}^{4p}. \tag{91}$$

Next, one can examine the relationship of M-gauge transformations with the G, H, and L-gauge transformations defined in the previous subsections. Specifically, considering the G-gauge symmetry generators, one finds

$$[\epsilon_{\mathfrak{g}} \cdot \hat{G}, \epsilon_{\mathfrak{m}} \cdot \hat{M}] = (\epsilon_{\mathfrak{g}} \triangleright \epsilon_{\mathfrak{m}}) \cdot \hat{M}, \tag{92}$$

obtaining the result:

$$\left[\hat{G}_\alpha, \hat{M}_\beta{}^\mu\right] = f_{\alpha\beta}{}^\gamma \hat{M}_\gamma{}^\mu. \tag{93}$$

Considering the H- and L-gauge transformations, one obtains

$$\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot\hat{H}}\mathrm{e}^{\epsilon_{\mathfrak{m}}\cdot\hat{M}} = \mathrm{e}^{\epsilon_{\mathfrak{m}}\cdot\hat{M}}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot\hat{H}}, \tag{94}$$

$$\mathrm{e}^{\epsilon_{\mathfrak{l}}\cdot\hat{L}}\mathrm{e}^{\epsilon_{\mathfrak{m}}\cdot\hat{M}} = \mathrm{e}^{\epsilon_{\mathfrak{m}}\cdot\hat{M}}\mathrm{e}^{\epsilon_{\mathfrak{l}}\cdot\hat{L}}, \tag{95}$$

leading to the conclusion that the generators of the M-gauge transformations commute with both the generators of H-gauge transformations and the generators of the L-gauge transformations:

$$\left[\hat{H}_a, \hat{M}_\alpha{}^\mu\right] = 0, \qquad \left[\hat{L}_A{}^{\mu\nu}, \hat{M}_\alpha{}^\rho\right] = 0. \tag{96}$$

Finally, examining the infinitesimal transformation corresponding to the parameter $\epsilon_{\mathfrak{n}}{}^{a}$, given by the form-variations as calculated in (E.2),

$$
\begin{aligned}
&\delta_0 \alpha^{\alpha}{}_{\mu} = 0, \\
&\delta_0 B^{\alpha}{}_{\mu\nu} = \beta_{b\mu\nu} \rhd_{\alpha' a}{}^{b} \epsilon_{\mathfrak{n}}{}^{a} g^{\alpha\alpha'}, \\
&\delta_0 \beta^{a}{}_{\mu\nu} = 0, \\
&\delta_0 C^{a}{}_{\mu} = -\nabla_{\mu} \epsilon_{\mathfrak{n}}{}^{a}, \\
&\delta_0 \gamma^{A}{}_{\mu\nu\rho} = 0, \\
&\delta_0 D^{A} = \delta^{A}{}_{a} \epsilon_{\mathfrak{n}}{}^{a}.
\end{aligned} \tag{97}
$$

one obtains the theorem 5, the symmetry transformations which will be referred to as *N*-gauge transformations. Note that the *N*-gauge transformations are simultaneously the transformations in the whole spacetime, since the parameter does not carry spacetime indices.

Theorem 5 (N-gauge transformations). *In the* $3BF$ *theory for the two-crossed module* $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_,\_\}_{\mathrm{pf}})$*, the following transformation is a symmetry*

$$
\begin{aligned}
\alpha &\to \alpha' = \alpha, \\
B &\to B' = B - \beta \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}}, \\
\beta &\to \beta' = \beta, \\
C &\to C' = C - \nabla \epsilon_{\mathfrak{n}}, \\
\gamma &\to \gamma' = \gamma, \\
D^{A} &\to D'^{A} = D^{A} + \delta^{A}{}_{a} \epsilon_{\mathfrak{n}}{}^{a},
\end{aligned} \tag{98}
$$

where $\epsilon_{\mathfrak{n}} : \mathcal{M}_4 \to \mathfrak{h}$ *is an arbitrary* $\mathfrak{h}$*-valued zero-form.*

Proof. Under the transformations defined in theorem 5, the action is transformed as follows:

$$
\begin{aligned}
S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}x^4 \epsilon^{\mu\nu\rho\sigma} \bigg(&\frac{1}{4} \beta_{b\mu\nu} \rhd_{\alpha a}{}^{b} \epsilon_{\mathfrak{n}}{}^{a} \mathcal{F}^{\alpha}{}_{\rho\sigma} - \frac{1}{3!} (\nabla_{\mu} \epsilon_{\mathfrak{n}}{}^{a}) \mathcal{G}_{a\nu\rho\sigma} \\
&+ \frac{1}{4!} \delta^{A}{}_{a} \epsilon_{\mathfrak{n}}{}^{a} \mathcal{H}_{A\,\mu\nu\rho\sigma} \bigg).
\end{aligned} \tag{99}
$$

Using the expressions for the three-curvature defined in (9), one obtains

$$
\begin{aligned}
S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}x^4 \epsilon^{\mu\nu\rho\sigma} \bigg(&\frac{1}{4} \beta_{b\mu\nu} \rhd_{\alpha a}{}^{b} \epsilon_{\mathfrak{n}}{}^{a} \left(F^{\alpha}{}_{\rho\sigma} - \partial_{\mathrm{c}}{}^{\alpha} \beta^{c}{}_{\rho\sigma} \right) \\
&- \frac{1}{3!} (\nabla_{\mu} \epsilon_{\mathfrak{n}}{}^{a}) \left(3 \nabla_{\nu} \beta_{a\rho\sigma} - \delta^{A}{}_{a} \gamma_{A\,\nu\rho\sigma} \right) \\
&+ \frac{1}{4!} \delta^{A}{}_{a} \epsilon^{a} \left(4 \nabla_{\mu} \gamma_{A\,\nu\rho\sigma} + 6 X_{(bc)A} \beta^{b}{}_{\mu\nu} \beta^{c}{}_{\rho\sigma} \right) \bigg).
\end{aligned} \tag{100}
$$

Here, after one partial integration the last term in the first row of the equation (100) cancels with the first term in the second row, while taking into account the identity

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\nabla_\nu\nabla_\mu\epsilon_{\mathfrak{n}}{}^a)\beta_{a\rho\sigma} = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\beta_{b\rho\sigma}\rhd_{\alpha a}{}^b\epsilon_{\mathfrak{n}}{}^aF^\alpha{}_{\mu\nu}, \tag{101}$$

the first term and the third term also cancel, leading to the following expression:

$$S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}x^4\epsilon^{\mu\nu\rho\sigma}\left(\frac{1}{4}\epsilon_{\mathfrak{n}a}\rhd_{\alpha(b|}{}^a\partial_{|c)}{}^\alpha\beta^b{}_{\mu\nu}\beta^c{}_{\rho\sigma}\right.$$
$$\left. + \frac{1}{4}\epsilon_{\mathfrak{n}a}\delta_A{}^aX_{(bc)}{}^A\beta^b{}_{\mu\nu}\beta^c{}_{\rho\sigma}\right). \tag{102}$$

Here, the remaining two terms vanish because of the symmetrized form of the identity (A.6):

$$\rhd_{\alpha(b|}{}^a\partial_{|c)}{}^\alpha + \delta_A{}^aX_{(bc)}{}^A = f_{(bc)}{}^a = 0,$$

as a consequence of the antisymmetry of the structure constants. One concludes that the S_{3BF} action is invariant under the transformations defined in theorem 5. □

The N-gauge transformations defined in theorem 5 define the group which will be denoted as $\tilde{N}$. Note that these transformations are also linear, and the composition of two N-gauge transformations gives one N-gauge transformation with the parameter $\epsilon_{\mathfrak{n}1} + \epsilon_{\mathfrak{n}2}$. The generators of the group $\tilde{N}$ will be denoted with $\hat{N}_a$, and one can write these results as:

$$\mathrm{e}^{\epsilon_{\mathfrak{n}1}\cdot\hat{N}}\mathrm{e}^{\epsilon_{\mathfrak{n}2}\cdot\hat{N}} = \mathrm{e}^{(\epsilon_{\mathfrak{n}1}+\epsilon_{\mathfrak{n}2})\cdot\hat{N}}, \tag{103}$$

where $\epsilon_{\mathfrak{n}}\cdot\hat{N} = \epsilon_{\mathfrak{n}}^a\hat{N}_a$, leading to the conclusion that:

$$[\hat{N}_a, \hat{N}_b] = 0. \tag{104}$$

It follows that the group $\tilde{N}$ is abelian, and the index structure of parameters and generators indicates that it is isomorphic to $\mathbb{R}^q$, where q is the dimension of the group H. Therefore,

$$\tilde{N} \cong \mathbb{R}^q. \tag{105}$$

Next, one can examine the relationship of the N-gauge transformations with the G, H, L, and M-gauge transformations. First, considering the G-gauge transformations one obtains:

$$[\epsilon_{\mathfrak{g}}\cdot\hat{G}, \epsilon_{\mathfrak{n}}\cdot\hat{N}] = (\epsilon_{\mathfrak{g}}\rhd\epsilon_{\mathfrak{n}})\cdot\hat{N}, \tag{106}$$

from which it follows:

$$[\hat{G}_\alpha, \hat{N}_a] = \rhd_{\alpha a}{}^b\hat{N}_b. \tag{107}$$

Let us now examine the relationship between N-gauge transformations and H-gauge transformations, calculating the following expression:

$$\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot\hat{H}}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot\hat{N}} - \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot\hat{N}}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot\hat{H}} = -(\epsilon_{\mathfrak{n}}\wedge^{\mathcal{T}}\epsilon_{\mathfrak{h}})\cdot\hat{M}, \tag{108}$$

where the proof is given in appendix F. One obtains that the commutator between the generators of H-gauge transformation and N-gauge transformation is the generator of M-gauge transformation:

$$\left[\hat{H}_a{}^\mu, \hat{N}^b\right] = \rhd_{\alpha a}{}^b\hat{M}^{\alpha\mu}. \tag{109}$$

Analogously, one can check that the following is satisfied

$$e^{\epsilon_{\mathfrak{l}}\cdot\hat{L}}e^{\epsilon_{\mathfrak{n}}\cdot\hat{N}} = e^{\epsilon_{\mathfrak{n}}\cdot\hat{N}}e^{\epsilon_{\mathfrak{l}}\cdot\hat{L}}, \qquad e^{\epsilon_{\mathfrak{m}}\cdot\hat{M}}e^{\epsilon_{\mathfrak{n}}\cdot\hat{N}} = e^{\epsilon_{\mathfrak{n}}\cdot\hat{N}}e^{\epsilon_{\mathfrak{m}}\cdot\hat{M}}, \tag{110}$$

leading to the conclusion that the generators of *L*-gauge, *M*-gauge, and *N*-gauge transformations mutually commute, i.e.

$$\left[\hat{M}_\alpha{}^\mu, \hat{N}_a\right] = 0, \qquad \left[\hat{L}_A{}^{\mu\nu}, \hat{N}_a\right] = 0. \tag{111}$$

This concludes the calculation of the algebra of generators.

4.4. Structure of the symmetry group

Summarizing the results of the previous subsections, one can write the algebra of the generators of the full gauge symmetry group as follows.

- The algebra $\mathfrak{g}$ of the group G of the two-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{\_,\_\}_{\mathrm{pf}})$:

$$[\hat{G}_\alpha, \hat{G}_\beta] = f_{\alpha\beta}{}^\gamma \hat{G}_\gamma. \tag{112}$$

- The algebra of the group $\tilde{H}_L$ consisting of the generators of *H*- and *L*-gauge transformations:

$$\begin{aligned}
&[\hat{H}_a{}^\mu, \hat{H}_b{}^\nu] = 2X_{(ab)}{}^A \hat{L}_A{}^{\mu\nu},\\
&[\hat{L}_A{}^{\mu\nu}, \hat{L}_B{}^{\rho\sigma}] = 0,\\
&[\hat{H}_a{}^\mu, \hat{L}_A{}^{\nu\rho}] = 0.
\end{aligned} \tag{113}$$

- The algebra of the generators of *M*-gauge transformations:

$$[\hat{M}_\alpha{}^\mu, \hat{M}_\beta{}^\nu] = 0. \tag{114}$$

- The algebra of the generators of *N*-gauge transformations:

$$[\hat{N}_a, \hat{N}_b] = 0. \tag{115}$$

- The commutators between the generators of the groups $\tilde{M}$ and $\tilde{N}$:

$$[\hat{M}_\alpha{}^\mu, \hat{N}_a] = 0. \tag{116}$$

- The action of the generators of the group $\tilde{H}_L$ on the generators of *M*- and *N*-gauge transformations:

$$\begin{aligned}
&[\hat{H}_a{}^\mu, \hat{N}^b] \ = \triangleright_{\alpha a}{}^b \hat{M}^{\alpha\mu},\\
&[\hat{H}_a{}^\mu, \hat{M}_\alpha{}^\nu] \ = 0,\\
&[\hat{L}_A{}^{\nu\rho}, \hat{M}_\alpha{}^\mu] = 0,\\
&[\hat{L}_A{}^{\mu\nu}, \hat{N}_a] \ = 0.
\end{aligned} \tag{117}$$

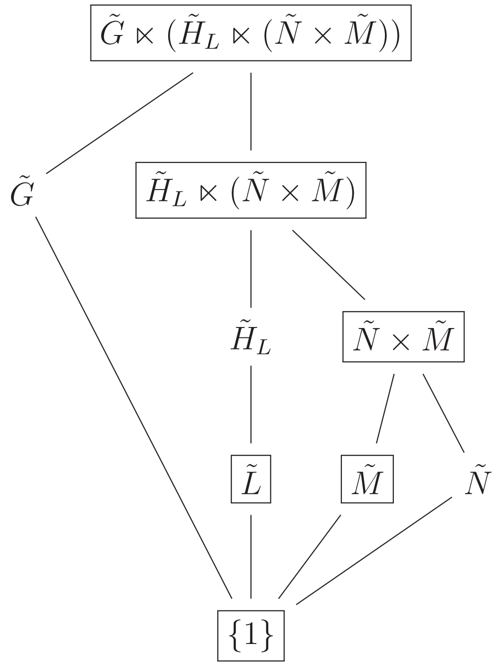

Figure 1. Relevant subgroups of the symmetry group $\mathcal{G}_{3BF}$. The invariant subgroups are boxed.

- The action of the generators of the group G on the generators of H-, L-, M-, and N-gauge transformations:

$$\begin{aligned} \left[\hat{G}_\alpha, \hat{H}_a{}^\mu\right] &= \triangleright_{\alpha a}{}^b\, \hat{H}_b{}^\mu, \\ \left[\hat{G}_\alpha, \hat{L}_A{}^{\mu\nu}\right] &= \triangleright_{\alpha A}{}^B \hat{L}_B{}^{\mu\nu}, \\ \left[\hat{G}_\alpha, \hat{M}_\beta{}^\mu\right] &= f_{\alpha\beta}{}^\gamma \hat{M}_\gamma{}^\mu, \\ \left[\hat{G}_\alpha, \hat{N}_a\right] &= \triangleright_{\alpha a}{}^b\, \hat{N}_b. \end{aligned} \tag{118}$$

Based on the equations (112)–(118), one can investigate the symmetry group structure. On the Hesse-like diagram shown in figure 1, we have included only the relevant subgroups of the whole symmetry group $\mathcal{G}_{3BF}$, where the invariant subgroups are boxed.

Let us remember that the subgroup is an *invariant subgroup*, or equivalently a *normal subgroup*, if it is invariant under conjugation by members of the group of which it is a subgroup. Formally, one says the group H is an invariant subgroup of the group G if H is a subgroup of G, i.e. $H \leqslant G$, and for all $h \in H$ and $g \in G$, the conjugation of the element of H with the element of G is an element of H, i.e. $\exists h' \in H$ such that $ghg^{-1} = h'$. On the level of algebra, the corresponding object is an *ideal*. Formally written, an algebra A is a subalgebra of an algebra L with respect to the multiplication in L, i.e. $[A, A] \subset A$. Then, a subalgebra A of L is an *ideal* in L if its elements, multiplied with any element of the algebra, give again an element of the subalgebra, i.e. $[A, L] \subset A$.

With the above definitions in mind, note first that the groups $\tilde{L}$, $\tilde{M}$, and $\tilde{N}$, are subgroups of the full symmetry group $\mathcal{G}_{3BF}$. The groups $\tilde{L}$ and $\tilde{M}$ are invariant subgroups, since the only nontrivial commutators between the generators $\hat{L}_A{}^{\mu\nu}$, and $\hat{M}_\alpha{}^{\mu}$, are with the generators of the group $\tilde{G}$, and are equal to some linear combinations of the generators of $\tilde{L}$, and $\tilde{M}$, respectively. The group $\tilde{N}$ is not an invariant subgroup, since the commutator between the generators $\hat{N}_a$ and $\hat{H}_a{}^{\mu}$ are linear combinations of the generators $\hat{M}_\alpha{}^{\mu}$. However, the generators of the groups $\tilde{N}$ and $\tilde{M}$ are mutually commuting, and the group $\tilde{N}$ is an invariant subgroup of the product of the groups $\tilde{M}$ and $\tilde{N}$, which makes this product a direct product. The obtained group $\tilde{N} \times \tilde{M}$ is an invariant subgroup of the whole symmetry group.

On the other hand, we saw that the H-gauge transformations together with the L-gauge transformations form the group $\tilde{H}_L$. This group is not an invariant subgroup of the whole symmetry group $\mathcal{G}_{3BF}$, because of the commutator of the generators $\hat{H}_a{}^{\mu}$ and $\hat{N}_b$. Similarly as before, one can join these two subgroups, of which one is invariant and one is not, using a semidirect product, to obtain a subgroup $\tilde{H}_L \ltimes (\tilde{N} \times \tilde{M})$, that will as a result be an invariant subgroup of the complete symmetry group $\mathcal{G}_{3BF}$. Here, the product is semidirect because the group $\tilde{H}_L$ is not an invariant subgroup of the group $\tilde{H}_L \ltimes (\tilde{N} \times \tilde{M})$, due to the commutator between the generators $\hat{H}_a{}^{\mu}$ and $\hat{N}_b$.

Finally, following the same line of reasoning, one adds the G-gauge transformations and obtains the complete gauge symmetry group $\mathcal{G}_{3BF}$ as:

$$\mathcal{G}_{3BF} = \tilde{G} \ltimes (\tilde{H}_L \ltimes (\tilde{N} \times \tilde{M})). \tag{119}$$

This concludes the analysis of the group of gauge symmetries for the $3BF$ action.

5. Conclusions

5.1. Summary of the results

Let us summarize the results of the paper. In section 2, we have introduced a generalization of the BF theory in the framework of higher category theory, the $3BF$ theory. Section 3 contains the Hamiltonian analysis for the $3BF$ theory. In subsection 3.1, the basic canonical structure and the total Hamiltonian are obtained, while in subsection 3.2 the complete Hamiltonian analysis of the $3BF$ theory is performed, resulting in the first-class and second-class constraints of the theory, as well as their Poisson brackets. In the subsection 3.3 we have discussed the BI and also the generalized BI, since they enforce restrictions and reduce the number of independent first-class constraints present in the theory, and having those identities in mind, the counting of the dynamical degrees of freedom has been performed. As expected, it was established that the considered $3BF$ action is a topological theory. Finally, this section concludes with the subsection 3.4 where we have constructed the generator of the gauge symmetries for the topological theory, based on the calculations done in section 3.2, and we have found the form-variations for all the variables and their canonical momenta, listed in the appendix E, equation (E.2).

In section 4, the main results of our paper are presented. With the material of the subsection 3.2 in hand, after obtaining the form variations of all variables and their canonical momenta, we proceeded to find all the gauge symmetries of the theory. The subsection 4.1 examined the gauge group G, and the G-gauge transformations. In subsection 4.2 we discussed the gauge group $\tilde{H}_L$ which gives the H-gauge and L-gauge transformations, while in the subsection 4.3 we analyzed the M-gauge and N-gauge transformations which represent a novel result. The results of the subsections 4.1–4.3 are summarized in subsection 4.4, where the complete structure of the symmetry group had been presented. The known G-, H-, and L-gauge transformations have been rigorously defined in theorems 1–3, while the two novel M- and N-gauge transformations, have been defined in theorems 4 and 5. The Lie algebra of the full gauge symmetry group $\mathcal{G}_{3BF}$ has also been obtained.

5.2. Discussion

One of the most important consequences of our results is the relationship between a two-crossed module and a symmetry group of the corresponding $3BF$ action, which we denoted as a *duality*. In particular, from the Lie algebra of the symmetry group $\mathcal{G}_{3BF}$ one sees that the structure constants depend on the choices of groups G, H, and L of the two-crossed module, on the action $\triangleright$, and on the symmetric part of the Peiffer lifting. However, $\mathcal{G}_{3BF}$ does not depend on the antisymmetric part of the Peiffer lifting, nor on the homomorphisms ∂ and δ. This means that in principle one can have several different two-crossed modules dual to the same symmetry group. Therefore, the term 'duality' is used in a loose sense, since there is no one-to-one correspondence between a two-crossed module and a symmetry group of the corresponding $3BF$ action. In addition, this result allows one to implement a strategy for the construction of a two-crossed module, by first specifying the choice of the group $\mathcal{G}_{3BF}$, and then supplying the additional information about the homomorphisms and the antisymmetric part of the Peiffer lifting, in a way that satisfies all axioms in the definition of a two-crossed module.

Another important topic for discussion is the following. From the fact that the $3BF$ action is formulated in a manifestly covariant way, using differential forms, it should be obvious that the diffeomorphisms are a symmetry of the theory. However, by looking at the structure of the gauge group $\mathcal{G}_{3BF}$, one does not immediately see whether $\text{Diff}(\mathcal{M}_4, \mathbb{R})$ is its subgroup. In fact, this issue is subtle, and it deserves some discussion.

It is easy to see that every action, which depends on at least two fields $\phi_1(x)$ and $\phi_2(x)$, is invariant under the following transformation, determined by the Henneaux–Teitelboim (HT) parameter ϵ^{HT} (see [35] for details and naming),

$$\delta_0{}^{\text{HT}}\phi_1 = \epsilon^{\text{HT}}(x)\frac{\delta S}{\delta\phi_2}, \qquad \delta_0{}^{\text{HT}}\phi_2 = -\epsilon^{\text{HT}}(x)\frac{\delta S}{\delta\phi_1}, \tag{120}$$

which can be easily verified by calculating the variation of the action:

$$\delta^{\text{HT}}S[\phi_1,\phi_2] = \frac{\delta S}{\delta\phi_1}\delta_0{}^{\text{HT}}\phi_1 + \frac{\delta S}{\delta\phi_2}\delta_0{}^{\text{HT}}\phi_2 = 0. \tag{121}$$

Since this invariance is present even in theories with no gauge symmetry, it is not associated with constraints, and thus not present in the generator of gauge symmetries (55), see [35] for details.

Now, let us consider the diffeomorphism transformation

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \tag{122}$$

where the parameter $\xi^\mu(x)$ is an arbitrary function, which we will consider to be infinitesimal. Also, let us denote all parameters of the gauge group collectively as $\epsilon_i(x)$. If diffeomorphisms are a symmetry of the action, then for every field $\phi(x)$ in the theory, and every parameter of the diffeomorphisms $\xi^{\mu}(x)$, there should exist a choice of the parameters $\epsilon_i(x)$ and $\epsilon^{\mathrm{HT}}(x)$, such that:

$$(\delta_0{}^{\mathrm{gauge}} + \delta_0{}^{\mathrm{HT}} + \delta_0{}^{\mathrm{diff}})\,\phi = 0. \tag{123}$$

In other words, if the diffeomorphisms are a symmetry of the theory, their form variations should be expressible as gauge form variations combined with HT form variations:

$$\delta_0{}^{\mathrm{diff}}\,\phi = -\delta_0{}^{\mathrm{gauge}}\phi - \delta_0{}^{\mathrm{HT}}\phi. \tag{124}$$

In our case, the 3BF action depends on the fields $\alpha^{\alpha}{}_{\mu}$, $\beta^{a}{}_{\mu\nu}$, $\gamma^{A}{}_{\mu\nu\rho}$, $B^{\alpha}{}_{\mu\nu}$, $C^{a}{}_{\mu}$, and D^{A}. The HT parameters $\epsilon^{\mathrm{HT}\alpha\beta}{}_{\mu\nu\rho}$, $\epsilon^{\mathrm{HT}ab}{}_{\mu\nu\rho}$, and $\epsilon^{\mathrm{HT}AB}{}_{\mu\nu\rho}$ are defined via the following form variations, analogous to (120):

$$\begin{aligned}
\delta_0{}^{\mathrm{HT}}\alpha^{\alpha}{}_{\mu} &= \frac{1}{2}\epsilon^{\mathrm{HT}\alpha\beta}{}_{\mu\nu\rho}\frac{\delta S}{\delta B^{\beta}{}_{\nu\rho}},\\
\delta_0{}^{\mathrm{HT}}B^{\alpha}{}_{\mu\nu} &= -\epsilon^{\mathrm{HT}\alpha\beta}{}_{\rho\mu\nu}\frac{\delta S}{\delta\alpha^{\beta}{}_{\rho}},\\
\delta_0{}^{\mathrm{HT}}\beta^{a}{}_{\mu\nu} &= \epsilon^{\mathrm{HT}ab}{}_{\mu\nu\rho}\frac{\delta S}{\delta C^{b}{}_{\rho}},\\
\delta_0{}^{\mathrm{HT}}C^{a}{}_{\mu} &= -\frac{1}{2}\epsilon^{\mathrm{HT}ab}{}_{\nu\rho\mu}\frac{\delta S}{\delta\beta^{b}{}_{\nu\rho}},\\
\delta_0{}^{\mathrm{HT}}\gamma^{A}{}_{\mu\nu\rho} &= \epsilon^{\mathrm{HT}AB}{}_{\mu\nu\rho}\frac{\delta S}{\delta D^{B}},\\
\delta_0{}^{\mathrm{HT}}D^{A} &= -\frac{1}{3!}\epsilon^{\mathrm{HTAB}}{}_{\mu\nu\rho}\frac{\delta S}{\delta\gamma^{B}{}_{\mu\nu\rho}},
\end{aligned} \tag{125}$$

while the gauge parameters $\epsilon_{\mathfrak{g}}{}^{\alpha}$, $\epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}$, $\epsilon_{\mathfrak{l}}{}^{A}{}_{\mu\nu}$, $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu}$, and $\epsilon_{\mathfrak{n}}^{a}$ are defined in theorems 1–5. Given these, there indeed exists a choice of these parameters, such that (123) is satisfied for all fields. Specifically, if one chooses the gauge parameters as

$$\begin{aligned}
\epsilon_{\mathfrak{g}}{}^{\alpha} &= -\xi^{\lambda}\alpha^{\alpha}{}_{\lambda},\\
\epsilon_{\mathfrak{h}}{}^{a}{}_{\mu} &= \xi^{\lambda}\beta^{a}{}_{\mu\lambda},\\
\epsilon_{\mathfrak{l}}^{A}{}_{\mu\nu} &= \xi^{\lambda}\gamma^{A}{}_{\mu\nu\lambda},\\
\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu} &= \xi^{\lambda}B^{\alpha}{}_{\mu\lambda},\\
\epsilon_{\mathfrak{n}}{}^{a} &= -\xi^{\lambda}C^{a}{}_{\lambda},
\end{aligned} \tag{126}$$

and the HT parameters as

$$\begin{aligned}
\epsilon^{\mathrm{HT}\alpha\beta}{}_{\mu\nu\rho} &= \xi^{\lambda}g^{\alpha\beta}\epsilon_{\mu\nu\rho\lambda},\\
\epsilon^{\mathrm{HT}ab}{}_{\mu\nu\rho} &= \xi^{\lambda}g^{ab}\epsilon_{\lambda\mu\nu\rho},\\
\epsilon^{\mathrm{HT}AB}{}_{\mu\nu\rho} &= \xi^{\lambda}g^{AB}\epsilon_{\mu\nu\rho\lambda},
\end{aligned} \tag{127}$$

one can obtain, using (124), precisely the standard form variations corresponding to diffeomorphisms:

$$
\begin{aligned}
\delta_0{}^{\text{diff}}\alpha^{\alpha}{}_{\mu} &= -\partial_\mu\xi^\lambda\alpha^{\alpha}{}_{\lambda} - \xi^\lambda\partial_\lambda\alpha^{\alpha}{}_{\mu},\\
\delta_0{}^{\text{diff}}\beta^{a}{}_{\mu\nu} &= -\partial_\mu\xi^\lambda\beta^{a}{}_{\lambda\nu} - \partial_\nu\xi^\lambda\beta^{a}{}_{\mu\lambda} - \xi^\lambda\partial_\lambda\beta^{a}{}_{\mu\nu},\\
\delta_0{}^{\text{diff}}\gamma^{A}{}_{\mu\nu\rho} &= -\partial_\mu\xi^\lambda\gamma^{A}{}_{\lambda\nu\rho} - \partial_\nu\xi^\lambda\gamma^{A}{}_{\mu\lambda\rho} - \partial_\rho\xi^\lambda\gamma^{A}{}_{\mu\nu\lambda} - \xi^\lambda\partial_\lambda\gamma^{A}{}_{\mu\nu\rho},\\
\delta_0{}^{\text{diff}}B^{\alpha}{}_{\mu\nu} &= -\partial_\mu\xi^\lambda B^{\alpha}{}_{\lambda\nu} - \partial_\nu\xi^\lambda B^{\alpha}{}_{\mu\lambda} - \xi^\lambda\partial_\lambda B^{\alpha}{}_{\mu\nu},\\
\delta_0{}^{\text{diff}}C^{a}{}_{\mu} &= -\partial_\mu\xi^\lambda C^{a}{}_{\lambda} - \xi^\lambda\partial_\lambda C^{a}{}_{\mu},\\
\delta_0{}^{\text{diff}}D^{A} &= -\xi^\lambda\partial_\lambda D^{A}.
\end{aligned}
\tag{128}
$$

This establishes that diffeomorphisms are indeed contained in the full gauge symmetry group $\mathcal{G}_{3BF}$, up to the HT transformations, which are always a symmetry of the theory.

5.3. Future lines of investigation

Based on the results obtained in this work, one can imagine various additional topics for further research.

First, since we have obtained that the pure $3BF$ theory is a topological theory, it does not describe a realistic physical theory which ought to contain local propagating degrees of freedom. To build a realistic physical theory, one introduces the degrees of freedom by imposing the simplicity constraints on the topological action. In our previous work [13], we have formulated the classical actions that manifestly distinguish the topological sector from the simplicity constraints, for all the fields present in the standard model coupled to Einstein–Cartan gravity. Specifically, we have defined the constrained $2BF$ actions describing the Yang–Mills field and Einstein–Cartan gravity, and also the constrained $3BF$ actions describing the Klein–Gordon, Dirac, Weyl and Majorana fields coupled to gravity in the standard way. The natural continuation of this line of research would be the Hamiltonian analysis of all such constrained $3BF$ models of gravity coupled to various matter fields, and the study of their canonical quantization.

On the other hand, as an alternative to the canonical quantization, one may choose the spinfoam quantization approach, and define the path integral of the theory as the state sum for the Regge-discretized $3BF$ action. The topological nature of the $3BF$ action, together with the structure of the gauge three-group, should ensure that such a sum is a topological invariant, i.e. that it is triangulation independent. This construction was recently carried out in [14], where the $3BF$ state sum for a general two-crossed module and a closed and orientable four-dimensional manifold $\mathcal{M}_4$ is defined. Unfortunately, in order to rigorously define this state sum, one needs the higher category generalizations of the Peter–Weyl and Plancherel theorems, from ordinary groups to the cases of two-groups and three-groups. These theorems ought to determine the domains of various labels living on simplices of the triangulation, as a consequence of the representation theory of three-groups. Until these mathematical results are obtained, one can try to guess the appropriate structure of the irreducible representations of a three-group and construct the topological invariant Z for the $3BF$ topological action, in analogy with what was done in the case of $2BF$ theory, see [25, 27]. Once the topological state sum is obtained, one can proceed to impose the simplicity constraints, and thus construct the state sum corresponding to the tentative quantum theory of gravity with matter. The classical action for gravity and matter is formulated in [13] in a way that explicitly distinguishes between the topological sector and the simplicity constraints sector of the action, making the procedure of imposing the constraints straightforward.

Next, it would be useful to investigate in more depth the mathematical structure and properties of the simplicity constraints, in particular their role as the gauge fixing conditions for the symmetry group $\mathcal{G}_{3BF}$. The simplicity constraints should explicitly break the symmetry group $\mathcal{G}_{3BF}$ to the subgroup corresponding to the constrained $3BF$ theory, which may then be further spontaneously broken by the Higgs mechanism.

One of the results obtained in this work is a duality between the gauge symmetry group of the $3BF$ action, $\mathcal{G}_{3BF}$, and the underlining three-group, i.e. the two-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{\_,\_\}_{\text{pf}})$. This duality should be better understood. On one hand, the group $\mathcal{G}_{3BF}$ can provide further insight into the construction of the TQFT state sum, i.e. a topological invariant corresponding to the underlining three-group structure. On the other hand, this duality is interesting from the perspective of pure mathematics, since it can provide deeper insight in the structure of three-groups. In addition, one could expect that the $3BF$ theory would have a three-group of higher gauge symmetries, but it is not obvious if the five types of gauge transformations can form a three-group structure or not. This is an important topic for future research.

Finally, in [31] it was pointed out that it may be useful to make one more step in the categorical generalization, and consider a $4BF$ theory as a description of a quantum gravity model with matter fields. One could then calculate the gauge group of the $4BF$ action, and compare the results with the results obtained for the $3BF$ theory.

The list is not conclusive, and there may be many other interesting topics to study.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Two-crossed module

Definition (Differential two-crossed module). A differential two-crossed module is given by an exact sequence of Lie algebras:

$$\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g},$$

together with left action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$, by derivations, and on itself via adjoint representation, and a $\mathfrak{g}$-equivariant bilinear map called the **Peiffer lifting**:

$$\{\_,\_\}_{\text{pf}} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{l}.$$

Fixing the basis in the algebras as $T_A \in \mathfrak{l}$, $t_a \in \mathfrak{h}$ and $\tau_\alpha \in \mathfrak{g}$:

$$[T_A, T_B] = f_{AB}{}^C\, T_C, \qquad [t_a, t_b] = f_{ab}{}^c\, t_c, \qquad [\tau_\alpha, \tau_\beta] = f_{\alpha\beta}{}^\gamma\, \tau_\gamma,$$

one defines the maps ∂ and δ as:

$$\partial(t_a) = \partial_a{}^\alpha\, \tau_\alpha, \qquad \delta(T_A) = \delta_A{}^a\, t_a,$$

and the action of $\mathfrak{g}$ on the generators of $\mathfrak{l}$, $\mathfrak{h}$, and $\mathfrak{g}$ is, respectively:

$$\tau_\alpha \triangleright T_A = \triangleright_{\alpha A}{}^B\, T_B, \qquad \tau_\alpha \triangleright t_a = \triangleright_{\alpha a}{}^b\, t_b, \qquad \tau_\alpha \triangleright \tau_\beta = \triangleright_{\alpha\beta}{}^\gamma\, \tau_\gamma.$$

The coefficients $X_{ab}{}^A$ are introduced as:

$$\{t_a,\, t_b\}_{\mathrm{pf}} = X_{ab}{}^A T_A.$$

The maps ∂ and δ satisfy the following identity:

$$\partial_a{}^\alpha\, \delta_A{}^a = 0. \tag{A.1}$$

Note that when η is a $\mathfrak{g}$-valued differential form and ω is $\mathfrak{l}$-, $\mathfrak{h}$-, or $\mathfrak{g}$-valued differential form, the previous action is defined as:

$$\eta \wedge^\triangleright \omega = \eta^\alpha \wedge \omega^A \triangleright_{\alpha A}{}^B\, T_B,$$
$$\eta \wedge^\triangleright \omega = \eta^\alpha \wedge \omega^a \triangleright_{\alpha a}{}^b\, t_b,$$
$$\eta \wedge^\triangleright \omega = \eta^\alpha \wedge \omega^\beta f_{\alpha\beta}{}^\gamma\, \tau_\gamma,$$

where the forms are multiplied via the wedge product $\wedge$, while the generators of G act on the generators of the three groups via the action $\triangleright$.

The following identities are satisfied:

(i) In the differential crossed module $(L \xrightarrow{\delta} H, \triangleright')$ the action $\triangleright'$ of $\mathfrak{h}$ on $\mathfrak{l}$ is defined for each $\underline{h} \in \mathfrak{h}$ and $\underline{l} \in \mathfrak{l}$ as:

$$\underline{h} \triangleright' \underline{l} = -\{\delta(\underline{l}),\, \underline{h}\}_{\mathrm{pf}},$$

or written in the basis where $t_a \triangleright' T_A = \triangleright'{}_{aA}{}^B T_B$ the previous identity becomes:

$$\triangleright'{}_{aA}{}^B = -\delta_A{}^b X_{ba}{}^B; \tag{A.2}$$

(ii) The action of $\mathfrak{g}$ on itself is via adjoint representation:

$$\triangleright_{\alpha\beta}{}^\gamma = f_{\alpha\beta}{}^\gamma; \tag{A.3}$$

(iii) The action of $\mathfrak{g}$ on $\mathfrak{h}$ and $\mathfrak{l}$ is equivariant, i.e. the following identities are satisfied:

$$\partial_a{}^\beta f_{\alpha\beta}{}^\gamma = \triangleright_{\alpha a}{}^b \partial_b{}^\gamma, \qquad \delta_A{}^a\, \triangleright_{\alpha a}{}^b = \triangleright_{\alpha A}{}^B \delta_B{}^b; \tag{A.4}$$

(iv) The Peiffer lifting is $\mathfrak{g}$-equivariant, i.e. for each $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ and $\underline{g} \in \mathfrak{g}$:

$$\underline{g} \triangleright \{\underline{h}_1, \underline{h}_2\}_{\mathrm{pf}} = \{\underline{g} \triangleright \underline{h}_1, \underline{h}_2\}_{\mathrm{pf}} + \{\underline{h}_1,\, \underline{g} \triangleright \underline{h}_2\}_{\mathrm{pf}},$$

or written in the basis:

$$X_{ab}{}^{B}\rhd_{\alpha B}{}^{A} = \rhd_{\alpha a}{}^{c}X_{cb}{}^{A} + \rhd_{\alpha b}{}^{c}X_{ac}{}^{A}; \tag{A.5}$$

(v) $\delta(\{\underline{h}_1, \underline{h}_2\}_{\rm pf}) = \langle \underline{h}_1, \underline{h}_2\rangle_{\rm p}, \ \forall\ \underline{h}_1, \underline{h}_2 \in \mathfrak{h}$.

The map $(\underline{h}_1, \underline{h}_2) \in \mathfrak{h}\times\mathfrak{h} \to \langle \underline{h}_1, \underline{h}_2\rangle_{\rm p} \in \mathfrak{h}$ is bilinear $\mathfrak{g}$-equivariant map called the **Peiffer paring**, i.e. all $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ and $\underline{g} \in \mathfrak{g}$ satisfy the following identity:

$$\underline{g} \rhd \langle \underline{h}_1, \underline{h}_2\rangle_{\rm p} = \langle \underline{g}\rhd \underline{h}_1, \underline{h}_2\rangle_{\rm p} + \langle \underline{h}_1, \underline{g}\rhd \underline{h}_2\rangle_{\rm p}.$$

Fixing the basis the identity becomes:

$$X_{ab}{}^{A}\delta_{A}{}^{c} = f_{ab}{}^{c} - \partial_{a}{}^{\alpha}\rhd_{\alpha b}{}^{c}; \tag{A.6}$$

(vi) $[\underline{l}_1, \underline{l}_2] = \{\delta(\underline{l}_1), \delta(\underline{l}_2)\}_{\rm pf}, \ \forall\ \underline{l}_1, \underline{l}_2 \in \mathfrak{l}$, i.e.

$$f_{AB}{}^{C} = \delta_{A}{}^{a}\delta_{B}{}^{b}X_{ab}{}^{C}; \tag{A.7}$$

(vii) $\{[\underline{h}_1, \underline{h}_2], \underline{h}_3\}_{\rm pf} = \partial(\underline{h}_1)\rhd\{\underline{h}_2, \underline{h}_3\}_{\rm pf} + \{\underline{h}_1, [\underline{h}_2, \underline{h}_3]\}_{\rm pf} - \partial(\underline{h}_2)\rhd\{\underline{h}_1, \underline{h}_3\}_{\rm pf} - \{\underline{h}_2, [\underline{h}_1, \underline{h}_3]\}_{\rm pf}, \ \forall\ \underline{h}_1, \underline{h}_2, \underline{h}_3 \in \mathfrak{h}$, i.e.

$$\begin{aligned}\{[\underline{h}_1, \underline{h}_2], \underline{h}_3\}_{\rm pf} = {}& \{\partial(\underline{h}_1)\rhd \underline{h}_2, \underline{h}_3\}_{\rm pf} - \{\partial(\underline{h}_2)\rhd \underline{h}_1, \underline{h}_3\}_{\rm pf} \\ & - \{\underline{h}_1, \delta\{\underline{h}_2, \underline{h}_3\}_{\rm pf}\}_{\rm pf} + \{\underline{h}_2, \delta\{\underline{h}_1, \underline{h}_3\}_{\rm pf}\}_{\rm pf},\end{aligned} \tag{A.8}$$

$$f_{ab}{}^{d}X_{dc}{}^{B} = \partial_{a}{}^{\alpha}X_{bc}{}^{A}\rhd_{\alpha A}{}^{B} + X_{ad}{}^{B}f_{bc}{}^{d} - \partial_{b}{}^{\alpha}\rhd_{\alpha A}{}^{B}X_{ac}{}^{A} - X_{bd}{}^{B}f_{ac}{}^{d}; \tag{A.9}$$

(viii) $\{\underline{h}_1, [\underline{h}_2, \underline{h}_3]\}_{\rm pf} = \left\{\delta\{\underline{h}_1, \underline{h}_2\}_{\rm pf}, \underline{h}_3\right\}_{\rm pf} - \left\{\delta\{\underline{h}_1, \underline{h}_3\}_{\rm pf}, \underline{h}_2\right\}_{\rm pf}, \ \forall\ \underline{h}_1, \underline{h}_2, \underline{h}_3 \in \mathfrak{h}$, i.e.

$$X_{ad}{}^{A}f_{bc}{}^{d} = X_{ab}{}^{B}\delta_{B}{}^{d}X_{dc}{}^{A} - X_{ac}{}^{B}\delta_{B}{}^{d}X_{db}{}^{A}; \tag{A.10}$$

(ix) $\{\delta(\underline{l}), \underline{h}\}_{\rm pf} + \{\underline{h}, \delta(\underline{l})\}_{\rm pf} = -\partial(\underline{h})\rhd \underline{l}, \ \forall\ \underline{l} \in \mathfrak{l}, \ \forall\ \underline{h} \in \mathfrak{h}$, i.e.

$$\delta_{A}{}^{a}X_{ab}{}^{B} + \delta_{A}{}^{a}X_{ba}{}^{B} = -\partial_{b}{}^{\alpha}\rhd_{\alpha A}{}^{B}. \tag{A.11}$$

A reader interested in more details about three-groups is referred to [21, 30].

The structure constants satisfy the Jacobi identities

$$\begin{aligned} f_{\alpha\gamma}{}^{\delta} f_{\beta\epsilon}{}^{\gamma} &= 2f_{\alpha[\beta|}{}^{\gamma} f_{\gamma|\epsilon]}{}^{\delta}, \\ f_{ad}{}^{c} f_{be}{}^{d} &= 2f_{a[b|}{}^{d} f_{d|e]}{}^{c}, \\ f_{AD}{}^{C} f_{BE}{}^{D} &= 2f_{A[B|}{}^{D} f_{D|E]}{}^{C}. \end{aligned} \tag{A.12}$$

Also, the following relations are useful:

$$f_{\beta\gamma}{}^{\alpha}\rhd_{\alpha b}{}^{a} = 2\rhd_{[\beta|c}{}^{a}\rhd_{|\gamma]b}{}^{c}, \qquad f_{\beta\gamma}{}^{\alpha}\rhd_{\alpha B}{}^{A} = 2\rhd_{[\beta|C}{}^{A}\rhd_{|\gamma]B}{}^{C}. \tag{A.13}$$

Appendix B. Additional relations of the constraint algebra

In this appendix the useful technical results used in the subsection 3.2 are given. First, since the secondary constraints, given by the equation (30), must be preserved during the evolution of the system, the consistency conditions of secondary constraints must be enforced. However, no tertiary constraints arise from these conditions, since one obtains the following PB:

$$
\begin{aligned}
\{\mathcal{S}(\mathcal{F})^{\alpha i}, H_{\mathrm{T}}\} &= f_{\beta\gamma}{}^{\alpha}\mathcal{S}(\mathcal{F})^{\beta i}\alpha^{\gamma}{}_{0}, \\
\{\mathcal{S}(\nabla B)_{\alpha}, H_{\mathrm{T}}\} &= f_{\beta\gamma\alpha}B^{\gamma}{}_{0k}\mathcal{S}(\mathcal{F})^{\beta k} + f_{\beta\alpha}{}^{\gamma}\alpha^{\beta}{}_{0}\mathcal{S}(\nabla B)_{\gamma} + C_{a0}\rhd_{\alpha b}{}^{a}\mathcal{S}(\mathcal{G})^{b} \\
&\quad - \rhd_{\alpha a}{}^{b}\beta^{a}{}_{0k}\mathcal{S}(\nabla C)_{b}{}^{k} + \frac{1}{2}\rhd_{\alpha}{}^{B}{}_{A}\gamma^{A}{}_{0jk}\mathcal{S}(\nabla D)_{B}{}^{jk}, \\
\{\mathcal{S}(\mathcal{G})^{a}, H_{\mathrm{T}}\} &= \rhd_{\alpha b}{}^{a}\beta^{b}{}_{0k}\mathcal{S}(\mathcal{F})^{\alpha k} - \alpha^{\alpha}{}_{0}\rhd_{\alpha b}{}^{a}\mathcal{S}(\mathcal{G})^{b}, \\
\{\mathcal{S}(\nabla C)_{a}{}^{i}, H_{\mathrm{T}}\} &= C_{b0}\rhd_{\alpha}{}^{b}{}_{a}\mathcal{S}(\mathcal{F})^{\alpha i} + \rhd_{\alpha a}{}^{b}\alpha^{\alpha}{}_{0}\mathcal{S}(\nabla C)_{b}{}^{i} + 2X_{(ab)}{}^{A}\beta^{b}{}_{0j}\mathcal{S}(\nabla D)_{A}{}^{ij}, \\
\{\mathcal{S}(\nabla D)_{A}{}^{ij}, H_{\mathrm{T}}\} &= \alpha^{\alpha}{}_{0}\rhd_{\alpha A}{}^{B}\mathcal{S}(\nabla D)_{B}{}^{ij}.
\end{aligned}
\tag{B.1}
$$

The PB between the first-class constraints, given by the equation (34), and the second-class constraints, given by the equation (35), are given by:

$$
\begin{aligned}
\{\,\Phi(\mathcal{F})^{\alpha i}(\vec{x})\,,\,\chi(\alpha)_{\beta j}(\vec{y})\,\} &= -f_{\beta\gamma}{}^{\alpha}\,\chi(B)^{\gamma ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\mathcal{G})^{a}(\vec{x})\,,\,\chi(\alpha)_{\alpha}{}^{i}(\vec{y})\,\} &= \rhd_{\alpha b}{}^{a}\,\chi(C)^{bi}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\mathcal{G})^{a}(\vec{x})\,,\,\chi(\beta)_{b}{}^{ij}(\vec{y})\,\} &= -\rhd_{\alpha b}{}^{a}\,\chi(B)^{\alpha ij}(\vec{x})\,\delta^{(3)}(x-y), \\
\{\,\Phi(\nabla C)^{ai}(\vec{x})\,,\,\chi(\alpha)_{\alpha}{}^{j}(\vec{y})\,\} &= -\rhd_{\alpha b}{}^{a}\,\chi(\beta)^{bij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\nabla C)^{ai}(\vec{x})\,,\,\chi(\beta)_{b}{}^{jk}(\vec{y})\,\} &= 2X^{(ac)A}g_{bc}\,\chi(\gamma)_{A}{}^{ijk}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\nabla C)^{ai}(\vec{x})\,,\,\chi(C)_{b}{}^{j}(\vec{y})\,\} &= \rhd_{\alpha b}{}^{a}\,\chi(B)^{\alpha ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\nabla C)^{ai}(\vec{x})\,,\,\chi(D)_{A}(\vec{y})\,\} &= 2X^{(ab)}{}_{A}\,\chi(C)_{b}{}^{i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(\alpha)_{\beta}{}^{i}(\vec{y})\,\} &= f_{\beta\gamma}{}^{\alpha}\,\chi(\alpha)^{\gamma i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(\beta)_{a}{}^{ij}(\vec{y})\,\} &= g^{\alpha\beta}\rhd_{\beta a}{}^{b}\,\chi(\beta)_{b}{}^{ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(\gamma)_{A}{}^{ijk}(\vec{y})\,\} &= g^{\alpha\beta}\rhd_{\beta A}{}^{B}\,\chi(\gamma)_{B}{}^{ijk}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(B)_{\beta}{}^{ij}(\vec{y})\,\} &= f\beta\gamma^{\alpha}\,\chi(B)^{\gamma ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}). \\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(C)_{a}{}^{i}(\vec{y})\,\} &= -\rhd^{\alpha b}{}_{a}\,\chi(C)_{b}{}^{i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}). \\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(D)_{A}(\vec{y})\,\} &= g^{\alpha\beta}\rhd_{\beta A}{}^{B}\,\chi(D)_{B}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\nabla D)^{Aij}(\vec{x})\,,\,\chi(\alpha)_{\alpha}{}^{k}(\vec{y})\,\} &= \rhd_{\alpha B}{}^{A}\chi(\gamma)^{Bijk}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \\
\{\,\Phi(\nabla D)^{Aij}(\vec{x})\,,\,\chi(D)_{B}(\vec{y})\,\} &= -\rhd_{\alpha B}{}^{A}\chi(B)^{\alpha ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}).
\end{aligned}
\tag{B.2}
$$

Finally, it is useful to calculate PB between the first-class constraints, given by the equation (34), and the total Hamiltonian, given by the equation (33):

$$\begin{aligned}
\{\Phi(\mathcal{F})^{\alpha i},\, H_{\rm T}\} &= f_{\beta\gamma}{}^{\alpha}\Phi(\mathcal{F})^{\beta i}\alpha^{\gamma}{}_{0},\\
\{\Phi(\nabla B)_{\alpha},\, H_{\rm T}\} &= f_{\beta\gamma\alpha}B^{\gamma}{}_{0k}\Phi(\mathcal{F})^{\beta k} + f_{\beta\alpha}{}^{\gamma}\alpha^{\beta}{}_{0}\Phi(\nabla B)_{\gamma} + C_{a0}\rhd_{\alpha b}{}^{a}\Phi(\mathcal{G})^{b}\\
&\quad - \rhd_{\alpha a}{}^{b}\beta^{a}{}_{0k}\Phi(\nabla C)_{b}{}^{k} + \frac{1}{2}\rhd_{\alpha}{}^{B}{}_{A}\gamma^{A}{}_{0jk}\Phi(\nabla D)_{B}{}^{jk},\\
\{\Phi(\mathcal{G})^{a},\, H_{\rm T}\} &= \rhd_{\alpha b}{}^{a}\beta^{b}{}_{0k}\Phi(\mathcal{F})^{\alpha k} - \alpha^{\alpha}{}_{0}\rhd_{\alpha b}{}^{a}\Phi(\mathcal{G})^{b},\\
\{\Phi(\nabla C)_{a}{}^{i},\, H_{\rm T}\} &= C_{b0}\rhd_{\alpha}{}^{b}{}_{a}\Phi(\mathcal{F})^{\alpha i} + \rhd_{\alpha a}{}^{b}\alpha^{\alpha}{}_{0}\Phi(\nabla C)_{b}{}^{i} + 2X_{(ab)}{}^{A}\beta^{b}{}_{0j}\Phi(\nabla D)_{A}{}^{ij},\\
\{\Phi(\nabla D)_{A}{}^{ij},\, H_{\rm T}\} &= \alpha^{\alpha}{}_{0}\rhd_{\alpha A}{}^{B}\Phi(\nabla D)_{B}{}^{ij}.
\end{aligned} \tag{B.3}$$

The calculated PB brackets given by the equation (B.3) will be useful for calculation of the generator of gauge symmetries (55). With these results one can proceed to the construction of the gauge symmetry generator on one hypersurface Σ_3 given in the equation (55), and ultimately obtain the finite gauge symmetry of the whole spacetime.

The PB algebra of gauge symmetry generators $(\tilde{M}_0)_{\alpha}{}^{i}$, $(\tilde{M}_1)_{\alpha}{}^{i}$, $(\tilde{G}_0)_{\alpha}$, $(\tilde{G}_1)_{\alpha}$, $(\tilde{H}_0)_{a}{}^{i}$, $(\tilde{H}_1)_{a}{}^{i}$, $(\tilde{N}_0)_{a}$, $(\tilde{N}_1)_{a}$, $(\tilde{L}_0)_{A}{}^{ij}$, and $(\tilde{L}_1)_{A}{}^{ij}$, as defined in (56), is:

$$\{(\tilde{G}_0)_{\alpha}(\vec{x}),(\tilde{G}_0)_{\beta}(\vec{y})\} = f_{\alpha\beta}{}^{\gamma}(\tilde{G}_0)_{\gamma}\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.4}$$

$$\{(\tilde{H}_0)_{a}{}^{i}(\vec{x}),(\tilde{H}_0)_{b}{}^{j}(\vec{y})\} = 2X_{(ab)}{}^{A}(\tilde{L}_0)_{A}{}^{ij}\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.5}$$

$$\{(\tilde{H}_0)_{a}{}^{i}(\vec{x}),(\tilde{H}_1)_{b}{}^{j}(\vec{y})\} = 2X_{(ab)}{}^{A}(\tilde{L}_1)_{A}{}^{ij}\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.6}$$

$$\{(\tilde{H}_0)_{a}{}^{i}(\vec{x}),(\tilde{N}_0)^{b}(\vec{y})\} = \rhd_{\alpha a}{}^{b}(\tilde{M}_0)^{\alpha i}\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.7}$$

$$\{(\tilde{H}_1)_{a}{}^{i}(\vec{x}),(\tilde{N}_0)^{b}(\vec{y})\} = \rhd_{\alpha a}{}^{b}(\tilde{M}_1)^{\alpha i}\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.8}$$

$$\{(\tilde{H}_0)_{a}(\vec{x}),(\tilde{N}_1)^{bi}(\vec{y})\} = \rhd_{\alpha a}{}^{b}(\tilde{M}_1)^{\alpha i}\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.9}$$

$$\{(\tilde{G}_0)_{\alpha}(\vec{x}),(\tilde{M}_0)_{\beta}{}^{i}(\vec{y})\} = f_{\alpha\beta}{}^{\gamma}(\tilde{M}_0)_{\gamma}{}^{i}\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.10}$$

$$\{(\tilde{G}_0)_{\alpha}(\vec{x}),(\tilde{M}_1)_{\beta}{}^{i}(\vec{y})\} = f_{\alpha\beta}{}^{\gamma}(\tilde{M}_1)_{\gamma}{}^{i}\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.11}$$

$$\{(\tilde{G}_0)_{\alpha}(\vec{x}),\,(\tilde{H}_1)_{a}{}^{i}(\vec{y})\} = \rhd_{\alpha a}{}^{b}\,(\tilde{H}_1)_{b}{}^{i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.12}$$

$$\{(\tilde{G}_0)_{\alpha}(\vec{x}),\,(\tilde{H}_0)_{a}{}^{i}(\vec{y})\} = \rhd_{\alpha a}{}^{b}\,(\tilde{H}_0)_{b}{}^{i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.13}$$

$$\{(\tilde{G}_0)_{\alpha}(\vec{x}),\,(\tilde{N}_1)_{a}(\vec{y})\} = \rhd_{\alpha a}{}^{b}\,(\tilde{N}_1)_{b}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.14}$$

$$\{(\tilde{G}_0)_{\alpha}(\vec{x}),\,(\tilde{N}_0)_{a}(\vec{y})\} = \rhd_{\alpha a}{}^{b}\,(\tilde{N}_0)_{b}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}), \tag{B.15}$$

$$\{(\tilde{G}_0)_{\alpha}(\vec{x}),\,(\tilde{L}_0)_{A}{}^{ij}(\vec{y})\} = \rhd_{\alpha A}{}^{B}(\tilde{L}_0)_{B}{}^{ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y}). \tag{B.16}$$

The gauge symmetry group has the following structure. First, the groups $\tilde{M}_1\times\tilde{M}_0$, $\tilde{N}_1\times\tilde{N}_0$ and $\tilde{L}_1\times\tilde{L}_0$ with the corresponding algebras $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$, respectively, where:

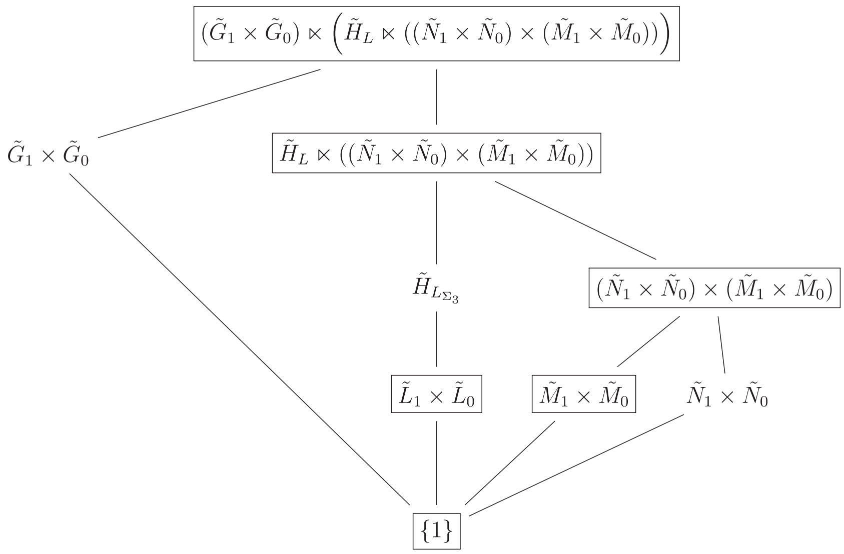

Figure B1. The symmetry group $\mathcal{G}_{\Sigma_3}$ of the Poisson bracket algebra in the phase space. The invariant subgroups are boxed.

$$
\begin{aligned}
\mathfrak{a}_1 &= \text{span}\{(\tilde{M}_1)_\alpha{}^i\} \oplus \text{span}\{(\tilde{M}_0)_\alpha{}^i\}, \\
\mathfrak{a}_2 &= \text{span}\{(\tilde{N}_1)_a\} \oplus \text{span}\{(\tilde{N}_0)_a\}, \\
\mathfrak{a}_3 &= \text{span}\{(\tilde{L}_1)_A{}^{ij}\} \oplus \text{span}\{(\tilde{L}_0)_A{}^{ij}\},
\end{aligned}
\tag{B.17}
$$

are the subgroups of the full symmetry group $\tilde{\mathcal{G}}_{\Sigma_3}$. Besides, the subgroups $\tilde{L}_1 \times \tilde{L}_0$ and $\tilde{M}_1 \times \tilde{M}_0$ are the invariant subgroups. The group $\tilde{N}_1 \times \tilde{N}_0$ is not an invariant subgroup of the whole symmetry group, since the Poisson brackets $\{(\tilde{H}_0)_a{}^i(\vec{x}), (\tilde{N}_0)^b(\vec{y})\}$ and $\{(\tilde{H}_1)_a{}^i(\vec{x}), (\tilde{N}_0)^b(\vec{y})\}$ are equal to some linear combinations of the generators of $\tilde{M}_1 \times \tilde{M}_0$. Nevertheless, one can form a direct product $(\tilde{N}_1 \times \tilde{N}_0) \times (\tilde{M}_1 \times \tilde{M}_0)$, since the generators of these groups are mutually commuting, giving a group which is an invariant subgroup of the complete symmetry group.

Next, consider a subgroup $\tilde{H}_{L_{\Sigma_3}}$ determined by the algebra spanned by the generators $(\tilde{L}_1)_A{}^{ij}$, $(\tilde{L}_0)_A{}^{ij}$, $(\tilde{H}_1)_a{}^i$, and $(\tilde{H}_0)_a{}^i$. This group is not invariant subgroup of the whole symmetry group, because of the PB $\{(\tilde{H}_0)_a{}^i(\vec{x}), (\tilde{N}_0)^b(\vec{y})\}$ and $\{(\tilde{H}_1)_a{}^i(\vec{x}), (\tilde{N}_0)^b(\vec{y})\}$, due to the same argument as before. Now, one can join these two subgroups, of which one is invariant and one is not, using a semidirect product into an invariant subgroup $H_L \ltimes ((N_1 \times N_0) \times (M_1 \times M_0))$, determined by the algebra $\mathfrak{a}_4$:

$$
\mathfrak{a}_4 = \text{span}\{(\tilde{M}_0)_\alpha{}^i, (\tilde{M}_1)_\alpha{}^i, (\tilde{H}_0)_a{}^i, (\tilde{H}_1)_a{}^i, (\tilde{N}_0)_a, (\tilde{N}_1)_a, (\tilde{L}_0)_A{}^{ij}, (\tilde{L}_1)_A{}^{ij}\}.
$$

Finally, following the same line of reasoning, one adds the group $\tilde{G}_1 \times \tilde{G}_0$ and obtains the full gauge symmetry group $\tilde{\mathcal{G}}_{\Sigma_3}$ to be equal to:

$$\tilde{\mathcal{G}}_{\Sigma_3} = (\tilde{G}_1 \times \tilde{G}_0) \ltimes \left(\tilde{H}_L \ltimes ((\tilde{N}_1 \times \tilde{N}_0) \times (\tilde{M}_1 \times \tilde{M}_0))\right).$$

The complete symmetry group structure is shown in the figure B1 appendix B. Here, the invariant subgroups of the whole symmetry group are boxed.

Appendix C. Construction of the symmetry generator

When one substitutes the generators (56) into the equation (55), one obtains the gauge generator of the 3*BF* theory in the following form

$$\begin{aligned} G = -\int_{\Sigma_3} \mathrm{d}^3\vec{x} \Bigg(& (\nabla_0 \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i)\Phi(B)_{\alpha}{}^i - \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i \Phi(\mathcal{F})_{\alpha}{}^i + (\nabla_0 \epsilon_{\mathfrak{g}}{}^{\alpha})\Phi(\alpha)_{\alpha} \\ & + \epsilon_{\mathfrak{g}}{}^{\alpha} \left(f_{\alpha\gamma}{}^{\beta} B_{\beta 0i}\Phi(B)^{\gamma i} + C_{a0} \triangleright_{\alpha b}{}^{a} \Phi(C)^{b0} + \beta_{a0i} \triangleright_{\alpha b}{}^{a}\Phi(\beta)^{b0i} \right. \\ & \left. - \frac{1}{2}\gamma^{A}{}_{0ij} \triangleright_{\alpha A}{}^{B} \Phi(\gamma)_B{}^{ij} - \Phi(\nabla B)_{\alpha} \right) + (\nabla_0 \epsilon_{\mathfrak{n}}{}^{a})\Phi(C)_a \\ & - \epsilon_{\mathfrak{n}}{}^{a} \left(\beta_{b0i} \triangleright_{\alpha a}{}^{b} \Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a + (\nabla_0 \epsilon_{\mathfrak{h}}{}^{a}{}_i)\Phi(\beta)_a{}^i \right) \\ & - \epsilon_{\mathfrak{h}}{}^{a}{}_i \left(C_{b0} \triangleright_{\alpha a}{}^{b} \Phi(B)^{\alpha i} - 2\beta^{b}{}_{0j} X_{(ab)}{}^{A} \Phi(\gamma)_A{}^{ij} + \Phi(\nabla C)_a{}^i \right) \\ & - \frac{1}{2}(\nabla_0 \epsilon_{\mathfrak{l}}{}^{A}{}_{ij})\Phi(\gamma)_A{}^{ij} + \frac{1}{2}\epsilon_{\mathfrak{l}}{}^{A}{}_{ij}\Phi(\nabla D)_A{}^{ij} \Bigg), \end{aligned} \tag{C.1}$$

where $\epsilon_{\mathfrak{g}}{}^{\alpha}$, $\epsilon_{\mathfrak{h}}{}^{a}{}_{i}$, $\epsilon_{\mathfrak{l}}{}^{A}{}_{ij}$, $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{i}$, and $\epsilon_{\mathfrak{n}}{}^{a}$ are the independent parameters of the gauge transformations.

The generator of gauge transformations (C.1) in 3*BF* theory given by the action (15), is obtained by the Castellani's procedure, requiring the following requirements to be met

$$G_1 = C_{\mathrm{PFC}}, \tag{C.2}$$

$$G_0 + \{G_1, H_{\mathrm{T}}\} = C_{\mathrm{PFC}}, \tag{C.3}$$

$$\{G_0, H_{\mathrm{T}}\} = C_{\mathrm{PFC}}, \tag{C.4}$$

where C_{PFC} denotes some first-class constraints, and assuming that the generator has the following structure:

$$\begin{aligned} G = \int_{\Sigma_3} \mathrm{d}^3\,\vec{x} \Bigg(& \dot{\epsilon}_{\mathfrak{m}}{}^{\alpha}{}_i (G_1)_{\mathfrak{m}\alpha}{}^i + \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i (G_0)_{\mathfrak{m}\alpha}{}^i + \dot{\epsilon}_{\mathfrak{g}}{}^{\alpha}(G_1)_{\mathfrak{g}\alpha} + \epsilon_{\mathfrak{g}}{}^{\alpha}(G_0)_{\mathfrak{g}\alpha} \\ & + \dot{\epsilon}_{\mathfrak{h}}{}^{a}{}_i (G_1)_{\mathfrak{h}a}{}^i + \epsilon_{\mathfrak{h}}{}^{a}{}_i (G_0)_{\mathfrak{h}a}{}^i + \dot{\epsilon}_{\mathfrak{n}}{}^{a}(G_1)_{\mathfrak{n}a} + \epsilon_{\mathfrak{n}}{}^{a}(G_0)_{\mathfrak{n}a} \\ & + \frac{1}{2}\dot{\epsilon}_{\mathfrak{l}}{}^{A}{}_{ij}(G_1)_{\mathfrak{l}A}{}^{ij} + \frac{1}{2}\epsilon_{\mathfrak{l}}{}^{A}{}_{ij}(G_0)_{\mathfrak{l}A}{}^{ij} \Bigg). \end{aligned} \tag{C.5}$$

The first step of Castellani's procedure, imposing the set of conditions

$$
\begin{aligned}
(G_1)_{\mathfrak{m}\alpha}{}^{i} &= C_{\mathrm{PFC}}, \\
(G_1)_{\mathfrak{g}\alpha} &= C_{\mathrm{PFC}}, \\
(G_1)_{\mathfrak{h}a}{}^{i} &= C_{\mathrm{PFC}}, \\
(G_1)_{\mathfrak{n}a} &= C_{\mathrm{PFC}}, \\
(G_1)_{\mathfrak{l}A}{}^{ij} &= C_{\mathrm{PFC}},
\end{aligned}
\tag{C.6}
$$

is satisfied with a natural choice:

$$
\begin{aligned}
(G_1)_{\mathfrak{m}\alpha}{}^{i} &= -\Phi(B)_{\alpha}{}^{i}, \\
(G_1)_{\mathfrak{g}\alpha} &= -\Phi(\alpha)_{\alpha}, \\
(G_1)_{\mathfrak{h}a}{}^{i} &= -\Phi(C)_{\alpha}{}^{i}, \\
(G_1)_{\mathfrak{n}a} &= -\Phi(\beta)_{a}, \\
(G_1)_{\mathfrak{l}A}{}^{ij} &= \Phi(\gamma)_{A}{}^{ij}.
\end{aligned}
\tag{C.7}
$$

It remains to determine the five generators G_0.

The Castellani's second condition for the generator $(G_0)_{\mathfrak{m}\alpha}{}^{i}$ gives:

$$
\begin{aligned}
(G_0)_{\mathfrak{m}\alpha}{}^{i} - \{\Phi(B)_{\alpha}{}^{i}, H_{\mathrm{T}}\} &= (C_{\mathrm{PFC}})_{\alpha}{}^{i}, \\
(G_0)_{\mathfrak{m}\alpha}{}^{i} - \Phi(\mathcal{F})_{\alpha}{}^{i} &= (C_{\mathrm{PFC}})_{\alpha}{}^{i},
\end{aligned}
\tag{C.8}
$$

that is $(G_0)_{\mathfrak{m}\alpha}{}^{i} = (C_{\mathrm{PFC}})_{\alpha}{}^{i} + \Phi(\mathcal{F})_{\alpha}{}^{i}$. Subsequently, from the Castellani's third condition it follows

$$
\begin{aligned}
&\{(G_0)_{\mathfrak{m}\alpha}{}^{i}, H_{\mathrm{T}}\} = (C_{\mathrm{PFC1}})_{\alpha}{}^{i}, \\
&\{(C_{\mathrm{PFC}})_{\alpha}{}^{i} + \Phi(\mathcal{F})_{\alpha}{}^{i}, H_{\mathrm{T}}\} = (C_{\mathrm{PFC1}})_{\alpha}{}^{i}, \\
&\{(C_{\mathrm{PFC}})_{\alpha}{}^{i}, H_{\mathrm{T}}\} - f_{\beta\gamma\alpha}\alpha^{\beta}{}_{0}\Phi(\mathcal{F})^{\gamma i} = (C_{\mathrm{PFC1}})_{\alpha}{}^{i},
\end{aligned}
\tag{C.9}
$$

which gives

$$
(C_{\mathrm{PFC}})_{\alpha}{}^{i} = f_{\beta\gamma\alpha}\alpha^{\beta}{}_{0}\Phi(B)^{\gamma i}.
$$

It follows that the generator is:

$$
(G_0)_{\mathfrak{m}\alpha}{}^{i} = f_{\beta\gamma\alpha}\alpha^{\beta}{}_{0}\Phi(B)^{\gamma i} + \Phi(\mathcal{F})_{\alpha}{}^{i}. \tag{C.10}
$$

The Castellani's second condition for the generator $(G_0)_{\mathfrak{g}\alpha}$ gives:

$$
\begin{aligned}
(G_0)_{\mathfrak{g}\alpha} - \{\Phi(\alpha)_{\alpha}, H_{\mathrm{T}}\} &= (C_{\mathrm{PFC}})_{\alpha}, \\
(G_0)_{\mathfrak{g}\alpha} - \Phi(\nabla B)_{\alpha} &= (C_{\mathrm{PFC}})_{\alpha},
\end{aligned}
\tag{C.11}
$$

that is $(G_0)_{\mathfrak{g}\alpha} = (C_{\text{PFC}})_\alpha + \Phi(\nabla B)_\alpha$. Subsequently, from the Castellani's third condition it follows

$$\begin{aligned} &\{(G_0)_{\mathfrak{g}\alpha}, H_{\text{T}}\} = (C_{\text{PFC1}})_\alpha, \\ &\{(C_{\text{PFC}})_\alpha + \Phi(\nabla B)_\alpha, H_{\text{T}}\} = (C_{\text{PFC1}})_\alpha, \\ &\{(C_{\text{PFC}})_\alpha, H_{\text{T}}\} + B_{\beta 0i} f_{\alpha\gamma}{}^\beta \Phi(\mathcal{F})^{\gamma i} - \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma \Phi(\nabla B)_\gamma + C_{a0} \rhd_{\alpha b}{}^a \Phi(\mathcal{G})^b \\ &\qquad + \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\nabla C)^{bi} - \frac{1}{2}\gamma^A{}_{0ij} \rhd_{\alpha A}{}^B \Phi(\nabla D)_B{}^{ij} = (C_{\text{PFC1}})_\alpha, \end{aligned} \tag{C.12}$$

which gives

$$\begin{aligned} (C_{\text{PFC}})_\alpha = &-B_{\beta 0i} f_{\alpha\gamma}{}^\beta \Phi(B)^{\gamma i} + \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma \Phi(\alpha)_\gamma - C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^b \\ &- \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)^{bi} + \frac{1}{2}\gamma^A{}_{0ij} \rhd_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij}. \end{aligned}$$

It follows that the generator is:

$$\begin{aligned} (G_0)_{\mathfrak{g}\alpha} = &-B_{\beta 0i} f_{\alpha\gamma}{}^\beta \Phi(B)^{\gamma i} + \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma \Phi(\alpha)_\gamma - C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^b \\ &- \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)^{bi} + \frac{1}{2}\gamma^A{}_{0ij} \rhd_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij} + \Phi(\nabla B)_\alpha. \end{aligned} \tag{C.13}$$

The Castellani's second condition for the generator $(G_0)_{\mathfrak{n}a}$ gives

$$\begin{aligned} (G_0)_{\mathfrak{n}a} - \{\Phi(C)_a, H_{\text{T}}\} &= (C_{\text{PFC}})_a, \\ (G_0)_{\mathfrak{n}a} - \Phi(\mathcal{G})_a &= (C_{\text{PFC}})_a, \end{aligned} \tag{C.14}$$

that is $(G_0)_{\mathfrak{n}a} = (C_{\text{PFC}})_a + \Phi(\mathcal{G})_a$. Subsequently, from the Castellani's third condition it follows

$$\begin{aligned} &\{(G_0)_{\mathfrak{n}a}, H_{\text{T}}\} = (C_{\text{PFC1}})_a, \\ &\{(C_{\text{PFC}})_a + \Phi(\mathcal{G})_a, H_{\text{T}}\} = (C_{\text{PFC1}})_a, \\ &\{(C_{\text{PFC}})_a, H_{\text{T}}\} + \alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(\mathcal{G})_b - \beta_{b0i} \rhd_{\alpha a}{}^b \Phi(\mathcal{F})^{\alpha i} = (C_{\text{PFC1}})_a, \end{aligned} \tag{C.15}$$

which gives

$$(C_{\text{PFC}})_a = -\alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(C)_b + \beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i}.$$

It follows that the generator is:

$$(G_0)_{\mathfrak{n}a} = -\alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(C)_b + \beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a.$$

The Castellani's second condition for the generator $(G_0)_{\mathfrak{h}a}{}^i$ gives:

$$\begin{aligned} &(G_0)_{\mathfrak{h}a}{}^i - \{\Phi(\beta)_a{}^i, H_{\text{T}}\} = (C_{\text{PFC}})_a{}^i, \\ &(G_0)_{\mathfrak{h}a}{}^i - \Phi(\nabla C)_a{}^i = (C_{\text{PFC}})_a{}^i, \end{aligned} \tag{C.16}$$

that is $(G_0)_{\mathfrak{h}a}{}^i = (C_{\text{PFC}})_a{}^i + \Phi(\nabla C)_a{}^i$. Subsequently, from the Castellani's third condition it follows

$$\begin{aligned}
&\{(G_0)_{\mathfrak{h}a}{}^i, H_{\text{T}}\} = (C_{\text{PFC1}})_a{}^i,\\
&\{(C_{\text{PFC}})_a{}^i + \Phi(\nabla C)_a{}^i, H_{\text{T}}\} = (C_{\text{PFC1}})_a{}^i,\\
&\{(C_{\text{PFC}})_a{}^i, H_{\text{T}}\} + \alpha^\alpha{}_0 \triangleright_{\alpha a}{}^b \Phi(\nabla C)_b{}^i - C_{b0} \triangleright_{\alpha a}{}^b \Phi(\mathcal{F})^{\alpha i} + 2\beta^b{}_{0j} X_{(ab)}{}^A \Phi(\nabla D)_A{}^{ij} = (C_{\text{PFC1}})_a{}^i,
\end{aligned}$$

which gives

$$(C_{\text{PFC}})_a{}^i = -\alpha^\alpha{}_0 \triangleright_{\alpha a}{}^b \Phi(\beta)_b{}^i + C_{b0} \triangleright_{\alpha a}{}^b \Phi(B)^{\alpha i} - 2\beta^b{}_{0j} X_{(ab)}{}^A \Phi(\gamma)_A{}^{ij}.$$

It follows that the generator is:

$$(G_0)_{\mathfrak{h}a}{}^i = -\alpha^\alpha{}_0 \triangleright_{\alpha a}{}^b \Phi(\beta)_b{}^i + C_{b0} \triangleright_{\alpha a}{}^b \Phi(B)^{\alpha i} - 2\beta^b{}_{0j} X_{(ab)}{}^A \Phi(\gamma)_A{}^{ij} + \Phi(\nabla C)_a{}^i.$$

The Castellani's second condition for the generator $(G_0)_{\mathfrak{l}A}{}^{ij}$ gives:

$$\begin{aligned}
&(G_0)_{\mathfrak{l}A}{}^{ij} + \{\Phi(\gamma)_A{}^{ij}, H_{\text{T}}\} = (C_{\text{PFC}})_A{}^{ij},\\
&(G_0)_{\mathfrak{l}A}{}^{ij} + \Phi(\nabla D)_A{}^{ij} = (C_{\text{PFC}})_A{}^{ij},
\end{aligned} \tag{C.17}$$

that is $(G_0)_{\mathfrak{l}A}{}^{ij} = (C_{\text{PFC}})_A{}^{ij} - \Phi(\nabla D)_A{}^{ij}$. Subsequently, from the Castellani's third condition it follows:

$$\begin{aligned}
&\{(G_0)_{\mathfrak{l}A}{}^{ij}, H_{\text{T}}\} = (C_{\text{PFC1}})_A{}^{ij},\\
&\{(C_{\text{PFC}})_A{}^{ij} - \Phi(\nabla D)_A{}^{ij}, H_{\text{T}}\} = (C_{\text{PFC1}})_A{}^{ij},\\
&\{(C_{\text{PFC}})_A{}^{ij}, H_{\text{T}}\} - \alpha^\alpha{}_0 \triangleright_{\alpha A}{}^B \Phi(\nabla D)_B{}^{ij} = (C_{\text{PFC1}})_A{}^{ij},
\end{aligned} \tag{C.18}$$

which gives

$$(C_{\text{PFC}})_A{}^{ij} = \alpha^\alpha{}_0 \triangleright_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij}.$$

It follows that the generator is:

$$(G_0)_{\mathfrak{l}A}{}^{ij} = \alpha^\alpha{}_0 \triangleright_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij} - \Phi(\nabla D)_A{}^{ij}. \tag{C.19}$$

At this point, it is useful to summarize the results, and introduce the new notation:

$$\begin{aligned}
\dot{\epsilon}_{\mathfrak{m}}{}^\alpha{}_i (G_1)_{\mathfrak{m}\alpha}{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i (G_0)_{\mathfrak{m}\alpha}{}^i &= -\nabla_0 \epsilon_{\mathfrak{m}}{}^\alpha{}_i \Phi(B)_\alpha{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i \Phi(\mathcal{F})_\alpha{}^i\\
&= \nabla_0 \epsilon_{\mathfrak{m}}{}^\alpha{}_i (\tilde{M}_1)_\alpha{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i (\tilde{M}_0)_\alpha{}^i.
\end{aligned} \tag{C.20}$$

Note that the time derivative of the parameter combines with some of the other terms into a covariant derivative in the time directions.

For the second part of the total generator one obtains:

$$\begin{aligned}
&{}^{\alpha}\dot{\epsilon}_{\mathfrak{g}}(G_1)_{\mathfrak{g}\alpha} + \epsilon_{\mathfrak{g}}{}^{\alpha}(G_0)_{\mathfrak{g}\alpha} \\
&= -{}^{\alpha}\dot{\epsilon}_{\mathfrak{g}}\Phi(\alpha)_\alpha - \epsilon_{\mathfrak{g}}{}^{\alpha}\bigg(B_{\beta 0i}f_{\alpha\gamma}{}^{\beta}\Phi(B)^{\gamma i} - \alpha^{\beta}{}_0 f_{\alpha\beta}{}^{\gamma}\Phi(\alpha)_\gamma + C_{a0}\rhd_{\alpha b}{}^{a}\Phi(C)^b \\
&+ \beta_{a0i}\rhd_{\alpha b}{}^{a}\Phi(\beta)_b{}^{i} - \frac{1}{2}\gamma^{A}{}_{0ij}\rhd_{\alpha A}{}^{B}\Phi(\gamma)_B{}^{ij} - \Phi(\nabla B)_\alpha\bigg) \\
&= -\nabla_0\epsilon_{\mathfrak{g}}{}^{\alpha}\Phi(\alpha)_\alpha - \epsilon_{\mathfrak{g}}{}^{\alpha}\bigg(B_{\beta 0i}f_{\alpha\gamma}{}^{\beta}\Phi(B)^{\gamma i} + C_{a0}\rhd_{\alpha b}{}^{a}\Phi(C)^b \\
&+ \beta_{a0i}\rhd_{\alpha b}{}^{a}\Phi(\beta)_b{}^{i} - \frac{1}{2}\gamma^{A}{}_{0ij}\rhd_{\alpha A}{}^{B}\Phi(\gamma)_B{}^{ij} - \Phi(\nabla B)_\alpha\bigg) \\
&= \nabla_0\epsilon_{\mathfrak{g}}{}^{\alpha}(\tilde{G}_1)_\alpha + \epsilon_{\mathfrak{g}}{}^{\alpha}(\tilde{G}_0)_\alpha.
\end{aligned} \tag{C.21}$$

Furthermore, it follows:

$$\begin{aligned}
\dot{\epsilon}_{\mathfrak{h}i}^{a}(G_1)_{\mathfrak{h}a}{}^{i} + \epsilon_{\mathfrak{h}}{}^{a}{}_{i}(G_0)_{\mathfrak{h}a}{}^{i} &= -\nabla_0\epsilon_{\mathfrak{h}}{}^{a}{}_{i}\Phi(\beta)_\alpha{}^{i} + \epsilon_{\mathfrak{h}}{}^{a}{}_{i}\big(C_{b0}\rhd_{\alpha a}{}^{b}\Phi(B)^{\alpha i} \\
&\quad - 2\beta^{b}{}_{0j}X_{(ab)}{}^{A}\Phi(\gamma)_A{}^{ij} + \Phi(\nabla C)_a{}^{i}\big) \\
&= \nabla_0\epsilon_{\mathfrak{h}}{}^{a}{}_{i}(\tilde{H}_1)_a{}^{i} + \epsilon_{\mathfrak{h}}{}^{a}{}_{i}(\tilde{H}_0)_a{}^{i},
\end{aligned} \tag{C.22}$$

$$\begin{aligned}
\dot{\epsilon}_{\mathfrak{n}}{}^{a}(G_1)_{\mathfrak{n}a} + \epsilon_{\mathfrak{n}}{}^{a}(G_0)_{\mathfrak{n}a} &= -\nabla_0\epsilon_{\mathfrak{n}}{}^{a}\Phi(C)_a + \epsilon_{\mathfrak{n}}{}^{a}(\beta_{b0i}\rhd_{\alpha a}{}^{b}\Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a) \\
&= \nabla_0\epsilon_{\mathfrak{n}}{}^{a}(\tilde{N}_1)_a + \epsilon_{\mathfrak{n}}{}^{a}(\tilde{N}_0)_a.
\end{aligned} \tag{C.23}$$

Finally, one gets:

$$\begin{aligned}
\frac{1}{2}\dot{\epsilon}_{\mathfrak{l}ij}^{A}(G_1)_{\mathfrak{l}A}{}^{ij} + \frac{1}{2}\epsilon_{\mathfrak{l}}^{A}{}_{ij}(G_0)_{\mathfrak{l}A}{}^{ij} &= \frac{1}{2}\dot{\epsilon}_{\mathfrak{l}ij}^{A}\Phi(\gamma)_A{}^{ij} + \frac{1}{2}\epsilon_{\mathfrak{l}}^{A}{}_{ij}\alpha^{\alpha}{}_0\rhd_{\alpha A}{}^{B}\Phi(\gamma)_B{}^{ij} \\
&\quad - \frac{1}{2}\epsilon_{\mathfrak{l}}^{A}{}_{ij}\Phi(\nabla D)_A{}^{ij} \\
&= \frac{1}{2}\nabla_0\epsilon_{\mathfrak{l}}^{A}{}_{ij}\Phi(\gamma)_A{}^{ij} - \frac{1}{2}\epsilon_{\mathfrak{l}}^{A}{}_{ij}\Phi(\nabla D)_A{}^{ij} \\
&= \frac{1}{2}\nabla_0\epsilon_{\mathfrak{l}}^{A}{}_{ij}(\tilde{L}_1)_A{}^{ij} + \frac{1}{2}\epsilon_{\mathfrak{l}}^{A}{}_{ij}(\tilde{L}_0)_A{}^{ij}.
\end{aligned} \tag{C.24}$$

Appendix D. Definitions of maps $\mathcal{T}$, $\mathcal{S}$, $\mathcal{D}$, $\mathcal{X}_1$, and $\mathcal{X}_2$

Given G-invariant symmetric non-degenerate bilinear forms in $\mathfrak{g}$ and $\mathfrak{h}$, one can define a bilinear antisymmetric map $\mathcal{T} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ by the rule:

$$\langle \mathcal{T}(\underline{h}_1, \underline{h}_2), \underline{g}\rangle_{\mathfrak{g}} = -\langle \underline{h}_1, \underline{g} \rhd \underline{h}_2\rangle_{\mathfrak{h}}, \quad \forall\ \underline{h}_1, \underline{h}_2 \in \mathfrak{h},\ \forall\ \underline{g} \in \mathfrak{g}.$$

Written in basis:

$$\mathcal{T}(t_a, t_b) = \mathcal{T}_{ab}{}^{\alpha}\tau_\alpha,$$

where the components of the map $\mathcal{T}$ are:

$$\mathcal{T}_{ab}{}^{\alpha} = -g_{ac}\rhd_{\beta b}{}^{c} g^{\alpha\beta}.$$

See [26] for more properties and the construction of 2*BF* invariant topological action using this map.

The transformations of the Lagrange multipliers and the 3*BF* invariant topological action is defined via maps

$$\mathcal{S} : \mathfrak{l}\times\mathfrak{l}\to\mathfrak{g}, \qquad \mathcal{X}_1 : \mathfrak{l}\times\mathfrak{h}\to\mathfrak{h}, \qquad \mathcal{X}_2 : \mathfrak{l}\times\mathfrak{h}\to\mathfrak{h}, \quad \mathcal{D} : \mathfrak{h}\times\mathfrak{h}\times\mathfrak{l}\to\mathfrak{g},$$

as it is defined in [13]. The map $\mathcal{S} : \mathfrak{l}\times\mathfrak{l}\to\mathfrak{g}$ is defined by the rule:

$$\langle \mathcal{S}(\underline{l}_1,\, \underline{l}_2),\, \underline{g}\rangle_{\mathfrak{g}} = -\langle \underline{l}_1,\, \underline{g}\rhd\underline{l}_2\rangle_{\mathfrak{l}}, \quad \forall\ \underline{l}_1,\ \forall\ \underline{l}_2\in\mathfrak{l},\ \forall\ \underline{g}\in\mathfrak{g}.$$

Written in the basis:

$$\mathcal{S}(T_A, T_B) = \mathcal{S}_{AB}{}^{\alpha}\tau_\alpha,$$

the defining relation for $\mathcal{S}$ becomes:

$$\mathcal{S}_{AB}{}^{\alpha} = -\rhd_{\beta[B}{}^{C}\ g_{A]C}g^{\alpha\beta}.$$

Given two $\mathfrak{l}$-valued forms η and ω, one can define a $\mathfrak{g}$-valued form:

$$\omega\wedge^{\mathcal{S}}\eta = \omega^A\wedge\eta^B\mathcal{S}_{AB}{}^{\alpha}\tau_\alpha.$$

Using this map, the transformations of the Lagrange multipliers under *L*-gauge are defined in [13].

Further, to define the transformations of the Lagrange multipliers under *H*-gauge transformations the bilinear map $\mathcal{X}_1 : \mathfrak{l}\times\mathfrak{h}\to\mathfrak{h}$ is defined:

$$\langle \mathcal{X}_1(\underline{l},\, \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}} = -\langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}, \quad \forall\ \underline{h}_1,\, \underline{h}_2\in\mathfrak{h},\ \forall\ \underline{l}\in\mathfrak{l},$$

and bilinear map $\mathcal{X}_2 : \mathfrak{l}\times\mathfrak{h}\to\mathfrak{h}$ by the rule:

$$\langle \mathcal{X}_2(\underline{l},\, \underline{h}_2),\, \underline{h}_1\rangle_{\mathfrak{h}} = -\langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}, \quad \forall\ \underline{h}_1,\, \underline{h}_2\in\mathfrak{h},\ \forall\ \underline{l}\in\mathfrak{l}.$$

As far as the bilinear maps $\mathcal{X}_1$ and $\mathcal{X}_2$ one can define the coefficients in the basis as:

$$\mathcal{X}_1(T_A, t_a) = \mathcal{X}_{1\ Aa}{}^b\, t_b, \qquad \mathcal{X}_2(T_A, t_a) = \mathcal{X}_{2\ Aa}{}^b\, t_b.$$

When written in the basis the defining relations for the maps $\mathcal{X}_1$ and $\mathcal{X}_2$ become:

$$\mathcal{X}_{1Ab}{}^c = -X_{ba}{}^B g_{AB} g^{ac}, \qquad \mathcal{X}_{2Ab}{}^c = -X_{ab}{}^B g_{AB} g^{ac}.$$

Given $\mathfrak{l}$-valued differential form ω and $\mathfrak{h}$-valued differential form η, one defines a $\mathfrak{h}$-valued form as:

$$\omega \wedge^{\mathcal{X}_1} \eta = \omega^A \wedge \eta^a \mathcal{X}_{1Aa}{}^b t_b, \qquad \omega \wedge^{\mathcal{X}_2} \eta = \omega^A \wedge \eta^a \mathcal{X}_{2Aa}{}^b t_b.$$

Finally, a trilinear map $\mathcal{D} : \mathfrak{h} \times \mathfrak{h} \times \mathfrak{l} \to \mathfrak{g}$ is needed:

$$\langle \mathcal{D}(\underline{h}_1, \underline{h}_2, \underline{l}), \underline{g} \rangle_{\mathfrak{g}} = -\langle \underline{l}, \{\underline{g} \triangleright \underline{h}_1, \underline{h}_2\} \rangle_{\mathfrak{l}}, \quad \forall\, \underline{h}_1, \underline{h}_2 \in \mathfrak{h},\ \forall\, \underline{l} \in \mathfrak{l},\ \forall\, \underline{g} \in \mathfrak{g},$$

One can define the coefficients of the trilinear map as:

$$\mathcal{D}(t_a, t_b, T_A) = \mathcal{D}_{abA}{}^{\alpha} \tau_{\alpha},$$

and the defining relation for the map $\mathcal{D}$ expressed in terms of coefficients becomes:

$$\mathcal{D}_{abA}{}^{\beta} = -\triangleright_{\alpha a}{}^c X_{cb}{}^B g_{AB} g^{\alpha\beta}.$$

Given two $\mathfrak{h}$-valued forms ω and η, and $\mathfrak{l}$-valued form ξ, the g-valued form is given by the formula:

$$\omega \wedge^{\mathcal{D}} \eta \wedge^{\mathcal{D}} \xi = \omega^a \wedge \eta^b \wedge \xi^A \mathcal{D}_{abA}{}^{\beta} \tau_{\beta}.$$

With these maps in hand, the transformations of the Lagrange multipliers under H-gauge transformations are defined, see [13].

Appendix E. Form-variations of all fields and momenta

The obtained gauge generator (55) is employed to calculate the form variations of variables and their corresponding canonical momenta, denoted as $A(t, \vec{x})$, using the following equation,

$$\delta_0 A(t, \vec{x}) = \{A(t, \vec{x}), G\}. \tag{E.1}$$

The computed form variations are given as follows:

$$
\begin{aligned}
\delta_0 B^{\alpha}{}_{0i} &= -\nabla_0 \epsilon_{\mathfrak{m}i}{}^{\alpha} + f_{\beta\gamma}{}^{\alpha} \epsilon_{\mathfrak{g}}{}^{\beta} B^{\gamma}{}_{0i} \\
&\quad + \epsilon_{\mathfrak{n}}{}^{a} \rhd_{\alpha a}{}^{b} \beta_{b0i} + \epsilon_{\mathfrak{h}i}{}^{a} \rhd_{\alpha a}{}^{b} C_{b0}, \\
\delta_0 B^{\alpha}{}_{ij} &= -2\nabla_{[i|} \epsilon_{\mathfrak{m}|j]}{}^{\alpha} + f_{\beta\gamma}{}^{\alpha} \epsilon_{\mathfrak{g}}{}^{\beta} B^{\gamma}{}_{ij} - \epsilon_{\mathfrak{l}ij}{}^{A} \rhd_{\alpha A}{}^{B} D_B \\
&\quad + \epsilon_{\mathfrak{n}}{}^{a} \rhd_{\alpha a}{}^{b} \beta_{bij} + 2\epsilon_{\mathfrak{h}}{}^{a}{}_{[j|} \rhd_{\alpha a}{}^{b} C_{b|i]}, \\
\delta_0 \alpha^{\alpha}{}_{0} &= -\nabla_0 \epsilon_{\mathfrak{g}}{}^{\alpha}, \\
\delta_0 \alpha^{\alpha}{}_{i} &= -\nabla_i \epsilon_{\mathfrak{g}}{}^{\alpha} - \partial_a{}^{\alpha} \epsilon_{\mathfrak{h}i}{}^{a}, \\
\delta_0 C^{a}{}_{0} &= -\nabla_0 \epsilon_{\mathfrak{n}}{}^{a} + \epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha b}{}^{a} C^{b}{}_{0}, \\
\delta_0 C^{a}{}_{i} &= -\nabla_i \epsilon_{\mathfrak{n}}{}^{a} + \epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha b}{}^{a} C^{b}{}_{i} \\
&\quad - \epsilon_{\mathfrak{m}i}{}^{\alpha} \partial^{a}{}_{\alpha} + 2\epsilon_{\mathfrak{h}}{}^{b}{}_{i} D_A X_{(bc)}{}^{A} g^{ac}, \\
\delta_0 \beta^{a}{}_{0i} &= -\nabla_0 \epsilon_{\mathfrak{h}i}{}^{a} + \epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha b}{}^{a} \beta_{b0i}, \\
\delta_0 \beta^{a}{}_{ij} &= -2\nabla_{[i|} \epsilon_{\mathfrak{h}|j]}{}^{a} + \epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha b}{}^{a} \beta^{b}{}_{ij} + \epsilon_{\mathfrak{l}ij}{}^{A} \delta_A{}^{a}, \\
\delta_0 \gamma^{A}{}_{0ij} &= \epsilon_{\mathfrak{g}}{}^{\alpha} \gamma^{B}{}_{0ij} \rhd_{\alpha B}{}^{A} + \nabla_0 \epsilon_{\mathfrak{l}ij}{}^{A} \\
&\quad - 4\epsilon_{\mathfrak{h}}{}^{a}{}_{[i|} \beta^{b}{}_{0|j]} X_{(ab)}{}^{A}, \\
\delta_0 \gamma^{A}{}_{ijk} &= \epsilon_{\mathfrak{g}}{}^{\alpha} \gamma^{B}{}_{ijk} \rhd_{\alpha B}{}^{A} + \nabla_i \epsilon_{\mathfrak{l}jk}{}^{A} \\
&\quad - \nabla_j \epsilon_{\mathfrak{l}ik}{}^{A} + \nabla_k \epsilon_{\mathfrak{l}ij}{}^{A} + 3! \epsilon_{\mathfrak{h}[i}{}^{a} \beta^{b}{}_{jk]} X_{(ab)}{}^{A}, \\
\delta_0 D^{A} &= \epsilon_{\mathfrak{n}}{}^{a} \delta_a{}^{A} + \epsilon_{\mathfrak{g}}{}^{\alpha} D^{B} \rhd_{\alpha B}{}^{A},
\end{aligned}
$$

$$
\begin{aligned}
\delta_0 \pi(B)_{\alpha}^{0i} &= f_{\alpha\beta}{}^{\gamma} \epsilon_{\mathfrak{g}}{}^{\beta} \pi(B)_{\gamma}{}^{0i}, \\
\delta_0 \pi(B)_{\alpha}{}^{ij} &= f_{\alpha\beta}{}^{\gamma} \epsilon_{\mathfrak{g}}{}^{\beta} \pi(B)_{\gamma}{}^{ij}, \\
\delta_0 \pi(\alpha)_{\alpha}^{0} &= f_{\alpha\beta}{}^{\gamma} \epsilon_{\mathfrak{m}}{}^{\beta}{}_{i} \pi(B)_{\gamma}{}^{0i} + f_{\alpha\beta}{}^{\gamma} \epsilon_{\mathfrak{g}}{}^{\beta} \pi(\alpha)_{\gamma}{}^{0} \\
&\quad + \rhd_{\alpha b}{}^{a} \epsilon_{\mathfrak{n}}{}^{b} \pi(C)_{a}^{0} + \rhd_{\alpha b}{}^{a} \epsilon_{\mathfrak{h}}{}^{b}{}_{i} \pi(\beta)_{a}^{i} \\
&\quad - \frac{1}{2} \rhd_{\alpha B}{}^{A} \epsilon_{\mathfrak{l}}{}^{B}{}_{ij} \pi(\gamma)_{A}{}^{0ij}, \\
\delta_0 \pi(\alpha)_{\alpha}^{i} &= f_{\alpha\beta}{}^{\gamma} \epsilon_{\mathfrak{m}}{}^{\beta}{}_{j} \pi(B)_{\gamma}{}^{ij} + f_{\alpha\beta}{}^{\gamma} \epsilon_{\mathfrak{g}}{}^{\beta} \pi(\alpha)_{\gamma}{}^{i} \\
&\quad + \rhd_{\alpha b}{}^{\alpha} \epsilon_{\mathfrak{n}}{}^{b} \pi(C)_{a}^{i} + \rhd_{\alpha b}{}^{\alpha} \epsilon_{\mathfrak{h}j}{}^{b} \pi(\beta)_{a}^{ij} \\
&\quad - \frac{1}{2} \rhd_{\alpha B}{}^{A} \epsilon_{\mathfrak{l}jk}{}^{B} \pi(\gamma)_{A}^{ijk} - \epsilon^{0ijk} \nabla_j \epsilon_{\mathfrak{m}\alpha k}, \\
&\quad - \frac{1}{2} \epsilon^{0ijk} \epsilon_{\mathfrak{n}}{}^{a} \rhd_{\alpha b}{}^{a} \beta^{b}{}_{jk}, \\
\delta_0 \pi(C)_{a}^{0} &= -\epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha a}{}^{b} \pi(C)_{b}^{0} + \epsilon_{\mathfrak{h}bi} \rhd_{\alpha a}{}^{b} \pi(B)^{\alpha 0i}, \\
\delta_0 \pi(C)_{a}^{i} &= -\epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha a}{}^{b} \pi(C)_{b}^{i} + \epsilon_{\mathfrak{h}bj} \rhd_{\alpha a}{}^{b} \pi(B)^{\alpha ij}, \\
\delta_0 \pi(\beta)_{a}{}^{0i} &= -\epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha a}{}^{b} \pi(\beta)_{b}{}^{0i} + \epsilon_{\mathfrak{n}b} \rhd_{\alpha a}{}^{b} \pi(B)^{\alpha 0i} \\
&\quad - 2\epsilon_{\mathfrak{h}j}{}^{b} X_{(ab)}{}^{A} \pi(\gamma)_{A}^{0ij}, \\
\delta_0 \pi(\beta)_{a}{}^{ij} &= -\epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha a}{}^{b} \pi(\beta)_{b}{}^{ij} + \epsilon_{\mathfrak{n}b} \rhd_{\alpha a}{}^{b} \pi(B)^{\alpha ij} \\
&\quad - 2\epsilon_{\mathfrak{h}k}{}^{b} X_{(ab)}{}^{A} \pi(\gamma)_{A}{}^{ijk} \\
&\quad + \epsilon^{0ijk} \nabla_k \epsilon_{\mathfrak{n}a} + \epsilon^{0ijk} \epsilon_{\mathfrak{h}k}{}^{a} \partial_{a\alpha}, \\
\delta_0 \pi(\gamma)_{A}{}^{0ij} &= -\epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha A}{}^{B} \pi(\gamma)_{B}{}^{0ij}, \\
\delta_0 \pi(\gamma)_{A}{}^{ijk} &= -\epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha A}{}^{B} \pi(\gamma)_{B}{}^{ijk} + \epsilon^{oijk} \delta_{aA} \epsilon_{\mathfrak{n}}{}^{a}, \\
\delta_0 \pi(D)_{A} &= -2\epsilon_{\mathfrak{h}i}{}^{a} X_{(ab)A} \pi(C)^{bi} \\
&\quad - \frac{1}{2} \epsilon_{\mathfrak{l}B}{}^{ij} \rhd_{\alpha A}{}^{B} \pi(B)_{0ij}^{\alpha} \\
&\quad - \epsilon_{\mathfrak{g}}{}^{\alpha} \rhd_{\alpha A}{}^{B} \pi(D)_{B}
\end{aligned}
\tag{E.2}
$$

Appendix F. Symmetry algebra calculations

To obtain the structure of the symmetry group of the 3*BF* action, as presented in the subsection 4.4, one has to calculate the commutators between the generators of all the symmetries, i.e. the *G*-, *H*-, *L*-, *M*-, and *N*-gauge symmetries. This process is described in the subsections 4.1–4.3, while details of the calculation which are not straightforward will be given in the following.

F.1. Commutator [H, H]

Let us derive the commutator of the generators of the *H*-gauge transformations, i.e. the equation (76). After transforming the variables under *H*-gauge transformations for the parameter $\epsilon_{\mathfrak{h}1}$ one obtains the following

$$\alpha' = \alpha - \partial\epsilon_{\mathfrak{h}1}, \tag{F.1}$$

$$\beta' = \beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1}, \tag{F.2}$$

$$\gamma' = \gamma + \{\beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1}, \epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}} + \{\epsilon_{\mathfrak{h}1}, \beta\}_{\mathrm{pf}}, \tag{F.3}$$

$$B' = B - (C - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}1})\wedge^{\mathcal{T}}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge^{\mathcal{D}}\epsilon_{\mathfrak{h}1}\wedge^{\mathcal{D}}D, \tag{F.4}$$

$$C' = C - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}1}, \tag{F.5}$$

$$D' = D, \tag{F.6}$$

and transforming the variables once more for the parameter $\epsilon_{\mathfrak{h}2}$ one obtains:

$$\begin{aligned}
\alpha'' &= \alpha - \partial\epsilon_{\mathfrak{h}1} - \partial\epsilon_{\mathfrak{h}2},\\
\beta'' &= \beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1} - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}-\partial\epsilon_{\mathfrak{h}2}}{\nabla}\epsilon_{\mathfrak{h}2} - \epsilon_{\mathfrak{h}2}\wedge\epsilon_{\mathfrak{h}2},\\
\gamma'' &= \gamma + \{\beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1}, \epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}} + \{\epsilon_{\mathfrak{h}1}, \beta\}_{\mathrm{pf}}\\
&\quad + \{\beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1} - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}-\partial\epsilon_{\mathfrak{h}2}}{\nabla}\epsilon_{\mathfrak{h}2} - \epsilon_{\mathfrak{h}2}\wedge\epsilon_{\mathfrak{h}2}, \epsilon_{\mathfrak{h}2}\}_{\mathrm{pf}}\\
&\quad + \{\epsilon_{\mathfrak{h}2}, \beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}},\\
B'' &= B - (C - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}1})\wedge^{\mathcal{T}}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge^{\mathcal{D}}\epsilon_{\mathfrak{h}1}\wedge^{\mathcal{D}}D\\
&\quad - (C - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}2} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}2})\wedge^{\mathcal{T}}\epsilon_{\mathfrak{h}2}\\
&\quad - \epsilon_{\mathfrak{h}2}\wedge^{\mathcal{D}}\epsilon_{\mathfrak{h}2}\wedge^{\mathcal{D}}D,\\
C'' &= C - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}2} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}2},\\
D'' &= D.
\end{aligned} \tag{F.7}$$

It is easy to see that for variables $\alpha^\alpha{}_\mu$, $C^a{}_\mu$ and D^A the following is obtained:

$$\begin{aligned} e^{\epsilon_{\mathfrak{h}1}\cdot H}e^{\epsilon_{\mathfrak{h}2}\cdot H}\alpha^\alpha{}_\mu &= e^{\epsilon_{\mathfrak{h}2}\cdot H}e^{\epsilon_{\mathfrak{h}1}\cdot H}\alpha^\alpha{}_\mu, \\ e^{\epsilon_{\mathfrak{h}1}\cdot H}e^{\epsilon_{\mathfrak{h}2}\cdot H}C^a{}_\mu &= e^{\epsilon_{\mathfrak{h}2}\cdot H}e^{\epsilon_{\mathfrak{h}1}\cdot H}C^a{}_\mu, \\ e^{\epsilon_{\mathfrak{h}1}\cdot H}e^{\epsilon_{\mathfrak{h}2}\cdot H}D^A &= e^{\epsilon_{\mathfrak{h}2}\cdot H}e^{\epsilon_{\mathfrak{h}1}\cdot H}D^A. \end{aligned} \tag{F.8}$$

For the remaining variables, $\beta^a{}_{\mu\nu}$, $\gamma^A{}_{\mu\nu\rho}$ and $B^\alpha{}_{\mu\nu}$, after subtracting (appendix F.1) and the corresponding equation where $\epsilon_{\mathfrak{h}1} \leftrightarrow \epsilon_{\mathfrak{h}2}$, one obtains:

$$\begin{aligned} \left(e^{\epsilon_{\mathfrak{h}1}\cdot H}e^{\epsilon_{\mathfrak{h}2}\cdot H} - e^{\epsilon_{\mathfrak{h}2}\cdot H}e^{\epsilon_{\mathfrak{h}1}\cdot H}\right)\frac{1}{2}\beta^a{}_{\mu\nu} &= \partial_b{}^\alpha\epsilon_{\mathfrak{h}2}{}^b{}_{[\mu|}\epsilon_{\mathfrak{h}1}{}^c{}_{|\nu]}\triangleright_{\alpha c}{}^a - \partial_b{}^\alpha\epsilon_{\mathfrak{h}1}{}^b{}_{[\mu|}\epsilon_{\mathfrak{h}2}{}^c{}_{|\nu]}\triangleright_{\alpha c}{}^a \\ &= 2\delta_A{}^a X_{(bc)}{}^A\epsilon_{\mathfrak{h}1}{}^b{}_{[\mu|}\epsilon_{\mathfrak{h}2}{}^c{}_{|\nu]} \\ &= \delta_A{}^a(\{\epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}2}\}_{\text{pf}} - \{\epsilon_{\mathfrak{h}2}\wedge\epsilon_{\mathfrak{h}1}\}_{\text{pf}})^A_{\mu\nu}, \\ \left(e^{\epsilon_{\mathfrak{h}1}\cdot H}e^{\epsilon_{\mathfrak{h}2}\cdot H} - e^{\epsilon_{\mathfrak{h}2}\cdot H}e^{\epsilon_{\mathfrak{h}1}\cdot H}\right)\frac{1}{3!}\gamma^A{}_{\mu\nu\rho} &= 2(\partial_{[\mu}\epsilon_{\mathfrak{h}1}{}^a{}_{\nu})\epsilon_{\mathfrak{h}2}{}^b{}_{\rho]}X_{(ab)}{}^A + 2\epsilon_{\mathfrak{h}1}{}^a{}_{[\nu}(\partial_\mu\epsilon_{\mathfrak{h}2}{}^b{}_{\rho]})X_{(ab)}{}^A \\ &\quad + 2\alpha^\alpha{}_{[\mu}\epsilon_{\mathfrak{h}1}{}^a{}_{\nu}\epsilon_{\mathfrak{h}2}{}^b{}_{\rho]}X_{(ab)}{}^B\triangleright_{\alpha B}{}^A \\ &= \nabla_{[\mu}(\{\epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}2}\}_{\text{pf}} - \{\epsilon_{\mathfrak{h}2}\wedge\epsilon_{\mathfrak{h}1}\}_{\text{pf}})^A{}_{\nu\rho]}, \\ \left(e^{\epsilon_{\mathfrak{h}1}\cdot H}e^{\epsilon_{\mathfrak{h}2}\cdot H} - e^{\epsilon_{\mathfrak{h}2}\cdot H}e^{\epsilon_{\mathfrak{h}1}\cdot H}\right)\frac{1}{2}B^\alpha{}_{\mu\nu} &= D^A\epsilon_{\mathfrak{h}2}{}^a_{[\mu|}\epsilon_{\mathfrak{h}1}{}^b_{|\nu]}(X_{1Aa}{}^c + X_{2Aa}{}^c)\mathcal{T}_{cb}{}^\alpha \\ &\quad - D^A\epsilon_{\mathfrak{h}1}{}^b_{[\mu|}\epsilon_{\mathfrak{h}2}{}^a_{|\nu]}(X_{1Ab}{}^c + X_{2Ab}{}^c)\mathcal{T}_{ca}{}^\alpha \\ &= -2D_A\epsilon_{\mathfrak{h}1}{}^a_{[\mu|}\epsilon_{\mathfrak{h}2}{}^b_{|\nu]}\,(X_{(ac)}{}^A\triangleright_{\alpha b}{}^c + X_{(bc)}{}^A\triangleright_{\alpha a}{}^c) \\ &= -2D_A\epsilon_{\mathfrak{h}1}{}^a_{[\mu|}\epsilon_{\mathfrak{h}2}{}^b_{|\nu]}X_{(ab)}{}^B\triangleright_{\alpha B}{}^A \\ &= \left(D\wedge^{\mathcal{S}}(\{\epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}2}\}_{\text{pf}} - \{\epsilon_{\mathfrak{h}2}\wedge\epsilon_{\mathfrak{h}1}\}_{\text{pf}}\right)^\alpha_{\mu\nu}. \end{aligned} \tag{F.9}$$

Comparing (F.8) and (F.9) with (72), one concludes that the commutator of two H-gauge transformations is the L-gauge transformation with the parameter $\epsilon_{\mathfrak{l}}^A{}_{\mu\nu} = 4\epsilon_{\mathfrak{h}1}{}^a{}_{[\mu|}\epsilon_{\mathfrak{h}2}{}^b{}_{|\nu]}X_{(ac)}{}^A$:

$$e^{\epsilon_{\mathfrak{h}1}\cdot H}e^{\epsilon_{\mathfrak{h}2}\cdot H} - e^{\epsilon_{\mathfrak{h}2}\cdot H}e^{\epsilon_{\mathfrak{h}1}\cdot H} = 2\left(\{\epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}2}\}_{\text{pf}} - \{\epsilon_{\mathfrak{h}2}\wedge\epsilon_{\mathfrak{h}1}\}_{\text{pf}}\right)\cdot\hat{L}. \tag{F.10}$$

F.2. Commutator [H, N]

Let us calculate the commutator between the generators of H-gauge transformation and N-gauge transformation, i.e. derive the equation (109). This is done by calculating the expressions

$$\left(e^{\epsilon_{\mathfrak{h}}\cdot H}e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N}e^{\epsilon_{\mathfrak{h}}\cdot H}\right)A, \tag{F.11}$$

for all variables A present in the theory. It is easy to see that for variables $\alpha^\alpha{}_\mu$, $\beta^a{}_{\mu\nu}$, $\gamma^A{}_{\mu\nu\rho}$, and D^A the following is obtained:

$$\begin{aligned} e^{\epsilon_{\mathfrak{h}}\cdot H}e^{\epsilon_{\mathfrak{n}}\cdot N}\alpha^{\alpha}{}_{\mu} &= e^{\epsilon_{\mathfrak{n}}\cdot N}e^{\epsilon_{\mathfrak{h}}\cdot H}\alpha^{\alpha}{}_{\mu},\\ e^{\epsilon_{\mathfrak{h}}\cdot H}e^{\epsilon_{\mathfrak{n}}\cdot N}\beta^{a}{}_{\mu\nu} &= e^{\epsilon_{\mathfrak{n}}\cdot N}e^{\epsilon_{\mathfrak{h}}\cdot H}\beta^{a}{}_{\mu\nu},\\ e^{\epsilon_{\mathfrak{h}}\cdot H}e^{\epsilon_{\mathfrak{n}}\cdot N}\gamma^{A}{}_{\mu\nu\rho} &= e^{\epsilon_{\mathfrak{n}}\cdot N}e^{\epsilon_{\mathfrak{h}}\cdot H}\gamma^{A}{}_{\mu\nu\rho},\\ e^{\epsilon_{\mathfrak{h}}\cdot H}e^{\epsilon_{\mathfrak{n}}\cdot N}D^{A} &= e^{\epsilon_{\mathfrak{n}}\cdot N}e^{\epsilon_{\mathfrak{h}}\cdot H}D^{A}. \end{aligned} \tag{F.12}$$

For the remaining variables, $B^{\alpha}{}_{\mu\nu}$ and $C^{a}{}_{\mu}$, after the H-gauge transformation one obtains the following:

$$B' = B - (C - D\wedge^{\chi_1}\epsilon_{\mathfrak{h}} - D\wedge^{\chi_2}\epsilon_{\mathfrak{h}})\wedge^{\tau}\epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}}\wedge^{\mathcal{D}}\epsilon_{\mathfrak{h}}\wedge^{\mathcal{D}}D, \tag{F.13}$$

$$C' = C - D\wedge^{\chi_1}\epsilon_{\mathfrak{h}} - D\wedge^{\chi_2}\epsilon_{\mathfrak{h}}. \tag{F.14}$$

Next, transforming those variables with N-gauge transformation one obtains:

$$\begin{aligned} B'' &= B' - \beta'\wedge^{\mathcal{T}}\epsilon_{\mathfrak{n}}\\ &= B - (C - D\wedge^{\chi_1}\epsilon_{\mathfrak{h}} - D\wedge^{\chi_2}\epsilon_{\mathfrak{h}})\wedge^{\tau}\epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}}\wedge^{\mathcal{D}}\epsilon_{\mathfrak{h}}\wedge^{\mathcal{D}}D\\ &\quad - (\beta - \overset{\{\alpha^{\alpha}-\partial_a{}^{\alpha}\epsilon_{\mathfrak{h}}^{a}\}}{\nabla}\epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}}\wedge\epsilon_{\mathfrak{h}})\wedge^{\mathcal{T}}\epsilon_{\mathfrak{n}},\\ C'' &= C' - \overset{\{\alpha^{\alpha}-\partial_a{}^{\alpha}\epsilon_{\mathfrak{h}}^{a}\}}{\nabla}\epsilon_{\mathfrak{n}}\\ &= C - D\wedge^{\chi_1}\epsilon_{\mathfrak{h}} - D\wedge^{\chi_2}\epsilon_{\mathfrak{h}} - \overset{\{\alpha^{\alpha}-\partial_a{}^{\alpha}\epsilon_{\mathfrak{h}}^{a}\}}{\nabla}\epsilon_{\mathfrak{n}}. \end{aligned} \tag{F.15}$$

Let us now exchange the order of transformations, and first transform the variables with N-gauge transformation,

$$B^{\cdot} = B - \beta\wedge^{\mathcal{T}}\epsilon_{\mathfrak{n}}, \tag{F.16}$$

$$C^{\cdot} = C - \nabla\epsilon_{\mathfrak{n}}, \tag{F.17}$$

and then with H-gauge transformation:

$$\begin{aligned} B^{\cdot\cdot} &= B^{\cdot} - (C^{\cdot} - D^{\cdot}\wedge^{\chi_1}\epsilon_{\mathfrak{h}} - D^{\cdot}\wedge^{\chi_2}\epsilon_{\mathfrak{h}})\wedge^{\tau}\epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}}\wedge^{\mathcal{D}}\epsilon_{\mathfrak{h}}\wedge^{\mathcal{D}}D^{\cdot}\\ &= B - \beta\wedge^{\mathcal{T}}\epsilon_{\mathfrak{n}} - \big(C - \nabla\epsilon_{\mathfrak{n}} - (D+\delta\epsilon_{\mathfrak{n}})\wedge^{\chi_1}\epsilon_{\mathfrak{h}}\\ &\quad - (D+\delta\epsilon_{\mathfrak{n}})\wedge^{\chi_2}\epsilon_{\mathfrak{h}}\big)\wedge^{\tau}\epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}}\wedge^{\mathcal{D}}\epsilon_{\mathfrak{h}}\wedge^{\mathcal{D}}(D+\delta\epsilon_{\mathfrak{n}}),\\ C^{\cdot\cdot} &= C^{\cdot} - D^{\cdot}\wedge^{\chi_1}\epsilon_{\mathfrak{h}} - D^{\cdot}\wedge^{\chi_2}\epsilon_{\mathfrak{h}}\\ &= C - \nabla\epsilon_{\mathfrak{n}} - (D+\delta\epsilon_{\mathfrak{n}})\wedge^{\chi_1}\epsilon_{\mathfrak{h}} - (D+\delta\epsilon_{\mathfrak{n}})\wedge^{\chi_2}\epsilon_{\mathfrak{h}}. \end{aligned} \tag{F.18}$$

After subtracting (F.15) and (F.18) one obtains:

$$
\begin{aligned}
\left(e^{\epsilon_{\mathfrak{h}}\cdot H}e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N}e^{\epsilon_{\mathfrak{h}}\cdot H}\right) B^{\alpha} &= \nabla\epsilon_{\mathfrak{n}}^{a} \wedge \epsilon_{\mathfrak{h}}{}^{b}\mathcal{T}_{ab}{}^{\alpha} + \delta^{A}{}_{a}\epsilon_{\mathfrak{n}}^{a}\epsilon_{\mathfrak{h}}{}^{b} \wedge \epsilon_{\mathfrak{h}}{}^{d}X_{1Ab}{}^{c}\mathcal{T}_{cd}{}^{\alpha} \\
&\quad + \delta^{A}{}_{a}\epsilon_{\mathfrak{n}}^{a}\epsilon_{\mathfrak{h}}{}^{b} \wedge \epsilon_{\mathfrak{h}}{}^{d}X_{2Ab}{}^{c}\mathcal{T}_{cd}{}^{\alpha} - \epsilon_{\mathfrak{h}}{}^{a} \wedge \epsilon_{\mathfrak{h}}{}^{b}\delta_{A}{}^{c}\epsilon_{\mathfrak{n}}{}^{c}D_{Aab}{}^{\alpha}, \\
&\quad - \nabla\epsilon_{\mathfrak{h}}^{a} \wedge \epsilon_{\mathfrak{n}}^{b}\mathcal{T}_{ab}{}^{\alpha} + \partial_{a}{}^{\beta}\epsilon_{\mathfrak{h}}^{a} \rhd_{\beta c}{}^{b}\epsilon_{\mathfrak{h}}^{c}\epsilon_{\mathfrak{n}}^{d}\mathcal{T}_{bd}{}^{\alpha} - \epsilon_{\mathfrak{h}}^{a} \wedge \epsilon_{\mathfrak{h}}^{b} f_{ab}{}^{c}\epsilon_{\mathfrak{n}}^{d}\mathcal{T}_{cd}{}^{\alpha}, \\
\left(e^{\epsilon_{\mathfrak{h}}\cdot H}e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N}e^{\epsilon_{\mathfrak{h}}\cdot H}\right) C^{c} &= -(\delta^{A}{}_{a}\epsilon_{\mathfrak{n}}^{a}) \wedge \epsilon_{\mathfrak{h}}^{b}X_{1Ab}{}^{c} - (\delta^{A}{}_{a}\epsilon_{\mathfrak{n}}^{a}) \wedge \epsilon_{\mathfrak{h}}^{b}X_{2Ab}{}^{c} - \partial_{a}{}^{\beta}\epsilon_{\mathfrak{h}}^{a} \rhd_{\beta b}{}^{c}\epsilon_{\mathfrak{n}}^{b},
\end{aligned}
\tag{F.19}
$$

where after using the definitions of the maps $\mathcal{T}$, $\mathcal{D}$, χ_1, and χ_2 one obtains the result

$$
\begin{aligned}
\left(e^{\epsilon_{\mathfrak{h}}\cdot H}e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N}e^{\epsilon_{\mathfrak{h}}\cdot H}\right) B^{\alpha} &= \nabla\epsilon_{\mathfrak{n}}^{a} \wedge \epsilon_{\mathfrak{h}}{}^{b}\mathcal{T}_{ab}{}^{\alpha} - \nabla\epsilon_{\mathfrak{h}}^{a} \wedge \epsilon_{\mathfrak{n}}^{b}\mathcal{T}_{ab}{}^{\alpha} \\
&= \nabla(\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}})^{\alpha}, \\
\left(e^{\epsilon_{\mathfrak{h}}\cdot H}e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N}e^{\epsilon_{\mathfrak{h}}\cdot H}\right) C^{c} &= \partial^{c}{}_{\alpha}(\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}})^{\alpha},
\end{aligned}
\tag{F.20}
$$

Comparing (F.12) and (F.20) with (85), one obtains that:

$$
\left(e^{\epsilon_{\mathfrak{h}}\cdot H}e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N}e^{\epsilon_{\mathfrak{h}}\cdot H}\right) = -(\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}}) \cdot M. \tag{F.21}
$$

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Construction and examples of higher gauge theories*

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Abstract

We provide several examples of higher gauge theories, constructed as generalizations of a BF model to $2BF$ and $3BF$ models with constraints. Using the framework of higher category theory, we introduce appropriate 2-groups and 3-groups, and construct the actions for the corresponding constrained $2BF$ and $3BF$ theories. In this way, we can construct actions which describe the correct dynamics of Yang-Mills, Klein-Gordon, Dirac, Weyl, and Majorana fields coupled to Einstein-Cartan gravity. Each action is naturally split into a topological sector and a sector with simplicity constraints. The properties of the higher gauge group structure opens up a possibility of a nontrivial unification of all fields.

1. Introduction

The quantization of the gravitational field is one of the fundamental open problems in modern physics. There are various approaches to this problem, some of which have developed into vast research frameworks. One of such frameworks is the Loop Quantum Gravity approach, which aims to establish a nonperturbative quantization of gravity, both canonically and covariantly [1, 2, 3]. The covariant approach is slightly more general, and

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focuses on providing a possible rigorous definition of the path integral for the gravitational field,

$$Z = \int \mathcal{D}g \; e^{iS[g]} \,. \tag{1}$$

This is done by considering a triangulation of a spacetime manifold, and defining the path integral as a discrete state sum of the gravitational field configurations living on the simplices in the triangulation. This quantization technique is known as the *spinfoam* quantization method, and is performed via the following three steps:

(1) one writes the classical action $S[g]$ as a constrained BF action;

(2) one uses the Lie group structure, underlying the topological sector of the action, to define a triangulation-independent state sum Z;

(3) one imposes the simplicity constraints on the state sum, promoting it into a triangulation-dependent state sum, which serves as a definition for the path integral (1).

So far, this quantization prescription has been implemented for various choices of the gravitational action, of the Lie group, and of the spacetime dimension. For example, in 3 dimensions, historically the first spinfoam model is known as the Ponzano-Regge model [4]. In 4 dimensions there are multiple models, depending on the choice of the Lie group and the way one imposes the simplicity constraints [5, 6, 7, 8, 9]. While these models do give a definition for the gravitational path integral, none of them are able to consistently include matter fields. Including the matter fields has so far had limited success [10], mainly due to the absence of the tetrad fields from the topological sector of the theory.

In order to resolve this issue, a new approach has been developed, using the framework of *higher gauge theory* (see [11] for a review). In particular, one uses the idea of a *categorical ladder* to generalize the BF action (based on a Lie group) into a $2BF$ action (based on the so-called 2-group structure). A suitable choice of the *Poincaré 2-group* introduces the needed tetrad fields into the topological sector of the action [12]. While this result opened up a possibility to couple matter fields to gravity, the matter fields could not be naturally expressed using the underlying algebraic structure of a 2-group, rendering the spinfoam quantization method inapplicable. Namely, the matter sector could indeed be added to the classical action, but could not be expressed itself as a constrained $2BF$ theory, which means that the steps 1–3 above could not be performed for the matter sector of the action, but only for gravity.

This final issue has recently been resolved in [13], by passing from the 2-group structure to the 3-group structure, generalizing the action one step further in the categorical ladder. This generalization naturally gives rise to the so-called $3BF$ action, which turns out to be suitable for a unified description of both gravity and matter fields. The steps of the categorical ladder and their corresponding structures are summarized as follows:

categorical structure	algebraic structure	linear structure	topological action	degrees of freedom
Lie group	Lie group	Lie algebra	BF theory	gauge fields
Lie 2-group	Lie crossed module	differential Lie crossed module	$2BF$ theory	tetrad fields
Lie 3-group	Lie 2-crossed module	differential Lie 2-crossed module	$3BF$ theory	scalar and fermion fields

The purpose of this paper is to give a systematic overview of the constructions of classical BF, $2BF$ and $3BF$ actions, both pure and constrained, in order to demonstrate the categorical ladder procedure and the construction of higher gauge theories. In other words, we focus on the step 1 of the spinfoam quantization programme.

The layout of the paper is as follows. Section 2 deals with models based on a BF theory. First we discuss the pure, topological BF theory, and then pass on to the the physically more interesting Yang-Mills theory in Minkowski spacetime and the Plebanski formulation of general relativity. In Section 3 we study the first step in the categorical ladder, namely models based on the $2BF$ theory. After introducing the pure $2BF$ theory, we study the relevant formulation of general relativity [12], and then the coupled Einstein-Yang-Mills theory. Then, in Section 4 we perform the second step in the categorical ladder, passing on to models based on the $3BF$ theory. After the introduction of the pure $3BF$ model, we construct constrained $3BF$ actions for the cases of Klein-Gordon, Dirac, Weyl and Majorana fields, all coupled to the Einstein-Cartan gravity in the standard way. As we shall see, the scalar and fermion fields will be *naturally associated to a new gauge group*, generalizing the purpose of a gauge group in the Yang-Mills theory, which opens up a possibility of an algebraic classification of matter fields. Finally, Section 5 contains a discussion and conclusions.

The notation and conventions are as follows. The local Lorentz indices are denoted by the Latin letters $a, b, c, \dots$, take values $0, 1, 2, 3$, and are raised and lowered using the Minkowski metric η_{ab} with signature $(-,+,+,+)$. Spacetime indices are denoted by the Greek letters $\mu, \nu, \dots$, and are raised and lowered by the spacetime metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ are the tetrad fields. The inverse tetrad is denoted as $e^\mu{}_a$. All other indices that appear in the paper are dependent on the context, and their usage is explicitly defined in the text where they appear. We work in the natural system of units where $c = \hbar = 1$, and $G = l_p^2$, where l_p is the Planck length.

2. BF theory

We begin with a short review of BF theories. See [14, 15, 16] for additional information.

2.1. Pure BF theory

Given a Lie group G, and denoting its corresponding Lie algebra as $\mathfrak{g}$, one introduces the pure BF action as follows (we limit ourselves to the physically relevant case of 4-dimensional spacetime manifolds $\mathcal{M}_4$):

$$S_{BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} \,. \tag{2}$$

Here, $\mathcal{F} \equiv \mathrm{d}\alpha + \alpha \wedge \alpha$ is the curvature 2-form for the algebra-valued connection 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$, and $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ is a Lagrange multiplier 2-form, while $\langle \_\,, \_ \rangle_{\mathfrak{g}}$ denotes a G-invariant bilinear symmetric nondegenerate form.

One can see from (2) that the action is diffeomorphism invariant, and it is also gauge invariant with respect to G, provided that B transforms as a scalar with respect to G.

Varying the action (2) with respect to B^β and α^β, where the index β is the group G index (which counts the generators of $\mathfrak{g}$), one obtains the following equations of motion,

$$\mathcal{F}^\beta = 0\,, \qquad \nabla B^\beta \equiv \mathrm{d}B^\beta + f_{\gamma\delta}{}^\beta \alpha^\gamma \wedge B^\delta = 0\,, \tag{3}$$

where $f_{\gamma\delta}{}^\beta$ are the structure constants of the Lie group G. From the first equation of motion, one immediately sees that α is a flat connection, meaning that $\alpha = 0$ up to gauge transformations. Given this, the second equation of motion implies that B is constant. Therefore, there are no local propagating degrees of freedom, and the theory is called *topological*.

2.2. Yang-Mills theory

In physics one is usually interested in theories which are not topological, i.e., which have local propagating degrees of freedom. As a rule of thumb, one recognizes that the theory does have local propagating degrees of freedom if one of the equations of motion is a second-order partial differential equation, usually featuring a D'Alambertian operator $\Box$ in some form. In order to transform the pure BF action into such a theory, one adds an additional term to the action, commonly called the *simplicity constraint.* The resulting action is called a *constrained BF theory.* A nice example is the Yang-Mills theory for the $SU(N)$ group in Minkowski spacetime, which can be rewritten as a constrained BF theory in the following way:

$$\begin{aligned} S \;=\; & \int B_I \wedge F^I + \lambda^I \wedge \Big(B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b \Big) \\ & + \zeta^{abI} \Big(M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - g_{IJ} F^J \wedge \delta_a \wedge \delta_b \Big)\,. \end{aligned} \tag{4}$$

Here $F \equiv \mathrm{d}A + A \wedge A$ is again the curvature 2-form for the connection $A \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{su}(N))$, and $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{su}(N))$ is the Lagrange multiplier

2-form. The Killing form $g_{IJ} \equiv \langle \tau_I \,, \tau_J \rangle_{\mathfrak{su}(N)} \propto f_{IK}{}^L f_{JL}{}^K$ is used to raise and lower the indices $I, J, \dots$ which count the generators of $SU(N)$, while $f_{IJ}{}^K$ are the structure constants for the $\mathfrak{su}(N)$ algebra. In addition to the topological $B \wedge F$ term, there are also two simplicity constraint terms present, featuring two Lagrange multipliers, a 2-form λ^I and a 0-form ζ^{abI}. The 0-form M_{abI} is also a Lagrange multiplier, while g is the coupling constant for the Yang-Mills theory.

Finally, δ^a is a nondynamical 1-form, such that there exists a global coordinate frame in which its components are equal to the Kronecker symbol $\delta^a{}_\mu$ (hence the notation δ^a). The 1-form δ^a plays the role of a background field, and defines the global spacetime metric, via the equation

$$\eta_{\mu\nu} = \eta_{ab} \delta^a{}_\mu \delta^b{}_\nu \,, \tag{5}$$

where $\eta_{ab} \equiv \text{diag}(-1, +1, +1, +1)$ is the Minkowski metric. Since the coordinate system is global, the spacetime manifold $\mathcal{M}_4$ is understood to be flat. The indices $a, b, \dots$ are local Lorentz indices, taking values $0, \dots, 3$. Note that the field δ^a has all the properties of the tetrad 1-form e^a in the flat Minkowski spacetime. Also note that the action (4) is manifestly diffeomorphism invariant and gauge invariant with respect to $SU(N)$, but not background independent, due to the presence of δ^a.

Varying the action (4) with respect to the variables ζ^{abI}, M_{abI}, A^I, B_I, and λ^I, respectively (but not with respect to the background field δ^a), we obtain the equations of motion:

$$M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - F_I \wedge \delta_a \wedge \delta_b = 0 \,, \tag{6}$$

$$-\frac{12}{g} \lambda^I \wedge \delta^a \wedge \delta^b + \zeta^{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f = 0 \,, \tag{7}$$

$$-\mathrm{d}B_I + f_{JI}{}^K B_K \wedge A^J + \mathrm{d}(\zeta^{ab}{}_I \delta_a \wedge \delta_b) - f_{JI}{}^K \zeta^{ab}{}_K \delta_a \wedge \delta_b \wedge A^J = 0 \,, \tag{8}$$

$$F_I + \lambda_I = 0 \,, \tag{9}$$

$$B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b = 0 \,, \tag{10}$$

From the equations (6), (7), (9) and (10) one obtains the multipliers as algebraic functions of the field strength $F^I{}_{\mu\nu}$ for the dynamical field A^I:

$$\begin{aligned} M_{abI} &= \frac{1}{48} \varepsilon_{abcd} F_I{}^{cd} \,, & \zeta^{abI} &= \frac{1}{4g} \varepsilon^{abcd} F^I{}_{cd} \,, \\ \lambda_{Iab} &= F_{Iab} \,, & B_{Iab} &= \frac{1}{2g} \varepsilon_{abcd} F_I{}^{cd} \,. \end{aligned} \tag{11}$$

Here we used the notation $F_{Iab} = F_{I\mu\nu}\delta_a{}^\mu\delta_b{}^\nu$, and similarly for other variables, where we exploited the fact that $\delta^a{}_\mu$ is invertible. Using these equations and the differential equation (8) one obtains the equation of motion for gauge field $A^I{}_\mu$,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0\,. \tag{12}$$

This is precisely the classical equation of motion for the free Yang-Mills theory. Note that this is a second-order partial differential equation for the field $A^I{}_\mu$, and moreover contains the $\Box$ operator in the first term.

In addition to the Yang-Mills theory, one can easily extend the action (4) in order to describe the massive vector field and obtain the Proca equation of motion. This is done by adding a mass term

$$-\frac{1}{4!}m^2 A_{I\mu}A^I{}_\nu\eta^{\mu\nu}\varepsilon_{abcd}\delta^a\wedge\delta^b\wedge\delta^c\wedge\delta^d \tag{13}$$

to the action (4). Of course, this term explicitly breaks the $SU(N)$ gauge symmetry of the action.

2.3. Plebanski general relativity

The second example of the constrained BF theory is the Plebanski action for general relativity [16, 14]. Using the Lorentz group $SO(3,1)$ as a gauge group, one constructs a constrained BF action as

$$S = \int_{\mathcal{M}_4} B_{ab}\wedge R^{ab} + \phi_{abcd}B^{ab}\wedge B^{cd}\,. \tag{14}$$

Here R^{ab} is the curvature 2-form for the spin connection ω^{ab}, B_{ab} is the usual Lagrange multiplier 2-form, while ϕ_{abcd} is the additional Lagrange multiplier 0-form multiplying the term $B^{ab}\wedge B^{cd}$ to form a simplicity constraint. It can be shown that the variation of this action with respect to B_{ab}, ω^{ab} and ϕ_{abcd} gives rise to the equations of motion of vacuum general relativity. However, in this model the tetrad fields appear only as a solution of the simplicity constraint equation of motion $B^{ab}\wedge B^{cd} = 0$. Therefore, being intrinsically on-shell objects, the tetrad fields are not present in the action itself and cannot be quantized. This renders the Plebanski model unsuitable for coupling of matter fields to gravity [10, 12, 20]. Nevertheless, regarded as a model for pure gravity, the Plebanski model has been successfully quantized in the context of spinfoam models [8, 9, 1, 2].

3. $2BF$ theory

In this section we perform the first step of the *categorical ladder*, generalizing the algebraic notion of a group to the notion of a 2-group. This leads to the generalization of the BF theory to the $2BF$ theory, also sometimes called $BFCG$ theory [11, 17, 18, 19].

3.1. Pure $2BF$ theory

In order to circumvent the issue of tetrad fields not being present in the Plebanski action, in the context of higher category theory [11] a recent promising approach has been developed [12, 21, 22, 23, 20, 24]. As an essential ingredient, let us first give a short review of the 2-group formalism.

Within the framework of category theory, the group as an algebraic structure can be understood as a category with only one object and invertible morphisms [11]. Additionally, the notion of a category can be generalized to the so-called *higher categories*, which have not only objects and morphisms, but also 2-morphisms (morphisms between morphisms), and so on. This process of generalization is called the *categorical ladder*. Using this process, one can introduce the notion of a 2*-group* as a 2-category consisting of only one object, where all the morphisms and all 2-morphisms are invertible. It has been shown that every strict 2-group is equivalent to a *crossed module* $(H \stackrel{\partial}{\to} G\,, \rhd)$, see [13] for detailed definitions. Here G and H are groups, ∂ is a homomorphism from H to G, while $\rhd : G \times H \to H$ is an action of G on H.

Similarly to the case of an ordinary Lie group G which has a naturally associated notion of a connection α, giving rise to a BF theory, the 2-group structure has a naturally associated notion of a 2-connection $(\alpha\,,\beta)$, described by the usual $\mathfrak{g}$-valued 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$ and an $\mathfrak{h}$-valued 2-form $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$, where $\mathfrak{h}$ is a Lie algebra of the Lie group H. The 2-connection gives rise to the so-called *fake* 2*-curvature* $(\mathcal{F},\mathcal{G})$, given as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = d\beta + \alpha \wedge^{\rhd} \beta\,. \tag{15}$$

Here $\alpha \wedge^{\rhd} \beta$ means that α and β are multiplied as forms using $\wedge$, and simultaneously multiplied as algebra elements using $\rhd$, see [13]. The curvature pair $(\mathcal{F},\mathcal{G})$ is called "fake" because of the presence of the additional term $\partial\beta$ in the definition of $\mathcal{F}$ [11].

Using the structure of a 2-group, or equivalently the crossed module, one can generalize the BF action to the so-called $2BF$ action, defined as follows [17, 18]:

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}}\,. \tag{16}$$

Here the 2-form $B \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{g})$ and the 1-form $C \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{h})$ are Lagrange multipliers. Also, $\langle\_\,,\_\rangle_{\mathfrak{g}}$ and $\langle\_\,,\_\rangle_{\mathfrak{h}}$ denote the G-invariant bilinear symmetric nondegenerate forms for the algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. As a consequence of the axiomatic structure of a crossed module (see [13]), the bilinear form $\langle\_\,,\_\rangle_{\mathfrak{h}}$ is H-invariant as well. See [17, 18] for review and references.

Similarly to the BF action, the $2BF$ action is also topological, which can be seen from equations of motion. Varying with respect to B^{α} and C^{a} one obtains

$$\mathcal{F}^{\alpha} = 0\,, \qquad \mathcal{G}^{a} = 0\,, \tag{17}$$

where indices a count the generators of the group H. Varying with respect to α^α and β^a one obtains the equations for the multipliers,

$$\mathrm{d}B_\alpha + f_{\alpha\beta}{}^\gamma B_\gamma \wedge \alpha^\beta - \rhd_{\alpha a}{}^b C_b \wedge \beta^a = 0\,, \tag{18}$$

$$\mathrm{d}C_a - \partial_a{}^\alpha B_\alpha + \rhd_{\alpha a}{}^b C_b \wedge \alpha^\alpha = 0\,. \tag{19}$$

We can again see that the equations of motion are only first-order and have only very simple solutions (note that this is not a sufficient argument for the absence of local propagating degrees of freedom — a counterexample is the Dirac equation, being a first-order partial differential equation which *does* have propagating degrees of freedom). One can additionally use the Hamiltonian analysis to rigorously demonstrate that there are no local propagating degrees of freedom [22, 23]. Thus the $2BF$ theory is also topological.

3.2. General relativity

An important example of a crossed module structure is a vector space V equipped with an isometry group O. Namely, V can be regarded as an Abelian Lie group with addition as a group operation, so that a representation of O on V is an action $\rhd$ of O on the group V, giving rise to the crossed module $(V \overset{\partial}{\to} O\,, \rhd)$, where the homomorphism ∂ is chosen to be trivial (it maps every element of V into a unit of O).

We can employ this construction to introduce the *Poincaré 2-group*. One constructs a crossed module by choosing

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,. \tag{20}$$

The map ∂ is trivial, while $\rhd$ is a natural action of $SO(3,1)$ on $\mathbb{R}^4$, defined by the equation

$$M_{ab} \rhd P_c = \eta_{[bc} P_{a]}\,, \tag{21}$$

where M_{ab} and P_a are the generators of groups $SO(3,1)$ and $\mathbb{R}^4$, respectively. The action $\rhd$ of $SO(3,1)$ on itself is given via conjugation. At the level of the algebra, conjugation reduces to the action via the adjoint representation, so that

$$M_{ab} \rhd M_{cd} = [\,M_{ab}\,,\, M_{cd}\,] \equiv \eta_{ad}M_{bc} - \eta_{ac}M_{bd} + \eta_{bc}M_{ad} - \eta_{bd}M_{ac}\,. \tag{22}$$

The 2-connection (α, β) is given by the algebra-valued differential forms

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \tag{23}$$

where ω^{ab} is called the spin connection. The corresponding 2-curvature in this case is given by

$$\begin{aligned} \mathcal{F} &= (\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}) M_{ab} &&\equiv R^{ab} M_{ab}\,, \\ \mathcal{G} &= (\mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b) P_a &&\equiv \nabla \beta^a P_a \equiv G^a P_a\,, \end{aligned} \tag{24}$$

Note that, since ∂ is trivial, the fake curvature is the same as ordinary curvature. Introducing the bilinear forms

$$\langle M_{ab}\,, M_{cd}\rangle_{\mathfrak{g}} = \eta_{a[c}\eta_{bd]}\,, \qquad \langle P_a\,, P_b\rangle_{\mathfrak{h}} = \eta_{ab}\,, \tag{25}$$

one can show that 1-forms C^a transform in the same way as the tetrad 1-forms e^a under the Lorentz transformations and diffeomorphisms, so the fields C^a can be identified with the tetrads. Then one can rewrite the pure $2BF$ action (16) for the Poincaré 2-group as

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a\,. \tag{26}$$

Note that the above step of recognizing that $C^a \equiv e^a$ was crucial, since we now see that the tetrad fields are explicitly present in the $2BF$ action for the Poincaré 2-group.

In order to promote (26) to an action for general relativity, we add a convenient simplicity constraint term:

$$S = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a - \lambda_{ab} \wedge \Big(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c \wedge e_d\Big)\,. \tag{27}$$

Here λ_{ab} is a Lagrange multiplier 2-form associated to the simplicity constraint term, and l_p is the Planck length. Note that the term "simplicity constraint" derives its name from the fact that the constraint imposes the property of *simplicity* on B^{ab} — a 2-form is said to be *simple* if it can be written as an exterior product of two 1-forms.

Varying the action (27) with respect to B_{ab}, e_a, ω_{ab}, β_a and λ_{ab}, we obtain the following equations of motion:

$$R_{ab} - \lambda_{ab} = 0\,, \tag{28}$$

$$\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d = 0\,, \tag{29}$$

$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} = 0\,, \tag{30}$$

$$\nabla e_a = 0\,, \tag{31}$$

$$B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c \wedge e_d = 0\,. \tag{32}$$

Given this system of equations, all fields can be algebraically determined in terms of the tetrads ${e^a}_\mu$, as follows. From the equations (31) and (32) we obtain that $\nabla B^{ab} = 0$, from which it follows, using the equation (30), that

$e_{[a} \wedge \beta_{b]} = 0$. Assuming that the tetrads are nondegenerate, $e \equiv \det(e^a{}_\mu) \neq 0$, it can be shown that this is equivalent to $\beta^a = 0$ [12]. Therefore, from the equations (28), (30), (31) and (32) we obtain

$$\lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu}\,, \quad \beta^a{}_{\mu\nu} = 0\,, \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \quad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,. \tag{33}$$

Here the Ricci rotation coefficients are defined as

$$\triangle^{ab}{}_\mu \equiv \frac{1}{2}(c^{abc} - c^{cab} + c^{bca})e_{c\mu}\,, \tag{34}$$

where

$$c^{abc} = e^\mu{}_b e^\nu{}_c \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu\right)\,. \tag{35}$$

The last equation establishes that the spin connection 1-form ω^{ab} is expressed as a function of the tetrads, which then implies the same for the curvature 2-form R^{ab}. Finally, the remaining equation (29) then reduces to

$$\varepsilon_{abcd}R^{bc} \wedge e^d = 0\,, \tag{36}$$

which is nothing but the vacuum Einstein field equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\,.$$

Therefore, the action (27) is classically equivalent to general relativity.

3.3. Einstein-Yang-Mills theory

As we have already mentioned above, the main advantage of the action (27) over the Plebanski model lies in the fact that the tetrad fields are explicitly present in the topological sector of the action. This allows one to couple matter fields in a straightforward way [12]. However, one can do even more [13], and couple the $SU(N)$ Yang-Mills fields to gravity within a unified framework of 2-group formalism.

Namely, we can modify the Poincaré 2-group structure to include the $SU(N)$ gauge group, as follows. We choose the two Lie groups as

$$G = SO(3,1) \times SU(N)\,, \qquad H = \mathbb{R}^4\,, \tag{37}$$

and we define the action $\triangleright$ of the group G in the following fashion. As in the case of the Poincaré 2-group, it acts on itself via conjugation. Next, it acts on H such that the $SO(3,1)$ subgroup acts on $\mathbb{R}^4$ via the vector representation (21), while the action of the $SU(N)$ subgroup is trivial,

$$\tau_I \triangleright P_a = 0\,, \tag{38}$$

where τ_I are the $SU(N)$ generators. The map ∂ also remains trivial, as before. The form of the 2-connection (α, β) now reflects the structure of the group G,

$$\alpha = \omega^{ab} M_{ab} + A^I \tau_I \,, \qquad \beta = \beta^a P_a \,, \tag{39}$$

where A^I is the gauge connection 1-form. Next, the curvature for α then becomes

$$\mathcal{F} = R^{ab} M_{ab} + F^I \tau_I \,, \qquad F^I \equiv \mathrm{d}A^I + f_{JK}{}^I A^J \wedge A^K \,. \tag{40}$$

The curvature for β remains the same as before, because of (38). Finally, the product structure of the group G implies that its Killing form $\langle \_ , \_ \rangle_{\mathfrak{g}}$ reduces to the Killing forms for the $SO(3,1)$ and $SU(N)$, along with the identity $\langle M_{ab} \,, \tau_I \rangle_{\mathfrak{g}} = 0$.

Given a crossed module defined in this way, its corresponding pure $2BF$ action (16) becomes

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a \,, \tag{41}$$

where $B^I \in \mathcal{A}^2(\mathcal{M}_4 \,, \mathfrak{su}(N))$ is the new Lagrange multiplier. The action (41) is topological, and again we add appropriate simplicity constraint terms, in order to transform it into action with nontrivial dynamics. The constraint giving rise to gravity is the same as in (27), while the constraint for the gauge fields is given as in the action (4) with the substitution $\delta^a \to e^a$. Putting everything together, we obtain:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a \\ & - \lambda_{ab} \wedge \Big(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \Big) + \lambda^I \wedge \Big(B_I - \frac{12}{g} M_{abI} e^a \wedge e^b \Big) \\ & + \zeta^{abI} \Big(M_{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - g_{IJ} F^J \wedge e_a \wedge e_b \Big) \,. \end{aligned} \tag{42}$$

It is crucial to note that the Yang-Mills simplicity constraints in (42) are obtained from the Yang-Mills action (4) by substituting the nondynamical background field δ^a from (4) with a dynamical field e^a. The relationship between these fields has already been hinted at in the equation (5), which describes the connection between δ^a and the flat spacetime metric $\eta_{\mu\nu}$. Once promoted to e^a, this field becomes dynamical due to the presence of gravitational terms, while the equation (5) becomes the usual relation between the tetrad and the metric,

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \,, \tag{43}$$

further confirming the identification $C^a = e^a$. Moreover, the total action (42) now becomes background independent, as expected in general relativity. All this is a consequence of the fact that the tetrad field is explicitly present in the topological sector of the action (27), and represents a clear improvement over the Plebanski model.

Taking the variations of the action (42) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, ζ^{abI}, M_{abI}, B_I, λ^I, A^I, and e^a, we obtain equations of motion. Similarly as before, all variables can be algebraically expressed as functions of A^I and e^a and their derivatives:

$$\begin{gathered} \lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \beta_{a\mu\nu} = 0\,, \qquad \omega_{ab\mu} = \triangle_{ab\mu}\,, \qquad \lambda_{abI} = F_{abI}\,, \\ B_{\mu\nu I} = -\frac{e}{2g}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}{}_I\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \\ M_{abI} = -\frac{1}{4eg}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma\,, \qquad \zeta^{abI} = \frac{1}{4eg}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma\,. \end{gathered} \tag{44}$$

In addition, we obtain two differential equations — An equation for A^I,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + \Gamma^\rho{}_{\lambda\rho}F^{I\lambda\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0\,, \tag{45}$$

where $\Gamma^\lambda{}_{\mu\nu}$ is the standard Levi-Civita connection, and an equation for e^a,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \tag{46}$$

where

$$T^{\mu\nu} \equiv -\frac{1}{4g}\left(F_{\rho\sigma}{}^I F^{\rho\sigma}{}_I g^{\mu\nu} + 4F^{\mu\rho}{}_I F_\rho{}^{\nu I}\right)\,. \tag{47}$$

In this way, we see that both gravity and gauge fields can be successfully represented within a unified framework of higher gauge theory, based on a 2-group structure. A generalization from $SU(N)$ Yang-Mills case to more complicated cases such as $SU(3)\times SU(2)\times U(1)$ is completely straightforward.

4. $3BF$ theory

While the structure of a 2-group can successfully describe both gravitational and gauge fields, unfortunately it cannot accommodate other matter fields, such as scalars or fermions. In order to remedy this drawback, we make one further step in the categorical ladder, passing from the notion of a 2-group to the notion of a 3-group. As it turns out, the 3-group structure is excellent for the description of all fields that are present in the Standard Model, coupled to gravity. Moreover, a 3-group contains one more gauge group, which is novel and corresponds to the choice of the scalar and fermion fields present in the theory. This is an unexpected and beautiful result, not present in ordinary gauge theory.

As before, we will begin by introducing the notion of a 3-group, and constructing the corresponding $3BF$ action. Afterwards, we will modify this action by adding appropriate simplicity constraints, giving rise to theories with expected nontrivial dynamics. Along the way, we shall see that scalar and fermion fields are being treated pretty much on an equal footing with gravity and gauge fields.

4.1. Pure $3BF$ theory

Similarly to the concepts of a group and a 2-group, one can introduce the notion of a 3-group in the framework of higher category theory, as a 3-category with only one object where all the morphisms, 2-morphisms and 3-morphisms are invertible. Also, in the same way as a 2-group is equivalent to a crossed module, it was proved that a strict 3-group is equivalent to a 2-*crossed module* [25].

A Lie 2-crossed module, denoted as $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$, is an algebraic structure specified by three Lie groups G, H and L, together with the homomorphisms δ and ∂, an action $\triangleright$ of the group G on all three groups, and a G-equivariant map

$$\{\_\,,\_\} : H \times H \to L\,.$$

called the Peiffer lifting. The maps ∂, δ, $\triangleright$ and the Peiffer lifting satisfy certain axioms, so that the resulting structure is equivalent to a 3-group [13].

Like in the cases of BF and $2BF$ actions, we can introduce a gauge invariant topological $3BF$ action over the manifold $\mathcal{M}_4$ for a given 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$. Denoting $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ as Lie algebras corresponding to the groups G, H and L, respectively, one can introduce a 3-connection (α, β, γ) given by the algebra-valued differential forms $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{h})$ and $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,, \mathfrak{l})$. The corresponding fake 3-curvature $(\mathcal{F}\,, \mathcal{G}\,, \mathcal{H})$ is then defined as

$$\begin{gathered}\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \\ \mathcal{H} = \mathrm{d}\gamma + \alpha \wedge^{\triangleright} \gamma + \{\beta \wedge \beta\}\,,\end{gathered} \tag{48}$$

see [25, 26] for details. Note that γ is a 3-form, while its corresponding field strength $\mathcal{H}$ is a 4-form, necessitating that the spacetime manifold be at least 4-dimensional. Then, a $3BF$ action is defined as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}\,, \tag{49}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$, $C \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ and $D \in \mathcal{A}^0(\mathcal{M}_4, \mathfrak{l})$ are Lagrange multipliers. Note that in precisely 4 spacetime dimensions the Lagrange multiplier D corresponding to $\mathcal{H}$ is a 0-form, i.e. a scalar function. The functionals $\langle \_ , \_ \rangle_{\mathfrak{g}}$, $\langle \_ , \_ \rangle_{\mathfrak{h}}$ and $\langle \_ , \_ \rangle_{\mathfrak{l}}$ are G-invariant bilinear symmetric non-degenerate forms on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, respectively. Under certain conditions, the forms $\langle \_ , \_ \rangle_{\mathfrak{h}}$ and $\langle \_ , \_ \rangle_{\mathfrak{l}}$ are also H-invariant and L-invariant.

One can see that varying the action with respect to the variables B^α, C^a and D^A (where indices A count the generators of the group L), one obtains the equations of motion

$$\mathcal{F}^\alpha = 0 \,, \qquad \mathcal{G}^a = 0 \,, \qquad \mathcal{H}^A = 0 \,, \tag{50}$$

while varying with respect to α^α, β^a, γ^A one obtains

$$\mathrm{d}B_\alpha + f_{\alpha\beta}{}^\gamma B_\gamma \wedge \alpha^\beta - \rhd_{\alpha a}{}^b C_b \wedge \beta^a + \rhd_{\alpha B}{}^A D_A \wedge \gamma^B = 0 \,, \tag{51}$$

$$\mathrm{d}C_a - \partial_a{}^\alpha B_\alpha + \rhd_{\alpha a}{}^b C_b \wedge \alpha^\alpha + 2X_{\{ab\}}{}^A D_A \wedge \beta^b = 0 \,, \tag{52}$$

$$\mathrm{d}D_A - \rhd_{\alpha A}{}^B D_B \wedge \alpha^\alpha + \delta_A{}^a C_a = 0 \,. \tag{53}$$

4.2. Klein-Gordon theory

Now we proceed to demonstrate that one can use the 3-group structure and the corresponding $3BF$ theory to describe the Klein-Gordon field coupled to general relativity. We begin by specifying a 2-crossed module, which is used to construct the topological $3BF$ theory, and then we impose appropriate simplicity constraints to obtain the desired equations of motion.

We specify a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G \,, \rhd \,, \{\_ , \_\})$, as follows. The groups are given as

$$G = SO(3,1) \,, \qquad H = \mathbb{R}^4 \,, \qquad L = \mathbb{R} \,. \tag{54}$$

The group G acts on itself via conjugation, on H via the vector representation, and on L via the trivial representation. This specifies the definition of the action $\rhd$. The map ∂ is chosen to be trivial, as before. The map δ is also trivial, that is, every element of L is mapped to the identity element of H. Finally, the Peiffer lifting is trivial as well, mapping every ordered pair of elements in H to an identity element in L. This specifies one concrete 2-crossed module which, as we shall see below, corresponds to gravity and one real scalar field.

Given this choice of a 2-crossed module, the 3-connection $(\alpha \,, \beta \,, \gamma)$ takes the form

$$\alpha = \omega^{ab} M_{ab} \,, \qquad \beta = \beta^a P_a \,, \qquad \gamma = \gamma \mathbb{I} \,, \tag{55}$$

where $\mathbb{I}$ is the sole generator of the Lie group $\mathbb{R}$. Since the homomorphisms ∂ and δ are trivial, as well as the Peiffer lifting, the fake 3-curvature (48) reduces to the ordinary 3-curvature,

$$\mathcal{F} = R^{ab} M_{ab} \,, \qquad \mathcal{G} = \nabla \beta^a P_a \,, \qquad \mathcal{H} = \mathrm{d}\gamma \,, \tag{56}$$

where we used the fact that G acts trivially on L, that is, $M_{ab} \triangleright \mathbb{I} = 0$. This means that the 3-form γ transforms as a scalar with respect to Lorentz symmetry. Consequently, its Lagrange multiplier D also transforms as a scalar, since it also belongs to the algebra $\mathfrak{l}$. Since D is also a 0-form, it transforms as a scalar with respect to diffeomorphisms as well. In other words, D completely behaves as a real scalar field, so we relabel it into more traditional notation, $D \equiv \phi$, and write the pure $3BF$ action (49) as:

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma \,, \tag{57}$$

where the bilinear form for L is $\langle \mathbb{I} , \mathbb{I} \rangle_{\mathfrak{l}} = 1$.

The existence of a scalar field in the $3BF$ action is a crucial property of a 3-group in a 4-dimensional spacetime, just like identifying the Lagrange multiplier C^a with a tetrad field e^a was a crucial property of the $2BF$ action and the Poincaré 2-group. We can also see that the choice of the third gauge group, L, dictates the number and the structure of the matter fields present in the action. In this case, $L = \mathbb{R}$ implies that we have only one real scalar field, corresponding to a single generator $\mathbb{I}$ of $\mathbb{R}$. The trivial nature of the action $\triangleright$ of $SO(3,1)$ on $\mathbb{R}$ implies that ϕ transforms as a scalar field. Finally, the scalar field appears in the topological sector of the action, making the quantization procedure feasible.

As in the case of BF and $2BF$ theories, we need to add appropriate simplicity constraints to the action (57). In order to obtain the Klein-Gordon field ϕ of mass m coupled to gravity in the standard way, the action takes the form:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma \\ & - \lambda_{ab} \wedge \Big(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \Big) \\ & + \lambda \wedge \Big(\gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c \Big) \\ & + \Lambda^{ab} \wedge \Big(H_{abc} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b \Big) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi^2 \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,. \end{aligned} \tag{58}$$

The first row is the topological sector (57), the second row is the familiar simplicity constraint for gravity from the action (27), the third and fourth rows contain the new simplicity constraints featuring the Lagrange multiplier 1-forms λ and Λ^{ab} and the 0-form H_{abc}, while the fifth row is the mass term for the scalar field.

The variation of (58) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, Λ_{ab}, γ, λ, H_{abc}, ϕ and e^a gives us the equations of motion. As before, all variables can be algebraically expressed in terms of the tetrads e^a and the scalar field ϕ:

$$\begin{gathered}\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \omega^{ab}{}_{\mu} = \triangle^{ab}{}_{\mu}\,, \qquad \gamma_{\mu\nu\rho} = -\frac{e}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^{\sigma}\phi\,, \\ \beta^{a}{}_{\mu\nu} = 0\,, \qquad \Lambda^{ab}{}_{\mu} = \frac{1}{12e}g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\partial_{\nu}\phi e^{a}{}_{\rho}e^{b}{}_{\sigma}\,, \qquad \lambda_{\mu} = \partial_{\mu}\phi\,, \\ H^{abc} = \frac{1}{6e}\varepsilon^{\mu\nu\rho\sigma}\partial_{\mu}\phi e^{a}{}_{\nu}e^{b}{}_{\rho}e^{c}{}_{\sigma}\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^{c}{}_{\mu}e^{d}{}_{\nu}\,.\end{gathered} \tag{59}$$

The equations of motion for e^a and ϕ, however, are differential equations. The equation for the scalar field becomes the covariant Klein-Gordon equation,

$$\left(\nabla_{\mu}\nabla^{\mu} - m^2\right)\phi = 0\,, \tag{60}$$

while the equation for the tetrads is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \tag{61}$$

where

$$T^{\mu\nu} \equiv \partial^{\mu}\phi\partial^{\nu}\phi - \frac{1}{2}g^{\mu\nu}\left(\partial_{\rho}\phi\partial^{\rho}\phi + m^2\phi^2\right) \tag{62}$$

is the stress-energy tensor for a single real scalar field.

4.3. Einstein-Cartan-Dirac theory

In order to describe the Dirac field coupled to Einstein-Cartan gravity, we follow the same procedure as for the case of the scalar field, but now we choose the 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$ in a different way, as follows. The groups are:

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^8(\mathbb{G})\,, \tag{63}$$

where $\mathbb{G}$ is the algebra of complex Grassmann numbers. The maps ∂, δ and the Peiffer lifting are trivial, as before. The action of the group G on itself is given via conjugation, on H via vector representation, and on L via spinor representation, in the following way. Denoting the 8 generators of the Lie group $\mathbb{R}^8(\mathbb{G})$ as P_α and P^α, where the index α takes the values $1, \dots, 4$, the action $\triangleright$ of G on L is thus given explicitly as

$$M_{ab} \triangleright P_{\alpha} = \frac{1}{2}(\sigma_{ab})^{\beta}{}_{\alpha}P_{\beta}\,, \qquad M_{ab} \triangleright P^{\alpha} = -\frac{1}{2}(\sigma_{ab})^{\alpha}{}_{\beta}P^{\beta}\,, \tag{64}$$

where $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, and γ_a are the usual Dirac matrices, satisfying the anticommutation rule $\{\gamma_a\,, \gamma_b\} = -2\eta_{ab}$.

As in the case of the scalar field, the choice of the group L dictates the matter content of the theory, while the action $\triangleright$ of G on L specifies its transformation properties.

Let us now proceed to construct the $3BF$ action. The 3-connection $(\alpha\,,\beta\,,\gamma)$ takes the form

$$\alpha=\omega^{ab}M_{ab}\,,\qquad \beta=\beta^a P_a\,,\qquad \gamma=\gamma^\alpha P_\alpha+\bar{\gamma}_\alpha P^\alpha\,, \tag{65}$$

while the 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is given as

$$\begin{aligned} &\mathcal{F}=R^{ab}M_{ab}\,,\qquad \mathcal{G}=\nabla\beta^a P_a\,,\\ \mathcal{H}&=\Big(\mathrm{d}\gamma^\alpha+\frac{1}{2}\omega^{ab}(\sigma_{ab})^\alpha{}_\beta\gamma^\beta\Big)P_\alpha+\Big(\mathrm{d}\bar{\gamma}_\alpha-\frac{1}{2}\omega^{ab}\bar{\gamma}_\beta(\sigma_{ab})^\beta{}_\alpha\Big)P^\alpha\\ &\equiv(\overrightarrow{\nabla}\gamma)^\alpha P_\alpha+(\bar{\gamma}\overleftarrow{\nabla})_\alpha P^\alpha\,, \end{aligned} \tag{66}$$

where we have used (64). The bilinear form $\langle\_\,,\_\rangle_{\mathfrak{l}}$ is defined via its action on the generators:

$$\begin{aligned} \langle P_\alpha\,,P_\beta\rangle_{\mathfrak{l}}=0\,,\qquad \langle P^\alpha\,,P^\beta\rangle_{\mathfrak{l}}=0\,,\\ \langle P_\alpha\,,P^\beta\rangle_{\mathfrak{l}}=-\delta^\beta_\alpha\,,\qquad \langle P^\alpha\,,P_\beta\rangle_{\mathfrak{l}}=\delta^\alpha_\beta\,. \end{aligned} \tag{67}$$

Note that the bilinear form defined in this way is antisymmetric, rather than symmetric, when it acts on the generators. The reason for this is the following. For general $A,B\in\mathfrak{l}$, we want the bilinear form to be symmetric. Expanding A and B into components, we can write

$$\langle A\,,B\rangle_{\mathfrak{l}}=A^I B^J g_{IJ}\,,\qquad \langle B\,,A\rangle_{\mathfrak{l}}=B^J A^I g_{JI}\,. \tag{68}$$

Since we require the bilinear form to be symmetric, the two expressions must be equal. However, since the coefficients in $\mathfrak{l}$ are Grassmann numbers, we have $A^I B^J=-B^J A^I$, so it follows that $g_{IJ}=-g_{JI}$. Hence the antisymmetry of (67) — it compensates for the anticommutativity property of the Grassman coefficients, making the bilinear form symmetric for general algebra elements $A,B\in\mathfrak{l}$.

Now we employ the action $\triangleright$ of G on L to determine the transformation properties of the Lagrange multiplier D in (49). Indeed, the choice of the group L dictates that D contains 8 independent complex Grassmannian matter fields as its components. Moreover, due to the fact that D is a 0-form and that it transforms according to the spinorial representation of $SO(3,1)$, we can identify its components with the Dirac bispinor fields, and write

$$D=\psi^\alpha P_\alpha+\bar{\psi}_\alpha P^\alpha\,. \tag{69}$$

This is again an illustration of the fact that information about the structure of the matter sector in the theory is specified by the choice of the group L

in the 2-crossed module, and its transformation properties with respect to the Lorentz group are fixed by the action $\triangleright$.

Given all of the above, we write the corresponding pure $3BF$ action as:

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + (\bar{\gamma}\overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla}\gamma)^\alpha \,. \tag{70}$$

In order to obtain the action that gives us the dynamics of Einstein-Cartan theory of gravity coupled to a Dirac field, we add the following simplicity constraints:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + (\bar{\gamma}\overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla}\gamma)^\alpha \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & - \lambda^\alpha \wedge \left(\bar{\gamma}_\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar{\psi}\gamma^d)_\alpha \right) \\ & + \bar{\lambda}_\alpha \wedge \left(\gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^\alpha \right) \\ & - \frac{1}{12} m \, \bar{\psi}\psi \, \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d + 2\pi i l_p^2 \, \bar{\psi}\gamma_5 \gamma^a \psi \, \varepsilon_{abcd} e^b \wedge e^c \wedge \beta^d . \end{aligned} \tag{71}$$

Similarly to the previous case of the scalar field, we recognize the topological sector in the first row, the gravitational simplicity constraint in the second row, while the third and fourth rows contain the new simplicity constraints for the Dirac field, featuring the Lagrange multiplier 1-forms λ^α and $\bar{\lambda}_\alpha$. The fifth row contains the mass term for the Dirac field, and a term which ensures the correct coupling between the torsion and the spin of the Dirac field. In particular, we want to obtain

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a \,, \tag{72}$$

as one of the equations of motion, where

$$s_a = i \varepsilon_{abcd} e^b \wedge e^c \bar{\psi} \gamma_5 \gamma^d \psi \tag{73}$$

is the Dirac spin 2-form. Of course, other alternative coupling choices are possible, but we choose this one since this is the traditional coupling most often discussed in textbooks.

The variation of the action (71) with respect to B_{ab}, λ^{ab}, $\bar{\gamma}_\alpha$, γ^α, λ^α, $\bar{\lambda}_\alpha$, $\bar{\psi}_\alpha$, ψ^α, e^a, β^a and ω^{ab}, again gives us equations of motion, which can be algebraically solved for all fields as functions of e^a, ψ and $\bar{\psi}$:

$$B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \qquad \lambda^\alpha{}_\mu = (\overrightarrow{\nabla}_\mu\psi)^\alpha\,, \qquad \bar{\lambda}_{\alpha\mu} = (\bar{\psi}\overleftarrow{\nabla}_\mu)_\alpha\,,$$
$$\bar{\gamma}_{\alpha\mu\nu\rho} = i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\bar{\psi}\gamma^d)_\alpha\,, \qquad \gamma^\alpha{}_{\mu\nu\rho} = -i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\gamma^d\psi)^\alpha\,,$$
$$\beta^a{}_{\mu\nu} = 0\,, \qquad \lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu + K^{ab}{}_\mu\,. \tag{74}$$

Here $K^{ab}{}_\mu$ is the contorsion tensor, constructed in the standard way from the torsion tensor. In addition, we also obtain

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a\,, \tag{75}$$

which is precisely the desired equation (72) for the torsion. Finally, the differential equations of motion for ψ and $\bar{\psi}$ are the standard covariant Dirac equation,

$$(i\gamma^a e^\mu{}_a\overrightarrow{\nabla}_\mu - m)\psi = 0\,, \tag{76}$$

and its conjugate,

$$\bar{\psi}(i\overleftarrow{\nabla}_\mu e^\mu{}_a\gamma^a + m) = 0\,, \tag{77}$$

where $e^\mu{}_a$ is the inverse tetrad. The differential equation of motion for e^a is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \tag{78}$$

where

$$T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\gamma^a\overleftrightarrow{\nabla}^\nu e^\mu{}_a\psi - \frac{1}{2}g^{\mu\nu}\bar{\psi}\Big(i\gamma^a\overleftrightarrow{\nabla}_\rho e^\rho{}_a - 2m\Big)\psi\,, \tag{79}$$

Here, we used the notation $\overleftrightarrow{\nabla} = \overrightarrow{\nabla} - \overleftarrow{\nabla}$. As expected, the equations of motion (75), (76), (77) and (78) are precisely the equations of motion of the Einstein-Cartan-Dirac theory.

4.4. Weyl and Majorana fields coupled to Einstein-Cartan gravity

As is well known, the Dirac fermions are not an irreducible representation of the Lorentz group, and one can rewrite them as left-chiral and right-chiral irreducible Weyl fermion fields. Hence, it is useful to construct the 2-crossed module and a constrained $3BF$ action for left and right Weyl spinors. For simplicity, we will discuss only the left-chiral spinor field (the right-chiral can be studied analogously). Additionally, we can also describe Majorana fermions using the same formalism, the only difference being the presence of an additional mass term in the Majorana action.

We soecify a 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \rhd\,, \{\_\,,\_\})$, in a way similar to the Dirac case, as follows. The groups are:

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^4(\mathbb{G})\,. \tag{80}$$

The maps ∂, δ and the Peiffer lifting are trivial. The action $\rhd$ of the group G on G, H and L is given in the same way as for the Dirac case, whereas the spinorial representation reduces to

$$M_{ab} \rhd P^\alpha = \frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta\,, \qquad M_{ab} \rhd P_{\dot\alpha} = \frac{1}{2}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha} P_{\dot\beta}\,, \tag{81}$$

where $\sigma^{ab} = -\bar\sigma^{ab} = \frac{1}{4}(\sigma^a\bar\sigma^b - \sigma^b\bar\sigma^a)$, for $\sigma^a = (1,\vec\sigma)$ and $\bar\sigma^a = (1,-\vec\sigma)$, in which $\vec\sigma$ denotes the set of three Pauli matrices. The four generators of the group L are denoted as P^α and $P_{\dot\alpha}$, where the Weyl indices $\alpha, \dot\alpha$ take values $1, 2$.

The 3-connection $(\alpha\,,\beta\,,\gamma)$ takes the form

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma_\alpha P^\alpha + \bar\gamma^{\dot\alpha}P_{\dot\alpha}\,, \tag{82}$$

while the 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is

$$\begin{aligned} &\mathcal{F} = R^{ab}M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,,\\ \mathcal{H} &= \big(\mathrm{d}\gamma_\alpha + \frac{1}{2}\omega^{ab}(\sigma^{ab})^\beta{}_\alpha\gamma_\beta\big)P^\alpha + \big(\mathrm{d}\bar\gamma^{\dot\alpha} + \frac{1}{2}\omega_{ab}(\bar\sigma^{ab})^{\dot\alpha}{}_{\dot\beta}\bar\gamma^{\dot\beta}\big)P_{\dot\alpha}\\ &\equiv (\overrightarrow{\nabla}\gamma)_\alpha P^\alpha + (\bar\gamma\overleftarrow{\nabla})^{\dot\alpha}P_{\dot\alpha}\,. \end{aligned} \tag{83}$$

The Lagrange multiplier D now contains as coefficients the spinor fields ψ_α and $\bar\psi^{\dot\alpha}$,

$$D = \psi_\alpha P^\alpha + \bar\psi^{\dot\alpha}P_{\dot\alpha}\,, \tag{84}$$

and the bilinear form $\langle\_\,,\_\rangle_{\mathfrak{l}}$ for the group L is

$$\begin{aligned} \langle P^\alpha\,, P^\beta\rangle_{\mathfrak{l}} = \varepsilon^{\alpha\beta}\,, \qquad \langle P_{\dot\alpha}\,, P_{\dot\beta}\rangle_{\mathfrak{l}} = \varepsilon_{\dot\alpha\dot\beta}\,,\\ \langle P^\alpha\,, P_{\dot\beta}\rangle_{\mathfrak{l}} = 0\,, \qquad \langle P_{\dot\alpha}\,, P^\beta\rangle_{\mathfrak{l}} = 0\,, \end{aligned} \tag{85}$$

where $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot\alpha\dot\beta}$ are the usual two-dimensional antisymmetric Levi-Civita symbols.

The pure $3BF$ action (49) now becomes

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + e_a \wedge \nabla\beta^a + \psi^\alpha \wedge (\overrightarrow{\nabla}\gamma)_\alpha + \bar\psi_{\dot\alpha}\wedge(\bar\gamma\overleftarrow{\nabla})^{\dot\alpha}\,. \tag{86}$$

In order to obtain the suitable equations of motion for the Weyl spinors, we again introduce appropriate simplicity constraints, to obtain:

$$
\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \psi^\alpha \wedge (\overrightarrow{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}} \wedge (\bar{\gamma}\overleftarrow{\nabla})^{\dot{\alpha}} \\
& - \lambda_{ab} \wedge (B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d) \\
& - \lambda^\alpha \wedge (\gamma_\alpha + \frac{i}{6}\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}}) \\
& - \bar{\lambda}_{\dot{\alpha}} \wedge (\bar{\gamma}^{\dot{\alpha}} + \frac{i}{6}\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta) \\
& - 4\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\bar{\psi}_{\dot{\alpha}} \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta) \, .
\end{aligned} \tag{87}
$$

The new simplicity constraints, in the third and fourth rows, feature the Lagrange multiplier 1-forms λ_α and $\bar{\lambda}^{\dot{\alpha}}$. Also, in analogy to the coupling between the spin and the torsion in Einstein-Cartan-Dirac theory, the term in the fifth row is chosen to ensure that the coupling between the Weyl spin tensor

$$
s_a \equiv i\varepsilon_{abcd} e^b \wedge e^c \; \psi^\alpha \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} \tag{88}
$$

and torsion is given as:

$$
T_a = 4\pi l_p^2 s_a \, . \tag{89}
$$

The action for the Majorana field is precisely the same, but for an additional mass term in the action:

$$
-\frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d (\psi^\alpha \psi_\alpha + \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}) \, . \tag{90}
$$

The variation of the action (87) with respect to the variables B_{ab}, λ^{ab}, γ_α, $\bar{\gamma}^{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, e^a, β^a and ω^{ab} gives us the equations of motion, which can be algebraically solved for all variables as functions of ψ_α, $\bar{\psi}^{\dot{\alpha}}$ and e^a:

$$
\begin{gathered}
\beta^a{}_{\mu\nu} = 0 \, , \quad \lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu} \, , \quad \lambda_{\alpha\mu} = \nabla_\mu \psi_\alpha \, , \quad \bar{\lambda}^{\dot{\alpha}}{}_\mu = \nabla_\mu \bar{\psi}^{\dot{\alpha}} \, , \\
B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu \, , \quad \omega_{ab\mu} = \triangle_{ab\mu} + K_{ab\mu} \, , \\
\gamma_{\alpha\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} \, , \quad \bar{\gamma}^{\dot{\alpha}}{}_{\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta \, .
\end{gathered} \tag{91}
$$

In addition, one also obtains (89). Finally, the differential equations of motion for the spinor and tetrad fields are

$$
\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta = 0 \, , \qquad \sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} = 0 \, , \tag{92}
$$

and

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2 \, T^{\mu\nu} \,, \tag{93}$$

where

$$\begin{aligned} T^{\mu\nu} \equiv\; & \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} \\ & -\frac{1}{2} g^{\mu\nu} \Big(i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi} \Big) \,. \end{aligned} \tag{94}$$

Here we have suppressed the spinor indices, for simplicity. In the case of the Majorana field, the equations of motion (91) remain the same. The equations of motion for ψ_α and $\bar{\psi}^{\dot{\alpha}}$ obtain the additional mass term,

$$i\sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} - m\psi_\alpha = 0 \,, \qquad i\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta - m\bar{\psi}^{\dot{\alpha}} = 0 \,, \tag{95}$$

while the stress-energy tensor becomes

$$\begin{aligned} T^{\mu\nu} \equiv\; & \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} \\ & -g^{\mu\nu}\frac{1}{2} \left[i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi} - \frac{1}{2} m \left(\psi\psi + \bar{\psi}\bar{\psi} \right) \right] \,. \end{aligned} \tag{96}$$

5. Conclusions

Let us summarize the results of the paper. In Section 2 we have introduced the BF theory and discussed models based on constrained BF action, in particular the Yang-Mills theory in Minkowski spacetime and the Plebanski formulation of general relativity. Section 3 was devoted to the first step in the categorical ladder and the $2BF$ theory. After introducing the notions of a 2-group, a crossed module, and the corresponding $2BF$ theory, we have studied the $2BF$ formulation of general relativity and the Einstein-Yang-Mills theory. Then, in Section 4 we have performed one more step in the categorical ladder, and introduced the notions of a 3-group, 2-crossed module, and the $3BF$ theory. This structure was employed to construct the constrained $3BF$ actions for the cases of Klein-Gordon, Dirac, Weyl and Majorana fields, each coupled to the Einstein-Cartan gravity in the standard way. In those descriptions, it turned out that the scalar and fermion fields are associated to a *new gauge group*, similar to the gauge fields being associated to a gauge group in the Yang-Mills theory. This opens up a possibility of a classification of matter fields based on an algebraic structure of a 3-group.

All the obtained results serve to complete the first step of the spinfoam quantization programme, as outlined in the Introduction. This paves the way to the study of steps 2 and 3 of the programme. Namely, the full action for gravity, gauge fields and matter is written completely in the langulage of differential forms, which can be easily adapted to a triangulated spacetime manifold, in the sense of Regge calculus. This can be seen in the following table:

d	triangulation	dual triangulation	form	fields	field strengths
0	vertex	4-polytope	0-form	ϕ, $\psi_{\tilde{\alpha}}$, $\bar{\psi}^{\tilde{\alpha}}$	
1	edge	3-polyhedron	1-form	ω^{ab}, A^I, e^a	
2	triangle	face	2-form	β^a, B^{ab}	R^{ab}, F^I, T^a
3	tetrahedron	edge	3-form	γ, $\gamma_{\tilde{\alpha}}$, $\bar{\gamma}^{\tilde{\alpha}}$	$\mathcal{G}^a$
4	4-simplex	vertex	4-form		$\mathcal{H}$, $\mathcal{H}_{\tilde{\alpha}}$, $\bar{\mathcal{H}}^{\tilde{\alpha}}$

This data can be utilized to construct a Regge-discretized topological $3BF$ action, and from that a state sum Z, giving rise to a rigorous definition of the path integral

$$Z = \int \mathcal{D}g \int \mathcal{D}\phi \; e^{iS[g,\phi]} \,, \tag{97}$$

which is a generalization of (1) in the sense that it adds matter fields (including the gauge boson sector) to gravity at the quantum level. Being a topological theory, and given the underlying structure of the 3-group, a pure $3BF$ action ought to ensure the topological invariance of the state sum Z, i.e., Z should be triangulation independent. This step, however, requires the generalizations of the Peter-Weyl and Plancharel theorems to 2-groups and 3-groups, which are unfortunately still missing (though there are some attempts to circumvent them at least in the 2-group case [27, 28]). Namely, the purpose of the Peter-Weyl and Plancharel theorems is to provide a decomposition of a function on a group into a sum over the corresponding irreducible representations, which then specifies the spectrum of labels for the simplices in the triangulation, and fixes the domain of values for the fields living on those simplices. In the absence of the two theorems, one can still try to *guess* the irreducible representations of the 2- and 3-groups, as was done for example in the *spincube model* of quantum gravity [12], or to try to construct the state sum using other techniques, as was done in [27, 28]).

Of course, when building a realistic theory, we are not interested in a topological theory, but instead in one which contains local propagating degrees of freedom. Thus the state sum Z need not be a topological invariant. This is obtained via the step 3 of the spinfoam quantization programme, by imposing the simplicity constraints on Z. The classical actions discussed in this paper manifestly distinguish the topological sector from the simplicity constraints, which have been explicitly determined. Imposing them should thus be a straightforward procedure for a given Z. Completing this programme would ultimately lead us to a tentative state sum describing both gravity and matter at a quantum level, which is a topic for future research.

In addition to the construction of a full quantum theory of gravity, there are also many additional possible studies of the classical constrained $3BF$ action. For example, a Hamiltonian analysis of the theory could be interesting for the canonical quantization programme, and some work has begun in this area [29]. Also, it is worth looking into the idea of imposing the simplicity constraints using a spontaneous symmetry breaking mechanism. Finally, one can also study in more depth the mathematical structure and properties of the simplicity constraints. The list is not conclusive, and there may be many other interesting topics to study.

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Quantum gravity and elementary particles from higher gauge theory

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Abstract

We give a brief overview how to couple general relativity to the Standard Model of elementary particles, within the higher gauge theory framework, suitable for the spinfoam quantization procedure. We begin by providing a short review of all relevant mathematical concepts, most notably the idea of a categorical ladder, 3-groups and generalized parallel transport. Then, we give an explicit construction of the algebraic structure which describes the full Standard Model coupled to Einstein-Cartan gravity, along with the classical action, written in the form suitable for the spinfoam quantization procedure. We emphasize the usefulness of the 3-group concept as a superior tool to describe gauge symmetry, compared to an ordinary Lie group, as well as the possibility to employ this new structure to classify matter fields and study their spectrum, including the origin of fermion families.

1 Introduction

The quantization of the gravitational field is one of the most fundamental open problems of modern theoretical physics. Since the inceptions of general relativity (GR) and quantum field theory (QFT), many attempts have been made over the years to unify the two into a self-consistent description of gravitational and matter fields as basic building blocks of nature. Some of the attempts have developed into vast research areas, such as String Theory, Loop Quantum Gravity, Causal Set Theory, and so on. One of the prominent approaches is Loop Quantum Gravity (LQG) [1, 2], which has branched into the canonical and covariant frameworks, the latter known as the *spinfoam* approach [3].

The spinfoam approach to the quantization of the gravitational field revolves around the idea of providing a precise mathematical definition to the Feynman path integral for the gravitational field,

$$Z = \int \mathcal{D}g \; e^{iS_{GR}[g]} \,,$$

where g denotes the gravitational degrees of freedom, and $S_{GR}[g]$ is the GR action expressed in terms of variables g. The strategy of defining the path integral can be roughly expressed in three main steps, called the *spinfoam quantization procedure*:

1. Choose convenient variables g and rewrite the classical action in the form

$$S_{GR}[g] = S_{\text{topological}}[g] + S_{\text{simp}}[g] \,, \tag{1}$$

where the first term represents a topological theory (with no propagating degrees of freedom), while the second term corresponds to the so-called *simplicity constraint* terms, whose purpose is to transform the full action into a realistic non-topological action with propagating degrees of freedom.

2. Employ the methods of topological quantum field theory (TQFT) to define the path integral for the topological part of the action. This is typically implemented by passing from a smooth spacetime manifold to a simplicial complex (triangulation), and writing the path integral in the form of a discrete state sum,

$$Z = \sum_g \prod_v \mathcal{A}_v(g) \prod_\epsilon \mathcal{A}_\epsilon(g) \prod_\Delta \mathcal{A}_\Delta(g) \prod_\tau \mathcal{A}_\tau(g) \prod_\sigma \mathcal{A}_\sigma(g) \,.$$

Here g represents the gravitational field variables living on the vertices v, edges ϵ, triangles Δ, terahedra τ, and 4-simplices σ of the simplicial complex, describing its geometry, while the corresponding amplitudes $\mathcal{A}_v(g), \dots, \mathcal{A}_\sigma(g)$ are chosen to render the whole state sum Z independent of the particular choice of the triangulation of the spacetime manifold.

3. Enforce the simplicity constraints of the theory by a suitable deformation of the amplitudes $\mathcal{A}$ and the set of independent variables g, thereby obtaining a modified state sum Z which corresponds to one possible rigorous definition of the realistic gravitational path integral.

Since its inception, the spinfoam quantization procedure has been formulated and implemented for various choices of the classical action, leading to a plethora of *spinfoam models* of quantum gravity, starting from the Ponzano-Regge model for $3D$ gravity [4], and leading up to the currently most sophisticated EPRL/FK model for the realistic $4D$ case [5, 6]. However, one property common to all spinfoam models is the fact that they all describe pure gravity, without matter fields. This is due to the common choice of the classical action — it is the well known BF theory [7], which is usually defined for the Lorentz group $SO(3,1)$, with some form of the simplicity constraint terms. The prototype description of GR in this form is the Plebanski action [8]. The reason why matter fields are absent from all such models lies in the fact that the BF action does not feature tetrad fields at the fundamental level. Instead, the tetrads appear as a consequence of classical equations of motion, and are thus inherently classical, on-shell quantities. This renders the approach based on the BF theory incapable of adding matter fields at the quantum level, since matter is coupled to gravity using precisely the tetrad fields.

The issue of the absence of the tetrad fields at the fundamental level has been successfully resolved in [9], where a categorical generalization has been made, and the $2BF$ action (introduced in [10, 11]) has been employed to build an action for GR, featuring tetrads explicitly in the topological sector of the action. The categorical generalization is based on a concept of a *categorical ladder*, an abstraction scheme introducing a chain of new objects: from categories to 2-categories to 3-categories and so forth. This powerful mathematical language gave rise to the idea that the notion of gauge symmetry in physics may be described by objects other than Lie groups. The new approach is called *higher gauge theory* (HGT), see [12] for an introduction. In the context of the spinfoam quantization procedure, HGT has been successfully applied to build a quantum gravity model, based on the Poincaré 2-group [13] as a gauge symmetry structure, and the corresponding $2BF$ action, leading to the so-called *spincube model* of quantum gravity [9]. Having the tetrads as fundamental fields in the $2BF$ action, the new model could be extended to include matter fields in a straightforward way. Nevertheless, the matter field action does not have the form analogous to (1), which renders the steps 2 and 3 of the spinfoam quantization procedure moot, since they can be applied only to the gravitational sector of the theory.

Thus, a natural need appeared to generalize the theory once more, in order to include the matter fields into the topological sector of the theory, in a similar way that was done to include the tetrad fields. The basic idea was to pass from the notion of a 2-group to a notion of a 3-group as a mathematical descriptor of gauge symmetry [12, 14, 15], giving rise to a topological $3BF$ action. With suitable simplicity constraint terms added, a $3BF$ action perfectly fits together all fields necessary for a unified description of quantum gravity coupled to matter fields — it features tetrads, spin connection, gauge fields, scalar fields and fermions. The explicit construction was done in [16], where the full Standard Model (SM) coupled to GR in the Einstein-Cartan formulation was rewritten in the form (1), suitable for the implementation of the spinfoam quantization procedure and building a full quantum theory. This demostrates the power and expressiveness of the HGT approach, and it provides us with novel mathematical tools to study the algebraic properties of the matter sector of the SM, in analogy to the gauge field sector which is being described in terms of ordinary Lie groups. In this paper we will review the essential properties of the new approach.

The layout of the paper is the following. In section 2 we give a brief introduction to the category theory, categorical ladder, and the notion of n-groups. Our attention focuses on 3-groups, in particular their representation in terms of 2-crossed modules. Section 3 reviews the construction and general properties of the $3BF$ action, and its relationship with the 3-group structure. Then, in section 4 we apply this developed formalism to construct the *Standard Model 3-group*, and explicitly build the action for the Standard Model coupled to Einstein-Cartan gravity in the form of the $3BF$ action with suitable simplicity constraints. Section 5 contains our concluding remarks.

2 Category theory and 3-groups

Let us begin by giving a short introduction to the category theory, and in particular the notion of *category theory ladder*, a concept used in higher gauge theory to generalize the notion of gauge symmetry. A nice introduction to this topic can be found in [12] and further technical details in [14, 15].

A category $\mathcal{C} = (Obj, Mor)$ is a structure which has objects and morphisms between them,

$$X, Y, Z, \dots \in Obj\,, \qquad f, g, h, \dots \in Mor\,,$$

where

$$f : X \to Y, \quad g : Z \to X, \quad h : X \to Y, \dots$$

such that certain rules are respected, like the associativity of composition of morphisms, and similar. Similarly, a 2-category $\mathcal{C}_2 = (Obj, Mor_1, Mor_2)$ is a structure which has objects, morphisms between them, and morphisms between morphisms, called 2-morphisms,

$$X, Y, Z, \dots \in Obj\,, \qquad f, g, h, \dots \in Mor_1\,, \qquad \alpha, \beta, \dots \in Mor_2\,,$$

where

$$f : X \to Y, \quad g : Z \to X, \quad h : X \to Y, \dots \qquad \alpha : f \to h\,, \dots$$

such that similar rules about compositions are respected. Then, a 3-category $\mathcal{C}_3 = (Obj, Mor_1, Mor_2, Mor_3)$ additionally has morphisms between 2-morphisms, called 3-morphisms,

$$\Theta, \Phi, \dots \in Mor_3\,, \qquad \Theta : \alpha \to \beta\,, \dots$$

again with a certain set of axioms about compositions of various n-morphisms. One can further generalize these structures to introduce 4-categories, n-categories, ∞-categories, etc. The process of raising the "dimensionality" of a categorical structure is called a *categorical ladder*.

It is useful to understand other algebraic structures as special cases of categories. As a particularly important example, the algebraic structure of a *group* is a special case of a category — it is a category with only one object, while all morphisms (i.e., group elements) are invertible. It is straightforward to verify that axioms of a group follow from this definition and the axioms of a category. Any group can be represented in this way, for example finite groups, Lie groups, and so on.

The notion of a categorical ladder then provides us with a natural way to introduce novel, more general algebraic structures, by extending the above definition to 2-categories, 3-categories, etc. In particular,

- a 2-group is a 2-category with only one object, while all 1-morphisms and 2-morphisms are invertible;
- a 3-group is a 3-category with only one object, while all 1-morphisms, 2-morphisms and 3-morphisms are invertible.

It is important to emphasize that an n-group is not a particular type of group. Instead, it is a different algebraic structure, which shares some of the features of groups, but is governed by a qualitatively different set of axioms.

The framework of higher gauge theory is centered around the idea that gauge symmetries in physics can be better described using these alternative algebraic structures than using the ordinary Lie groups. To that end, our attention will mostly focus on the so-called Lie 3-groups and their corresponding Lie 3-algebras. While the abstract definition in terms of n-category theory is particularly appealing from the conceptual point of view, for applications in physics there exists a more practical way to talk about 3-group. Namely, every strict Lie 3-group is known to be equivalent to a so-called 2-*crossed module*, defined as an exact sequence of three Lie groups G, H and L,

$$L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \tag{2}$$

and equipped with two "boundary homomorphisms" δ and ∂, an action $\triangleright$ of G onto G, H and L,

$$\triangleright : G \times G \to G\,, \qquad \triangleright : G \times H \to H\,, \qquad \triangleright : G \times L \to L\,,$$

and a bracket operation called *Peiffer lifting* over H to L,

$$\{\_\,,\_\} : H \times H \to L\,.$$

Certain set of axioms is assumed to hold true among all these maps. In particular, for all $g \in G$, $h \in H$ and $l \in L$, we have:

- the axiom stating that (2) is an exact sequence,

$$\partial\delta = 1_G\,, \tag{3}$$

- the axiom specifying that the action of G onto itself is conjugation,

$$g \triangleright g_0 = g\, g_0\, g^{-1} \,, \tag{4}$$

- the axioms stating that the action of G on H and L is equivariant with respect to homomorphisms ∂ and δ and the Peiffer lifting,

$$\begin{array}{lcl} g \triangleright \partial h & = & \partial(g \triangleright h)\,, \\ g \triangleright \delta l & = & \delta(g \triangleright l)\,, \\ g \triangleright \{h_1, h_2\} & = & \{g \triangleright h_1, g \triangleright h_2\}\,, \end{array} \tag{5}$$

- and finally the axioms determining the properties of the Peiffer lifting,

$$\begin{array}{lcl} \delta\,\{h_1, h_2\} & = & h_1 h_2 h_1^{-1} (\partial h_1) \triangleright h_2^{-1}\,, \\ \{\delta l_1, \delta l_2\} & = & l_1 l_2 l_1^{-1} l_2^{-1}\,, \\ \{h_1 h_2, h_3\} & = & \{h_1, h_2 h_3 h_2^{-1}\}\, \partial h_1 \triangleright \{h_2, h_3\}\,, \\ \{\delta l, h\}\,\{h, \delta l\} & = & l(\partial h \triangleright l^{-1})\,. \end{array} \tag{6}$$

Since it is constructed from three Lie groups, a Lie 3-group has a corresponding Lie 3-algebra, also called a *differential* 2-*crossed module*,

$$\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}\,,$$

where $\mathfrak{l}$, $\mathfrak{h}$, $\mathfrak{g}$ are Lie algebras of L, H, G, the maps δ, ∂, $\triangleright$ and $\{\_\ ,\_\}$ are inherited from the 3-group via natural linearization, and finally, the set of corresponding axioms applies. In addition to all this, Lie algebras have their own usual Lie structure — the generators,

$$T_A \in \mathfrak{l}\,, \qquad t_a \in \mathfrak{h}\,, \qquad \tau_\alpha \in \mathfrak{g}$$

the corresponding structure constants,

$$[T_A, T_B] = f_{AB}{}^C T_C\,, \qquad [t_a, t_b] = f_{ab}{}^c t_c\,, \qquad [\tau_\alpha, \tau_\beta] = f_{\alpha\beta}{}^\gamma \tau_\gamma\,,$$

and G-invariant nondegenerate symmetric bilinear forms (for example Killing forms),

$$\langle T_A, T_B \rangle_{\mathfrak{l}} = g_{AB}\,, \qquad \langle t_a, t_b \rangle_{\mathfrak{h}} = g_{ab}\,, \qquad \langle \tau_\alpha, \tau_\beta \rangle_{\mathfrak{g}} = g_{\alpha\beta}\,.$$

The main purpose of the 3-group structure is to *generalize the notion of parallel transport* from curves to surfaces to volumes. Namely, given a 4-dimensional manifold $\mathcal{M}$, one defines a 3-connection (α, β, γ) as a triple of 3-algebra-valued differential forms,

$$\begin{array}{lcll} \alpha & = & \alpha^\alpha{}_\mu(x)\, \tau_\alpha\, \mathbf{d}x^\mu & \in \Lambda^1(\mathcal{M}, \mathfrak{g})\,, \\ \beta & = & \dfrac{1}{2}\beta^a{}_{\mu\nu}(x)\, t_a\, \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu & \in \Lambda^2(\mathcal{M}, \mathfrak{h})\,, \\ \gamma & = & \dfrac{1}{3!}\gamma^A{}_{\mu\nu\rho}(x)\, T_A\, \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \wedge \mathbf{d}x^\rho & \in \Lambda^3(\mathcal{M}, \mathfrak{l})\,. \end{array}$$

Then one can introduce the line, surface and volume holonomies,

$$g = \mathcal{P}\mathrm{exp} \int_{\mathcal{P}_1} \alpha \,, \qquad h = \mathcal{S}\mathrm{exp} \int_{\mathcal{S}_2} \beta \,, \qquad l = \mathcal{V}\mathrm{exp} \int_{\mathcal{V}_3} \gamma \,,$$

and corresponding curvature forms,

$$\begin{array}{rcl} \mathcal{F} & = & \mathbf{d}\alpha + \alpha \wedge \alpha - \partial\beta \,, \\ \mathcal{G} & = & \mathbf{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma \,, \\ \mathcal{H} & = & \mathbf{d}\gamma + \alpha \wedge^{\triangleright} \gamma - \{\beta \wedge \beta\} \,. \end{array}$$

The 3-group structure ensures that all these quantities are well defined, in particular the surface- and volume-ordered exponentials and the respective holonomies.

3 Higher gauge theories

The basic idea behind the higher gauge theory approach is to employ the structure of n-groups as a mathematical representation of gauge symmetries in physics, generalizing the ordinary notion of gauge symmetry described via a Lie group. Namely, in ordinary gauge theory, the prototype action functional was the so-called BF action [7], based on a chosen gauge group G. In the HGT approach, one generalizes the BF action in accord with the chosen n-group structure, leading to the nBF action. For the case of 3-groups, one defines a $3BF$ action as:

$$S_{3BF} = \int_{\mathcal{M}} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,.$$

Here B, C, and D are Lagrange multipliers, in particular a $\mathfrak{g}$-valued 2-form, an $\mathfrak{h}$-valued 1-form, and an $\mathfrak{l}$-valued 0-form, respectively.

As in the case of a BF theory, one can demonstrate that $3BF$ theory is a topological gauge theory, having no local propagating degrees of freedom. Nevertheless, it can be transformed into a physically relevant action by adding the so-called *simplicity constraint terms* to the action, changing the dynamical structure of the theory. The prototype of this procedure is represented by transforming the topological BF theory based on the Lorentz group $SO(3,1)$ into a Plebanski action [8], which describes general relativity.

One can even do more, and provide a physical interpretation of the Lagrange multipliers C and D in the $3BF$ action, as follows:

- the $\mathfrak{h}$-valued 1-form C can be interpreted as the tetrad field, if $H = \mathbb{R}^4$ is the spacetime translation group,

$$C \to e = e^a{}_\mu(x)\, t_a \, \mathbf{d}x^\mu \,,$$

- the $\mathfrak{l}$-valued 0-form D can be interpreted as the set of real-valued matter fields, given some Lie group L,

$$D \to \phi = \phi^A(x)\, T_A \,.$$

An interested reader can see [16] for further details.

4 The Standard Model

One natural question that can be asked is what choice of a 3-group can be relevant for physics. There are various answers to this question, but perhaps the most illustrative example is a choice of the 3-group which reproduces the Standard Model of elementary particles, coupled to general relativity in the Einstein-Cartan version. This is called the *Standard Model 3-group*, and in the remainder of this section we will demonstrate how it can be constructed, step by step.

The first step is to specify the groups G and H as the usual Lorentz, internal, and translational symmetries:

$$G = SO(3,1) \times SU(3) \times SU(2) \times U(1) \,, \qquad H = \mathbb{R}^4 \,.$$

Note that the Poincaré group has been broken into the separate Lorentz and translational parts, and these have been associated with two different groups within the 3-group structure.

The next step is to define the homomorphisms δ and ∂, as well as the Peiffer lifting, to be trivial,

$$\delta l = 1_H = 0 \,, \qquad \partial \vec{v} = 1_G \,,$$

and

$$\{\vec{u}, \vec{v}\} = 1_L \,,$$

for all $l \in L$ and $\vec{u}, \vec{v} \in H$. Additionally, we define the action of the group G on H via vector representation for the $SO(3,1)$ sector and via trivial representation for the $SU(3) \times SU(2) \times U(1)$ sector. Finally, the choice of the group L and the action of G on L will be discussed below. But already now one can verify that all axioms (3)–(6) are satisfied, thus making sure that these choices represent one genuine 3-group.

The next step is to choose the group L. One general property of L that can be determined immediately comes from the second axiom in (6). Namely, due to the trivial choices for the Peiffer lifting and the homomorphism δ, the axiom implies that L must be Abelian. Aside from this, the choice of the group L is guided by physical requirements, as follows.

Begin by rewriting the $3BF$ action in the form

$$S_{3BF} = \int_{\mathcal{M}} B^\alpha \wedge \mathcal{F}^\beta g_{\alpha\beta} + e^a \wedge \mathcal{G}^b g_{ab} + \phi^A \mathcal{H}^B g_{AB} \,.$$

Since the group G is a direct product of the Lorentz and internal groups, the corresponding indices α of G split according to this structure, as $\alpha = (ab\,,\, i)$, leading to the corresponding splitting of the connection α and its curvature $\mathcal{F}$,

$$\alpha = \omega^{ab} J_{ab} + A^i \tau_i \,, \qquad \mathcal{F} = R^{ab} J_{ab} + F^i \tau_i \,.$$

Here ω^{ab} is the ordinary spin connection 1-form, J_{ab} are Lorentz generators, while A^i are internal gauge potential 1-forms and τ_i the generators of $SU(3) \times SU(2) \times U(1)$. Also, R^{ab} and F^i are the Riemann curvature and gauge field strength 2-forms, respectively. Also, given that the action of $SO(3,1)$ onto $H = \mathbb{R}^4$ is via vector representation, and given that the bilinear symmetric nondegenerate form for H must be G-invariant, the only available choice is

$$g_{ab} = \eta_{ab} \equiv \text{diag}(-1, +1, +1, +1) \,.$$

Finally, given that the matter fields are elements in the Lie algebra $\mathfrak{l}$ of the group L, namely $\phi = \phi^A T_A$, we observe that there should be precisely one real-valued field $\phi^A(x)$ for each generator $T_A \in \mathfrak{l}$. This information allows us to determine the dimension of the algebra $\mathfrak{l}$, by counting the total number of real-valued components of all matter fields in the Standard Model. The matter fields have two sectors — fermions and the Higgs.

The number of the real-valued components of all fermion fields can be counted according to the following scheme:

$$
\left.
\begin{array}{cccc}
\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L & \begin{pmatrix} u_r \\ d_r \end{pmatrix}_L & \begin{pmatrix} u_g \\ d_g \end{pmatrix}_L & \begin{pmatrix} u_b \\ d_b \end{pmatrix}_L \\[2ex]
(\nu_e)_R & (u_r)_R & (u_g)_R & (u_b)_R \\[2ex]
(e^-)_R & (d_r)_R & (d_g)_R & (d_b)_R
\end{array}
\right\} = 16 \ \frac{\text{Weyl spinors}}{\text{family}} \times
$$

$$
\times 3 \text{ families } \times 4 \ \frac{\text{real-valued fields}}{\text{Weyl spinor}} \ = 192 \text{ real-valued fields } \phi^A .
$$

Similarly, the Higgs sector gives us:

$$
\left. \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} \right\} = 2 \text{ complex scalar fields } = 4 \text{ real-valued fields } \phi^A .
$$

This suggests the structure for L in the form:

$$
L = L_{\text{fermion}} \times L_{\text{Higgs}} , \qquad \dim L_{\text{fermion}} = 192 , \qquad \dim L_{\text{Higgs}} = 4 .
$$

The structure of L can be further understood by looking at the action of the gauge group G on various components of fields ϕ^A. This is fixed by the choice of the action of G on L, chosen as follows. Given that G is constructed from Lorentz and internal gauge symmetry groups, the action $\triangleright : G \times L \to L$ specifies the transformation properties of each real-valued field ϕ^A with respect to those symmetries. For example, if we look at a Weyl spinor u_b that sits in the doublet

$$
\begin{pmatrix} u_b \\ d_b \end{pmatrix}_L ,
$$

the action $g \triangleright u_b$ (where $g \in SO(3,1) \times SU(3) \times SU(2) \times U(1)$) encodes that u_b consists of 4 real-valued fields which transform as:

- a left-handed spinor with respect to $SO(3,1)$,
- as a “blue” component of the fundamental representation of $SU(3)$,
- and as “isospin $+\frac{1}{2}$” of the left doublet with respect to $SU(2) \times U(1)$.

The action $\triangleright : G \times L \to L$ similarly defines the transformation properties for all other fermions in the theory, as well as for the Higgs field.

From such a definition of the action $\triangleright$, one can observe that G acts on L in precisely the same way across the three fermion families. This implies that L_{fermion} can be written as

$$
L_{\text{fermion}} = L_{\text{1st family}} \times L_{\text{2nd family}} \times L_{\text{3rd family}} , \qquad \dim L_{k\text{-th family}} = 64 .
$$

Ultimately, given that the components of Weys spinors mutually anticommute, given that the group L is Abelian, and given that it has the structure and dimension as given above, we can fix the choice of the group L which corresponds to the Standard Model as

$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \,,$$

where $\mathbb{G}$ is the algebra of Grassmann numbers. This completes the construction of the Standard Model 3-group.

The final step in specifying the theory is to spell out its classical action. As was previously discussed, the action has the form of a $3BF$ action, with the addition of appropriate simplicity constraints which will transform it into a non-topological theory, i.e., a theory with local propagating degrees of freedom. The choice of the Standard Model 3-group completely fixes the structure of the $3BF$ action, and the only thing left to do is to add the appropriate simplicity constraints. The details of the construction of these terms is given in detail in [16], and will not be repeated here. We will only quote the result,

$$S_{SM+EC} = S_{3BF} + S_{\text{simp}} \,,$$

where

$$S_{3BF} = \int B_{\hat{\alpha}} \wedge \mathcal{F}^{\hat{\alpha}} + e_{\hat{a}} \wedge \mathcal{G}^{\hat{a}} + \phi_{\hat{A}} \wedge \mathcal{H}^{\hat{A}} \,,$$

and

$$\begin{aligned} S_{\text{simp}} = & \left(B_{\hat{\alpha}} - C_{\hat{\alpha}}{}^{\hat{\beta}} M_{cd\hat{\beta}} e^c \wedge e^d\right) \wedge \lambda^{\hat{\alpha}} - \left(\gamma_{\hat{A}} - e^a \wedge e^b \wedge e^c C_{\hat{A}}{}^{\hat{B}} M_{abc\hat{B}}\right) \wedge \lambda^{\hat{A}} \\ & -4\pi i \, l_p^2 \, \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c \phi_{\hat{A}} T^{d\hat{A}}{}_{\hat{B}} \phi^{\hat{B}} \\ & +\zeta^{ab}{}_{\hat{\alpha}} \wedge \left(M_{ab}{}^{\hat{\alpha}} \varepsilon^{cdef} e_c \wedge e_d \wedge e_e \wedge e_f - F^{\hat{\alpha}} \wedge e_c \wedge e_d\right) \\ & +\zeta^{ab}{}_{\hat{A}} \wedge \left(M_{abc}{}^{\hat{A}} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - F^{\hat{A}} \wedge e_a \wedge e_b\right) \\ & -\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \, \left(\Lambda + M_{\hat{A}\hat{B}} \phi^{\hat{A}} \phi^{\hat{B}} + Y_{\hat{A}\hat{B}\hat{C}} \phi^{\hat{A}} \phi^{\hat{B}} \phi^{\hat{C}} + L_{\hat{A}\hat{B}\hat{C}\hat{D}} \phi^{\hat{A}} \phi^{\hat{B}} \phi^{\hat{C}} \phi^{\hat{D}}\right) \,. \end{aligned}$$

See [16] for details and notation.

By varying the action with respect to all variables, and with a little technical effort, one can demonstrate that the corresponding equations of motion are precisely the classical equations of the Standard Model, coupled to general relativity in the Einstein-Cartan formulation.

5 Conclusions

Let us summarize the results of the paper. In section 2 we have given a short introduction into the category theory, introduced the notions of categorical ladder and n-categories, and in the resulting framework, provided a definition for the notion of an n-group. Our attention focused on the case of 3-groups, which are relevant for applications in physics, and the equivalent notion of a 2-crossed module, which is more convenient for practical applications. Section 3 was devoted to introducing the higher gauge theory formalism and the $3BF$ action corresponding to a choice of a 3-group, as a generalization of the well known BF action in terms of the categorical ladder. Also, we have interpreted the additional Lagrange multipliers appearing in the $3BF$ action as the tetrad and matter fields, providing the setup for the application in physics. This application was then demonstrated in detail in section 4, where the Standard Model 3-group has been defined, and utilized to construct a physically relevant constrained $3BF$ action, which is classically equivalent to the Standard Model of elementary particles coupled to general relativity in the Einstein-Cartan formulation. This is the main result, which successfully establishes the first step of the spinfoam quantization procedure, and opens up a possibility of straightforward implementation of the second and third steps, hopefully leading to a full model of quantum gravity with matter.

It should be noted that the most important feature of the higher gauge theory framework is its ability to treat gravity, gauge fields, fermions and scalar fields on completely equal footing, describing all of them via the underlying algebraic structure of a 3-group. The 3-group also provides us with a natural geometric description of a generalized notion of parallel transport, namely along a surface and along a volume, in addition to the standard notion of parallel transport along a curve. This relationship opens up a possibility for a fully geometric interpretation of all fields present in physics.

Moreover, just as the gauge group dictates the number and properties of gauge fields in Yang-Mills theories, the sector of the 3-group described by the Lie group L determines the number and properties of the fermion and scalar fields. This fact enables us to classify the spectrum of matter fields in terms of group theory, generalizing the constructions present in the Standard Model, where only gauge fields are classified in such terms. The choice of the group L thus opens up novel avenues for research on the unification of all fields, and specifically the origin of particle families, Higgs and fermion sectors, and so on.

Finally, the higher gauge theory framework may have applications in other areas of physics and mathematics as well, and various possible research directions are yet to be explored.

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Rezime

U ovoj tezi je predstavljena kategorijska generalizacija BF teorije na $2BF$ i $3BF$ teorije, prelaskom sa pojma gejdž grupe simetrija na pojmove gejdž 2-grupe i gejdž 3-grupe, u okviru formalizma više gejdž teorije. Razmatrana su $2BF$ dejstva sa vezama koja opisuju teoriju gravitacije i Jang-Milsovog polja i $3BF$ dejstva sa vezama koja opisuju teoriju Klajn-Gordonovog, Dirakovog, Vajlovog i Majorana polja kuplovanog sa Ajnštajn-Kartanovom gravitacijom. Ova klasična dejstva napisana su u obliku koji je prilagođen kvantizacionoj proceduri spinske pene, tj. prirodno su podeljena na dva sektora: topološki sektor i sektor sa vezama. U okviru ovih $3BF$ teorija, struktura 3-grupe dovodi do pojave nove gejdž grupe koja određuje spektar polja materije prisutne u teoriji, na sličan način kao što obična gejdž grupa određuje spektar gejdž bozona u Jang-Milsovoj teoriji. Ovakva formulacija polja materije i gravitacije nam omogućava da prepišemo ceo Standardni Model kuplovan sa gravitacijom kao $3BF$ dejstvo sa vezama i dovodi nas korak bliže konstruisanju unifikovanog opisa i neperturbativne kvantizacije polja gravitacije i materije. U nastavku je određena kompletna gejdž grupa simetrije topološkog $3BF$ dejstva sprovođenjem kompletne Hamiltonove analize $3BF$ dejstva za proizvoljnu semistriktnu Lijevu 3-grupu, koristeći Dirakovu proceduru. Određivanje ukupne grupe simetrija je važan korak u kanonskoj kvantizaciji teorije kompletnog Standardnog Modela elementarnih čestica kuplovanog sa Ajnštajn-Kartanovom gravitacijom, formulisanog kao $3BF$ dejstvo sa vezama. Rezultujuća gejdž grupa simetrije se sastoji od pet vrsta transformacija: G-, H-, L-, M- i N-gejdž transformacija. Pokazuje se da je razmatrana teorija invarijantna na difeomorfizme, jer je $3BF$ dejstvo sa vezama manifestno kovarijantno, a grupa lokalnih translacija dobijena je kao podgrupa direktnog proizvoda ukupne gejdž simetrije i Eno-Tajtelboim transformacija. Kao važan korak u kovarijantnoj kvantizacionoj proceduri spinske pene, razmatrana je topološka suma po stanjima Z, konstruisana za klasično $3BF$ dejstvo za generalnu 3-grupu i 4-dimenzionalnu prostorvremensku mnogostrukost $\mathcal{M}_4$. Konstruisana suma po stanjima specijalan je slučaj Porterove topološke kvantne teorije polja za $d = 4$ i $n = 3$ i topološka je invarijanta 4-dimenzionalne mnogostrukosti, što je utvrđeno ispitivanjem njene invarijantnosti pri Pahnerovim potezima. Ova suma po stanjima je generalizacija sume po stanjima koju su formulisali Žireli, Pfajfer i Popesku za slučaj $2BF$ dejstva sa odgovarajućom strukturom 2-grupe.

Ključne reči: BF teorija, teorija kategorija, više gradijentne teorije, $3BF$ teorija.
Naučna oblast: Fizika
Uža naučna oblast: Teorijska fizika visokih energija
UDK broj: 539.120.226 (043.3)

Abstract

In this thesis we study the categorical generalizations of a BF theory to $2BF$ and $3BF$ theories, by passing from the notion of a gauge group to the notions of a gauge 2-group and a gauge 3-group in the framework of higher gauge theory. In particular, we construct the constrained $2BF$ actions describing the dynamics of the gravitational and the Yang-Mills fields, and the the constrained $3BF$ actions describing the dynamics of Klein-Gordon, Dirac, Weyl and Majorana fields coupled to Einstein-Cartan gravity. The action is naturally split into a topological sector and a sector with simplicity constraints, adapted to the spinfoam quantization program. In addition, the structure of the 3-group gives rise to a novel gauge group which specifies the spectrum of matter fields present in the theory, just like the ordinary gauge group specifies the spectrum of gauge bosons in Yang-Mills theory. This allows us to rewrite the whole Standard Model coupled to gravity as a constrained $3BF$ action, facilitating the nonperturbative quantization of both gravity and matter fields. We determine the full gauge symmetry of the $3BF$ action by carrying out the complete Hamiltonian analysis of the $3BF$ action for an arbitrary semistrict Lie 3-group, using the Dirac procedure. This analysis is an important step in the canonical quantization of the complete Standard Model of elementary particles coupled to Einstein-Cartan gravity, formulated as a $3BF$ action with suitable simplicity constraints. We show that the resulting gauge symmetry group consists of the already familiar G-, H-, and L-gauge transformations, as well as additional M- and N-gauge transformations, which have not been discussed in the existing literature. As expected, since the $3BF$ action is formulated in a manifestly covariant way, we establish that diffeomorphisms are a symmetry of the theory, and are obtained as a subgroup of the direct product of the full gauge symmetry group and the Henneaux-Teitelboim transformations. As an important step in the covariant spinfoam quantization of the theory, we construct a triangulation independent topological state sum Z, based on the classical $3BF$ action for a general 3-group and a 4-dimensional spacetime manifold $\mathcal{M}_4$. The obtained state sum coincides with Porter's TQFT for $d = 4$ and $n = 3$. In order to verify that the constructed state sum is a topological invariant of the underlying 4-dimensional manifold, we analyze its behavior under Pachner moves, and we obtain that the state sum Z remains the same. The constructed state sum is a generalization of the work done by Girelli, Pfeiffer, and Popescu for the case of state sum based on the topological $2BF$ action with the underlying 2-group structure.

Key words: BF theory, category theory, higher gauge theory, $3BF$ theory.
Scientific field: Physics
Research area: Theoretical high energy physics
UDC number: 539.120.226 (043.3)

Sadržaj

Slike

Tabele

Deo I

Klasična teorija

Glava 1

Uvod

U okviru teorije *Kvantne Gravitacije na Petljama*, moguće je proučavati neperturbativnu kvantizaciju gravitacije, kovarijantnim ili kanonskim pristupom (za detaljan uvod u pristupe kvantovanja gravitacije u okviru ove teorije pogledati [1]–[4]). *Kovarijantni pristup* ima za primarni cilj definisanje konfiguracionog integrala gravitacionog polja,

$$Z = \int \mathcal{D}g \; e^{iS[g]} \,, \tag{1.1}$$

razmatranjem triangulacije $\mathcal{T}(\mathcal{M}_D)$ prostorvremenske mnogostrukosti $\mathcal{M}_D$ i definisanjem integrala kao diskretizovane *sume po stanjima* konfiguracija gravitacionog polja na simpleksima koji čine triangulaciju. Suma po stanjima triangulacije $\mathcal{T}(\mathcal{M}_D)$ mnogostrukosti $\mathcal{M}_D$ u D dimenzija je definisana kao

$$Z = \sum_{\{\phi\}} \prod_{v\in\mathcal{T}} \mathcal{A}_v(\phi) \prod_{\epsilon\in\mathcal{T}} \mathcal{A}_\epsilon(\phi) \cdots \prod_{\sigma_D\in\mathcal{T}} \mathcal{A}_{\sigma_D}(\phi), \tag{1.2}$$

gde su proizvodi po svim verteksima v, ivicama ϵ, trouglovima Δ, tetraedrima τ, sve do D-simpleksa σ_D koje triangulacija sadrži. Svaka ova ćelija obojena je bojama ϕ koje opisuju fundamentalne varijable modela, a svakoj ćeliji dodeljena je amplituda $\mathcal{A}$ koja opisuje dinamiku varijabli ϕ. Na ovaj način možemo definisati *konfiguracioni integral* (1.1). Primetimo da amplitude $\mathcal{A}_v(\phi)$, $\mathcal{A}_\epsilon(\phi)\,, \dots \,, \mathcal{A}_{\sigma_D}(\phi)$ delom ulaze u definiciju mere konfiguracionog integrala, delom u definiciju dejstva $S[\phi]$.

Ova tehnika kvantizacije je poznata kao *kvantizacioni metod spinske pene.* U okviru formalizma *spinskih pena*[1], a zatim i njegove generalizacije u formalizmu teorije kategorija – *spin-kub modela*[2], podrazumeva se *kovarijantni pristup* kvantovanju gravitacije u kom se konfiguracioni integral definiše na isti način na koji je to urađeno u *Fajnmanovoj definiciji integrala po putanjama*, a motivisan je *kanonskom kvantizacijom na petljama.* Ovaj pristup se može podeliti na tri glavna koraka.

1. Prvo, definiše se klasično dejstvo $S[g]$ koje čine dva sektora: sektor topološke BF teorije i sektor koji sadrži veze.

2. Zatim, koristeći algebarsku strukturu, tj. Lijevu grupu G koja odgovara topološkom sektoru teorije, definiše se suma po stanjima Z_{BF} nezavisna od triangulacije $\mathcal{T}(\mathcal{M}_4)$.

3. Najzad, nametanjem veza na topološku sumu po stanjima Z_{BF} dobija se suma po stanjima koja odgovara pravoj fizičkoj teoriji.

[1]eng. *spin foam model.*
[2]eng. *spin cube model.*

Ovaj metod kvantovanja gravitacije uspešno je primenjen za različite izbore dejstva $S[g]$, Lijeve grupe G, kao i dimenzije prostorvremenske mnogostrukosti D.

Topološka BF teorija definiše se zadavanjem gejdž invarijantnog BF dejstva koje je definisano za Lijevu grupu G, odnosno njenu odgovarajuću Lijevu algebru $\mathfrak{g}$:

$$S_{BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} \,. \tag{1.3}$$

Ovde je $\mathcal{F} \equiv \mathrm{d}\alpha + \alpha \wedge \alpha$ 2-forma element algebre $\mathfrak{g}$, krivina za 1-formu koneksiju $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$, definisanu na glavnom G-raslojenju neke 4-dimenzionalne prostorvremenske mnogostrukosti $\mathcal{M}_4$. Lagranžev množitelj $B \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{g})$ je 2-forma, dok $\langle \_\,,\_ \rangle_{\mathfrak{g}}$ označava G-invarijantnu bilinearnu simetričnu nedegenerisanu formu.

Kratak pregled klasične BF teorije biće prikazan u poglavlju 4. Sprovešćemo kompletnu Hamiltonovu analizu topološkog BF dejstva, a zatim ćemo prebrojati stepene slobode u teoriji. Kao što i očekujemo, dobićemo da BF dejstvo opisuje teoriju bez lokalnih propagirajućih stepeni slobode. Kastelanijevom procedurom biće izračunat generator gejdž simetrije u BF teoriji, a zatim ćemo izračunati varijacije formi za sve varijable u teoriji i njihove konjugovane impulse. Na osnovu ovih rezultata, dobićemo dva tipa gejdž transformacija simetrije u BF teoriji — G-gejdž i M-gejdž transformacije, koje su već poznate u literaturi, kao i komutacione relacije ukupne grupe gejdž simetrije BF dejstva, grupe $\mathcal{G}_{BF}$. Demonstriraćemo da je BF teorija invarijantna na difeomorfizme. Zatim, razmatraćemo dva za fiziku relevantna BF modela, dobijena dodavanjem odgovarajućih članova, *veza*, u BF dejstvo — *Jang-Milsovu teoriju* za $SU(N)$ grupu u prostoru Minkovskog i *Plebanski model* za Opštu relativnost.

U okviru formalizma BF teorije, trodimenzionalna kvantna gravitacija prvi put je definisana u *Ponzano-Redže modelu* za triangulaciju trodimenzionalne mnogostrukosti za grupu $SU(2)$ [5]. Svakoj ivici triangulacije dodeljena je jedna ireducibilna reprezentacija grupe $SU(2)$, pa su amplitude sumirane po svim mogućim konfiguracijama ovih spinova. U 4-dimenzionalnom slučaju definisani su raznovrsni modeli, od kojih su jedni od najpoznatijih *Baret-Krejn model* [6], [7] i *Oguri model* [8]. Najzad, najsofisticiraniji - tzv. *EPRL/FK model* je model $4D$ kvantne gravitacije koji su nezavisno razvili Dž. Engl, R. Pereira, E. Livajn i K. Roveli [9] i L. Fridel i K. Krasnov [10], u okviru formalizma *spinskih pena.* Konkretan izbor polja u $EPRL/FK$ modelu motivisan je rezultatom iz kanonske kvantizacije na petljama [1], gde je stanje gravitacionog polja opisano tzv. *spinskim mrežama*, koje su obojene polucelim brojevima $i, j \in \mathbb{N}/2$. Ovi modeli predstavljaju nezavisne pokušaje da se definiše kvantna teorija gravitacije sa različitim stepenima uspeha, pri čemu su svi fokusirani na definisanje teorije čiste gravitacije bez materije. Pokušaji da se u teoriju uključi materija imali su ograničenog uspeha [11], uglavnom zbog činjenice da maseni članovi ne mogu biti izraženi u okviru ove teorije. Razlog leži u tome što polja tetrada nisu prisutna u topološkom sektoru BF teorije.

Kovarijantno kvantovanje gravitacije u okviru BF teorija, odnosno konstrukcija topološke BF sume po stanjima u slučajevima trodimenzionalne i četvorodimenzionalne mnogostrukosti uobičajenom kvantizacionom procedurom spinske pene, biće razmatrani u poglavlju 7. U trodimenzionalnom slučaju, polazeći od topološke BF teorije konstruisaćemo sumu po stanjima koja opisuje *Ponzano-Redže model*, što je posledica činjenice da na klasičnom nivou teorija trodimenzionalne gravitacije nema lokalne propagirajuće stepene slobode. Zatim, u četvorodimenzionalnom slučaju konstruisaćemo BF topološku sumu po stanjima koja odgovara *Ouguri modelu.* Međutim, u realnom slučaju četvorodimenzionalne prostorvremenske mnogostrukosti, situacija je komplikovanija. Naime, u $4D$ teorija gravitacije nije topološka teorija, pa dobijena suma po stanjima ne odgovara fizičkoj teoriji, a kvantna teorija gravitacije može se dobiti tek modifikacijom amplituda topološke sume po stanjima. Ipak, poslednjim, trećim korakom kvantizacione procedure spinske pene se nećemo baviti u okviru ove disertacije.

U cilju prevazilaženja problema sa kuplovanjem materije u BF modelima kvantne gravitacije, u okviru formalizma *teorije kategorija* razvija se nov pristup koji koristi kategorijsku generalizaciju BF dejstva, u kontekstu *viših gejdž teorija* [12]. Kratak uvod u formalizam teorije viših kategorija dat je u poglavlju 2. U teoriji kategorija, grupa se definiše kao kategorija sa samo jednim objektom gde su svi morfizmi invertibilni. Pojam kategorije može se generalizovati na takozvane *više kategorije*, koje osim objekata i morfizama, kao elemente imaju i 2-morfizme (morfizme između morfizama) itd. Tada je moguće definisati tzv. 2-*grupu* kao 2-kategoriju sa samo jednim objektom gde su svi morfizmi i 2-morfizmi invertibilni. Konkretno, koristi se ideja *kategorijskih lestvica*, videti Tabelu 1.1. Kategorijskom generalizacijom BF dejstva, koje je definisano za neku Lijevu grupu, dolazimo do $2BF$ dejstva, koje je definisano za određenu 2-grupu. Pokazano je da je svaka striktna 2-grupa ekvivalentna nekom *ukrštenom modulu* $(H \stackrel{\partial}{\to} G\,, \triangleright)$, gde su G i H Lijeve grupe, $\partial : H \to G$ homomorfizam iz H u G, dok je $\triangleright : G \times H \to H$ dejstvo grupe G na grupu H.

kategorijska struktura	algebarska struktura	linearna struktura	topološko dejstvo	stepeni slobode
Lijeva grupa	Lijeva grupa	Lijeva algebra	BF teorija	gejdž polja
Lijeva 2-grupa	Lijev ukršteni modul	diferencijalni Lijev ukršteni modul	$2BF$ teorija	tetrade
Lijeva 3-grupa	Lijev 2-ukršteni modul	diferencijalni Lijev 2-ukršteni modul	$3BF$ teorija	skalarna i fermionska polja

Tabela 1.1: Kategorijske lestvice.

Kao što je to slučaj kod Lijeve grupe G za koju se prirodno definiše koneksija α na glavnom G-raslojenju neke 4-dimenzionalne prostorvremenske mnogostrukosti $\mathcal{M}_4$, a zatim i BF dejstvo, struktura 2-grupe prirodno daje uređeni par 2-*koneksiju* $(\alpha\,,\beta)$. Ovde je $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$ uobičajna 1-forma element algebre $\mathfrak{g}$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$ nova koneksija 2-forma element algebre $\mathfrak{h}$, gde je $\mathfrak{h}$ je Lijeva algebra za Lijevu grupu H. Za 2-koneksiju definiše se tzv. *lažna* 2-*krivina*, urđeni par $(\mathcal{F},\mathcal{G})$, na sledeći način [12]:

$$\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = d\beta + \alpha \wedge^{\triangleright} \beta\,. \tag{1.4}$$

Ovde $\alpha \wedge^{\triangleright} \beta$ označava da su varijable α i β pomnožene kao forme $\wedge$-proizvodom i istovremeno kao elementi algebre dejstvom $\triangleright$ algebre $\mathfrak{g}$ na algebru $\mathfrak{h}$. Uređeni par krivina $(\mathcal{F},\mathcal{G})$ se naziva lažnom zbog prisustva člana $\partial\beta$ u definiciji $\mathcal{F}$, videti poglavlje 5 za detalje.

Koristeći ove varijable, može se definisati $2BF$ dejstvo, koje je gejdž invarijantno pri 2-gejdž transformacijama [13], [14],

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}}\,, \tag{1.5}$$

gde su 2-forma $B \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{g})$ i 1-forma $C \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{h})$ Lagranževi množitelji. Takođe, $\langle \_\,,\_ \rangle_{\mathfrak{g}}$ i $\langle \_\,,\_ \rangle_{\mathfrak{h}}$ redom označavaju G-invarijantnu bilinearnu simetičnu nedegenerisanu formu za algebre $\mathfrak{g}$ i $\mathfrak{h}$.

Kada se izabere na pogodan način, struktura 2-grupe uvodi tetradna polja u topološko $2BF$ dejstvo, kao što je to uspešno urađeno u [15]. U ovom radu, dejstvo za Opštu relativnost napisano je kao $2BF$ dejstvo za vezama za određeni izbor 2-grupe, tako da su polja tetrade prisutna u topološkom sektoru teorije. Ovaj rezultat otvorio je mogućnost kuplovanja materije sa gravitacijom na pravolinijski način.

U poglavlju 5 ćemo analizirati klasičnu $2BF$ teoriju. Slično kao i za BF dejstvo, Hamiltonovom analizom i prebrojavanjem stepeni slobode ćemo pokazati da je $2BF$ dejstvo takođe topološko, tj. da opisuje teoriju bez lokalnih propagirajućih stepeni slobode. Dobićemo kako glase konačne transformacije simetrije za $2BF$ dejstvo: G-gejdž, H-gejdž, M-gejdž i N-gejdž transformacija i komutacione relacije ukupne gejdž grupe simetrija $\mathcal{G}_{2BF}$. Pokazaćemo difeomorfizam invarijantnost $2BF$ teorije. Zatim, pogodnim izborom 2-grupe prikazaćemo *Opštu relativnost* kao $2BF$ teoriju sa vezama kao što je to učinjeno u [15]. Na kraju, poslednji odeljak poglavlja 5 biće posvećen diskusiji *Ajnštajn-Jang-Milsove teorije*, odnosno teoriji gravitacije i gejdž polja formulisanoj kao $2BF$ teorija sa vezama [16].

U poglavlju 8 ćemo sprovesti drugi korak kovarijantne kvantizacione procedure spinske pene za $2BF$ teoriju [13]. Demonstriraćemo kako se konstruiše suma po stanjima Z koja je nezavisna od triangulacije, na osnovu klasičnog $2BF$ dejstva za opštu striktnu 2-grupu i bilo koju triangulaciju bilo koje glatke d-dimenzionalne prostorvremenske mnogostrukosti, za slučaje $d \in \{3, 4\}$. Za $d = 3$, kontruisana suma po stanjima je upravo Jeterov model, dok se za $d = 4$ poklapa sa Porterovom TKTP za $d = 4$ i $n = 2$. Analiziraćemo ponašanje konstruisane sume po stanjima pri Pahnerovim potezima, lokalnim promenama triangulacije koje čuvaju topologiju, tako da su bilo koje dve triangulacije iste mnogostrukosti povezane konačnim brojem Pahnerovih poteza. U trodimenzionalnom slučaju postoje četiri Pahnerova poteza — potezi $1 \leftrightarrow 4$ i $2 \leftrightarrow 3$ i njihovi inverzi, dok u 4 dimenzije postoji pet različitih Pahnerovih poteza — potezi $3 \leftrightarrow 3$, $4 \leftrightarrow 2$ i $5 \leftrightarrow 1$ i njihovi inverzi. Postavku analize ponašanja konstruisane sume po stanjima pri ovim Pahnerovim potezima predstavićemo u odeljku 8.3, dok su detalji računa dati u Dodatku E.1. Dobićemo da $2BF$ suma po stanjima ostaje nepromenjena pri ovim transformacijama triangulacije, što dokazuje da je to jedna *topološka invarijanta* mnogostrukosti.

Ipak, dok struktura 2-grupe može prirodno da opiše gravitaciono i vektorsko polje, polja materije ne mogu biti prirodno izražena u okviru algebarske strukture 2-grupe, tj. sektor materije u dejstvu ne može biti napisan kao topološki član plus veza. To čini ovako napisano dejstvo tek polovično pripremljenim za kvantizacionu proceduru spinske pene, pa drugi i treći korak kvantizacione procedure ne mogu biti sprovedeni za celokupnu teoriju gravitacije i materije. Upravo to je problem na koji ćemo se fokusirati u okviru ove doktorske disertacije.

U cilju konstruisanja unifikovanog opisa gravitacije i materije, predložen je još jedan korak generalizacije primenom kategorijskih lestvica, tj. generalizacija algebarske strukture sa 2-grupe na 3-grupu [16]. Nivoi kategorijskih lestvica su prikazani u Tabeli 1.1. Kako se ispostavlja, struktura 3-grupe uspešno daje opis svih polja prisutnih u Standardnom Modelu, u interakciji sa gravitacijom. Pored toga, ova struktura uvodi novu gejdž grupu, koja odgovara skalarnim i fermionskim poljima prisutnim u teoriji [16]. Ovo je nov i neočekivan rezultat, koji ima potencijal da otvori novi pravac istraživanja i ponudi objašnjenje za strukturu sektora materije prisutne u Standardnom Modelu, kao i izvan.

Struktura 3-grupe definiše se u okviru teorije kategorija kao 3-kategorija sa samo jednim objektom gde su svi morfizmi, 2-morfizmi i 3-morfizmi invertibilni. Slično kao što je striktna 2-grupa ekvivalentna ukrštenom modulu, pokazano je da je svaka semistriktna 3-grupa ekvivalentna nekom 2-*ukrštenom modulu* [17]. Lijev 2-ukršteni modul $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\}_{\rm pf})$ je algebarska struktura zadata trima Lijevim grupama G, H i L, zajedno sa homomorfizmima $\delta : L \to H$ i $\partial : H \to G$, dejstvom $\triangleright$ grupe G na sve tri grupe, kao i G-ekvivarijantnim preslikavanjem

$$\{\_\,,\_\}_{\rm pf} : H \times H \to L\,,$$

koje se naziva Pajferovo podizanje.

Analogno konstrukciji BF i $2BF$ topološkog dejstva, može se definisati topološko $3BF$ dejstvo za mnogostrukost $\mathcal{M}_4$ i 2-ukršteni modul $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\}_{\rm pf})$. Za Lijeve algebre $\mathfrak{g}$, $\mathfrak{h}$ i $\mathfrak{l}$ asocirane sa Lijevim grupama G, H i L, prirodno se uvodi uređena trojka 3-*koneksija* (α, β, γ), gde su diferencijalne forme elementi algebri $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{h})$

i $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,,\mathfrak{l})$. Odgovarajuća *lažna* 3-*krivina* $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ je:

$$\mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha\wedge^{\rhd}\beta - \delta\gamma\,, \qquad \mathcal{H} = \mathrm{d}\gamma + \alpha\wedge^{\rhd}\gamma + \beta\wedge^{\{,\}}\beta\,. \tag{1.6}$$

Videti [17], [18] za detalje. Sada, moguće je definisati $3BF$ dejstvo:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}} + \langle C\wedge\mathcal{G}\rangle_{\mathfrak{h}} + \langle D\wedge\mathcal{H}\rangle_{\mathfrak{l}}\,, \tag{1.7}$$

gde su $B\in\mathcal{A}^2(\mathcal{M}_4,\mathfrak{g})$, $C\in\mathcal{A}^1(\mathcal{M}_4,\mathfrak{h})$ i $D\in\mathcal{A}^0(\mathcal{M}_4,\mathfrak{l})$ Lagranževi množitelji. Forme $\langle\_\,,\_\rangle_{\mathfrak{g}}$, $\langle\_\,,\_\rangle_{\mathfrak{h}}$ i $\langle\_\,,\_\rangle_{\mathfrak{l}}$ su G-invarijantne bilinearne simetrične nedegenerisane forme na $\mathfrak{g}$, $\mathfrak{h}$ i $\mathfrak{l}$, redom.

Za teoriju gravitacije i materije formulisanu kao $3BF$ dejstvo sa vezama za određenu 3-grupu potrebno je sprovesti kovarijantnu kvantizacionu proceduru i kanonsku kvantizacionu proceduru, pri čemu ćemo se mi za sada fokusirati na prve korake.

Kanonska kvantizaciona procedura. Prvi korak ka kanonskoj kvatizaciji teorije je Hamiltonova analiza, koja rezultira algebrom veza prve klase i veza druge klase prisutnih u teoriji. Veze prve klase postaju uslovi na fizička stanja koja određuju Hilbertov prostor, dok Hamiltonova veza određuje dinamiku. Sa tim ciljem, u prvom koraku kanonske kvantizacije $3BF$ teorije, fokusirali smo se na pronalaženje kompletne gejdž grupe simetrije topološkog $3BF$ dejstva [19]. Kompletna Hamiltonova analiza $3BF$ dejstva za opštu semistriktnu Lijevu 3-grupu korišćenjem Dirakove procedure biće prikazana u poglavlju 6. Ovaj postupak je generalizacija Hamiltonove analize $2BF$ dejstva izvedene u [20]–[23] i Hamiltonove analize za poseban slučaj 2-ukrštenog modula koja odgovara teoriji skalarne elektrodinamike, sprovedene u [24]. Analiza Hamiltonove strukture teorije daje nam algebru veza prve klase i veza druge klase prisutnih u teoriji. Kao i obično, veze prve klase generišu gejdž transformacije, koje ne menjaju fizičko stanje sistema. Nakon izračunavanja veza prve klase, korišćenjem Kastelanijeve procedure izračunaćemo generator gejdž transformacija na prostornoj hiperpovrši, a zatim će rezultati dobijeni ovom metodom biti generalizovani na čitavo prostorvreme. Dobićemo kompletnu gejdž simetriju topološke $3BF$ teorije, koja se sastoji od pet vrsta konačnih gejdž transformacija — G-gejdž, H-gejdž i L-gejdž transformacije koje su već poznate iz prethodne literature, kao i dodatne M-gejdž i N-gejdž transformacije koje su jedan od naših glavnih rezultata. Uzimajući u obzir ovaj rezultat, analiziraćemo strukturu kompletne gejdž grupe simetrija $\mathcal{G}_{3BF}$. Dobijeni rezultati dovode do veze između grupe gejdž simetrije $3BF$ dejstva i strukture 3-grupe na kojoj je zasnovano $3BF$ dejstvo. Pokazaćemo da difeomorfizam invarijantnost $3BF$ teorije. Ova analiza je važan korak ka proučavanju gejdž grupe simetrije teorije gravitacije sa materijom, formulisane kao $3BF$ dejstvo sa vezama [16], kao i njene kanonske kvantizacije.

Prvi korak kvantizacione procedure spinske pene. Nakon što odredimo odgovarajuće 3-grupe i konstruišemo odgovarajuća $3BF$ dejstva, potrebno je nametnuti odgovarajuće veze na stepene slobode prisutne u topološkom sektoru $3BF$ dejstva, tako da dobijemo željenu klasičnu dinamiku polja materije i gravitacije. Nametanjem odgovarajućih veza na dejstvo (1.7) moguće je dobiti dejstva za polja materije u interakciji sa gravitacijom. Pored prethodno definisanih $2BF$ dejstva sa vezama za *Jang-Milsovo* i *Proka vektorsko polje*, u poglavlju 6 ćemo konstruisati odgovarajuća $3BF$ dejstva sa vezama za slučajeve *Klajn-Gordonovog*, *Dirakovog*, *Vejlovog* i *Majorana polja* u interakciji sa Ajnštajn-Kartanovom gravitacijom. Ova konstrukcija će nas dovesti do neočekivanog novog rezultata. Kao što ćemo videti, skalarno i fermionsko polje će biti *prirodno pridruženo novoj gejdž grupi*, na taj način generalizujući pojam gejdž grupe Jang Milsove teorije. Nova gejdž grupa otvara mogućnost klasifikacije polja materije i opisa struktura poput familije kvarkova i leptona itd. No, s obzirom na složenost algebarskih svojstava 3-grupa u okviru novog formalizma teorije kategorija, u ovom koraku našeg istraživanja fokusirali smo se samo na rekonstrukciju već poznatih teorija, poput Standardnog Modela. U tom smislu, svako potencijalno objašnjenje spektra polja materije u SM ostavljeno je za budući rad.

Drugi korak kvantizacione procedure spinske pene. Sproveden prvi korak kvantizacione procedure spinske pene, nagoveštaj je mogućnosti realizovanja drugog i trećeg koraka takođe, imajući u vidu da je dejstvo napisano preko diferencijalnih formi, što nam dozvoljava da ga prilagodimo na prostorvremensku deo-po-deo ravnu mnogostrukost korišćenjem Redže računa [25]. Konkretno, sva polja i jačine polja prisutna u $3BF$ dejstvu mogu se prirodno dodeliti nekom d-dimenzionalnom simpleksu 4-dimenzionalne triangulacije, dodeljivanjem 0-forme verteksu, odnosno 0-dimenzionalnom objektu, 1-forme ivicama, itd. Ovo nas dovodi do Tabele 1.2.

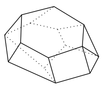
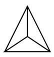
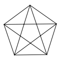
d	triangulacija	dualna triangulacija	forma	polja	jačine polja
0	verteks •	4-kompleks	0-forma	ϕ, $\psi_{\tilde{\alpha}}$, $\bar{\psi}^{\tilde{\alpha}}$	
1	ivica ⁄	3-poliedar 	1-forma	ω^{ab}, A^I, e^a	
2	trogao △	poligon 	2-forma	β^a, B^{ab}	R^{ab}, F^I, T^a
3	tetraedar 	ivica ⁄	3-forma	γ, $\gamma_{\tilde{\alpha}}$, $\bar{\gamma}^{\tilde{\alpha}}$	$\mathcal{G}^a$
4	4-simpleks 	verteks •	4-forma		$\mathcal{H}$, $\mathcal{H}_{\tilde{\alpha}}$, $\bar{\mathcal{H}}^{\tilde{\alpha}}$

Tabela 1.2: Korespodencija između polja i jačina polja i elemenata triangulacije $\mathcal{T}(\mathcal{M}_4)$.

Jednom kada je klasično Redže-diskretizovano $3BF$ dejstvo konstruisano, sledeći korak kvantizacione procedure je definisanje sume po stanjima Z koja definiše konfiguracioni integral teorije. Topološka priroda $3BF$ dejstva, zajedno sa strukturom gejdž 3-grupe, obezbeđuje da takva suma bude topološka invarijanta, odnosno da je nezavisna od triangulacije. Za klasično $3BF$ dejstvo u slučaju generalne semistriktne 3-grupe i 4-dimenzionalne prostorvremenske mnogostrukosti $\mathcal{M}_4$, u poglavlju 9 ćemo formulisati topološku sumu po stanjima Z nezavisnu od trijagulacije. Ova suma po stanjima podudara se sa Porterovom apstraktnom definicijom topološke kvantne teorije polja (TKTP) [26] za slučaj $d = 4$ i $n = 3$, gde je d predstavlja dimenziju mnogostrukosti $\mathcal{M}$, a n nivo kategorijskih lestvica. Ova definicija je generalizacija Jeterove definicije sume po stanjima. Kako bismo proverili da li je konstruisana suma po stanjima zaista topološke prirode, analiziraćemo njeno ponašanje pri Pahnerovim potezima [27]. Analizom Pahnerovih poteza dobijeno je da suma po stanjima Z ostaje ista, tj. da je zaista topološka invarijanta 4-dimenzionalne mnogostrukosti. Postavka dokaza invarijantnosti biće predstavljena u poglavlju 9, dok je detaljna analiza Pahnerovih poteza prikazana u Dodatku E.

Nažalost, da bismo ovaj korak procedure precizno sproveli do kraja, neophodna je generalizacija Piter-Vejlove i Planšarelove teoreme za slučajeve 2-grupa i 3-grupa. Cilj Piter-Vejlove teoreme je da obezbedi dekompoziciju funkcija na grupi na sumu po odgovarajućim ireducibilnim reprezentacijama, što nam omogućava preciziranje spektra oznaka d-simpleksa u triangulaciji, utvrđujući domen vrednosti polja koja žive na tom d-simpleksu. U slučaju 2-grupa i 3-grupa, teorija reprezentacija nije dovoljno razvijena da bi mogla da obezbedi takvu konstrukciju, pa su teoreme analogne Piter-Vajlovoj, odnosno Planšarelovoj, matematički rezultati koji i dalje nedostaju. Sa druge strane, dok se ove formulacije čekaju, moguće je pokušati *pogoditi* odgovarajuću strukturu ireducibilnih reprezentacija 2-grupa, odnosno 3 grupa, kao što je to urađeno u [15] gde je konstruisan *spin-kub model* kvantne gravitacije.

Treći korak kvantizacione procedure spinske pene. Najzad, kako za potrebe fizičke teorije nismo zainteresovani za topološku teoriju, već za teoriju sa lokalnim propagirajućim stepenima slobode, naš cilj nije konstruisanje topološke invarijante – sume po stanjima Z, već sume po stanjima koja daje netrivijalnu dinamiku. Da bismo to dobili, neophodno je da nametnemo veze na topološku sumu po stanjima Z, odnosno sprovedemo treći korak kvantizacione procedure spinske pene. Imajući to u vidu, glavna motivacija prvog dela našeg istraživanja u radu [16], bio je da prepišemo dejstvo za gravitaciju i materiju na način koji eksplicitno odvaja topološki deo od veza, što čini nametanje veza pravolinijskim postupkom. Rezultat ovog poslednjeg trećeg koraka kvantizacione procedure bio bi suma po stanjima koja opisuje teoriju kvantne gravitacije sa materijom.

Oznake i konvencije

Lokalni Lorencovi indeksi su označeni latiničnim slovima $a, b, c, \ldots$, koji uzimaju vrednosti $0, 1, 2, 3$, a njihovo podizanje i spuštanje vrši se metrikom Minkovskog η_{ab} sa signaturom $(-, +, +, +)$. Prostorvremenski indeksi su označeni grčkim slovima $\mu, \nu, \ldots$, a za njihovo podizanje i spuštanje koristi se metrika prostorvremena $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, gde $e^a{}_\mu$ označava tetradna polja. Njihov prostorni deo je označen malim latiničnim indeksima polovine alfabeta $i, j, \ldots$, a vremenska komponenta je označena sa 0. Indeksi koji prebrojavaju generatore grupa G, H i L su redom označeni početnim slovima grčkog alfabeta $\alpha, \beta, \ldots$, malim početnim latiničnim slovima $a, b, c, \ldots$ i velikim latiničnim slovima $A, B, C, \ldots$. Antisimetrizacija tenzora po dva indeksa je označena kao

$$A_{[a_1|a_2\ldots a_{n-1}|a_n]} = \frac{1}{2}\left(A_{a_1 a_2\ldots a_{n-1} a_n} - A_{a_n a_2 \ldots a_{n-1} a_1}\right) ,$$

dok je ukupna antisimetrizacija po svim indeksima označena kao

$$A_{[a_1\ldots a_n]} = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{\mathrm{sgn}(\sigma)} A_{a_{\sigma(1)}\ldots a_{\sigma(n)}} .$$

Slično, simetrizacija tenzora po dva indeksa se označava kao

$$A_{(a_1|a_2\ldots a_{n-1}|a_n)} = \frac{1}{2}\left(A_{a_1 a_2\ldots a_{n-1} a_n} + A_{a_n a_2 \ldots a_{n-1} a_1}\right) ,$$

dok je ukupna simetrizacija po svim indeksima označena kao

$$A_{(a_1\ldots a_n)} = \frac{1}{n!}\sum_{\sigma\in S_n} A_{a_{\sigma(1)}\ldots a_{\sigma(n)}} .$$

Radimo u prirodnom sistemu jedinica, gde je $c = \hbar = 1$ i $G = l_p^2$, a l_p je Plankova dužina.

Ako je G konačna grupa, $\int_G dg = 1/|G| \sum_{g\in G}$ označava normalizovanu sumu po svim elementima grupe, dok δ_G označava odgovarajuću δ-distribuciju na G. Ova distribucija definisana je za svaki element $g \in G$ tako da $\delta_G(g) = |G|$ ako je g jedinični element grupe $g = e$, a ako nije tj. ako $g \neq e$ onda je definisana kao $\delta_G(g) = 0$. Ako je G Lijeva grupa, $\int_G dg$ i δ_G označavaju redom Harovu meru i δ-distribuciju na G. Set svih k-simpleksa, pri čemu je indeks k takav da $0 < k < d$, označen je sa Λ_k. Skup svih verteksa Λ_0 je konačan i uređen, a svaki k-simpleks je označen sa $(k+1)$-torkama verteksa $(i_0 \ldots i_k)$, gde je $i_0, \ldots, i_k \in \Lambda_0$ tako da $i_0 < \cdots < i_k$. Broj k-simpleksa je označen sa $|\Lambda_k|$.

Za svaku triangulaciju $\mathcal{T}(\mathcal{M}_d)$ d-dimenzionalne mnogostrukosti $\mathcal{M}_d$ možemo definisati dualnu triangulaciju $\mathcal{T}(\mathcal{M}_d)^*$. *Dualna triangulacija*, ili *dualna rešetka* je definisana na sledeći način [1]. Postavljamo verteks dualne rešetke u centar svakog d-simpleksa triangulacije. Povezivanjem verteksa koji odgovaraju susednim d-simpleksima dobijamo ivice dualne triangulacije.

Svaka ova ivica dualne triangulacije odgovara jednom $d-1$-simpleksu triangulacije koji je zajednički za dva susedna d-simpleksa. Mnogougao, 2-kompleks koji dobijamo povezivanjem odgovarajućih ivica $\epsilon \in \mathcal{T}(\mathcal{M}_d)^*$ odgovara $d-2$-simpleksu iz $\mathcal{T}(\mathcal{M}_d)$. Na ovaj način dodeljujemo i ostale elemente dualne triangulacije koji su dualni odgovarajućim elementima triangulacije, sve do d-kompleksa koji je dualan verteksu $v \in \mathcal{T}(\mathcal{M}_d)$. Set svih k-kompleksa dualne triangulacije, pri čemu je indeks k takav da $0 < k < d$, označen je sa Λ_k^*. Ukupan broj k-kompleksa u dualnoj rešetki je označen sa $|\Lambda_k^*|$.

Sve dodatne oznake i konvencije koje se koriste su eksplicitno definisane u tekstu gde se pojavljuju.

Glava 2

Više gejdž teorije

Obična gejdž teorija opisuje kako se 0-dimenzionalne čestice transformišu pri paralelnom prenosu duž 1-dimenzionalne putanje koja pripada bazi raslojenog prostora[1], pri čemu svakoj putanji mnogostrukosti prirodno odgovara neki element grupe. Na sličan način, u okviru 2-gejdž teorije definiše se paralelni transport 1-dimenzionalnih objekata definisanjem 2-koneksije na 2-raslojenom prostoru[2]. 2-Raslojeni prostor je generalizacija raslojenog prostora u 2-gejdž teoriji – vlakna nisu mnogostrukost već kategorija sa odgovarajućom glatkom strukturom.

Kategorija se sastoji od elemenata koje nazivamo *objekti* i *morfizama* - preslikavanja između tih objekata. Grupu onda možemo posmatrati kao kategoriju sa samo jednim objektom gde su svi morfizmi invertibilni. Više gejdž teorije umesto grupe simetrije koja određuje gejdž teoriju, oređene su višom kategorijskom generalizacijom pojma grupe – 2-grupom.

2.1 Gejdž teorija

Gejdž teorija je teorija polja u kojoj Lagranžijan teorije koji određuje dinamiku sistema ostaje invarijantan pri *lokalnim transformacijama*[3], koje nazivamo *gejdž transformacijama*. Gejdž teorije se odlikuju prisustvom redundantnih stepeni slobode u fizičkom sistemu, tj. postojanjem nefizičkih varijabli u dejstvu, pored varijabli koje opisuju fiziku sistema. Gejdž teorije poseduju *gejdž simetriju*[4] koja predstavlja nefizičke transformacije dinamičkih promenljivih. Dinamički sistemi ovog tipa se nazivaju i singularnim, a njihova analiza zahteva primenu *Hamiltonove analize* sistema sa vezama o kojoj ćemo više govoriti u narednom poglavlju. U Hamiltonovom formalizmu ove teorije karakteriše prisustvo veza u dejstvu.

Gejdž transformacije formiraju Lijevu grupu koja se naziva *grupa simetrije* ili *gejdž grupa* teorije. Svakom generatoru ove grupe odgovara polje koje se naziva kalibraciono polje, tj. *gejdž polje*.

Matematički gledano, gejdž teorije su opisane jezikom diferencijalne geometrije, tj. definisane su na raslojenom prostoru. Opšta teorija relativnosti takođe je definisana na raslojenom

[1] *Raslojeni prostor* (eng. *fibre bundle*) $E(B, F, \pi)$ je mnogostrukost E, na kojoj je definisano glatko preslikavanje π (*projekcija*) na mnogostrukost B (*baza*), takvo da je za svaku tačku baze b, *sloj nad njom* $\pi^{-1}(b) \equiv \{e \in E \,|\, \pi(e) = b\}$ difeomorfan sa mnogostrukošću F (*tipični sloj*) i postoji okolina U_b za koju je $\pi^{-1}(U_b)$ difeomorfan sa direktnim proizvodom $U_b \times F$.

[2] 2-*Raslojeni prostor* (eng. *fibre* 2-*bundle*)

[3] Razlikujemo *globalnu* i *lokalnu* simetriju fizičkog sistema. Kada je Lagranžijan invarijantan pri transformacijama koje izgledaju identično u svakoj tački prostorvremena u kojoj se dešavaju fizički procesi, tj. kada parametar transformacije ne zavisi od tačke prostorvremena, kažemo da ima *globalnu simetriju*. *Lokalna simetrija* je simetrija teorije kod koje Lagranžijan ostaje invarijantan pri transformacijama čiji parametar zavisi od prostorvremenske koordinate.

[4] *Gejdž simetrija* (eng. *gauge symmetry*) se naziva još i baždarna, kalibraciona ili gradijentna simetrija.

prostoru, konkretno *tangetnom raslojenju*[5], dok je u slučaju gejdž teorija reč o *glavnom raslojenju*[6].

U gejdž teoriji definišemo 1-formu *koneksije*[7] $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$, element algebre $\mathfrak{g}$, koju u fizici nazivamo i *gejdž potencijal*. Koneksija se zadaje definisanjem diferencijalnog operatora odstupanja tenzorskih polja u nekoj tački mnogostrukosti od tenzorskih polja dobijenih *paralelnim prenosom*[8] u susedne tačke, tj. *kovarijantnim izvodom*[9] ∇.

Gejdž *krivina* je matematički koncept u *gejdž* teorijama koji opisuje kako se gejdž potencijal menja pri kretanju duž zatvorene petlje mnogostrukosti. Za koneksiju α, 2-forma *krivine* definisana je izrazom:

$$F = \mathrm{d}\alpha + \alpha \wedge \alpha\,. \tag{2.1}$$

Gejdž krivina je fundamentalna veličina u gejdž teorijama i igra ključnu ulogu u određivanju dinamike i interakcija čestica sa gejdž poljima.

Gejdž teorije su izuzetno uspešne u opisivanju i predviđanju ponašanja elementarnih čestica i njihovih interakcija. *Standardni Model*, koji je definisan $SU(3) \times SU(2) \times U(1)$ gejdž simetrijom, testiran je i potvrđen brojnim eksperimentima, uključujući i onim izvedenim na visokoenergetskim akceleratorima čestica kao što je Veliki hadronski sudarač (LHC). Međutim, i dalje postoje otvorena pitanja i izazovi, kao što je objedinjeni opis svih fundamentalnih sila, koji je predmet stalnih istraživanja u teorijskoj fizici.

2.2 2-Gejdž teorija

Uopštenje kategorije – 2-kategorija sastoji se od objekata, morfizama i morfizama između morfizama – 2-morfizama [12].

Viša kategorijska generalizacija gejdž teorije – 2-gejdž teorija je teorija u kojoj su fundamentalne simetrije teorije date 2-gejdž grupom. U okviru 2-gejdž teorije osim što su putanjama dodeljeni elementi grupe $g \in G$ kao u običnoj gejdž teoriji, površinama su dodeljeni elementi $h \in H$. Pritom, oznake ovih elementarnih elemenata mnogostrukosti ne mogu biti proizvoljne, tj. moraju biti zadovoljeni sledeći uslovi.

1. Za svaku površinu označenu sa $h \in H$, oznake izvorne krive g_1 i ciljne krive g_2 zadovoljavaju relaciju $\partial(h) = g_2 g_1^{-1}$.

2. Za svaku zapreminu, površinska holonomija oko nje je trivijalna.

U ovom odeljku razmatraćemo načine na koje možemo vršiti kompoziciju označenih putanja i površina, kako bismo izračunali kompoziciju elementarnih do proizvoljno velikih elemenata mnogostrukosti.

[5]Svakoj tački m mnogostrukosti $\mathcal{M}$ dimenzije D možemo pridružiti po jedan D-dimenzionalni tangentni prostor $T_m(\mathcal{M})$ koji čine tangentni vektori dobijeni za različite krive kroz tačku m. Pridruživanjem svakoj tački m mnogostrukosti $\mathcal{M}$ vektorskog prostora $T_m(\mathcal{M})$ dobijeno je *tangentno raslojenje* $T(M)$, tj. vektorsko raslojenje sa bazom $\mathcal{M}$, slojem $\mathbb{R}^D$ i projekcijom $\pi : T_m(\mathcal{M}) \to m$. Svi tangentni prostori su medusobno izomorfni, ali da bismo uspostavili korespodenciju između vektora tangentnih prostora koji odgovaraju različitim tačkama neophodno je uvesti pojam *koneksije*.

[6]Kada je tipični sloj $F = G$ neka Lijeva grupa raslojenje se naziva *glavno raslojenje* (eng. *principal bundle*) P grupe G.

[7]*Koneksija* (eng. *connection*) svakoj putanji γ koja povezuje tačke x i y mnogostrukosti $\mathcal{M}$ dodeljuje preslikavanje $\rho(\gamma) : T_x(\mathcal{M}) \to T_y(\mathcal{M})$.

[8]*Paralelni prenos* (eng. *parallel transport*).

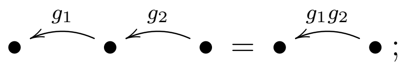
[9]*Kovarijantni izvod* (eng. *covariant derivative*).

2.2.1 2-Grupa

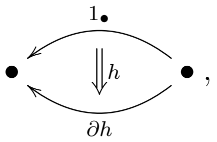
U okviru teorije kategorija, 2-grupa se definiše kao 2-kategorija sa samo jednim objektom kod koje su svi morfizmi i 2-morfizmi invertibilni. Pokazano je da je svaka striktna 2-grupa ekvivalentna ukrštenom modulu $(H \overset{\partial}{\to} G\,, \triangleright)$.

Definicija 2.2.1 (Pre-ukršteni modul i ukršteni modul) *Pre-ukršteni modul* $(H \overset{\partial}{\to} G\,, \triangleright)$ *čine:*

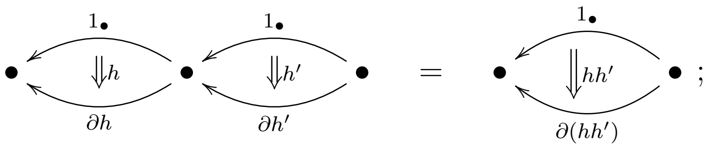
- *grupa* G *koju čine morfizmi sa kompozicijom kao grupnom operacijom*



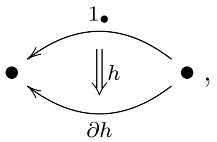
- *grupa* H *koju čine svi* 2*-morfizmi čiji je izvor identitet*



gde je horizontalna kompozicija grupna operacija



- *grupni homomorfizam* $\partial : H \to G$ *koji preslikava svaki* 2*-morfizam iz* $h \in H$ *u metu* $\partial h \in G$

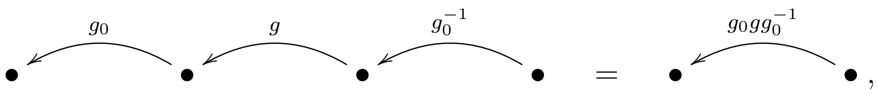


pri čemu imamo grupni homomorfizam

$$\partial(hh') = \partial(h)\partial(h')\,; \tag{2.2}$$

- *dejstvo* $\triangleright$ *grupe* G *na obe grupe, pri čemu*

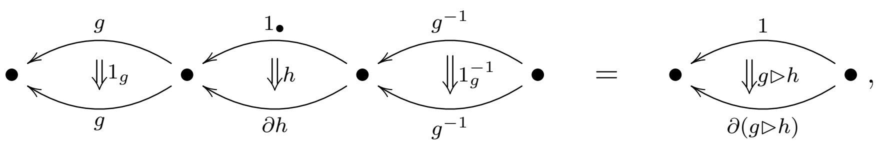
  – *grupa* G *deluje na samu sebe horizontalnom konjugacijom:*



  odnosno formalno zapisano za sve $g_0, g \in G$*:*

$$g_0 \triangleright g = g_0 g g_0^{-1}\,, \tag{2.3}$$

  – *element grupe* $g \in G$ *na element grupe* $h \in H$ *deluje horizontalnom konjugacijom njegovim jediničnim* 2*-morfizmom* 1_g*, što rezultuje* 2*-morfizmom* $g \triangleright h \in H$*,*



  tj. formalno zapisano za sve $g \in G$ *i* $h \in H$ *imamo*

$$g \partial h g^{-1} = \partial(g \triangleright h)\,, \tag{2.4}$$

– *dejstvo* $\triangleright$ *je* G*-ekvivarijantno, tj. za svako* $h, h' \in H$ *i* $g, g' \in G$ *važi*

$$(gg') \triangleright h = g \triangleright (g' \triangleright h)\,, \qquad g \triangleright (hh') = (g \triangleright h)(g \triangleright h')\,. \tag{2.5}$$

U pre-ukrštenom modulu definišemo ***Pajferov komutator*** $\langle h_1\,, h_2\rangle_{\rm pf}$*, za sve* $h_1\,, h_2 \in H$ *i* $g \in G$*, kao:*

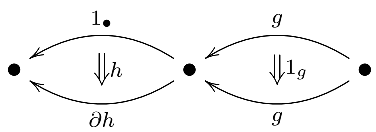
$$\langle h_1\,, h_2\rangle_{\rm pf} = h_1 h_2 h_1^{-1} \partial(h_1) \triangleright h_2^{-1}\,. \tag{2.6}$$

Pre-ukršteni modul kod kojeg su svi Pajferovi komutatori trivijalni naziva se ***ukršten modul****. Formalno, to znači da važi* ***Pajferov identitet****, tj. za sve* $h, h' \in H$*:*

$$(\partial h) \triangleright h' = hh'h^{-1}\,. \tag{2.7}$$

Za svaki element $g \in G$ operacija $\triangleright$ daje jedan automorfizam H. Kompozicija dva automorfizma daje jedan automorfizam, pa automorfizmi grupe H formiraju grupu $\mathrm{Aut}(H)$, a $\triangleright$ nam preslikava svaki element $g \in G$ u jedan element ove grupe tj. $\triangleright : G \to \mathrm{Aut}(H)$.

Između pojma 2-grupe i pojma ukrštenog modula postoji ekvivalencija, pa je svakom 2-morfizmu $\alpha \in \mathcal{G}$ koji je element 2-grupe $\mathcal{G}$, definisan parom (h, g),



pridružen jedan element 2-grupe, 2-morfizam:

$$g \to (\partial h)\, g\,. \tag{2.8}$$

2.2.2 Lijeva 2-algebra

Slično definicijama pre-ukrštenog modula i ukrštenog modula, na jeziku Lijevih 2-algebri analogno se definiše *diferencijalni pre-ukršten modul* i *diferencijalni ukršten modul.*

Definicija 2.2.2 (Diferencijalni pre-ukršteni modul i diferencijalni ukršteni modul) *Diferencijalni pre-ukršten modul* $(\mathfrak{h} \overset{\partial}{\to} \mathfrak{g}\,, \triangleright)$*, zadat je algebrama* $\mathfrak{g}$ *i* $\mathfrak{h}$*, preslikavanjem* $\partial : \mathfrak{h} \to \mathfrak{g}$ *i dejstvom* $\triangleright$ *algebre* $\mathfrak{g}$ *na algebre* $\mathfrak{g}$ *i* $\mathfrak{h}$*.*

Kod diferencijalog pre-ukrštenog modula definišemo ***Pajferov komutator*** *dva elementa* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ *algebre* $\mathfrak{h}$ *kao:*

$$\langle \underline{h}_1,\, \underline{h}_2\rangle_{\rm pf} = [\underline{h}_1, \underline{h}_2] - \partial(\underline{h}_1) \triangleright \underline{h}_2\,. \tag{2.9}$$

Diferencijalni pre-ukršten modul je ***diferencijalni ukršten modul*** *kada su svi njegovi Pajferovi komutatori jednaki nuli, odnosno kada za sve* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ *važi* ***Pajferov identitet****:*

$$\partial(\underline{h}_1) \triangleright \underline{h}_2 = [\underline{h}_1,\, \underline{h}_2]\,. \tag{2.10}$$

Dejstvo $\triangleright$ elemenata algebre $\mathfrak{g}$ na elemente algebri $\mathfrak{h}$ i $\mathfrak{g}$ definisano je delovanjem generatora algebre $\mathfrak{g}$ na generatore odgovarajućih algebri, kao:

$$\tau_\alpha \triangleright \tau_\beta = \triangleright_{\alpha\beta}{}^{\gamma}\, \tau_\gamma\,, \qquad \tau_\alpha \triangleright t_a = \triangleright_{\alpha a}{}^{b}\, t_b\,, \tag{2.11}$$

Komponente diferencijalnog ukrštenog modula poseduju sledeće osobine.

1. Dejstvo $\triangleright$ elemenata algebre $\mathfrak{g}$ na elemente iste algebre je po definiciji u svojstvenoj reprezentaciji, tj. formalno zapisano, za svako $\underline{g}_0, \underline{g} \in \mathfrak{g}$:

$$\underline{g}_0 \triangleright \underline{g} = \left[\underline{g}_0, \underline{g}\right]\,. \tag{2.12}$$

odnosno u bazisu:

$$\triangleright_{\alpha\beta}{}^{\gamma} = f_{\alpha\beta}{}^{\gamma}\,. \tag{2.13}$$

2. Preslikavanje $\partial : H \to G$ je $\mathfrak{g}$-ekvivarijantno preslikavanje, odnosno za sve $\underline{h} \in \mathfrak{h}$ i $\underline{g} \in \mathfrak{g}$ važi:
$$\partial(\underline{g} \rhd \underline{h}) = [\underline{g}, \partial(\underline{h})]\,, \tag{2.14}$$
odnosno izraženo u bazisu:
$$\partial_a{}^\beta f_{\alpha\beta}{}^\gamma = \rhd_{\alpha a}{}^b \partial_b{}^\gamma\,. \tag{2.15}$$

3. Dejstvo $\rhd$ algebre $\mathfrak{g}$ na algebru $\mathfrak{h}$ je $\mathfrak{g}$-ekvivarijantno preslikavanje, tako da, za svako $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ važi sledeće pravilo
$$\underline{g} \rhd [\underline{h}_1,\, \underline{h}_2] = [\underline{g} \rhd \underline{h}_1,\, \underline{h}_2] + [\underline{h}_1,\, \underline{g} \rhd \underline{h}_2]\,, \tag{2.16}$$
odnosno izraženo u bazisu:
$$f_{ab}{}^c \rhd_{\alpha c}{}^d = 2 \rhd_{\alpha[a|}{}^c f_{c|b]}{}^d\,. \tag{2.17}$$

4. Pajferov komutator $(\underline{h}_1, \underline{h}_2) \in \mathfrak{h} \times \mathfrak{h} \to \langle \underline{h}_1, \underline{h}_2 \rangle_{\rm pf} \in \mathfrak{h}$ je bilinearno $\mathfrak{g}$-ekvivarijantno preslikavanje, tj. svi $\underline{h}_1\,, \underline{h}_2 \in \mathfrak{h}$ i $\underline{g} \in \mathfrak{g}$ zadovoljavaju sledeći identitet,
$$\underline{g} \rhd \langle \underline{h}_1\,, \underline{h}_2 \rangle_{\rm pf} = \langle \underline{g} \rhd \underline{h}_1\,, \underline{h}_2 \rangle_{\rm pf} + \langle \underline{h}_1\,, \underline{g} \rhd \underline{h}_2 \rangle_{\rm pf}\,, \tag{2.18}$$
odnosno u bazisu:
$$\rhd_{\alpha c}{}^d (f_{ab}{}^c - \partial_a{}^\beta \rhd_{\beta b}{}^c) = \rhd_{\alpha a}{}^c (f_{cb}{}^d - \partial_c{}^\beta \rhd_{\beta b}{}^d) + \rhd_{\alpha b}{}^c (f_{ac}{}^d - \partial_a{}^\beta \rhd_{\beta c}{}^d)\,. \tag{2.19}$$

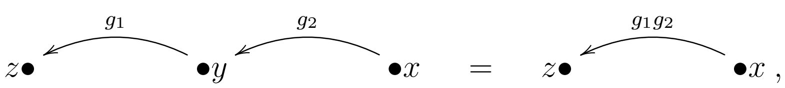
5. *Pajferov identitet* definisan u (2.10) na jeziku algebri postaje:
$$f_{ab}{}^c = \partial_a{}^\alpha \rhd_{\alpha b}{}^c\,. \tag{2.20}$$

2.2.3 Kompozicija morfizama

U 2-gejdž teoriji geometrijski objekti su obojeni na dva nivoa, krive su označene elementima $g \in G$, a površine elementima grupe $h \in H$. Na jeziku 2-gejdž teorije kompozicija i promena orijentacije krivih definisana je kao u standardnoj gejdž teoriji.

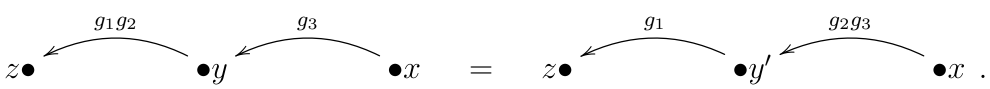
Da bismo diskutovali kompoziciju morfizama, pogodno je da uvedemo sledeću notaciju. Označimo izvor i metu morfizma g kao $\partial_1^-(g)$ i $\partial_1^+(g)$, respektivno.

- Horizontalna kompozicija morfizama g_1 i g_2, kada su oni kompozibilni, odnosno kada je $\partial_1^-(g_1) = \partial_1^+(g_2)$, je morfizam $g_1 \#_1 g_2$:



  odnosno formalno zapisano
$$g_1 \#_1 g_2 = g_1 g_2\,. \tag{2.21}$$

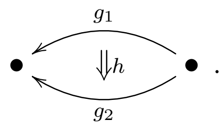
- Kompozicija morfizama je asocijativna operacija, odnosno za morfizme g_1, g_2 i g_3, kada je $\partial_1^-(g_1) = \partial_1^+(g_2)$ i $\partial_1^-(g_2) = \partial_1^+(g_3)$, važi:



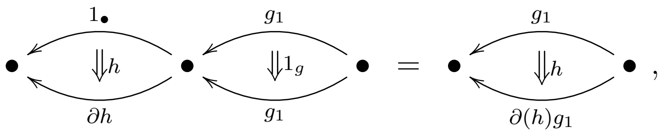
2.2.4 Kompozicija 2-morfizama

U ovom odeljku ćemo opisati kako se na jeziku 2-gejdž teorije definiše kompozicija elementarnih površina. Površine su označene elementima grupe $h \in H$, a možemo definisati njihovu *vertikalnu i horizontalnu kompoziciju.* U okviru 2-gejdž teorije definišemo i *nadovezivanje* 2-morfizma sa 1-morfizmom *sa leve* i nadovezivanje 2-morfizma sa 1-morfizmom sa desne strane.

Za svaku površinu možemo izabrati dve referentne tačke na granici i podeliti granicu na dve krive, na izvornu krivu označenu sa $g_1 \in G$ i ciljnu krivu označenu sa $g_2 \in G$, kao što je prikazano na dijagramu:



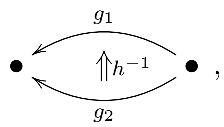
Pri tome 2-morfizam $h \in H$ preslikava krivu $g_1 \in G$ u krivu $\partial(h)g_1 \in G$,



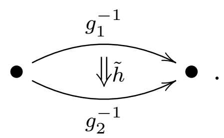
tako da $h \in H$ je zadovoljena relacija:

$$\partial(h) = g_2 g_1^{-1} . \tag{2.22}$$

Orijentacija površine može biti obrnuta, pri čemu površinu označavamo inverznim elementom,



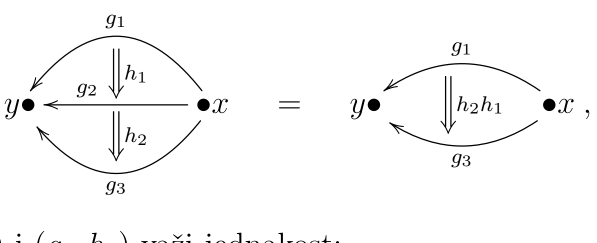
dok promena orijentacije krivih vodi do površinskog elementa označenog sa $\tilde{h} = g_1^{-1} \triangleright h^{-1}$:



Kako bi diskutovali kompozibilnost dva 2-morfizma uvedimo sledeću notaciju. Označimo izvor i metu k-strelice $(k = 1, 2)$ 2-morfizma h kao $\partial_k^-(h)$ i $\partial_k^+(h)$, respektivno. Sada možemo definisati načine na koje se mogu vršiti kompozicije elementarnih površina i koji su uslovi koji moraju biti zadovoljeni da bi određene kompozicije bile definisane.

Vertikalna kompozicija 2-morfizama

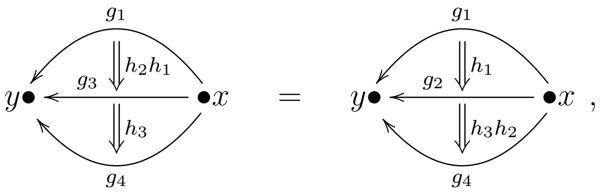
- Vertikalna kompozicija 2-morfizma (g_1, h_1) i 2-morfizma (g_2, h_2), kada su oni kompozibilni, odnosno kada je $\partial_2^+(h_1) = \partial_2^-(h_2)$, daje 2-morfizam $(g_1, h_2 h_1)$,



  odnosno za par (g_1, h_1) i (g_2, h_2) važi jednakost:

$$(g_2, h_2)\#_2(g_1, h_1) = (g_1, h_2 h_1) . \tag{2.23}$$

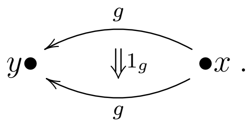
- Vertikalna kompozicija je asocijativna, tj. za 2-morfizme (g_1,h_1), (g_2,h_2) i (g_3,h_3), kada je $\partial_1^+(h_1)=\partial_1^-(h_2)$ i $\partial_1^+(h_2)=\partial_1^-(h_3)$,



važi:

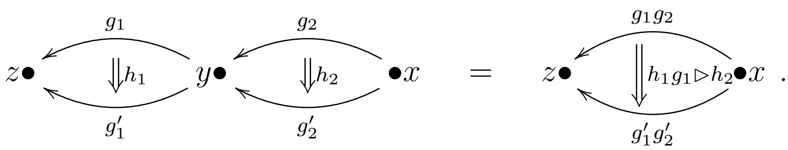
$$(g_3,h_3)\#_2(g_1,h_2h_1)=(g_2,h_3h_2)\#_2(g_1,h_1)\,. \tag{2.24}$$

- Za svaki morfizam g postoji 2-morfizam $(g,1_g)$ koji je identitet za vertikalnu kompoziciju 2-morfizama:



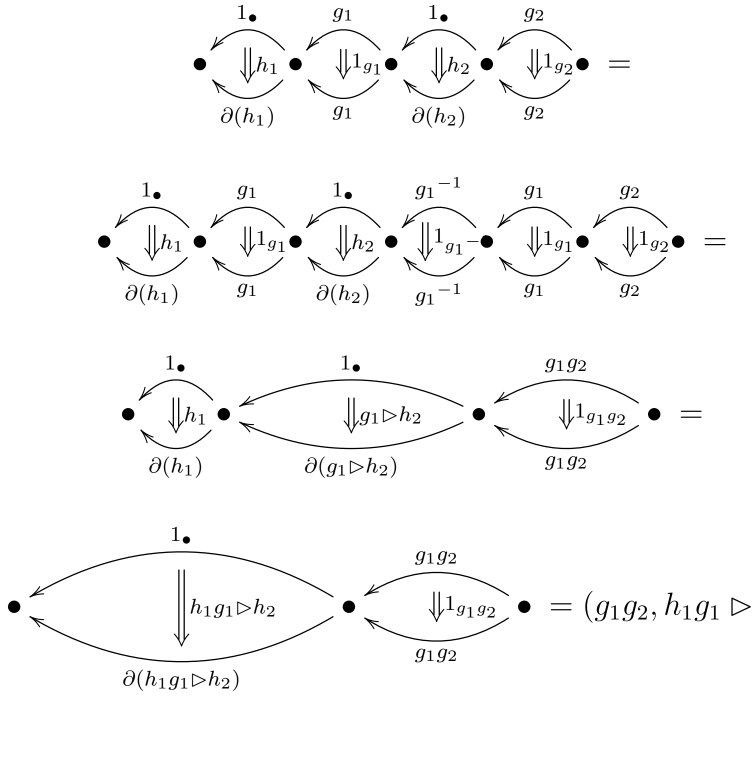
Horizontalna kompozicija 2-morfizama

- Horizontalna kompozicija 2-morfizama (g_1,h_1) i (g_2,h_2), kada su oni kompozibilni, odnosno kada je $\partial_1^-(h_1)=\partial_1^+(h_2)$, daje 2-morfizam $(g_1g_2, h_1g_1\rhd h_2)$:



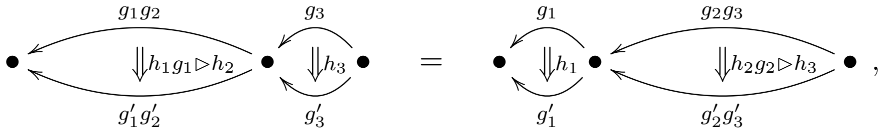
Horizontalna kompozicija 2-morfizama je grupna operacija u $G\times H$ grupi:

$$(g_1,h_1)\#_1(g_2,h_2)=$$



$$=(g_1g_2,h_1g_1\rhd h_2)\,.$$

- Horizontalna kompozicija 2-morfizama je asocijativna, odnosno za 2-morfizme (g_1, h_1), (g_2, h_2) i (g_3, h_3) za koje je $\partial_1^-(h_1) = \partial_1^+(h_2)$ i $\partial_1^-(h_2) = \partial_1^+(h_3)$, važi



odnosno formalno zapisano

$$(g_1g_2, h_1g_1 \rhd h_2)\#_1(g_3, h_3) = (g_1, h_1)\#_1(g_2g_3, h_2g_2 \rhd h_3)\,. \tag{2.25}$$

Dokaz. Proverimo ovu jednakost. Leva strana jednačine daje:

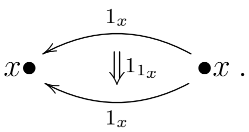
$$(g_1g_2, h_1g_1 \rhd h_2)\#_1(g_3, h_3) = (g_1g_2g_3, h_1\, g_1 \rhd h_2\, (g_1g_2) \rhd h_3)\,. \tag{2.26}$$

Desna strana jednačine daje takođe

$$\begin{aligned}(g_1, h_1)\#_1(g_2g_3, h_2g_2 \rhd h_3) &= (g_1g_2g_3, h_1g_1 \rhd (h_2g_2 \rhd h_3))\\ &= (g_1g_2g_3, h_1\, g_1 \rhd h_2\, g_1 \rhd (g_2 \rhd h_3)\\ &= (g_1g_2g_3, h_1\, g_1 \rhd h_2\, (g_1g_2) \rhd h_3)\,,\end{aligned} \tag{2.27}$$

primenom identiteta (2.5). ■

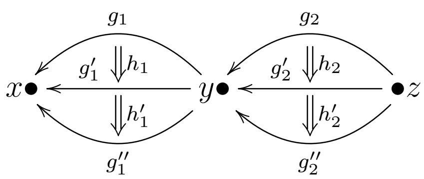
- Postoji 2-morfizam koji služi kao identitet za horizontalnu kompoziciju 2-morfizama:



- Kompozicija morfizama je invarijantna na redosled vršenja horizontalnih i vertikalnih kompozicija, tj. važi relacija

$$\Big((g_1',h_1')\#_2(g_1,h_1)\Big)\#_1\Big((g_2',h_2')\#_2(g_2,h_2)\Big)=\Big((g_1',h_1')\#_1(g_2',h_2')\Big)\#_2\Big((g_1,h_1)\#_1(g_2,h_2)\Big), \tag{2.28}$$

što je lepo ilustrovano u dijagramskoj notaciji, gde dijagram oblika



jednoznačno određuje 2-morfizam. Pravilo (2.28) se zove "izmenski zakon"[10].

Dokaz. Dokažimo jednakost (2.28). Leva strana jednačine jednaka je:

$$\begin{aligned}\big((g_1', h_1')\#_2(g_1, h_1)\big)\#_1\big((g_2', h_2')\#_2(g_2, h_2)\big) &= \big(g_1, h_1'h_1\big)\#_1\big(g_2, h_2'h_2\big)\\ &= \big(g_1g_2, h_1'h_1\, g_1 \rhd (h_2'h_2)\big)\,.\end{aligned} \tag{2.29}$$

[10] *izmenski zakon* (eng. *interchange law*).

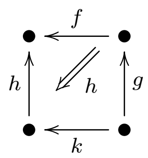
Desna strana jednačine (2.28) postaje:

$$
\begin{aligned}
\big((g_1',h_1')\#_1(g_2',h_2')\big)\#_2\big((g_1,h_1)\#_1(g_2,h_2)\big) &= \big(g_1'g_2', h_1'g_1' \rhd h_2'\big)\#_2\big(g_1g_2, h_1g_1 \rhd h_2\big)\\
&= \big(g_1g_2, h_1'g_1' \rhd h_2'\, h_1\, g_1 \rhd h_2\big)\\
&= \big(g_1g_2, h_1'(\partial(h_1)g_1) \rhd h_2'\, h_1\, g_1 \rhd h_2\big)\\
&= \big(g_1g_2, h_1'\, \partial(h_1) \rhd (g_1 \rhd h_2')\, h_1\, g_1 \rhd h_2\big)\\
&= \big(g_1g_2, h_1'\, h_1(g_1 \rhd h_2')h_1^{-1}\, h_1\, g_1 \rhd h_2\big)\\
&= \big(g_1g_2, h_1'\, h_1\, g_1 \rhd h_2'\, g_1 \rhd h_2\big)\\
&= \big(g_1g_2, h_1'\, h_1\, g_1 \rhd (h_2'h_2)\big)\,.
\end{aligned}
\tag{2.30}
$$

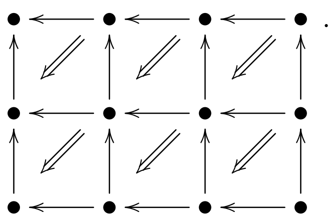
U prethodnom postupku primenili smo identitete (2.5) i (2.7). Dokazali smo identitet (2.28). ■

Kompozicija 2-morfizma i 1-morfizma

Izračunajmo holonomiju koja odgovara nekoj površini ako imamo 1-formu $\alpha \in \mathfrak{g}$ i 2-formu $\beta \in \mathfrak{h}$. Površinu

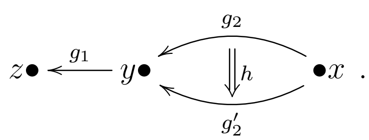


interpretiramo kao 2-morfizam $h : fg \to hk$. Veću površinu možemo dobiti nadovezivanjem[11]:

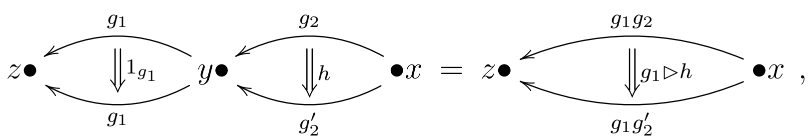


Koristeći *nadovezivanje* možemo da nađemo holonomiju $\mathrm{hol}(\Sigma)$ koja odgovara nekoj površini Σ, svodeći problem na traženje holonomije malih kvadrata na koje je ta površina izdeljena, formiranjem njihove kompozicije na prethodno definisan način, a zatim uzimanjem limesa kada ti kvadrati postaju infinitezimalno mali.

- *Nadovezivanje* je način na koji se vrši kompozicija jednog 2-morfizma h i jednog 1-morfizma g_1 koji se nalazi *sa leve strane*, odnosno kada je $\partial_1^-(g_1) = \partial_1^+(h)$:



Ovako formiran 2-morfizam naziva se *nadovezivanje* 2-morfizma h i 1-morfizma g_1 sa leve strane[12] $g_1\#_1(g_2, h)$, osnosno 2-morfizam h *proširen sa leve strane* sa 1-morfizmom g_1. Ova kompozicija formirana je kao horizontalna kompozicija dva 2-morfizma $(g_1, 1_{g_1})\#_1(g_2, h)$,

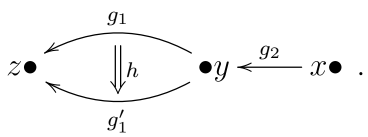


[11]eng. *whiskering*.
[12]eng. *left-whiskered*.

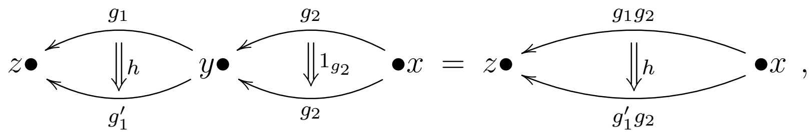
odnosno formalno zapisano:

$$g_1 \#_1 (g_2, h) = (g_1, 1_{g_1}) \#_1 (g_2, h) = (g_1 g_2, g_1 \rhd h)\,. \tag{2.31}$$

- Na sličan način možemo formirati i *nadovezivanje* 2-morfizma h i 1-morfizma g_2 *sa desne strane*, kada je $\partial_1^-(h) = \partial_1^+(g_2)$, čiju kompoziciju obeležavamo sa $(g_1, h)\#_1 g_2$:



Ovako formiran 2-morfizam naziva se *nadovezivanje* 2-morfizma h i 1-morfizma g_2 sa desne strane[13], odnosno 2-morfizam h *proširen sa desne strane* sa 1-morfizmom g_2. Ova kompozicija formirana je kao horizontalna kompozicija dva 2-morfizma $(g_1, h)\#_1(g_2, 1_{g_2})$,

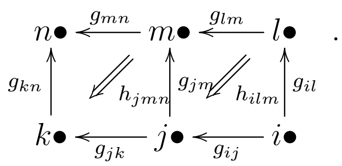


odnosno formalno zapisano:

$$(g_1, h)\#_1 g_2 = (g_1, h)\#_1(g_2, 1_{g_2}) = (g_1 g_2, h g_1 \rhd 1_{g_2}) = (g_1 g_2, h)\,. \tag{2.32}$$

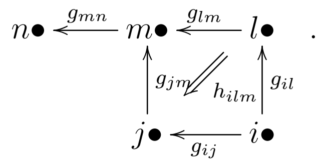
Uz pomoć ovog trika, možemo da kombinujemo dva 2-morfizma na način demonstriran u narednom primeru.

Primer 2.2.1 *Nađimo holonomiju koja odgovara površini dva kvadrata:*



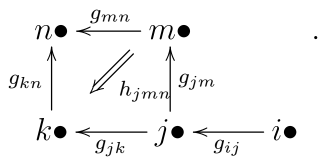
Ovo se radi iz dva koraka. Prvo se formira kompozicija 2-morfizma h_{ilm} sa 1-morfizmom g_{mn} koji deluje sa leve strane, pri čemu se dobija 2-morfizam:

$$g_{mn} \#_1 (g_{lm} g_{il}, h_{ilm}) : \; g_{mn} g_{lm} g_{il} \;\rightarrow\; g_{mn} g_{jm} g_{ij}\,, \tag{2.33}$$



Zatim se formira $(g_{mn} g_{jm}, h_{jmn})\#_1 g_{ij}$ kompozicija 2-morfizma h_{jmn} i 1-morfizma g_{ij} koji deluje sa desne strane:

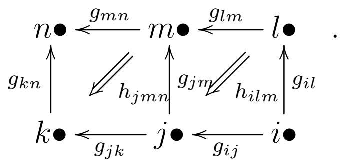
$$(g_{mn} g_{jm}, h_{jmn})\#_1 g_{ij} : \; g_{mn} g_{jm} g_{ij} \;\rightarrow\; g_{kn} g_{jk} g_{ij}\,, \tag{2.34}$$



[13] eng. *right-whiskered.*

Najzad vertikalnom kompozicijom ova dva 2-morfizma dobijamo 2-morfizam:

$$\big((g_{mn}g_{jm},h_{jmn})\#_1 g_{ij}\big)\#_2\big(g_{mn}\#_1(g_{lm}g_{il},h_{ilm})\big)\,:\;g_{mn}g_{lm}g_{il}\;\rightarrow\;g_{kn}g_{jk}g_{ij}\,, \tag{2.35}$$



Na sličan način kao u prethodnom primeru formiraćemo površinsku holonomiju koja odgovara tetraedru – relevantan rezultat u slučaju *triangulacije prostorvremena.*

2.2.5 2-koneksija i 2-krivina

Kao što Lijeva grupa G generiše koneksiju α, koju koristimo da formulišemo BF teoriju, 2-grupa generiše 2-koneksiju, uređeni par $(\alpha\,,\beta)$, zadat 1-formom elementom algebre $\mathfrak{g}$, $\alpha\in\mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$, i 2-formom elementom algebre $\mathfrak{h}$, $\beta\in\mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$, gde je $\mathfrak{h}$ Lijeva algebra koja odgovara Ljevoj grupi H. Za 2-koneksiju se definiše tzv. *lažna* 2-*krivina*, uređeni par $(\mathcal{F},\mathcal{G})$, na sledeći način

$$\mathcal{F}=\mathrm{d}\alpha+\alpha\wedge\alpha-\partial\beta\,,\qquad \mathcal{G}=d\beta+\alpha\wedge^{\triangleright}\beta\,. \tag{2.36}$$

Ovde $\alpha\wedge^{\triangleright}\beta$ predstavlja istovremeno ukršteni proizvod diferencijalnih formi α i β i njihov proizvod kao elemenata algebri dejstvom $\triangleright$, videti [16]. Uređeni par 2-krivine $(\mathcal{F},\mathcal{G})$ ima epitet *"lažni"* zbog prisustva dodatnog člana $\partial\beta$ u definiciji $\mathcal{F}$ [12].

Krivine možemo raspisati u bazisima odgovarajućih algebri i diferencijalnih formi:

$$\mathcal{F}=\frac{1}{2}\mathcal{F}^{\alpha}{}_{\mu\nu}\tau_{\alpha}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\,,\quad \mathcal{G}=\frac{1}{3!}\mathcal{G}^{a}{}_{\mu\nu\rho}t_{a}\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\wedge\mathrm{d}x^{\rho}\,,$$

gde su koeficijenti:

$$\begin{aligned}
\mathcal{F}^{\alpha}{}_{\mu\nu}&=\partial_{\mu}\alpha^{\alpha}{}_{\nu}-\partial_{\nu}\alpha^{\alpha}{}_{\mu}+f_{\beta\gamma}{}^{\alpha}\alpha^{\beta}{}_{\mu}\alpha^{\gamma}{}_{\nu}-\beta^{a}{}_{\mu\nu}\partial_{a}{}^{\alpha}\,,\\
\mathcal{G}^{a}{}_{\mu\nu\rho}&=\partial_{\mu}\beta^{a}{}_{\nu\rho}+\partial_{\nu}\beta^{a}{}_{\rho\mu}+\partial_{\rho}\beta^{a}{}_{\mu\nu}+\alpha^{\alpha}{}_{\mu}\beta^{b}{}_{\nu\rho}\triangleright_{\alpha b}{}^{a}+\alpha^{\alpha}{}_{\nu}\beta^{b}{}_{\rho\mu}\triangleright_{\alpha b}{}^{a}+\alpha^{\alpha}{}_{\rho}\beta^{b}{}_{\mu\nu}\triangleright_{\alpha b}{}^{a}\,.
\end{aligned} \tag{2.37}$$

Lažna krivina

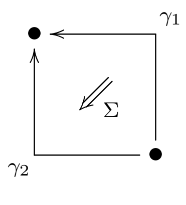
Prema Baezu [28], pridev *"lažna"* potiče iz rada Brina i Mesinga i koristi se za razlikovanje dve vrste krivine, 2-*krivine* i *lažne* 2-*krivine.* Obična 2-krivina, uređen par (F,G) za 2-koneksiju (α,β), $\alpha\in\mathcal{A}^1(\mathfrak{g},\mathcal{M}_4)$ i $\beta\in\mathcal{A}^2(\mathfrak{h},\mathcal{M}_4)$, definiše se na standardan način:

$$\begin{aligned}
F&=\mathrm{d}\alpha+\alpha\wedge\alpha,\\
G&=\mathrm{d}\beta+\alpha\wedge^{\triangleright}\beta\,.
\end{aligned} \tag{2.38}$$

Lažna 3-*krivina*, uređen par $(\mathcal{F},\mathcal{G})$ ima dodatni član:

$$\begin{aligned}
\mathcal{F}&=F-\partial\beta,\\
\mathcal{G}&=G\,.
\end{aligned} \tag{2.39}$$

Pojam lažne 2-krivine se uvodi iz sledećeg razloga. Posmatrajmo površinu Σ:



Ovaj kvadrat predstavlja 2-morfizam $\Sigma : \gamma_1 \to \gamma_2$, gde su γ_1 i γ_2 1-morfizmi prikazani na slici. Želimo da izračunamo:

$$\mathrm{hol}(\Sigma) : \mathrm{hol}(\gamma_1) \to \mathrm{hol}(\gamma_2)\,. \tag{2.40}$$

Holonomija hol(Σ) zavisi od 2-forme koneksije β. Njen izvor i meta zavise samo od 1-forme α. Zahtev da ovaj 2-morfizam ima dobro definisan izvor i metu dovodi do relacije koja povezuje koneksije α i β. Ovde ćemo demonstrirati kako se ona izvodi.

Izraženo preko 1-koneksije α, važi:

$$\mathrm{hol}(\gamma_1) = \mathcal{P}\exp\left(\int_{\gamma_1} \alpha\right), \qquad \mathrm{hol}(\gamma_2) = \mathcal{P}\exp\left(\int_{\gamma_2} \alpha\right). \tag{2.41}$$

Takođe, 2-morfizam $h : g_1 \to g_2$ je određen elementom $h \in H$, pri čemu je $g_2 = \partial(h)g_1$. To znači da je 2-morfizam $\mathrm{hol}(\Sigma) : \mathrm{hol}(\gamma_1) \to \mathrm{hol}(\gamma_2)$ određen elementom $h \in H$, pri čemu

$$\mathcal{P}\exp\left(\int_{\gamma_2} \alpha\right) = \partial(h)\, \mathcal{P}\exp\left(\int_{\gamma_1} \alpha\right), \tag{2.42}$$

odnosno $\partial(h) = \mathcal{P}\exp\left(\int_{\partial\Sigma} \alpha\right)$. U prethodnom izrazu integracija se vrši po granici površine $\partial\Sigma = \gamma_2\gamma_1^{-1}$. Imajući u vidu da je kvadrat mali, a koristeći Stoksovu teoremu dobijamo:

$$\partial(h) = \mathcal{P}\exp\left(\int_{\partial\Sigma} \alpha\right) \approx \exp\left(\int_{\Sigma} F\right). \tag{2.43}$$

Sa druge strane 2-morfizam h izražen preko 2-forme β je:

$$h \approx \exp\left(\int_{\Sigma} \beta\right). \tag{2.44}$$

Zamenjujući prethodni izraz za h u jednačini (2.43) dobijamo vezu između 1-koneksije α i 2-koneksije β

$$\partial(\exp\left(\int_{\Sigma} \beta\right)) \approx \exp\left(\int_{\Sigma} F\right), \tag{2.45}$$

odnosno identitet:

$$\partial(\beta) = F = \mathrm{d}\alpha + \alpha \wedge \alpha\,. \tag{2.46}$$

Sada, možemo uvesti pojam lažne krivine $\mathcal{F} = F - \partial(h)$.

Transformacije 2-koneksije i 2-krivine

U teoriji kategorija, 2-gejdž transformacije generisane su elementima grupa G i H. Pri G-gejdž transformacijama, 2-koneksija se transformiše po zakonu transformacije

$$\alpha' = g\alpha g^{-1} + g\mathrm{d}g^{-1}\,, \qquad \beta' = g \rhd \beta\,, \tag{2.47}$$

gde je parametar transformacija $g : \mathcal{M}_4 \to G$ element G-glavnog raslojenja $\mathcal{M}_4$. Zatim, pri H-gejdž transformacijama, generisanim parametrom $\eta \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{h})$, 2-koneksija se transformiše:

$$\alpha' = \alpha + \partial\eta\,, \qquad \beta' = \beta + \mathrm{d}\eta + \alpha' \wedge^{\rhd} \eta - \eta \wedge \eta\,. \tag{2.48}$$

Teorema 1 *Kompozicija G-gejdž i H-gejdž transformacija dovodi do transformacije 2-koneksije*

$$\begin{aligned} \alpha'' &= g\alpha g^{-1} + g\mathrm{d}g^{-1} + \partial(\eta)\,, \\ \beta'' &= g \rhd \beta + \mathrm{d}\eta + \alpha'' \wedge^{\rhd} \eta - \eta \wedge \eta\,, \end{aligned} \tag{2.49}$$

gde su $g : \mathcal{M}_4 \to G$ i $\eta \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ redom parametri G- i H-gejdž transformacija.

Na osnovu definicije 2-krivine (1.4), a primenom gore navedenih transformacionih pravila, dobijamo zakon transformacije 2-krivine.

Teorema 2 *Pri G-gejdž transformacijama 2-krivina $(\mathcal{F}, \mathcal{G})$ se transformiše na sledeći način*

$$\mathcal{F} \to g\mathcal{F}g^{-1}\,, \qquad \mathcal{G} \to g \rhd \mathcal{G}\,, \tag{2.50}$$

dok se pri H-gejdž transformacijama transformiše po zakonu transformacije:

$$\mathcal{F} \to \mathcal{F}\,, \qquad \mathcal{G} \to \mathcal{G} + \mathcal{F} \wedge^{\rhd} \eta\,, \tag{2.51}$$

gde su $g : \mathcal{M}_4 \to G$ i $\eta \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ redom parametri G- i H-gejdž transformacija.

Za više detalja o transformaciji 2-krivine pri 2-gejdž transformacijama pogledati [29] i Dodatak A.

2.3 3-gejdž teorija

U okviru teorije kategorija definiše se viša kategorijska generalizacija pojma 2-kateogije, 3-kategorija, koja se sastoji od objekata, morfizama, zatim morfizama između morfizama koje nazivamo 2-morfizmima, i morfizama između 2-morfizama – 3-morfizama. Pogledati [16] za više detelja.

U okviru 3-gejdž teorije elementarne strukture mnogostrukosti su označene na tri nivoa – krive su obojene elementima grupe $g \in G$, površine elementima $h \in H$, a zapremine elementima $l \in L$. Pritom, da bi konfiguracija 3-gejdž teorije bila dobro definisana, oznake ovih elementarnih struktura mnogostrukosti ne mogu biti proizvoljne, tj. moraju biti zadovoljeni sledeći uslovi.

1. Za svaku površinu označenu sa $h \in H$, oznake izvorne krive g_1 i ciljne krive g_2 zadovoljavaju relaciju $\partial(h) = g_2 g_1^{-1}$.

2. Za svaku zapreminu, oznaka $l \in L$ zadovoljava identitet $\delta(l) = h_2 h_1^{-1}$, gde su h_1 i h_2 izvorna i ciljna površina, respektivno.

3. Za svaku 4-dimenzionalnu hiperpovršinu mnogostrukosti zapreminska holonomija oko nje je trivijalna.

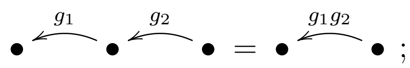
U ovom odeljku razmatraćemo brojne operacije pomoću kojih možemo kombinovati označene putanje, površine i zapremine, kako bismo izračunali kompoziciju elementarnih do proizvoljno velikih struktura mnogostrukosti. Definisane konfiguracije se mogu posmatrati kao klasične konfiguracije 3-gejdž teorije, dok će kasnije u kvantnoj teoriji ovo predstavljati konfiguracije po kojima sabiramo u sumi po stanjima.

2.3.1 3-Grupa

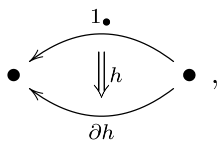
Analogno definiciji grupe i 2-grupe u formalizmu teorije kategorija, može se definisati pojam 3-grupe kao 3-kategorije sa samo jednim objektom, gde su svi morfizmi, 2-morfizmi i 3-morfizmi invertibilni. Takođe, slično kao što je striktna 2-grupa ekvivalentna ukrštenom modulu, može se pokazati da je semistriktna 3-grupa – Grejeva grupa, ekvivalentna strukturi 2-ukrštenog modula [17], [30].

Definicija 2.3.1 (2-ukršten modul) *2-ukršten modul $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{-,-\})$ čine:*

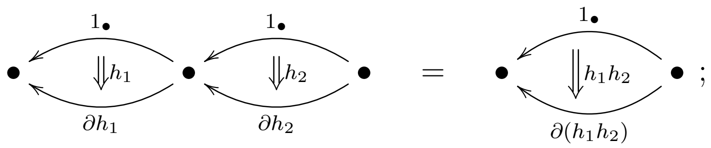
- *grupa G koju čine morfizmi sa kompozicijom kao grupnom operacijom*



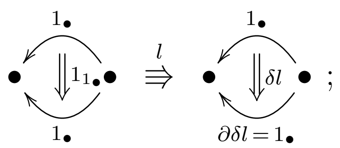
- *grupa H koju čine svi 2-morfizmi čiji je izvor identitet*



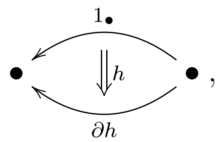
  gde je horizontalna kompozicija grupna operacija



- *grupa L koju čine 3-morfizmi čiji je izvor identitet*



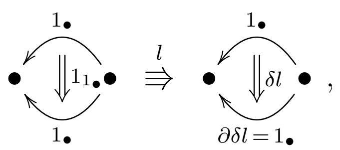
- *grupni homomorfizam $\partial : H \to G$ koji preslikava svaki 2-morfizam $h \in H$ u metu $\partial h \in G$*



  pri čemu imamo grupni homomorfizam, odnosno za svaki $h_1, h_2 \in H$ važi

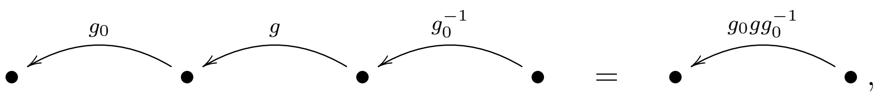
$$\partial(h_1 h_2) = \partial(h_1)\partial(h_2)\,;$$

- *grupni homomorfizam $\delta : L \to H$ koji mapira svaki 3-morfizam $l \in L$ u metu $\delta l \in H$:*



  pri čemu je za $\forall l \in L$ element $\partial(\delta l) = e$ neutralni element grupe G;

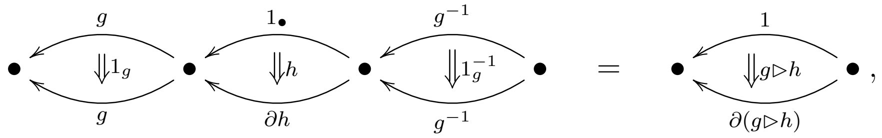
- *dejstvo $\triangleright$ grupe G na sve tri grupe, pri čemu je*
  - *dejstvo grupe G na samu sebe zadato horizontalnom konjugacijom*



  odnosno formalno zapisano za sve $g_0, g \in G$

$$g_0 \triangleright g = g_0 g g_0^{-1}\,,$$

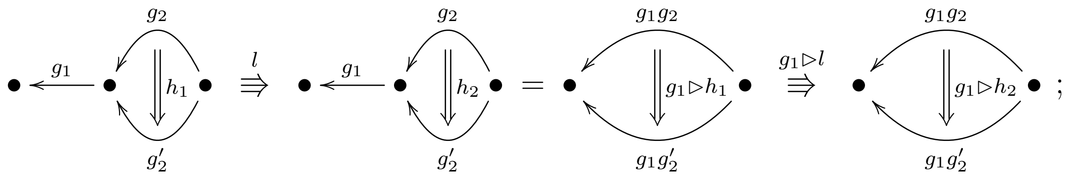
– *element grupe G deluje na element grupe H dejstvom $\triangleright$ kao horizontalnom konjugacijom, tačnije njegovim jediničnim 2-morfizmom 1_g, što rezultuje 2-morfizmom $g \triangleright h \in H$,*



tj. formalno zapisano za sve $g \in G$ i $h \in H$ imamo

$$g\partial h g^{-1} = \partial(g \triangleright h),$$

– *element grupe G deluje na element grupu L dejstvom $\triangleright$, što rezultuje 3-morfizmom $g \triangleright l \in L$,*



- *G-ekvivarijantno preslikavanje koje se naziva **Pajferovo podizanje***

$$\{-\,,-\}_{\mathrm{pf}} : H \times H \to L\,.$$

Komponente 2-ukrštenog modula poseduju sledeće osobine.

1. *Homomorfizmi ∂ i δ su G-ekvivarijantni, tj. za svako $g \in G$ i $h \in H$:*

$$g \triangleright \partial(h) = \partial(g \triangleright h)\,, \qquad g \triangleright \delta(l) = \delta(g \triangleright l)\,, \tag{2.52}$$

dejstvo grupe G na grupe H, odnosno L, je glatko dejstvo sa leva, i daje jedan automorfizam grupe H, odnosno automorfizam grupe L, tj. $\triangleright : G \to \mathrm{Aut}(H)$, odnosno $\triangleright : G \to \mathrm{Aut}(L)$. Za svako $g, g_1, g_2 \in G$, $h_1, h_2 \in H$, $l_1, l_2 \in L$ i $e \in H, L$ važi:

$$g_1 \triangleright (g_2 \triangleright e) = (g_1 g_2) \triangleright e\,, \qquad g \triangleright (h_1 h_2) = (g \triangleright h_1)(g \triangleright h_2)\,, \qquad g \triangleright (l_1 l_2) = (g \triangleright l_1)(g \triangleright l_2)\,. \tag{2.53}$$

Pajferovo podizanje je G-ekvivarijantno bilinearno preslikavanje, odnosno za sve h_1, $h_2 \in H$ i $g \in G$:

$$g \triangleright \{h_1\,, h_2\}_{\mathrm{pf}} = \{g \triangleright h_1,\, g \triangleright h_2\}_{\mathrm{pf}}\,. \tag{2.54}$$

2. *Za sve $h_1, h_2 \in H$ Pajferov komutator $\langle h_1\,, h_2\rangle_{\mathrm{pf}}$ definisan u (2.6) je identički jednak:*

$$\delta(\{h_1, h_2\}_{\mathrm{pf}}) = \langle h_1\,, h_2\rangle_{\mathrm{pf}}\,. \tag{2.55}$$

3. *Za sve $l_1\,, l_2 \in L$ zadovoljen je identitet:*

$$[l_1, l_2] = \{\delta(l_1)\,, \delta(l_2)\}_{\mathrm{pf}}\,, \tag{2.56}$$

gde je korišćena notacija $[l, k] = lkl^{-1}k^{-1}$.

4. *Pajferovo podizanje zadovoljava sledeće identitete, za sve $h_1, h_2, h_3 \in H$:*

$$\{h_1 h_2, h_3\}_{\mathrm{pf}} = \{h_1, h_2 h_3 h_2^{-1}\}_{\mathrm{pf}} \partial(h_1) \triangleright \{h_2, h_3\}_{\mathrm{pf}}\,, \tag{2.57}$$

odnosno

$$\{h_1, h_2 h_3\}_{\mathrm{pf}} = \{h_1, h_2\}_{\mathrm{pf}} \{h_1, h_3\}_{\mathrm{pf}} \{\langle h_1, h_3\rangle_{\mathrm{pf}}^{-1}, \partial(h_1) \triangleright h_2\}_{\mathrm{pf}}\,. \tag{2.58}$$

5. *Za sve $h \in H$ i $l \in L$ važi identitet:*

$$\{\delta(l), h\}_{\mathrm{pf}} \{h, \delta(l)\}_{\mathrm{pf}} = l(\partial(h) \triangleright l^{-1})\,. \tag{2.59}$$

Podstruktura ukršteni modul

Teorema 3 *Podstruktura 2-ukrštenog modula* $(L \xrightarrow{\delta} H, \triangleright')$ *formira ukršteni modul, gde je* $\triangleright'$ *dejstvo grupe* H *na grupu* L*, takvo da za svako* $h \in H$ *i* $l \in L$*:*

$$h \triangleright' l = l\,\{\delta(l)^{-1},\, h\}_{\mathrm{pf}}\,. \tag{2.60}$$

Dokaz ove teoreme izdelićemo na četiri dela. Pogledati [30] za više detalja.

Lema 1 *Za svako* $h_1, h_2 \in H$ *i* $l \in L$ *važi:*

$$(h_1 h_2) \triangleright' l = h_1 \triangleright' (h_2 \triangleright l)\,.$$

Dokaz. Za svako $h_1, h_2 \in H$ i $l \in L$ važi:

$$\begin{aligned}(h_1h_2) \triangleright' l &= l\{\delta(l)^{-1}, h_1h_2\}_{\mathrm{pf}}\\ &= l\{\delta(l)^{-1}, h_1\}_{\mathrm{pf}}\{\delta(l)^{-1}, h_2\}_{\mathrm{pf}}\{\langle\delta(l)^{-1}, h_2\rangle_{\mathrm{pf}}^{-1}, \partial(\delta(l))^{-1} \triangleright h_1\}_{\mathrm{pf}}\\ &= l\{\delta(l)^{-1}, h_1\}_{\mathrm{pf}}\{\delta(l)^{-1}, h_2\}_{\mathrm{pf}}\{\langle\delta(l)^{-1}, h_2\rangle_{\mathrm{pf}}^{-1}, h_1\}_{\mathrm{pf}}\,,\end{aligned}$$

gde smo u prvom redu koristili definiciju dejstva $\triangleright'$ datu teoremom (3), u drugom redu identitet (2.58) i u trećem redu identitet $\partial(\delta) = 1$. Sa druge strane,

$$\begin{aligned}h_1 \triangleright' (h_2 \triangleright' l) &= h_1 \triangleright' (l\{\delta(l)^{-1}, h_2\}_{\mathrm{pf}})\\ &= (h_1 \triangleright' l)(h_1 \triangleright' \{\delta(l)^{-1}, h_2\}_{\mathrm{pf}})\\ &= l\{\delta(l)^{-1}, h_1\}_{\mathrm{pf}}\{\delta(l)^{-1}, h_2\}_{\mathrm{pf}}\{\delta(\{\delta(l)^{-1}, h_2\}_{\mathrm{pf}})^{-1}, h_1\}_{\mathrm{pf}}\\ &= l\{\delta(l)^{-1}, h_1\}_{\mathrm{pf}}\{\delta(l)^{-1}, h_2\}_{\mathrm{pf}}\{\langle\delta(l)^{-1}, h_2\}_{\mathrm{pf}}\rangle_{\mathrm{pf}}^{-1}, h_1\}_{\mathrm{pf}}\,,\end{aligned}$$

gde smo u poslednjem koraku iskoristili identitet (2.56). ■

Lema 2 *Za svako* $h \in \mathfrak{h}$ *i* $l_1, l_2 \in \mathfrak{l}$ *važi:*

$$h \triangleright' (l_1 l_2) = (h \triangleright' l_1)(h \triangleright' l_2)\,.$$

Dokaz. Za svako $h \in \mathfrak{h}$ i $l_1, l_2 \in \mathfrak{l}$ važi:

$$\begin{aligned}h\triangleright'(l_1l_2) &= (l_1l_2)\{\delta(l_1l_2)^{-1},h\}_{\mathrm{pf}}\\ &= (l_1l_2)\{\delta(l_2)^{-1}\delta(l_1)^{-1},h\}_{\mathrm{pf}}\\ &= l_1l_2\{\delta(l_2)^{-1},\delta(l_1)^{-1}h\delta(l_1)\}_{\mathrm{pf}}\partial(\delta(l_2)^{-1})\triangleright\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}\\ &= l_1l_2\{\delta(l_2)^{-1},\delta(l_1)^{-1}h\delta(l_1)\}_{\mathrm{pf}}\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}\\ &= l_1l_2\{\delta(l_2)^{-1},[\delta(l_1)^{-1},h]h\}_{\mathrm{pf}}\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}\\ &= l_1l_2\{\delta(l_2)^{-1},\delta(\{\delta(l_1)^{-1},h\}_{\mathrm{pf}})h\}_{\mathrm{pf}}\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}\\ &= l_1l_2\{\delta(l_2)^{-1},\delta(\{\delta(l_1)^{-1},h\}_{\mathrm{pf}})\}_{\mathrm{pf}}\{\delta(l_2)^{-1},h\}_{\mathrm{pf}}\{\langle\delta(l_2)^{-1},h\rangle_{\mathrm{pf}}^{-1},\partial(\delta(l_2)^{-1})\triangleright\delta(\{\delta(l_1)^{-1},h\}_{\mathrm{pf}})\}_{\mathrm{pf}}\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}\\ &= l_1l_2[l_2^{-1},\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}]\{\delta(l_2)^{-1},h\}_{\mathrm{pf}}\{\delta(\{\delta(l_2)^{-1},h\}_{\mathrm{pf}})^{-1},\delta(\{\delta(l_1)^{-1},h\}_{\mathrm{pf}})\}_{\mathrm{pf}}\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}\\ &= l_1l_2l_2^{-1}\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}l_2\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}{}^{-1}\{\delta(l_2)^{-1},h\}_{\mathrm{pf}}[\{\delta(l_2)^{-1},h\}_{\mathrm{pf}}^{-1},\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}]\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}\\ &= l_1\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}l_2\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}{}^{-1}\{\delta(l_2)^{-1},h\}_{\mathrm{pf}}\{\delta(l_2)^{-1},h\}_{\mathrm{pf}}^{-1}\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}\{\delta(l_2)^{-1},h\}_{\mathrm{pf}}\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}{}^{-1}\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}\\ &= l_1\{\delta(l_1)^{-1},h\}_{\mathrm{pf}}l_2\{\delta(l_2)^{-1},h\}_{\mathrm{pf}}\\ &= (h\triangleright'l_1)(h\triangleright'l_2),\end{aligned}$$

gde smo u trećem redu iskoristili identitet (2.57), u četvrtom i osmom redu identitet $\partial(\delta) = 1$, u šestom i devetom redu jednačinu (2.56) i u sedmom redu identitet (2.58). ■

Lema 3 *Zadovoljen je **Pajferov identitet** za sve* l_1*,* $l_2 \in L$*:*

$$\delta(l_1) \triangleright' l_2 = l_1\, l_2\, l_1^{-1}\,.$$

Dokaz. Za sve l_1, $l_2 \in L$:

$$\delta(l_1) \triangleright' l_2 = l_2\{\delta(l_2)^{-1}, \delta(l_1)\}_{\rm pf} = l_2 l_2^{-1} l_1 l_2 l_1^{-1} = l_1\, l_2\, l_1^{-1}\,,$$

ovde smo kod prve jednakosti koristili definiciju dejstva $\triangleright'$ datu teoremom (3), kod druge identitet (2.56). ■

Lema 4 *Za sve $h \in H$ i $l \in L$ važi:*

$$\delta(h \triangleright' l) = h \triangleright' \delta(l)\,.$$

Dokaz. Za sve $h \in H$ i $l \in L$ važi:

$$\begin{aligned}
\delta(h \triangleright' l) &= \delta\left(l\{\delta(l)^{-1}, h\}_{\rm pf}\right)\\
&= \delta(l)\delta(\{\delta(l)^{-1}, h\}_{\rm pf})\\
&= \delta(l)\langle\delta(l)^{-1}, h\rangle_{\rm pf}\\
&= \delta(l)\delta(l)^{-1}h\delta(l)\,\partial(\delta(l))^{-1} \triangleright' h^{-1}\\
&= h\delta(l)h^{-1}\\
&= h \triangleright' \delta(l)\,,
\end{aligned}$$

gde smo u prvom redu koristili definiciju (3) dejstva $\triangleright'$, u trećem redu identitet (2.55), u četvrtom redu definiciju Pajferovog komutatora (2.6), u petom identitet $\partial(\delta) = 1$ i u poslednjem činjenicu da grupa H deluje na samu sebe konjugacijom. ■

Sa druge strane, podstruktura $(H \xrightarrow{\partial} G\,, \triangleright)$ je u opštem slučaju pre-ukršteni modul, tj. nije zadovoljen Pajferov identitet. Međutim, u slučaju kada je preslikavanje ∂ trivijalno, a grupa H Abelova, svi Pajferovi komutatori su trivijalni i Pajferov identitet je zadovoljen, tj. za sve h_1, $h_2 \in H$:

$$\partial(h_1) \triangleright h_2 = h_1\, h_2\, h_1^{-1}\,. \tag{2.61}$$

Važni identiteti

Elementi grupe H zadovoljavaju sledeće identitete $h_1, h_2, h_3 \in H$ [17]:

$$\{h_1h_2, h_3\}_{\rm pf} = (h_1 \triangleright' \{h_2, h_3\}_{\rm pf})\{h_1, \partial(h_2) \triangleright h_3\}_{\rm pf}\,, \tag{2.62}$$

$$\{h_1, h_2h_3\}_{\rm pf} = \{h_1, h_2\}_{\rm pf}(\partial(h_1) \triangleright h_2) \triangleright' \{h_1, h_3\}_{\rm pf}\,. \tag{2.63}$$

Koristeći peti uslov 2-ukrštenog modula dobijamo da za svako $h \in H$ i $l \in L$ važi

$$\{h, \delta(l)^{-1}\}_{\rm pf} = (h \triangleright' l^{-1})(\partial(h) \triangleright l)\,, \tag{2.64}$$

tj. za sve elemente $h_1, h_2 \in H$ važe sledeći identiteti:

$$\{h_1, h_2\}_{\rm pf}^{-1} = e \triangleright' \{h_1^{-1}, \partial(h_1) \triangleright h_2\}_{\rm pf}\,, \tag{2.65}$$

$$\{h_1, h_2\}_{\rm pf}^{-1} = \partial(h_1) \triangleright \{h_1^{-1}, h_1h_2h_1^{-1}\}_{\rm pf}\,, \tag{2.66}$$

$$\{h_1, h_2\}^{-1}{}_{\rm pf} = (h_1h_2h_1^{-1}) \triangleright' \{h_1, h_2^{-1}\}_{\rm pf}\,, \tag{2.67}$$

$$\{h_1, h_2\}_{\rm pf}^{-1} = (\partial(h_1) \triangleright h_2) \triangleright' \{h_1, h_2^{-1}\}_{\rm pf}\,. \tag{2.68}$$

2.3.2 Lijeva 3-algebra

Slično definiciji 2-ukrštenog modula, na jeziku Lijevih 3-algebri analogno se definiše *diferencijalni 2-ukršten modul.*

Definicija 2.3.2 (Diferencijalni 2-ukršten modul) *Diferencijalni 2-ukršten modul zadat je Lijevim algebrama $\mathfrak{g}$, $\mathfrak{h}$ i $\mathfrak{l}$, kao i preslikavanjima $\partial : \mathfrak{h} \to \mathfrak{g}$ i $\delta : \mathfrak{l} \to \mathfrak{h}$*

$$\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}\,, \tag{2.69}$$

zajedno sa dejstvom $\triangleright$ algebre $\mathfrak{g}$ na sve tri algebre i $\mathfrak{g}$-ekvivarijantnim bilinearnim preslikavanjem, koje se naziva **Pajferovo podizanje***:*

$$\{-\,,\,-\}_{\rm pf} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{l}\,. \tag{2.70}$$

Izborom bazisa $T_A \in \mathfrak{l}$, $t_a \in \mathfrak{h}$ i $\tau_\alpha \in \mathfrak{g}$,

$$[T_A, T_B] = f_{AB}{}^C\, T_C\,, \qquad [t_a, t_b] = f_{ab}{}^c\, t_c\,, \qquad [\tau_\alpha, \tau_\beta] = f_{\alpha\beta}{}^\gamma\, \tau_\gamma\,, \tag{2.71}$$

preslikavanja ∂ i δ možemo da definišemo u bazisima algebri:

$$\partial(t_a) = \partial_a{}^\alpha\, \tau_\alpha\,, \qquad \delta(T_A) = \delta_A{}^a\, t_a\,. \tag{2.72}$$

Pritom, važi identitet:

$$\delta_A{}^a \partial_a{}^\alpha = 0\,. \tag{2.73}$$

Dejstvo $\triangleright$ elemenata algebre $\mathfrak{g}$ na elemente algebri $\mathfrak{l}$, $\mathfrak{h}$ i $\mathfrak{g}$ definisano je delovanjem generatora algebre $\mathfrak{g}$ na generatore odgovarajućih algebri, kao:

$$\tau_\alpha \triangleright T_A = \triangleright_{\alpha A}{}^B\, T_B\,, \qquad \tau_\alpha \triangleright t_a = \triangleright_{\alpha a}{}^b\, t_b\,, \qquad \tau_\alpha \triangleright \tau_\beta = \triangleright_{\alpha\beta}{}^\gamma\, \tau_\gamma\,. \tag{2.74}$$

*Koeficijenti $X_{ab}{}^A$ koji određuju **Pajferovo podizanje** definisani su relacijom:*

$$\{t_a,\, t_b\}_{\rm pf} = X_{ab}{}^A T_A\,. \tag{2.75}$$

Komponente diferencijalnog 2-ukrštenog modula poseduju sledeće osobine.

1. *Dejstvo algebre $\mathfrak{g}$ na samu sebe je preko pridružene reprezentacije, tj. formalno zapisano, za svako $\underline{g}_0, \underline{g} \in \mathfrak{g}$*

$$\underline{g}_0 \triangleright \underline{g} = \left[\underline{g}_0, \underline{g}\right]\,, \tag{2.76}$$

odnosno u bazisu:

$$\triangleright_{\alpha\beta}{}^\gamma = f_{\alpha\beta}{}^\gamma\,. \tag{2.77}$$

2. *Preslikavanja $\partial : H \to G$ i $\delta : L \to H$ su $\mathfrak{g}$-ekvivarijantna preslikavanja, odnosno za sve $\underline{l} \in \mathfrak{l}$, $\underline{h} \in \mathfrak{h}$ i $\underline{g} \in \mathfrak{g}$ važi:*

$$\partial(\underline{g} \triangleright \underline{h}) = [\underline{g}, \partial(\underline{h})]\,, \qquad \delta(\underline{g} \triangleright \underline{l}) = \underline{g} \triangleright \delta(\underline{l})\,, \tag{2.78}$$

odnosno izraženo u bazisu:

$$\partial_a{}^\beta f_{\alpha\beta}{}^\gamma = \triangleright_{\alpha a}{}^b \partial_b{}^\gamma\,, \qquad \partial_B{}^a \triangleright_{\alpha A}{}^B = \triangleright_{\alpha b}{}^a \delta_A{}^b\,. \tag{2.79}$$

3. *Dejstvo* $\rhd$ *algebre* $\mathfrak{g}$ *na algebre* $\mathfrak{h}$ *i* $\mathfrak{l}$ *je* $\mathfrak{g}$*-ekvivarijantno preslikavanje, tako da, za svako* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$, $\underline{l}_1, \underline{l}_2$ *i svako* $\underline{g} \in \mathfrak{g}$, *važe sledeća pravila*

$$\underline{g} \rhd [\underline{h}_1,\, \underline{h}_2] = [\underline{g} \rhd \underline{h}_1,\, \underline{h}_2] + [\underline{h}_1,\, \underline{g} \rhd \underline{h}_2]\,, \tag{2.80}$$

$$\underline{g} \rhd [\underline{l}_1,\, \underline{l}_2] = [\underline{g} \rhd \underline{l}_1,\, \underline{l}_2] + [\underline{l}_1,\, \underline{g} \rhd \underline{l}_2]\,, \tag{2.81}$$

odnosno izraženo u bazisu:

$$f_{ab}{}^c \rhd_{\alpha c}{}^d = 2 \rhd_{\alpha [a|}{}^c f_{c|b]}{}^d\,, \tag{2.82}$$

$$f_{AB}{}^C \rhd_{\alpha C}{}^D = 2 \rhd_{\alpha [A|}{}^C f_{C|B]}{}^D\,. \tag{2.83}$$

4. *Pajferovo podizanje je* $\mathfrak{g}$*-ekvivarijantno preslikavanje, tj. za sve* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ *i* $\underline{g} \in \mathfrak{g}$

$$\underline{g} \rhd \{\underline{h}_1, \underline{h}_2\}_{\mathrm{pf}} = \{\underline{g} \rhd \underline{h}_1, \underline{h}_2\}_{\mathrm{pf}} + \{\underline{h}_1,\, \underline{g} \rhd \underline{h}_2\}_{\mathrm{pf}}\,, \tag{2.84}$$

tj. izraženo preko odgovarajućih koeficijenata definisanih u bazisu:

$$X_{ab}{}^B \rhd_{\alpha B}{}^A = \rhd_{\alpha a}{}^c X_{cb}{}^A + \rhd_{\alpha b}{}^c X_{ac}{}^A\,. \tag{2.85}$$

5. *Za sve* $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$ *Pajferov komutator definisan u (2.9) na jeziku algebri postaje*

$$\delta(\{\underline{h}_1,\, \underline{h}_2\}_{\mathrm{pf}}) = -\langle \underline{h}_1,\, \underline{h}_2 \rangle_{\mathrm{pf}}\,, \tag{2.86}$$

tj. izraženo u bazisu:

$$X_{ab}{}^A \delta_A{}^c = f_{ab}{}^c - \partial_a{}^\alpha \rhd_{\alpha b}{}^c\,. \tag{2.87}$$

6. *Za sve* $\underline{l}_1,\, \underline{l}_2 \in \mathfrak{l}$ *zadovoljen je identitet*

$$[\underline{l}_1,\, \underline{l}_2] = \{\delta(\underline{l}_1),\, \delta(\underline{l}_2)\}_{\mathrm{pf}}\,, \tag{2.88}$$

tj. važi relacija:

$$f_{AB}{}^C = \delta_A{}^a \delta_B{}^b X_{ab}{}^C\,. \tag{2.89}$$

7. *Pajferovo podizanje za sve* $\underline{h}_1,\, \underline{h}_2,\, \underline{h}_3 \in \mathfrak{h}$ *zadovoljava sledeće identitete:*

$$\{[\underline{h}_1,\underline{h}_2],\underline{h}_3\}_{\mathrm{pf}} = \partial(\underline{h}_1) \rhd \{\underline{h}_2,\underline{h}_3\}_{\mathrm{pf}} + \{\underline{h}_1,[\underline{h}_2,\underline{h}_3]\}_{\mathrm{pf}} - \partial(\underline{h}_2) \rhd \{\underline{h}_1,\underline{h}_3\}_{\mathrm{pf}} - \{\underline{h}_2,[\underline{h}_1,\underline{h}_3]\}_{\mathrm{pf}}, \tag{2.90}$$

$$\{[\underline{h}_1,\underline{h}_2],\underline{h}_3\}_{\mathrm{pf}} = \{\partial(\underline{h}_1) \rhd \underline{h}_2,\underline{h}_3\}_{\mathrm{pf}} - \{\partial(\underline{h}_2) \rhd \underline{h}_1,\underline{h}_3\}_{\mathrm{pf}} - \{\underline{h}_1,\delta\{\underline{h}_2,\underline{h}_3\}_{\mathrm{pf}}\}_{\mathrm{pf}} + \{\underline{h}_2,\delta\{\underline{h}_1,q,\underline{h}_3\}_{\mathrm{pf}}\}_{\mathrm{pf}}. \tag{2.91}$$

Izražen u bazisu ovaj identitet postaje:

$$f_{ab}{}^d X_{dc}{}^B = \partial_a{}^\alpha X_{bc}{}^A \rhd_{\alpha A}{}^B + X_{ad}{}^B f_{bc}{}^d - \partial_b{}^\alpha \rhd_{\alpha A}{}^B X_{ac}{}^A - X_{bd}{}^B f_{ac}{}^d\,. \tag{2.92}$$

Pored toga, za sve $\underline{h}_1,\, \underline{h}_2,\, \underline{h}_3 \in \mathfrak{h}$ *važi identitet*

$$\{\underline{h}_1,\, [\underline{h}_2,\, \underline{h}_3]\}_{\mathrm{pf}} = \{\delta\,\{\underline{h}_1,\, \underline{h}_2\}_{\mathrm{pf}}, \underline{h}_3\}_{\mathrm{pf}} - \{\delta\,\{\underline{h}_1,\, \underline{h}_3\}_{\mathrm{pf}}, \underline{h}_2\}_{\mathrm{pf}}\,, \tag{2.93}$$

tj. u bazisu:

$$X_{ad}{}^A f_{bc}{}^d = X_{ab}{}^B \delta_B{}^d X_{dc}{}^A - X_{ac}{}^B \delta_B{}^d X_{db}{}^A\,. \tag{2.94}$$

8. *Svako* $\underline{h} \in \mathfrak{h}$ *i* $\underline{l} \in \mathfrak{l}$ *zadovoljavaju identitet*

$$\{\delta(\underline{l}),\, \underline{h}\} + \{\underline{h},\, \delta(\underline{l})\} = -\partial(\underline{h}) \rhd \underline{l}\,, \tag{2.95}$$

tj. relacija:

$$\delta_A{}^a X_{ab}{}^B + \delta_A{}^a X_{ba}{}^B = -\partial_b{}^\alpha \rhd_{\alpha A}{}^B\,. \tag{2.96}$$

Podstruktura diferencijalni ukršteni modul

Teorema 4 *Podstruktura diferencijalnog 2-ukrštenog modula, formira diferencijalni modul* $(L \xrightarrow{\delta} H\,, \triangleright')$, *gde je dejstvo* $\triangleright'$ *algebre* $\mathfrak{h}$ *na algebru* $\mathfrak{l}$ *takvo da za sve* $\underline{h} \in \mathfrak{h}$ *i* $\underline{l} \in \mathfrak{l}$

$$\underline{h} \triangleright' \underline{l} = -\{\delta(\underline{l}),\, \underline{h}\}_{\mathrm{pf}}\,, \tag{2.97}$$

tj. u bazisu:

$$t_a \triangleright' T_A = \triangleright'{}_{aA}{}^{B} T_B\,, \qquad \triangleright'{}_{aA}{}^{B} = -\delta_A{}^{b} X_{ba}{}^{B}\,. \tag{2.98}$$

Dokaz ove teoreme podelićemo na četiri dela.

Lema 5 *Za svako* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ *i* $\underline{l} \in \mathfrak{l}$ *važi:*

$$[\underline{h}_1, \underline{h}_2] \triangleright' \underline{l} = \underline{h}_1 \triangleright' (\underline{h}_2 \triangleright' l) - \underline{h}_2 \triangleright' (\underline{h}_1 \triangleright' l)\,.$$

Dokaz. Za svako $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ i $\underline{l} \in \mathfrak{l}$ važi:

$$\begin{aligned}
[\underline{h}_1, \underline{h}_2] \triangleright' \underline{l} &= -\{\delta(l), [\underline{h}_1, \underline{h}_2]\}_{\mathrm{pf}} \\
&= -\{\delta(\{\delta(l), \underline{h}_1\}_{\mathrm{pf}}), \underline{h}_2\}_{\mathrm{pf}} + \{\delta(\{\delta(l), \underline{h}_2\}_{\mathrm{pf}}), \underline{h}_1\}_{\mathrm{pf}} \\
&= -\underline{h}_2 \triangleright' (\underline{h}_1 \triangleright' \underline{l}) + \underline{h}_1 \triangleright' (\underline{h}_2 \triangleright' \underline{l})\,,
\end{aligned}$$

gde smo koristili definiciju (2.97) u prvom i trećem redu i identitet (2.93) u drugom redu. ■

Lema 6 *Za svako* $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$ *i* $\underline{h} \in \mathfrak{h}$ *važi:*

$$\underline{h} \triangleright' [\underline{l}_1, \underline{l}_2] = [\underline{h} \triangleright' \underline{l}_1, \underline{l}_2] + [\underline{l}_1, \underline{h} \triangleright' \underline{l}_2]\,.$$

Dokaz. Za svako $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$ i $\underline{h} \in \mathfrak{h}$ važi:

$$\begin{aligned}
\underline{h} \triangleright' [\underline{l}_1, \underline{l}_2] &= -\{\delta([\underline{l}_1, \underline{l}_2]), \underline{h}\}_{\mathrm{pf}} \\
&= -\{[\delta(\underline{l}_1), \delta(\underline{l}_2)], \underline{h}\}_{\mathrm{pf}} \\
&= -\partial(\delta(\underline{l}_1)) \triangleright \{\delta(\underline{l}_2),\, \underline{h}\}_{\mathrm{pf}} - \{\delta(\underline{l}_1),\, [\delta(\underline{l}_2),\, \underline{h}]\}_{\mathrm{pf}} \\
&\quad + \partial(\delta(\underline{l}_2)) \triangleright \{\delta(\underline{l}_1),\, \underline{h}\}_{\mathrm{pf}} + \{\delta(\underline{l}_2),\, [\delta(\underline{l}_1),\, \underline{h}]\}_{\mathrm{pf}} \\
&= -\{\delta(\underline{l}_1),\, [\delta(\underline{l}_2),\, \underline{h}]\}_{\mathrm{pf}} + \{\delta(\underline{l}_2),\, [\delta(\underline{l}_1),\, \underline{h}]\}_{\mathrm{pf}} \\
&= -\{\delta\left(\{\delta(\underline{l}_1), \delta(\underline{l}_2)\}_{\mathrm{pf}}\right), \underline{h}\}_{\mathrm{pf}} + \{\delta\left(\{\delta(\underline{l}_1), h\}_{\mathrm{pf}}\right), \delta(\underline{l}_2)\}_{\mathrm{pf}} \\
&\quad + \{\delta\left(\{\delta(\underline{l}_2), \delta(\underline{l}_1)\}_{\mathrm{pf}}\right), \underline{h}\}_{\mathrm{pf}} - \{\delta\left(\{\delta(\underline{l}_2), h\}_{\mathrm{pf}}\right), \delta(\underline{l}_1)\}_{\mathrm{pf}} \\
&= -\{\delta([\underline{l}_1, \underline{l}_2], \underline{h}\}_{\mathrm{pf}} + [\{\delta(\underline{l}_1), h\}_{\mathrm{pf}}, \underline{l}_2] + \{\delta([\underline{l}_2, \underline{l}_1], \underline{h}\}_{\mathrm{pf}} - [\{\delta(\underline{l}_2), h\}_{\mathrm{pf}}, \underline{l}_1] \\
&= +\underline{h} \triangleright' [\underline{l}_1, \underline{l}_2] - [\underline{h} \triangleright' \underline{l}_1, \underline{l}_2] - \underline{h} \triangleright' [\underline{l}_2, \underline{l}_1] + [\underline{h} \triangleright' \underline{l}_2, \underline{l}_1]\,,
\end{aligned}$$

odakle sledi tvrđenje (6). Ovde smo primenili identitet (2.90) u trećem redu, zatim $\partial(\delta) = 0$ u četvrtom redu, identitet (2.93) u petom redu, identitet (2.88) u šestom redu i definiciju $\triangleright'$ (2.97) u sedmom redu. ■

Lema 7 *Za sve* $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$ *važi* ***Pajferov identitet****:*

$$\delta(\underline{l}_1) \triangleright' \underline{l}_2 = [\underline{l}_1, \underline{l}_2]\,.$$

Dokaz. Za sve $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$:

$$\delta(\underline{l}_1) \triangleright' \underline{l}_2 = -\{\delta(\underline{l}_2), \delta(\underline{l}_1)\}_{\mathrm{pf}} = -[\underline{l}_2, \underline{l}_1]\,,$$

odakle sledi tvrđenje (7), pri čemu smo iskoristili definiciju (2.97) u prvoj jednakosti i identitet (2.88) u drugoj jednakosti. ■

Lema 8 *Za sve* $\underline{h} \in \mathfrak{h}$ *i* $\underline{l} \in \mathfrak{l}$ *važi:*

$$\delta(\underline{h} \rhd' \underline{l}) = [\underline{h}, \delta(\underline{l})] \,.$$

Dokaz. Za sve $\underline{h} \in \mathfrak{h}$ i $\underline{l} \in \mathfrak{l}$ važi:

$$\delta(\underline{h} \rhd' \underline{l}) = -\delta(\{\delta(\underline{l}), \underline{h}\}_{\mathrm{pf}}) = -\langle \delta(\underline{l}), \underline{h} \rangle_{\mathrm{pf}} = -[\delta(\underline{l}), \underline{h}] + \partial(\delta(\underline{l})) \rhd \underline{h} \,,$$

odakle sledi tvrđenje (8) kad primenimo $\partial\delta = 0$. Ovde smo iskoristili definiciju (2.97) kod prve jednakosti, identitet (2.86) kod druge jednakosti i definiciju Pajferovog komutatora (2.9) kod treće jednakosti. ■

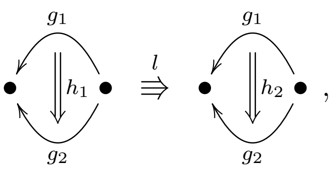
Na osnovu osobine 5. možemo da primetimo da trivijalno preslikavanje δ, ili trivijalno Pajferovo podizanje, povlače da L mora da bude Abelova grupa. Odnosno, ako je L Abelova grupa, makar jedno od ova dva preslikavanja, δ ili Pajferovo podizanje, mora da bude trivijalno. Ovo nam nagoveštava da komponente koje formiraju strukturu 3-grupe nisu međusobno nezavisne, tj. da je pri formiranju 3-grupe potrebno pažljivo birati grupe G, H i L i preslikavanja ∂, δ i $\{\_ ,\ \_\}_{\mathrm{pf}}$, tako da zajedno zaista formiraju strukturu 3-grupe.

Detaljnija analiza strukture 3-grupe izložena je u [18].

2.3.3 Kompozicija 3-morfizama

U ovom odeljku ćemo opisati kako se na jeziku 3-gejdž teorije definišu kompozicije elementarnih putanja, površina i zapremina. U 3-gejdž teoriji geometrijski objekti su obojeni na tri nivoa. Krive su označene elementima grupe $g \in G$, a njihova kompozicija i promena orijentacije definisana je kao u standardnoj gejdž teoriji. Površine su označene elementima grupe $h \in H$, a njihova vertikalna kompozicija definisana je na isti način kao u 2-gejdž teoriji diskutovanoj u prethodnom odeljku. Nadovezivanje 2-morfizma sa 1-morfizmom sa leve i desne strane takođe je definisano na isti način kao u 2-gejdž teoriji, dok to nije slučaj sa horizontalnom kompozicijom koja sada u 3-gejdž teoriji rezultuje *izmenskim 3-morfizmom.* Zapremine su označene elementima grupe $l \in L$.

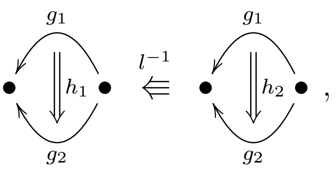
Za svaku zapreminu, podelimo granicu na dve površine, pri čemu je izvorna površina označena sa $\partial_3^-(l) = h_1$ i ciljna površina označena sa $\partial_3^+(l) = h_2$. Na zajedničkoj granici površine izvora i mete biramo dve referentne tačke i delimo granicu na dve krive, pri čemu je izvorna kriva označena sa $\partial_2^-(l) = g_1$ i ciljna kriva označena sa $\partial_2^+(l) = g_2$, kao što je prikazano na dijagramu ispod,



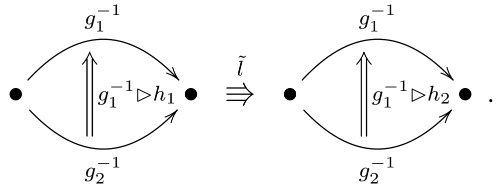
tako da element $l \in L$ zadovoljava relaciju:

$$\delta(l) = h_2 h_1^{-1} \,. \tag{2.99}$$

Orijentacija zapremine se može obrnuti, pri čemu je zapremina promenjene orijentacija označena inverznim elementom l^{-1},



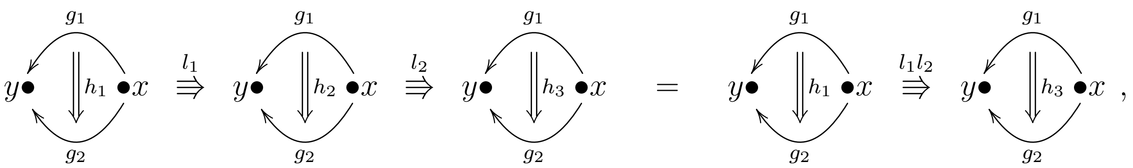
dok promena orijentacije krivih i površina dovodi do površinskog elementa označenog sa $\tilde{l} = g_1^{-1} \triangleright l$:



Da bismo definisali kada su elementarne zapremine kompozibilne označimo izvor i metu k-strelice ($k = 1, 2, 3$) 3-morfizma l kao $\partial_k^-(l)$ i $\partial_k^+(l)$, respektivno.

Kompozicija 3-morfizama prema gore

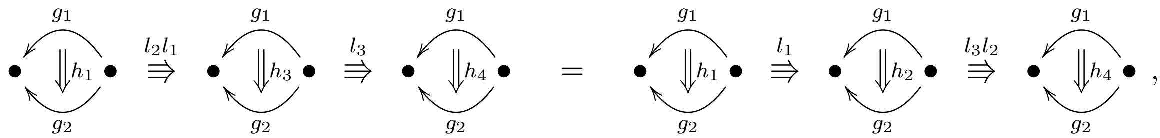
- *Kompozicija* dva 3-morfizama *prema gore* daje 3-morfizam, kada su oni kompozibilni, odnosno kada $\partial_3^+(l_1) = \partial_3^-(l_2)$,



  odnosno za dva 3-morfizma (g_1, h_1, l_1) i (g_1, h_2, l_2) važi:

$$(g_1, h_2, l_2)\#_3(g_1, h_1, l_1) = (g_1, h_1, l_2 l_1)\,. \tag{2.100}$$

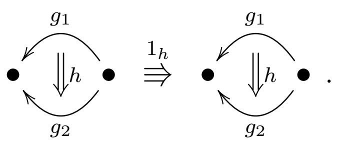
- Kompozicija 3-morfizama prema gore je asocijativna operacija, odnosno za $l_1, l_2, l_3 \in L$ koji zadovoljavaju $\partial_3^-(l_3) = \partial_3^+(l_2)$ i $\partial_3^-(l_3) = \partial_3^+(l_2)$,



  formalno zapisano:

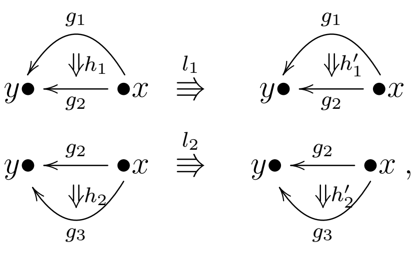
$$(g_1, h_3, l_3)\#_3(g_1, h_1, l_2 l_1) = (g_1, h_2, l_3 l_2)\#_3(g_1, h_1, l_1)\,. \tag{2.101}$$

- Za svaki element $h \in H$ postoji 3-morfizam koji je identitet za kompoziciju 3-morfizama prema gore:

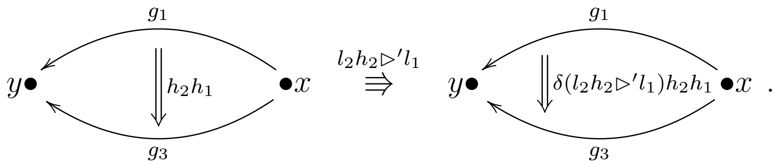


Vertikalna kompozicija 3-morfizama

- Vertikalna kompozicija dva 3-morfizama, kada su oni kompozibilni, odnosno kada je $\partial_2^+(l_1) = \partial_2^-(l_2)$,



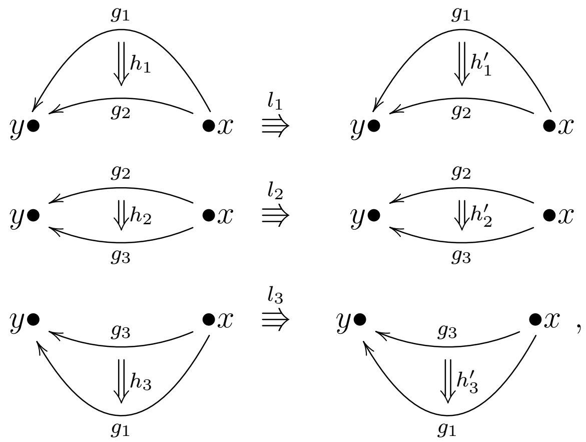
daje 3-morfizam:



Formalno zapisano, vertikalna kompozicija (g_1, h_1, l_1) i (g_2, h_2, l_2) daje 3-morfizam:

$$(g_2, h_2, l_2)\#_2(g_1, h_1, l_1) = (g_1, h_2h_1, l_2h_2 \rhd' l_1)\,. \tag{2.102}$$

- Vertikalna kompozicija je asocijativna operacija, odnosno za $l_1, l_2, l_3 \in L$ za koje je $\partial_2^+(l_1) = \partial_2^-(l_2)$ i $\partial_2^+(l_2) = \partial_2^-(l_1)$,



važi:

$$(g_3, h_3, l_3)\#_2(g_1, h_2h_1, l_2h_2 \rhd' l_1) = (g_2, h_3h_2, l_3h_3 \rhd' l_2)\#_2(g_1, h_1, l_1)\,. \tag{2.103}$$

Dokaz. Ovu jednakost dokazujemo sličnim postupkom kao u (2.25). Leva strana jednačine (2.103) daje:

$$(g_3, h_3, l_3)\#_2(g_1, h_2h_1, l_2h_2 \rhd' l_1) = (g_1, h_3h_2h_1, l_3h_3 \rhd' (l_2h_2 \rhd' l_1))\,. \tag{2.104}$$

Desna strana jednačine (2.103) daje takođe

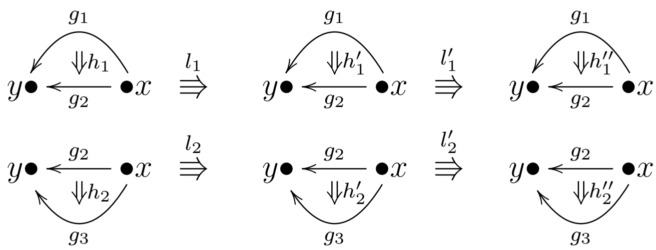
$$\begin{aligned}(g_2, h_3h_2, l_3h_3 \rhd' l_2)\#_2(g_1, h_1, l_1) &= (g_1, h_3h_2h_1, l_3\, h_3 \rhd' l_2\, (h_3h_2) \rhd' l_1)\\ &= (g_1, h_3h_2h_1, l_3\, h_3 \rhd' l_2\, h_3 \rhd' (h_2 \rhd' l_1))\\ &= (g_1, h_3h_2h_1, l_3\, h_3 \rhd' (l_2\, h_2 \rhd' l_1))\,,\end{aligned} \tag{2.105}$$

primenom Leme 2. Ovim smo dokazali jednakost (2.103). ∎

- Kompozicija 3-morfizama je invarijantna na redosled vršenja vertikalne kompozicije 3-morfizama i kompozicije 3-morfizama prema gore, odnosno važi:

$$\big((g_2,h'_2,l'_2)\#_3(g_2,h_2,l_2)\big)\#_2\big((g_1,h'_1,l'_1)\#_3(g_1,h_1,l_1)\big)=\big((g_2,h'_2,l'_2)\#_2(g_1,h'_1,l'_1)\big)\#_3\big((g_2,h_2,l_2)\#_2(g_1,h_1,l_1)\big), \tag{2.106}$$

što se lepo vidi u dijagramskoj notaciji, gde dijagram oblika



jednoznačno određuje 3-morfizam.

Dokaz. Dokažimo jednakost (2.106). Leva strana jednačine je jednaka:

$$\begin{aligned}\big((g_2,h'_2,l'_2)\#_3(g_2,h_2,l_2)\big)\#_2\big((g_1,h'_1,l'_1)\#_3(g_1,h_1,l_1)\big) &= \big(g_2,h_2,l'_2l_2\big)\#_2\big(g_1,h_1,l'_1l_1\big)\\ &= \big(g_1,h_2h_1,l'_2l_2\,h_2\rhd'(l'_1l_1)\big)\,.\end{aligned} \tag{2.107}$$

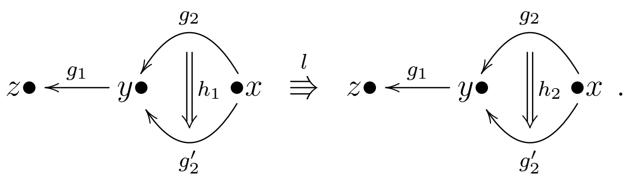
Desna strana jednačine (2.106) je jednaka:

$$\begin{aligned}\big((g_2,h'_2,l'_2)\#_2(g_1,h'_1,l'_1)\big)\#_3\big((g_2,h_2,l_2)\#_2(g_1,h_1,l_1)\big) &= \big(g_1,h'_2h'_1,l'_2h'_2\rhd' l'_1\big)\#_3\big(g_1,h_2h_1,l_2h_2\rhd' l_1\big)\\ &= \big(g_1,h_2h_1,l'_2h'_2\rhd' l'_1l_2h_2\rhd' l_1\big) && (l'_2=\delta(l_2)h_2)\\ &= \big(g_1,h_2h_1,l'_2(\delta(l_2)h_2)\rhd' l'_1l_2h_2\rhd' l_1\big) && \text{(Lema 1)}\\ &= \big(g_1,h_2h_1,l'_2\delta(l_2)\rhd'(h_2\rhd' l'_1)l_2h_2\rhd' l_1\big) && \text{(Pajferov id.)}\\ &= \big(g_1,h_2h_1,l'_2l_2(h_2\rhd' l'_1)l_2^{-1}l_2h_2\rhd' l_1\big) && (l_2^{-1}l_2=e)\\ &= \big(g_1,h_2h_1,l'_2l_2h_2\rhd' l'_1h_2\rhd' l_1\big) && \text{(Lema 2)}\\ &= \big(g_1,h_2h_1,l'_2l_2h_2\rhd'(l'_1l_1)\big).\end{aligned} \tag{2.108}$$

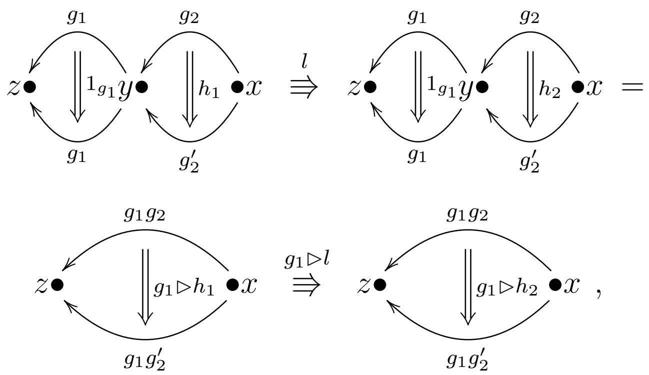
Ovim smo dokazali jednakost (2.106). ■

Kompozicija 3-morfizma i 1-morfizma

- *Nadovezivanje* je način na koji se vrši kompozicija 3-morfizma l i 1-morfizma g_1 koji se nalazi sa leve strane, tj. kada je $\partial_1^+(l)=\partial_1^-(g_1)$:



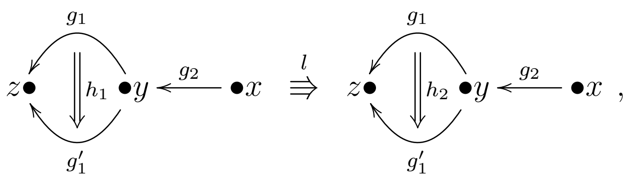
Ovako formiran 3-morfizam naziva se *nadovezivanje* 3-morfizma l i 1-morfizma g_1 *sa leve strane* $g_1\#_1 l$, osnosno 3-morfizam l *proširen sa leve strane* sa 1-morfizmom g_1. Ova kompozicija formirana je kao $(g_1,1_{g_1})\#_1(g_2,h_1,l)$



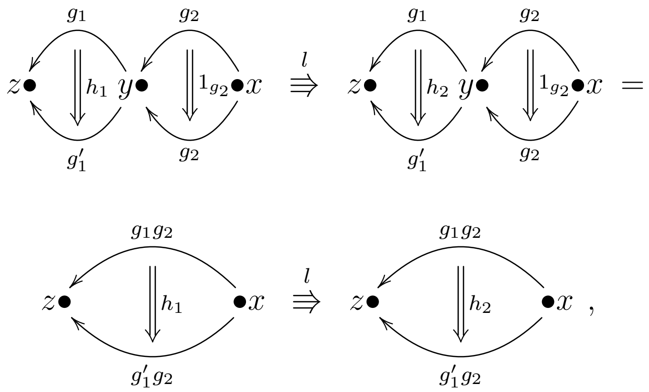
odnosno formalno zapisano:

$$(g_1,1_{g_1})\#_1(g_2,h_1,l)=(g_1g_2,g_1\rhd h,g_1\rhd l)\,. \tag{2.109}$$

- Na sličan način možemo formirati i *nadovezivanje* 3-morfizma l i 1-morfizma g_2 *sa desne strane*, kada je $\partial_1^-(l)=\partial_1^+(g_2)$, osnosno 3-morfizam l *proširen sa desne strane* sa 1-morfizmom g_2,



koju formiramo kao $(g_1, h_1, l)\#_1(g_2, 1_{g_2})$,

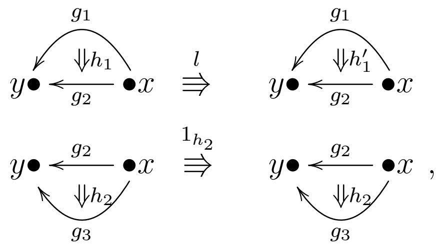


tj. formalno zapisano:

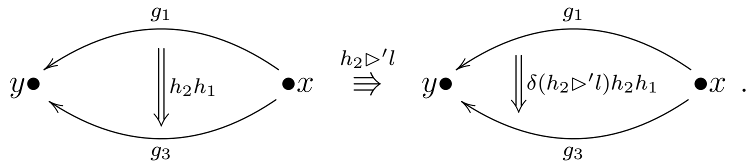
$$(g_1, h_1, l)\#_1(g_2, 1_{g_2}) = (g_1 g_2, h_1, l)\,. \tag{2.110}$$

Kompozicija 3-morfizma i 2-morfizma

- *Nadovezivanje* 3-morfizma l i 2-morfizma h_2 *sa gornje strane*, odnosno 3-morfizam l *proširen sa gornje strane* sa 2-morfizmom h_1, kada su oni kompozibilni, odnosno kada $\partial_2^+(l) = \partial_2^-(h_2)$, posmatramo kao vertikalnu kompoziciju (g_1, h_1, l) i $(g_2, h_2, 1_{h_2})$



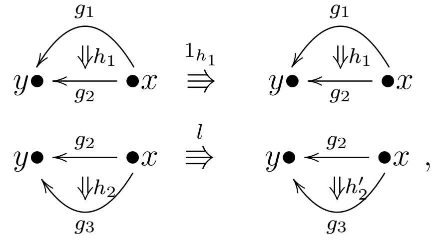
koja rezultuje 3-morfizmom:



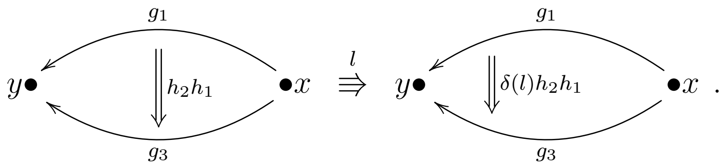
Formalno zapisano:

$$(g_1, h_1, l)\#_2(g_2, h_2, 1_{h_2}) = (g_1, h_2 h_1, h_2 \triangleright' l)\,. \tag{2.111}$$

- *Nadovezivanje* 3-morfizma l i 2-morfizma h_1 *sa donje strane*, odnosno 3-morfizam l *proširen sa donje strane* sa 2-morfizmom h_1, kada su oni kompozibilni, odnosno kada $\partial_2^-(l) = \partial_2^+(h_1)$, posmatramo kao vertikalnu kompoziciju $(g_1, h_1, 1_{h_1})$ i (g_2, h_2, l)



koja rezultuje sa 3-morfizmom:

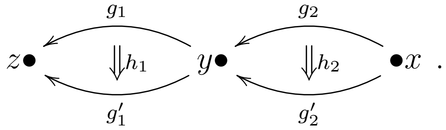


Formalno zapisano:

$$(g_1, h_1, 1_{h_1})\#_2(g_2, h_2, l) = (g_1, h_2h_1, l)\,. \tag{2.112}$$

2.3.4 Horizontalna kompozicija 2-morfizama - izmenski 3-morfizam

- *Horizontalna kompozicija 2-morfizma* h_1 i h_2 kada važi $\partial_1^-(h_1) = \partial_1^+(h_2)$ daje *izmenski 3-morfizam*[14].

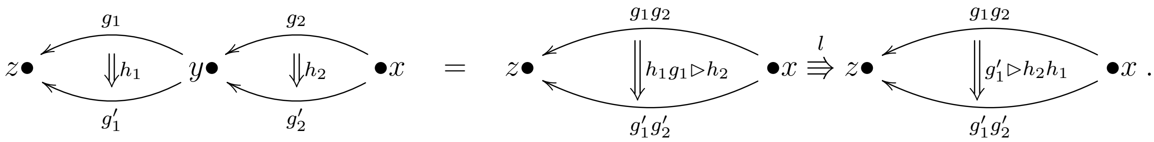


Kompozicija rezultuje 3-morfizmom, čiji je izvor 2-morfizam

$$\partial_3^-(l) = \big((g_1, h_1)\#_1 g'_2\big)\#_2\big(g_1\#_1(g_2, h_2)\big)\,,$$

a meta 2-morfizam

$$\partial_3^+(l) = \big(g'_1\#_1(g_2, h_2)\big)\#_2\big((g_1, h_1)\#_1 g_2\big)\,,$$



Formalno zapisano:

$$(g_1, h_1)\#_1(g_2, h_2) = (g_1g_2, h_1g_1 \rhd h_2, l)\,. \tag{2.113}$$

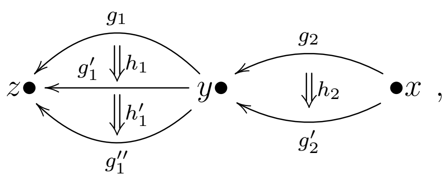
U jednačini (2.113) 3-morfizam l jednak je Pajferovom podizanju $\{h_1, g_1\rhd h_2\}_{\mathrm{pf}}^{-1}$. Koristeći identitet (2.99), dobijamo:

$$(\partial(h_1)g_1) \rhd h_2h_1 = \delta(l)h_1\big(g_1 \rhd h_2\big)\,. \tag{2.114}$$

Koristeći definiciju Pajferovog komutatora, tj. identitet (2.55) i prvu osobinu 2-ukrštenog modula, tj. $\delta(\{h_1, h_2\}_{\mathrm{p}}) = \langle h_1\,, h_2\rangle_{\mathrm{pf}}$, dobijamo da je 3-morfizam l:

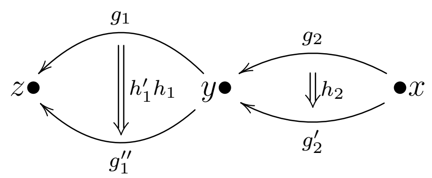
$$\delta(l)^{-1} = h_1g_1 \rhd h_2h_1^{-1}(\partial(h_1)g_1) \rhd h_2{}^{-1} = \langle h_1, g_1 \rhd h_2\rangle_{\mathrm{pf}} = \delta(\{h_1, g_1 \rhd h_2\}_{\mathrm{p}})\,. \tag{2.115}$$

- Horizontalna kompozicija vertikalne kompozicije 2-morfizma h_1 i h'_1 i 2-morfizma h_2 *sa desne strane*, kada su kompozibilni, tj. kada je $\partial_1^-(h_1) = \partial_1^-(h'_1) = \partial_1^+(h_2)$ i $\partial_2^+(h_1) = \partial_2^-(h'_1)$,



[14]eng. *the interchanging 3-arrow.*

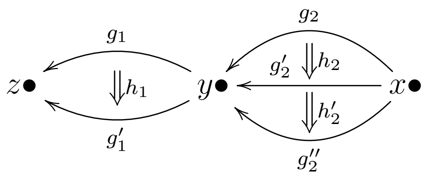
dobija se kao izmenski 3-morfizam $(g_1, h_1' h_1)\#_1(g_2, h_2)$,



$$= (g_1, h_1' h_1)\#_1(g_2, h_2) = (g_1 g_2, h_1' h_1 g_1 \rhd h_2, l)\,.$$

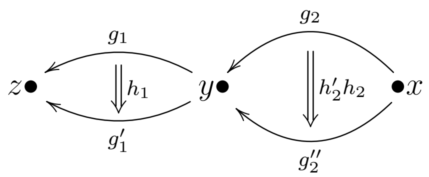
U prethodnoj formuli $l = \{h_1' h_1, g_1 \rhd h_2\}_{\mathrm{pf}}^{-1}$, gde je površina $h_1' h_1 g_1 \rhd h_2$ izvor, a površina $g_1'' \rhd h_2 h_1' h_1$ meta 3-morfizma.

- Horizontalna kompozicija vertikalne kompozicije 2-morfizma h_2 i h_2' i 2-morfizma h_1 *sa leve strane*, kada su kompozibilni, tj. kada je $\partial_1^+(h_2) = \partial_1^+(h_2') = \partial_1^-(h_1)$ i $\partial_2^+(h_2) = \partial_2^-(h_2')$,



,

dobija se kao izmenski 3-morfizam $(g_1, h_1)\#_1(g_2, h_2' h_2)$,



$$= (g_1, h_1)\#_1(g_2, h_2' h_2) = (g_1 g_2, h_1 g_1 \rhd (h_2' h_2), l)\,.$$

U prethodnoj formuli $l = \{h_1, g_1 \rhd (h_2' h_2)\}_{\mathrm{pf}}^{-1}$, gde je površina $h_1 g_1 \rhd (h_2' h_2)$ izvor, a površina $g_1' \rhd (h_2' h_2) h_1$ meta 3-morfizma.

2.3.5 3-koneksija i 3-krivina

Neka su $\mathfrak{g}$, $\mathfrak{h}$ i $\mathfrak{l}$ Lijeve algebre koje odgovaraju grupama G, H i L. Možemo definisati 3-koneksiju, uređenu trojku (α, β, γ), gde su diferencijalne forme elementi algebre $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{h})$ i $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,, \mathfrak{l})$. Odgovarajuća *lažna*[15] *3-krivina* $(\mathcal{F}\,, \mathcal{G}\,, \mathcal{H})$ se definiše kao:

$$\begin{gathered} \mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\rhd} \beta - \delta\gamma\,, \\ \mathcal{H} = \mathrm{d}\gamma + \alpha \wedge^{\rhd} \gamma + \beta \wedge^{\{,\}} \beta\,. \end{gathered} \tag{2.118}$$

Primetimo da je koneksija γ diferencijalna 3-forma, odnosno da je njena odgovarajuća jačina polja $\mathcal{H}$ diferencijalna 4-forma, što povlači da prostorvremenska mnogostrukost $\mathcal{M}$ za koju

[15]Pravimo razliku između 3-*krivine* i *lažne* 3-*krivine*. Obična 3-krivina, uređena trojka (F, G, H) za koneksiju $\alpha \in \mathcal{A}^1(\mathfrak{g}, \mathcal{M}_4)$, $\beta \in \mathcal{A}^2(\mathfrak{h}, \mathcal{M}_4)$ i $\gamma \in \mathcal{A}^3(\mathfrak{l}, \mathcal{M}_4)$ definiše se na standardan način:

$$\begin{aligned} F &= \mathrm{d}\alpha + \alpha \wedge \alpha, \\ G &= \mathrm{d}\beta + \alpha \wedge^{\rhd} \beta, \\ H &= \mathrm{d}\gamma + \alpha \wedge^{\rhd} \gamma\,. \end{aligned} \tag{2.116}$$

Lažna 3-*krivina*, uređena trojka $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ ima dodatne članove i definisana je kao:

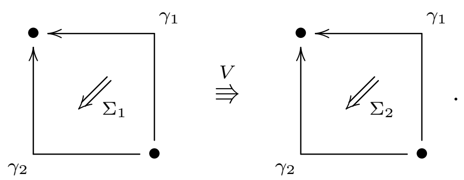
$$\begin{aligned} \mathcal{F} &= F - \partial\beta, \\ \mathcal{G} &= G - \delta\gamma, \\ \mathcal{H} &= H + \beta \wedge^{\{,\}} \beta\,. \end{aligned} \tag{2.117}$$

definišemo $3BF$ dejstvo mora biti najmanje 4-dimenzionalna. Ako uporedimo definiciju 3-krivine u 3-gejdž teoriji sa definicijom 2-krivine u 2-gejdž teoriji, primećujemo da je krivina $\mathcal{G}$ definisana sa dodatnim članom[16].

Krivine možemo raspisati u bazisima odgovarajućih algebri i diferencijalnih formi:

$$\mathcal{F}{=}\frac{1}{2}\mathcal{F}^{\alpha}{}_{\mu\nu}\tau_{\alpha}\mathrm{d}x^{\mu}{\wedge}\mathrm{d}x^{\nu}, \quad \mathcal{G}{=}\frac{1}{3!}\mathcal{G}^{a}{}_{\mu\nu\rho}t_{a}\mathrm{d}x^{\mu}{\wedge}\mathrm{d}x^{\nu}{\wedge}\mathrm{d}x^{\rho}, \quad \mathcal{H}{=}\frac{1}{4!}\mathcal{H}^{A}{}_{\mu\nu\rho\sigma}T_{A}\mathrm{d}x^{\mu}{\wedge}\mathrm{d}x^{\nu}{\wedge}\mathrm{d}x^{\rho}{\wedge}\mathrm{d}x^{\sigma},$$

[16]Posmatrajmo zapreminu V:



Ovaj kocka predstavlja 3-morfizam

$$V : \Sigma_1 \to \Sigma_2 \,, \tag{2.119}$$

gde su Σ_1 i Σ_2 2-morfizmi prikazani na slici. Želimo da izračunamo:

$$\mathrm{hol}(V) : \mathrm{hol}(\Sigma_1) \to \mathrm{hol}(\Sigma_2) \,. \tag{2.120}$$

Holonomija $\mathrm{hol}(V)$ zavisi od 3-koneksije γ. Njen izvor i meta zavise samo od 2-forme β. Dobra definisanost izvora i mete 3-morfizma V osigurava se vezom između 2-koneksije β i 3-koneksije γ. Sa jedne strane, imamo da je:

$$\mathrm{hol}(\Sigma_1) = \mathcal{P}\mathrm{exp}\left(\int_{\Sigma_1} \beta\right) , \qquad \mathrm{hol}(\Sigma_2) = \mathcal{P}\mathrm{exp}\left(\int_{\Sigma_2} \beta\right) . \tag{2.121}$$

Takođe, znamo da je 3-morfizam $l : h_1 \to h_2$ određen elementom $l \in L$ takvim da $h_2 = \delta(l)h_1$. To znači da je 3-morfizam $\mathrm{hol}(V) : \mathrm{hol}(\Sigma_1) \to \mathrm{hol}(\Sigma_2)$ određen elementom $l \in L$, pri čemu

$$\mathcal{P}\mathrm{exp}\left(\int_{\Sigma_2} \beta\right) = \delta(l)\, \mathcal{P}\mathrm{exp}\left(\int_{\Sigma_1} \beta\right) , \tag{2.122}$$

odnosno

$$\delta(l) = \mathcal{P}\mathrm{exp}\left(\int_{\partial V} \beta\right) , \tag{2.123}$$

gde je granična površina zapremine $\partial V = \Sigma_2 \Sigma_1^{-1}$. Imajući u vidu da je kvadrat mali i koristeći Stoksovu teoremu dobijamo:

$$\mathcal{P}\mathrm{exp}\left(\int_{\partial V} \beta\right) \approx \mathrm{exp}\left(\int_{V} G\right) . \tag{2.124}$$

Sa druge strane, 3-morfizam l izražen preko 3-forme γ je:

$$l \approx \mathrm{exp}\left(\int_{V} \gamma\right) . \tag{2.125}$$

Da bi ovi izrazi bili jednaki potrebno je da važi jednakost

$$\delta(\mathrm{exp}\left(\int_{V} \gamma\right)) \approx \mathrm{exp}\left(\int_{V} G\right) , \tag{2.126}$$

odnosno:

$$\delta(\gamma) = G = \mathrm{d}\beta + \alpha \wedge \beta \,. \tag{2.127}$$

gde su koeficijenti:

$$
\begin{aligned}
\mathcal{F}^{\alpha}{}_{\mu\nu} &= \partial_\mu \alpha^{\alpha}{}_{\nu} - \partial_\nu \alpha^{\alpha}{}_{\mu} + f_{\beta\gamma}{}^{\alpha} \alpha^{\beta}{}_{\mu} \alpha^{\gamma}{}_{\nu} - \beta^{a}{}_{\mu\nu} \partial_a{}^{\alpha} \,, \\
\mathcal{G}^{a}{}_{\mu\nu\rho} &= \partial_\mu \beta^{a}{}_{\nu\rho} + \partial_\nu \beta^{a}{}_{\rho\mu} + \partial_\rho \beta^{a}{}_{\mu\nu} + \alpha^{\alpha}{}_{\mu} \beta^{b}{}_{\nu\rho} \rhd_{\alpha b}{}^{a} + \alpha^{\alpha}{}_{\nu} \beta^{b}{}_{\rho\mu} \rhd_{\alpha b}{}^{a} + \alpha^{\alpha}{}_{\rho} \beta^{b}{}_{\mu\nu} \rhd_{\alpha b}{}^{a} - \gamma^{A}{}_{\mu\nu\rho} \delta_A{}^{a} \,, \\
\mathcal{H}^{A}{}_{\mu\nu\rho\sigma} &= \partial_\mu \gamma^{A}{}_{\nu\rho\sigma} - \partial_\nu \gamma^{A}{}_{\rho\sigma\mu} + \partial_\rho \gamma^{A}{}_{\sigma\mu\nu} - \partial_\sigma \gamma^{A}{}_{\mu\nu\rho} \\
&\quad + 2 \beta^{a}{}_{\mu\nu} \beta^{b}{}_{\rho\sigma} X_{\{ab\}}{}^{A} - 2 \beta^{a}{}_{\mu\rho} \beta^{b}{}_{\nu\sigma} X_{\{ab\}}{}^{A} + 2 \beta^{a}{}_{\mu\sigma} \beta^{b}{}_{\nu\rho} X_{\{ab\}}{}^{A} \\
&\quad + \alpha^{\alpha}{}_{\mu} \gamma^{B}{}_{\nu\rho\sigma} \rhd_{\alpha B}{}^{A} - \alpha^{\alpha}{}_{\nu} \gamma^{B}{}_{\rho\sigma\mu} \rhd_{\alpha B}{}^{A} + \alpha^{\alpha}{}_{\rho} \gamma^{B}{}_{\sigma\mu\nu} \rhd_{\alpha B}{}^{A} - \alpha^{\alpha}{}_{\sigma} \gamma^{B}{}_{\mu\nu\rho} \rhd_{\alpha B}{}^{A} \,.
\end{aligned}
$$

pogledati [17], [18] za više detalja.

Transformacije 3-koneksije i 3-krivine

U teoriji kategorija, 3-grupa daje tri tipa gejdž transformacija generisanih grupama G, H i L. Pri G-gejdž transformacijama, 3-koneksija se transformiše na sledeći način,

$$\alpha' = g\alpha g^{-1} + g\mathrm{d}g^{-1} \,, \qquad \beta' = g \rhd \beta \,, \qquad \gamma' = g \rhd \gamma \,, \tag{2.128}$$

gde je $g : \mathcal{M}_4 \to G$ element G-glavnog raslojenja mnogostrukosti $\mathcal{M}_4$. Zatim, pri H-gejdž transformacijama, generisanih elementom $\eta \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{h})$, 3-koneksija se transformiše po zakonu transformacije:

$$\alpha' = \alpha + \partial\eta \,, \qquad \beta' = \beta + \mathrm{d}\eta + \alpha' \wedge^{\rhd} \eta - \eta \wedge \eta \,, \qquad \gamma' = \gamma - \beta' \wedge^{\{,\}} \eta - \eta \wedge^{\{,\}} \beta \,. \tag{2.129}$$

Na kraju, pri L-gejdž transformacijama, generisanih sa $\theta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{l})$, 3-koneksija se transformiše po zakonu transformacije:

$$\alpha' = \alpha \,, \qquad \beta' = \beta - \delta\theta \,, \qquad \gamma' = \gamma - \mathrm{d}\theta - \alpha \wedge^{\rhd} \theta \,. \tag{2.130}$$

Teorema 5 *Kompozicija G-gejdž, H-gejdž i L-gejdž transformacija dovodi do transformacije 3-koneksije:*

$$
\begin{aligned}
\tilde{\alpha} &= g\alpha g^{-1} + g\mathrm{d}g^{-1} + \partial(\eta) \,, \\
\tilde{\beta} &= g \rhd \beta + \mathrm{d}\eta + \tilde{\alpha} \wedge^{\rhd} \eta - \eta \wedge \eta - \delta(\theta) \,, \\
\tilde{\gamma} &= g \rhd \gamma - \mathrm{d}\theta - \tilde{\alpha} \wedge \theta - \tilde{\beta} \wedge^{\{,\}} \eta - \eta \wedge^{\{,\}} (g \rhd \beta) + \eta \wedge^{\rhd'} \theta \,,
\end{aligned} \tag{2.131}
$$

gde su $g : \mathcal{M}_4 \to G$, $\eta \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ i $\theta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{l})$ redom parametri G-, H- i L-gejdž transformacija.

Na osnovu transformacija 3-koneksije, dobijamo zakon transformacije 3-krivine definisane izrazom (2.118) pri 3-gejdž transformacijama.

Teorema 6 *Pri G-gejdž transformacijama 3-krivina $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ se transformiše na sledeći način*

$$\mathcal{F} \to g\mathcal{F}g^{-1} \,, \qquad \mathcal{G} \to g \rhd \mathcal{G} \,, \qquad \mathcal{H} \to g \rhd \mathcal{H} \,, \tag{2.132}$$

pri H-gejdž transformacijama na sledeći način

$$\mathcal{F} \to \mathcal{F} \,, \qquad \mathcal{G} \to \mathcal{G} + \mathcal{F} \wedge^{\rhd} \eta \,, \qquad \mathcal{H} \to \mathcal{H} - \mathcal{G}' \wedge^{\{,\}} \eta + \eta \wedge^{\{,\}} \mathcal{G} \,, \tag{2.133}$$

a pri L-gejdž transformacijama kao:

$$\mathcal{F} \to \mathcal{F} \,, \qquad \mathcal{G} \to \mathcal{G} \,, \qquad \mathcal{H} \to \mathcal{H} - \mathcal{F} \wedge^{\rhd} \theta \,, \tag{2.134}$$

gde su $g : \mathcal{M}_4 \to G$, $\eta \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ i $\theta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{l})$ redom parametri G-, H- i L-gejdž transformacija.

Za više detalja i dokaze ovih teorema pogledati [18], odnosno dodatak A.

Glava 3

Hamiltonova analiza

Hamiltonova analiza teorije je neophodan prvi korak kanonske kvantizacione procedure koju je formulisao Pol Dirak za Hamiltonove sisteme sa vezama [31]. Ova procedura nam dozvoljava da formulišemo kvantnu teoriju za sisteme koju poseduju gejdž simetriju. Ovaj pristup može se podeliti na dva glavna koraka.

1. Prvo, neophodno je izvršiti Hamiltonovu analizu sistema, čiji je rezultat algebra veza prve klase i veza druge klase prisutnih u teoriji. Veze prve klase generišu nefizičke transformacije dinamičkih varijabli, gejdž transformacije, koje ne menjaju fizičko stanje sistema. Veze prve klase F i veze druge klase S, zajedno definišu potprostor Γ^* faznog prostora Γ dimenzije

$$2n = 2N - (2F + S)\,, \tag{3.1}$$

   u kome se odigrava dinamika nezavisnih stepeni slobode.

   U ovom poglavlju predstavljene su osnovne ideje Dirakovog metoda. Zatim, prikazan je kratak pregled sistemskog pristupa konstrukciji *generatora gejdž transformacija* na osnovu poznate Hamiltonove strukture – *Kastelanijeve procedure.*

2. Prelaz sa klasične na kvantnu teoriju sa gejdž stepenima slobode u koordinatnoj reprezentaciji postiže se na sledeći način:

   - Dinamičke varijable koordinata i njihovih kanonskih impulsa postaju operatori:

$$q(x) \to \hat{q}(x)\,,$$

$$\pi(x) \to \hat{\pi}(x)\,;$$

   - Zatim, pomoću Poasonove zagrade uvede se Dirakova zagrada $\{\_,\,\_\}_D$, čime se iz teorije eliminišu veze druge klase, ukoliko ih ima u teoriji;
   - Zatim, Dirakova zagrada operatora q i π, tzv. *Hajzenbergove algebre*, postaje komutator:

$$\{q(x),\, \pi(x)\} \to -i[\hat{q}(x),\, \hat{\pi}(x)]\,;$$

   - Veze prve klase postaju uslovi na fizička stanja:

$$\hat{\Phi}\,|\psi\rangle = 0\,.$$

     Ovi uslovi su analogni Gupta-Blojlerovim uslovima iz kvantne elektrodinamike.

3.1 Lagranžev i Hamiltonov formalizam

U ovom odeljku prikazan je kratak osvrt na Lagranžev i Hamiltonov formalizam. Neka je n *broj stepeni slobode* nekog fizičkog sistema, koji predstavlja minimalan broj veličina potrebnih da se potpuno opiše položaj svih čestica, odnosno konfiguracija sistema. Lagranževa funkcija, ili *Lagranžijan* je funkcija oblika:

$$L(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q_N}, t)\,, \tag{3.2}$$

gde su veličine q_i i $\dot{q}_i$, $i = 1, \ldots, N$, generalisane koordinate i njihovi vremenski izvodi. *Hamiltonovo dejstvo* je po definiciji jednako:

$$S = \int_{t_1}^{t_2} \mathrm{d}t\, L(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q_N}, t)\,. \tag{3.3}$$

Ako na posmatrani fizički sistem deluju samo potencijalne sile i ako se u Lagranzijanu pojavljuju samo prvi izvodi generalisanih koordinata po vremenu, *Ojler-Lagranževe jednačine kretanja* imaju oblik

$$\frac{\delta S}{\delta q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,, \qquad i = 1, \ldots, N\,, \tag{3.4}$$

gde je $\frac{\delta S}{\delta q_i}$ funkcionalni izvod dejstva po generalisanoj koordinati q_i. Jednačine kretanja su posledica *Hamiltonovog principa najmanjeg dejstva.* Ojler-Lagranževe jednačine možemo napisati u obliku

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j\,, \qquad i = 1, \ldots, N\,, \tag{3.5}$$

gde je matrica $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ Hesijan Lagranžijana L. Ako je Hesijan invertibilan, tj.

$$\det H^{ij} = \det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) \neq 0\,, \tag{3.6}$$

jednačine (3.5), ako su zadovoljeni uslovi egzistencije i jedinstvenosti rešenja sistema diferencijalnih jednačina, a za date inicijalne uslove, možemo rešiti po ubrzanjima $\ddot{q}_i$. Ovakve teorije nazivamo *nesingularnim.* Kažemo da je teorija *singularna* ako uslov (3.6) nije zadovoljen, tj. ako je

$$\det H^{ij} = 0\,. \tag{3.7}$$

Kanonski impulsi su definisani na sledeći način:

$$p^i = \pi(q_i) = \frac{\delta L}{\delta \dot{q}_i}\,, \qquad i = 1, \ldots, N\,. \tag{3.8}$$

U ovom trenutku možemo da preformulišemo definiciju nesingularne teorije – vidimo da Hesijan možemo izraziti kao $\frac{\partial p^i}{\partial \dot{q}_j}$, pa uslov (3.6) možemo razumeti kao zahtev da se brzine $\dot{q}_i$ mogu izraziti kao funkcija generalisanih koordinata i njihovih kanonskih impulsa.

U slučaju nesingularne teorije, definišemo *Hamiltonijan*,

$$H = \sum_i p^i\, \dot{q}_i - L\,. \tag{3.9}$$

Jednačine kretanja u Hamiltonovom formalizmu su:

$$\dot{p^i} = \frac{\partial H}{\partial q_i}\,, \qquad \dot{q}_i = -\frac{\partial H}{\partial q_i}\,, \qquad i = 1, \ldots, N\,. \tag{3.10}$$

Poasonova zagrada funkcija $f = f(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t)$ i $g = g(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t)$ je definisana na sledeći način:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial p^i} \frac{\partial f}{\partial q_i} \right) . \tag{3.11}$$

Koristeći Poasonovu zagradu, definišemo vremensku evoluciju proizvoljne funkcije f na faznom prostoru

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t} , \tag{3.12}$$

pa Hamiltonove jednačine kretanja možemo napisati u obliku:

$$\dot{p}^i = \{p^i, H\} , \qquad \dot{q}_i = \{q_i, H\} , \qquad i = 1, \ldots, N . \tag{3.13}$$

Definicije date u ovom odeljku pravolinijski se generalizuju na slučaj teorije sa beskonačno mnogo stepeni slobode. U slučaju teorije koja opisuje polje ϕ^α u prostorvremenu čije su koordinate x^μ, *gustina Lagranžijana* $\mathcal{L}$ je funkcija oblika:

$$L = \int_{\Sigma_3} \mathrm{d}^3 x \, \mathcal{L}(\phi^\alpha, \partial_\mu \phi^\alpha) , \tag{3.14}$$

pa *dejstvo* definišemo na sledeći način:

$$S = \int \mathrm{d}^4 x \, \mathcal{L}(\phi^\alpha, \partial_\mu \phi^\alpha) . \tag{3.15}$$

Kanonski impulsi su u ovom slučaju funkcije koordinata

$$p^\alpha(x) = \pi(\phi_\alpha) = \frac{\delta L}{\delta \dot{\phi}_\alpha(x)} , \tag{3.16}$$

a *Hamiltonijan*:

$$H = \int_{\Sigma_3} \mathrm{d}^3 x \left[\sum_\alpha p^\alpha(x) \, \dot{\phi}_\alpha(x) \right] - L . \tag{3.17}$$

U ovom odeljku napravili smo kratak pregled Lagranževog i Hamiltonovog formalizma, pri čemu smo našu pažnju ograničili isključivo na nesingularne teorije, tj. one koji ispunjavaju uslov (3.6). U narednom odeljku razmatraćemo *singularne teorije*.

3.2 Sistemi sa vezama

Singularne teorije se odlikuju prisustvom redundantnih stepeni slobode u fizičkom sistemu, tj. postojanjem dodatnih, nefizičkih varijabli u dejstvu, tako da je ukupan broj varijabli u teoriji veći od broja varijabli koje opisuju fiziku sistema. Za analizu ovih fizičkih sistema neophodna je *Hamiltonova analiza sistema sa vezama*, tj. *Dirakova procedura*.

3.2.1 Dirakova teorija

Primarne veze

Ako nije moguće invertovati relacije (3.8) i izraziti brzine kao funkcije generalisanih koordinata i njihovih konjugovanih impulsa, tj. ako je zadovoljen uslov (3.7), teorija je singularna. Sve relevantne fizičke teorije, kao što su Standardni Model elementarnih čestica i Opšta teorija relativnosti, spadaju u ovu klasu. *Singularne teorije* odlikuje postojanje seta relacija između varijabli i impulsa, koje nazivamo *primarnim vezama*

$$P_m(q,p) \approx 0\,, \qquad m = 1, \ldots, P\,, \tag{3.18}$$

gde P predstavlja ukupan broj primarnih veza u teoriji, a znak $\approx$ obeležava slabu jednakost[1]. Iz jednačine (3.8) dobijamo oblik primarnih veza:

$$P(q_i) \equiv \pi(q_i) - \frac{\delta L}{\delta \dot{q}_i}\,. \tag{3.19}$$

Primarne veze određuju potprostor Γ_1 ukupnog faznog prostora Γ u kome se odigrava dinamika sistema. Primarne veze P_m ne moraju biti sve međusobno nezavisne. Ako je broj nezavisnih primarnih veza $P' \leq P$, onda je dimenzija faznog potprostora $2N - P'$. Tada je Hesijan $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ matrica ranga $N - P'$.

Kanonski i totalni Hamiltonijan

Definišemo *kanonski Hamiltonijan*

$$H_c = \sum_{i=1}^{N} p^i\,\dot{q}_i - L\,, \tag{3.20}$$

koji je dobro definisan na površini određenoj primarnim vezama. U slučaju singularne teorije definišemo *totalni hamiltonijan* koji je definisan na celom faznom prostoru Γ

$$H_T = H_c + \sum_m \lambda^m P_m\,, \tag{3.21}$$

gde su $\lambda^m(q)$ proizvoljni Lagranževi množitelji. Jednačine kretanja za kanonske varijable i njihove impulse dobijene totalnim Hamiltonijanom

$$\dot{p}_i = -\frac{\partial H_T}{\partial q_i}\,, \qquad \dot{q}_i = \frac{\partial H_T}{\partial p_i}\,, \qquad i = 1, \ldots, N\,, \tag{3.22}$$

primenom jednačine (3.21) postaju:

$$\dot{p}_i = -\frac{\partial H_c}{\partial q_i} - \lambda^m \frac{\partial P_m}{\partial q_i}\,, \qquad \dot{q}_i = \frac{\partial H_c}{\partial p_i} + \lambda^m \frac{\partial P_m}{\partial p_i}\,, \qquad i = 1, \ldots, N\,. \tag{3.23}$$

Ako je $A(q,p)$ neka proizvoljna dinamička varijabla u teoriji, njena jednačina kretanja je

$$\dot{A} = \frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i\,, \tag{3.24}$$

pa primenom jednačina (3.22) dobijamo:

$$\dot{A} = \frac{\partial A}{\partial q_i}\frac{\partial H_T}{\partial p_i} - \frac{\partial A}{\partial q_i}\frac{\partial H_T}{\partial q_i} = \{A, H_T\}\,. \tag{3.25}$$

[1] Neka je $F(q,p)$ funkcija koja je definisana i diferencibilna u okolini $\mathcal{O} \subseteq \Gamma$ koja sadrži podprostor Γ_1. Ako je funkcija $F(q,p)$ na Γ_1 jednaka nuli, kažemo da je F *slabo jednaka* nuli:

$$F(q,p) \approx 0 \quad \Leftrightarrow \quad F(q,p)|_{\Gamma_1} = 0\,.$$

Ako su funkcija F i svi njeni prvi izvodi jednaki nula na Γ_1, kažemo da je *jako jednaka* nuli:

$$F(q,p) = 0 \quad \Leftrightarrow \quad F(q,p)|_{\Gamma_1} = \frac{\partial F(q,p)}{\partial q}|_{\Gamma_1} = \frac{\partial F(q,p)}{\partial p}|_{\Gamma_1} = 0\,.$$

Uslovi konzistentnosti i sekundarne veze

Uslovi konzistentnosti primarnih veza zahtevaju da primarne veze budu sačuvane tokom dinamičke evolucije sistema:

$$\dot{P}(q_i) \equiv \{\, P(q_i)\,,\, H_T\,\} \approx 0\,. \tag{3.26}$$

Uslovi konzistentnosti rezultuju u jednom od tri ishoda.

1. Poasonova zagrada (3.26) jednaka je nekoj linearnoj kombinaciji primarnih veza, pa kao takva je slabo jednaka nuli – uslov konzistentnosti je automatski zadovoljen.

2. Uslovi konzistentnosti rezultuju u dobijanju *sekundardnih veza* u teoriji:

   $$\mathcal{S}(q_i) = \{\, P(q_i)\,,\, H_T\,\} \approx 0\,. \tag{3.27}$$

   Sekundardne veze koje su se pojavile u teoriji takođe moraju da zadovolje *uslove konzistentnosti*:

   $$\dot{\mathcal{S}}(q_i) \equiv \{\, \mathcal{S}(q_i)\,,\, H_T\,\} \approx 0\,. \tag{3.28}$$

   Uslovi konzistentnosti sekundarnih veza mogu dovesti do pojave novih veza u teoriji – *tercijarnih veza.* Tada moramo zahtevati da tercijarne veze ostaju nepromenjene, tj. da zadovoljavaju uslove konzistentnosti, što takođe rezultuje nekim od ovde navedenih ishodom. Ovaj proces se nastavlja sve dok uslovi konzistentnosti ne prestanu da uvode nove veze.

3. Konačno, uslovi konzistentnosti mogu odrediti neke Lagranževe množitelje λ^m.

Zajedno P primarnih veza $P(q,p)$ i K novih veza $\mathcal{S}(q,p)$ (sekundarnih, tercijarnih...), dobijenih uslovima konzistentnosti, čine skup svih veza u teoriji:

$$\phi_m(q,p) \approx 0\,, \qquad m = 1,\ldots,P,P+1,\ldots,P+K\,. \tag{3.29}$$

Može se pokazati da je neka funkcija na faznom prostoru $F(q,p)$ slabo jednaka nuli, ako i samo ako je jednaka linearnoj kombinaciji veza:

$$F \approx 0 \quad \Leftrightarrow \quad F = \sum_m \lambda^m \phi_m\,.$$

Primenom definicije totalnog Hamiltonijana, uslovi konzistentnosti za sve veze u teoriji postaju:

$$\{\, \phi_m\,,\, H_c\,\} + \sum_{m'} \lambda^{m'} \{\, \phi_m\,,\, \phi_{m'}\,\} \approx 0\,. \tag{3.30}$$

Neka je L broj uslova konzistentnosti koji nisu automatski zadovoljeni, tj. koji nameću ograničenja na Lagranževe množitelje. Tada sistem od L diferencijalnih jednačina (3.30) možemo rešiti po λ^m

$$\lambda^m = U^m + \sum_a v^a(t) V_a^m\,, \tag{3.31}$$

gde su $v^a(t)$ arbitrarne funkcije vremena, V_a^m homogeni

$$\sum_{m'} \sum_a v^a(t) V_a^{m'} \{\, \phi_m\,,\, \phi_{m'}\,\} \approx 0\,, \tag{3.32}$$

i U^m partikularni deo rešenja sistema diferencijalnih jednačina. Dakle, zakljuučujemo da u totalnom Hamiltonijanu postoje proizvoljne funkcije vremena. Na kraju, primetimo da, kako su funkcije $v^a(t)$ proizvoljne funkcije vremena, važi relacija:

$$\sum_{m'} V_a^{m'} \{\, \phi_m\,,\, \phi_{m'}\,\} \approx 0\,. \tag{3.33}$$

Veze prve i veze druge klase

Dinamička varijabla $R(q,p)$ je *prve klase* ako njene Poasonove zagrade sa svim vezama u teoriji slabo nestaju:

$$\{R(q,p), \phi_m(q,p)\} \approx 0\,.$$

Kao i svaka funkcija koja je slabo jednaka nuli, ova Poasonova zagrada je jednaka linearnoj kombinaciji veza. Možemo primetiti, na osnovu ove definicije, da je totalni Hamiltonijan H_T veličina prve klase. Ako promenljiva $R(q,p)$ nije prve klase, ona je *druge klase*. Dok je razlika između primarnih i sekundarnih veza od malog značaja za Hamiltonovu analizu, razlika između veza prve i veza druge klase je suštinska za dinamičku interpretaciju veza u okviru Hamiltonove teorije. Veze prve klase su definisane jednačinom,

$$\{\Phi(q,p), \phi_m(q,p)\} \approx 0\,, \tag{3.34}$$

dok veze druge klase zadovoljavaju uslov

$$\{\chi(q,p), \phi_m(q,p)\} \not\approx 0\,, \tag{3.35}$$

gde su $\phi_m(q,p)$ sve veze, primarne i sekunardne (tercijarne ako postoje itd), u teoriji.

Na kraju prethodnog dela zaključili smo da u totalnom Hamiltonijanu figurišu proizvoljne funkcije vremena $v^a(t)$. Prisustvo ovih proizvoljnih funkcija u Hamiltonijanu, pa time i u jednačinama kretanja i njihovim rešenjima, znači da varijable $(q(t), p(t))$ ne mogu biti jednoznačno određene pri poznatim početnim uslovima $(q(t=0), p(t=0))$. Ako varijable $(q(t), p(t))$ ne mogu biti jednoznačno određene u svakom trenutku, znači da nemaju direktnu fizičku interpretaciju, tj. to nisu fizičke varijable u teoriji. Neka je f neka varijabla u teoriji, njena dinamička evolucija je:

$$\begin{aligned} f(\delta t) &= f(t=0) + \delta t\, \dot{f} \\ &= f(t=0) + \delta t\, \{f, H_T\} \\ &= f(t=0) + \delta t\, \{f, H_c\} + \delta t \sum_m U^m \{f, \phi_m\} + \delta t \sum_m \sum_a v^a(t)\, V_a^m \{f, \phi_m\} \\ &= f(t=0) + \delta t\, \{f, H'\} + \delta t \sum_a v^a(t)\, \{f, \phi_a\}\,, \end{aligned} \tag{3.36}$$

gde su $H' = H_c + \sum_m U^m \phi_m$ i $\phi_a = \sum_m V_a^m \phi_m$. Na osnovu činjenice da je totalni Hamiltonijan veličina prve klase i na osnovu relacije (3.33) zaključujemo da su i veličine H' i ϕ_a prve klase, tj. da njihove Poasonove zagrade sa svim vezama u teoriji slabo nestaju. U poslednjem članu u izrazu (3.36) figurišu proizvoljne funkcije vremena, pa stoga funkcija

$$f_1(\delta t) = f(t=0) + \delta t\, \{f, H'\}\,,$$

i funkcija

$$f_2(\delta t) = f(t=0) + \delta t\, \{f, H'\} + \delta t \sum_a v^a(t)\, \{f, \phi_a\}\,,$$

odgovaraju istom fizičkom stanju. Transformacija

$$\delta f(\delta t) = \delta t \sum_a v^a(t)\, \{f, \phi_a\} = \sum_a \epsilon^a(t)\, \{f, \phi_a\}\,, \tag{3.37}$$

je nefizička. Dakle, zaljučujemo da primarne veze prve klase ϕ_a generišu nefizičke transformacije dinamičkih varijabli koje ne menjaju fizičko stanje sistema – *gejdž transformacije*.

Kako je uzastopna primena, kao i razlika, dve nefizičke transformacije takođe nefizička transformacija,

$$\delta_1(\delta_2 f) - \delta_2(\delta_1 f) = \sum_{a,b} \epsilon_1^b(t)\epsilon_2^a(t) \left(\{\{f,\phi_a\},\phi_b\} - \{\{f,\phi_b\},\phi_a\}\right) = \sum_{a,b} \epsilon_1^b(t)\epsilon_2^a(t) \{f,\{\phi_a,\phi_b\}\},$$

zaključujemo da je Poasonova zagrada dve primarne veze prve klase $\{\phi_a, \phi_b\}$ takođe generator nefizičkih transformacija. Ova Poasonova zagrada svakako mora biti jednaka nekoj linearnoj kombinaciji veza, pa možemo očekivati da u njoj figurišu i sekunarne veze. Ove sekundarne veze su prve klase, jer Poasonova zagrada veza prve klase daje vezu prve klase. To znači da očekujemo da su neke sekundarne veze prve klase generatori nefizičkih transformacija u teoriji, a u narednom delu ćemo demonstrirati da ovo važi za sve veze prve klase.

Fizičke informacije o sistemu u nekom proizvoljnom trenutku t mogu se dobiti iz funkcija koje su definisane na potprostoru Γ' ukupnog faznog prostora Γ, definisanim vezama u teoriji - tj. iz *fizičkih observabli* u teoriji.

Veze druge klase i fizičke observable u teoriji

Neka su Φ_f veze prve klase u teoriji, $f = 1, \ldots, F$, i χ_s veze druge klase, gde je $s = 1, \ldots, S$. Kako su veze χ_s druge klase, matrica $\Delta_{rs} = \{\chi_r, \chi_s\}$ je nesingularna i dakle invertibilna. Kako svaka antisimetrična matrica od neparane dimenzije ima determinantu jednaku nuli, a matrica Δ_{rs} je antisimetrična i invertibilna, zaključujemo da broj vaza druge klase mora biti paran.

Možemo definisati novu Poasonovu zagradu – tzv. *Dirakovu zagradu*

$$\{f,g\}_D = \{f,g\} - \{f,\chi_r\}\,\Delta^{-1}_{rs}\{\chi_s,g\}\,, \tag{3.38}$$

koja je, kao i Poasonova zagrada, antisimetrično bilinearno preslikavanje. Na osnovu definicije vidimo da je Dirakova zagrada bilo koje veze druge klase sa proizvoljnom promenljivom jednaka nuli:

$$\{\chi_p,g\}_D = \{\chi_p,g\} - \{\chi_p,\chi_r\}\,\Delta^{-1}_{rs}\{\chi_s,g\} = \{\chi_p,g\} - \Delta_{pr}\,\Delta^{-1}_{rs}\{\chi_s,g\} = \{\chi_p,g\} - \delta_{ps}\{\chi_s,g\} = 0\,. \tag{3.39}$$

Dakle, nakon konstrukcije Dirakovih zagrada veze druge klase postaju jako jednake nuli, a jednačina kretanja za proizvoljnu varijablu g u teoriji je:

$$\dot{g} \approx \{g, H_T\}_D\,. \tag{3.40}$$

Razlika između veza prve klase i veza druge klase definisana je uz pomoć Poasonove zagrade, jednačinama (3.34) i (3.35). Jednom kada uvedemo Dirakovu zagradu, prestaje potreba za korišćenjem Poasonovih zagrada – veze druge klase postaju jake jednakosti i celokupna teorija se može formulisati u terminima Dirakovih zagrada. Kao što smo videli, prisustvo veza druge klase znači da postoje dinamički stepeni slobode u teoriji koji nisu od značaja. Eliminisanje ovih varijabli se postiže definisanjem Dirakove zagrade koja se odnosi samo na fizički relevantne stepene slobode. Ipak, u praksi u teorijama koje poseduju veliki broj veza druge klase to može biti izuzetno teško.

Broj stepeni slobode

U opštem slučaju, za N inicijalnih polja u teoriji, F nezavisnih veza prve klase i S nezavisnih veza druge klase, *broj lokanih propagirajućih stepeni slobode* dat je relacijom:

$$n = N - F - \frac{S}{2}\,. \tag{3.41}$$

Jednačina (3.41) je posledica činjenice da je postojanje veza druge klase S ekvivalentno iščezavanju $S/2$ kanonskih koordinata i $S/2$ njihovih impulsa. Postojanje F veza prve klase u teoriji je ekvivalentno nestajanju F kanonskih koordinata, a kako veze prve klase generišu gejdž simetrije, možemo nametnuti F uslova za fiksiranje gejdža za odgovarajućih F kanonskih momenata. Prema tome, postoji $2N - 2F - S$ nezavisnih kanonskih koordinata i momenata i stoga je $2n = 2N - 2F - S$, što dovodi do jednačine (3.41).

3.2.2 Generator gejdž transformacija

U ovom delu prikazaćemo algoritam za konstruisanje generatora gejdž simetrija u teoriji – tzv. *Kastelanijevu proceduru.*

Razmotrimo sistem koji je određen ukupnim Hamiltonijanom H_T i kompletnim skupom veza u teoriji ϕ_m, gde je $m = 1, \ldots, M, M+1, \ldots, M+K$. Neka je $(q(t), p(t))$ neka trajektorija u faznom prostoru određena parametrom t, pri čemu početna tačka $(q(t=0), p(t=0))$ leži na hiperpovrši određenoj vezama. Jednačine kretanja dobijene totalnim Hamiltonijanom (3.22) nemaju isti oblik kao jednačine kretanja dobijene primenom (3.10), ali je njihova razlika nefizička. Primenom jednačine $H_T = H' + \sum_a v^a(t)\phi_a$ dobija se

$$\dot{p}_i = -\frac{\partial H'}{\partial q_i} - v^a(t)\frac{\partial \phi_a}{\partial q_i}\,, \qquad \dot{q}_i = \frac{\partial H'}{\partial p_i} + v^a(t)\frac{\partial \phi_a}{\partial p_i}\,, \qquad i = 1, \ldots, N\,, \tag{3.42}$$

dok su jednačine kretanja za $v^a(t)$:

$$\phi_a(q,p) = 0\,. \tag{3.43}$$

Posmatrajmo sada novu trajektoriju koja počinje u istoj tački $(q(t=0), p(t=0))$ na hiperpovrši određenoj vezama, pri čemu $(q(t)+\delta_0 q(t), p(t)+\delta_0 p(t))$ takođe zadovoljavaju jednačine kretanja za funkciju $v^a(t) + \delta_0 v^a(t)$:

$$\begin{aligned}
\delta_0\dot{p}_i &= -\sum_{j=1}^{N}\left(\delta_0 q_j\frac{\partial}{\partial q_j} + \delta_0 p_j\frac{\partial}{\partial p_j}\right)\frac{\partial H_T}{\partial q_i} - \delta_0 v^a(t)\,\frac{\partial \phi_a}{\partial q_i}\,,\\
\delta_0\dot{q}_i &= \sum_{j=1}^{N}\left(\delta_0 q_j\frac{\partial}{\partial q_j} + \delta_0 p_j\frac{\partial}{\partial p_j}\right)\frac{\partial H_T}{\partial p_i} + \delta_0 v^a(t)\,\frac{\partial \phi_a}{\partial p_i}\,,\\
&\sum_{j=1}^{N}\left(\delta_0 q_j\frac{\partial}{\partial q_j} + \delta_0 p_j\frac{\partial}{\partial p_j}\right)\phi_a(q,p) = 0\,.
\end{aligned} \tag{3.44}$$

Razlika ove dve trajektorije u nekom fiksnom trenutku t predstavlja nefizičku transformaciju.

Sa druge strane, varijacije formi pri gejdž transformacijama određene su generatorom:

$$\begin{aligned}
\delta_0 q_i &= \epsilon(t)\{\, q_i\,,\, G\,\} = \epsilon(t)\frac{\partial G}{\partial p_i}\,,\\
\delta_0 p_i &= \epsilon(t)\{\, p_i\,,\, G\,\} = -\epsilon(t)\frac{\partial G}{\partial q_i}\,,
\end{aligned} \tag{3.45}$$

gde smo primenili definiciju Poasonove zagrade. Dalje, dobijamo:

$$\begin{aligned}
\delta_0\dot{q}_i &= \dot{\epsilon}(t)\frac{\partial G}{\partial p_i} + \epsilon(t)\{\,\frac{\partial G}{\partial p_i}\,,\, H_T\,\}\,,\\
\delta_0\dot{p}_i &= -\,\dot{\epsilon}(t)\frac{\partial G}{\partial q_i} - \epsilon(t)\{\,\frac{\partial G}{\partial q_i}\,,\, H_T\,\}\,.
\end{aligned} \tag{3.46}$$

Upoređivanjem jednačina (3.44) i (3.46) dobijamo:

$$\begin{aligned}\sum_{j=1}^{N}\left(\delta_0 q_j\frac{\partial}{\partial q_j}+\delta_0 p_j\frac{\partial}{\partial p_j}\right)\frac{\partial H_T}{\partial p_i}+\delta_0 v^a(t)\,\frac{\partial \phi_a}{\partial p_i} &=\dot\epsilon(t)\frac{\partial G}{\partial p_i}+\epsilon(t)\{\,\frac{\partial G}{\partial p_i}\,,\,H_T\,\}\,,\\ \sum_{j=1}^{N}\left(\delta_0 q_j\frac{\partial}{\partial q_j}+\delta_0 p_j\frac{\partial}{\partial p_j}\right)\frac{\partial H_T}{\partial q_i}+\delta_0 v^a(t)\,\frac{\partial \phi_a}{\partial q_i} &=\dot\epsilon(t)\frac{\partial G}{\partial q_i}+\epsilon(t)\{\,\frac{\partial G}{\partial q_i}\,,\,H_T\,\}\,,\end{aligned}\tag{3.47}$$

tj.

$$\begin{aligned}\epsilon(t)\{\,\tfrac{\partial H_T}{\partial p_i}\,,\,G\,\}+\delta_0 v^a(t)\,\tfrac{\partial \phi_a}{\partial p_i} &= \dot\epsilon(t)\frac{\partial G}{\partial p_i}+\epsilon(t)\{\,\frac{\partial G}{\partial p_i}\,,\,H_T\,\}\,,\\ \epsilon(t)\{\,\tfrac{\partial H_T}{\partial q_i}\,,\,G\,\}+\delta_0 v^a(t)\,\tfrac{\partial \phi_a}{\partial q_i} &= \dot\epsilon(t)\frac{\partial G}{\partial q_i}+\epsilon(t)\{\,\frac{\partial G}{\partial q_i}\,,\,H_T\,\}\,,\\ \epsilon(t)\{\,\phi_a(q,p)\,,\,G\,\} &= 0\,.\end{aligned}\tag{3.48}$$

Ove jednačine dalje možemo prepisati na sledeći način,

$$\begin{aligned}\frac{\partial}{\partial p_i}\Big(\dot\epsilon(t)G+\epsilon(t)\{\,H_T\,,\,G\,\}-v^a(t)\,\phi_a\Big) &= 0\,,\\ \frac{\partial}{\partial q_i}\Big(\dot\epsilon(t)G+\epsilon(t)\{\,H_T\,,\,G\,\}-v^a(t)\,\phi_a\Big) &= 0\,,\\ \epsilon(t)\{\,\phi_a(q,p)\,,\,G\,\} &= 0\,,\end{aligned}\tag{3.49}$$

pa zaključujemo da za bilo koju dinamičku varijablu $F(q,p)$ važi:

$$\{\,F(q,p)\,,\,\dot\epsilon(t)G+\epsilon(t)\{\,H_T\,,\,G\,\}-v^a(t)\,\phi_a\,\}=0\,.\tag{3.50}$$

Sledi da je izraz $\dot\epsilon(t)G+\epsilon(t)\{\,H_T\,,\,G\,\}-v^a(t)\,\phi_a$ trivijalna funkcija, jednaka nuli ili nekom stepenu veza, pa dobijamo da su zadovoljene relacije:

$$G=\alpha^a\phi_a\,,\qquad \{\,G\,,\,H_T\,\}=0\,.\tag{3.51}$$

U opštem slučaju, generatori gejdž transformacija su oblika

$$G=\sum_{n=0}^{k}\epsilon^{(n)}G_n\,,\tag{3.52}$$

gde $\epsilon^{(n)}$ označava n-ti vremenski izvod parametra ϵ po vremenu $\epsilon^{(n)}=\dfrac{\partial^n\epsilon}{\partial t^n}$. Generatori G_n, gde je $n=1,\ldots,k$, su određeni rekurzivno *Kastelanijevom procedurom* i moraju da zadovoljavaju sledeće relacije:

$$\begin{aligned}G_k &= C_{PFC}\,,\\ &\vdots\\ G_1+\{G_2,H_T\} &= C_{PFC}\,,\\ G_0+\{G_1,H_T\} &= C_{PFC}\,,\\ \{G_0,H_T\} &= C_{PFC}\,,\end{aligned}\tag{3.53}$$

gde C_{PFC} predstavlja neku primarnu vezu prve klase. Ukupan broj generatora k je jednak broju generacija sekundarnih veza – ako u teoriji postoje samo primarne i sekundarne veze sledi da je $k = 1$. Tada, generator ima oblik:

$$G = \epsilon(t)G_0 + \dot{\epsilon}(t)G_1 \,. \tag{3.54}$$

U tom slučaju, rekurzivni algoritam Kastelanijeve procedure opisan relacijama (3.53) dobija jednostavniji oblik:

$$\begin{aligned} G_1 &= C_{PFC} \,, \\ G_0 + \{G_1, H_T\} &= C_{PFC} \,, \\ \{G_0, H_T\} &= C_{PFC} \,, \end{aligned} \tag{3.55}$$

gde C_{PFC} predstavlja neku primarnu vezu prve klase.

Varijacija forme pri transformacijama simetrije proizvoljne varijable A definisane na faznom prostoru se računa primenom formule:

$$\delta_0 A = \{A, G\} \,. \tag{3.56}$$

Glava 4

BF teorija

U ovom poglavlju predstavljena je BF teorija, definisana za neku opštu Lijevu grupu i n-dimenzionalnu mnogostrukost. Kao topološka teorija, BF teorija je teorija bez lokalnih propagirajućih stepeni slobode, pa se opis fizički relevantnih sistema dobija dodavanjem određenih veza na topološko BF dejstvo, tj. definisanjem BF dejstva sa vezama. Najpoznatiji primer ove konstrukcije je Plebanski dejstvo [32]. Slično, simetriju BF dejstva moguće je narušiti dodavanjem člana koji narušava simetriju, kao što je to slučaj u Mekdauel-Mansuri teoriji [33].

Kovarijantna kvantizaciona procedura, sprovedena za različite izbore klasičnog BF dejstva, daje *topološke kvantne teorije polja* u smislu da zadovoljavaju Atijine aksiome. Procedura je rezultovala u mnogobrojnim *modelima spinske pene* kvantne gravitacije, počevši od Ponzano-Redže modela trodimenzionalne gravitacije [5], pa sve do trenutno najsofisticiranijeg $EPRL/FK$ modela definisanog u četiri prostorvremenske dimenzije [9], [10]. Međutim, svim modelima spinske pene može se zameriti da opisuju teorije čiste gravitacije, bez prisustva polja materije, što je posledica činjenice da BF dejstvo ne sadrži tetradna polja u eksplicitnom obliku. Umesto toga, tetrade se pojavljuju kao posledica klasičnih jednačina kretanja, pa stoga predstavljaju "on-shell" varijable. Iz tog razloga u okviru BF teorija nemoguće je dodati materiju na kvantnom nivou. Pogledati [32], [34], [35] za više informacija o BF teoriji.

Poglavlje pred nama je organizovano na sledeći način. U odeljku 4.1 dat je kratak pregled topološke BF teorije. Pododeljak 4.1.1 sadrži Hamiltonovu analizu BF teorije i rezultujuću kanonsku strukturu, uključujući veze prve klase i veze druge klase prisutne u teoriji, kao i algebru veza. U istom pododeljku predstavljen je Bjankijev identitet koji zadovoljavaju veze prve klase, koji smanjuje broj nezavisnih veza prve klase prisutnih u teoriji. Na osnovu ovih rezultata izvršeno je brojanje fizičkih stepeni slobode u BF teoriji. U pododeljku 4.1.1 predstavljen je oblik generatora gejdž transformacija teorije, kao i varijacije svih varijabli i njihovih kanonskih impulsa u teoriji, dok je Kastelanijeva procedura kojom je dobijen ovaj generator data u Dodatku D.1. Ovaj rezultat koristimo u pododeljku 4.1.2 gde su dobijene sve gejdž simetrije BF teorije, grupe G-gejdž i M-gejdž transformacija. Sumiranjem ovih rezultata predstavljena je kompletna struktura grupe simetrije, uključujući njenu Lijevu algebru, kao i konkretan izbor parametara kojim se dobijaju difeomorfizam transformacije u BF teoriji. Konačno, ova glava se završava odeljkom 4.2, gde diskutujemo Jang-Milsovu teoriju u ravnom prostorvremenu predstavljenu kao topološku BF teoriju sa vezama i odeljkom 4.3, gde je predstavljeno poznato Plebanski dejstvo za Opštu relativnost.

4.1 Topološka BF teorija

Za Lijevu grupu G, njenu odgovarajuću Lijevu algebru $\mathfrak{g}$ i neku 4-dimenzionalnu prostorvremensku mnogostrukost $\mathcal{M}_4$, možemo definisati topološko BF dejstvo na sledeći način:

$$S_{BF} = \int_{\mathcal{M}_4} \langle B \wedge F \rangle_{\mathfrak{g}} \,. \tag{4.1}$$

Ovde je 2-forma F krivina za 1-formu koneksije $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$ elementa algebre $\mathfrak{g}$ kao što je definisana jednačinom (2.1), $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ je Lagranžev množitelj 2-forma, dok oznaka $\langle \_\,, \_ \rangle_{\mathfrak{g}}$ označava G-invarijantnu simetričnu nedegenerisanu bilinearnu formu. Dejstvo (4.1) napisano je u manifestno difeomorfizam invarijantnom obliku i invarijantno je na gejdž transformacije generisane grupom G. Variranjem dejstva (4.1) po varijablama B^β i α^β dobijaju se jednačine kretanja:

$$F^\beta = 0\,, \qquad \nabla B^\beta \equiv \mathrm{d}B^\beta + f_{\gamma\delta}{}^\beta \alpha^\gamma \wedge B^\delta = 0\,. \tag{4.2}$$

Ovde je indeks β grupni indeks G grupe koji prebrojava generatore $\mathfrak{g}$, a $f_{\gamma\delta}{}^\beta$ označava strukturne konstante Lijeve grupe G.

Na osnovu prve jednačine kretanja u (4.2) zaključujemo da je koneksija α ravna, odnosno $\alpha = 0$ (do na gejdž transformacije). Koristeći ovaj rezultat u drugoj jednačini kretanja dobijamo da je Lagranžev množitelj B konstantan. Stoga, dejstvo (4.1) opisuje teoriju bez lokalnih propagirajućih stepeni slobode, odnosno *topološku teoriju*[1].

Formalno se broj stepeni slobode dobija Hamiltonovom analizom dejstva, što je urađeno u odeljku 4.1.1.

4.1.1 Hamiltonova analiza topološke BF teorije

Topološka BF teorija zadata je dejstvom (4.1), odakle raspisivanjem po prostornovremenskim indeksima diferencijalnih formi $\mu, \nu...$ i grupnim indeksima $\alpha, \beta...$ grupe G dobijamo:

$$S_{BF} = \int_{\mathcal{M}_4} \mathrm{d}^4x \, \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B^\alpha{}_{\mu\nu} F^\beta{}_{\rho\sigma} g_{\alpha\beta}\,. \tag{4.7}$$

[1] *Topološka kvantna teorija polja* je kvantna teorija polja koja se bavi izračunavanjem topoloških invarijanti. U topološkoj teoriji konfiguracioni integral je

$$Z^I = \int \mathcal{D}\phi \, \exp\big[iS_{TOP}[\phi]\big], \tag{4.3}$$

gde je dejstvo jednako nekoj konstanti $\chi(\mathcal{M}_D)$ koja zavisi samo od topologije:

$$S_{TOP}[\phi] = \int_{\mathcal{M}_D} \mathrm{d}^D x \, \mathcal{L}^{TOP}(\phi, \partial\phi) = \chi(\mathcal{M}_D). \tag{4.4}$$

Dakle, u konfiguracionom integralu konstanta $\exp[i\chi]$ tada izlazi ispred integrala i dobijamo:

$$Z^I = \exp\big[i\chi(\mathcal{M}_D)\big] \int \mathcal{D}\phi = \mathrm{const} \cdot \exp\big[i\chi(\mathcal{M}_D)\big]. \tag{4.5}$$

Ovaj konfiguracioni integral nazivamo prvom kvantizacijom. Drugu kvantizaciju, tj. konfiguracioni integral Z^{II} diskretizovane teorije dobijamo sabiranjem po svim triangulacijama mnogostrukosti. Diskretizovana teorija je *topološka* ako partitivna funkcija Z^I ne zavisi od triangulacije mnogostrukosti. U tom slučaju su I i II kvantizacija identične

$$Z^{II} = \sum_{T(\mathcal{M}_D)} Z^I = Z^I \sum_{T(\mathcal{M}_D)} 1 = \mathrm{const} \cdot Z^I, \tag{4.6}$$

gde Z^I izlazi ispred sume jer ne zavisi od triangulacije. Kako nam triangulacija definiše broj stepeni slobode u teoriji, a fizičke opservable ne zavise od triangulacije, zaključujemo da fizičkih stepeni slobode nema.

Pretpostavljajući da je prostorvreme globalno hiperbolično možemo da izvršimo folijaciju prostorvremena na prostorne hiperpovrši Σ_3 i napišemo Lagranžijan za BF teoriju:

$$L_{3BF} = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} B^{\alpha}{}_{\mu\nu} F^{\beta}{}_{\rho\sigma} g_{\alpha\beta}\,. \tag{4.8}$$

Kanonski impulsi, definisani jednačinom (3.16), koji odgovaraju varijablama $B^{\alpha}{}_{\mu\nu}$ i α^{α}, dobijeni variranjem Lagranžijana po vremenskim izvodima varijabli su:

$$\begin{aligned} \pi(B)_{\alpha}{}^{\mu\nu} &= \frac{\delta L}{\delta \partial_0 B^{\alpha}{}_{\mu\nu}} = 0\,, \\ \pi(\alpha)_{\alpha}{}^{\mu} &= \frac{\delta L}{\delta \partial_0 \alpha^{\alpha}{}_{\mu}} = \frac{1}{2}\epsilon^{0\mu\nu\rho} B_{\alpha\nu\rho}\,. \end{aligned} \tag{4.9}$$

Kako ove jednačine ne možemo rešiti po vremenskim izvodima varijabli one daju *primarne veze* (3.19):

$$\begin{aligned} P(B)_{\alpha}{}^{\mu\nu} &\equiv \pi(B)_{\alpha}{}^{\mu\nu} \approx 0\,, \\ P(\alpha)_{\alpha}{}^{\mu} &\equiv \pi(\alpha)_{\alpha}{}^{\mu} - \frac{1}{2}\epsilon^{0\mu\nu\rho} B_{\alpha\nu\rho} \approx 0\,. \end{aligned} \tag{4.10}$$

Koristeći fundamentalnu Poasonovu zagradu definisanu na sledeći način:

$$\begin{aligned} \{\, B^{\alpha}{}_{\mu\nu}(\vec{x})\,,\, \pi(B)_{\beta}{}^{\rho\sigma}(\vec{y}) \,\} &= 2\delta^{\alpha}_{\beta}\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\, \alpha^{\alpha}{}_{\mu}(\vec{x})\,,\, \pi(\alpha)_{\beta}{}^{\nu}(\vec{y}) \,\} &= \delta^{\alpha}_{\beta}\delta^{\nu}_{\mu}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \end{aligned} \tag{4.11}$$

izračunavamo Poasonovu zagradu između primarnih veza:

$$\{\, P(B)_{\alpha}{}^{jk}(\vec{x})\,,\, P(\alpha)_{\beta}{}^{i}(\vec{y}) \,\} = \epsilon^{0ijk}\, g_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{y})\,. \tag{4.12}$$

Kanonski Hamiltonijan, definisan jednačinom (3.20), glasi:

$$H_c = \int_{\Sigma_3} \mathrm{d}^3\vec{x} \left[\frac{1}{2}\pi(B)_{\alpha}{}^{\mu\nu}\, \partial_0 B^{\alpha}{}_{\mu\nu} + \pi(\alpha)_{\alpha}{}^{\mu}\, \partial_0 \alpha^{\alpha}{}_{\mu} \right] - L\,. \tag{4.13}$$

Koristeći definiciju krivine $F^{\alpha}{}_{\mu\nu}$, možemo da prepišemo Hamiltonijan (4.13) tako da vremenski izvodi varijabli množe primarne veze, pa su u *"on-shell"* objektu ti članovi identički jednaki nuli:

$$H_c = -\int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \epsilon^{0ijk} \left[\frac{1}{2} B_{\alpha 0i}\, F^{\alpha}{}_{jk} + \frac{1}{2}\alpha^{\alpha}{}_{0} \nabla_i B_{\alpha\, jk} \right]. \tag{4.14}$$

Ovako napisan kanonski Hamiltonijan ne zavisi od kanonskih impulsa i sadrži samo polja i njihove vremenske izvode. Uvodeći Lagranževe množitelje λ koji odgovaraju dvema primarnim vezama možemo da definišemo *"off-shell" totalni Hamiltonijan*, definisan jednačinom (3.21):

$$H_T = H_c + \int_{\Sigma_3} \mathrm{d}^3\vec{x} \left[\frac{1}{2}\lambda(B)^{\alpha}{}_{\mu\nu} P(B)_{\alpha}{}^{\mu\nu} + \lambda(\alpha)^{\alpha}{}_{\mu} P(\alpha)_{\alpha}{}^{\mu} \right]. \tag{4.15}$$

Kako primarne veze moraju biti konstantne, *uslovi konzistentnosti* (3.26) za sve primarne veze moraju biti zadovoljeni. Za primarne veze $P(B)_{\alpha}{}^{0i}$ i $P(\alpha)_{\alpha}{}^{0}$ ovi uslovi dovode do sekundarnih veza (3.27) u teoriji $\mathcal{S}$:

$$\begin{aligned} \mathcal{S}(F)_{\alpha}{}^{i} &\equiv \frac{1}{2}\epsilon^{0ijk} F_{\alpha jk} \approx 0\,, \\ \mathcal{S}(\nabla B)_{\alpha} &\equiv \frac{1}{2}\epsilon^{0ijk} \nabla_{[i} B_{\alpha\, j]k} \approx 0\,, \end{aligned} \tag{4.16}$$

dok u slučaju primarnih veza $P(\alpha)_\alpha{}^k$ i $P(B)_\alpha{}^{jk}$ uslov konzistentnosti određuje Lagranževe množitelje:

$$\begin{aligned} \lambda(B)_{\alpha ij} &\approx \nabla_i B_{\alpha 0j} - \nabla_j B_{\alpha 0i} + g_{\alpha\gamma}{}^{\beta} \alpha_{\beta 0} B^{\gamma}{}_{ij}\,, \\ \lambda(\alpha)^{\alpha}{}_i &\approx \nabla_i \alpha^{\alpha}{}_0\,. \end{aligned} \tag{4.17}$$

Primetimo da Lagranževi množitelji

$$\lambda(B)^{\alpha}{}_{0i}\,, \qquad \lambda(\alpha)^{\alpha}{}_0\,, \tag{4.18}$$

ostaju neodređeni. *Uslovi konzistentnosti sekundarnih veza* (3.28) ne dovode do pojave novih veza,

$$\begin{aligned} \{\mathcal{S}(F)^{\alpha i}\,, H_T\} &= f_{\beta\gamma}{}^{\alpha} \mathcal{S}(F)^{\beta i} \alpha^{\gamma}{}_0 && \approx 0\,, \\ \{\mathcal{S}(\nabla B)_{\alpha}\,, H_T\} &= -f_{\beta\gamma\alpha} B^{\gamma}{}_{0k} \mathcal{S}(F)^{\beta k} + f_{\beta\alpha}{}^{\gamma} \alpha^{\beta}{}_0 \mathcal{S}(\nabla B)_{\gamma} && \approx 0\,. \end{aligned} \tag{4.19}$$

Zamenjujući izraze za Lagranževe množitelje, totalni Hamiltonijan se svodi na

$$H_T = \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big[\lambda(B)_{\alpha 0i}\, \Phi(B)^{\alpha i} + \lambda(\alpha)_{\alpha}\, \Phi(\alpha)^{\alpha} - B_{\alpha 0i}\, \Phi(F)^{\alpha i} - \alpha_{\alpha 0}\, \Phi(\nabla B)^{\alpha} \Big]\,, \tag{4.20}$$

gde su *veze prve klase* (3.34)

$$\begin{aligned} \Phi(B)^{\alpha i} &= P(B)^{\alpha 0i}\,, \\ \Phi(\alpha)^{\alpha} &= P(\alpha)^{\alpha 0}\,, \\ \Phi(F)^{\alpha i} &= \mathcal{S}(F)^{\alpha i} - \nabla_j P(B)^{\alpha ij}\,, \\ \Phi(\nabla B)^{\alpha} &= \mathcal{S}(\nabla B)^{\alpha} + \nabla_i P(\alpha)^{\alpha i} - \frac{1}{2} f_{\gamma\beta}{}^{\alpha} B^{\beta}{}_{ij} P(B)^{\gamma ij}\,, \end{aligned} \tag{4.21}$$

dok su *veze druge klase* (3.35):

$$\chi(B)_{\alpha}{}^{jk} = P(B)_{\alpha}{}^{jk} \qquad \chi(\alpha)_{\alpha}{}^{i} = P(\alpha)_{\alpha}{}^{i}\,. \tag{4.22}$$

Poasonova zagrada veza prve klase glasi:

$$\begin{aligned} \{\, \Phi(F)^{\alpha}{}_i(\vec{x})\,, \Phi(\nabla B)_{\beta}(\vec{y}) \,\} &= f_{\beta\gamma}{}^{\alpha}\, \Phi(F)^{\gamma}{}_i(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\, \Phi(\nabla B)_{\alpha}(\vec{x})\,, \Phi(\nabla B)_{\beta}(\vec{y}) \,\} &= f_{\alpha\beta}{}^{\gamma}\, \Phi(\nabla B)_{\gamma}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y})\,, \end{aligned} \tag{4.23}$$

dok je Poasonova zagrada veza prve sa vezama druge klase daje relacije:

$$\begin{aligned} \{\, \Phi(F)^{\alpha i}(\vec{x})\,, \chi(\alpha)_{\beta}{}^{j}(\vec{y}) \,\} &= -f_{\beta\gamma}{}^{\alpha}\, \chi(B)^{\gamma ij}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\, \Phi(\nabla B)^{\alpha}(\vec{x})\,, \chi(\alpha)_{\beta}{}^{i}(\vec{y}) \,\} &= f_{\beta\gamma}{}^{\alpha}\, \chi(\alpha)^{\gamma i}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\, \Phi(\nabla B)^{\alpha}(\vec{x})\,, \chi(B)_{\beta}{}^{ij}(\vec{y}) \,\} &= -f_{\beta\gamma}{}^{\alpha}\, \chi(B)^{\gamma ij}(\vec{x})\, \delta^{(3)}(\vec{x}-\vec{y})\,. \end{aligned} \tag{4.24}$$

Komutatori veza prve klase sa Hamiltonijanom su,

$$\begin{aligned}
\{\Phi(B)_\alpha{}^i\,, H_T\} &= \Phi(F)_\alpha{}^i\,,\\
\{\Phi(\alpha)_\alpha\,, H_T\} &= \Phi(\nabla B)_\alpha\,,\\
\{\Phi(F)^\alpha{}_i\,, H_T\} &= -\alpha^\beta{}_0\, f_{\beta\gamma}{}^\alpha\, \Phi(F)^\gamma{}_i\,,\\
\{\Phi(\nabla B)_\alpha\,, H_T\} &= B_{\beta 0i}\, f_{\alpha\gamma}{}^\beta\, \Phi(F)^{\gamma i} - \alpha^\beta{}_0\, f_{\alpha\beta}{}^\gamma\, \Phi(\nabla B)_\gamma\,,
\end{aligned} \tag{4.25}$$

rezultat koji ćemo kasnije iskoristiti da konstruišemo generator gejdž simetrija BF teorije i odredimo ukupnu grupu simetrija.

Broj stepeni slobode topološke BF teorije

Da bi smo izračunali broj nezavisnih komponenti primarnih veza neophodna je primena Bjankijevih identiteta.

Teorema 7 (BI za BF teoriju) *Bjankijev identitet (BI) za 1-formu koneksije α, odnosno odgovarajuću 2-formu krivine*

$$F^\alpha = \mathrm{d}\alpha^\alpha + f_{\beta\gamma}{}^\alpha\, \alpha^\beta \wedge \alpha^\gamma\,, \tag{4.26}$$

glasi

$$\epsilon^{\lambda\mu\nu\rho}\, \nabla_\mu F^\alpha{}_{\nu\rho} = 0\,. \tag{4.27}$$

$\alpha^\alpha{}_\mu$	$B^\alpha{}_{\mu\nu}$
$4p$	$6p$

Tabela 4.1: Ukupan broj inicijalnih polja u BF teoriji.

Prebrojavanjem inicijalnih polja u teoriji prikazanih u tabeli 4.1 zaključujemo da je $N = 10p$, gde je p dimenzionalnost grupe G. Slično, broj veza druge klase možemo da pročitamo iz tabele (4.2) $S = 6p$. Posmatrajući veze prve klase zaključujemo da one nisu sve međusobno nezavisne

$\chi(B)_\alpha{}^{jk}$	$\chi(\alpha)_\alpha{}^i$
$3p$	$3p$

Tabela 4.2: Ukupan broj veza druge klase u BF teoriji.

i zadovoljavaju sledeći identitet,

$$\nabla_i \Phi(F)_\alpha{}^i = \epsilon^{ijk} \nabla_i F_{\alpha jk}\,, \tag{4.28}$$

što postaje

$$\nabla_i \Phi(F)_\alpha{}^i = 0\,, \tag{4.29}$$

kada iskoristimo da je desna strana jednačine (4.28) $\epsilon^{ijk}\, \nabla_i F^a_{jk} = 0$ komponenta $\lambda = 0$ BI (4.27).

Imajući ovu vezu između veza prve klase u vidu, broj nezavisnih komponenti veza prve klase možemo da pročitamo iz tabele 4.3. Vidimo da je broj nezavisnih veza prve klase:

$$F = 8p - p = 7p\,.$$

U prethodnom izrazu smo od ukupnog broja veza prve klase navedenih u tabeli (4.3) oduzeli p relacija (4.29). Dakle, broj stepeni slobode u BF teoriji, definisan jednačinom (3.41) je

$$n = 10p - 7p - \frac{6p}{2} = 0\,, \tag{4.30}$$

odnosno *BF teorija nema lokalne propagirajuće stepene slobode.*

$\Phi(B)_\alpha{}^i$	$\Phi(\alpha)_\alpha$	$\Phi(F)^{\alpha i}$	$\Phi(\nabla B)^\alpha$
$3p$	p	$3p-p$	p

Tabela 4.3: Ukupan broj veza prve klase u BF teoriji.

Generator gejdž transformacija za BF teoriju

Generator gejdž transformacija za BF teoriju glasi:

$$G = \int \nabla_0\epsilon^\alpha{}_i\Phi(B)_\alpha{}^i - \epsilon^\alpha{}_i\Phi(F)_\alpha{}^i + \nabla_0\epsilon^\alpha\Phi(\alpha)_\alpha + \epsilon^\alpha\left(f_{\alpha\gamma}{}^\beta B_{\beta 0i}\Phi(B)^{\gamma i} - \Phi(\nabla B)_\alpha\right) . \tag{4.31}$$

Postupak izvođenja generatora Kastelanijevom procedurom prikazan je u dodatku D.1. Varijacije formi varijabli i njihovih konjugovanih impulsa računamo primenom jednačine (3.56), što daje:

$$\begin{aligned}
\delta_0 B^\alpha{}_{0i} &= \nabla_0\epsilon^\alpha{}_i - f_{\beta\gamma}{}^\alpha\epsilon^\beta B^\gamma{}_{0i}\,, & \delta_0\pi(B)_\alpha{}^{0i} &= -f_{\alpha\beta}{}^\gamma\epsilon^\beta\pi(B)_\gamma{}^{0i}\,,\\
\delta_0 B^\alpha{}_{ij} &= 2\nabla_{[i}\epsilon^\alpha{}_{j]} - f_{\beta\gamma}{}^\alpha\epsilon^\beta B^\gamma{}_{ij}\,, & \delta_0\pi(B)_\alpha{}^{ij} &= -f_{\alpha\beta}{}^\gamma\epsilon^\beta\pi(B)_\gamma{}^{ij}\,,\\
\delta_0\alpha^\alpha{}_0 &= \nabla_0\epsilon^\alpha\,, & \delta_0\pi(\alpha)_\alpha{}^0 &= -f_{\alpha\beta}{}^\gamma\epsilon^\beta{}_i\pi(B)_\gamma{}^{0i} - f_{\alpha\beta}{}^\gamma\epsilon^\gamma\pi(\alpha)_\gamma\,,\\
\delta_0\alpha^\alpha{}_i &= \nabla_i\epsilon^\alpha\,, & \delta_0\pi(\alpha)_\alpha{}^i &= -f_{\alpha\beta}{}^\gamma\epsilon^\beta{}_j\pi(B)_\gamma{}^{ij} - f_{\alpha\beta}{}^\gamma\epsilon^\beta\pi(\alpha)_\gamma{}^i + \epsilon^{0ijk}\nabla_j\epsilon_{\alpha k}\,.
\end{aligned} \tag{4.32}$$

4.1.2 Simetrije BF dejstva

Grupa simetrije G

Posmatrajući varijacije formi varijabli (4.32) primetimo da članovi odgovaraju transformaciji definisanoj u Teoremi 8,

$$\delta_0\alpha^\alpha{}_\mu = \partial_\mu\epsilon_{\mathfrak{g}}{}^\alpha + f_{\beta\gamma}{}^\alpha\alpha^\beta{}_\mu\epsilon_{\mathfrak{g}}{}^\gamma\,, \qquad \delta_0 B^\alpha{}_{\mu\nu} = -f_{\beta\gamma}{}^\alpha\epsilon_{\mathfrak{g}}{}^\beta B^\gamma{}_{\mu\nu}\,, \tag{4.33}$$

gde je slobodan parametar $\epsilon_{\mathfrak{g}}{}^\alpha = \epsilon^\alpha$.

Teorema 8 (G-gejdž transformacije) *U BF teoriji nad proizvoljnom Lijevom grupom G, sledeća transformacija je simetrija:*

$$\begin{aligned}
\alpha &\to \alpha' = \mathrm{Ad}_g\alpha + g\mathrm{d}g^{-1}\,, & B &\to B' = gBg^{-1}\,,\\
\beta &\to \beta' = g\rhd\beta\,, & C &\to C\,,
\end{aligned} \tag{4.34}$$

gde je $g = \exp(\epsilon_{\mathfrak{g}}\cdot\hat{G}) = \exp(\epsilon_{\mathfrak{g}\alpha}\hat{G}^\alpha) \in G$, a $\epsilon_{\mathfrak{g}} : \mathcal{M}_4 \to \mathfrak{g}$ je parametar transformacija.

Dokaz. Pri G-gejdž transformacijama definisanim u Teoremi 8 za parametar $g \in G$, krivina F se transformiše na sledeći način:

$$\mathcal{F} \to \mathcal{F}' = g\mathcal{F}g^{-1}\,. \tag{4.35}$$

Teoremu dokazujemo direktnom proverom:

$$\begin{aligned}
S_{BF} \to S'_{BF} &= \frac{1}{4}\int_{\mathcal{M}_4} \mathrm{d}^4x\,\epsilon^{\mu\nu\rho\sigma}\left(B^\alpha{}_{\mu\nu} - f_{\gamma\delta}{}^\alpha\epsilon_{\mathfrak{g}}^\gamma B^\delta{}_{\mu\nu}\right)\left(F^\beta{}_{\rho\sigma} - f_{\epsilon\tau}{}^\beta\epsilon_{\mathfrak{g}}^\epsilon F^\tau{}_{\rho\sigma}\right) g_{\alpha\beta}\\
&= S_{BF} - \frac{1}{4}\int_{\mathcal{M}_4} \mathrm{d}^4x\,\epsilon^{\mu\nu\rho\sigma}\left(f_{\gamma\delta}{}^\alpha\epsilon_{\mathfrak{g}}^\gamma B^\delta{}_{\mu\nu}F^\beta{}_{\rho\sigma} + f_{\epsilon\tau}{}^\beta\epsilon_{\mathfrak{g}}^\epsilon B^\alpha{}_{\mu\nu}F^\tau{}_{\rho\sigma}\right) g_{\alpha\beta}\,,
\end{aligned} \tag{4.36}$$

gde je drugi član jednak nuli. Takođe, posmatrajući konačnu transformaciju, možemo da pišemo

$$\langle B, F\rangle_{\mathfrak{g}} \to \langle g^{-1}Bg, g^{-1}Fg\rangle_{\mathfrak{g}} = \langle B, F\rangle_{\mathfrak{g}}\,, \tag{4.37}$$

budući da je Kilingova forma $\langle \_, \_\rangle_{\mathfrak{g}}$ G-invarijantna. ■

Posmatrajmo dve infinitezimalne G-gejdž transformacije, određene infinitezimalnim parametrima ${\epsilon_{\mathfrak{g}}}^{\alpha}{}_1$ i ${\epsilon_{\mathfrak{g}}}^{\beta}{}_2$. Za izračunavanje komutatora između generatora G-gejdž transformacija koristićemo Baker-Kampbel-Hausdorf (BCH)[2] formulu u slučaju kada su parametri transformacija mali

$$\mathrm{e}^{{\epsilon_{\mathfrak{g}}}^{\alpha}{}_1\hat{G}_\alpha}\mathrm{e}^{{\epsilon_{\mathfrak{g}}}^{\beta}{}_2\hat{G}_\beta} = \mathrm{e}^{{\epsilon_{\mathfrak{g}}}^{\alpha}{}_1\hat{G}_\alpha + {\epsilon_{\mathfrak{g}}}^{\beta}{}_2\hat{G}_\beta + \frac{1}{2}{\epsilon_{\mathfrak{g}}}^{\alpha}{}_1\,{\epsilon_{\mathfrak{g}}}^{\beta}{}_2\,[\hat{G}_\alpha,\hat{G}_\beta] + O({\epsilon_{\mathfrak{g}}}^3)}\,, \tag{4.38}$$

iz čega sledi:

$$\mathrm{e}^{{\epsilon_{\mathfrak{g}}}^{\alpha}{}_1\hat{G}_\alpha}\mathrm{e}^{{\epsilon_{\mathfrak{g}}}^{\beta}{}_2\hat{G}_\beta} - \mathrm{e}^{{\epsilon_{\mathfrak{g}}}^{\beta}{}_2\hat{G}_\beta}\mathrm{e}^{{\epsilon_{\mathfrak{g}}}^{\alpha}{}_1\hat{G}_\alpha} = {\epsilon_{\mathfrak{g}}}^{\alpha}{}_1\,{\epsilon_{\mathfrak{g}}}^{\beta}{}_2\,[\hat{G}_\alpha, \hat{G}_\beta] + O({\epsilon_{\mathfrak{g}}}^3)\,. \tag{4.39}$$

Koristeći jednačinu (4.39), dobijamo da generatori G-gejdž transformacija definisanih u Teoremi 8 zadovoljavaju sledeće komutacione relacije:

$$[\hat{G}_\alpha, \hat{G}_\beta] = f_{\alpha\beta}{}^{\gamma}\hat{G}_\gamma\,, \tag{4.40}$$

gdje su $f_{\alpha\beta}{}^{\gamma}$ strukturne konstante algebre $\mathfrak{g}$. Primećujući da postoji izomorfizam između generatora $\hat{G}_\alpha \cong \tau_\alpha$, zaključujemo da je G-gejdž grupa transformacija iz Teoreme 8 isto što i grupa G koja odgovara formiranom BF dejstvu.

Grupa simetrija $\tilde{M}$

Posmatrajući transformacije varijabli (4.32) uočavamo da preostali članovi oblika $\delta_0 B^{\alpha}{}_{0i} = \nabla_0\epsilon_i$ i $\delta_0 B^{\alpha}{}_{ij} = 2\nabla_{[i|}\epsilon_{|j]}$ odgovaraju transformaciji definisanoj u Teoremi 9:

$$\delta_0\alpha^{\alpha}{}_{\mu} = 0\,, \qquad \delta_0 B^{\alpha}{}_{\mu\nu} = 2\nabla_{[\mu|}\epsilon^{\alpha}_{\mathfrak{m}|\nu]}\,, \tag{4.41}$$

gde je slobodan parametar $\epsilon^{\alpha}_{\mathfrak{m}\mu} = \epsilon^{\alpha}{}_{\mu}$.

Teorema 9 (M-gejdž transformacije) *U BF teoriji nad proizvoljnom Lijevom grupom G, sledeća transformacija je simetrija:*

$$\alpha \to \alpha' = \alpha\,, \qquad B \to B' = B + \nabla\epsilon_{\mathfrak{m}}\,, \tag{4.42}$$

gde je $\epsilon^{\alpha}_{\mathfrak{m}\mu}$ proizvoljna 1-forma element algebre $\mathfrak{g}$, a ∇ kovarijantan spoljašnji izvod definisan na standardni način, tj.

$$\nabla\epsilon_{\mathfrak{m}} = \mathrm{d}\epsilon_{\mathfrak{m}} + [\alpha \wedge \epsilon_{\mathfrak{m}}]\,. \tag{4.43}$$

Dokaz. Teoremu dokazujemo direktnom proverom:

$$\begin{aligned} S_{BF} \to S'_{BF} &= \frac{1}{4}\int_{\mathcal{M}_4} \mathrm{d}^4x\,\epsilon^{\mu\nu\rho\sigma}\left(B^{\alpha}{}_{\mu\nu} + 2\partial_{[\mu|}\epsilon^{\alpha}_{\mathfrak{m}|\nu]} + 2f_{\gamma\delta}{}^{\alpha}\alpha^{\gamma}{}_{[\mu|}\epsilon^{\delta}_{\mathfrak{m}|\nu]}\right)F^{\beta}{}_{\rho\sigma}g_{\alpha\beta} \\ &= S_{BF} - \frac{1}{2}\int_{\mathcal{M}_4}\mathrm{d}^4x\,\epsilon^{\mu\rho\sigma\nu}\left(\partial_\mu F^{\tau}{}_{\rho\sigma} + f_{\gamma\beta}{}^{\tau}\alpha^{\gamma}{}_{\mu}F^{\beta}{}_{\rho\sigma}\right)\epsilon^{\delta}_{\mathfrak{m}\nu}\,g_{\tau\delta} + \text{član na granici}\,, \end{aligned} \tag{4.44}$$

gde je drugi član jednak nuli jer je $\epsilon^{\mu\rho\sigma\nu}\left(\partial_\mu F^{\tau}{}_{\rho\sigma} + f_{\gamma\beta}{}^{\tau}\alpha^{\gamma}{}_{\mu}F^{\beta}{}_{\rho\sigma}\right) = 0$ na osnovu BI (7). ■

Primetimo da su transformacije definisane u Teoremi 9 linearne transformacije, a dve uzastopne

[2] eng. *Baker-Campbell-Hausdorff formula.*

M-gejdž transformacije daju jednu M-gejdž transformaciju sa parametrom $\epsilon_{\mathfrak{m}1} + \epsilon_{\mathfrak{m}2}$. Ako označimo generatore M-gejdž transformacija kao $\hat{M}_\alpha{}^\mu$,

$$\mathrm{e}^{\epsilon_{\mathfrak{m}1}\cdot\hat{M}}\mathrm{e}^{\epsilon_{\mathfrak{m}2}\cdot\hat{M}} = \mathrm{e}^{(\epsilon_{\mathfrak{m}1}+\epsilon_{\mathfrak{m}2})\cdot\hat{M}} , \tag{4.45}$$

gde je $\epsilon_{\mathfrak{m}} \cdot \hat{M} = \epsilon^\alpha_{\mathfrak{m}\mu}\hat{M}_\alpha{}^\mu$, iz čega sledi da je komutator generatora trivijalan,

$$[\hat{M}_\alpha{}^\mu, \hat{M}_\beta{}^\nu] = 0 . \tag{4.46}$$

Dakle, M-gejdž transformacije formiraju Abelovu grupu, koju ćemo u daljem tekstu označavati $\tilde{M}$. Prema indeksnoj strukturi njenih parametara i generatora, vidimo da je ova grupa izomorfna grupi $\mathbb{R}^{4p}$, gde je p dimenzija grupe G:

$$\tilde{M} \cong \mathbb{R}^{4p} . \tag{4.47}$$

Zatim, može se ispitati odnos M-gejdž transformacija i G-gejdž transformacija definisanih u prethodnom odeljku[3],

$$[\epsilon_{\mathfrak{g}} \cdot \hat{G}, \epsilon_{\mathfrak{m}} \cdot \hat{M}] \;=\; (\epsilon_{\mathfrak{g}} \rhd \epsilon_{\mathfrak{m}}) \cdot \hat{M} , \tag{4.48}$$

na osnovu čega dobijamo komutator:

$$[\hat{G}_\alpha \,, \hat{M}_\beta{}^\mu] \;=\; f_{\alpha\beta}{}^\gamma \hat{M}_\gamma{}^\mu . \tag{4.49}$$

Ovim smo završili izračunavanje algebra generatora gejdž transformacija u BF teoriji.

Ukupna gejdž grupa simetrije *BF* dejstva

Sumirajući rezultate prethodnih pododeljaka, može se zapisati algebra generatora ukupne gejdž grupe simetrije.

- Algebra $\mathfrak{g}$ grupe G data je komutacionim relacijama,

$$[\hat{G}_\alpha, \hat{G}_\beta] \;=\; f_{\alpha\beta}{}^\gamma \hat{G}_\gamma . \tag{4.50}$$

- Algebra generatora M-gejdž transformacija:

$$[\hat{M}_\alpha{}^\mu, \hat{M}_\beta{}^\nu] \;=\; 0 . \tag{4.51}$$

- Dejstvo generatora grupe G na generatore M-gejdž transformacija:

$$[\hat{G}_\alpha \,, \hat{M}_\beta{}^\mu] = f_{\alpha\beta}{}^\gamma \hat{M}_\gamma{}^\mu . \tag{4.52}$$

Na osnovu komutacionih relacija (4.51) zaključujemo da je grupa $\tilde{M}$ *invarijantna podgrupa*[4] ukupne grupe simetrija $\mathcal{G}_{BF}$. Na osnovu komutacionih relacija (4.52) dobijamo da semidirektan proizvod podgrupa G i $\tilde{M}$ daje ukupnu grupu simetrija BF dejstva:

$$\mathcal{G}_{BF} = G \ltimes \tilde{M} . \tag{4.53}$$

[3]Dejstvo parametra $\epsilon_{\mathfrak{g}}$ na parametar $\epsilon_{\mathfrak{m}}$, $\epsilon_{\mathfrak{g}} \rhd \epsilon_{\mathfrak{m}}$, je definisano kao $\epsilon_{\mathfrak{g}} \rhd \epsilon_{\mathfrak{m}} \equiv \rhd_{\alpha\beta}{}^\gamma\, \epsilon_{\mathfrak{g}}{}^\alpha \epsilon_{\mathfrak{m}}{}^\beta{}_\mu \mathrm{d}x^\mu$, pri čemu je $\rhd_{\alpha\beta}{}^\gamma = f_{\alpha\beta}{}^\gamma$. Sledi da je

$$(\epsilon_{\mathfrak{g}} \rhd \epsilon_{\mathfrak{m}}) \cdot \hat{M} = f_{\alpha\beta}{}^\gamma\, \epsilon_{\mathfrak{g}}{}^\alpha \epsilon_{\mathfrak{m}}{}^\beta{}_\mu \tilde{M}_\gamma{}^\mu .$$

[4]Podgrupa je *invarijantna podgrupa* neke grupe, ili ekvivalentno *normalna podgrupa*, ako je invarijantna pri konjugaciji elemenata podgrupe elementima grupe. Formalno, kažemo da je grupa H invarijantna podgrupa grupe G, ako je H podgrupa od G, tj. $H \leq G$, i za sve elemente $h \in H$ i $g \in G$, konjugacija elementa H elementom G je element H, tj. $\exists h' \in H$ takav da $ghg^{-1} = h'$. Na nivou algebre, odgovarajući objekt je *ideal*. Odnosno, algebra A je podalgebra algebre L u odnosu na operaciju množenje u L, tj. $[A, A] \subset A$. Zatim, podalgebra A algebre L je *ideal* u L ako njeni elementi, pomnoženi sa bilo kojim elementom algebre, daju ponovo element podalgebre, tj. $[A, L] \subset A$.

Difeomorfizmi

Druga važna tema za diskusiju je invarijantnost teorije na difeomorfizme. Iz činjenice da je BF dejstvo formulisano na manifestno kovarijantni način preko diferencijalnih formi, očigledno je da su difeomorfizmi simetrija teorije. Međutim, gledajući strukturu gejdž grupe $\mathcal{G}_{BF}$, ne vidi se odmah da li je $Diff(\mathcal{M}_4, \mathbb{R})$ njena podgrupa.

Razmotrimo difeomorfizam transformacije

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)\,, \tag{4.54}$$

gde je parametar $\xi^\mu(x)$ proizvoljna funkcija koordinata. Takođe, označimo parametre transformacija gejdž simetrija $\epsilon_i(x)$. Ako su difeomorfizmi simetrija dejstva, onda za svako polje $\phi(x)$ u teoriji i svaki parametar difeomorfizam transformacija $\xi^\mu(x)$, postoji izbor gejdž $\epsilon_i(x)$ i Eno-Taitelboim[5] parametara $\epsilon^{\rm HT}(x)$, tako da:

$$({\delta_0}^{\rm gauge} + {\delta_0}^{\rm HT} + {\delta_0}^{\rm diff})\,\phi = 0\,. \tag{4.57}$$

Drugim rečima, ako su difeomorfizmi simetrija teorije, njihove varijacije forme se mogu izraziti kao zbir varijacija formi varijabli pri gejdž transformacijama i varijacija formi pri HT transformacijama:

$${\delta_0}^{\rm diff}\,\phi = -{\delta_0}^{\rm gauge}\phi - {\delta_0}^{\rm HT}\phi\,. \tag{4.58}$$

Konkretno, BF dejstvo zavisi od varijabli $\alpha^\alpha{}_\mu$ i $B^\alpha{}_{\mu\nu}$. Parametri HT transformacija $\epsilon^{{\rm HT}\alpha\beta}{}_{\mu\nu\rho}$ su definisani relacijama (4.55)

$${\delta_0}^{\rm HT}\alpha^\alpha{}_\mu \;=\; \frac{1}{2}\epsilon^{{\rm HT}\alpha\beta}{}_{\mu\nu\rho}\frac{\delta S}{\delta B^\beta{}_{\nu\rho}}\,, \qquad {\delta_0}^{\rm HT}B^\alpha{}_{\mu\nu} \;=\; -\epsilon^{{\rm HT}\alpha\beta}{}_{\rho\mu\nu}\frac{\delta S}{\delta \alpha^\beta{}_\rho}\,, \tag{4.59}$$

dok su gejdž parametri $\epsilon_{\mathfrak{g}}{}^\alpha$ i $\epsilon_{\mathfrak{m}}{}^\alpha{}_\mu$ definisani u Teoremama 8 i 9. Možemo pokazati da zaista postoji izbor ovih parametara, tako da je jednačina (4.57) zadovoljena za sva polja. Konkretno, ako odaberemo gejdž parametre kao

$$\epsilon_{\mathfrak{g}}{}^\alpha = -\xi^\lambda\alpha^\alpha{}_\lambda\,, \quad \epsilon_{\mathfrak{m}}{}^\alpha{}_\mu = \xi^\lambda B^\alpha{}_{\mu\lambda}\,, \tag{4.60}$$

a HT parametre kao

$$\epsilon^{{\rm HT}\alpha\beta}{}_{\mu\nu\rho} = \xi^\lambda g^{\alpha\beta}\epsilon_{\mu\nu\rho\lambda}\,, \tag{4.61}$$

primenom jednačine (4.58) dobijamo upravo standardne varijacije formi koje odgovaraju difeomorfizmima:

$$\begin{array}{rcl} {\delta_0}^{\rm diff}\alpha^\alpha{}_\mu &=& -\partial_\mu\xi^\lambda\alpha^\alpha{}_\lambda - \xi^\lambda\partial_\lambda\alpha^\alpha{}_\mu\,, \\ {\delta_0}^{\rm diff}B^\alpha{}_{\mu\nu} &=& -\partial_\mu\xi^\lambda B^\alpha{}_{\lambda\nu} - \partial_\nu\xi^\lambda B^\alpha{}_{\mu\lambda} - \xi^\lambda\partial_\lambda B^\alpha{}_{\mu\nu}\,. \end{array} \tag{4.62}$$

Ovim se utvrđuje da su difeomorfizmi zaista simetrija teorije, čak i ako nisu sadržani u ukupnoj gejdž grupi simetrija $\mathcal{G}_{BF}$, već u direktnom proizvodu ukupne grupe simetrija i HT grupe simetrija.

[5]Lako je videti da je svako dejstvo, koje zavisi od najmanje dva polja $\phi_1(x)$ i $\phi_2(x)$, invarijantno na Eno-Taitelboim (HT) transformaciju [36], određenu parametrom $\epsilon^{\rm HT}$

$${\delta_0}^{\rm HT}\phi_1 = \epsilon^{\rm HT}(x)\frac{\delta S}{\delta\phi_2}\,, \qquad {\delta_0}^{\rm HT}\phi_2 = -\epsilon^{\rm HT}(x)\frac{\delta S}{\delta\phi_1}\,, \tag{4.55}$$

što se može lako proveriti izračunavanjem varijacije dejstva:

$$\delta^{\rm HT}S[\phi_1,\phi_2] = \frac{\delta S}{\delta\phi_1}{\delta_0}^{\rm HT}\phi_1 + \frac{\delta S}{\delta\phi_2}{\delta_0}^{\rm HT}\phi_2 = 0\,. \tag{4.56}$$

Ova simetrija prisutna je čak i u teorijama koje nemaju gejdž simetriju, ali se ne vidi se u generatoru gejdž simetrija (4.31) dobijenim Kastelanijevom procedurom. Razlog za to sastoji se u tome da je HT-varijacija polja ϕ_1 i ϕ_2 uvek jednaka nuli on-shell, tj. varijacije su uvek linearne kombinacije veza, pa prema tome slabo jednake nuli.

4.2 Jang-Milsova teorija

U fizici smo najčešće zainteresovani za teorije koje nisu toploške, odnosno teorije koje poseduju lokalne propagirajuće stepene slobode. Kako bi topološko BF dejstvo transformisali u dejstvo sa propagirajućim stepenima slobode dodaje se dodatni član u dejstvo, tzv. *veza jednostavnosti*. Rezultujuće dejstvo daje *BF teoriju sa vezama*.

Jedan primer takvog dejstva je *Jang-Milsova teorija* za $SU(N)$ grupu u prostoru Minkovskog, koju možemo da napišemo kao BF dejstvo sa vezama na sledeći način:

$$\begin{aligned} S \;=\; \int & B_I \wedge F^I + \lambda^I \wedge \Big(B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b \Big) \\ & + \zeta^{abI} \Big(M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - g_{IJ} F^J \wedge \delta_a \wedge \delta_b \Big) . \end{aligned} \tag{4.63}$$

Ovde je $F \equiv \mathrm{d}A + A \wedge A$ 2-forma krivine za 1-formu koneksije $A \in \mathcal{A}^1(\mathcal{M}_4 , \mathfrak{su}(N))$, a $B \in \mathcal{A}^2(\mathcal{M}_4 , \mathfrak{su}(N))$ 2-forma Lagranžev množitelj. Kilingova forma $g_{IJ} \equiv \langle \tau_I , \tau_J \rangle_{\mathfrak{su}(N)} \propto f_{IK}{}^L f_{JL}{}^K$ definiše podizanje i spuštanje indeksa $I, J, \dots$ koji prebrojavaju generatore $SU(N)$ grupe, dok $f_{IJ}{}^K$ označava strukturne konstante $\mathfrak{su}(N)$ algebre. Dejstvo (4.63) dobijeno je nametanjem veza topološkom dejstvu (4.1), dodavanjem dva dodatna člana u obliku proizvoda Lagranževih množitelja, 2-forme λ^I i 0-forme ζ^{abI}, i odgovarajućih veza. Funkcija, odnosno 0-forma, M_{abI} je takođe Lagranžev množitelj, dok g označava kapling konstantu za Jang-Milsovo polje. Najzad, δ^a je nedinamičko polje 1-forma, takvo da postoji globalni koordinatni sistem u kome su njegove komponente jednake Kronekerovoj delti $\delta^a{}_\mu$. Dakle, ova 1-forma predstavlja pozadinsko polje, i definiše globalnu prostorvremensku metriku jednačinom

$$\eta_{\mu\nu} = \eta_{ab} \delta^a{}_\mu \delta^b{}_\nu \,, \tag{4.64}$$

gde je $\eta_{ab} \equiv \mathrm{diag}(-1, +1, +1, +1)$ metrika Minkovskog. Stoga, prostorvremenska mnogostrukost $\mathcal{M}_4$ je ravna. Indeksi $a, b, \dots$ su lokalni Lorencovi indeksi, koji uzimaju vrednosti $0, \dots , 3$. Polje δ^a ima sve osobine 1-forme tetrade e^a u ravnom prostorvremenu Minkovskog. Primetimo da je dejstvo (4.63) manifestno difeomorfizam invarijantno i invarijantno pri gejdž transformcijama grupe simetrija $SU(N)$, ali nije nezavisno od pozadine zbog prisustva pozadinskog polja δ^a. Variranjem dejstva (4.63) po varijablama ζ^{abI}, M_{abI}, A^I, B_I, i λ^I, dobijamo jednačine kretanja:

$$M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - F_I \wedge \delta_a \wedge \delta_b = 0 \,, \tag{4.65}$$

$$-\frac{12}{g} \lambda^I \wedge \delta^a \wedge \delta^b + \zeta^{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f = 0 \,, \tag{4.66}$$

$$-\mathrm{d}B_I + f_{JI}{}^K B_K \wedge A^J + \mathrm{d}(\zeta^{ab}{}_I \delta_a \wedge \delta_b) - f_{JI}{}^K \zeta^{ab}{}_K \delta_a \wedge \delta_b \wedge A^J = 0 \,, \tag{4.67}$$

$$F_I + \lambda_I = 0 \,, \tag{4.68}$$

$$B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b = 0 \,. \tag{4.69}$$

Primetimo da nismo varirali po pozadinskom polju δ^a. Iz jednačina (4.65), (4.66), (4.68) i (4.69) možemo da izrazimo Lagranževe množitelje kao algebarske funkcije jačine polja $F^I{}_{\mu\nu}$ za dinamičko polje A^I:

$$\begin{aligned} M_{abI} &= \frac{1}{48} \varepsilon_{abcd} F_I{}^{cd} \,, & \zeta^{abI} &= \frac{1}{4g} \varepsilon^{abcd} F^I{}_{cd} \,, \\ \lambda_{Iab} &= F_{Iab} \,, & B_{Iab} &= \frac{1}{2g} \varepsilon_{abcd} F_I{}^{cd} \,. \end{aligned} \tag{4.70}$$

Ovde je korišćena notacija $F_{Iab} = F_{I\mu\nu}\delta_a{}^\mu\delta_b{}^\nu$, analogno za sve varijable. Takođe, podrazumeva se da je $\delta^a{}_\mu$ invertibilna matrica. Na osnovu jednačina (4.70) i diferencijalne jednačine kretanja (4.67) dobija se jednačina kretanja za gejdž polje $A^I{}_\mu$:

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0\,. \tag{4.71}$$

Ovo je upravo klasična jednačina kretanja za slobodno Jang-Milsovo polje. Dejstvo (4.63) se može transformisati u dejstvo koje opisuje *maseno vektosko polje*, sa dinamikom zadatom Proka jednačinom kretanja, dodavanjem masenog člana:

$$-\frac{1}{4!}m^2 A_{I\mu}A^I{}_\nu \eta^{\mu\nu}\varepsilon_{abcd}\delta^a\wedge\delta^b\wedge\delta^c\wedge\delta^d\,. \tag{4.72}$$

Prisustvo ovog člana u dejstvu naravno eksplicitno narušava $SU(N)$ gejdž simetriju dejstva.

4.3 Plebanski dejstvo za Opštu relativnost

Najpoznatiji primer BF teorije sa vezama je *Plebanski dejstvo za Opštu relativnost* [32], [34]. Plebanski je 1977. godine pokazao kako umesto metrike možemo koristiti polje B kao fundamentalnu varijablu kojom opisujemo gravitaciono polje.

Izborom Lorencove grupe $G = SO(3,1)$ kao gejdž grupe, BF dejstvo definišemo kao

$$S = \int_{\mathcal{M}_4} B_{ab}\wedge R^{ab}\,. \tag{4.73}$$

Ovde je R^{ab} 2-forma krivine za spinsku koneksiju ω^{ab}, B_{ab} je 2-forma Lagranžev množitelj, dok je Kilingova forma definisana kao

$$g_{ab,cd} = \frac{1}{2}(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc})\,,$$

gde je η_{ab} metrika Minkovskog. Antisimetrični par indeksa prebrojava generatore J_{ab} Lorencove grupe $SO(3,1)$. Dodavanjem odgovarajućih veza dobijamo BF dejstvo sa vezama - *Plebanski dejstvo*:

$$S = \int_{\mathcal{M}_4} B_{ab}\wedge R^{ab} + \frac{1}{2}\phi_{abcd}B^{ab}\wedge B^{cd} + \mu\phi_{abcd}\left(a_1 g^{ab,cd} + a_2\epsilon^{abcd}\right)\,. \tag{4.74}$$

gde je ϕ_{abcd} Lagranžev množitelj 0-forma koji množi vezu $B^{ab}\wedge B^{cd}$, μ je Lagranžev množitelj 4-forma, $a_1, a_2 \in \mathbb{R}$ su realni parametri, a $g^{ab,cd}$ je inverzna Kilingova forma. Može se pokazati da se variranjem ovog dejstva po varijablama B_{ab}, ω^{ab}, ϕ_{abcd} i μ dobijaju jednačine kretanja:

$$R_{ab} - \phi_{abcd}B^{cd} = 0\,, \tag{4.75}$$

$$\nabla B_{ab} = 0\,, \tag{4.76}$$

$$\frac{1}{2}B^{ab}\wedge B^{cd} + \mu\left(a_1 g^{ab,cd} + a_2\varepsilon^{abcd}\right) = 0\,, \tag{4.77}$$

$$\phi_{abcd}\left(a_1 g^{ab,cd} + a_2\varepsilon^{abcd}\right) = 0\,. \tag{4.78}$$

Rešavanjem jednačine (4.77) može se pokazati da rešenje za Lagranžev množitelj ima oblik

$$B_{ab} = \frac{\alpha}{2}\varepsilon_{abcd}e^c\wedge e^d + \beta e_a\wedge e_b\,, \tag{4.79}$$

gde koeficijenti α i β zadovoljavaju jednačinu:

$$a_2\alpha\beta = \frac{a_1}{4}(\alpha^2 - \beta^2)\,.$$

Sada, za uobičajeni izbor u savremenoj literaturi $a_1 = 0$ i $a_2 = 1$ imamo dva moguća rešenja:

$$B_{ab} = \frac{\alpha}{2}\varepsilon_{abcd}e^c \wedge e^d\,, \quad \beta = 0\,, \tag{4.80}$$

$$B_{ab} = \beta e_a \wedge e_b\,, \quad \alpha = 0\,. \tag{4.81}$$

Oba rešenja podrazumevaju dakle da je diferencijalna forma B_{ab} *prosta*[6], pa se i veza

$$\frac{1}{2}B^{ab} \wedge B^{cd} + \mu\left(a_1 g^{ab,cd} + a_2\epsilon^{abcd}\right)\,,$$

koja dovodi do ovog rešenja zove *veza jednostavnosti*.

Rešavanjem sistema jednačina (4.75)-(4.78) dobija se da je jednačina (4.75) ekvivalentna Ajnštajnovoj vakuumskoj jednačini Opšte relativnosti:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\,. \tag{4.82}$$

Primetimo da u ovom modelu polja tetrade nisu eksplicitno prisutna u modelu, već se pojavljuju samo kao rešenja jednačine kretanja. Na osnovu toga sledi da su tetrade *on-shell objekti*, odnosno da se ne mogu kvantovati. Ovo čini model Plebanskog nepovoljnim za kuplovanje polja materije sa gravitacijom [11], [15], [37]. Ipak, uspešno je sproverena kvantizacija Plebanski modela kao modela Opšte relativnosti, u kontekstu *modela spinske pene* [1], [2], [9], [10].

[6]Diferencijalna 2-forma je *prosta* (eng. *simple, decomposable differential form*) ako može da se napiše kao spoljašnji proizvod dve 1-forme.

Glava 5

$2BF$ teorija

Kako bi se rešio problem kvantizacije polja materije prisutne u Standardnom Modelu kuplovanih sa gravitacijom koji postoji u modelu Plebanskog, razvija se novi pravac istraživanja – generalizovani BF modeli u kontekstu teorije kategorija [12], videti [15], [22], [23], [37], [38]. Prvi korak ove kategorijske generalizacije – tzv. *kategorijskih lestvica*, je kategorijska generalizacija pojma grupe na pojam 2-grupe. Ovaj pristup se zasniva na ideji da gejdž simetrije u fizici osim Lijevim grupama možemo opisati i drugim objektima. Generalizacijom BF teorije koja je definisana za neku Lijevu grupu, na teoriju koja je definisana za neku opštu semistriktnu 2-grupu, dolazimo do *$2BF$ teorije*, takođe poznatu i pod nazivom *$BFCG$ teorija* [12], [13], [17], [39].

U kontekstu kvantizacione procedure spinske pene, teorija viših kategorija je uspešno primenjena u formulaciji kvantnog gravitacionog modela, zasnovanog na Poenkareovoj 2-grupi [39] i odgovarajućem $2BF$ dejstvu, tzv. *spinkub modela kvantne gravitacije*. Kako su u spinkub modelu tetrade prisutne u topološkom sektoru teorije kao fundamentalna polja u $2BF$ dejstvu, ovaj model bi u principu mogao biti proširen tako da teorija uključuje sva polja materije prisutna u Standardnom Modelu. Ipak, da bi to bilo ostvareno na kvantnom nivou, neophodno je da i dejstva koja opisuju polja materije budu napisana u obliku prilagođenom za kvantizacionu proceduru spinske pene, za šta je kako se ispostavlja neophodan još jedan korak kategorijske generalizacije – formulacija $3BF$ teorije koja će biti opisana u narednom poglavlju.

Najpre, u ovom poglavlju, u odeljku 5.1 ćemo definisati i analizirati simetrije $2BF$ topološkog dejstva. Pritom, pratićemo sličnu liniju izlaganja kao u poglavlju 4. U odeljku 5.1 dat je kratak pregled topološke $2BF$ teorije. Pododeljak 5.1.1 sadrži Hamiltonovu analizu $2BF$ teorije i rezultujuću kanonsku strukturu, analizu Bjankijevih identiteta koje zadovoljavaju veze prve klase, a koji smanjuju broj nezavisnih veza prve klase prisutnih u teoriji, kao i brojanje fizičkih stepeni slobode u $2BF$ teoriji. Kao što je i očekivano, ovom analizom je dobijeno da je $2BF$ teorija topološka, tj. teorija bez lokalnih propagirajućih stepeni slobode. Na kraju pododeljka 5.1.1, dat je konačan oblik generatora gejdž transformacija teorije, kao i varijacije formi svih varijabli i njihovih kanonskih impulsa u teoriji, dok je Kastelanijeva procedura kojom je dobijen ovaj generator predstavljena u Dodatku D.2.2.

Dobijene varijacije formi varijabli koristimo u pododeljku 5.1.2 kako bismo dobili oblik konačnih transformacija svih gejdž simetrija $2BF$ teorije. Poglavlje 5.1.2 je podeljeno na četiri dela. Najpre, diskutujemo gejdž grupu G i odgovarajuće G-gejdž transformacije. U drugom delu, predstavljena je grupa $\tilde{M}$, tj. M-gejdž transformacije, treći deo sadrži analizu grupe $\tilde{H}$ koja se sastoji od H-gejdž transformacija koje su već poznate iz prethodne literature, a četvrti deo sadrži analizu grupe $\tilde{N}$ i N-gejdž transformacija koje takođe nastaju u teoriji. U ovom poglavlju dati su i komutatori generatora ovih transformacija, dok su računski detalji dati u Dodatku D.2.3. Sumiranjem ovih rezultata predstavljena je kompletna struktura gejdž grupe simetrije $2BF$ dejstva, uključujući njenu Lijevu algebru, kao i konkretan izbor parametara kojim se dobijaju difeomorfizam transformacije u $2BF$ teoriji.

Konačno, modifikacijom $2BF$ topološkog dejstva dodavanjem odgovarajućih veza u odeljcima 5.2 i 5.3 formirana su $2BF$ dejstva sa vezama koja opisuju teorije sa netrivijalnom dinamikom – *Opštu relativnost* i *Ajnštajn-Jang-Milsovu teoriju*. Pokazano je kako se gravitaciono i Jang-Milsovo polje u zakrivljenom prostoru mogu zapisati u formi $2BF$ dejstva sa vezama. Ovo nas dovodi jedan korak bliže zapisivanju ukupnog dejstva koje opisuje svu materiju prisutnu u Standardnom Modelu i gravitacije u obliku prilagođenom za kovarijantnu kvantizacionu proceduru spinske pene.

5.1 Topološka 2BF teorija

Koristeći definicije 2-grupe, odnosno ukrštenog modula, 2-koneksije i 2-krivine, definiše se generalizacija BF dejstva – tzv. $2BF$ dejstvo [13], [17]:

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} \,. \tag{5.1}$$

Ovde su 2-forma $\mathcal{F} \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ i 3-forma $\mathcal{G} \in \mathcal{A}^3(\mathcal{M}_4\,, \mathfrak{h})$ komponente 2-krivine definisane jednačinom (2.36), 2-forma $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ i 1-forma $C \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{h})$ Lagranževi množitelji, dok $\langle \_\,,\_ \rangle_{\mathfrak{g}}$ i $\langle \_\,,\_ \rangle_{\mathfrak{h}}$ označavaju G-invarijantne bilinarne simetrične nedegenerisane forme algebri $\mathfrak{g}$ i $\mathfrak{h}$. Kao posledica strukture ukrštenog modula (videti [16]), bilinearna forma $\langle \_\,,\_ \rangle_{\mathfrak{h}}$ je takođe H-invarijantna. Videti [13], [17] za detaljni pregled teorije i relevantne reference.

Slično kao BF dejstvo, $2BF$ dejstvo je takođe topološko, kao što se može videti iz jednačina kretanja. Variranjem dejstva (5.1) po varijablama B^α i C^a dobijamo jednačine kretanja

$$\mathcal{F}^\alpha = 0\,, \qquad \mathcal{G}^a = 0\,, \tag{5.2}$$

gde indeks α prebrojava generatore grupe G, a a prebrojava generatore grupe H. Variranjem po 2-koneksiji, varijablama α^α i β^a, dobijamo jednačine kretanja za Lagranževe množitelje:

$$\mathrm{d}B_\alpha - f_{\alpha\beta}{}^\gamma B_\gamma \wedge \alpha^\beta - \rhd_{\alpha a}{}^b C_b \wedge \beta^a = 0\,, \tag{5.3}$$
$$\mathrm{d}C_a - \partial_a{}^\alpha B_\alpha + \rhd_{\alpha a}{}^b C_b \wedge \alpha^\alpha = 0\,. \tag{5.4}$$

Primetimo da su jednačine kretanja diferencijalne jednačine prvog reda i da opisuju teoriju bez propagirajućih stepeni slobode. Da je zaista u pitanju teorija sa propagirajućim stepenima slobode rigorozno se pokazuje primenom Hamiltonove analize, kao što je to urađeno u radovima [22], [23]. Na osnovu rezultata Hamiltonove analize sledi da je $2BF$ teorija *topološka teorija*.

5.1.1 Hamiltonova analiza topološke 2BF teorije

Topološko $2BF$ dejstvo (5.1) daje Lagranžijan:

$$L_{2BF} = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \epsilon^{\mu\nu\rho\sigma} \Big(\frac{1}{4}\, B^\alpha{}_{\mu\nu}\, \mathcal{F}^\beta{}_{\rho\sigma}\, g_{\alpha\beta} + \frac{1}{3!}\, C^a{}_\mu\, \mathcal{G}^b{}_{\nu\rho\sigma}\, g_{ab} \Big)\,. \tag{5.5}$$

Kanonski impulsi za varijable $B^\alpha{}_{\mu\nu}$, $\alpha^\alpha{}_\mu$, $C^a{}_\mu$ i $\beta^a{}_{\mu\nu}$ nalaze se variranjem dejstva po vremenskim izvodima varijabli:

$$\begin{aligned}
\pi(B)_\alpha{}^{\mu\nu} &= \frac{\delta L}{\delta \partial_0 B^a{}_{\mu\nu}} = 0\,, \\
\pi(\alpha)_\alpha{}^\mu &= \frac{\delta L}{\delta \partial_0 \alpha^\alpha{}_\mu} = \frac{1}{2}\epsilon^{0\mu\nu\rho} B_{\alpha\nu\rho}\,, \\
\pi(C)_a{}^\mu &= \frac{\delta L}{\delta \partial_0 C^a{}_\mu} = 0\,, \\
\pi(\beta)_a{}^{\mu\nu} &= \frac{\delta L}{\delta \partial_0 \beta^a{}_{\mu\nu}} = -\epsilon^{0\mu\nu\rho} C_{a\rho}\,.
\end{aligned} \tag{5.6}$$

Dakle, *primarne veze* u teoriji su izračunavaju se primenom jednačine (3.19):

$$\begin{aligned} P(B)_\alpha{}^{\mu\nu} &\equiv \pi(B)_\alpha{}^{\mu\nu} \approx 0\,, \\ P(\alpha)_\alpha{}^{\mu} &\equiv \pi(\alpha)_\alpha{}^{\mu} - \frac{1}{2}\epsilon^{0\mu\nu\rho}B_{\alpha\nu\rho} \approx 0\,, \\ P(C)_a{}^{\mu} &\equiv \pi(C)_a{}^{\mu} \approx 0\,, \\ P(\beta)_a{}^{\mu\nu} &\equiv \pi(\beta)_a{}^{\mu\nu} + \epsilon^{0\mu\nu\rho}C_{a\rho} \approx 0\,. \end{aligned} \tag{5.7}$$

Fundamentalna Poasonova zagrada varijabli i njihovih kanonskih impulsa definiše se na sledeći način:

$$\begin{aligned} \{\, B^\alpha{}_{\mu\nu}(\vec{x})\,,\, \pi(B)_\beta{}^{\rho\sigma}(\vec{y})\,\} &= 2\delta^\alpha_\beta \delta^\rho_{[\mu}\delta^\sigma_{\nu]}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\, \alpha^\alpha{}_{\mu}(\vec{x})\,,\, \pi(\alpha)_\beta{}^{\nu}(\vec{y})\,\} &= \delta^\alpha_\beta \delta^\nu_\mu\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\, C^a{}_{\mu}(\vec{x})\,,\, \pi(C)_b{}^{\nu}(\vec{y})\,\} &= \delta^a_b \delta^\nu_\mu\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\, \beta^a{}_{\mu\nu}(\vec{x})\,,\, \pi(\beta)_b{}^{\rho\sigma}(\vec{y})\,\} &= 2\delta^a_b\, \delta^\rho_{[\mu}\delta^\sigma_{\nu]}\,\delta^{(3)}(\vec{x}-\vec{y})\,. \end{aligned} \tag{5.8}$$

Na osnovu ove definicije, izračunavamo algebru primarnih veza

$$\begin{aligned} \{\, P(B)_\alpha{}^{jk}(\vec{x})\,,\, P(\alpha)_\beta{}^{i}(\vec{y})\,\} &= \epsilon^{0ijk}\, g_{\alpha\beta}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\, P(C)_a{}^{k}(\vec{x})\,,\, P(\beta)_b{}^{ij}(\vec{y})\,\} &= -\epsilon^{0ijk}\, g_{ab}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \end{aligned} \tag{5.9}$$

dok su ostale Poasonove zagrade jednake nuli. Kanonski *on-shell* Hamiltonijan je definisan jednačinom (3.20)

$$H_c = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\left[\frac{1}{2}\pi(B)_\alpha{}^{\mu\nu}\,\partial_0 B^\alpha{}_{\mu\nu} + \pi(\alpha)_\alpha{}^{\mu}\,\partial_0\alpha^\alpha{}_\mu + \pi(C)_a{}^{\mu}\,\partial_0 C^a{}_\mu + \frac{1}{2}\pi(\beta)_a{}^{\mu\nu}\,\partial_0\beta^a{}_{\mu\nu}\right] - L\,, \tag{5.10}$$

odnosno:

$$\begin{aligned} H_c \approx -\int_{\Sigma_3} \mathrm{d}^3\vec{x}\,\epsilon^{0ijk}\Big[&\frac{1}{2}B_{\alpha 0i}\,\mathcal{F}^\alpha{}_{jk} + \frac{1}{6}C_{a0}\,\mathcal{G}^a{}_{ijk} \\ &+ \beta^a{}_{0i}\Big(\nabla_j C_{ak} - \frac{1}{2}\partial_a{}^\alpha B_{\alpha\, jk}\Big) + \frac{1}{2}\alpha^\alpha{}_0\Big(\nabla_i B_{\alpha\, jk} - C_{ai} \rhd_{\alpha b}{}^a\, \beta^b{}_{jk}\Big)\Big]\,. \end{aligned} \tag{5.11}$$

U prethodnom izrazu primenili smo da su primarne veze slabo jednake nuli. Dodavanjem proizvoda Lagranževih množitelja λ i primarnih veza, definišemo *totalni off-shell Hamiltonijan* definisan jednačinom (3.21):

$$H_T = H_c + \int_{\Sigma_3}\mathrm{d}^3\vec{x}\left[\frac{1}{2}\lambda(B)^\alpha{}_{\mu\nu}P(B)_\alpha{}^{\mu\nu} + \lambda(\alpha)^\alpha{}_\mu P(\alpha)_\alpha{}^\mu + \lambda(C)^a{}_\mu P(C)_a{}^\mu + \frac{1}{2}\lambda(\beta)^a{}_{\mu\nu}P(\beta)_a{}^{\mu\nu}\right]. \tag{5.12}$$

Uslovi konzistentnosti (3.26) za primarne veze moraju biti zadovoljeni, pa za primarne veze $P(B)_\alpha{}^{0i}$, $P(\alpha)_\alpha{}^0$, $P(C)_a{}^0$ i $P(\beta)_a{}^{0i}$ ovaj uslov dovodi do pojave *sekundarnih veza* $\mathcal{S}$,

$$\begin{aligned} \mathcal{S}(\mathcal{F})_\alpha{}^i &\equiv \frac{1}{2}\epsilon^{0ijk}\mathcal{F}_{\alpha jk} &&\approx 0\,, \\ \mathcal{S}(\nabla B)_\alpha &\equiv \frac{1}{2}\epsilon^{0ijk}\big(\nabla_{[i}B_{\alpha\, j]k} - C_{a[i} \rhd_{\alpha b}{}^a\,\beta^b{}_{j]k}\big) &&\approx 0\,, \\ \mathcal{S}(\mathcal{G})_a &\equiv \frac{1}{6}\epsilon^{0ijk}\mathcal{G}_{aijk} &&\approx 0\,, \\ \mathcal{S}(\nabla C)_a{}^i &\equiv \epsilon^{0ijk}\big(\nabla_{[j}C_{ak]} - \frac{1}{2}\partial_a{}^\alpha B_{\alpha\, jk}\big) &&\approx 0\,, \end{aligned} \tag{5.13}$$

dok u slučaju primarnih veza $P(\alpha)_\alpha{}^k$, $P(B)_\alpha{}^{jk}$, $P(\beta)_a{}^{jk}$ i $P(C)_a{}^k$ uslovi konzistentnosti određuju Lagranževe množitelje:

$$
\begin{aligned}
\lambda(B)_{\alpha ij} &\approx \nabla_i B_{\alpha 0j} - \nabla_j B_{\alpha 0i} + C_{a0}\beta^b{}_{ij} \rhd_{\alpha b}{}^a + C_{bi} \rhd_\alpha{}^b{}_a \beta^a{}_{0j} \\
&\quad - C_{bj} \rhd_\alpha{}^b{}_a \beta^a{}_{0i} + g_{\beta\gamma}{}^\alpha \alpha^\beta{}_0 B^\gamma{}_{ij}, \\
\lambda(\alpha)^\alpha{}_i &\approx \nabla_i \alpha^\alpha{}_0 + \partial_a{}^\alpha \beta^a{}_{0i}, \\
\lambda(C)^a{}_i &\approx \nabla_i C^a{}_0 + C^b{}_i \rhd_\alpha{}^a{}_b \alpha^\alpha{}_0 + B_{\alpha 0i}\partial^{a\alpha}, \\
\lambda(\beta)^a{}_{ij} &\approx \nabla_i \beta^a{}_{0j} - \nabla_j \beta^a{}_{0i} - \beta^b{}_{ij} \rhd_{\alpha b}{}^a \alpha^\alpha{}_0.
\end{aligned}
\tag{5.14}
$$

Preostali Lagranževi množitelji

$$
\lambda(B)^\alpha{}_{0i}\,, \qquad \lambda(\alpha)^\alpha{}_0\,, \qquad \lambda(C)^a{}_0\,, \qquad \lambda(\beta)^a{}_{0i}\,,
\tag{5.15}
$$

ostaju neodređeni. Uslovi konzistentnosti sekundarnih veza ne dovode do pojave tercijarnih veza, odnosno dobijamo:

$$
\begin{aligned}
\{\mathcal{S}(\mathcal{F})^{\alpha i}\,,\, H_T\} &= f_{\beta\gamma}{}^\alpha \mathcal{S}(\mathcal{F})^{\beta i}\alpha^\gamma{}_0\,, \\
\{\mathcal{S}(\nabla B)_\alpha\,,\, H_T\} &= f_{\beta\gamma\alpha}B^\gamma{}_{0k}\mathcal{S}(\mathcal{F})^{\beta k} + f_{\beta\alpha}{}^\gamma \alpha^\beta{}_0 \mathcal{S}(\nabla B)_\gamma + C_{a0} \rhd_{\alpha b}{}^a \mathcal{S}(\mathcal{G})^b \\
&\quad - \rhd_{\alpha a}{}^b \beta^a{}_{0k}\mathcal{S}(\nabla C)_b{}^k\,, \\
\{\mathcal{S}(\mathcal{G})^a\,,\, H_T\} &= \rhd_{\alpha b}{}^a \beta^b{}_{0k}\mathcal{S}(\mathcal{F})^{\alpha k} - \alpha^\alpha{}_0 \rhd_{\alpha b}{}^a \mathcal{S}(\mathcal{G})^b\,, \\
\{\mathcal{S}(\nabla C)_a{}^i\,,\, H_T\} &= C_{b0} \rhd_\alpha{}^b{}_a \mathcal{S}(\mathcal{F})^{\alpha i} + \rhd_{\alpha a}{}^b \alpha^\alpha{}_0 \mathcal{S}(\nabla C)_b{}^i\,,
\end{aligned}
\tag{5.16}
$$

Totalni Hamiltonijan možemo da napišemo u sledećem obliku:

$$
\begin{aligned}
H_T = \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big[&\lambda(B)^\alpha{}_{0i}\,\Phi(B)_\alpha{}^i + \lambda(\alpha)^\alpha\,\Phi(\alpha)_\alpha + \lambda(C)^a{}_0\,\Phi(C)_a + \lambda(\beta)^a{}_{0i}\,\Phi(\beta)_a{}^i \\
&- B_{\alpha 0i}\,\Phi(\mathcal{F})^{ai} - \alpha_{\alpha 0}\,\Phi(\nabla B)^\alpha - C_{a0}\,\Phi(\mathcal{G})^a - \beta_{a0i}\,\Phi(\nabla C)^{ai} \Big]\,,
\end{aligned}
\tag{5.17}
$$

gde su

$$
\begin{aligned}
\Phi(B)_\alpha{}^i &= P(B)_\alpha{}^{0i}\,, \\
\Phi(\alpha)_\alpha &= P(\alpha)_\alpha{}^0\,, \\
\Phi(C)_a &= P(C)_a{}^0\,, \\
\Phi(\beta)_a{}^i &= P(\beta)_a{}^{0i}\,, \\
\Phi(\mathcal{F})^{\alpha i} &= \mathcal{S}(\mathcal{F})^{\alpha i} - \nabla_j P(B)^{\alpha ij} - P(C)_a{}^i \partial^{a\alpha}\,, \\
\Phi(\mathcal{G})_a &= \mathcal{S}(\mathcal{G})_a + \nabla_i P(C)_a{}^i - \tfrac{1}{2}\beta_{bij} \rhd_\alpha{}^b{}_a P(B)^{\alpha ij}\,, \\
\Phi(\nabla C)_a{}^i &= \mathcal{S}(\nabla C)_a{}^i - \nabla_j P(\beta)_a{}^{ij} + C_{bj} \rhd{}_\alpha{}^b{}_a P(B)^{\alpha ij} - \partial_a{}^\alpha P(\alpha)_\alpha{}^i\,, \\
\Phi(\nabla B)_\alpha &= \mathcal{S}(\nabla B)_\alpha + \nabla_i P(\alpha)_\alpha{}^i - \tfrac{1}{2}B_{\beta ij}f_{\alpha\gamma}{}^\beta P(B)^{\gamma ij} \\
&\quad - C_{bi} \rhd_{\alpha a}{}^b P(C)^{ai} - \tfrac{1}{2}\beta_{bij} \rhd_{\alpha a}{}^b P(\beta)^{aij}\,,
\end{aligned}
\tag{5.18}
$$

veze prve klase, dok su veze druge klase u teoriji:

$$\chi(B)_\alpha{}^{jk} = P(B)_\alpha{}^{jk}\,, \quad \chi(C)_a{}^i = P(C)_a{}^i\,, \quad \chi(\alpha)_\alpha{}^i = P(\alpha)_\alpha{}^i\,, \quad \chi(\beta)_a{}^{ij} = P(\beta)_a{}^{ij}\,. \tag{5.19}$$

Možemo da izračunamo Poasonovu algebru veza prve klase:

$$\begin{aligned}
\{\,\Phi(\mathcal{G})^a(\vec{x})\,,\,\Phi(\nabla C)_b{}^i(\vec{y})\,\} &= -\rhd_{\alpha b}{}^a\,\Phi(\mathcal{F})^{\alpha i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\mathcal{G})^a(\vec{x})\,,\,\Phi(\nabla B)_\alpha(\vec{y})\,\} &= \rhd_{\alpha b}{}^a\,\Phi(\mathcal{G})^b(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla C)_a{}^i(\vec{x})\,,\,\Phi(\nabla B)_\alpha(\vec{y})\,\} &= \rhd_{\alpha a}{}^b\,\Phi(\nabla C)_b{}^i(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\mathcal{F})^\alpha{}_i(\vec{x})\,,\,\Phi(\nabla B)_\beta(\vec{y})\,\} &= f_{\beta\gamma}{}^\alpha\,\Phi(\mathcal{F})^\gamma{}_i(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)_\alpha(\vec{x})\,,\,\Phi(\nabla B)_\beta(\vec{y})\,\} &= f_{\alpha\beta}{}^\gamma\,\Phi(\nabla B)_\gamma(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,
\end{aligned} \tag{5.20}$$

kao i Poasonovu algebru veza prve klase i veza druge klase:

$$\begin{aligned}
\{\,\Phi(\mathcal{F})^{\alpha i}(\vec{x})\,,\,\chi(\alpha)_\beta{}^j(\vec{y})\,\} &= -f_{\beta\gamma}{}^\alpha\,\chi(B)^{\gamma ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\mathcal{G})^a(\vec{x})\,,\,\chi(\alpha)_\alpha{}^i(\vec{y})\,\} &= -\rhd_{\alpha b}{}^a\,\chi(C)^{bi}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\mathcal{G})^a(\vec{x})\,,\,\chi(\beta)_b{}^{ij}(\vec{y})\,\} &= \rhd_{\alpha b}{}^a\,\chi(B)^{\alpha ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla C)^{ai}(\vec{x})\,,\,\chi(\alpha)_\alpha{}^j(\vec{y})\,\} &= -\rhd_{\alpha b}{}^a\,\chi(\beta)^{bij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla C)^{ai}(\vec{x})\,,\,\chi(C)_b{}^j(\vec{y})\,\} &= \rhd^a{}_{\alpha b}\,\chi(B)^{\alpha ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^\alpha(\vec{x})\,,\,\chi(\alpha)_\beta{}^i(\vec{y})\,\} &= f_{\beta\gamma}{}^\alpha\,\chi(\alpha)^{\gamma i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^\alpha(\vec{x})\,,\,\chi(\beta)_a{}^{ij}(\vec{y})\,\} &= \rhd_{\alpha a}{}^b\,\chi(\beta)_b{}^{ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^\alpha(\vec{x})\,,\,\chi(B)_\beta{}^{ij}(\vec{y})\,\} &= -f_{\beta\gamma}{}^\alpha\,\chi(B)^{\gamma ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,.\\
\{\,\Phi(\nabla B)^\alpha(\vec{x})\,,\,\chi(C)_a{}^i(\vec{y})\,\} &= -\rhd^{\alpha b}{}_a\,\chi(C)_b{}^i(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned} \tag{5.21}$$

Možemo da izračunamo i komutator veza prve klase sa Hamiltonijanom:

$$\begin{aligned}
\{\Phi(B)_\alpha{}^i\,, H_T\} &= \Phi(F)_\alpha{}^i\,,\\
\{\Phi(\alpha)_\alpha\,, H_T\} &= \Phi(\nabla B)_\alpha\,,\\
\{\Phi(F)^\alpha{}_i\,, H_T\} &= -\alpha^\beta{}_0\,f_{\beta\gamma}{}^\alpha\,\Phi(F)^\gamma{}_i\,,\\
\{\Phi(\nabla B)_\alpha\,, H_T\} &= -B_{\beta 0i}\,f_{\alpha\gamma}{}^\beta\,\Phi(F)^{\gamma i} - \alpha^\beta{}_0\,f_{\alpha\beta}{}^\gamma\,\Phi(\nabla B)_\gamma\,,\\
\{\Phi(C)_a,\, H_T\} &= \Phi(\mathcal{G})_a\,,\\
\{\Phi(\beta)_a{}^i,\, H_T\} &= \Phi(\nabla C)_a{}^i\,,\\
\{\Phi(\mathcal{G})_a,\, H_T\} &= \alpha^\alpha{}_0\rhd_{\alpha a}{}^b\Phi(\mathcal{G})_b - \beta_{b0i}\rhd_{\alpha a}{}^b\Phi(F)^{\alpha i}\,,\\
\{\Phi(\nabla C)_a{}^i,\, H_T\} &= \alpha^\alpha{}_0\rhd_{\alpha a}{}^b\Phi(\nabla C)_b{}^i - C_{b0}\rhd_{\alpha a}{}^b\Phi(F)^{\alpha i}\,.
\end{aligned} \tag{5.22}$$

Broj stepeni slobode topološke $2BF$ teorije

Za određivanje broja stepeni slobode topološke $2BF$ teorije koristićemo sledeće Bjankijeve identitete (BI).

Lema 9 (BI za 1-forme α i C.) *Odgovarajuće 2-forme krivina ovih polja*

$$F^\alpha = \mathrm{d}\alpha^\alpha + f_{\beta\gamma}{}^\alpha\, \alpha^\beta \wedge \alpha^\gamma\,, \qquad T^a = \mathrm{d}C^a + \triangleright_{\alpha b}{}^a\, \alpha^\alpha \wedge C^b\,, \tag{5.23}$$

zadovoljavaju Bjankijeve identitete:

$$\epsilon^{\lambda\mu\nu\rho}\, \nabla_\mu F^\alpha{}_{\nu\rho} = 0\,, \tag{5.24}$$

$$\epsilon^{\lambda\mu\nu\rho} \left(\nabla_\mu T^a{}_{\nu\rho} - \triangleright_{\alpha b}{}^a F^\alpha{}_{\mu\nu} C^b{}_\rho\right) = 0\,. \tag{5.25}$$

Lema 10 (BI za 2-forme B i β.) *Odgovarajuće 3-krivine su date izrazima*

$$S^\alpha = \mathrm{d}B^\alpha + f_{\beta\gamma}{}^\alpha\, \alpha^\beta \wedge B^\gamma\,, \qquad G^a = \mathrm{d}\beta^a + \triangleright_{\alpha b}{}^a\, \alpha^\alpha \wedge \beta^\beta\,, \tag{5.26}$$

i zadovoljavaju Bjankijeve identitete:

$$\epsilon^{\lambda\mu\nu\rho} \left(\frac{2}{3}\nabla_\lambda\, S^\alpha{}_{\mu\nu\rho} - f_{\beta\gamma}{}^\alpha F^\beta{}_{\lambda\mu}\, B^\gamma{}_{\nu\rho}\right) = 0\,, \tag{5.27}$$

$$\epsilon^{\lambda\mu\nu\rho} \left(\frac{2}{3}\nabla_\lambda\, G^a{}_{\mu\nu\rho} - \triangleright_{\alpha b}{}^a\, F^\alpha{}_{\lambda\mu}\, \beta^b{}_{\nu\rho}\right) = 0\,. \tag{5.28}$$

U slučaju $2BF$ teorije, inicijalan broj polja u teoriji N može se odrediti iz tabele (5.1). Ovde je p dimenzionalnost Lijeve grupe G i q je dimenzionalnost Lijeve grupe H.

$\alpha^\alpha{}_\mu$	$\beta^a{}_{\mu\nu}$	$B^\alpha{}_{\mu\nu}$	$C^a{}_\mu$
$4p$	$6q$	$6p$	$4q$

Tabela 5.1: Broj inicijalnih polja u $2BF$ teoriji.

Prebrojavanjem polja u tabeli (5.1) nalazimo da je $N = 10(p+q)$. Broj nezavisnih komponenata veza druge klase određen je prebrojavanjem veza prikazanih u tabeli (5.2). Dobija se da je $S = 6(p+q)$.

$\chi(B)_\alpha{}^{jk}$	$\chi(C)_a{}^i$	$\chi(\alpha)_\alpha{}^i$	$\chi(\beta)_a{}^{ij}$
$3p$	$3q$	$3p$	$3q$

Tabela 5.2: Veze druge klase u $2BF$ teoriji.

Veze prve klase nisu sve međusobno nezavisne i zadovoljavaju relacije

$$\nabla_i \Phi(\mathcal{F})_\alpha{}^i + \frac{1}{2}\partial_{a\alpha}\Phi(\mathcal{G})^a - \frac{1}{2}\partial^a{}_\alpha \nabla_i \chi(C)_a{}^i - \frac{1}{2} f_{\beta\gamma\alpha}\partial_a{}^\beta \beta^a{}_{ij}\chi(B)^{\gamma ij} = \epsilon^{ijk}\nabla_i F_{\alpha jk}\,, \tag{5.29}$$

odnosno kada iskoristimo da je $\epsilon^{ijk}\,\nabla_i F^a_{jk} = 0$ kao $\lambda = 0$ komponentu BI (5.24) ovaj izraz se svodi na:

$$\nabla_i \Phi(\mathcal{F})_\alpha{}^i + \frac{1}{2}\partial_{a\alpha}\Phi(\mathcal{G})^a - \frac{1}{2}\partial^a{}_\alpha \nabla_i \chi(C)_a{}^i - \frac{1}{2} f_{\beta\gamma\alpha}\partial_a{}^\beta \beta^a{}_{ij}\chi(B)^{\gamma ij} = 0\,. \tag{5.30}$$

Slično, veze prve klase zadovoljavaju relacije

$$\begin{aligned} & \nabla_i \Phi(\nabla C)_a{}^i - \frac{1}{2} C_{bi} \rhd{}_{\alpha a}{}^b \Phi(\mathcal{F})^{\alpha i} + \partial_{a\alpha} S(\nabla B)^\alpha + \\ & + \frac{1}{2} F^\beta{}_{ij} \rhd_{\beta c}{}^b \chi(\beta)^{cij} + T^b{}_{jk} \rhd_{\alpha a}{}^b \chi(B)^{\alpha jk} - \partial_{a\alpha} \nabla_i \chi(\alpha)^{\alpha i} \\ = \; & \epsilon^{ijk} \left(\nabla_i T_{ajk} - \rhd_{\alpha b}{}^a F^\alpha_{jk} C^b_i \right) , \end{aligned} \tag{5.31}$$

odnosno, kako je desna strana jednačine $\lambda = 0$ komponenta (5.25), dobijamo da (5.31) daje vezu:

$$\begin{aligned} & \nabla_i \Phi(\nabla C)_a{}^i - \frac{1}{2} C_{bi} \rhd{}_{\alpha a}{}^b \Phi(\mathcal{F})^{\alpha i} + \partial_{a\alpha} S(\nabla B)^\alpha + \\ & + \frac{1}{2} F^\beta{}_{ij} \rhd_{\beta c}{}^b \chi(\beta)^{cij} + T^b{}_{jk} \rhd_{\alpha a}{}^b \chi(B)^{\alpha jk} - \partial_{a\alpha} \nabla_i \chi(\alpha)^{\alpha i} = 0 \, . \end{aligned} \tag{5.32}$$

Broj veza prve klase može biti određen prebrojavanjem veza u tabeli (5.3). Dobija se da je broj veza prve klase dat izrazom

$$F = 8(p+q) - (p+q) = 7(p+q) \, ,$$

gde smo oduzeli p relacija (5.30) i q relacija (5.32). Dakle, na osnovu definicije broja stepeni slobode (3.41), sledi:

$$n = 10(p+q) - 7(p+q) - \frac{6(p+q)}{2} = 0 \, . \tag{5.33}$$

Zaključujemo da $2BF$ teorija nema lokalne propagrajuće stepene slobode, odnosno da je *topološka teorija*.

$\Phi(B)_\alpha{}^i$	$\Phi(C)_a$	$\Phi(\alpha)_\alpha$	$\Phi(\beta)_a{}^i$	$\Phi(\mathcal{F})^{\alpha i}$	$\Phi(\mathcal{G})^a$	$\Phi(\nabla C)^{ai}$	$\Phi(\nabla B)^\alpha$
$3p$	q	p	$3q$	$3p-p$	q	$3q-q$	p

Tabela 5.3: Veze prve klase u $2BF$ teoriji.

Generator gejdž transformacija za $2BF$ teoriju

Generator gejdž transformacija u $2BF$ teoriji dat je izrazom:

$$\begin{aligned} G = & \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big((\nabla_0 \epsilon^\alpha{}_i) \Phi(B)_\alpha{}^i - \epsilon^\alpha{}_i \Phi(\mathcal{F})_\alpha{}^i + (\nabla_0 \epsilon^\alpha) \Phi(\alpha)_\alpha \\ & + \epsilon^\alpha \big(f_{\alpha\gamma}{}^\beta B_{\beta 0i} \Phi(B)^{\gamma i} + C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^{b0} + \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)^{b0i} - \Phi(\nabla B)_\alpha \big) \\ & + (\nabla_0 \epsilon^a) \Phi(C)_a - \epsilon^a \left(\beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a \right) \\ & + (\nabla_0 \epsilon^a{}_i) \Phi(\beta)_a{}^i - \epsilon^a{}_i \left(C_{b0} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\nabla C)_a{}^i \right) \Big) \, . \end{aligned} \tag{5.34}$$

Ovde su $\epsilon^{ab}{}_i$, ϵ^{ab}, ϵ_i, ϵ^a i $\epsilon^a{}_i$ nezavisni parametri gejdž transformacija. Postupak izvođenja generatora (5.34) prikazan je u Dodatku D.2.

Varijaciju forme varijabla i njihovih konjugovanih impulsa računamo primenom (3.56):

$$
\begin{aligned}
\delta_0 B^\alpha{}_{0i} &= \nabla_0 \epsilon^\alpha{}_i - f_{\beta\gamma}{}^\alpha \epsilon^\beta B^\gamma{}_{0i} & \delta_0 \pi(B)_\alpha{}^{0i} &= f_{\alpha\beta}{}^\gamma \epsilon^\beta \pi(B)_\gamma{}^{0i}\,,\\
&\quad -\epsilon^a \rhd_{\alpha a}{}^b \beta_{b0i} - \epsilon^a{}_i \rhd_{\alpha a}{}^b C_{b\,0}\,, & & \\
\delta_0 B^\alpha{}_{ij} &= 2\nabla_{[i}\epsilon^\alpha{}_{j]} - f_{\beta\gamma}{}^\alpha \epsilon^\beta B^\gamma{}_{ij} & \delta_0 \pi(B)_\alpha{}^{ij} &= f_{\alpha\beta}{}^\gamma \epsilon^\beta \pi(B)_\gamma{}^{ij} - \epsilon^a{}_k \epsilon^{0ijk}\partial_{a\alpha}\,,\\
&\quad -\epsilon^a \rhd_{\alpha a}{}^b \beta_{bij} - 2\epsilon^a{}_{[j|} \rhd_{\alpha a}{}^b C_{b\,|i]}\,, & & \\
\delta_0 \alpha^\alpha{}_0 &= \nabla_0 \epsilon^\alpha\,, & \delta_0 \pi(\alpha)_\alpha{}^0 &= -f_{\alpha\beta}{}^\gamma \epsilon^\beta{}_i \pi(B)_\gamma{}^{0i} - f_{\alpha\beta}{}^\gamma \epsilon^\gamma \pi(\alpha)_\gamma{}^0\\
& & &\quad -\epsilon^b \rhd_{\alpha b}{}^a \pi(C)_a - \epsilon^b{}_i \rhd_{\alpha b}{}^a \pi(\beta)_a{}^i\,,\\
\delta_0 \alpha^\alpha{}_i &= \nabla_i \epsilon^\alpha + \partial_a{}^\alpha \epsilon^a{}_i\,, & \delta_0 \pi(\alpha)_\alpha{}^i &= -f_{\alpha\beta}{}^\gamma \epsilon^\beta{}_j \pi(B)_\gamma{}^{ij} - f_{\alpha\beta}{}^\gamma \epsilon^\beta \pi(\alpha)_\gamma{}^i\,,\\
\\
\delta_0 C^a{}_0 &= \nabla_0 \epsilon^a - \epsilon^\alpha \rhd_{\alpha b}{}^a C^b{}_0\,, & \delta_0 \pi(C)_a{}^0 &= \epsilon^\alpha \rhd_{\alpha a}{}^b \pi(C)_b{}^0 + \epsilon_{bi} \rhd_{\alpha a}{}^b \pi(B)^{\alpha 0i}\,,\\
\delta_0 C^a{}_i &= \nabla_i \epsilon^a - \epsilon^\alpha \rhd_{\alpha b}{}^a C^b{}_i\,, & \delta_0 \pi(C)_a{}^i &= \epsilon^\alpha \rhd_{\alpha a}{}^b \pi(C)_b{}^i + \epsilon_{bj} \rhd_{\alpha a}{}^b \pi(B)^{\alpha ij}\,,\\
\delta_0 \beta^a{}_{0i} &= \nabla_0 \epsilon^a{}_i - \epsilon^\alpha \rhd_{\alpha b}{}^a \beta^b{}_{0i}\,, & \delta_0 \pi(\beta)_a{}^{0i} &= \epsilon^\alpha \rhd_{\alpha a}{}^b \pi(\beta)_b{}^{0i} + \epsilon_b \rhd_{\alpha a}{}^b \pi(B)^{\alpha 0i}\,,\\
\delta_0 \beta^a{}_{ij} &= 2\nabla_{[i}\epsilon^a{}_{j]} - \epsilon^\alpha \rhd_{\alpha b}{}^a \beta^b{}_{ij}\,, & \delta_0 \pi(\beta)_a{}^{ij} &= \epsilon^\alpha \rhd_{\alpha a}{}^b \pi(\beta)_b{}^{ij} + \epsilon_b \rhd_{\alpha a}{}^b \pi(B)^{\alpha ij}\\
& & &\quad -\epsilon^\alpha{}_k \epsilon^{0ijk} \partial_{a\alpha}\,.
\end{aligned}
\tag{5.35}
$$

5.1.2 Simetrije $2BF$ dejstva

Grupa simetrija G

Dejstvo (5.1) poseduje dodatne simetrije u odnosu na transformacije simetrija definisane za BF dejstvo u Teoremama 8 i 9.

Transformacije generisane gejdž parametrom $\epsilon_{\mathfrak{g}}{}^\alpha$, na osnovu varijacija formi varijabli (5.35), date su izrazima

$$
\begin{aligned}
\delta_0 \alpha^\alpha{}_\mu &= -\,\partial_\mu \epsilon_{\mathfrak{g}}{}^\alpha - f_{\beta\gamma}{}^\alpha \alpha^\beta{}_\mu \epsilon_{\mathfrak{g}}{}^\gamma\,, & \delta_0 B^\alpha{}_{\mu\nu} &= f_{\beta\gamma}{}^\alpha \epsilon_{\mathfrak{g}}{}^\beta B^\gamma{}_{\mu\nu}\,,\\
\delta_0 \beta^a{}_{\mu\nu} &= \rhd_{\alpha b}{}^a \epsilon_{\mathfrak{g}}{}^\alpha \beta^b{}_{\mu\nu}\,, & \delta_0 C^a{}_\mu &= \rhd_{\alpha b}{}^a \epsilon_{\mathfrak{g}}{}^\alpha C^b{}_\mu\,,
\end{aligned}
\tag{5.36}
$$

što analogno možemo zapisati

$$
\begin{aligned}
\alpha &\to \alpha' = \alpha - \nabla\epsilon_{\mathfrak{g}}\,, & B &\to B' = B - [B, \epsilon_{\mathfrak{g}}]\,,\\
\beta &\to \beta' = \beta + \epsilon_{\mathfrak{g}} \rhd \beta\,, & C &\to C' = C + \epsilon_{\mathfrak{g}} \rhd C\,,
\end{aligned}
\tag{5.37}
$$

Na osnovu ovih infinitezimalnih transformacija, možemo ekstrapolirati konačne transformacije, definisane Teoremom 10.

Teorema 10 (G-gejdž transformacije) *U $2BF$ teoriji konstruisanoj za proizvoljni ukršteni modul $(H \xrightarrow{\partial} G, \rhd)$, sledeća transformacija je transformacija simetrije*

$$
\begin{aligned}
\alpha &\to \alpha' = \mathrm{Ad}_g \alpha + g\mathrm{d}g^{-1}\,, & B &\to B' = gBg^{-1}\,,\\
\beta &\to \beta' = g \rhd \beta\,, & C &\to C' = g \rhd C\,,
\end{aligned}
\tag{5.38}
$$

gde je $g = \exp(\epsilon_{\mathfrak{g}} \cdot \hat{G}) = \exp(\epsilon_{\mathfrak{g}\alpha} \hat{G}^\alpha) \in G$, a $\epsilon_{\mathfrak{g}} : \mathcal{M}_4 \to \mathfrak{g}$ parametar transformacija.

Dokaz. Pri ovim transformacijama, 2-krivina se transformiše na sledeći način:

$$\mathcal{F} \;\to\; \mathcal{F}' = g\mathcal{F}g^{-1}\,, \qquad \mathcal{G} \;\to\; \mathcal{G}' = g \rhd \mathcal{G}\,. \tag{5.39}$$

Invarijantnost $2BF$ dejstva pri ovoj transformaciji sledi na osnovu G-invarijantnosti bilinearnih formi $\langle \_ , \_ \rangle_{\mathfrak{g}}$ i $\langle \_ , \_ \rangle_{\mathfrak{h}}$:

$$S_{2BF} = \int_{\mathcal{M}_4} \Big(\langle B, \mathcal{F} \rangle_{\mathfrak{g}} + \langle C, G \rangle_{\mathfrak{h}} \Big) \to S'_{2BF} = \int_{\mathcal{M}_4} \Big(\langle g^{-1}Bg, g^{-1}\mathcal{F}g \rangle_{\mathfrak{g}} + \langle g^{-1} \rhd C, g^{-1} \rhd G \rangle_{\mathfrak{h}} \Big)\,, \tag{5.40}$$

odakle dobijamo da je $2BF$ dejstvo invarijantno. Invarijantnost se može takođe pokazati na sličan način kao u Teoremi 8. ■

Prethodna teorema je generalizacija Teoreme 8 za slučaj $2BF$ teorije.

Grupa simetrija $\tilde{M}$

Zatim, posmatrajući transformacije varijabli uočavamo da članovi oblika $\delta_0 B^{\alpha}{}_{0i} = \nabla_0 \epsilon_i$ i $\delta_0 B^{\alpha}{}_{ij} = 2\nabla_{[i}\epsilon_{j]}$ odgovaraju transformaciji definisanoj u Teoremi 11, kao u slučaju BF teorije.

Teorema 11 (M-gejdž transformacije) *U $2BF$ teoriji nad proizvoljnim ukrštenim modulom $(H \overset{\partial}{\to} G, \rhd)$, sledeća transformacija je simetrija:*

$$\begin{array}{ccccccccc} \alpha & \to & \alpha' & = & \alpha\,, & B & \to & B' & = & B - \nabla\epsilon_{\mathfrak{m}}\,, \\ \beta & \to & \beta' & = & \beta\,, & C^a & \to & C'^a & = & C^a - \partial^a{}_{\alpha}\epsilon_{\mathfrak{m}}{}^{\alpha}\,, \end{array} \tag{5.41}$$

gde je $\epsilon_{\mathfrak{m}}$ proizvoljna 1-forma element algebre $\mathfrak{g}$, a ∇ kovarijantan spoljasnji izvod definisan na standardni način, tj.

$$\nabla\epsilon_{\mathfrak{m}} = \mathrm{d}\epsilon_{\mathfrak{m}} + [\alpha \wedge \epsilon_{\mathfrak{m}}]\,. \tag{5.42}$$

Dokaz. Varijacija $2BF$ dejstva pri M-gejdž transformacijama je

$$S'_{2BF} = S_{2BF} + \int_{\mathcal{M}_4} \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} \left(-\frac{1}{2}(\nabla_{\mu}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\nu})\mathcal{F}_{\alpha\rho\sigma} - \frac{1}{3!}\partial^a{}_{\alpha}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu}G_{a\nu\rho\sigma} \right)\,. \tag{5.43}$$

Primenom definicije 2-krivine (2.37), dobijamo:

$$S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} \Big(-\frac{1}{2}(\nabla_{\mu}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\nu})\,(F_{\alpha\rho\sigma} - \partial^a{}_{\alpha}\beta_{a\rho\sigma}) - \frac{1}{3!}\partial^a{}_{\alpha}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu}\, 3\nabla_{\nu}\beta_{a\rho\sigma} \Big)\,. \tag{5.44}$$

Drugi član u zagradi i treći član se krate, pa se izraz svodi na:

$$S'{}_{3BF} = S_{3BF} - \frac{1}{2}\int_{\mathcal{M}_4} \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu}\nabla_{\nu}F_{\alpha\rho\sigma}\,. \tag{5.45}$$

Član $\epsilon^{\mu\nu\rho\sigma}\nabla_{\nu}F_{\alpha\rho\sigma} = 0$ je BI (5.24). Zaključujemo da je $2BF$ dejstvo S_{2BF} invarijantno na M-gejdž transformacije definisane Teoremom 11. ■

Ova teorema je generalizacija Teoreme 9 u slučaju $2BF$ teorije. Komutatori između generatora G-gejdž transformacija, između generatora M-gejdž transformacija, kao i komutatori između generatora G- i M-gejdž transformacija izračunati su istim postupkom kao i u slučaju BF-teorije i dobijeni su isti rezultati, odnosno jednačine (4.40), (4.46) i (4.49). Slično kao u slučaju simetrija BF dejstva, postoji izomorfizam između generatora $\hat{G}_{\alpha} \cong \tau_{\alpha}$, tj. možemo zaključiti da je grupa G-gejdž transformacija iz Teoreme 10 upravo grupa G iz ukrštenog modula $(H \overset{\partial}{\to} G, \rhd)$. Ovo je važan rezultat, koji neće važiti za preostale transformacije simetrije $2BF$ dejstva, kao što ćemo videti u nastavku.

Grupa simetrija $\tilde{H}$

Uočimo u jednačinama varijacija formi (5.35) varijacije varijabli na prostornoj hiperpovrši Σ_3 koje odgovaraju parametru $\epsilon^a{}_i$. Na osnovu njih možemo da ekstrapoliramo varijacije formi varijabli koje odgovaraju parametru $\epsilon^a{}_\mu$ i definišemo H-gejdž transformacije Teoremom 12,

$$\begin{array}{rclrcl} \delta_0 \alpha^\alpha{}_\mu &=& \partial_a{}^\alpha \epsilon_{\mathfrak{h}}{}^a{}_\mu \,, & \delta_0 B^\alpha{}_{\mu\nu} &=& -2C_{a[\mu|}\epsilon_{\mathfrak{h}}{}^b{}_{|\nu]} \rhd_{\beta b}{}^a g^{\alpha\beta} \,, \\ \delta_0 \beta^a{}_{\mu\nu} &=& 2\nabla_{[\mu|}\epsilon_{\mathfrak{h}}{}^a{}_{|\nu]} \,, & \delta_0 C^a{}_\mu &=& 0 \,, \end{array} \tag{5.46}$$

pri čemu identifikujemo $\epsilon_{\mathfrak{h}}{}^a{}_i = \epsilon^a{}_i$ i $\epsilon_{\mathfrak{h}}{}^a{}_0 = 0$.

Teorema 12 (H-gejdž transformacije) *U $2BF$ teoriji nad proizvoljnim ukrštenim modulom $(H \xrightarrow{\partial} G, \rhd)$, sledeća transformacija je simetrija*

$$\begin{array}{rclcl} \alpha &\to& \alpha' &=& \alpha - \partial\epsilon_{\mathfrak{h}} \,, \\ \beta &\to& \beta' &=& \beta - \nabla'\epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge \epsilon_{\mathfrak{h}} \,, \end{array} \qquad \begin{array}{rclcl} B &\to& B' &=& B - C' \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}} \,, \\ C &\to& C' &=& C \,, \end{array} \tag{5.47}$$

gde je $\epsilon_{\mathfrak{h}}$ proizvoljna 1-forma element algebre $\mathfrak{h}$, a oznaka ∇' je kovarijantni izvod sa koneksijom α'. Preslikavanje τ definisano je u Dodatku A jednačinom (A.13) i predstavlja rešenje jednačine:

$$\langle C \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}}, \mathcal{F} \rangle_{\mathfrak{g}} + \langle C, \mathcal{F} \wedge^{\rhd} \epsilon_{\mathfrak{h}} \rangle_{\mathfrak{h}} = 0 \,.$$

Dokaz. Invarijantnost se može pokazati direktnom proverom. Transformacija 3-krivine je

$$\begin{array}{rclcl} \mathcal{F} &\to& \mathcal{F}' &=& \mathcal{F} \,, \\ \mathcal{G} &\to& \mathcal{G}' &=& \mathcal{G} - \mathcal{F} \wedge^{\rhd} \epsilon_{\mathfrak{h}} \,. \end{array} \tag{5.48}$$

Pri transformacijama 3-krivine (5.48) i transformacijama Lagranževih množitelja, dejstvo S_{3BF} se transformiše

$$S'{}_{2BF} = S_{2BF} + \int_{\mathcal{M}_4} \Big(- \langle C' \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}}, \mathcal{F} \rangle_{\mathfrak{g}} - \langle C', \mathcal{F} \wedge^{\rhd} \epsilon_{\mathfrak{h}} \rangle_{\mathfrak{h}} \Big) \,. \tag{5.49}$$

Definicija preslikavanja $\mathcal{T}$ data jednačinom (A.13) osigurava da se članovi u zagradi poništavaju, odnosno da dejstvo ostaje invarijantno. ■

Označimo generatore H-gejdž transformacija datih Teoremom 12 sa $\hat{H}_a{}^\mu$. Istim postupkom kojim su izvedeni komutatori G- i M-gejdž transformacija sada nalazimo komutatore H-gejdž transformacija. Ako se izvedu dve uzastopne infinitezimalne H-gejdž transformacije, definisane parametrima $\epsilon_{\mathfrak{h}1}$ i $\epsilon_{\mathfrak{h}2}$, dobijamo

$$\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot\hat{H}}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot\hat{H}} - \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot\hat{H}}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot\hat{H}} = 0 \,, \tag{5.50}$$

gde je $\epsilon_{\mathfrak{h}} \cdot \hat{H} = \epsilon_{\mathfrak{h}}{}^a{}_\mu \hat{H}_a{}^\mu$. Na osnovu prethodne jednačine komutator H-gejdž transformacija je

$$[\hat{H}_a{}^\mu, \hat{H}_b{}^\nu] = 0 \,. \tag{5.51}$$

Ovaj komutator izračunat je u Dodatku D.2.3. Dakle, H-gejdž transformacije formiraju Abelovu grupu, koju ćemo u daljem tekstu označavati $\tilde{H}$. Prema indeksnoj strukturi njenih parametara i generatora, vidimo da je ova grupa izomorfna grupi $\mathbb{R}^{4q}$, gde je q dimenzija grupe H:

$$\tilde{H} \cong \mathbb{R}^{4q} \,. \tag{5.52}$$

Zatim, komutatori generatora grupa G i $\tilde{M}$ i generatora H-gejdž transformacija su

$$[\hat{G}_\alpha, \hat{H}_a{}^\mu] = \rhd_{\alpha a}{}^b \hat{H}_b{}^\mu \,, \qquad [\hat{H}_a, \hat{M}_\alpha{}^\mu] = 0 \,. \tag{5.53}$$

Grupa simetrija $\tilde{N}$

Teorema 13 (N-gejdž transformacije) *U $2BF$ teoriji nad proizvoljnim ukrštenim modulom $(H \xrightarrow{\partial} G, \triangleright)$, sledeća transformacija je simetrija:*

$$\begin{array}{rclcrcl} \alpha & \to & \alpha' = \alpha\,, & & B & \to & B' = B - \beta \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}}\,, \\ \beta & \to & \beta' = \beta\,, & & C & \to & C' = C - \nabla \epsilon_{\mathfrak{n}}\,, \end{array} \tag{5.54}$$

gde je $\epsilon_{\mathfrak{n}}$ proizvoljna 0-forma element algebre $\mathfrak{h}$.

Dokaz. Dokaz je pravolinijski. Pri transformacijama definisanim u Teoremi 13 $2BF$ dejstvo se transformiše na sledeći način:

$$\begin{aligned} S_{2BF} \to S'_{2BF} &= \int_{\mathcal{M}_4} \mathrm{d}x^4 \epsilon^{\mu\nu\rho\sigma} \left(\frac{1}{4} \left(B_{\alpha\,\mu\nu} - \beta_{b\mu\nu} \triangleright_{\alpha a}{}^{b} \epsilon_{\mathfrak{n}}{}^{a} \right) \mathcal{F}^{\alpha}{}_{\rho\sigma} + \frac{1}{3!} \left(C^{a}{}_{\mu} + \nabla_{\mu} \epsilon_{\mathfrak{n}}{}^{a} \right) G_{a\nu\rho\sigma} \right) \\ &= S_{2BF} + \int_{\mathcal{M}_4} \mathrm{d}x^4 \epsilon^{\mu\nu\rho\sigma} \left(-\frac{1}{4} \beta_{b\mu\nu} \triangleright_{\alpha a}{}^{b} \epsilon_{\mathfrak{n}}{}^{a} \mathcal{F}^{\alpha}{}_{\rho\sigma} - \frac{1}{2} \nabla_{\nu} \nabla_{\mu} \epsilon_{\mathfrak{n}}{}^{a} \beta_{a\rho\sigma} \right) \\ &= S_{2BF} + \int_{\mathcal{M}_4} \mathrm{d}x^4 \epsilon^{\mu\nu\rho\sigma} \left(-\frac{1}{4} \beta_{b\mu\nu} \triangleright_{\alpha a}{}^{b} \epsilon_{\mathfrak{n}}{}^{a} \mathcal{F}^{\alpha}{}_{\rho\sigma} + \frac{1}{4} \triangleright_{\alpha b}{}^{a} F^{\alpha}{}_{\mu\nu} \epsilon_{\mathfrak{n}}{}^{b} \beta_{a\rho\sigma} \right) . \end{aligned} \tag{5.55}$$

Ovde smo iskoristili činjenicu da je član

$$\epsilon^{\mu\nu\rho\sigma} \triangleright_{\alpha a}{}^{b} \epsilon_{\mathfrak{n}}{}^{a} \beta_{b\mu\nu} \partial_{c}{}^{\alpha} \beta^{c}{}_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma} f_{ca}{}^{b} \epsilon_{\mathfrak{n}}{}^{a} \beta_{b\mu\nu} \beta^{c}{}_{\rho\sigma} = 0\,,$$

identički jednak nuli zbog Pajferovog identiteta (2.20) i antisimetričnosti strukturne konstante. ■

Grupu N-gejdž transformacija definisanih u Teoremi 13 obeležavamo sa $\tilde{N}$. Ove transformacije su linearne, a kompozicija dve N-gejdž transformacije daje jednu N-gejdž transformaciju sa parametrom $\epsilon_{\mathfrak{n}1} + \epsilon_{\mathfrak{n}2}$. Obeležavajući generatore grupe $\tilde{N}$ sa $\hat{N}_a$, dobijamo

$$\mathrm{e}^{\epsilon_{\mathfrak{n}1}\cdot \hat{N}} \mathrm{e}^{\epsilon_{\mathfrak{n}2}\cdot \hat{N}} = \mathrm{e}^{(\epsilon_{\mathfrak{n}1}+\epsilon_{\mathfrak{n}2})\cdot \hat{N}}\,, \tag{5.56}$$

gde je $\epsilon_{\mathfrak{n}} \cdot \hat{N} = \epsilon_{\mathfrak{n}}{}^{a} \hat{N}_a$, odnosno generatori gejdž tansformacija komutiraju:

$$[\hat{N}_a, \hat{N}_b] = 0\,. \tag{5.57}$$

Odatle sledi, da je grupa $\tilde{N}$ Abelova, a indeksna struktura parametara i generatora pokazuje da je ona izomorfna grupi realnih brojeva $\mathbb{R}^q$, gde je q dimenzija grupe H. Dakle,

$$\tilde{N} \cong \mathbb{R}^q\,. \tag{5.58}$$

Zatim se može ispitati komutator N-gejdž transformacija sa komutatorima G, H i M-gejdž transformacija. Razmatranjem G-gejdž transformacija, dobijamo[1]

$$[\epsilon_{\mathfrak{g}} \cdot \hat{G}, \epsilon_{\mathfrak{n}} \cdot \hat{N}] = (\epsilon_{\mathfrak{g}} \triangleright \epsilon_{\mathfrak{n}}) \cdot \hat{N}\,, \tag{5.59}$$

dakle komutator G- i N-gejdž transformacija je:

$$[\hat{G}_\alpha, \hat{N}_a] = \triangleright_{\alpha a}{}^{b} \hat{N}_b\,. \tag{5.60}$$

[1]Dejstvo paraetra $\epsilon_{\mathfrak{g}}$ na parametar $\epsilon_{\mathfrak{n}}$, $\epsilon_{\mathfrak{g}} \triangleright \epsilon_{\mathfrak{n}}$, je definisano kao $\epsilon_{\mathfrak{g}} \triangleright \epsilon_{\mathfrak{n}} \equiv \triangleright_{\alpha a}{}^{b} \epsilon_{\mathfrak{g}}{}^{\alpha} \epsilon_{\mathfrak{n}}{}^{a}$. Sledi da je

$$(\epsilon_{\mathfrak{g}} \triangleright \epsilon_{\mathfrak{n}}) \cdot \hat{N} = \triangleright_{\alpha a}{}^{b} \epsilon_{\mathfrak{g}}{}^{\alpha} \epsilon_{\mathfrak{n}}{}^{a} \tilde{N}_b\,.$$

Ispitivanjem odnosa između N-gejdž transformacija i H-gejdž transformacija dobijamo relaciju,

$$e^{\epsilon_{\mathfrak{h}}\cdot\hat{H}}e^{\epsilon_{\mathfrak{n}}\cdot\hat{N}} - e^{\epsilon_{\mathfrak{n}}\cdot\hat{N}}e^{\epsilon_{\mathfrak{h}}\cdot\hat{H}} = -(\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}}) \cdot \hat{M}\,, \tag{5.61}$$

gde je dokaz dat u Dodatku D.2.3. Dobija se da je komutator između generatora H-gejdž transformacija i N-gejdž transformacija linearna kombinacija generatora M-gejdž transformacije:

$$[\hat{H}_a{}^{\mu}, \hat{N}^b] = \rhd_{\alpha a}{}^b \hat{M}^{\alpha\mu}\,. \tag{5.62}$$

Analogno tome, može se proveriti da važi

$$e^{\epsilon_{\mathfrak{m}}\cdot\hat{M}}e^{\epsilon_{\mathfrak{n}}\cdot\hat{N}} = e^{\epsilon_{\mathfrak{n}}\cdot\hat{N}}e^{\epsilon_{\mathfrak{m}}\cdot\hat{M}}\,, \tag{5.63}$$

što dovodi do zaključka da generatori M-gejdž transformacija i N-gejdž transformacija komutiraju, tj.

$$[\hat{M}_{\alpha}{}^{\mu}, \hat{N}_a] = 0\,. \tag{5.64}$$

Ovim smo završili izračunavanje algebre generatora gejdž transformacija u $2BF$ teoriji.

Ukupna gejdž grupa simetrije $2BF$ dejstva

Sumirajući rezultate prethodnih pododeljaka, može se zapisati algebra generatora ukupne grupe gejdž simetrije na sledeći način.

- Algebra $\mathfrak{g}$ grupe G 2-ukrštenog modula $(H \stackrel{\partial}{\to} G\,, \rhd\,,)$ zadovoljava komutacione relacije:

$$[\hat{G}_{\alpha}, \hat{G}_{\beta}] = f_{\alpha\beta}{}^{\gamma}\hat{G}_{\gamma}\,. \tag{5.65}$$

- Algebra grupe $\tilde{H}$ koja se sastoji iz generatora H-gejdž transformacija,

$$[\hat{H}_a{}^{\mu}, \hat{H}_b{}^{\nu}] = 0\,, \tag{5.66}$$

- Algebra generatora M-gejdž transformacija:

$$[\hat{M}_{\alpha}{}^{\mu}, \hat{M}_{\beta}{}^{\nu}] = 0\,. \tag{5.67}$$

- Algebra generatora N-gejdž transformacija:

$$[\hat{N}_a, \hat{N}_b] = 0\,. \tag{5.68}$$

- Komutatori između generatora grupa $\tilde{M}$ i $\tilde{N}$ glase:

$$[\hat{M}_{\alpha}{}^{\mu}, \hat{N}_a] = 0\,. \tag{5.69}$$

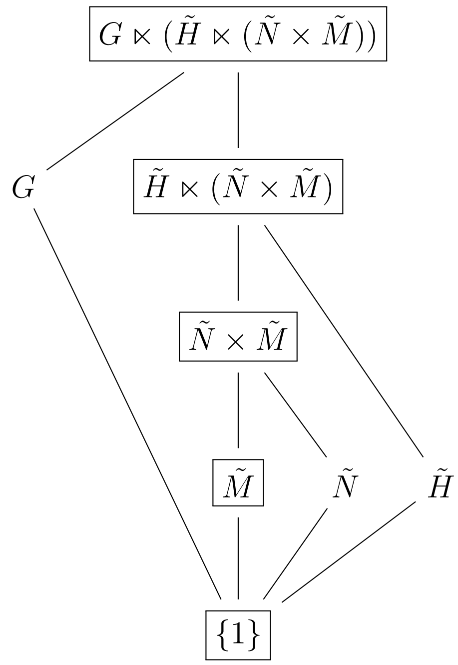
- Dejstvo generatora grupe $\tilde{H}$ na generatore M- i N-gejdž transformacija:

$$\begin{aligned} [\hat{H}_a{}^{\mu}, \hat{N}^b] &= \rhd_{\alpha a}{}^b \hat{M}^{\alpha\mu}\,, \\ [\hat{H}_a{}^{\mu}, \hat{M}_{\alpha}{}^{\nu}] &= 0\,, \end{aligned} \tag{5.70}$$

- Dejstvo generatora grupe G na generatore transformacija H-, M- i N-gejdž transformacija:

$$
\begin{aligned}
[\hat{G}_\alpha, \hat{H}_a{}^\mu] &= \triangleright_{\alpha a}{}^b \, \hat{H}_b{}^\mu \,, \\
[\hat{G}_\alpha \,, \hat{M}_\beta{}^\mu] &= f_{\alpha\beta}{}^\gamma \hat{M}_\gamma{}^\mu \,, \\
[\hat{G}_\alpha, \hat{N}_a] &= \triangleright_{\alpha a}{}^b \, \hat{N}_b \,.
\end{aligned} \tag{5.71}
$$

Na osnovu jednačina (5.65)-(5.71), može se analizirati struktura ukupne grupe simetrije. Na dijagramu Heseovog tipa prikazanom na slici 5.1, uključili smo samo relevantne podgrupe ukupne grupe simetrije $\mathcal{G}_{2BF}$, gde su *invarijantne podgrupe* uokvirene.



Slika 5.1: Relevantne podgrupe grupe simetrija $\mathcal{G}_{2BF}$. Invarijantne podgrupe su uokvirene.

Grupa M-gejdž transformacija $\tilde{M}$, grupa H-gejdž transformacija $\tilde{H}$ i grupa N-gejdž transformacija $\tilde{N}$ su podgrupe ukupne grupe simetrije $\mathcal{G}_{2BF}$. Grupa $\tilde{M}$ je invarijantna podgrupa, pošto su jedini netrivijalni komutatori između generatora $\hat{M}_\alpha{}^\mu$ i generatora grupe G, jednaki nekim linearnim kombinacijama generatora $\tilde{M}$. Grupa $\tilde{N}$ nije invarijantna podgrupa, pošto su komutator između generatora $\hat{N}_a$ i $\hat{H}_a{}^\mu$ linearne kombinacije generatora $\hat{M}_\alpha{}^\mu$. Međutim, generatori grupa $\tilde{N}$ i $\tilde{M}$ komutiraju, a grupa $\tilde{N}$ je invarijantna podgrupa direktnog proizvoda grupa $\tilde{M}$ i $\tilde{N}$. Dobijena grupa $\tilde{N} \times \tilde{M}$ je invarijantna podgrupa ukupne grupe simetrije.

Sa druge strane, grupa H-gejdž transformacija, zbog oblika komutatora generatora $\hat{H}_a{}^\mu$ i $\hat{N}_b$, nije invarijantna podgrupa ukupne grupe simetrija. Možemo pomnožiti ove dve podgrupe, od kojih je jedna invarijantna, a druga nije, koristeći semidirektan proizvod, pri čemu se dobija grupa $\tilde{H} \ltimes (\tilde{N} \times \tilde{M})$. Dobijena grupa je invarijantna podgrupa ukupne grupe simetrije $\mathcal{G}_{2BF}$.

Konačno, prateći istu liniju rezonovanja, dobijenu grupu i grupu G-gejdž transformacija možemo pomnožiti semidirektnim prozvodom, pri čemu dobijamo kompletnu grupu gejdž simetrija $\mathcal{G}_{2BF}$ kao:

$$
\mathcal{G}_{2BF} = G \ltimes (\tilde{H} \ltimes (\tilde{N} \times \tilde{M}))\,. \tag{5.72}
$$

Ovim je završena analiza grupe gejdž simetrija za 2BF teoriju.

Difeomorfizmi

Slično kao kod BF teorija, ako su difeomorfizmi simetrija teorije, njihove varijacije forme se mogu izraziti kao zbir varijacija formi varijabli pri gejdž transformacijama i varijacija formi pri HT transformacijama:

$$\delta_0{}^{\text{diff}}\,\phi = -\delta_0{}^{\text{gauge}}\phi - \delta_0{}^{\text{HT}}\phi\,. \tag{5.73}$$

Konkretno, $2BF$ dejstvo zavisi od parametara $\alpha^\alpha{}_\mu$, $\beta^a{}_{\mu\nu}$, $B^\alpha{}_{\mu\nu}$ i $C^a{}_\mu$. Parametri HT transformacija $\epsilon^{\text{HT}\alpha\beta}{}_{\mu\nu\rho}$ i $\epsilon^{\text{HT}ab}{}_{\mu\nu\rho}$ su definisani relacijama (4.55)

$$\begin{aligned}
\delta_0{}^{\text{HT}}\alpha^\alpha{}_\mu &= \frac{1}{2}\epsilon^{\text{HT}\alpha\beta}{}_{\mu\nu\rho}\frac{\delta S}{\delta B^\beta{}_{\nu\rho}}\,, & \delta_0{}^{\text{HT}}B^\alpha{}_{\mu\nu} &= -\epsilon^{\text{HT}\alpha\beta}{}_{\rho\mu\nu}\frac{\delta S}{\delta\alpha^\beta{}_\rho}\,,\\
\delta_0{}^{\text{HT}}\beta^a{}_{\mu\nu} &= \epsilon^{\text{HT}ab}{}_{\mu\nu\rho}\frac{\delta S}{\delta C^b{}_\rho}\,, & \delta_0{}^{\text{HT}}C^a{}_\mu &= -\frac{1}{2}\epsilon^{\text{HT}ab}{}_{\nu\rho\mu}\frac{\delta S}{\delta\beta^b{}_{\nu\rho}}\,,
\end{aligned} \tag{5.74}$$

dok su gejdž parametri $\epsilon_{\mathfrak{g}}{}^\alpha$, $\epsilon_{\mathfrak{h}}{}^a{}_\mu$, $\epsilon_{\mathfrak{m}}{}^\alpha{}_\mu$ i $\epsilon_{\mathfrak{n}}{}^a$ definisani u Teoremama 10–13. Možemo pokazati da zaista postoji izbor ovih parametara, tako da je jednačina (4.57) zadovoljena za sva polja. Konkretno, ako odaberemo gejdž parametre kao

$$\epsilon_{\mathfrak{g}}{}^\alpha = -\xi^\lambda\alpha^\alpha{}_\lambda\,,\quad \epsilon_{\mathfrak{h}}{}^a{}_\mu = \xi^\lambda\beta^a{}_{\mu\lambda}\,,\quad \epsilon_{\mathfrak{m}}{}^\alpha{}_\mu = \xi^\lambda B^\alpha{}_{\mu\lambda}\,,\quad \epsilon_{\mathfrak{n}}{}^a = -\xi^\lambda C^a{}_\lambda\,, \tag{5.75}$$

a HT parametre kao

$$\epsilon^{\text{HT}\alpha\beta}{}_{\mu\nu\rho} = \xi^\lambda g^{\alpha\beta}\epsilon_{\mu\nu\rho\lambda}\,,\qquad \epsilon^{\text{HT}ab}{}_{\mu\nu\rho} = \xi^\lambda g^{ab}\epsilon_{\lambda\mu\nu\rho}\,, \tag{5.76}$$

primenom jednačine (5.73) dobijamo upravo standardne varijacije formi koje odgovaraju difeomorfizmima:

$$\begin{aligned}
\delta_0{}^{\text{diff}}\alpha^\alpha{}_\mu &= -\partial_\mu\xi^\lambda\alpha^\alpha{}_\lambda - \xi^\lambda\partial_\lambda\alpha^\alpha{}_\mu\,,\\
\delta_0{}^{\text{diff}}\beta^a{}_{\mu\nu} &= -\partial_\mu\xi^\lambda\beta^a{}_{\lambda\nu} - \partial_\nu\xi^\lambda\beta^a{}_{\mu\lambda} - \xi^\lambda\partial_\lambda\beta^a{}_{\mu\nu}\,,\\
\delta_0{}^{\text{diff}}B^\alpha{}_{\mu\nu} &= -\partial_\mu\xi^\lambda B^\alpha{}_{\lambda\nu} - \partial_\nu\xi^\lambda B^\alpha{}_{\mu\lambda} - \xi^\lambda\partial_\lambda B^\alpha{}_{\mu\nu}\,,\\
\delta_0{}^{\text{diff}}C^a{}_\mu &= -\partial_\mu\xi^\lambda C^a{}_\lambda - \xi^\lambda\partial_\lambda C^a{}_\mu\,.
\end{aligned} \tag{5.77}$$

Ovim se utvrđuje da su difeomorfizmi zaista simetrija teorije, čak i ako nisu sadržani u ukupnoj gejdž grupi simetrija $\mathcal{G}_{2BF}$, već u direktnom proizvodu ukupne grupe simetrija i HT grupe simetrija.

5.2 Opšta relativnost

Bitan primer strukture ukrštenog modula je vektorski prostor V sa grupom izometrija prostora O. Vektorski prostor V možemo da posmatramo kao Abelovu Lijevu grupu sa sabiranjem vektora kao grupnom operacijom, pa reprezentacija grupe O na prostoru V postaje dejstvo $\triangleright$ grupe O na grupu V. Za definisanje ukrštenog modula $(V \xrightarrow{\partial} O\,,\triangleright)$, neophodno je definisati još homomorfizam $\partial : V \to O$, tako da bude trivijalan, odnosno da svaki element V preslikava u jedinični element grupe O). *Poenkareova 2-grupa*, odnosno njoj ekvivalentan ukršteni modul, je konstruisana na ovaj način. Izbor Lijevih grupa je

$$G = SO(3,1)\,,\qquad H = \mathbb{R}^4\,, \tag{5.78}$$

preslikavanje ∂ je trivijalno, a dejstvo $\triangleright$ je prirodno dejstvo grupe $SO(3,1)$ na $\mathbb{R}^4$, definisano jednačinom

$$M_{ab} \triangleright P_c = \eta_{[bc}P_{a]}\,, \tag{5.79}$$

gde su sa M_{ab} i P_a označeni generatori grupa $SO(3,1)$ i $\mathbb{R}^4$. Dejstvo $\triangleright$ grupe $SO(3,1)$ na samu sebe dato je konjugacijom, na osnovu definicije strukture ukrštenog modula. Na nivou algebre, dejstvo konjugacijom predstavlja pridruženu reprezentaciju, tako da je dejstvo zadato standardnim komutacionim relacijama za generatore $SO(3,1)$:

$$M_{ab} \triangleright M_{cd} = [M_{ab}, M_{cd}] \equiv \eta_{ad}M_{bc} - \eta_{ac}M_{bd} + \eta_{bc}M_{ad} - \eta_{bd}M_{ac} \,. \tag{5.80}$$

Zatim, 2-koneksija (α, β) je zadata parom diferencijalnih formi elementima algebri, 1-formom $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{so}(3,1))$ i 2-formom $\beta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{R}^4)$

$$\alpha = \omega^{ab}M_{ab} \,, \qquad \beta = \beta^a P_a \,, \tag{5.81}$$

gde je ω^{ab} spinska koneksija. Odgovarajuća 2-krivina, uređeni par $(\mathcal{F}, \mathcal{G})$ dat je izrazima:

$$\begin{aligned} \mathcal{F} &= (\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb})M_{ab} &\equiv R^{ab}M_{ab} \,, \\ \mathcal{G} &= (\mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b)P_a &\equiv \nabla\beta^a P_a \equiv G^a P_a \,, \end{aligned} \tag{5.82}$$

Primetimo da je, kako je homomorfizam ∂ trivijalan, "lažna" krivina jednaka običnoj krivini. Definisanjem bilinearnih formi

$$\langle M_{ab}, M_{cd}\rangle_{\mathfrak{g}} = \eta_{a[c}\eta_{bd]} \,, \qquad \langle P_a, P_b\rangle_{\mathfrak{h}} = \eta_{ab} \,, \tag{5.83}$$

može se pokazati da se 1-forma C^a tranformiše na isti način kao 1-forma tetrade e^a pri Lorencovim transformacijama i difeomorfizmima, tj. da polja C^a možemo identifikovati sa tetradom. Imajući sve ovo u vidu, $2BF$ dejstvo (5.1) za Poenkareovu 2-grupu definiše se kao:

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a \,. \tag{5.84}$$

Primetimo da je prepoznavanje tetrada tj. njihova identifikacija kao polja $C^a \equiv e^a$, bio krucijalan korak, omogućavajući da polja tetrade budu eksplicitno prisutna u $2BF$ dejstvu za Poenkareovu grupu. Kako bi u dejstvo (5.84) uveli stepene slobode koji odgovaraju teoriji opšte relativnosti, nephodno je napisati dodatni član:

$$S = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd}e_c \wedge e_d\right) . \tag{5.85}$$

Ovde je λ_{ab} 2-forma Lagranževog množitelja, a l_p označava Plankovu dužinu. Variranjem dejstva (5.85) redom po varijablama B_{ab}, e_a, ω_{ab}, β_a i λ_{ab}, dobijaju se jednačine kretanja:

$$R_{ab} - \lambda_{ab} = 0 \,, \tag{5.86}$$

$$\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d = 0 \,, \tag{5.87}$$

$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} = 0 \,, \tag{5.88}$$

$$\nabla e_a = 0 \,, \tag{5.89}$$

$$B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd}e_c \wedge e_d = 0 \,. \tag{5.90}$$

Iz jednačina (5.89) i (5.90) sledi da je $\nabla B^{ab} = 0$, što dalje povlači, primenom jednačine (5.88), jednakost $e_{[a} \wedge \beta_{b]} = 0$. Pretpostavljajući da su tetrade invertibilne, $e \equiv \det(e^a{}_\mu) \neq 0$, može se pokazati da je ovaj zahtev ekvivalentan jednačini $\beta^a = 0$ [15]. Dalje, iz jednačina (5.86), (5.88), (5.89) i (5.90) dobijamo:

$$\lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu}\,, \quad \beta^a{}_{\mu\nu} = 0\,, \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \quad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,. \tag{5.91}$$

Ovde su Ričijevi koeficijenti rotacije dati izrazom

$$\triangle^{ab}{}_\mu \equiv \frac{1}{2}(c^{abc} - c^{cab} + c^{bca})e_{c\mu}\,, \tag{5.92}$$

gde je

$$c^{abc} = e^\mu{}_b e^\nu{}_c \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu\right)\,. \tag{5.93}$$

Poslednja jednačina predstavlja spin koneksiju ω^{ab} izraženu kao funkciju tetrada, što dalje povlači da i 2-formu R^{ab} možemo napisati u tom obliku. Preostala jednačina (5.87) predstavlja jednačine kretanja za tetrade:

$$\varepsilon_{abcd}R^{bc} \wedge e^d = 0\,. \tag{5.94}$$

Lako se uočava da ovaj izraz predstavlja ništa drugo no Ajnštanove jednačine kretanja zapisane na jeziku diferencijalnih formi:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\,.$$

Zaključujemo da je dejstvo (5.85) klasično ekvivalentno *opštoj teoriji relativnosti.*

5.3 Ajnštajn-Jang-Milsova teorija

Kao što smo već naveli, osnovna prednost teorije (5.85) nad Plebanski modelom leži upravo u činjenici da su polja tetrade eksplicitno prisutna u topološkom sektoru teorije. Ova činjenica nam omogućava da kuplujemo polja materije sa gravitacijom, kao što je urađeno u radu [15]. Ipak, moguće je kuplovati i Jang-Milsovo polje materije u okviru formalizma 2-grupe [16]. Naime, Poenkareovu 2-grupu možemo proširiti tako da obuhvati $SU(N)$ gejdž polja. Da bi se to uradilo biramo sledeće Lijeve grupe kao elemente ukrštenog modula

$$G = SO(3,1) \times SU(N)\,, \qquad H = \mathbb{R}^4\,, \tag{5.95}$$

dok se dejstvo $\triangleright$ grupe G bira na sledeći način. Kao što je to bio slučaj kod Poenkareove 2-grupe, grupa G deluje na samu sebe konjugacijom. Dejstvo grupe G na grupu H je takvo da pogrupa $SO(3,1)$ deluje na $\mathbb{R}^4$ vektorskom reprezentacijom (5.79), kao što je to bio slučaj kod Ponkareove 2-grupe, dok je dejstvo podgrupe $SU(N)$ na grupu H trivijalno

$$\tau_I \triangleright P_a = 0\,, \tag{5.96}$$

gde su τ_I generatori $SU(N)$ grupe. Preslikavanje $\partial : H \to G$ ostaje trivijalno. Oblik 2-koneksije (α,β) oslikava strukturu grupe G, pa na osnovu direktnog proizvoda imamo razlaganje koneksije α na dva sabirka, od kojih svaki odgovara jednoj podgrupi:

$$\alpha = \omega^{ab}M_{ab} + A^I\tau_I\,, \qquad \beta = \beta^a P_a\,, \tag{5.97}$$

gde A^I označava 1-formu gejdž koneksije. Element $\mathcal{F}$ uređenog para 2-krivine $\mathcal{F},\mathcal{G}$ postaje

$$\mathcal{F} = R^{ab}M_{ab} + F^I\tau_I\,, \qquad F^I \equiv \mathrm{d}A^I + f_{JK}{}^I A^J \wedge A^K\,, \tag{5.98}$$

dok element $\mathcal{G}$ ostaje isti kao u slučaju Poenkareove 2-grupe, što sledi iz dejstva (5.96). Najzad, struktura direktnog proizvoda prisutna u grupi G znači da se Kilingova forma $\langle \_ , \_ \rangle_{\mathfrak{g}}$ razdvaja na dve podgrupe, odnosno na Kilingove forme za $SO(3,1)$ i $SU(N)$, odnosno da važi

$$\langle M_{ab}, M_{cd} \rangle = \eta_{a[c|}\eta_{b|d]} \,, \quad \langle \tau_I, \tau_J \rangle = \delta_{IJ} \,, \quad \langle M_{ab}, \tau_I \rangle_{\mathfrak{g}} = 0 \,. \tag{5.99}$$

Na osnovu definicije ukrštenog modula sledi da je Kilingova forma definisana na ovaj način invarijantna na G-gejdž transformacije i na H-gejdž transformacije. Topološko $2BF$ dejstvo (5.1) definisano za ovakav izbor ukrštenog modula je dato izrazom

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a \,, \tag{5.100}$$

gde je $B^I \in \mathcal{A}^2(\mathcal{M}_4 \,, \mathfrak{su}(N))$ novi Lagranžev množitelj. Dejstvo (5.100) je topološko dejstvo i neophodno je dodati odgovarajuće veze kako bismo ga transformisali u dejstvo koje opisuje teoriju sa odgovarajućom netrivijalnom dinamikom. Veza koja dovodi do jednačina kretanja za opštu relativnost data je izrazom (5.85), dok veza koja dovodi do odgovarajuće dimanike za gejdž polja data kao u dejstvu (4.63), pri čemu je učinjena smena $\delta^a \to e^a$. Dakle, dejstvo za Jang-Milsovo polje kulovano sa Ajnštajn-Kartanovom gravitacijom dato je izrazom:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a \\ & - \lambda_{ab} \wedge \Big(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \Big) + \lambda^I \wedge \Big(B_I - \frac{12}{g} M_{abI} e^a \wedge e^b \Big) \\ & + \zeta^{abI} \Big(M_{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - g_{IJ} F^J \wedge e_a \wedge e_b \Big) \,. \end{aligned} \tag{5.101}$$

Vidimo da je veza koja dovodi do pojave Jang-Milsovog polja u zakrivljenom prostorvremenu u dejstvu (5.101) dobijena iz dejstva za Jang-Milsovo polje u prostoru Minkovskog (4.63) zamenom nedinamičkog pozadinskog polja δ^a prisutnog u dejstvu (4.63) sa dinamičkim poljem tetrade e^a. Veza između ova dva polja već je nagoveštena jednačinom (4.64), koja opisuje vezu između δ^a i metrike ravnog prostora Minkovskog $\eta_{\mu\nu}$. Posle zamene ovog polja poljem e^a, ovo polje postaje dinamičko zbog gravitacionog sektora dejstva, dok jednačina (4.64) postaje uobičajena relacija koja povezuje polja tetrade i prostorvremensku metriku,

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \,. \tag{5.102}$$

Ovom smenom dejstvo (5.101) postaje nezavisno od pozadine, kao što je očekivano u opštoj relativnosti. Napomenimo još jednom da je ova konstrukcija moguća na osnovu prisustva tetrada u topološkom sektoru dejstva (5.85).

Varirajem dejstva (5.101) redom po varijablama B_{ab}, ω_{ab}, β_a, λ_{ab}, ζ^{abI}, M_{abI}, B_I, λ^I, A^I i

e^a, dobijaju se jednačine kretanja:

$$R^{ab} - \lambda^{ab} = 0\,, \tag{5.103}$$

$$\nabla B^{ab} - e^{[a} \wedge \beta^{b]} = 0\,, \tag{5.104}$$

$$\nabla e^a = 0\,, \tag{5.105}$$

$$B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd} e^c \wedge e^d = 0\,, \tag{5.106}$$

$$M_{abI}\varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F_I \wedge e_a \wedge e_b = 0\,, \tag{5.107}$$

$$-\frac{12}{g}\lambda^I \wedge e^a \wedge e^b + \zeta^{abI}\varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f = 0\,, \tag{5.108}$$

$$F_I + \lambda_I = 0\,, \tag{5.109}$$

$$B_I - \frac{12}{g} M_{abI} e^a \wedge e^b = 0\,, \tag{5.110}$$

$$-\mathrm{d}B_I + B_K \wedge f_{JI}{}^K A^J + \mathrm{d}(\zeta_I^{ab} e_a \wedge e_b) - \zeta_K^{ab} e_a \wedge e_b \wedge f_{JI}{}^K A^J = 0\,, \tag{5.111}$$

$$\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d - \frac{24}{g} M_{abI}\lambda^I \wedge e^b + 4\zeta^{efI} M_{ef\,I}\varepsilon_{abcd} e^b \wedge e^c \wedge e^d - 2\zeta_{ab}{}^I F_I \wedge e^b = 0\,. \tag{5.112}$$

Ovaj sistem jednačina opisuje dva dinamička polja e_a i A^I, dok sve ostale varijable možemo izraziti preko njih i njihovih izvoda, kao što sledi iz jednačina (5.103)–(5.110):

$$\begin{gathered}\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \quad \beta_{a\mu\nu} = 0\,, \quad \omega_{ab\mu} = \triangle_{ab\mu}\,, \quad \lambda_{abI} = F_{abI}\,, \quad B_{\mu\nu I} = -\frac{e}{2g}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}{}_I\,, \\ B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd} e^c{}_\mu e^d{}_\nu\,, \quad M_{abI} = -\frac{1}{4eg}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma\,, \quad \zeta^{abI} = \frac{1}{4eg}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma\,.\end{gathered} \tag{5.113}$$

Korišćenjem ovih izraza za varijable u jednačinama kretanja (5.111) i (5.112) dobijamo jednačinu kretanja za A^I:

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + \Gamma^\rho{}_{\lambda\rho} F^{I\lambda\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0\,, \tag{5.114}$$

gde je $\Gamma^\lambda{}_{\mu\nu}$ standardna oznaka za Levi-Čivita koneksiju, i jednačinu kretanja za e^a

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2\, T^{\mu\nu}\,, \tag{5.115}$$

gde je

$$T^{\mu\nu} \equiv -\frac{1}{4g}\left(F_{\rho\sigma}{}^I F^{\rho\sigma}{}_I g^{\mu\nu} + 4F^{\mu\rho}{}_I F_\rho{}^{\nu I}\right)\,. \tag{5.116}$$

Sistem jednačina (5.113)–(5.116) ekvivalentan je sistemu jednačina (5.103)–(5.112). Primetimo da smo ponovo dobili identitet $\beta^a = 0$, kao što je to bio slučaj kod čiste gravitacije.

Generalizacija izbora gejdž grupe za Jang-Milsovu teoriju sa $SU(N)$ na kompleksiji slučaj, recimo $SU(3) \times SU(2) \times U(1)$, je pravolinijska.

Glava 6

$3BF$ teorija

Premda je struktura 2-grupe uspešno primenjena za opisivanje gravitacionog i gejdž polja, nedovoljna je da opiše ostala polja materije, kao što su skalarno i fermionsko polje. Da bi opisali ova polja neophodno je izvršiti još jedan korak kategorijskih lestvica, kategorijskom generalizacijom algebarske strukture 2-grupe na strukturu 3-grupe. Ispostaviće se da struktura 3-grupe uspešno opisuje sva polja prisutna u Standardnom Modelu kuplovana sa gravitacijom. Pored toga, struktura 3-grupe poseduje i treći tip gejdž transformacija, koji je novitet u odnosu na strukturu 2-grupe i odgovara izboru skalarnih i fermionskih polja prisutnih u teoriji. Ovaj neočekivan i intrigantan rezultat detaljno je analiziran u [16].

Struktura ovog poglavlja prati strukturu prethodnih poglavlja 4 i 5 u kojima su razmatrane BF i $2BF$ teorija. Najpre, u odeljku 6.1 ćemo definisati i analizirati simetrije $3BF$ topološkog dejstva. Pododeljak 6.1.1 sadrži Hamiltonovu analizu za $3BF$ teoriju, koja rezultuje kompletnom kanonskom strukturom teorije, kao i vezama prve klase i vezama druge klase prisutnim u teoriji i njihovom algebrom. Zatim, na osnovu ovih rezultata, u nastavku pododeljka 6.1.1 analiziramo Bjankijeve identitete koje zadovoljavaju veze prve klase, koji smanjuju broj nezavisnih veza prve klase prisutnih u teoriji. Konačno, sumiranjem ovih rezultata dobijen je broj lokalnih propagirajućih stepeni slobode prisutnih u $3BF$ teoriji, tj. da je $3BF$ teorija topološka teorija. Konačno, ovaj pododeljak se završava konstrukcijom generatora gejdž simetrija za topološku teoriju, na osnovu Kastelanijeve procedure za konstrukciju generatora čiji su računski detalji prikazani u Dodatku D.3.2, i izračunavanjem varijacija formi za varijable prisutne u teoriji i njihove konjugovane impulse.

Pododeljak 6.1.2 sadrži glavne rezultate našeg rada i posvećen je analizi gejdž simetrija $3BF$ dejstva. Na osnovu rezultata prethodnog pododeljka, varijacija svih varijabli i njihovih kanonskih impulsa, a imajući u vidu da ove varijacije predstavljaju infinitezimalne transformacije gejdž simetrije na nekoj prostornoj hiperpovrši Σ_3 koje odgovaraju nultoj vremenskoj komponenti parametra transformacija, možemo ekstrapolirati infinitezimalnu transformaciju varijabli na čitavom prostorvremenu. Zatim, za ove infinitezimalne transformacije pogođen je oblik konačnih transformacija i na taj način je dobijeno pet vrsta gejdž transformacija u teoriji – već poznate G-gejdž, H-gejdž i L-gejdž transformacije, kao i M-gejdž i N-gejdž transformacije koje predstavljaju nov rezultat. Analiza transformacija simetrija, tj. izračunavanje komutatora generatora ovih transformacija, nam ukazuje na jednu bitnu razliku u odnosu na $2BF$ teoriju, a to je da u $3BF$-teoriji H-gejdž transformacije ne čine grupu. Videćemo da u strukturi $3BF$ teorije bitnu ulogu igra gejdž grupa $\tilde{H}_L$ koju čine H-gejdž i L-gejdž transformacije. Računski detalji izračunavanja komutatora prikazani su u Dodatku D.3.3. Rezultati ovog pododeljka su na kraju sumirani u kompletnoj strukturi gejdž grupe simetrije.

Zatim, modifikacijom topološkog $3BF$ dejstva dodavanjem odgovarajućih veza, formirana su $3BF$ *dejstva sa vezama* koja opisuju teorije sa netrivijalnom dinamikom. Videćemo u odeljcima 6.2 i 6.3 kako se *Klajn-Gordonovo* i *Dirakovo polje* u zakrivljenom prostoru mogu zapisati u formi 3BF dejsva sa vezama. Radi kompletnosti, u 6.4 analizirana su i *Vajlova* i *Majorana polja* u interakciji sa Ajnštajn-Kartanovom gravitacijom. U odeljku 6.5 videćemo kako se svi ovi rezultati mogu primeniti za konstrukciju 3BF dejstva sa vezama koje opisuje svu materiju prisutnu u Standardnom Modelu kuplovanu sa gravitacionim poljem. Na kraju ovog poglavlja, predstavljen je jednostavan model koji opisuje skalarnu elektrodinamiku kao 3BF teoriju sa vezama, dok je u Dodatku C urađena kompletna Hamiltonova analiza ove teorije.

Zaključujemo da su gravitacija, gejdž polja i polja materije uspešno obuhvaćena formalizmom 3-grupe. Klasična teorija time je uspešno zapisana u obliku prilagođenom za kovarijantnu kvantizacionu proceduru.

6.1 Topološka 3BF teorija

Slično kao kod BF i 2BF dejstva, definišemo gejdž invarijantno topološko 3BF dejstvo za mnogostrukost $\mathcal{M}_4$ i 2-ukršteni modul $(L \xrightarrow{\delta} H \xrightarrow{\partial} G \,, \triangleright \,, \{\_\,,\_\})$:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,. \tag{6.1}$$

U prethodnoj jednačini 2-forma $\mathcal{F} \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$, 3-forma $\mathcal{G} \in \mathcal{A}^3(\mathcal{M}_4\,, \mathfrak{h})$ i 4-forma $\mathcal{H} \in \mathcal{A}^4(\mathcal{M}_4\,, \mathfrak{l})$ označavaju komponente 3-krivine definisane jednačinom (2.118). Pored Lagranževih množitelja 2-forme $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ i 1-forme $C \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{h})$ prisutnih i u 2BF teoriji, u 3BF teoriji imamo i Lagranžev množitelj $D \in \mathcal{A}^0(\mathcal{M}_4\,, \mathfrak{l})$ koji ima interesantnu fizičku interpretaciju koju ćemo diskutovati kasnije. Zagrade $\langle \_\,,\_ \rangle_{\mathfrak{g}}$, $\langle \_\,,\_ \rangle_{\mathfrak{h}}$ i $\langle \_\,,\_ \rangle_{\mathfrak{l}}$ označavaju G-invarijantne bilinarne simetrične nedegenerisane forme algebri $\mathfrak{g}$, $\mathfrak{h}$ i $\mathfrak{l}$.

Variranjem 3BF topološkog dejstva (6.1) po varijablama B^α, C^a i D^A (gde indeksi A prebrojavaju generatore grupe L), dobijaju se jednačine kretanja:

$$\mathcal{F}^\alpha = 0 \,, \qquad \mathcal{G}^a = 0 \,, \qquad \mathcal{H}^A = 0 \,, \tag{6.2}$$

dok se variranjem dejstva po varijablama α^α, β^a i γ^A dobijaju:

$$\mathrm{d}B_\alpha - f_{\alpha\beta}{}^\gamma B_\gamma \wedge \alpha^\beta - \triangleright_{\alpha a}{}^b C_b \wedge \beta^a + \triangleright_{\alpha B}{}^A D_A \wedge \gamma^B = 0 \,, \tag{6.3}$$

$$\mathrm{d}C_a - \partial_a{}^\alpha B_\alpha + \triangleright_{\alpha a}{}^b C_b \wedge \alpha^\alpha + 2X_{\{ab\}}{}^A D_A \wedge \beta^b = 0 \,, \tag{6.4}$$

$$\mathrm{d}D_A - \triangleright_{\alpha A}{}^B D_B \wedge \alpha^\alpha + \delta_A{}^a C_a = 0 \,. \tag{6.5}$$

6.1.1 Hamiltonova analiza topološke 3BF teorije

U ovom odeljku prikazaćemo kompletnu Hamiltonovu analizu topološke 3BF teorije [19]. Pretpostavljajući da je prostorvremenska mnogostrukost $\mathcal{M}_4$ globalno hiperbolička možemo da definišemo Lagranžijan na prostornoj folijaciji Σ_3 za 3BF dejstvo:

$$L_{3BF} = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \epsilon^{\mu\nu\rho\sigma} \Big(\frac{1}{4}\, B^\alpha{}_{\mu\nu}\, \mathcal{F}^\beta{}_{\rho\sigma}\, g_{\alpha\beta} + \frac{1}{3!}\, C^a{}_\mu\, \mathcal{G}^b{}_{\nu\rho\sigma}\, g_{ab} + \frac{1}{4!} D^A \mathcal{H}^B{}_{\mu\nu\rho\sigma} g_{AB} \Big) \,. \tag{6.6}$$

Za Lagranžijan (6.6) konjugovani impulsi koji odgovaraju varijablama $B^\alpha{}_{\mu\nu}$, $\alpha^\alpha{}_\mu$, $C^a{}_\mu$, $\beta^a{}_{\mu\nu}$, D^A i $\gamma^A{}_{\mu\nu\rho}$, dobijeni varijacijom Lagranžijana po vremenskim izvodima varijabli, su:

$$\begin{aligned}
\pi(B)_\alpha{}^{\mu\nu} &= \frac{\delta L}{\delta \partial_0 B^a{}_{\mu\nu}} &&= 0\,,\\
\pi(\alpha)_\alpha{}^{\mu} &= \frac{\delta L}{\delta \partial_0 \alpha^\alpha{}_{\mu}} &&= \tfrac{1}{2}\epsilon^{0\mu\nu\rho}B_{\alpha\nu\rho}\,,\\
\pi(C)_a{}^{\mu} &= \frac{\delta L}{\delta \partial_0 C^a{}_{\mu}} &&= 0\,,\\
\pi(\beta)_a{}^{\mu\nu} &= \frac{\delta L}{\delta \partial_0 \beta^a{}_{\mu\nu}} &&= -\epsilon^{0\mu\nu\rho}C_{a\rho}\,,\\
\pi(D)_A &= \frac{\delta L}{\delta \partial_0 D^A} &&= 0\,,\\
\pi(\gamma)_A{}^{\mu\nu\rho} &= \frac{\delta L}{\delta \partial_0 \gamma^A{}_{\mu\nu\rho}} &&= \epsilon^{0\mu\nu\rho}D_A\,.
\end{aligned} \tag{6.7}$$

Kako relacije (6.7) ne mogu biti invertovane po vremenskim izvodima varijabli, zaključujemo da imamo sledeće primarne veze u teoriji:

$$\begin{aligned}
P(B)_\alpha{}^{\mu\nu} &\equiv \pi(B)_\alpha{}^{\mu\nu} \approx 0\,,\\
P(\alpha)_\alpha{}^{\mu} &\equiv \pi(\alpha)_\alpha{}^{\mu} - \tfrac{1}{2}\epsilon^{0\mu\nu\rho}B_{\alpha\nu\rho} \approx 0\,,\\
P(C)_a{}^{\mu} &\equiv \pi(C)_a{}^{\mu} \approx 0\,,\\
P(\beta)_a{}^{\mu\nu} &\equiv \pi(\beta)_a{}^{\mu\nu} + \epsilon^{0\mu\nu\rho}C_{a\rho} \approx 0\,,\\
P(D)_A &\equiv \pi(D)_A \approx 0\,,\\
P(\gamma)_A{}^{\mu\nu\rho} &\equiv \pi(\gamma)_A{}^{\mu\nu\rho} - \epsilon^{0\mu\nu\rho}D_A \approx 0\,.
\end{aligned} \tag{6.8}$$

Koristimo fundamentalnu Poasonovu zagradu definisanu na sledeći način,

$$\begin{aligned}
\{\, B^\alpha{}_{\mu\nu}(\vec{x})\,,\, \pi(B)_\beta{}^{\rho\sigma}(\vec{y})\,\} &= 2\delta^\alpha_\beta \delta^\rho_{[\mu}\delta^\sigma_{\nu]}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \alpha^\alpha{}_{\mu}(\vec{x})\,,\, \pi(\alpha)_\beta{}^{\nu}(\vec{y})\,\} &= \delta^\alpha_\beta \delta^\nu_\mu\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, C^a{}_{\mu}(\vec{x})\,,\, \pi(C)_b{}^{\nu}(\vec{y})\,\} &= \delta^a_b \delta^\nu_\mu\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \beta^a{}_{\mu\nu}(\vec{x})\,,\, \pi(\beta)_b{}^{\rho\sigma}(\vec{y})\,\} &= 2\delta^a_b\, \delta^\rho_{[\mu}\delta^\sigma_{\nu]}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, D^A(\vec{x})\,,\, \pi(D)_B(\vec{y})\,\} &= \delta^A_B\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \gamma^A{}_{\mu\nu\rho}(\vec{x})\,,\, \pi(\gamma)_B{}^{\sigma\tau\xi}(\vec{y})\,\} &= 3!\delta^A_B\, \delta^\sigma_{[\mu}\delta^\tau_\nu\delta^\xi_{\rho]}\,\delta^{(3)}(\vec{x}-\vec{y})\,,
\end{aligned} \tag{6.9}$$

da izračunamo *algebru primarnih veza*:

$$\begin{aligned}
\{\, P(B)_\alpha{}^{jk}(\vec{x})\,,\, P(\alpha)_\beta{}^{i}(\vec{y})\,\} &= \epsilon^{0ijk}\, g_{\alpha\beta}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, P(C)_a{}^{k}(\vec{x})\,,\, P(\beta)_b{}^{ij}(\vec{y})\,\} &= -\epsilon^{0ijk}\, g_{ab}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, P(D)_A(\vec{x})\,,\, P(\gamma)_B{}^{ijk}(\vec{y})\,\} &= \epsilon^{0ijk}\, g_{AB}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned} \tag{6.10}$$

Sve ostale Poasonove zagrade su jednake nuli. *Kanonski "on-shell" Hamiltonijan* je:

$$\begin{aligned} H_c = \int_\Sigma \mathrm{d}^3\vec{x}\Big[&\frac{1}{2}\pi(B)_\alpha{}^{\mu\nu}\,\partial_0 B^\alpha{}_{\mu\nu} + \pi(\alpha)_\alpha{}^\mu\,\partial_0\alpha^\alpha{}_\mu + \pi(C)_a{}^\mu\,\partial_0 C^a{}_\mu \\ &+ \frac{1}{2}\pi(\beta)_a{}^{\mu\nu}\,\partial_0\beta^a{}_{\mu\nu} + \pi(D)_A\,\partial_0 D^A + \frac{1}{3!}\pi(\gamma)_A{}^{\mu\nu\rho}\,\partial_0\gamma^A{}_{\mu\nu\rho}\Big] - L\,. \end{aligned} \tag{6.11}$$

Raspisivajem 3-krivine, odnosno zamenom relacija (2.3.5), Hamiltonijan (6.11) prepisujemo u formi članova koji su jednaki proizvodu primarnih veza i vremenskih izvoda varijabli i ostatka. Primarne veze su nula "on-shell" tako da kanonski Hamiltonijan postaje:

$$\begin{aligned} H_c \approx -\int_\Sigma \mathrm{d}^3\vec{x}\,\epsilon^{0ijk}\Big[&\frac{1}{2}B_{\alpha 0i}\,\mathcal{F}^\alpha{}_{jk} + \frac{1}{6}C_{a0}\,\mathcal{G}^a{}_{ijk} + \beta^a{}_{0i}\Big(\nabla_j C_{ak} - \frac{1}{2}\partial_a{}^\alpha B_{\alpha\,jk} + \beta^b{}_{jk}\,D_A\,X_{\{ab\}}{}^A\Big) \\ &+ \frac{1}{2}\alpha^\alpha{}_0\Big(\nabla_i B_{\alpha\,jk} - C_{ai}\,\rhd_{\alpha b}{}^a\,\beta^b{}_{jk} + \frac{1}{3}D_A \rhd_{\alpha B}{}^A\,\gamma^B{}_{ijk}\Big) + \frac{1}{2}\gamma^A{}_{0ij}\Big(\nabla_k D_A + C_{ak}\,\delta_A{}^a\Big)\Big]\,. \end{aligned} \tag{6.12}$$

Dodavanjem proizvoda Lagranževih množitelja λ i primarnih veza za svaku vezu možemo da dobijemo "off-shell" totalni Hamiltonijan:

$$\begin{aligned} H_T \;=\; H_c + \int \mathrm{d}^3\vec{x}\Big[&\frac{1}{2}\lambda(B)^\alpha{}_{\mu\nu}P(B)_\alpha{}^{\mu\nu} + \lambda(\alpha)^\alpha{}_\mu P(\alpha)_\alpha{}^\mu + \lambda(C)^a{}_\mu P(C)_a{}^\mu \\ &+\frac{1}{2}\lambda(\beta)^a{}_{\mu\nu}P(\beta)_a{}^{\mu\nu} + \lambda(D)^A P(D)_A + \frac{1}{3!}\lambda(\gamma)^A{}_{\mu\nu\rho}P(\gamma)_A{}^{\mu\nu\rho}\Big]\,. \end{aligned} \tag{6.13}$$

Kako bi primarne veze bile očuvane u toku evolucije sistema one moraju da zadovoljavaju uslove konzistentnosti

$$\dot{P} \equiv \{\,P\,,\,H_T\,\} \approx 0\,, \tag{6.14}$$

za svaku primarnu vezu P. Korišćenjem (6.14) za primarne veze $P(B)_\alpha{}^{0i}$, $P(\alpha)_\alpha{}^0$, $P(C)_a{}^0$, $P(\beta)_a{}^{0i}$ i $P(\gamma)_A{}^{0ij}$ dobijamo sekundarne veze $\mathcal{S}$:

$$\begin{aligned} \mathcal{S}(\mathcal{F})_\alpha{}^i &\equiv \tfrac{1}{2}\epsilon^{0ijk}\mathcal{F}_{\alpha jk} \approx 0\,, \\ \mathcal{S}(\nabla B)_\alpha &\equiv \frac{1}{2}\epsilon^{0ijk}\big(\nabla_{[i}B_{\alpha\,j]k} - C_{a[i}\,\rhd_{\alpha b}{}^a\,\beta^b{}_{j]k} + \frac{1}{3}D_A \rhd_{\alpha B}{}^A\,\gamma^B{}_{ijk}\big) \approx 0\,, \\ \mathcal{S}(\mathcal{G})_a &\equiv \frac{1}{6}\epsilon^{0ijk}\mathcal{G}_{aijk} \approx 0\,, \\ \mathcal{S}(\nabla C)_a{}^i &\equiv \epsilon^{0ijk}\big(\nabla_{[j}C_{ak]} - \frac{1}{2}\partial_a{}^\alpha B_{\alpha\,jk} + \beta^b{}_{jk}\,D_A\,X_{\{ab\}}{}^A\big) \approx 0\,, \\ \mathcal{S}(\nabla D)_A{}^{ij} &\equiv \epsilon^{0ijk}\big(\nabla_k D_A + C_{ak}\,\delta_A{}^a\big) \approx 0\,. \end{aligned} \tag{6.15}$$

dok u slučaju primarnih veza $P(\alpha)_\alpha{}^k$, $P(B)_\alpha{}^{jk}$, $P(\beta)_a{}^{jk}$, $P(C)_a{}^k$, $P(\gamma)_A{}^{ijk}$ i $P(D)_A$ uslovi konzistentnosti određuju Lagranževe množitelje:

$$
\begin{aligned}
\lambda(B)_{\alpha ij} &\approx \nabla_i B_{\alpha 0j} - \nabla_j B_{\alpha 0i} + C_{a0}\beta^b{}_{ij}\rhd_{\alpha b}{}^a + C_{bi}\rhd_\alpha{}^b{}_a\beta^a{}_{0j} \\
&\quad - C_{bj}\rhd_\alpha{}^b{}_a\beta^a{}_{0i} + f_{\beta\gamma}{}^\alpha\alpha^\beta{}_0 B^\gamma{}_{ij} + D_B\gamma^A{}_{0ij}\rhd_\alpha{}^B{}_A, \\
\lambda(\alpha)^\alpha{}_i &\approx \nabla_i\alpha^\alpha{}_0 + \partial_a{}^\alpha\beta^a{}_{0i}, \\
\lambda(C)^a{}_i &\approx \nabla_i C^a{}_0 + C^b{}_i\rhd_\alpha{}^a{}_b\alpha^\alpha{}_0 - 2\beta_{b0i}D_A X^{\{ba\}A} + B_{\alpha 0i}\partial^{a\alpha}, \\
\lambda(\beta)^a{}_{ij} &\approx \nabla_i\beta^a{}_{0j} - \nabla_j\beta^a{}_{0i} - \beta^b{}_{ij}\rhd_{\alpha b}{}^a\alpha^\alpha{}_0 + \gamma^A{}_{0ij}\delta_A{}^a, \\
\lambda(D)_A &\approx \alpha^\alpha{}_0 D_B\rhd_{\alpha A}{}^B - C_{a0}\delta_A{}^a, \\
\lambda(\gamma)^A{}_{ijk} &\approx -2\beta^a{}_{0i}\beta^b{}_{jk}X_{\{ab\}}{}^A + 2\beta^a{}_{0j}\beta^b{}_{ik}X_{\{ab\}}{}^A - 2\beta^a{}_{0k}\beta^b{}_{ij}X_{\{ab\}}{}^A \\
&\quad - \alpha^\alpha{}_0\rhd_{\alpha B}{}^A\gamma^B{}_{ijk} + \nabla_i\gamma^A{}_{0jk} - \nabla_j\gamma^A{}_{0ik} + \nabla_k\gamma^A{}_{0ij}.
\end{aligned}
\tag{6.16}
$$

Preostali Lagranževi množitelji

$$
\lambda(B)^\alpha{}_{0i}\,, \qquad \lambda(\alpha)^\alpha{}_0\,, \qquad \lambda(C)^a{}_0\,, \qquad \lambda(\beta)^a{}_{0i}\,, \qquad \lambda(\gamma)^A{}_{0ij} \tag{6.17}
$$

ostaju neodređeni iz uslova konzistentnosti primarnih veza. Sekundarne veze takođe moraju biti očuvane, pa se zahtevaju i uslovi konzistentnosti sekundarnih veza, koji u ovom slučaju ne dovode do pojave novih veza u teoriji:

$$
\begin{aligned}
\{\mathcal{S}(\mathcal{F})^{\alpha i}\,,\, H_T\} &= f_{\beta\gamma}{}^\alpha\mathcal{S}(\mathcal{F})^{\beta i}\alpha^\gamma{}_0\,, \\
\{\mathcal{S}(\nabla B)_\alpha\,,\, H_T\} &= f_{\beta\gamma\alpha}B^\gamma{}_{0k}\mathcal{S}(\mathcal{F})^{\beta k} + f_{\beta\alpha}{}^\gamma\alpha^\beta{}_0\mathcal{S}(\nabla B)_\gamma + C_{a0}\rhd_{\alpha b}{}^a\mathcal{S}(\mathcal{G})^b \\
&\quad - \rhd_{\alpha a}{}^b\beta^a{}_{0k}\mathcal{S}(\nabla C)_b{}^k + \tfrac{1}{2}\rhd_\alpha{}^B{}_A\gamma^A{}_{0jk}S(\nabla D)_B{}^{jk}\,, \\
\{\mathcal{S}(\mathcal{G})^a\,,\, H_T\} &= \rhd_{\alpha b}{}^a\beta^b{}_{0k}\mathcal{S}(\mathcal{F})^{\alpha k} - \alpha^\alpha{}_0\rhd_{\alpha b}{}^a\mathcal{S}(\mathcal{G})^b\,, \\
\{\mathcal{S}(\nabla C)_a{}^i\,,\, H_T\} &= C_{b0}\rhd_\alpha{}^b{}_a\mathcal{S}(\mathcal{F})^{\alpha i} + \rhd_{\alpha a}{}^b\alpha^\alpha{}_0\mathcal{S}(\nabla C)_b{}^i + 2X_{\{ab\}}{}^A\beta^b{}_{0j}\mathcal{S}(\nabla D)_A{}^{ij}\,, \\
\{\mathcal{S}(\nabla D)_A{}^{ij}\,,\, H_T\} &= \alpha^\alpha{}_0\rhd_{\alpha A}{}^B\mathcal{S}(\nabla D)_B{}^{ij}\,.
\end{aligned}
\tag{6.18}
$$

Najzad, totalni Hamiltonijan može da se zapiše u sledećem obliku:

$$
\begin{aligned}
H_T = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\Big[&\lambda(B)^\alpha{}_{0i}\,\Phi(B)_\alpha{}^i + \lambda(\alpha)^\alpha\,\Phi(\alpha)_\alpha + \lambda(C)^a{}_0\,\Phi(C)_a + \lambda(\beta)^a{}_{0i}\,\Phi(\beta)_a{}^i + \frac{1}{2}\lambda(\gamma)^A{}_{0ij}\Phi(\gamma)_A{}^{ij} \\
&- B_{\alpha 0i}\,\Phi(\mathcal{F})^{ai} - \alpha_{\alpha 0}\,\Phi(\nabla B)^\alpha - C_{a0}\,\Phi(\mathcal{G})^a - \beta_{a0i}\,\Phi(\nabla C)^{ai} - \frac{1}{2}\gamma_{A0ij}\,\Phi(\nabla D)^{Aij}\Big]\,,
\end{aligned}
\tag{6.19}
$$

gde su

$$
\begin{aligned}
\Phi(B)_\alpha{}^i &= P(B)_\alpha{}^{0i}\,, \\
\Phi(\alpha)_\alpha &= P(\alpha)_\alpha{}^0\,, \\
\Phi(C)_a &= P(C)_a{}^0\,, \\
\Phi(\beta)_a{}^i &= P(\beta)_a{}^{0i}\,, \\
\Phi(\gamma)_A{}^{ij} &= P(\gamma)_A{}^{0ij}\,, \\
\Phi(\mathcal{F})^{\alpha i} &= \mathcal{S}(\mathcal{F})^{\alpha i} - \nabla_j P(B)^{\alpha ij} - P(C)_a{}^i \partial^{a\alpha}\,, \\
\Phi(\mathcal{G})_a &= \mathcal{S}(\mathcal{G})_a + \nabla_i P(C)_a{}^i - \frac{1}{2}\beta_{bij} \rhd_\alpha{}^b{}_a P(B)^{\alpha ij} + P(D)^A \delta_{Aa}\,, \\
\Phi(\nabla C)_a{}^i &= \mathcal{S}(\nabla C)_a{}^i - \nabla_j P(\beta)_a{}^{ij} + C_{bj} \rhd{}_\alpha{}^b{}_a P(B)^{\alpha ij} \\
&\quad -\partial_a{}^\alpha P(\alpha)_\alpha{}^i + 2D_A X_{\{ab\}}{}^A P(C)^{bi} + \beta^b{}_{jk} X_{\{ab\}}{}^A P(\gamma)_A{}^{ijk}\,, \\
\Phi(\nabla B)_\alpha &= \mathcal{S}(\nabla B)_\alpha + \nabla_i P(\alpha)_\alpha{}^i - \frac{1}{2} f_{\alpha\gamma}{}^\beta B_{\beta ij} P(B)^{\gamma ij} \\
&\quad -C_{bi} \rhd_{\alpha a}{}^b P(C)^{ai} - \frac{1}{2}\beta_{bij} \rhd_{\alpha a}{}^b P(\beta)^{aij} \\
&\quad -P(D)^A D_B \rhd_{\alpha A}{}^B + \frac{1}{3!} P(\gamma)_A{}^{ijk} \gamma^B{}_{ijk} \rhd_{\alpha B}{}^A\,, \\
\Phi(\nabla D)_A{}^{ij} &= \mathcal{S}(\nabla D)_A{}^{ij} + \nabla_k P(\gamma)_A{}^{ijk} - P(\beta)_a{}^{ij} \delta_A{}^a - P(B)^{\alpha ij} \rhd_\alpha{}^B{}_A D_B\,,
\end{aligned}
\tag{6.20}
$$

veze prve klase, a

$$
\begin{aligned}
\chi(B)_\alpha{}^{jk} = P(B)_\alpha{}^{jk}\,, \qquad \chi(C)_a{}^i = P(C)_a{}^i\,, \qquad \chi(D)_A = P(D)_A\,, \\
\chi(\alpha)_\alpha{}^i = P(\alpha)_\alpha{}^i\,, \qquad \chi(\beta)_a{}^{ij} = P(\beta)_a{}^{ij}\,, \qquad \chi(\gamma)_A{}^{ijk} = P(\gamma)_A{}^{ijk}\,,
\end{aligned}
\tag{6.21}
$$

veze druge klase.

Poasonova algebra veza prve klase je:

$$
\begin{aligned}
\{\,\Phi(\mathcal{G})^a(\vec{x})\,,\,\Phi(\nabla C)_b{}^i(\vec{y})\,\} &= -\rhd{}_{\alpha b}{}^a\,\Phi(\mathcal{F})^{\alpha i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla C)_a{}^i(\vec{x})\,,\,\Phi(\nabla C)_b{}^j(\vec{y})\,\} &= -2X_{\{ab\}}{}^A\,\Phi(\nabla D)_A{}^{ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\mathcal{G})^a(\vec{x})\,,\,\Phi(\nabla B)_\alpha(\vec{y})\,\} &= \rhd_{\alpha b}{}^a\,\Phi(\mathcal{G})^b(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla C)_a{}^i(\vec{x})\,,\,\Phi(\nabla B)_\alpha(\vec{y})\,\} &= \rhd_{\alpha a}{}^b\,\Phi(\nabla C)_b{}^i(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\mathcal{F})^\alpha{}_i(\vec{x})\,,\,\Phi(\nabla B)_\beta(\vec{y})\,\} &= f_{\beta\gamma}{}^\alpha\,\Phi(\mathcal{F})^\gamma{}_i(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla B)_\alpha(\vec{x})\,,\,\Phi(\nabla B)_\beta(\vec{y})\,\} &= f_{\alpha\beta}{}^\gamma\,\Phi(\nabla B)_\gamma(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla B)_\alpha(\vec{x})\,,\,\Phi(\nabla D)_A{}^{ij}(\vec{y})\,\} &= \rhd_{\alpha A}{}^B\Phi(\nabla D)_B{}^{ij}(\vec{x})\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{6.22}
$$

Poasonova zagrada veza prve klase sa vezama druge klase je:

$$
\begin{aligned}
\{\,\Phi(\mathcal{F})^{\alpha i}(\vec{x})\,,\,\chi(\alpha)_{\beta}{}^{j}(\vec{y})\,\} &= -f_{\beta\gamma}{}^{\alpha}\,\chi(B)^{\gamma ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\mathcal{G})^{a}(\vec{x})\,,\,\chi(\alpha)_{\alpha}{}^{i}(\vec{y})\,\} &= -\rhd_{\alpha b}{}^{a}\,\chi(C)^{bi}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\mathcal{G})^{a}(\vec{x})\,,\,\chi(\beta)_{b}{}^{ij}(\vec{y})\,\} &= \rhd_{\alpha b}{}^{a}\,\chi(B)^{\alpha ij}(\vec{x})\,\delta^{(3)}(x-y)\,,\\
\{\,\Phi(\nabla B)^{ai}(\vec{x})\,,\,\chi(\alpha)_{\alpha}{}^{j}(\vec{y})\,\} &= -\rhd_{\alpha b}{}^{a}\,\chi(\beta)^{bij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{ai}(\vec{x})\,,\,\chi(\beta)_{b}{}^{jk}(\vec{y})\,\} &= -2X^{\{ac\}A}g_{bc}\,\chi(\gamma)_{A}{}^{ijk}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla C)^{ai}(\vec{x})\,,\,\chi(C)_{b}{}^{j}(\vec{y})\,\} &= \rhd^{a}{}_{\alpha b}\,\chi(B)^{\alpha ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{ai}(\vec{x})\,,\,\chi(D)_{A}(\vec{y})\,\} &= X^{\{ab\}}{}_{A}\,\chi(C)_{b}{}^{i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(\alpha)_{\beta}{}^{i}(\vec{y})\,\} &= f_{\beta\gamma}{}^{\alpha}\,\chi(\alpha)^{\gamma i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(\beta)_{a}{}^{ij}(\vec{y})\,\} &= \rhd_{\alpha a}{}^{b}\,\chi(\beta)_{b}{}^{ij}(\vec{x})\,\delta^{(3)}(x-y)\,,\\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(\gamma)_{A}{}^{ijk}(\vec{y})\,\} &= -\rhd_{\alpha A}{}^{B}\,\chi(\gamma)_{B}{}^{ijk}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(B)_{\beta}{}^{ij}(\vec{y})\,\} &= -f_{\beta\gamma}{}^{\alpha}\,\chi(B)^{\gamma ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,.\\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(C)_{a}{}^{i}(\vec{y})\,\} &= -\rhd^{\alpha b}{}_{a}\,\chi(C)_{b}{}^{i}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,.\\
\{\,\Phi(\nabla B)^{\alpha}(\vec{x})\,,\,\chi(D)_{A}(\vec{y})\,\} &= -\rhd^{\alpha B}{}_{A}\,\chi(D)_{B}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla D)^{Aij}(\vec{x})\,,\,\chi(\alpha)_{\alpha}{}^{k}\,\} &= -\rhd_{\alpha B}{}^{A}\chi(\gamma)^{Bijk}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla D)^{Aij}(\vec{x})\,,\,\chi(D)_{B}\,\} &= \frac{1}{2}\rhd_{\alpha B}{}^{A}\chi(B)^{\alpha ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{6.23}
$$

Najzad, korisno je izračunati Poasonovu zagradu između veza prve klase i Hamiltonijana:

$$
\begin{aligned}
\{\Phi(\mathcal{F})^{\alpha i}\,,\,H_T\} &= f_{\beta\gamma}{}^{\alpha}\Phi(\mathcal{F})^{\beta i}\alpha^{\gamma}{}_{0}\,,\\
\{\Phi(\nabla B)_{\alpha}\,,\,H_T\} &= f_{\beta\gamma\alpha}B^{\gamma}{}_{0k}\Phi(\mathcal{F})^{\beta k}+f_{\beta\alpha}{}^{\gamma}\alpha^{\beta}{}_{0}\Phi(\nabla B)_{\gamma}+C_{a0}\rhd_{\alpha b}{}^{a}\Phi(\mathcal{G})^{b}\\
&\quad-\rhd_{\alpha a}{}^{b}\beta^{a}{}_{0k}\Phi(\nabla C)_{b}{}^{k}+\frac{1}{2}\rhd_{\alpha}{}^{B}{}_{A}\gamma^{A}{}_{0jk}\Phi(\nabla D)_{B}{}^{jk}\,,\\
\{\Phi(\mathcal{G})^{a}\,,\,H_T\} &= \rhd_{\alpha b}{}^{a}\beta^{b}{}_{0k}\Phi(\mathcal{F})^{\alpha k}-\alpha^{\alpha}{}_{0}\rhd_{\alpha b}{}^{a}\Phi(\mathcal{G})^{b}\,,\\
\{\Phi(\nabla C)_{a}{}^{i}\,,\,H_T\} &= C_{b0}\rhd_{\alpha}{}^{b}{}_{a}\Phi(\mathcal{F})^{\alpha i}+\rhd_{\alpha a}{}^{b}\alpha^{\alpha}{}_{0}\Phi(\nabla C)_{b}{}^{i}+2X_{\{ab\}}{}^{A}\beta^{b}{}_{0j}\Phi(\nabla D)_{A}{}^{ij}\,,\\
\{\Phi(\nabla D)_{A}{}^{ij}\,,\,H_T\} &= \alpha^{\alpha}{}_{0}\rhd_{\alpha A}{}^{B}\Phi(\nabla D)_{B}{}^{ij}\,.
\end{aligned}
\tag{6.24}
$$

Broj stepeni slobode topološke 3*BF* teorije

Bjankijevi identiteti (BI) za 1-forme α i C i 2-forme β i B dati su izrazima kao u slučaju $2BF$ teorije (5.24), (5.25), (5.26) i (5.27). Osim njih, u $3BF$ teoriji postoji Bjankijev identitet koji odgovara 0-formi D.

Lema 11 (BI za 0-formu D) *Definisanjem varijable*

$$
Q^{A}=\mathrm{d}D^{A}+\rhd_{\alpha B}{}^{A}\alpha^{\alpha}\wedge D^{B}\,,
\tag{6.25}
$$

dobijamo da važi sledeći identitet:

$$\epsilon^{\lambda\mu\nu\rho}\left(\nabla_\nu Q^A{}_\rho - \rhd_{\alpha B}{}^A F^\alpha{}_{\nu\rho} D^B\right) = 0\,. \tag{6.26}$$

Bjankijevi identiteti igraju važnu ulogu u određivanju broja stepeni slobode u dejstvu.

Obeležimo sa p dimenzionalnost grupe G, sa q dimenzionalnost grupe H i sa r dimenzionalnost grupe L. Sada možemo dobiti broj inicijalnih polja u $3BF$ teoriji prebrojavanjem polja navedenih u tabeli 6.1.

$\alpha^\alpha{}_\mu$	$\beta^a{}_{\mu\nu}$	$\gamma^A{}_{\mu\nu\rho}$	$B^\alpha{}_{\mu\nu}$	$C^a{}_\mu$	D^A
$4p$	$6q$	$4r$	$6p$	$4q$	r

Tabela 6.1: Inicijalna polja u $3BF$ teoriji.

Dobijamo da je $N = 10(p+q) + 5r$. Slično se može odrediti broj nazavisnih komponenata veza druge klase prebrojavanjem komponenti veza prikazanih u tabeli (6.2).

$\chi(B)_\alpha{}^{jk}$	$\chi(C)_a{}^i$	$\chi(D)_A$	$\chi(\alpha)_\alpha{}^i$	$\chi(\beta)_a{}^{ij}$	$\chi(\gamma)_A{}^{ijk}$
$3p$	$3q$	r	$3p$	$3q$	r

Tabela 6.2: Veze druge klase u $3BF$ teoriji.

Dobijamo da je $S = 6(p+q) + 2r$.

Veze prve klase nisu sve međusobno nezavisne. Osim identiteta (5.30) i (5.32) koji su zadovoljeni i u slučaju $2BF$ teorije, u $3BF$ teoriji pojavljuje se novi identitet. Naime, veze prve klase zadovoljavaju:

$$\begin{aligned}\nabla_i\Phi(\nabla D)_A{}^{ij} + \frac{1}{2}\delta_A{}^a\mathcal{S}(\nabla C)_a{}^j - \nabla_i\nabla_k\chi(\gamma)_A{}^{ijk} + \frac{1}{2}\chi(\beta)_a{}^{ij}\delta_A{}^a - \frac{1}{2}\rhd_\alpha{}^B{}_A D_B\Phi(\mathcal{F})_\alpha{}^j \\ + \frac{1}{2}\rhd_\alpha{}^B{}_A D_B\partial_a{}^\alpha\chi(C)^{aj} = \frac{1}{2}\epsilon^{0ijk}(\nabla_i Q_{Ak} + \rhd_{\alpha A}{}^B F^\alpha{}_{ik} D_B)\,,\end{aligned} \tag{6.27}$$

odnosno, kako je desna strana upravo $\lambda = 0$ komponenta Bjankijevog identiteta (6.26), dobija se:

$$\begin{aligned}\nabla_i\Phi(\nabla D)_A{}^{ij} + \frac{1}{2}\delta_A{}^a\mathcal{S}(\nabla C)_a{}^j - \nabla_i\nabla_k\chi(\gamma)_A{}^{ijk} + \frac{1}{2}\chi(\beta)_a{}^{ij}\delta_A{}^a \\ - \frac{1}{2}\rhd_\alpha{}^B{}_A D_B\Phi(\mathcal{F})_\alpha{}^j + \frac{1}{2}\rhd_\alpha{}^B{}_A D_B\partial_a{}^\alpha\chi(C)^{aj} = 0\,.\end{aligned} \tag{6.28}$$

Broj nezavisnih komponenti veza prve klase može se odrediti prebrojavanjem veza prikazanih u tabeli 6.3, a zatim oduzimanjem broja nezavisnih Bjankijevih identiteta od tog broja.

$\Phi(B)_\alpha{}^i$	$\Phi(C)_a$	$\Phi(\alpha)_\alpha$	$\Phi(\beta)_a{}^i$	$\Phi(\gamma)_A{}^{ij}$	$\Phi(\mathcal{F})^{\alpha i}$	$\Phi(\mathcal{G})^a$	$\Phi(\nabla C)^{ai}$	$\Phi(\nabla B)^\alpha$	$\Phi(\nabla D)_A{}^{ij}$
$3p$	q	p	$3q$	$3r$	$3p-p$	q	$3q-q$	p	$3r-2r$

Tabela 6.3: Broj veza prve klase u $3BF$ teoriji.

Broj nezavisnih komponenata veza prve klase je:

$$F = 8(p+q) + 6r - p - q - 2r = 7(p+q) + 4r\,,$$

gde smo od broja komponenta veza prve klase navedenih u tabeli 6.3 oduzeli p relacija (5.30), q relacija (5.32) i $2r$ nezavisnih[1] relacija (6.28). Najzad, koristeći formulu za izračunavanje broja

[1]Jednačina (6.28) se sastoji od $3r$ identiteta, ali od njih su samo $2r$ međusobno nezavisni. Izračunavanjem divergencije izraza (6.28), dobijamo da je ona automatski jednaka nuli na osnovu Bjankijevog identiteta (5.24). Oduzimanjem od ukupnog broja relacija (6.28) ovih r relacija divergencije koje ne predstavljaju nove identitete koje veze u teoriji zadovoljavaju, dobijamo $2r$ nezavisnih identiteta (6.28).

stepeni slobode u teoriji (3.41), dobija se da je broj stepeni slobode u $3BF$ teoriji

$$n = 10(p+q) + 5r - 7(p+q) - 4r - \frac{6(p+q)+2r}{2} = 0\,, \tag{6.29}$$

odnosno da $3BF$ teorija nema propagirajućih stepeni slobode.

Generator gejdž transformacija za $3BF$ teoriju

Generator gejdž transformacija u $2BF$ teoriji dat je izrazom:

$$\begin{aligned} G \;=\; \int_{\Sigma_3} \mathrm{d}^3\vec{x}\Big(& (\nabla_0 \epsilon^\alpha{}_i)\Phi(B)_\alpha{}^i - \epsilon^\alpha{}_i \Phi(\mathcal{F})_\alpha{}^i + (\nabla_0\epsilon^\alpha)\Phi(\alpha)_\alpha + \epsilon^\alpha\big(f_{\alpha\gamma}{}^\beta B_{\beta 0 i}\Phi(B)^{\gamma i} \\ & + C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^{b0} + \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)^{b0i} - \frac{1}{2}\gamma^A{}_{0ij} \rhd_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij} - \Phi(\nabla B)_\alpha\big) \\ & + (\nabla_0 \epsilon^a)\Phi(C)_a - \epsilon^a\left(\beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a\right) \\ & + (\nabla_0\epsilon^a{}_i)\Phi(\beta)_a{}^i - \epsilon^a{}_i\left(C_{b0} \rhd_{\alpha a}{}^b\Phi(B)^{\alpha i} - 2\beta^b{}_{0j}X_{\{ab\}}{}^A\Phi(\gamma)_A{}^{ij} + \Phi(\nabla C)_a{}^i\right) \\ & + \frac{1}{2}(\nabla_0\epsilon^A{}_{ij})\Phi(\gamma)_A{}^{ij} - \frac{1}{2}\epsilon^A{}_{ij}\Phi(\nabla D)_A{}^{ij}\Big)\,. \end{aligned} \tag{6.30}$$

Ovde su $\epsilon^\alpha{}_i$, ϵ^α, ϵ^a, $\epsilon^a{}_i$ and $\epsilon^A{}_{ij}$ nezavisni parametri gejdž transformacija. Postupak izvođenja generatora (6.30) prikazan je u dodatku D.3.

Varijaciju forme varijabli i njihovih konjugovanih impulsa računamo primenom (3.56):

$$\begin{array}{rclrcl} \delta_0 B^\alpha{}_{0i} &=& \nabla_0\epsilon^\alpha{}_i - f_{\beta\gamma}{}^\alpha \epsilon^\beta B^\gamma{}_{0i} & \delta_0\pi(B)_\alpha{}^{0i} &=& -f_{\alpha\beta}{}^\gamma\epsilon^\beta\pi(B)_\gamma{}^{0i}\,, \\ && -\epsilon^a \rhd_{\alpha a}{}^b \beta_{b0i} - \epsilon^a{}_i \rhd_{\alpha a}{}^b C_{b0}\,, &&& \\ \delta_0 B^\alpha{}_{ij} &=& 2\nabla_{[i}\epsilon^\alpha{}_{j]} - f_{\beta\gamma}{}^\alpha\epsilon^\beta B^\gamma{}_{ij} - \epsilon^A{}_{ij} \rhd_{\alpha A}{}^B D_B & \delta_0\pi(B)_\alpha{}^{ij} &=& -f_{\alpha\beta}{}^\gamma\epsilon^\beta\pi(B)_\gamma{}^{ij}\,, \\ && -\epsilon^a \rhd_{\alpha a}{}^b\beta_{bij} - 2\epsilon^a{}_{[j|} \rhd_{\alpha a}{}^b C_{b|i]}\,, &&& \\ \delta_0\alpha^\alpha{}_0 &=& \nabla_0\epsilon^\alpha\,, & \delta_0\pi(\alpha)_\alpha{}^0 &=& -f_{\alpha\beta}{}^\gamma\epsilon^\beta{}_i\pi(B)_\gamma{}^{0i} - f_{\alpha\beta}{}^\gamma\epsilon^\beta\pi(\alpha)_\gamma{}^0 \\ &&&&& - \rhd_{\alpha b}{}^a\epsilon^b\pi(C)_a{}^0 - \rhd_{\alpha b}{}^a\epsilon^b{}_i\pi(\beta)_a{}^i \\ &&&&& -\frac{1}{2}\rhd_{\alpha B}{}^A\epsilon^B{}_{ij}\pi(\gamma)_A{}^{0ij}\,, \\ \delta_0\alpha^\alpha{}_i &=& \nabla_i\epsilon^\alpha + \partial_a{}^\alpha\epsilon^a{}_i\,, & \delta_0\pi(\alpha)_\alpha{}^i &=& -f_{\alpha\beta}{}^\gamma\epsilon^\beta{}_j\pi(B)_\gamma{}^{ij} - f_{\alpha\beta}{}^\gamma\epsilon^\beta\pi(\alpha)_\gamma{}^i \\ &&&&& - \rhd_{\alpha b}{}^\alpha\epsilon^b\pi(C)_a{}^i - \rhd_{\alpha b}{}^\alpha\epsilon^b{}_j\pi(\beta)_a{}^{ij} \\ &&&&& -\frac{1}{2}\rhd_{\alpha B}{}^A\epsilon^B{}_{jk}\pi(\gamma)_A{}^{ijk} + \epsilon^{0ijk}\nabla_j\epsilon_{\alpha k}\,, \\ &&&&& +\frac{1}{2}\epsilon^{0ijk}\epsilon^a \rhd_{\alpha b}{}^a\beta^b{}_{jk}\,, \\ \delta_0 C^a{}_0 &=& \nabla_0\epsilon^a - \epsilon^\alpha \rhd_{\alpha b}{}^a C^b{}_0\,, & \delta_0\pi(C)_a{}^0 &=& \epsilon^\alpha \rhd_{\alpha a}{}^b\pi(C)_b{}^0 - \epsilon_{bi} \rhd_{\alpha a}{}^b\pi(B)^{\alpha 0i}\,, \\ \delta_0 C^a{}_i &=& \nabla_i\epsilon^a - \epsilon^\alpha \rhd_{\alpha b}{}^a C^b{}_i & \delta_0\pi(C)_a{}^i &=& \epsilon^\alpha \rhd_{\alpha a}{}^b\pi(C)_b{}^i - \epsilon_{bj} \rhd_{\alpha a}{}^b\pi(B)^{\alpha ij}\,, \\ && +\epsilon^\alpha{}_i\partial^a{}_\alpha - 2\epsilon^b{}_i D_A X_{\{bc\}}{}^A g^{ac}\,, &&& \end{array} \tag{6.31}$$

$$
\begin{aligned}
\delta_0 \beta^a{}_{0i} &= \nabla_0 \epsilon^a{}_i - \epsilon^\alpha \rhd_{\alpha b}{}^a \beta_{b0i}\,, & \delta_0 \pi(\beta)_a{}^{0i} &= \epsilon^\alpha \rhd_{\alpha a}{}^b \pi(\beta)_b{}^{0i} - \epsilon_b \rhd_{\alpha a}{}^b \pi(B)^{\alpha 0i} \\
& & &\quad +2\epsilon^b{}_j X_{\{ab\}}{}^A \pi(\gamma)_A{}^{0ij}\,, \\
\delta_0 \beta^a{}_{ij} &= 2\nabla_{[i}\epsilon^a{}_{j]} - \epsilon^\alpha \rhd_{\alpha b}{}^a \beta^b{}_{ij} + \epsilon^A{}_{ij}\delta_A{}^a\,, & \delta_0 \pi(\beta)_a{}^{ij} &= \epsilon^\alpha \rhd_{\alpha a}{}^b \pi(\beta)_b{}^{ij} - \epsilon_b \rhd_{\alpha a}{}^b \pi(B)^{\alpha ij} \\
& & &\quad +2\epsilon^b{}_k X_{\{ab\}}{}^A \pi(\gamma)_A{}^{ijk} \\
& & &\quad -\epsilon^{0ijk}\nabla_k \epsilon_a - \epsilon^{0ijk}\epsilon^\alpha{}_k \partial_{\alpha a}\,, \\
\delta_0 \gamma^A{}_{0ij} &= -\epsilon^\alpha \gamma^B{}_{0ij} \rhd_{\alpha B}{}^A + \nabla_0 \epsilon^A{}_{ij} & \delta_0 \pi(\gamma)_A{}^{0ij} &= \epsilon^\alpha \rhd_{\alpha A}{}^B \pi(\gamma)_B{}^{0ij}\,, \\
&\quad +4\epsilon^a{}_{[i|}\beta^b{}_{0|j]} X_{\{ab\}}{}^A\,, \\
\delta_0 \gamma^A{}_{ijk} &= -\epsilon^\alpha \gamma^B{}_{ijk} \rhd_{\alpha B}{}^A + \nabla_i \epsilon^A{}_{jk} & \delta_0 \pi(\gamma)_A{}^{ijk} &= \epsilon^\alpha \rhd_{\alpha A}{}^B \pi(\gamma)_B{}^{ijk} - \epsilon^{oijk}\delta_{aA}\epsilon^a\,, \\
&\quad -\nabla_j \epsilon^A{}_{ki} + \nabla_k \epsilon^A{}_{ij} - 3!\epsilon^a{}_{[i}\beta^b{}_{jk]} X_{\{ab\}}{}^A\,, \\
\delta_0 D^A &= -\epsilon^a \delta_a{}^A - \epsilon^\alpha D^B \rhd_{\alpha B}{}^A\,, & \delta_0 \pi(D)_A &= 2\epsilon^a{}_i X_{\{ab\}A} \pi(C)^{bi} \\
& & &\quad -\frac{1}{2}\epsilon_B{}^{ij} \rhd_{\alpha A}{}^B \pi(B)^\alpha{}_{0ij} \\
& & &\quad +\epsilon^\alpha \rhd_{\alpha A}{}^B \pi(D)_B\,.
\end{aligned}
$$

6.1.2 Simetrije 3BF dejstva

Dejstvo (6.1) poseduje dodatne simetrije u odnosu na transformacije simetrija definisane za 2BF dejstvo u Teoremama 10, 11 i 12. Naime, važe sledeće teoreme [19].

Grupa G

Najpre, posmatrajmo infinitezimalne transformacije određene parametrom $\epsilon_{\mathfrak{g}}{}^\alpha$, date varijacijama formi

$$
\begin{aligned}
\delta_0 \alpha^\alpha{}_\mu &= -\,\partial_\mu \epsilon_{\mathfrak{g}}{}^\alpha - f_{\beta\gamma}{}^\alpha \alpha^\beta{}_\mu \epsilon_{\mathfrak{g}}{}^\gamma\,, & \delta_0 B^\alpha{}_{\mu\nu} &= f_{\beta\gamma}{}^\alpha \epsilon_{\mathfrak{g}}{}^\beta B^\gamma{}_{\mu\nu}\,, \\
\delta_0 \beta^a{}_{\mu\nu} &= \rhd_{\alpha b}{}^a \epsilon_{\mathfrak{g}}{}^\alpha \beta^b{}_{\mu\nu}\,, & \delta_0 C^a{}_\mu &= \rhd_{\alpha b}{}^a \epsilon_{\mathfrak{g}}{}^\alpha C^b{}_\mu\,, \\
\delta_0 \gamma^A{}_{\mu\nu\rho} &= \rhd_{\alpha B}{}^A \epsilon_{\mathfrak{g}}{}^\alpha \gamma^B{}_{\mu\nu\rho}\,, & \delta_0 D^A &= \rhd_{\alpha B}{}^A \epsilon_{\mathfrak{g}}{}^\alpha D^B\,,
\end{aligned}
\tag{6.32}
$$

koje analogno možemo da zapišemo na sledeći način:

$$
\begin{array}{lllll@{\qquad}lllll}
\alpha & \to & \alpha' & = & \alpha + \nabla\epsilon_{\mathfrak{g}}\,, & B & \to & B' & = & B + [B, \epsilon_{\mathfrak{g}}]\,, \\
\beta & \to & \beta' & = & \beta - \epsilon_{\mathfrak{g}} \rhd \beta\,, & C & \to & C' & = & C - \epsilon_{\mathfrak{g}} \rhd C\,, \\
\gamma & \to & \gamma' & = & \gamma - \epsilon_{\mathfrak{g}} \rhd \gamma\,, & D & \to & D' & = & D - \epsilon_{\mathfrak{g}} \rhd D\,,
\end{array}
\tag{6.33}
$$

Na osnovu ovih infinitezimalnih transformacija možemo ekstrapolirati konačnu transformaciju definisanu u Teoremi 14.

Teorema 14 (G-gejdž transformacije) *U 3BF teoriji nad proizvoljnim 2-ukrštenim modulom $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_\,,\_\}_{\rm pf})$, sledeća transformacija je simetrija,*

$$
\begin{array}{lll@{\qquad}lll}
\alpha & \to & \alpha' = \mathrm{Ad}_g \alpha + g \mathrm{d} g^{-1}\,, & B & \to & B' = gBg^{-1}\,, \\
\beta & \to & \beta' = g \rhd \beta\,, & C & \to & C' = g \rhd C\,, \\
\gamma & \to & \gamma' = g \rhd \gamma\,, & D & \to & D' = g \rhd D\,,
\end{array}
\tag{6.34}
$$

gde je $g = \exp(\epsilon_{\mathfrak{g}} \cdot \hat{G}) = \exp(\epsilon_{\mathfrak{g}\alpha} \hat{G}^\alpha) \in G$ i $\epsilon_{\mathfrak{g}} : \mathcal{M}_4 \to \mathfrak{g}$ parametar transformacije.

Dokaz. Transformacija 3-koneksije definisana u Teoremi 14 dovodi do sledeće transformacije 3-krivine:

$$\mathcal{F} \;\to\; \mathcal{F}' = g\mathcal{F}g^{-1}\,, \qquad \mathcal{G} \;\to\; \mathcal{G}' = g \rhd \mathcal{G}\,, \qquad \mathcal{H} \;\to\; \mathcal{H}' = g \rhd \mathcal{H}\,, \tag{6.35}$$

Primenom ove transformacije, $3BF$ dejstvo postaje:

$$\begin{aligned} S_{3BF} &= \int_{\mathcal{M}_4} \Big(\langle B, \mathcal{F} \rangle_{\mathfrak{g}} + \langle C, \mathcal{G} \rangle_{\mathfrak{h}} + \langle D, \mathcal{H} \rangle_{\mathfrak{h}} \Big) \\ \to S'_{BF} &= \int_{\mathcal{M}_4} \Big(\langle gBg^{-1}, g\mathcal{F}g^{-1} \rangle_{\mathfrak{g}} + \langle g \rhd C, g \rhd \mathcal{G} \rangle_{\mathfrak{h}} + \langle g \rhd D, g \rhd \mathcal{H} \rangle_{\mathfrak{l}} \Big)\,. \end{aligned} \tag{6.36}$$

Iz G-invarijantnosti bilinearnih formi $\langle \_\,, \_ \rangle_{\mathfrak{g}}$, $\langle \_\,, \_ \rangle_{\mathfrak{h}}$ i $\langle \_\,, \_ \rangle_{\mathfrak{l}}$ sledi da je $3BF$ dejstvo invarijantno. Invarijantnost se može takođe pokazati na sličan način kao u Teoremi 8. ■

Razmatranjem dve uzastopne infinitezimalne G-gejdž transformacije, određene malim parametrima $\epsilon_{\mathfrak{g}}{}^{\alpha}{}_1$ i $\epsilon_{\mathfrak{g}}{}^{\beta}{}_2$, izračunavamo komutator dva generatora G-gejdž transformacija na sličan način kao što je to urađeno u slučaju BF teorije. Dobijamo da generatori G-gejdž transformacija definisanih u Teoremi 14 zadovoljavaju komutacione relacije

$$[\hat{G}_{\alpha}, \hat{G}_{\beta}] = f_{\alpha\beta}{}^{\gamma} \hat{G}_{\gamma}\,, \tag{6.37}$$

gde su $f_{\alpha\beta}{}^{\gamma}$ strukturne konstante algebre $\mathfrak{g}$. Primetimo da, isto kao što je to bio slučaj kod BF i $2BF$ transformacija, postoji izomorfizam između generatora $\hat{G}_{\alpha} \cong \tau_{\alpha}$, tj. možemo zaključiti da je grupa G-gejdž transformacija iz Teoreme 14 upravo grupa G iz 2-ukrštenog modula $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_\,,\_\}_{\mathrm{pf}})$. Ovo je važan rezultat, koji neće važiti za preostale transformacije simetrije, kao što ćemo videti u nastavku.

Gejdž grupa $\tilde{H}_L$

Razmotrimo sada varijacije formi koje odgovaraju parametru transformacija $\epsilon_{\mathfrak{h}}{}^{a}{}_i$. Na primer, iz jednačina (6.31) se može videti da je varijacija formi promenljivih $\alpha^{\alpha}{}_0$ i $\alpha^{\alpha}{}_i$:

$$\delta_0 \alpha^{\alpha}{}_o = 0\,, \qquad \delta_0 \alpha^{\alpha}{}_i = -\partial_a{}^{\alpha} \epsilon_{\mathfrak{h}}{}^{a}{}_i\,. \tag{6.38}$$

Uzimajući u obzir da dejstvo generatora (6.30) daje transformacije simetrije na jednoj prostornoj hiperpovrši Σ_3 sa vremenskom komponentom parametra transformacije jednakom nuli $\epsilon_{\mathfrak{h}}{}^{a}{}_0 = 0$, može se ekstrapolirati transformacija za parametre prostorvremenskih gejdž transformacija $\epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}$. Dobijamo da je varijacija forme promenljive $\alpha^{\alpha}{}_{\mu}$

$$\delta_0 \alpha^{\alpha}{}_{\mu} = -\partial_a{}^{\alpha} \epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}\,, \tag{6.39}$$

a slično se može zaključiti i za preostale varijable. Tako je infinitezimalna transformacija u celom prostorvremenu koji odgovara parametru $\epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}$ data varijacijama formi:

$$\begin{aligned} \delta_0 \alpha^{\alpha}{}_{\mu} &= -\partial_a{}^{\alpha} \epsilon_{\mathfrak{h}}{}^{a}{}_{\mu}\,, & \delta_0 B^{\alpha}{}_{\mu\nu} &= 2C_{a[\mu|} \epsilon_{\mathfrak{h}}{}^{b}{}_{|\nu]} \rhd_{\beta b}{}^{a} g^{\alpha\beta}\,, \\ \delta_0 \beta^{a}{}_{\mu\nu} &= -2\nabla_{[\mu|} \epsilon_{\mathfrak{h}}{}^{a}{}_{|\nu]}\,, & \delta_0 C^{a}{}_{\mu} &= 2D_A X_{(ab)}{}^{A} \epsilon_{\mathfrak{h}}{}^{b}{}_{\mu}\,, \\ \delta_0 \gamma^{A}{}_{\mu\nu\rho} &= 3!\beta^{a}{}_{[\mu\nu} \epsilon_{\mathfrak{h}}{}^{b}{}_{\rho]} X_{(ab)}{}^{A}\,, & \delta_0 D &= 0\,. \end{aligned} \tag{6.40}$$

Za ove infinitezimalne transformacije dobijaju se konačne transformacije simetrije date u Teoremi 15.

Teorema 15 ($\boldsymbol{H}$-gejdž transformacije) *U 3BF teoriji nad proizvoljnim 2-ukrštenim modulom* $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_\,,\_\}_{\rm pf})$*, sledeća transformacija je simetrija:*

$$
\begin{array}{lllllllll}
\alpha & \to & \alpha' & = & \alpha - \partial\epsilon_{\mathfrak{h}}\,, & B & \to & B' & = & B - C' \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} D\,,\\
\beta & \to & \beta' & = & \beta - \nabla'\epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge \epsilon_{\mathfrak{h}}\,, & C & \to & C' & = & C - D \wedge^{\mathcal{X}_1} \epsilon_{\mathfrak{h}} - D \wedge^{\mathcal{X}_2} \epsilon_{\mathfrak{h}}\,,\\
\gamma & \to & \gamma' & = & \gamma + \{\beta', \epsilon_{\mathfrak{h}}\}_{\rm pf} + \{\epsilon_{\mathfrak{h}}, \beta\}_{\rm pf}\,, & D & \to & D' & = & D\,,
\end{array}
\tag{6.41}
$$

gde je parametar $\epsilon_{\mathfrak{h}} \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ *proizvoljna* 1*-forma element algebre* $\mathfrak{h}$*, a oznaka* ∇' *je kovarijantni izvod sa koneksijom* α'*. Preslikavanja* $\mathcal{T}$*,* $\mathcal{D}$*,* $\mathcal{X}_1$ *i* $\mathcal{X}_2$ *su definisana u Dodatku* A*, jednačinama* (A.13)*,* (A.51)*,* (A.47) *i* (A.48)*.*

Dokaz. Primetimo da se 3-krivina pri transformacijama simetrije definisanim u Teoremi 15 transformiše na sledeći način:

$$
\begin{array}{lll}
\mathcal{F} & \to & \mathcal{F}' = \mathcal{F}\,,\\
\mathcal{G} & \to & \mathcal{G}' = \mathcal{G} - \mathcal{F} \wedge^{\rhd} \epsilon_{\mathfrak{h}}\,,\\
\mathcal{H} & \to & \mathcal{H}' = \mathcal{H} + \{\mathcal{G}', \epsilon_{\mathfrak{h}}\}_{\rm pf} - \{\epsilon_{\mathfrak{h}}\,, \mathcal{G}\}_{\rm pf}\,.
\end{array}
\tag{6.42}
$$

Koristeći izraze za transformacije 3-krivine (6.42) i za transformacije Lagranževih množitelja, dobija se da je transformisano dejstvo S'_{3BF}:

$$
\begin{aligned}
S'{}_{3BF} =& S_{3BF} + \int_{\mathcal{M}_4} \Big(- \langle C' \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}}, \mathcal{F}\rangle_{\mathfrak{g}} - \langle \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} D, \mathcal{F}\rangle_{\mathfrak{g}} - \langle C', \mathcal{F} \wedge^{\rhd} \epsilon_{\mathfrak{h}}\rangle_{\mathfrak{h}} - \langle D \wedge^{\mathcal{X}_1} \epsilon_{\mathfrak{h}}, \mathcal{G}\rangle_{\mathfrak{h}}\\
& - \langle D \wedge^{\mathcal{X}_2} \epsilon_{\mathfrak{h}}, \mathcal{G}\rangle_{\mathfrak{h}} + \langle D, \{\mathcal{G}, \epsilon_{\mathfrak{h}}\}_{\rm pf}\rangle_{\mathfrak{l}} - \langle D, \{\mathcal{F} \wedge^{\rhd} \epsilon_{\mathfrak{h}}, \epsilon_{\mathfrak{h}}\}_{\rm pf}\rangle_{\mathfrak{l}} - \langle D, \{\epsilon_{\mathfrak{h}}\,, \mathcal{G}\}_{\rm pf}\rangle_{\mathfrak{l}}\Big)\,.
\end{aligned}
\tag{6.43}
$$

Koristeći definicije preslikavanja $\mathcal{T}$, $\mathcal{D}$, $\mathcal{X}_1$ i $\mathcal{X}_2$, date u Dodatku A, vidimo da se preostali članovi međusobno pokrate, tj. da dejstvo ostaje invarijantno pri ovim transformacijama $S'{}_{3BF} = S_{3BF}$. Konkretno, prvi član se skraćuje sa trećim, drugi sa sedmim, četvrti sa osmim i peti član se skraćuje sa šestim članom. ∎

Možemo pokazati da H-gejdž transformacije ne čine grupu. Naime, može se proveriti da dve uzastopne H-gejdž transformacije ne daju transformaciju iste vrste, tačnije, aksiom zatvaranja grupe nije zadovoljen. Analogan slučaj imamo kod dobro poznate strukture Lorencove grupe, gde bust transformacije nisu zatvorene tj. ne formiraju grupu. Zaista, moraju se uzeti u obzir i rotacije i bustovi da bi se dobio skup transformacija koje formiraju Lorencovu grupu. U slučaju H-gejdž transformacija, pokazaćemo da pored njih treba uzeti u obzir i transformacije koje odgovaraju parametru $\epsilon_{\mathfrak{l}}{}^A{}_{ij}$. Iz jednačina (6.31) vidimo varijacije formi na prostornoj hiperpovrši Σ_3 koje odgovaraju ovom parametru. Slično kao što smo to uradili u slučaju H-gejdž trasformacija, možemo ekstrapolirati prostorvremenske varijacije formi koje odgovaraju parametru $\epsilon_{\mathfrak{l}}{}^A{}_{\mu\nu}$:

$$
\begin{array}{lllllll}
\delta_0 \alpha^{\alpha}{}_{\mu} & = & 0\,, & \delta_0 B^{\alpha}{}_{\mu\nu} & = & -D_A \rhd_{\beta B}{}^{A} \epsilon_{\mathfrak{l}}{}^{B}{}_{\mu\nu} g^{\alpha\beta}\,,\\
\delta_0 \beta^{a}{}_{\mu\nu} & = & \delta_A{}^a \epsilon_{\mathfrak{l}}{}^A{}_{\mu\nu}\,, & \delta_0 C^a{}_{\mu} & = & 0\,,\\
\delta_0 \gamma^A{}_{\mu\nu\rho} & = & \nabla_\mu \epsilon_{\mathfrak{l}}{}^A{}_{\nu\rho} - \nabla_\nu \epsilon_{\mathfrak{l}}{}^A{}_{\mu\rho} + \nabla_\rho \epsilon_{\mathfrak{l}}{}^A{}_{\mu\nu}\,, & \delta_0 D^A & = & 0\,.
\end{array}
\tag{6.44}
$$

Ove infinitezimalne transformacije odgovaraju konačnim transformacijama simetrije definisanim u Teoremi 16.

Teorema 16 (L-gejdž transformacije) *U 3BF teoriji nad proizvoljnim 2-ukrštenim modulom* $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_\,,\_\}_{\mathrm{pf}})$, *sledeća transformacija je simetrija:*

$$\begin{array}{lllllllll} \alpha & \to & \alpha' & = & \alpha\,, & B & \to & B' & = & B + D\wedge^{\mathcal{S}}\epsilon_{\mathfrak{l}}\,, \\ \beta & \to & \beta' & = & \beta+\delta\epsilon_{\mathfrak{l}}\,, & C & \to & C' & = & C\,, \\ \gamma & \to & \gamma' & = & \gamma+\nabla\epsilon_{\mathfrak{l}}\,, & D & \to & D' & = & D\,, \end{array} \tag{6.45}$$

gde je parametar transformacije $\epsilon_{\mathfrak{l}} \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{l})$ *proizvoljna 2-forma element algebre* $\mathfrak{l}$, *a preslikavanje* $\mathcal{S}$ *je definisano u Dodatku A jednačinom (A.43).*

Dokaz. Pri transformacijama definisanim u Teoremi 16 3-krivina se transformiše na sledeći način:

$$\begin{array}{lllll} \mathcal{F} & \to & \mathcal{F}' & = & \mathcal{F}\,, \\ \mathcal{G} & \to & \mathcal{G}' & = & \mathcal{G}\,, \\ \mathcal{H} & \to & \mathcal{H}' & = & \mathcal{H}+\mathcal{F}\wedge^{\rhd}\epsilon_{\mathfrak{l}}\,. \end{array} \tag{6.46}$$

Uzimajući u obzir transformacije 3-krivine (6.46) i transformacije Lagranževih množitelja, 3BF dejstvo se transformiše:

$$S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \Big(\langle D\wedge^{\mathcal{S}}\epsilon_{\mathfrak{l}}, \mathcal{F}\rangle_{\mathfrak{g}} + \langle D, \mathcal{F}\wedge^{\rhd}\epsilon_{\mathfrak{l}}\rangle_{\mathfrak{l}} \Big)\,. \tag{6.47}$$

Primenjujući definiciju preslikavanja $\mathcal{S}$ datu u Dodatku A, članovi u zagradi se pokrate. ■

Označimo generatore H-gejdž transformacija definisanih u Teoremi 15 kao $\hat{H}_a{}^{\mu}$ i generatore L-gejdž transformacija definisanih u Teoremi 16 kao $\hat{L}_A{}^{\mu\nu}$. Sada možemo proveriti da li H-gejdž transformacije definisane u Teoremi 15 formiraju grupu. Ako izvršimo dve uzastopne H-gejdž transformacije, definisane parametrima $\epsilon_{\mathfrak{h}1}$ i $\epsilon_{\mathfrak{h}2}$, dobijamo

$$\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot\hat{H}}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot\hat{H}} - \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot\hat{H}}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot\hat{H}} = 2\left(\{\epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}2}\}_{\mathrm{pf}} - \{\epsilon_{\mathfrak{h}2}\wedge\epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}}\right)\cdot\hat{L}\,, \tag{6.48}$$

gde $\epsilon_{\mathfrak{h}}\cdot\hat{H} = \epsilon_{\mathfrak{h}}{}^a{}_{\mu}\hat{H}_a{}^{\mu}$ i $\epsilon_{\mathfrak{l}}\cdot\hat{L} = \frac{1}{2}\epsilon_{\mathfrak{l}}{}^A{}_{\mu\nu}\hat{L}_A{}^{\mu\nu}$. Koristeći jednačinu analognu BCH formuli (4.38), dobijamo da je komutator generatora dve H-gejdž transformacije generator L-gejdž transformacije (detalji računa dati su u Dodatku D):

$$[\hat{H}_a{}^{\mu}, \hat{H}_b{}^{\nu}] = 2X_{(ab)}{}^A\hat{L}_A{}^{\mu\nu}\,. \tag{6.49}$$

Transformacije definisane u Teoremi 16 su linearne transformacije, a dve uzastopne L-gejdž transformacije daju L-gejdž transformaciju sa parametrom $\epsilon_{\mathfrak{l}1}+\epsilon_{\mathfrak{l}2}$, tj. formalno zapisano:

$$\mathrm{e}^{\epsilon_{\mathfrak{l}1}\cdot\hat{L}}\mathrm{e}^{\epsilon_{\mathfrak{l}2}\cdot\hat{L}} = \mathrm{e}^{(\epsilon_{\mathfrak{l}1}+\epsilon_{\mathfrak{l}2})\cdot\hat{L}}\,, \tag{6.50}$$

Na osnovu ovoga zaključujemo da L-gejdž transformacije komutiraju:

$$[\hat{L}_A{}^{\mu\nu}, \hat{L}_B{}^{\rho\sigma}] = 0\,. \tag{6.51}$$

Stoga, L-gejdž transformacije čine Abelovu grupu $\tilde{L}$. Prema strukturi indeksa parametara i generatora, možemo zaključiti da je grupa $\tilde{L}$ izomorfna grupi $\mathbb{R}^{6r}$, gde je r red grupe L:

$$\tilde{L} \cong \mathbb{R}^{6r}\,. \tag{6.52}$$

Obratimo pažnju da Abelovu grupu $\tilde{L}$ ne treba pomešati sa ne-Abelovom grupom L koja je deo strukture 2-ukrštenog modula $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_\,,\_\}_{\mathrm{pf}})$.

Razmotrimo sada odnos između H-gejdž transformacija i L-gejdž transformacija. Na osnovu jednakosti,

$$\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot\hat{H}}\mathrm{e}^{\epsilon_{\mathfrak{l}}\cdot\hat{L}} = \mathrm{e}^{\epsilon_{\mathfrak{l}}\cdot\hat{L}}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot\hat{H}}\,, \tag{6.53}$$

možemo zaključiti da je komutator između generatora H-gejdž transformacija i generatora L-gejdž transformacija jednak nuli:

$$[\hat{H}_a{}^{\mu}, \hat{L}_A{}^{\nu\rho}] = 0\,. \tag{6.54}$$

Na osnovu relacija (6.49), (6.51) i (6.54) vidimo da H-gejdž transformacije zajedno sa L-gejdž transformacijama formiraju grupu, obeležimo je kao $\tilde{H}_L$. Na kraju, dejstvo grupe G na H-gejdž i L-gejdž transformacije dobijamo izračunavanjem sledećih izraza,

$$[\epsilon_{\mathfrak{g}} \cdot \hat{G}, \epsilon_{\mathfrak{h}} \cdot \hat{H}] \;=\; (\epsilon_{\mathfrak{g}} \rhd \epsilon_{\mathfrak{h}}) \cdot \hat{H}\,, \qquad [\epsilon_{\mathfrak{g}} \cdot \hat{G}, \epsilon_{\mathfrak{l}} \cdot \hat{L}] \;=\; (\epsilon_{\mathfrak{g}} \rhd \epsilon_{\mathfrak{l}}) \cdot \hat{L}\,, \tag{6.55}$$

na osnovu kojih dobijamo komutacione relacije

$$\begin{aligned} [\hat{G}_\alpha, \hat{H}_a{}^{\mu}] &= \rhd_{\alpha a}{}^{b}\, \hat{H}_b{}^{\mu}\,, \\ [\hat{G}_\alpha, \hat{L}_A{}^{\mu\nu}] &= \rhd_{\alpha A}{}^{B}\, \hat{L}_B{}^{\mu\nu}\,. \end{aligned} \tag{6.56}$$

Teoreme 14, 15 i 16 predstavljaju G-, H- i L-gejdž transformacije, za više informacija videti [18], [40]).

Grupe $\tilde{M}$ i $\tilde{N}$

Zatim, razmotrimo infinitezimalnu transformaciju sa parametrom $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{i}$, datu varijacijama formi (6.31). Slično kao što je to učinjeno u prethodnom delu, na osnovu varijacija formi varijabli dobijenih kao rezultat Hamiltonove analize tj. transformacija na jednoj prostornoj hiperpovrši Σ_3, možemo pogoditi transformacije u celom prostorvremenu. Imajući u vidu da varijacije na hiperpovrši imaju vremensku komponentu parametra jednaku nuli $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{0} = 0$, ekstrapoliramo varijacije formi na celom prostorvremenu koje odgovaraju parametru $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu}$:

$$\begin{aligned} \delta_0 \alpha^{\alpha}{}_{\mu} &= 0\,, & \delta_0 B^{\alpha}{}_{\mu\nu} &= -2\nabla_{[\mu|}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{|\nu]}\,, \\ \delta_0 \beta^{a}{}_{\mu\nu} &= 0\,, & \delta_0 C^{a}{}_{\mu} &= -\partial^{a}{}_{\alpha}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu}\,, \\ \delta_0 \gamma^{A}{}_{\mu\nu\rho} &= 0\,, & \delta_0 D^{A} &= 0\,. \end{aligned} \tag{6.57}$$

Na osnovu ovog rezultata, dobija se konačna transformacija simetrije u celom prostorvremenu, kao što definisano Teoremom 17, koje nazivamo M-gejdž transformacijama.

Teorema 17 (M-gejdž transformacije) *U 3BF teoriji nad proizvoljnim 2-ukrštenim modulom $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_\,,\_\}_{\mathrm{pf}})$, sledeća transformacija je simetrija*

$$\begin{aligned} \alpha &\to \alpha' = \alpha\,, & B &\to B' = B - \nabla\epsilon_{\mathfrak{m}}\,, \\ \beta &\to \beta' = \beta\,, & C^a &\to C'^a = C^a - \partial^a{}_{\alpha}\epsilon_{\mathfrak{m}}{}^{\alpha}\,, \\ \gamma &\to \gamma' = \gamma\,, & D &\to D' = D\,, \end{aligned} \tag{6.58}$$

gde je parametar transformacija $\epsilon_{\mathfrak{m}} \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$ proizvoljna 1-forma element algebre $\mathfrak{g}$.

Dokaz. Razmotrimo transformaciju 3BF dejstva pri transformacijama definisanim u Teoremi 17. Dobijamo:

$$S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} \left(-\frac{1}{2}(\nabla_{\mu}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\nu})\mathcal{F}_{\alpha\rho\sigma} - \frac{1}{3!}\partial^{a}{}_{\alpha}\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{\mu}\mathcal{G}_{a\nu\rho\sigma} \right). \tag{6.59}$$

Primenom definicije 3-krivine (2.3.5):

$$S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}^4x \, \epsilon^{\mu\nu\rho\sigma} \Big(-\frac{1}{2} (\nabla_\mu \epsilon_{\mathfrak{m}}{}^\alpha{}_\nu) \left(F_{\alpha\rho\sigma} - \partial^a{}_\alpha \beta_{a\rho\sigma} \right) - \frac{1}{3!} \partial^a{}_\alpha \epsilon_{\mathfrak{m}}{}^\alpha{}_\mu \left(3\nabla_\nu \beta_{a\rho\sigma} - \delta^A{}_a \gamma_{A\nu\rho\sigma} \right) \Big) . \tag{6.60}$$

U prethodnom izrazu drugi i treći član se pokrate, dok je poslednji član nula zbog identiteta (2.73), pa sledi:

$$S'{}_{3BF} = S_{3BF} - \frac{1}{2} \int_{\mathcal{M}_4} \mathrm{d}^4x \, \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mathfrak{m}}{}^\alpha{}_\mu \nabla_\nu F_{\alpha\rho\sigma} \, . \tag{6.61}$$

Na kraju, član $\epsilon^{\mu\nu\rho\sigma} \nabla_\nu F_{\alpha\rho\sigma} = 0$ je upravo BI (5.24). Zaključujemo da je S_{3BF} invarijantno pri transformacijama definisanim Teoremom 17. ■

Možemo pokazati da su transformacije definisane u Teoremi 17 linearne transformacije, tj. da dve uzastopne M-gejdž transformacije daju jednu M-gejdž transformaciju sa parametrom $\epsilon_{\mathfrak{m}1} + \epsilon_{\mathfrak{m}2}$. Ako generatore M-gejdž transformacija obeležimo sa $\hat{M}_\alpha{}^\mu$, možemo pisati

$$\mathrm{e}^{\epsilon_{\mathfrak{m}1} \cdot \hat{M}} \mathrm{e}^{\epsilon_{\mathfrak{m}2} \cdot \hat{M}} = \mathrm{e}^{(\epsilon_{\mathfrak{m}1} + \epsilon_{\mathfrak{m}2}) \cdot \hat{M}} \, , \tag{6.62}$$

gde je $\epsilon_{\mathfrak{m}} \cdot \hat{M} = \epsilon_{\mathfrak{m}}{}^\alpha{}_\mu \hat{M}_\alpha{}^\mu$. Dobijamo komutacionu relaciju:

$$[\hat{M}_\alpha{}^\mu, \hat{M}_\beta{}^\nu] = 0 \, . \tag{6.63}$$

Stoga, M-gejdž transformacije formiraju Abelovu grupu $\tilde{M}$. Prema strukturi indeksa njenih parametara i generatora, vidimo da je ova grupa izomorfna sa grupom $\mathbb{R}^{4p}$, gde je p dimenzija grupe G:

$$\tilde{M} \cong \mathbb{R}^{4p} \, . \tag{6.64}$$

Zatim se može ispitati odnos M-gejdž transformacija sa G, H i L-gejdž transformacijama definisanim u prethodnim delovima. Konkretno, za generatore G-gejdž transformacija važi relacija,

$$[\epsilon_{\mathfrak{g}} \cdot \hat{G}, \epsilon_{\mathfrak{m}} \cdot \hat{M}] \;=\; (\epsilon_{\mathfrak{g}} \rhd \epsilon_{\mathfrak{m}}) \cdot \hat{M} \, , \tag{6.65}$$

na osnovu čega dobijamo komutator:

$$[\hat{G}_\alpha \, , \hat{M}_\beta{}^\mu] \;=\; f_{\alpha\beta}{}^\gamma \hat{M}_\gamma{}^\mu \, . \tag{6.66}$$

Za generatore H- i L-gejdž transformacija, dobijamo relacije

$$\begin{aligned} \mathrm{e}^{\epsilon_{\mathfrak{h}} \cdot \hat{H}} \mathrm{e}^{\epsilon_{\mathfrak{m}} \cdot \hat{M}} &= \mathrm{e}^{\epsilon_{\mathfrak{m}} \cdot \hat{M}} \mathrm{e}^{\epsilon_{\mathfrak{h}} \cdot \hat{H}} \, , \\ \mathrm{e}^{\epsilon_{\mathfrak{l}} \cdot \hat{L}} \mathrm{e}^{\epsilon_{\mathfrak{m}} \cdot \hat{M}} &= \mathrm{e}^{\epsilon_{\mathfrak{m}} \cdot \hat{M}} \mathrm{e}^{\epsilon_{\mathfrak{l}} \cdot \hat{L}} \, , \end{aligned} \tag{6.67}$$

na osnovu čega zaključujemo da generatori M-gejdž transformacija komutiraju kako sa generatorima H-gejdž transformacija, tako i sa generatorima L-gejdž transformacija:

$$[\hat{H}_a, \hat{M}_\alpha{}^\mu] = 0 \, , \quad [\hat{L}_A{}^{\mu\nu}, \hat{M}_\alpha{}^\rho] = 0 \, . \tag{6.68}$$

Poslednji tip transformacija dobijamo razmatranjem varijacija formi varijabli dobijenih u (6.31) koje odgovaraju parametru transformacija $\epsilon_{\mathfrak{n}}{}^a$,

$$\begin{array}{llllll} \delta_0 \alpha^\alpha{}_\mu & = & 0 \, , & \delta_0 B^\alpha{}_{\mu\nu} & = & \beta_{b\mu\nu} \rhd_{\alpha' a}{}^b \epsilon_{\mathfrak{n}}{}^a g^{\alpha\alpha'} \, , \\ \delta_0 \beta^a{}_{\mu\nu} & = & 0 \, , & \delta_0 C^a{}_\mu & = & -\nabla_\mu \epsilon_{\mathfrak{n}}{}^a \, , \\ \delta_0 \gamma^A{}_{\mu\nu\rho} & = & 0 \, , & \delta_0 D^A & = & \delta^A{}_a \epsilon_{\mathfrak{n}}{}^a \, . \end{array} \tag{6.69}$$

Ove, N-gejdž transformacije, definisane su Teoremom 18. Primetimo da su N-gejdž transformacije transformacije u celom prostorvremenu, pošto parametar ne nosi prostorvremenske indekse.

Teorema 18 (N-gejdž transformacije) *U $3BF$ teoriji nad proizvoljnim 2-ukrštenim modulom $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{\_\,,\_\}_{\rm pf})$, sledeća transformacija je simetrija*

$$\begin{array}{llllllll} \alpha & \to & \alpha' & = & \alpha\,, & B & \to & B' = B - \beta\wedge^{\mathcal{T}}\epsilon_{\mathfrak{n}}\,,\\ \beta & \to & \beta' & = & \beta\,, & C & \to & C' = C - \nabla\epsilon_{\mathfrak{n}}\,,\\ \gamma & \to & \gamma' & = & \gamma\,, & D^A & \to & D'^A = D^A + \delta^A{}_a{\epsilon_{\mathfrak{n}}}^a\,, \end{array} \tag{6.70}$$

gde je parametar $\epsilon_{\mathfrak{n}} : \mathcal{M}_4 \to \mathfrak{h}$ proizvoljna 0-forma element algebre $\mathfrak{h}$.

Dokaz. Pri transformacijama definisanim Teoremom 18, $3BF$ dejstvo se transformiše na sledeći način:

$$S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}x^4 \epsilon^{\mu\nu\rho\sigma}\left(\frac{1}{4}\beta_{b\mu\nu}\triangleright_{\alpha a}{}^b{\epsilon_{\mathfrak{n}}}^a\mathcal{F}^\alpha{}_{\rho\sigma} - \frac{1}{3!}(\nabla_\mu{\epsilon_{\mathfrak{n}}}^a)\mathcal{G}_{a\nu\rho\sigma} + \frac{1}{4!}\delta^A{}_a{\epsilon_{\mathfrak{n}}}^a\mathcal{H}_{A\,\mu\nu\rho\sigma}\right)\,. \tag{6.71}$$

Primenom definicije 3-krivine (2.118), dobijamo:

$$\begin{aligned} S'_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}x^4 \epsilon^{\mu\nu\rho\sigma}\Big(&\frac{1}{4}\beta_{b\mu\nu}\triangleright_{\alpha a}{}^b{\epsilon_{\mathfrak{n}}}^a\left(F^\alpha{}_{\rho\sigma} - \partial_c{}^\alpha\beta^c{}_{\rho\sigma}\right) - \frac{1}{3!}(\nabla_\mu{\epsilon_{\mathfrak{n}}}^a)\left(3\nabla_\nu\beta_{a\rho\sigma} - \delta^A{}_a\gamma_{A\,\nu\rho\sigma}\right)\\ &+ \frac{1}{4!}\delta^A{}_a\epsilon^a\left(4\nabla_\mu\gamma_{A\,\nu\rho\sigma} + 6X_{(bc)A}\beta^b{}_{\mu\nu}\beta^c{}_{\rho\sigma}\right)\Big)\,. \end{aligned} \tag{6.72}$$

Nakon parcijalne integracije poslednji član u prvom redu jednačine (6.72) i prvim član u drugom redu se skraćuju. Takođe, koristeći identitet

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(\nabla_\nu\nabla_\mu{\epsilon_{\mathfrak{n}}}^a)\beta_{a\rho\sigma} = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\beta_{b\rho\sigma}\triangleright_{\alpha a}{}^b{\epsilon_{\mathfrak{n}}}^a F^\alpha{}_{\mu\nu}\,, \tag{6.73}$$

pokratiće će i prvi i treći član u prvom redu. Konačno, dobijamo izraz:

$$S'{}_{3BF} = S_{3BF} + \int_{\mathcal{M}_4} \mathrm{d}x^4 \epsilon^{\mu\nu\rho\sigma}\left(\frac{1}{4}\epsilon_{\mathfrak{n}a}\triangleright_{\alpha(b|}{}^a\partial_{|c)}{}^\alpha\beta^b{}_{\mu\nu}\beta^c{}_{\rho\sigma} + \frac{1}{4}\epsilon_{\mathfrak{n}a}\delta_A{}^aX_{(bc)}{}^A\beta^b{}_{\mu\nu}\beta^c{}_{\rho\sigma}\right)\,. \tag{6.74}$$

Zbir preostala dva člana jednak je nuli zbog simetrizovanog oblika identiteta (2.87),

$$\triangleright_{\alpha(b|}{}^a\partial_{|c)}{}^\alpha + \delta_A{}^aX_{(bc)}{}^A = f_{(bc)}{}^a = 0\,,$$

zbog antisimetričnosti strukturnih konstanti. Zaključujemo da dejstvo S_{3BF} ostaje invarijantno pri transformacijama definisanim u Teoremi 18. ∎

Transformacije definisane Teoremom 18 – N-gejdž transformacije, formiraju grupu koju obeležavamo sa $\tilde{N}$. Imamo na kraju da su ove transformacije takođe linearne, a dve N-gejdž transformacije daju N-gejdž transformaciju sa parametrom $\epsilon_{\mathfrak{n}1} + \epsilon_{\mathfrak{n}2}$. Ako generatore grupe $\tilde{N}$ obeležimo sa $\hat{N}_a$, možemo da pišemo

$$\mathrm{e}^{\epsilon_{\mathfrak{n}1}\cdot\hat{N}}\mathrm{e}^{\epsilon_{\mathfrak{n}2}\cdot\hat{N}} = \mathrm{e}^{(\epsilon_{\mathfrak{n}1}+\epsilon_{\mathfrak{n}2})\cdot\hat{N}}\,, \tag{6.75}$$

gde je $\epsilon_{\mathfrak{n}}\cdot\hat{N} = {\epsilon_{\mathfrak{n}}}^a\hat{N}_a$. Iz ovoga sledi da je komutator dva generatora N-gejdž transformacija,

$$[\hat{N}_a, \hat{N}_b] = 0\,, \tag{6.76}$$

tj. da je $\tilde{N}$ Abelova grupa. Pritom, indeksna struktura parametara i generatora ukazuje na to da je $\tilde{N}$ izomorfna sa grupom $\mathbb{R}^q$, gde je q dimenzija grupe H:

$$\tilde{N} \cong \mathbb{R}^q\,. \tag{6.77}$$

Zatim se može ispitati odnos N-gejdž transformacija i G, H i L-gejdž transformacijama definisanim u prethodnim delovima. Konkretno, za generatore G-gejdž transformacija važi relacija

$$[\epsilon_{\mathfrak{g}} \cdot \hat{G}, \epsilon_{\mathfrak{n}} \cdot \hat{N}] = (\epsilon_{\mathfrak{g}} \rhd \epsilon_{\mathfrak{n}}) \cdot \hat{N} \,, \tag{6.78}$$

tj. komutator G-gejdž i N-gejdž transformacija je:

$$[\hat{G}_\alpha, \hat{N}_a] = \rhd_{\alpha a}{}^b \hat{N}_b \,. \tag{6.79}$$

Ispitajmo sada odnos između N-gejdž i H-gejdž transformacija, izračunavajući sledeći izraz:

$$\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot\hat{H}}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot\hat{N}} - \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot\hat{N}}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot\hat{H}} = -(\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}}) \cdot \hat{M} \,. \tag{6.80}$$

Dokaz je dat u Dodatku D. Dobijamo da je komutator H- i N-gejdž transformacija linearna kombinacija M-gejdž generatora:

$$[\hat{H}_a{}^\mu, \hat{N}^b] = \rhd_{\alpha a}{}^b \hat{M}^{\alpha\mu} \,. \tag{6.81}$$

Analognim postupkom, dobijamo relacije

$$\mathrm{e}^{\epsilon_{\mathfrak{l}}\cdot\hat{L}}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot\hat{N}} = \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot\hat{N}}\mathrm{e}^{\epsilon_{\mathfrak{l}}\cdot\hat{L}} \,, \quad \mathrm{e}^{\epsilon_{\mathfrak{m}}\cdot\hat{M}}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot\hat{N}} = \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot\hat{N}}\mathrm{e}^{\epsilon_{\mathfrak{m}}\cdot\hat{M}} \,, \tag{6.82}$$

iz kojih sledi da generatori L- i M-gejdž transformacija komutiraju sa generatorima N-gejdž transformacija:

$$[\hat{M}_\alpha{}^\mu, \hat{N}_a] = 0 \,, \qquad [\hat{L}_A{}^{\mu\nu}, \hat{N}_a] = 0 \,. \tag{6.83}$$

Ukupna gejdž grupa simetrije 3BF dejstva

Sumirajući rezultate prethodnih delova, može se napisati ukupna algebra generatora grupe gejdž simetrija na sledeći način.

- Algebra $\mathfrak{g}$ koja odgovara grupi G iz 2-ukrštenog modula $(L \xrightarrow{\delta} H \xrightarrow{\partial} G \,, \rhd \,, \{\_\,,\_\}_{\mathrm{pf}})$ data je komutacionim relacijama:

$$[\hat{G}_\alpha, \hat{G}_\beta] = f_{\alpha\beta}{}^\gamma \hat{G}_\gamma \,. \tag{6.84}$$

- Algebra koja odgovara grupi $\tilde{H}_L$ sastoji se od generatora H- i L-gejdž transformacija koji zadovoljavaju komutacione relacije:

$$[\hat{H}_a{}^\mu, \hat{H}_b{}^\nu] = 2X_{(ab)}{}^A \hat{L}_A{}^{\mu\nu} \,, \qquad [\hat{L}_A{}^{\mu\nu}, \hat{L}_B{}^{\rho\sigma}] = 0 \,, \qquad [\hat{H}_a{}^\mu, \hat{L}_A{}^{\nu\rho}] = 0 \,. \tag{6.85}$$

- Algebra generatora M-gejdž transformacija određena je komutacionim relacijama:

$$[\hat{M}_\alpha{}^\mu, \hat{M}_\beta{}^\nu] = 0 \,. \tag{6.86}$$

- Algebra generatora N-gejdž transformacija određena je komutacionim relacijama:

$$[\hat{N}_a, \hat{N}_b] = 0 \,. \tag{6.87}$$

- Komutatori između generatora grupa $\tilde{M}$ i $\tilde{N}$:

$$[\hat{M}_\alpha{}^\mu, \hat{N}_a] = 0\,. \tag{6.88}$$

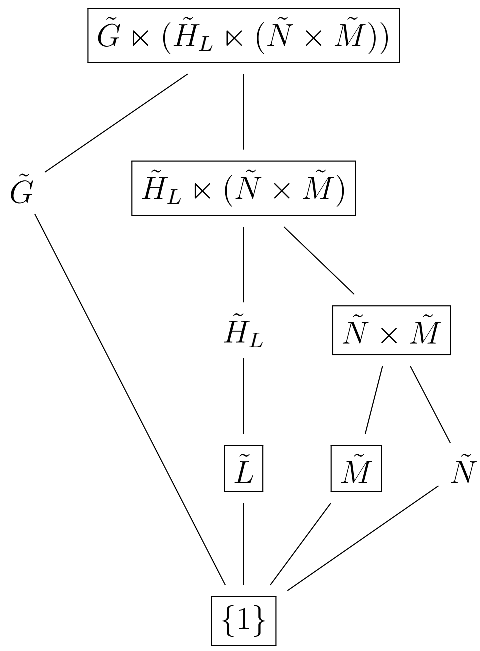
- Dejstvo generatora grupe $\tilde{H}_L$ na generatore M- i N-gejdž transformacija:

$$\begin{aligned} [\hat{H}_a{}^\mu, \hat{N}^b] &= \rhd_{\alpha a}{}^b \hat{M}^{\alpha\mu}\,, \\ [\hat{H}_a{}^\mu, \hat{M}_\alpha{}^\nu] &= 0\,, \\ [\hat{L}_A{}^{\nu\rho}, \hat{M}_\alpha{}^\mu] &= 0\,, \\ [\hat{L}_A{}^{\mu\nu}, \hat{N}_a] &= 0\,. \end{aligned} \tag{6.89}$$

- Dejstvo generatora grupe G na generatore H-, L-, M- i N-gejdž transformacija:

$$\begin{aligned} [\hat{G}_\alpha, \hat{H}_a{}^\mu] &= \rhd_{\alpha a}{}^b \hat{H}_b{}^\mu\,, \\ [\hat{G}_\alpha, \hat{L}_A{}^{\mu\nu}] &= \rhd_{\alpha A}{}^B \hat{L}_B{}^{\mu\nu}\,, \\ [\hat{G}_\alpha\,, \hat{M}_\beta{}^\mu] &= f_{\alpha\beta}{}^\gamma \hat{M}_\gamma{}^\mu\,, \\ [\hat{G}_\alpha, \hat{N}_a] &= \rhd_{\alpha a}{}^b \hat{N}_b\,. \end{aligned} \tag{6.90}$$

Na osnovu jednačina (6.84)-(6.90) dobijamo strukturu ukupne gejdž grupe simetrija. Na dijagramu Heseovog tipa prikazanom na slici 6.1, prikazane su sve relevantne podgrupe ukupne grupe simetrija $\mathcal{G}_{3BF}$, pri čemu su *invarijantne podgrupe* uokvirene.



Slika 6.1: Relevantne podgrupe ukupne grupe simetrija $\mathcal{G}_{3BF}$. Invarijantne podgrupe su uokvirene.

Na osnovu komutacionih relacija vidimo da su grupe $\tilde{L}$, $\tilde{M}$ i $\tilde{N}$ podgrupe ukupne grupe simetrija $\mathcal{G}_{3BF}$. Da su grupe $\tilde{L}$ i $\tilde{M}$ invarijantne podgrupe zaključujemo na osnovu toga što su jedini netrivijani komutatori generatora $\hat{L}_A{}^{\mu\nu}$, odnosno $\hat{M}_\alpha{}^\mu$, i generatora grupe G jednaki nekoj linearnoj kombinaciji generatora grupe $\tilde{L}$, odnosno $\tilde{M}$. Da grupa $\tilde{N}$ nije invarijantna podgrupa, zaključujemo na osnovu komutatora generatora $\hat{N}_a$ i generatora $\hat{H}_a{}^\mu$ koji su jednaki linearnim kombinacijama generatora $\hat{M}_\alpha{}^\mu$. Ipak, generatori $\tilde{N}$ i $\tilde{M}$ međusobno komutiraju, pa je grupa $\tilde{N}$ invarijantna podgrupa direktnog proizvoda grupa $\tilde{M}$ i $\tilde{N}$, tj. grupe $\tilde{N}\times\tilde{M}$. Grupa $\tilde{N}\times\tilde{M}$ je invarijantna podgrupa ukupne grupe simetrija.

Sa druge strane, u prethodnom delu smo videli da generatori H-gejdž transformacija zajedno sa generatorima L-gejdž transformacija formiraju grupu $\tilde{H}_L$. Da ova grupa nije invarijantna podgrupa ukupne grupe simetrija $\mathcal{G}_{3BF}$ vidimo iz oblika komutatora $\hat{H}_a{}^\mu$ i $\hat{N}_b$. Sada, ove dve podgrupe, $\tilde{N}\times\tilde{M}$ i $\tilde{H}_L$ formiraju semidirektan proizvod $\tilde{H}_L\ltimes(\tilde{N}\times\tilde{M})$. Proizvod je semidirektan jer grupa $\tilde{H}_L$ nije invarijantna podgrupa grupe $\tilde{H}_L\ltimes(\tilde{N}\times\tilde{M})$, zbog oblika komutatora između generatora $\hat{H}_a{}^\mu$ i $\hat{N}_b$, dok je grupa $\tilde{N}\times\tilde{M}$ invarijantna podgrupa iste grupe. Grupa $\tilde{H}_L\ltimes(\tilde{N}\times\tilde{M})$ je invarijantna podgrupa ukupne grupe simetrija $\mathcal{G}_{3BF}$.

Na kraju, uzimajući u obzir generatore G-gejdž transformacija, tj. komutacione relacije (6.90), a po istom principu zaključivanja, dobija se ukupna gejdž grupa simetrija $\mathcal{G}_{3BF}$:

$$\mathcal{G}_{3BF} = G\ltimes(\tilde{H}_L\ltimes(\tilde{N}\times\tilde{M}))\,. \tag{6.91}$$

Difeomorfizmi

Druga važna tema za diskusiju je invarijantnost $3BF$ teorije na difeomorfizme. Slično kao u slučaju $2BF$ teorije, ako su difeomorfizmi simetrija teorije, njihove varijacije forme se mogu izraziti kao zbir varijacija formi varijabli pri gejdž transformacijama i varijacija formi pri HT transformacijama:

$$\delta_0{}^{\text{diff}}\phi = -\delta_0{}^{\text{gauge}}\phi - \delta_0{}^{\text{HT}}\phi\,. \tag{6.92}$$

Konkretno, $3BF$ dejstvo zavisi od parametara $\alpha^\alpha{}_\mu$, $\beta^a{}_{\mu\nu}$, $\gamma^A{}_{\mu\nu\rho}$, $B^\alpha{}_{\mu\nu}$, $C^a{}_\mu$ i D^A. Parametri HT transformacija $\epsilon^{\text{HT}\alpha\beta}{}_{\mu\nu\rho}$, $\epsilon^{\text{HT}ab}{}_{\mu\nu\rho}$, i $\epsilon^{\text{HT}AB}{}_{\mu\nu\rho}$ su definisani relacijama (4.55)

$$\begin{array}{rclrcl}
\delta_0{}^{\text{HT}}\alpha^\alpha{}_\mu &=& \dfrac{1}{2}\epsilon^{\text{HT}\alpha\beta}{}_{\mu\nu\rho}\dfrac{\delta S}{\delta B^\beta{}_{\nu\rho}}\,, & \delta_0{}^{\text{HT}}B^\alpha{}_{\mu\nu} &=& -\epsilon^{\text{HT}\alpha\beta}{}_{\rho\mu\nu}\dfrac{\delta S}{\delta\alpha^\beta{}_\rho}\,,\\
\delta_0{}^{\text{HT}}\beta^a{}_{\mu\nu} &=& \epsilon^{\text{HT}ab}{}_{\mu\nu\rho}\dfrac{\delta S}{\delta C^b{}_\rho}\,, & \delta_0{}^{\text{HT}}C^a{}_\mu &=& -\dfrac{1}{2}\epsilon^{\text{HT}ab}{}_{\nu\rho\mu}\dfrac{\delta S}{\delta\beta^b{}_{\nu\rho}}\,,\\
\delta_0{}^{\text{HT}}\gamma^A{}_{\mu\nu\rho} &=& \epsilon^{\text{HT}AB}{}_{\mu\nu\rho}\dfrac{\delta S}{\delta D^B}\,, & \delta_0{}^{\text{HT}}D^A &=& -\dfrac{1}{3!}\epsilon^{\text{HT}AB}{}_{\mu\nu\rho}\dfrac{\delta S}{\delta\gamma^B{}_{\mu\nu\rho}}\,,
\end{array} \tag{6.93}$$

dok su gejdž parametri $\epsilon_{\mathfrak{g}}{}^\alpha$, $\epsilon_{\mathfrak{h}}{}^a{}_\mu$, $\epsilon_{\mathfrak{l}}{}^A{}_{\mu\nu}$, $\epsilon_{\mathfrak{m}}{}^\alpha{}_\mu$ i $\epsilon_{\mathfrak{n}}{}^a$ definisani u Teoremama 14–18. Možemo pokazati da zaista postoji izbor ovih parametara, tako da je jednačina (4.57) zadovoljena za sva polja. Konkretno, ako odaberemo gejdž parametre kao

$$\epsilon_{\mathfrak{g}}{}^\alpha = -\xi^\lambda\alpha^\alpha{}_\lambda\,,\quad \epsilon_{\mathfrak{h}}{}^a{}_\mu = \xi^\lambda\beta^a{}_{\mu\lambda}\,,\quad \epsilon_{\mathfrak{l}}{}^A{}_{\mu\nu} = \xi^\lambda\gamma^A{}_{\mu\nu\lambda}\,,\quad \epsilon_{\mathfrak{m}}{}^\alpha{}_\mu = \xi^\lambda B^\alpha{}_{\mu\lambda}\,,\quad \epsilon_{\mathfrak{n}}{}^a = -\xi^\lambda C^a{}_\lambda\,, \tag{6.94}$$

a HT parametre kao

$$\epsilon^{\text{HT}\alpha\beta}{}_{\mu\nu\rho} = \xi^\lambda g^{\alpha\beta}\epsilon_{\mu\nu\rho\lambda}\,,\qquad \epsilon^{\text{HT}ab}{}_{\mu\nu\rho} = \xi^\lambda g^{ab}\epsilon_{\lambda\mu\nu\rho}\,,\qquad \epsilon^{\text{HT}AB}{}_{\mu\nu\rho} = \xi^\lambda g^{AB}\epsilon_{\mu\nu\rho\lambda}\,, \tag{6.95}$$

primenom jednačine (6.92) dobijamo upravo standardne varijacije formi koje odgovaraju difeomorfizmima:

$$\begin{array}{rcl}
\delta_0{}^{\text{diff}}\alpha^\alpha{}_\mu &=& -\partial_\mu\xi^\lambda\alpha^\alpha{}_\lambda - \xi^\lambda\partial_\lambda\alpha^\alpha{}_\mu\,,\\
\delta_0{}^{\text{diff}}\beta^a{}_{\mu\nu} &=& -\partial_\mu\xi^\lambda\beta^a{}_{\lambda\nu} - \partial_\nu\xi^\lambda\beta^a{}_{\mu\lambda} - \xi^\lambda\partial_\lambda\beta^a{}_{\mu\nu}\,,\\
\delta_0{}^{\text{diff}}\gamma^A{}_{\mu\nu\rho} &=& -\partial_\mu\xi^\lambda\gamma^A{}_{\lambda\nu\rho} - \partial_\nu\xi^\lambda\gamma^A{}_{\mu\lambda\rho} - \partial_\rho\xi^\lambda\gamma^A{}_{\mu\nu\lambda} - \xi^\lambda\partial_\lambda\gamma^A{}_{\mu\nu\rho}\,,\\
\delta_0{}^{\text{diff}}B^\alpha{}_{\mu\nu} &=& -\partial_\mu\xi^\lambda B^\alpha{}_{\lambda\nu} - \partial_\nu\xi^\lambda B^\alpha{}_{\mu\lambda} - \xi^\lambda\partial_\lambda B^\alpha{}_{\mu\nu}\,,\\
\delta_0{}^{\text{diff}}C^a{}_\mu &=& -\partial_\mu\xi^\lambda C^a{}_\lambda - \xi^\lambda\partial_\lambda C^a{}_\mu\,,\\
\delta_0{}^{\text{diff}}D^A &=& -\xi^\lambda\partial_\lambda D^A\,.
\end{array} \tag{6.96}$$

Ovim se utvrđuje da su difeomorfizmi zaista simetrija teorije, čak i ako nisu sadržani u ukupnoj gejdž grupi simetrija $\mathcal{G}_{3BF}$, već u direktnom proizvodu ukupne grupe simetrija i HT grupe simetrija.

6.2 Klajn-Gordonova teorija

U ovom odeljku ćemo demonstrirati kako možemo da iskoristimo strukturu 3-grupe, odnosno odgovarajuću $3BF$ teoriju da opišemo Klajn-Gordonovo polje koje interaguje sa gravitacionim poljem [16]. Najpre, neophodno je precizirati 2-ukršteni modul za koji se definiše $3BF$ teorija, a zatim se teorija sa odgovarajućom dinamikom konstruiše dodavanjem odgovarajućih *veza* topološkom $3BF$ dejstvu. Definišimo 2-ukršteni modul $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\}_{\rm pf})$, na sledeći način. Lijeve grupe G, H i L su:

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}\,. \tag{6.97}$$

Grupa G deluje na samu sebe konjugacijom, na grupu H po vektorskoj reprezentaciji, dok je dejstvo grupe G na grupu L trivijalno. Ovim je definisano dejstvo $\triangleright$. Preslikavanje ∂ je trivijalno, kao što je to slučaj kod čiste gravitacije. Preslikavanje δ je takođe izabrano da bude trivijalno, odnosno svaki element grupe L se preslikava u jedinični element grupe H. Najzad, Pajferovo podizanje je takođe trivijalno, odnosno svaki uređeni par elemenata grupe H se preslikava u jedinični element grupe L. Ovo definiše jedan određeni izbor 2-ukrštenog modula, koji odgovara jednom skalarnom polju u interakciji sa gravitacionim poljem, kao što ćemo to demonstrirati u ovom odeljku.

Za ovaj izbor 2-ukrštenog modula, 3-koneksija $(\alpha\,,\beta\,,\gamma)$ je

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma \mathbb{I}\,, \tag{6.98}$$

gde $\mathbb{I}$ označava generator Lijeve grupe $\mathbb{R}$. Kako su preslikavanja ∂, δ i Pajferovo podizanje trivijalni, lažna 3-krivina (2.118) se svodi na običnu 3-krivinu,

$$\mathcal{F} = R^{ab}M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,, \qquad \mathcal{H} = {\rm d}\gamma\,. \tag{6.99}$$

Ovde je iskorišćena činjenica da je dejstvo grupe G na grupu L trivijalno, tj. $M_{ab} \triangleright \mathbb{I} = 0$. Ovo znači da se 3-forma γ transformiše kao skalar pri Lorencovim transformacijama. Dakle, odgovarajući Lagranžev množitelj D se transformiše na isti način, što vidimo na osnovu njegove indeksne strukture. Kako je D 0-forma, on se transformiše kao skalar i pri delovanju simetrije difeomorfizama. Na osnovu ovoga sledi da se Lagranžev množitelj D transformiše kao realno skalarno polje pri svim transformacijama, odnosno možemo ga označiti kao $D \equiv \phi$, i napisati topološko $3BF$ dejstvo (6.1) kao:

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \phi\,{\rm d}\gamma\,, \tag{6.100}$$

gde je bilinearna forma na L definisana kao $\langle \mathbb{I}, \mathbb{I} \rangle_{\mathfrak{l}} = 1$.

Da bi opisali skalarno polje mase m sa odgovarajućom dinamikom opisanom Klajn-Gordonovom jednačinom u interakciji sa gravitacionim poljem neophodno je dodati odgovarajuće veze topološkom dejstvu (6.100):

$$
\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma \\
& - \lambda_{ab} \wedge \Big(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \Big) \\
& + \lambda \wedge \Big(\gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c \Big) \\
& + \Lambda^{ab} \wedge \Big(H_{abc} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b \Big) \\
& - \frac{1}{2 \cdot 4!} m^2 \phi^2 \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \, .
\end{aligned} \tag{6.101}
$$

U prethodnom izrazu prvi red predstavlja topološki sektor (6.100), drugi red je poznata *simplicity veza* za gravitaciju uvedena u dejstvu (5.85), treći i četvrti red nove *simplicity veze* u kojima se pojavljuju 1-forme Lagranževi množitelji λ i Λ^{ab} i 0-forma Lagranžev množitelj H_{abc}, dok poslednji red obezbeđuje odgovarajuću masu m skalarnog polja ϕ. Variranjem dejstva (6.101) redom po varijablama B_{ab}, ω_{ab}, β_a, λ_{ab}, Λ_{ab}, γ, λ, H_{abc}, ϕ i e^a dobijamo jednačine kretanja:

$$ R^{ab} - \lambda^{ab} = 0 \, , \tag{6.102} $$

$$ \nabla B^{ab} - e^{[a} \wedge \beta^{b]} = 0 \, , \tag{6.103} $$

$$ \nabla e^a = 0 \, , \tag{6.104} $$

$$ B_{ab} - \frac{1}{16\pi l_p^2} \varepsilon_{abcd} e^c \wedge e^d = 0 \, , \tag{6.105} $$

$$ H_{abc} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b = 0 \, , \tag{6.106} $$

$$ \mathrm{d}\phi - \lambda = 0 \, , \tag{6.107} $$

$$ \gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c = 0 \, , \tag{6.108} $$

$$ -\frac{1}{2} \lambda \wedge e^a \wedge e^b \wedge e^c + \varepsilon^{cdef} \Lambda^{ab} \wedge e_d \wedge e_e \wedge e_f = 0 \, , \tag{6.109} $$

$$ \mathrm{d}\gamma - \mathrm{d}(\Lambda^{ab} \wedge e_a \wedge e_b) - \frac{1}{4!} m^2 \phi \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d = 0 \, , \tag{6.110} $$

$$
\begin{aligned}
\nabla \beta_a + \frac{1}{8\pi l_p^2} \varepsilon_{abcd} \lambda^{bc} \wedge e^d + \frac{3}{2} H_{abc} \lambda \wedge e^b \wedge e^c + 3 H^{def} \varepsilon_{abcd} \Lambda_{ef} \wedge e^b \wedge e^c \\
- 2 \Lambda_{ab} \wedge \mathrm{d}\phi \wedge e^b - 2 \frac{1}{4!} m^2 \phi \varepsilon_{abcd} e^b \wedge e^c \wedge e^d = 0 \, .
\end{aligned} \tag{6.111}
$$

Ovaj sistem jednačina opisuje dva dinamička polja, tetrade e^a i skalarno polje ϕ, dok se sve ostale varijable mogu izraziti kao funkcije njih i njihovih izvoda:

$$
\begin{gathered}
\lambda_{ab\mu\nu} = R_{ab\mu\nu} \, , \qquad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu \, , \qquad \gamma_{\mu\nu\rho} = -\frac{e}{2} \varepsilon_{\mu\nu\rho\sigma} \partial^\sigma \phi \, , \\
\beta^a{}_{\mu\nu} = 0 \, , \qquad \Lambda^{ab}{}_\mu = \frac{1}{12e} g_{\mu\lambda} \varepsilon^{\lambda\nu\rho\sigma} \partial_\nu \phi e^a{}_\rho e^b{}_\sigma \, , \qquad \lambda_\mu = \partial_\mu \phi \, , \\
H^{abc} = \frac{1}{6e} \varepsilon^{\mu\nu\rho\sigma} \partial_\mu \phi e^a{}_\nu e^b{}_\rho e^c{}_\sigma \, , \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu \, .
\end{gathered} \tag{6.112}
$$

Za razliku od jednačina kretanja za Lagranževe množitelje, jednačine kretanja za e^a i ϕ su diferencijalne jednačine kretanja, pri čemu jednačina kretanja za ϕ (6.110) daje Klajn-Gordonovu jednačinu kretanja,

$$\left(\nabla_\mu \nabla^\mu - m^2\right)\phi = 0\,, \tag{6.113}$$

dok je jednačina kretanja za polja tetrada (6.111)

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \tag{6.114}$$

gde je tenzor energije-impulsa realnog skalarnog polja

$$T^{\mu\nu} \equiv \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\left(\partial_\rho\phi\partial^\rho\phi + m^2\phi^2\right). \tag{6.115}$$

6.3 Ajnštajn-Kartan-Dirak teorija

Kako bismo opisali Dirakovo polje koje interaguje sa Ajnštajn-Kartanovom gravitacijom definišemo 2-ukršteni modul $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\}_{\mathrm{pf}})$ na sledeći način [16]. Lijeve grupe G, H i L su

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^8(\mathbb{G})\,, \tag{6.116}$$

gde je $\mathbb{G}$ oznaka za skup Grasmanovih brojeva. Preslikavanja ∂, δ i Pajferovo podizanje ostaju trivijalni, kao što je to bio slučaj kod skalarnog polja. Grupa G deluje na samu sebe konjugacijom, na grupu H po vektorskoj reprezentaciji, dok na grupu L deluje po spinorskoj reprezentaciji. Formalno zapisano, ako su P_α i P^α 8 generatora Lijeve grupe $\mathbb{R}^8(\mathbb{G})$, pri čemu indeks α uzima vrednosti $1,\dots,4$, dejstvo $\triangleright$ grupe G na grupu L definisano je na sledeći način:

$$M_{ab} \triangleright P_\alpha = \frac{1}{2}(\sigma_{ab})^\beta{}_\alpha P_\beta\,, \qquad M_{ab} \triangleright P^\alpha = -\frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta\,, \tag{6.117}$$

gde je korišćena standardna notacija za $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, gde su γ_a Dirakove matrice, koje zadovoljavaju antikomutacione relacije

$$\{\gamma_a, \gamma_b\} \equiv \gamma_a\gamma_b + \gamma_b\gamma_a = -2\eta_{ab}\,.$$

Kao što je to bio slučaj kod skalarnog polja, vidimo da izbor grupe L određuje polja materije prisutna u teoriji, dok dejstvo $\triangleright$ grupe G na grupu L osigurava odgovarajuće transformacione osobine polja.

Sada kada smo upotpunili definiciju 2-ukrštenog modula, možemo definisati odgovarajuće $3BF$ dejstvo. Uređena trojka 3-koneksije $(\alpha\,,\beta\,,\gamma)$ za ovaj izbor 3-grupe je:

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma^\alpha P_\alpha + \bar{\gamma}_\alpha P^\alpha\,, \tag{6.118}$$

dok je 3-krivina $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$:

$$\begin{aligned} &\mathcal{F} = R^{ab}M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,, \\ \mathcal{H} &= \left(\mathrm{d}\gamma^\alpha + \frac{1}{2}\omega^{ab}(\sigma_{ab})^\alpha{}_\beta\gamma^\beta\right)P_\alpha + \left(\mathrm{d}\bar{\gamma}_\alpha - \frac{1}{2}\omega^{ab}\bar{\gamma}_\beta(\sigma_{ab})^\beta{}_\alpha\right)P^\alpha \\ &\equiv (\vec{\nabla}\gamma)^\alpha P_\alpha + (\bar{\gamma}\overleftarrow{\nabla})_\alpha P^\alpha\,. \end{aligned} \tag{6.119}$$

U prethodnim izrazima korišćena je definicija dejstva $\triangleright$ (6.117). Bilinearna forma $\langle\_\,,\_\rangle_{\mathfrak{l}}$ je definisana delovanjem na generatore grupe L, na sledeći način

$$\begin{aligned} \langle P_\alpha, P_\beta\rangle_{\mathfrak{l}} = 0\,, \qquad \langle P^\alpha, P^\beta\rangle_{\mathfrak{l}} = 0\,, \\ \langle P_\alpha, P^\beta\rangle_{\mathfrak{l}} = -\delta_\alpha^\beta\,, \qquad \langle P^\alpha, P_\beta\rangle_{\mathfrak{l}} = \delta^\alpha_\beta\,. \end{aligned} \tag{6.120}$$

Primetimo da je bilinearna forma definisana na ovaj način antisimetrična kada deluje na generatore, za razliku od bilinearne simetrične koju smo imali u primerima do sada. Motivacija za ovako definisanom bilinearnom formom sastoji se u sledećem. Za elemente $A, B \in \mathfrak{l}$ bilinearna forma je simetrična bilinearna nedegenerisana forma. Razvijajući elemente A i B po bazisu algebre vidimo da je:

$$\langle A, B\rangle_{\mathfrak{l}} = A^I B^J g_{IJ}\,, \qquad \langle B, A\rangle_{\mathfrak{l}} = B^J A^I g_{JI}\,. \tag{6.121}$$

Kako bilinearna forma mora biti simetrična, dva izraza u prethodnoj jednačini moraju biti jednaka. Kako su koeficijenti u $\mathfrak{l}$ Grasmanovi brojevi, imamo da je $A^I B^J = -B^J A^I$, iz čega sledi da je $g_{IJ} = -g_{JI}$. Sada je jasna antisimetričnost (6.120) — ona kompenzuje antikomutirajuću prirodu Grasmanovih brojeva, osiguravajući da bilinearna forma bude simetrična za bilo koja dva elementa $A, B \in \mathfrak{l}$.

Dejstvo $\triangleright$ grupe G na grupu L se definiše tako da obezbedi spinorsku prirodu Lagranževog množitelja D u dejstvu (6.1). Grupa L određuje strukturu polja D tako da su njegove komponente 8 nezavisnih Grasmanovih polja materije. Dalje, na osnovu činjenice da je polje D diferencijalna 0-forma i da se transformiše po spinorskoj reprezentaciji pod dejstvom grupe $SO(3,1)$, možemo ga identifikovati sa Dirakovim bispinorom:

$$D = \psi^\alpha P_\alpha + \bar{\psi}_\alpha P^\alpha\,. \tag{6.122}$$

Kao u slučaju skalarnog polja, ovo je demonstracija kako struktura sektora materije prisutne u teoriji može biti zadata određenim izborom grupe L i dejstvom $\triangleright$ grupe G na nju, komponentama 2-ukrštenog modula. Transformacione osobine polja pri delovanju Lorencove grupe definišemo odgovarajućim izborom dejstva $\triangleright$.

Za ovaj izbor 2-ukrštenog modula, a nakon izvršene identifikacije polja, možemo definisati odgovarajuće $3BF$ dejstvo:

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + e_a \wedge \nabla\beta^a + (\bar{\gamma}\overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla}\gamma)^\alpha\,. \tag{6.123}$$

Kako bismo definisali spinorska polja sa odgovarajućom dinamikom kuplovana sa Ajnštajn-Kartanovom gravitacijom, neophodno je dejstvu (6.123) dodati odgovarajuće *simplicity veze*:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab}\wedge R_{ab} + e_a \wedge \nabla\beta^a + (\bar{\gamma}\overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla}\gamma)^\alpha \\ & - \lambda_{ab}\wedge\left(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c\wedge e_d\right) \\ & - \lambda^\alpha \wedge \left(\bar{\gamma}_\alpha - \frac{i}{6}\varepsilon_{abcd} e^a\wedge e^b\wedge e^c (\bar{\psi}\gamma^d)_\alpha\right) \\ & + \bar{\lambda}_\alpha \wedge \left(\gamma^\alpha + \frac{i}{6}\varepsilon_{abcd} e^a\wedge e^b\wedge e^c (\gamma^d\psi)^\alpha\right) \\ & - \frac{1}{12} m\,\bar{\psi}\psi\,\varepsilon_{abcd} e^a\wedge e^b\wedge e^c\wedge e^d + 2\pi i l_p^2\,\bar{\psi}\gamma_5\gamma^a\psi\,\varepsilon_{abcd} e^b\wedge e^c\wedge \beta^d. \end{aligned} \tag{6.124}$$

Analogno prethodnom slučaju skalarnog polja, prvi red je topološki sektor dejstva (6.123), drugi red je gravitaciona veza, dok su treći i četvrti red nove *simplicity veze* za Dirakovo polje, u kojima se pojavljuju 1-forme Lagranževih množitelja λ^α i $\bar{\lambda}_\alpha$. Peti red sadrži maseni član za Dirakovo polje i član koji osigurava odgovarajuću interakciju između torzije i spina Dirakovog polja. Na osnovu Ajnštajn-Kartanove teorije imamo da je

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a\,, \tag{6.125}$$

jedna jednačina kretanja, gde je

$$s_a = i\varepsilon_{abcd}e^b \wedge e^c\bar{\psi}\gamma_5\gamma^d\psi \tag{6.126}$$

2-forma Dirakovog spina. Naravno, alternativni izbori su mogući, ali ćemo se u ovom izlaganju ograničiti na ovaj.

Variranjem dejstva (6.124) redom po varijablama B_{ab}, λ^{ab}, $\bar{\gamma}_\alpha$, γ^α, λ^α, $\bar{\lambda}_\alpha$, $\bar{\psi}_\alpha$, ψ^α, e^a, β^a i ω^{ab}, dobijamo jednačine kretanja:

$$R^{ab} - \lambda^{ab} = 0\,, \tag{6.127}$$

$$B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd}e^c \wedge e^d = 0\,, \tag{6.128}$$

$$(\overrightarrow{\nabla}\psi)^\alpha - \lambda^\alpha = 0\,, \tag{6.129}$$

$$(\bar{\psi}\overleftarrow{\nabla})_\alpha - \bar{\lambda}_\alpha = 0\,, \tag{6.130}$$

$$\bar{\gamma}_\alpha - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c(\bar{\psi}\gamma^d)_\alpha = 0\,, \tag{6.131}$$

$$\gamma^\alpha + \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c(\gamma^d\psi)^\alpha = 0\,, \tag{6.132}$$

$$\begin{aligned} &\mathrm{d}\gamma^\alpha + \omega^\alpha{}_\beta \wedge \gamma^\beta + \frac{i}{6}\lambda^\beta \wedge \varepsilon_{abcd}e^a \wedge e^b \wedge e^c\gamma^{d\alpha}{}_\beta + \frac{1}{12}m\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d\psi^\alpha \\ &+ i2\pi l_p^2\varepsilon_{abcd}e^a \wedge e^b \wedge \beta^c(\gamma_5\gamma^d\psi)^\alpha = 0\,, \end{aligned} \tag{6.133}$$

$$\begin{aligned} &\mathrm{d}\bar{\gamma}_\alpha - \bar{\gamma}_\beta \wedge \omega^\beta{}_\alpha + \frac{i}{6}\bar{\lambda}_\beta \wedge \varepsilon_{abcd}e^a \wedge e^b \wedge e^c\gamma^{d\beta}{}_\alpha - \frac{1}{12}m\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d\bar{\psi}_\alpha \\ &- i2\pi l_p^2\varepsilon_{abcd}e^a \wedge e^b \wedge \beta^c(\bar{\psi}\gamma_5\gamma^d)_\alpha = 0\,, \end{aligned} \tag{6.134}$$

$$\begin{aligned} &\nabla\beta_a + 2\varepsilon_{abcd}\lambda^{bc} \wedge e^d - \frac{i}{2}\varepsilon_{abcd}\lambda^\alpha \wedge e^b \wedge e^c(\bar{\psi}\gamma^d)_\alpha + \frac{i}{2}\varepsilon_{abcd}\bar{\lambda}_\alpha \wedge e^b \wedge e^c(\gamma^d\psi)^\alpha \\ &- \frac{1}{3}\varepsilon_{abcd}e^b \wedge e^c \wedge e^d m\bar{\psi}\psi - 4\pi l_p^2 i\varepsilon_{abcd}e^b \wedge \beta^c\bar{\psi}\gamma_5\gamma^d\psi = 0\,, \end{aligned} \tag{6.135}$$

$$\nabla e_a - i2\pi l_p^2\varepsilon_{abcd}e^b \wedge e^c\bar{\psi}\gamma_5\gamma^d\psi = 0\,, \tag{6.136}$$

$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} + \bar{\gamma}\frac{1}{8}[\gamma_a, \gamma_b]\psi + \bar{\psi}\frac{1}{8}[\gamma_a, \gamma_b]\gamma = 0\,. \tag{6.137}$$

Jedina dinamička polja u teoriji su e^a, ψ i $\bar{\psi}$, dok se preostala mogu algebarski izraziti u funkciji dinamičkih polja i njihovih izvoda:

$$\begin{gathered} B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \qquad \lambda^\alpha{}_\mu = (\overrightarrow{\nabla}_\mu\psi)^\alpha\,, \qquad \bar{\lambda}_{\alpha\mu} = (\bar{\psi}\overleftarrow{\nabla}_\mu)_\alpha\,, \\ \bar{\gamma}_{\alpha\mu\nu\rho} = i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\bar{\psi}\gamma^d)_\alpha\,, \qquad \gamma^\alpha{}_{\mu\nu\rho} = -i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\gamma^d\psi)^\alpha\,, \\ \beta^a{}_{\mu\nu} = 0\,, \qquad \lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu + K^{ab}{}_\mu\,. \end{gathered} \tag{6.138}$$

Ovde je $K^{ab}{}_\mu$ tenzor kontorzije, definisan na standardan način kao funkcija tenzora torzije. Pored toga, vidimo dejstvo daje odgovarajuću torziju, kako dobijamo da je jedna jednačina kretanja upravo (6.125):

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a\,. \tag{6.139}$$

Najzad, jednačine kretanja za varijable ψ i $\bar{\psi}$ su standardne kovarijantne Dirakove jednačine kretanja

$$(i\gamma^a e^\mu{}_a \overrightarrow{\nabla}_\mu - m)\psi = 0\,, \tag{6.140}$$

i konjugovana,

$$\bar{\psi}(i\overleftarrow{\nabla}_\mu e^\mu{}_a \gamma^a + m) = 0\,, \tag{6.141}$$

gde je $e^\mu{}_a$ inverzna tetrada. Jednačina kretanja za polje tetrade e^a je

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \tag{6.142}$$

gde je tenzor energije-impulsa:

$$T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\gamma^a\overleftrightarrow{\nabla}^\nu e^\mu{}_a\psi - \frac{1}{2}g^{\mu\nu}\bar{\psi}\Big(i\gamma^a\overleftrightarrow{\nabla}_\rho e^\rho{}_a - 2m\Big)\psi\,, \tag{6.143}$$

Ovde je korišćena notacija $\overleftrightarrow{\nabla} = \overrightarrow{\nabla} - \overleftarrow{\nabla}$. Kao što je očekivano, jednačine (6.139), (6.140), (6.141) i (6.142) su upravo jednačine kretanja koje opisuje Dirakovo polje koje interaguje sa Ajnštajn-Kartanovom gravitacijom.

6.4 Vajlova i Majorana polja u interakciji sa Ajnštajn-Kartanovom gravitacijom

Kao što znamo, rešenje Dirakove jednačine nije ireducibilna reprezentacija Lorencove grupe. Dirakove fermione moguće je prepisati tako da razdvojimo polja leve kiralnosti i polja desne kiralnosti, koji su ireducibilne reprezentacije, odnosno koji pri Lorencovim transformacijama ne menjaju svoju kiralnost. Da bismo u okviru našeg pristupa razmatrali ove spinorske reprezentacije neophodno naprepisati teoriju levog i desnog Vajlovog polja kao $3BF$ dejstvo sa vezama. Radi jednostavnosti, ovde ćemo razmatrati samo levo kiralno polje, pri čemu se teorija desnog kiralnog polja definiše analogno. Vajlovi i Majorana fermioni mogu se tretirati na ovaj način, pri čemu je u slučaju Majorana fermiona dejstvu neophodno dodati dodatni maseni član.

Odgovarajući 2-ukršteni modul $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\}_{\text{pf}})$ se definiše na sledeći način [16]. Lijeve grupe G, H i L su:

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^4(\mathbb{G})\,. \tag{6.144}$$

Preslikavaja ∂, δ i Pajferovo podizanje su trivijalna. Dejstvo $\triangleright$ grupe G na grupe G, H i L je definisano na isti način kao u slučaju Dirakovih fermiona, pri čemu je spinorska reprezentacija za levo kiralno polje:

$$M_{ab} \triangleright P^\alpha = \frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta\,, \qquad M_{ab} \triangleright P_{\dot\alpha} = \frac{1}{2}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha} P_{\dot\beta}\,, \tag{6.145}$$

gde su $\sigma^{ab} = -\bar\sigma^{ab} = \frac{1}{4}(\sigma^a\bar\sigma^b - \sigma^b\bar\sigma^a)$, za $\sigma^a = (1,\vec\sigma)$ i $\bar\sigma^a = (1,-\vec\sigma)$, pri čemu oznaka $\vec\sigma$ označava tri Paulijeve matrice. Četiri generatora grupe L su označena sa P^α i $P_{\dot\alpha}$, gde Vajlovi indeksi $\alpha, \dot\alpha$ uzimaju vrednosti $1, 2$.

Odgovarajuća 3-koneksija $(\alpha\,,\beta\,,\gamma)$ za ovakav izbor 2-ukrštenog modula ima sledeći oblik

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma_\alpha P^\alpha + \bar\gamma^{\dot\alpha} P_{\dot\alpha}\,, \tag{6.146}$$

dok je odgovarajuća lažna 3-krivina $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ definisana jednačinom (2.118):

$$\mathcal{F} = R^{ab}M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,,$$
$$\mathcal{H} = \big(\mathrm{d}\gamma_\alpha + \frac{1}{2}\omega_{ab}(\sigma^{ab})^\beta{}_\alpha\gamma_\beta\big)P^\alpha + \big(\mathrm{d}\bar{\gamma}^{\dot{\alpha}} + \frac{1}{2}\omega_{ab}(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\gamma}^{\dot{\beta}}\big)P_{\dot{\alpha}} \equiv (\overset{\rightarrow}{\nabla}\gamma)_\alpha P^\alpha + (\bar{\gamma}\overset{\leftarrow}{\nabla})^{\dot{\alpha}}P_{\dot{\alpha}}\,. \tag{6.147}$$

Analogno slučaju Dirakovih spinora, Lagranžev množitelj D identifikuje se sa spinorskim poljima ψ_α i $\bar{\psi}^{\dot{\alpha}}$

$$D = \psi_\alpha P^\alpha + \bar{\psi}^{\dot{\alpha}}P_{\dot{\alpha}}\,, \tag{6.148}$$

dok se bilinearna forma $\langle\_,\_\rangle_{\mathfrak{l}}$ na grupi L definiše delovanjem na generatore

$$\langle P^\alpha, P^\beta\rangle_{\mathfrak{l}} = \varepsilon^{\alpha\beta}\,, \qquad \langle P_{\dot{\alpha}}, P_{\dot{\beta}}\rangle_{\mathfrak{l}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\,, \qquad \langle P^\alpha, P_{\dot{\beta}}\rangle_{\mathfrak{l}} = 0\,, \qquad \langle P_{\dot{\alpha}}, P^\beta\rangle_{\mathfrak{l}} = 0\,. \tag{6.149}$$

U prethodnim jednačinama $\varepsilon^{\alpha\beta}$ i $\varepsilon_{\dot{\alpha}\dot{\beta}}$ su standardne oznake za dvodimenzionalan Levi-Čivita simbol. Sada je moguće definisati topološko 3BF dejstvo (6.1) za spinorska polja i gravitaciju

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + e_a\wedge\nabla\beta^a + \psi^\alpha\wedge(\overset{\rightarrow}{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}}\wedge(\bar{\gamma}\overset{\leftarrow}{\nabla})^{\dot{\alpha}}\,. \tag{6.150}$$

Kako bismo dobili teoriju sa odgovarajućom dinamikom Vajlovih spinora, neophodno je dejstvu (6.150) dodati odgovarajuće *simplicity veze*, tako da je 3BF dejstvo sa vezama:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab}\wedge R_{ab} + e_a\wedge\nabla\beta^a + \psi^\alpha\wedge(\overset{\rightarrow}{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}}\wedge(\bar{\gamma}\overset{\leftarrow}{\nabla})^{\dot{\alpha}} \\ & - \lambda_{ab}\wedge(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd}e_c\wedge e_d) \\ & - \lambda^\alpha\wedge(\gamma_\alpha + \frac{i}{6}\varepsilon_{abcd}e^a\wedge e^b\wedge e^c\sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}) - \bar{\lambda}_{\dot{\alpha}}\wedge(\bar{\gamma}^{\dot{\alpha}} + \frac{i}{6}\varepsilon_{abcd}e^a\wedge e^b\wedge e^c\bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta) \\ & - 4\pi l_p^2\varepsilon_{abcd}e^a\wedge e^b\wedge\beta^c(\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta)\,. \end{aligned} \tag{6.151}$$

Treći red sadrži nove veze i 1-forme Lagranževih množitelja λ_α i $\bar{\lambda}^{\dot{\alpha}}$. Četvrti red osigurava odgovarajuću interakciju između torzije i spina Vajlovog polja. Ovde smo koristeći interakciju torzije i spina u slučaju Dirakovih čestica, definisali odgovarajuću interakciju spina Vajlovog polja

$$s_a \equiv i\varepsilon_{abcd}e^b\wedge e^c\;\psi^\alpha\sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}\,, \tag{6.152}$$

i torzije na sledeći način:

$$T_a = 4\pi l_p^2 s_a\,. \tag{6.153}$$

Majorana polja su definisana analogno, pri čemu se dejstvu dodaje još i maseni član :

$$-\frac{1}{12}m\varepsilon_{abcd}e^a\wedge e^b\wedge e^c\wedge e^d(\psi^\alpha\psi_\alpha + \bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}})\,. \tag{6.154}$$

Variranjem dejstva (6.151) redom po varijablama B_{ab}, λ^{ab}, γ_α, $\bar{\gamma}^{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, e^a, β^a i ω^{ab} dobijamo jednačine kretanja, koje su prikazane u dodatku B. Jedini dinamički stepeni slobode su polja ψ_α, $\bar{\psi}^{\dot{\alpha}}$ i e^a, dok je preostale varijable moguće algebarski izraziti kao funkcije ovih polja i njihovih izvoda:

$$\begin{gathered} \lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu}\,, \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \quad \lambda_{\alpha\mu} = \nabla_\mu\psi_\alpha\,, \quad \bar{\lambda}^{\dot{\alpha}}{}_\mu = \nabla_\mu\bar{\psi}^{\dot{\alpha}}\,, \\ \gamma_{\alpha\mu\nu\rho} = i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho\sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}\,, \quad \bar{\gamma}^{\dot{\alpha}}{}_{\mu\nu\rho} = i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho\bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta\,, \quad \omega_{ab\mu} = \triangle_{ab\mu} + K_{ab\mu}\,. \end{gathered} \tag{6.155}$$

Primetimo da je rezultat $\beta = 0$ nepromenjen. Najzad, jednačine kretanja dinamičkih polja u teoriji su

$$\bar{\sigma}^{a\dot{\alpha}\beta} e^{\mu}{}_{a} \nabla_{\mu} \psi_{\beta} = 0\,, \qquad \sigma^{a}{}_{\alpha\dot{\beta}} e^{\mu}{}_{a} \nabla_{\mu} \bar{\psi}^{\dot{\beta}} = 0\,,$$
$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2 \, T^{\mu\nu}\,, \tag{6.156}$$
$$T^{\mu\nu} \equiv \frac{i}{2} \bar{\psi} \bar{\sigma}^{b} e^{\nu}{}_{b} \nabla^{\mu} \psi + \frac{i}{2} \psi \sigma^{b} e^{\nu}{}_{b} \nabla^{\mu} \bar{\psi} - g^{\mu\nu} \frac{1}{2} \Big(i\bar{\psi} \bar{\sigma}^{a} e^{\lambda}{}_{a} \nabla_{\lambda} \psi + i\psi \sigma^{a} e^{\lambda}{}_{a} \nabla_{\lambda} \bar{\psi} \Big)\,.$$

U Majorana slučaju jednačine kretanja (6.155) ostaju ista, dok su jednačine kretanja za polja ψ_α i $\bar{\psi}^{\dot{\alpha}}$,

$$i\sigma^{a}{}_{\alpha\dot{\beta}} e^{\mu}{}_{a} \nabla_{\mu} \bar{\psi}^{\dot{\beta}} - m\psi_{\alpha} = 0\,, \qquad i\bar{\sigma}^{a\dot{\alpha}\beta} e^{\mu}{}_{a} \nabla_{\mu} \psi_{\beta} - m\bar{\psi}^{\dot{\alpha}} = 0\,, \tag{6.157}$$

a tenzor energije-impulsa ima oblik:

$$T^{\mu\nu} \equiv \frac{i}{2} \bar{\psi} \bar{\sigma}^{b} e^{\nu}{}_{b} \nabla^{\mu} \psi + \frac{i}{2} \psi \sigma^{b} e^{\nu}{}_{b} \nabla^{\mu} \bar{\psi} - g^{\mu\nu} \frac{1}{2} \Big[i\bar{\psi} \bar{\sigma}^{a} e^{\lambda}{}_{a} \nabla_{\lambda} \psi + i\psi \sigma^{a} e^{\lambda}{}_{a} \nabla_{\lambda} \bar{\psi} - \frac{1}{2} m \left(\psi\psi + \bar{\psi}\bar{\psi} \right) \Big]\,. \tag{6.158}$$

6.5 Standardni Model

Odgovarajuća 3-grupa koja opisuje sva polja prisutna u Standardnom Modelu sa odgovarajućom dinamikom dobija se izborom Lijevih grupa G, H i L na sledeći način [16]:

$$G = SO(3,1) \times SU(3) \times SU(2) \times U(1)\,, \quad H = \mathbb{R}^4\,, \quad L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G})\,, \tag{6.159}$$

gde je $\mathbb{C}$ oznaka za skup kompleksnih brojeva. Motivacija iza ovog izbora grupa postaje jasna analizirajući tabelu 6.4.

I generacija leptona	crvena boja I generacija kvarkova	zelena boja I generacija kvarkova	plava boja I generacija kvarkova
$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L$	$\begin{pmatrix} u_r \\ d_r \end{pmatrix}_L$	$\begin{pmatrix} u_g \\ d_g \end{pmatrix}_L$	$\begin{pmatrix} u_b \\ d_b \end{pmatrix}_L$
$(\nu_e)_R$	$(u_r)_R$	$(u_g)_R$	$(u_b)_R$
$(e^-)_R$	$(d_r)_R$	$(d_g)_R$	$(d_b)_R$

Tabela 6.4: Polja materije prisutna u Standardnom Modelu čestica (I generacija).

Prebrojavanjem polja u tabeli 6.4 zaključujemo da je neophodno definisati 16 spinora kako bismo definisati prvu generaciju spinorskih polja materije prisutnih u Standardnom Modelu čestica, odnosno da grupu L treba izabrati na sledeći način $L = \mathbb{R}^{64}(\mathbb{G})$. Kako postoje ukupno tri generacije materije, ukupna podgrupa grupe L koja opisuje fermionska polja u teoriji je $L = \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G})$. Da bismo definisali Higsov sektor neophodno je definisati dva kompleksna skalarna polja $\begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}$, odnosno pogrupa grupe L koja odgovara skalarnom sektoru Standardnog Modela je $L = \mathbb{R}^4(\mathbb{C})$.

Dalje, kako bi definisali 2-ukršteni modul kome odgovara $3BF$ dejstvo sa odgovarajućom dinamikom, neophodno je definisati preslikavanja ∂, δ i Pajferovo podizanje da budu trivijalna. Takođe, dejstvo grupe G na samu sebe je, po definiciji 2-ukrštenog modula, konjugacija. Dejstvo podgrupe $SO(3,1)$ grupe G na grupu H je po vektorskoj reprezentaciji, dok je dejstvo podgrupe $SU(3)\times SU(2)\times U(1)$ grupe G na grupu H trivijalno. Dejstvo podgrupe $SO(3,1)$ na podgrupu grupe L koja odgovara skalarnom sektoru materije, tj. $\mathbb{R}^4(\mathbb{C})$ podgrupu grupe L, je trivijalno, dok je zadato spinorskom reprezentacijom za svaku četvorku generatora koja opisuje jedno skalarno polje, kao što je to prikazano u odeljku 6.3. Transformacione osobine spinorskih polja pod dejstvom podgrupe $SU(3)\times SU(2)\times U(1)$ grupe G zadate su dejstvom ove podgrupe na grupu L.

6.5.1 Leptoni i elektroslaba interakcija

Demonstriraćemo proceduru definisanja 2-ukrštenog modula na jednostavnom primeru jedne leptonske familije i elektroslabe interakcije. Ostatak Standardnog Modela definiše se analogno.

Lijeve grupe G, H i L definišemo na sledeći način:

$$G = SO(3,1)\times SU(2)\times U(1)\,,\quad H=\mathbb{R}^4\,,\quad L^{\,\text{leptoni i Higsov bozon}} = \mathbb{R}^{16}(\mathbb{G})\times\mathbb{R}^4(\mathbb{C})\,. \tag{6.160}$$

Zatim, odgovarajuća 3-koneksija je:

$$\alpha = \omega^{ab}M_{ab} + W^I T_I + AY\,,\quad \beta=\beta^a P_a\,,\quad \gamma = \gamma_\alpha{}^{\tilde L}P^\alpha{}_{\tilde L} + \gamma^{\dot\alpha}{}_{\tilde L}P_{\dot\alpha}{}^{\tilde L} + \gamma_\alpha{}^{\tilde R}P^\alpha{}_{\tilde R} + \gamma^{\dot\alpha}{}_{\tilde R}P_{\dot\alpha}{}^{\tilde R} + \gamma^{\tilde a}P_{\tilde a}\,. \tag{6.161}$$

Ovde indeksi $I,J,\dots$ uzimaju vrednosti $1,2,3$ i prebrojavaju Paulijeve matrice, generatore grupe $SU(2)$, dok indeksi $\tilde L,\tilde L',\dots$ uzimaju vrednosti $1,2$ i prebrojavaju komponente levog dubleta, $\tilde R$ označava desni singlet $(e^-)_R$ i desni singlet $(\nu_e)_R$, dok indeksi $\tilde a,\tilde b,\dots$ uzimaju vrednosti $1,2$ i prebrojavaju komponente skalarnog dubleta. Takođe, definišimo indeks $\tilde i = (\tilde L,\tilde R)$ koji uzima vrednosti $1,\dots,4$.

Dejstvo grupe G na grupu L definiše se na sledeći način:

$$\begin{gathered}
M_{ab}\rhd P^\alpha{}_i = \frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta{}_i\,,\qquad M_{ab}\rhd P_{\dot\alpha i} = \frac{1}{2}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha}P_{\dot\beta i}\,,\qquad M_{ab}\rhd P_{\tilde a}=0\,,\\
T_I\rhd P^\alpha{}_{\tilde L} = \frac{1}{2}(\sigma_I)^{\tilde L'}{}_{\tilde L}P^\alpha{}_{\tilde L'}\,,\quad T_I\rhd P_{\dot\alpha\tilde L} = \frac{1}{2}(\sigma_I)^{\tilde L'}{}_{\tilde L}P_{\dot\alpha\tilde L'}\,,\\
T_I\rhd P^\alpha{}_{\tilde R}=0\,,\qquad T_I\rhd P_{\dot\alpha\tilde R}=0\,,\qquad T_I\rhd P_{\tilde a} = \frac{1}{2}(\sigma_I)^{\tilde b}{}_{\tilde a}P_{\tilde b}\,,\\
Y\rhd P^\alpha{}_{\tilde L} = -P^\alpha{}_{\tilde L}\,,\quad Y\rhd P^\alpha{}_{\tilde R} = -2P^\alpha{}_{\tilde R}\,,\quad Y\rhd P_{\tilde a} = P_{\tilde a}\,,\\
Y\rhd P_{\dot\alpha\tilde L} = -P_{\dot\alpha\tilde L}\,,\quad Y\rhd P_{\dot\alpha\tilde R} = -2P_{\dot\alpha\tilde R}\,.
\end{gathered} \tag{6.162}$$

Odgovarajuće 3-krivine za ovaj izbor 2-ukrštenog modula su:

$$\begin{gathered}
\mathcal{F} = R^{ab}M_{ab} + F^I T_I + FY\,,\qquad \mathcal{G} = \nabla\beta^a P_a\,,\\
\mathcal{H} = (\overrightarrow{\nabla}\gamma^{\tilde L})_\alpha P^\alpha{}_{\tilde L} + (\bar\gamma_{\tilde L}\overleftarrow{\nabla})^{\dot\alpha}P_{\dot\alpha}{}^{\tilde L} + (\overrightarrow{\nabla}\gamma^{\tilde R})_\alpha P^\alpha{}_{\tilde R} + (\bar\gamma_{\tilde R}\overleftarrow{\nabla})^{\dot\alpha}P_{\dot\alpha}{}^{\tilde R} + \mathrm{d}\gamma^{\tilde a}P_{\tilde a}\,.
\end{gathered} \tag{6.163}$$

Topološko $3BF$ dejstvo je:

$$S = \int B_{ab}R^{ab} + B_I F^I + BF + e_a\nabla\beta^a + \psi^\alpha{}_{\tilde i}(\overrightarrow{\nabla}\gamma^{\tilde i})_\alpha + \bar\psi_{\dot\alpha}{}^{\tilde i}(\bar\gamma_{\tilde i}\overleftarrow{\nabla})^{\dot\alpha} + \phi^{\tilde a}\mathrm{d}\gamma_{\tilde a}\,. \tag{6.164}$$

Sada možemo pojednostaviti notaciju uvođenjem indeksa $\hat\alpha$ koji prebrojavaju generatore grupe G, indeksa $\hat a$ grupe H i indeksa $\hat A$ grupe L. Kako bismo u teoriju uveli odgovarajuće stepene slobode koji opisuju teoriju prve leptonske familije u interakciji sa elektroslabim gejdž poljima, Higsovim poljem i gravitacijom, neophodno je topološkom dejstvu (6.164) dodati odgovarajuće *simplicity veze*, na sledeći način

$$
\begin{aligned}
S = \int & B_{\hat\alpha} \wedge \mathcal{F}^{\hat\alpha} + e_{\hat a} \wedge \mathcal{G}^{\hat a} + D_{\hat A} \wedge \mathcal{H}^{\hat A} \\
& + \left(B_{\hat\alpha} - C_{\hat\alpha}{}^{\hat\beta} M_{cd\hat\beta} e^c \wedge e^d \right) \wedge \lambda^{\hat\alpha} - \left(\gamma_{\hat A} - e^a \wedge e^b \wedge e^c C_{\hat A}{}^{\hat B} M_{abc\hat B} \right) \wedge \lambda^{\hat A} \\
& + \zeta^{ab}{}_{\hat\alpha} \wedge \left(M_{ab}{}^{\hat\alpha} \varepsilon^{cdef} e_c \wedge e_d \wedge e_e \wedge e_f - F^{\hat\alpha} \wedge e_c \wedge e_d \right) \\
& + \zeta^{ab}{}_{\hat A} \wedge \left(M_{abc}{}^{\hat A} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - F^{\hat A} \wedge e_a \wedge e_b \right) \\
& - \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \left(Y_{\hat A\hat B\hat C} D^{\hat A} D^{\hat B} D^{\hat C} + M_{\hat A\hat B} D^{\hat A} D^{\hat B} + L_{\hat A\hat B\hat C\hat D} D^{\hat A} D^{\hat B} D^{\hat C} D^{\hat D} \right) \\
& - 4\pi i\, l_p^2\, \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c D_{\hat A} T^{d\hat A}{}_{\hat B} D^{\hat B} \,,
\end{aligned}
\tag{6.165}
$$

gde su:

$$
B_{\hat\alpha} = \begin{bmatrix} B_{ab} & B_I & B \end{bmatrix}, \quad \mathcal{F}^{\hat\alpha} = \begin{bmatrix} R_{ab} & F_I & F \end{bmatrix}^T, \quad D_{\hat A} = \begin{bmatrix} \psi^\alpha{}_{\tilde L} & \bar\psi_{\dot\alpha\tilde L} & \psi^\alpha{}_R & \bar\psi_{\dot\alpha R} & \phi_{\tilde a} \end{bmatrix},
$$

$$
\mathcal{H}^{\hat A} = \begin{bmatrix} (\vec\nabla \gamma_{\tilde L})_\alpha & (\bar\gamma_{\tilde L} \overleftarrow\nabla)^{\dot\alpha} & (\vec\nabla \gamma_{\tilde R})_\alpha & (\bar\gamma_{\tilde R} \overleftarrow\nabla)^{\dot\alpha} & \mathrm{d}\gamma_{\tilde a} \end{bmatrix}^T, \quad \gamma_{\hat A} = \begin{bmatrix} \gamma^\alpha{}_{\tilde L} & \bar\gamma_{\dot\alpha\tilde L} & \gamma^\alpha{}_{\tilde R} & \bar\gamma_{\dot\alpha\tilde R} & \gamma_{\tilde a} \end{bmatrix},
$$

$$
\lambda^{\hat\alpha} = \begin{bmatrix} -\lambda^{ab} & \lambda^I & \lambda \end{bmatrix}^T, \quad M_{cd\hat\alpha} = \begin{bmatrix} \varepsilon_{abcd} & M_{cdI} & M_{cd} \end{bmatrix},
$$

$$
\lambda^{\hat A} = \begin{bmatrix} \lambda_{\alpha L} & \bar\lambda^{\dot\alpha}{}_L & \lambda_{\alpha R} & \bar\lambda^{\dot\alpha}{}_R & \lambda^{\tilde a} \end{bmatrix}^T, \quad \zeta^{cd}{}_{\hat\alpha} = \begin{bmatrix} 0 & \zeta^{cd}{}_I & \zeta^{cd} \end{bmatrix}, \quad \zeta^{ab}{}_{\hat A} = \begin{bmatrix} \zeta^{ab} & 0 & 0 \end{bmatrix},
$$

$$
M_{abc\hat A} = \begin{bmatrix} \varepsilon_{abcd} \sigma^d{}_{\alpha\dot\beta} \bar\psi^{\dot\beta}{}_L & \varepsilon_{abcd} \bar\sigma^{d\dot\alpha\beta} \psi_{\beta L} & \varepsilon_{abcd} \sigma^d{}_{\alpha\dot\beta} \bar\psi^{\dot\beta}{}_R & \varepsilon_{abcd} \bar\sigma^{d\dot\alpha\beta} \psi_{\beta R} & M_{abc\tilde a} \end{bmatrix}.
$$

Matrice $C^{\hat\alpha}{}_{\hat\beta}$, $C^{\hat A}{}_{\hat B}$, $M_{\hat A\hat B}$, $Y_{\hat A\hat B\hat C}$, $L_{\hat A\hat B\hat C\hat D}$ i $T^{d\hat A}{}_{\hat B}$ su konstantne matrice koje nose informaciju o odgovarajućim konstantama interakcije, masi Higsovog polja, Jukava kaplingu, uglovima mešanja, Higsovoj konstanti samointerakcije i torziji.

6.6 Skalarna elektrodinamika kao $3BF$ teorija sa vezama

Kao prvi korak ka proučavanju Hamiltonove strukture $3BF$ teorija, razmatran je najjednostavniji netrivijalni primer – teorija skalarne elektrodinamike kuplovane sa gravitacijom [24].

Standardni način da se definiše skalarna elektrodinamika kuplovana sa gravitacijom je dejstvom:

$$
S = \int \mathrm{d}^4k \sqrt{-g} \left[-\frac{1}{16\pi l_p^2} R - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + g^{\mu\nu} \nabla_\mu \phi^* \nabla_\nu \phi - m^2 \phi^* \phi \right] . \tag{6.166}
$$

Ovde je $g_{\mu\nu}$ metrika prostorvremena, $g \equiv \det(g_{\mu\nu})$ je njena determinanta, R je Ričijev skalar, a l_p je Plankova dužina. Kovarijantni izvod ∇_μ kompleksnog skalarnog polja ϕ je definisan izrazom $\nabla_\mu \phi = (\partial_\mu + ikA_\mu)\phi$, gde je A_μ elektromagnetni potencijal, a k označava konstantu interakcije, tj. električni naboj polja ϕ. U ovom odeljku ćemo preformulisati ovaj model kao $3BF$ teoriju sa vezama za određenu 3-grupu. Razmatrana je Hamiltonova struktura teorije, koja je neophodan korak njene kanonske kvantizacije. Radi jednostavnosti, Hamiltonova analiza je za sada urađena samo za topološki sektor teorije, zanemarujući sektor sa vezama, videti Dodatak C.

Kako bi se dobila teorija skalarne elektrodinamike u interakciji sa Ajnštajn-Kartanovom gravitacijom ukršteni modul se bira na sledeći način. Lijeve grupe G, H i L su:

$$G = SO(3,1) \times U(1)\,, \quad H = \mathbb{R}^4\,, \quad L = \mathbb{R}^2\,.$$

Preslikavanja ∂ i δ su trivijalna. Dejstvo algebre $\mathfrak{g}$ na algebre $\mathfrak{h}$ i $\mathfrak{l}$ definisano delovanjem na generatore:

$$\begin{array}{ll} M_{ab} \rhd P_c = \rhd_{ab,c}{}^d\, P_d = \delta_{[a|}{}^d \eta_{|b]c}\, P_d = \eta_{[b|c}\, P_{|a]}\,, & T \rhd P_a = 0\,, \\ M_{ab} \rhd P_A = 0\,, & T \rhd P_A = \rhd_A{}^B\, P_B \end{array} \tag{6.167}$$

gde su M_{ab} šest generatora $\mathfrak{so}(3,1)$, T je generator $\mathfrak{u}(1)$, P_a su četiri generatora $\mathbb{R}^4$ i P_A su dva generatora $\mathbb{R}^2$. U prethodnom izrazu dejstvo $\rhd_A{}^B$ algebre $\mathfrak{u}(1)$ na algebru $\mathbb{R}^2$ je definisano kao

$$\rhd_A{}^B = iq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Dejstvo algebre $\mathfrak{g}$ na samu sebe zadato je pridruženom reprezentacijom i za izbor $\mathfrak{g} = \mathfrak{so}(3,1) \times \mathfrak{u}(1)$ glasi

$$\begin{gathered} M_{ab} \rhd M_{cd} = \rhd_{ab\,,cd}{}^{ef}\, M_{ef} = f_{ab\,,cd}{}^{ef}\, M_{ef} = \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}\,, \\ M_{ab} \rhd T = 0\,, \quad T \rhd M_{ab} = 0\,, \quad T \rhd T = 0\,, \end{gathered} \tag{6.168}$$

kao posledica strukture direktnog proizvoda i toga što je grupa $U(1)$ Abelova podgrupa. Pajferovo podizanje,

$$\{\_\,,\_\}_{\rm pf} : H \times H \to L\,,$$

je takođe trivijalno, tj. svi koeficijenti $X_{ab}{}^A$ su jednaki nuli:

$$\{P_a\,, P_b\}_{\rm pf} \equiv X_{ab}{}^A T_A = 0\,. \tag{6.169}$$

Odgovarajuća 3-krivina za ovaj izbor 2-ukrštenog modula dobija najpre definisanjem koneksije (α, β, γ), a zatim primenom formule (2.118). Na osnovu strukture direktnog proizvoda, koneksija $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$ se može zapisati kao $\alpha = \omega + A$, gde su $\omega \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{so}(3,1))$ i $A \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{u}(1))$ diferencijalne 1-forme elementi odgovarajućih algebri. Definišemo i koneksije $\beta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathbb{R}^4)$ i $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,, \mathbb{R}^2)$. Sada možemo naći odgovarajuću 3-krivinu $(\mathcal{F}\,, \mathcal{G}\,, \mathcal{H})$ primenom formule (2.118):

$$\begin{array}{rclcl} \mathcal{F} &=& R^{ab} M_{ab} + F T &=& \big(\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\big) M_{ab} + \mathrm{d}A\, T\,, \\ \mathcal{G} &=& \mathcal{G}^a P_a &=& \big(\mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b\big) P_a\,, \\ \mathcal{H} &=& \mathcal{H}^A P_A &=& \big(\mathrm{d}\gamma^A + \rhd_B{}^A A \wedge \gamma^B\big) P_A\,. \end{array} \tag{6.170}$$

Primetimo da koneksija ω^{ab} nije prisutna u poslednjem izrazu, što sledi na osnovu definicija dejstva $\rhd$ i Pajferovog podizanja $\{\_\,,\_\}_{\rm pf}$, videti (6.167) i (6.169):

$$\begin{array}{rcl} \mathcal{H} &=& \mathrm{d}\gamma + \alpha \wedge^{\rhd} \gamma + \{\beta \wedge \beta\} \\ &=& \mathrm{d}\gamma^A P_A + (\omega^{ab} M_{ab} + AT) \wedge^{\rhd} (\gamma^A P_A) \\ &=& \mathrm{d}\gamma^A P_A + \omega^{ab} \wedge \gamma^A M_{ab} \rhd P_A + A \wedge \gamma^A T \rhd P_A \\ &=& \mathrm{d}\gamma^A P_A + A \wedge \gamma^A \rhd_A{}^B P_B \\ &=& (\mathrm{d}\gamma^A + \rhd_B{}^A A \wedge \gamma^B) P_A\,. \end{array} \tag{6.171}$$

Koeficijenti diferencijalnih 2-formi F i R^{ab}, 3-forme $\mathcal{G}$, i 4-forme $\mathcal{H}$ su:

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \,, \\
R^{ab}{}_{\mu\nu} &= \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^a{}_{c\mu}\omega^{cb}{}_\nu - \omega^a{}_{c\nu}\omega^{cb}{}_\mu \,, \\
\mathcal{G}^a{}_{\mu\nu\rho} &= \partial_\mu \beta^a{}_{\nu\rho} + \partial_\nu \beta^a{}_{\rho\mu} + \partial_\rho \beta^a{}_{\mu\nu} + \omega^a{}_{b\mu}\,\beta^b{}_{\nu\rho} + \omega^a{}_{b\nu}\,\beta^b{}_{\rho\mu} + \omega^a{}_{b\rho}\,\beta^b{}_{\mu\nu} \,, \\
\mathcal{H}^A{}_{\mu\nu\rho\sigma} &= \partial_\mu \gamma^A{}_{\nu\rho\sigma} - \partial_\nu \gamma^A{}_{\rho\sigma\mu} + \partial_\rho \gamma^A{}_{\sigma\mu\nu} - \partial_\sigma \gamma^A{}_{\mu\nu\rho} \\
&\quad + \rhd_B{}^A A_\mu \gamma^B{}_{\nu\rho\sigma} - \rhd_B{}^A A_\nu \gamma^B{}_{\rho\sigma\mu} + \rhd_B{}^A A_\rho \gamma^B{}_{\sigma\mu\nu} - \rhd_B{}^A A_\sigma \gamma^B{}_{\mu\nu\rho} \,.
\end{aligned} \tag{6.172}$$

Sada možemo definisati 3BF dejstvo:

$$S_{3BF} = \int_{\mathcal{M}_4} \Big(\langle B, \mathcal{F} \rangle_{\mathfrak{g}} + \langle C, \mathcal{G} \rangle_{\mathfrak{h}} + \langle D, \mathcal{H} \rangle_{\mathfrak{l}} \Big) \,, \tag{6.173}$$

gde su $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{so}(3,1))$, $C \in \mathcal{A}^1(\mathcal{M}_4\,, \mathbb{R}^4)$ i $D \in \mathcal{A}^0(\mathcal{M}_4\,, \mathbb{R}^2)$ Lagranževi množitelji. Forme $\langle \_\,, \_ \rangle_{\mathfrak{g}}$, $\langle \_\,, \_ \rangle_{\mathfrak{h}}$ i $\langle \_\,, \_ \rangle_{\mathfrak{l}}$ su G-invarijantne bilinearne simetrične nedegenerisane forme na $\mathfrak{g}$, $\mathfrak{h}$ i $\mathfrak{l}$, redom, definisane delovanjem na generatore

$$\langle M_{ab}, M_{cd} \rangle_{\mathfrak{g}} = g_{ab,\,cd} \,, \quad \langle T, T \rangle_{\mathfrak{g}} = 1 \,, \quad \langle M_{ab}, T \rangle_{\mathfrak{g}} = 0 \,, \quad \langle P_a, P_b \rangle_{\mathfrak{h}} = \eta_{ab} \,, \quad \langle P_A, P_B \rangle_{\mathfrak{l}} = g_{AB} \,,$$

gde su

$$g_{ab,\,cd} = \eta_{a[c|}\eta_{b|d]} \,, \qquad \eta_{ab} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \qquad g_{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Identifikovanjem Lagranževog množitelja C^a kao tetrade e^a, i Lagranževog množitelja D^A kao dubleta skalarnih polja ϕ^A,

$$\phi = \phi^A P_A = \phi P_1 + \phi^* P_2 \,,$$

na osnovu njihovih transformacionih osobina, kao što je diskutovano u odeljku (6.2), dejstvo (6.173) možemo zapisati u sledećem obliku:

$$S_{3BF} = \int_{\mathcal{M}_4} \mathrm{d}^4x \; \epsilon^{\mu\nu\rho\sigma} \Big(\frac{1}{4} \, B^{ab}{}_{\mu\nu} \, R^{cd}{}_{\rho\sigma} \, g_{ab,\,cd} + \frac{1}{4} \, B_{\mu\nu} F_{\rho\sigma} + \frac{1}{3!} \, e^a{}_\mu \, \mathcal{G}^b{}_{\nu\rho\sigma} \, \eta_{ab} + \frac{1}{4!} \, \phi^A \mathcal{H}^B{}_{\mu\nu\rho\sigma} \, g_{AB} \Big) \,. \tag{6.174}$$

Variranjem dejstva (6.174) dobijamo jednačine kretanja:

$$\begin{aligned}
\delta B^{ab} &: & 2R_{ab} &= 0 \,, \\
\delta \omega^{ab} &: & \nabla B_{ab} - e_{[a|} \wedge \beta_{|b]} &= 0 \,, \\
\delta B &: & F &= 0 \,, \\
\delta A &: & \mathrm{d}B + \phi_A \rhd_B{}^A \gamma^B &= 0 \,, \\
\delta e^a &: & \mathcal{G}_a &= 0 \,, \\
\delta \beta^a &: & \nabla e_a &= 0 \,, \\
\delta \phi^A &: & \nabla \gamma_A &= 0 \,, \\
\delta \gamma^A &: & \nabla \phi_A &= 0 \,.
\end{aligned} \tag{6.175}$$

Kako želimo da dobijemo teoriju koja opisuje dublet skalarnih polja ϕ^A mase m i naelektrisanja q minimalno kuplovanih sa gravitacijom i elektromagnetnim poljem, dejstvu (6.174) je neophodno dadati odgovarajuće veze kako bismo dobili jednačine kretanja ekvivalentne jednačinama kretanja dobijenih variranjem dejstva (6.166):

$$
\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B \wedge F + e_a \wedge \nabla\beta^a + \phi_A \nabla\gamma^A \\
& - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\
& + \lambda^A \wedge \left(\gamma_A - \frac{1}{2} H_{abcA} e^a \wedge e^b \wedge e^c \right) + \Lambda^{abA} \wedge \left(H_{abcA} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \nabla\phi_A A \wedge e_a \wedge e_b \right) \\
& + \lambda \wedge \left(B - \frac{12}{q} M_{ab} e^a \wedge e^b \right) + \zeta^{ab} \left(M_{ab} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b \right) \\
& - \frac{1}{2 \cdot 4!} m^2 \phi_A \phi^A \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,.
\end{aligned}
\tag{6.176}
$$

Variranjem dejstva (6.176) redom po varijablama B_{ab}, B, ω_{ab}, β_a, λ_{ab}, Λ^{abA}, γ^A, λ^A, H_{abcA}, ζ^{ab}, M_{ab}, λ, A, ϕ^A i e^a dobijaju se jednačine kretanja:

$$ R^{ab} - \lambda^{ab} = 0 \,, \tag{6.177} $$

$$ F + \lambda = 0 \,, \tag{6.178} $$

$$ \nabla B^{ab} - e^{[a|} \wedge \beta^{|b]} = 0 \,, \tag{6.179} $$

$$ \nabla e^a = 0 \,, \tag{6.180} $$

$$ B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d = 0 \,, \tag{6.181} $$

$$ H_{abcA} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \nabla\phi_A \wedge e_a \wedge e_b = 0 \,, \tag{6.182} $$

$$ \nabla\phi_A - \lambda_A = 0 \,, \tag{6.183} $$

$$ \gamma_A - \frac{1}{2} H_{abcA} e^a \wedge e^b \wedge e^c = 0 \,, \tag{6.184} $$

$$ -\frac{1}{2} \lambda^A \wedge e^a \wedge e^b \wedge e^c + \varepsilon^{cdef} \Lambda^{abA} \wedge e_d \wedge e_e \wedge e_f = 0 \,, \tag{6.185} $$

$$ M_{ab} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b = 0 \,, \tag{6.186} $$

$$ -\frac{12}{q} \lambda \wedge e^a \wedge e^b + \zeta^{ab} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f = 0 \,, \tag{6.187} $$

$$ B - \frac{12}{g} M_{ab} e^a \wedge e^b = 0 \,, \tag{6.188} $$

$$ -\mathrm{d}B + \mathrm{d}(\zeta^{ab} e_a \wedge e_b) - \phi_A \rhd_B{}^A \gamma^B - \Lambda^{abA} \rhd^B{}_A \phi_B \wedge e_a \wedge e_b = 0 \,, \tag{6.189} $$

$$ \nabla\gamma_A - \nabla(\Lambda^{ab}{}_A \wedge e_a \wedge e_b) - \frac{1}{4!} m^2 \phi_A \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d = 0 \,, \tag{6.190} $$

(6.191)

$$
\begin{aligned}
&\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc}\wedge e^d + \frac{3}{2}H_{abcA}\lambda^A\wedge e^b\wedge e^c + 3H^{defA}\varepsilon_{abcd}\Lambda_{efA}\wedge e^b\wedge e^c \\
&\quad - 2\Lambda_{abA}\wedge\nabla\phi^A\wedge e^b - 2\frac{1}{4!}m^2\phi_A\,\phi^A\varepsilon_{abcd}e^b\wedge e^c\wedge e^d \\
&\quad - \frac{24}{q}M_{ab}\lambda\wedge e^b + 4\zeta^{ef}M_{ef}\varepsilon_{abcd}e^b\wedge e^c\wedge e^d - 2\zeta_{ab}F\wedge e^b = 0\,.
\end{aligned}
\tag{6.192}
$$

Dinamički stepeni slobode su tetrade e^a, skalarno polje ϕ^A i elektromagnetni potencijal A, dok preostale varijable mogu biti određene kao funkcije dinamičkih varijabli i njihovih izvoda. Jednačine (6.177)–(6.188) daju izraze za nedinamičke varijable:

$$
\begin{gathered}
\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,, \qquad \gamma^A{}_{\mu\nu\rho} = -\frac{1}{2e}\varepsilon_{\mu\nu\rho\sigma}\,\nabla^\sigma\phi^A\,, \\
\Lambda^{abA}{}_\mu = \frac{1}{12e}g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\nabla_\nu\phi^A\,e^a{}_\rho e^b{}_\sigma\,, \qquad \beta^a{}_{\mu\nu} = 0\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \\
H^{abcA} = \frac{1}{6e}\varepsilon^{\mu\nu\rho\sigma}\,\nabla_\mu\phi^A\,e^a{}_\nu e^b{}_\rho e^c{}_\sigma\,, \qquad \lambda^A{}_\mu = \nabla_\mu\phi^A\,, \\
\lambda_{\mu\nu} = F_{\mu\nu}\,, \quad B_{\mu\nu} = -\frac{1}{2eq}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,, \\
M^{ab} = -\frac{1}{4e}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,e^a{}_\rho e^b{}_\sigma\,, \quad \zeta^{ab} = \frac{1}{4eq}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,e^a{}_\rho e^b{}_\sigma\,.
\end{gathered}
\tag{6.193}
$$

Primetimo da na osnovu jednačina (6.179), (6.180) i (6.181) sledi da je koneksija $\beta^a = 0$, kao što je to bio slučaj kod čiste gravitacije. Jednačina kretanja (6.190) daje kovarijantnu Klajn-Gordonovu jednačinu za skalarno polje kuplovano sa elektromagnetnim potencijalom A,

$$
\left(\nabla_\mu\nabla^\mu - m^2\right)\phi_A = 0\,. \tag{6.194}
$$

Jednačina (6.189) daje diferencijalnu jednačinu kretanja za elektromagnetni potencijal A:

$$
\nabla_\mu F^{\mu\nu} = j^\nu\,, \qquad j^\mu \equiv \frac{1}{2}\Big(\nabla^\nu\phi^A \rhd^B{}_A\phi_B - \phi_A \rhd_B{}^A\nabla^\nu\phi^B\Big) = iq\left(\nabla\phi^*\,\phi - \phi^*\nabla\phi\right). \tag{6.195}
$$

Najzad, jednačina kretanja (6.191) za e^a nakon sređivanja daje:

$$
\begin{gathered}
R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\,T^{\mu\nu}\,, \\
T^{\mu\nu} \equiv \nabla^\mu\phi_A\,\nabla^\nu\phi^A - \frac{1}{2}g^{\mu\nu}\left(\nabla_\rho\phi_A\,\nabla^\rho\phi^A + m^2\phi_A\,\phi^A\right) - \frac{1}{4q}\left(F_{\rho\sigma}F^{\rho\sigma}g^{\mu\nu} + 4F^{\mu\rho}F_\rho{}^\nu\right)\,.
\end{gathered}
\tag{6.196}
$$

Sistem jednačina (6.177)–(6.191) je ekvivalentan sistemu jednačina (6.193)–(6.196).

Kompletna Hamiltonova analiza topološkog sektora skalarne elektrodinamike data je u Dodatku C.

Deo II

Kvantna teorija

Glava 7

Modeli spinske pene: $\boldsymbol{BF}$ teorija

Ajnštajnova opšta teorija relativnosti dovela je do našeg shvatanja prirode prostora i vremena kao manifestacije gravitacionog polja. Kao što i ostala fizička polja ispoljavaju njihova kvantna svojstva na određenoj skali, prirodno je očekivati da i gravitaciono polje, pa time i prostorvreme, poseduju određena kvantna svojstva. Stoga, neophodno je modifikovati naše razumevanje prirode prostora i vremena, kako bismo uzeli u obzir ove kvantne osobine. Problem leži u tome da sadašnje teorije, opšta teorija relativnosti i kvantna teorija polja, ne mogu da opišu kvantno ponašanje gravitacionog polja. Neophodna je *kvantna teorija gravitacije* koja ima prediktivnu moć da opiše fenomene gde i gravitacija i kvantna teorija igraju ulogu, kao što je to slučaj kod crnih rupa, ranog univerzuma, fizike na malim rastojanjima itd.

Godine 1936. Bronštajn je ponovio Bor-Rozenfeldovu analizu za elektromagnetno polje u slučaju gravitacionog polja i pokazao da kvantna teorija zabranjuje određivanje polja u proizvoljno maloj oblasti prostorvremena. Ako merimo polje u tački x, koju želimo da odredimo sa preciznošću L, zbog postojanja *Hajzenbergove relacije neodređenosti* koja povezuje poziciju i impuls čestice, sledi da neodređenost impulsa mora biti $\Delta p \geqslant \hbar/L$. U ultrarelativističkom limitu imamo da je energija $E \sim cp$, pa vidimo da oštra lokalizacija zahteva veliku energiju. Na osnovu opšte teorije relativnosti znamo da energija zakrivljuje prostor, a krivina raste kako je energija koncentrisanija u prostoru, sve do tačke formiranja crne rupe kada je masa $M \sim E/c^2$ koncentrisana u radijusu $R \sim GM/c^2$. Zahtevanjem bolje lokalizacije, dolazimo do tačke gde je $L_{Plank} = R$ ispod koje je nemoguće ići, jer bi tada lokalizacija bila sakrivena horizontom crne rupe[1]. Na osnovu prethodnih relacija dobija se:

$$L_{Plank} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33}\mathrm{cm}\,.$$

Na skalama većim od one određene *Plankovom dužinom* L_{Plank}, prostorvreme možemo posmatrati kao glatku mnogostrukost, dok ispod nje kvantne fluktuacije prostorvremena postaju nezanemarljive i više nema smisla pričati o dužini.

Prethodna analiza sugeriše da kvantna teorija polja, formalizam u kome su kvantna polja definisana na nekoj prostorvremenskoj mnogostrukosti, nije dobra slika sveta u teoriji kvantne gravitacije. Neophodan je *kvantni opis geometrije*, gde je geometrija opisana kvantnim stanjima, a prostorvreme je semiklasična aproksimacija takve kvantne konfiguracije. Jedan mogući opis kvantnih stanja geometrije, tj. gravitacionog polja, nam obezbeđuje formalizam teorije *kvantne gravitacije na petljama*[2].

[1] Pri tom se podrazumeva da Opšta teorija relativnosti važi u neizmenjenom obliku i na skalama manjim od Plankove, tj. da postoji rešenje Ajnštajnovih jednačina koje opisuje tako malu crnu rupu.

[2] eng. *Loop Quantum Gravity.*

Kvatna gravitacija na petljama

Kvantna gravitacija na petljama je pristup kvantovanju gravitacije star preko trideset i pet godina, započet Aštekarovim radom 1986. godine. Kao teorija čiste gravitacije ne nastoji da reši *problem unifikacije*, tj. da objedini interakcije i smanji broj stepeni slobode *Standardnog Modela*. Kvantizacija teorije u okviru *kanonske kvantizacione procedure* podrazumeva izbor algebre polja koja postaju kvantni operatori, što je u ovom slučaju algebra zasnovana na *holonomijama gravitacione koneksije*. Holonomija postaje operator koji formira *stanje petlje*. Teorija je nezavisna od pozadine, i stanje petlje je relevantno samo u odnosu na druge petlje i infinitezimalni pomeraj petlje ne proizvodi novo stanje, već stanje ekvivalentno do na gradijentnu transformaciju. Prostor stanja teorije je separabilan Hilbertov prostor sa bazisom stanja petlji, gde konačne linearne kombinacije stanja petlji zovemo *spinskim mrežama*.

Kvantne osobine se manifestuju diskretnim spektrom svojstvenih vrednosti operatora koji odgovaraju veličinama koje opisuju lokalne osobine gravitacionog polja, kao što je na primer operator pridružen svakom linku graničnog grafa dualne triangulacije

$$\vec{E}_l = 8\pi\gamma\hbar G\vec{L}_l\,, \tag{7.1}$$

koji su normale na granične trouglove triangulacije kojima odgovara operator površine:

$$\hat{A}_l = 8\pi\gamma\hbar G|\vec{L}|^2\,. \tag{7.2}$$

Ovaj operator ima diskretni spektar,

$$A = 8\pi\gamma\hbar G\sqrt{j(j+1)}\,, \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

gde je γ *Barbero-Imirci parametar* bezdimenziona konstanta u teoriji, čija je vrednost određena tako da broj mikrostanja u KGP odgovara semiklasično izračunatoj entropiji crne rupe [41], [42]. Takođe, još jedna opservabla u ovoj teoriji je orijentisana zapremina tetraedra,

$$V^2 = \frac{2}{9}(\vec{E}_1 \cdot \vec{E}_2) \times \vec{E}_3 = \frac{2}{9}\epsilon_{ijk}E_1^i E_2^j E_3^k\,, \tag{7.3}$$

odnosno operator zapremine $\hat{V}$

$$\hat{V} = \frac{\sqrt{2}}{3}(8\pi\gamma\hbar G)^{\frac{3}{2}}\sqrt{(\vec{L}_1 \cdot \vec{L}_2) \times \vec{L}_3} \tag{7.4}$$

koji takođe ima diskretan spektar:

$$\hat{V}\,|i_v\rangle = V\,|i_v\rangle\ . \tag{7.5}$$

To znači da prostor možemo posmatrati kao sastavljen od *ćelija* prostora, tj. tetraedara koji imaju zapreminu određenu ovim spektrom.

Četiri operatora površine $\hat{A}_a$ i operator zapremine $\hat{V}$ formiraju maksimalno komutirajući set operatora koji opisuje kvantno stanje jednog tetraedra, pa se istovremeno mogu dijagonalizovati, a kvantna stanja geometrije tetraedra su jedinstveno određena njihovim svojstvenim vrednostima $|j_a, V\rangle$. Kako je za jedinstven klasičan opis tetraedra inače potrebno šest brojeva, recimo šest dužina njegovih ivica, vidimo da, kao što je i očekivano u kvantnoj teoriji, kvantno stanje tetraedra poseduje izvesnu kvantnu neodređenost na Planovoj skali.

Ova kvalitativna slika kvantne strukture prostorvremena je očuvana u *kovarijantnoj kvantizaciji*, tj. *kvantizacionoj proceduri spinske pene*. U okviru kovarijantne kvantizacione procedure *spinske pene* podrazumeva se *funkcionalni pristup* kvantovanju gravitacije u kom se konfiguracioni integral definiše na isti način na koji je to urađeno u *Fajnmanovoj definiciji integrala po putanjama*.

Po definiciji Majkla Atije, $(n+1)$-dimenzionalna *topološka kvantna teorija polja* (TKTP) je funktorijalno pridruživanje konačnodimenzionalnog Hilbertovog prostora $\mathcal{H}_\Sigma$ svakoj zatvorenoj orijentisanoj n-mnogostrukosti Σ i vektora $\mathcal{Z}_\mathcal{M} \in \mathcal{H}_\Sigma$ svakoj orijentisanoj $(n+1)$-mnogostrukosti $\mathcal{M}$ koja ima Σ kao svoju granicu. Ako posmatramo kvantno stanje prostora formirano od $|\Lambda_3|$ tetraedara u KGP teoriji, možemo ga predstaviti kao graf u dualnoj triangulaciji, gde verteksi odgovaraju tetraedrima, a linkovi između njih trouglovima koje razdvajaju dva susedna tetraedra. Analogno Atijinoj opštoj definiciji, *model spinske pene* svakom orijentisanom grafu[3] Γ pridružuje Hilbertov prostor $\mathcal{H}_\Gamma$, a svakoj peni[4] $\mathcal{C}$, koja ima graf Γ kao svoju granicu[5], vektor $\mathcal{Z}_\mathcal{C} \in \mathcal{H}_\Gamma$. Pritom su zadovoljene sledeće aksiome [43]:

1. (multiplikativnost) $\mathcal{H}_{\Gamma_1 \cup \Gamma_2} = \mathcal{H}_{\Gamma_1} \otimes \mathcal{H}_{\Gamma_2}$,

2. (dualnost) $\mathcal{H}_{\bar{\Gamma}} = \mathcal{H}^*_\Gamma$, $\qquad \mathcal{Z}_{\bar{\Gamma}} = \mathcal{Z}^\dagger_\Gamma$,

3. (funkcionalnost)[6] $\mathcal{Z}_{\mathcal{C}_1 \cup_\Gamma \mathcal{C}_2} = \langle \mathcal{Z}_{\bar{\mathcal{C}}_2} | \mathcal{Z}_{\mathcal{C}_1} \rangle_{\mathcal{H}_\Gamma} = \langle \mathcal{Z}_{\bar{\mathcal{C}}_1} | \mathcal{Z}_{\mathcal{C}_2} \rangle_{\mathcal{H}_{\bar{\Gamma}}}$,

4. $\mathcal{H}_\varnothing = \mathbb{C}$,

5. $\mathcal{Z}_{\mathbb{1}_\Gamma} = \mathrm{id}_{\mathcal{H}_\Gamma}$.

Dinamika kvantne gravitacije na petljama odgovara ovoj definiciji, pri čemu je Hilbertov prostor pridružen grafu Γ rešetkasti $SU(2)$ Jang-Millsov prostor $L^2(SU(2))^{|L_\Gamma|}/SU(2)^{|N_\Gamma|}$, gde je svaki verteks v u dualnoj triangulaciji prostorvremena obojen sa i_v koji odgovara svojstvenoj vrednosti operatora zapremine V za tu ćeliju, a svaki link ϵ u dualnoj triangulaciji prostorvremena je obojen sa j_ϵ koji odgovara svojstvenoj vrednosti površine A koja spaja te dve ćelije prostora. Ovakvo kvantno stanje prostorvremena nazivamo *spinskom mrežom*. Dalje, ako posmatramo *evoluciju spinske mreže* u vremenu, dobijamo da i_v koji je bojio verteks spinske mreže sada boji ivicu *spinske pene*, dok j_ϵ sada boji stranu. Amplitude pene $\mathcal{Z}_\mathcal{C}$ su definisane kao sumiranje amplituda spinske pene $\mathcal{Z}_\mathcal{C}(\sigma)$ po bojama stranica triangulacije $\sigma = \{j_f\}$, odnosno po irreducibilnim reprezentacijama j_f grupe $SU(2)$,

$$\mathcal{Z}_\mathcal{C} = \Sigma_\sigma \mathcal{Z}_\mathcal{C}(\sigma)\,,$$

kao što je to pokazano u narednim odeljcima.

[3] *Graf* je uređeni par $\Gamma = (N_\Gamma, L_\Gamma)$, gde je N_Γ konačan skup čvorova i L_Γ skup uređenih parova čvorova, tj. linkova grafa Γ. Čvorovi n i n' linka $l = (n, n')$ se nazivaju izvor i meta linka l i označavaju $\partial^-(l)$ i $\partial^+(l)$, redom. Inverzni link je definisan kao $l^{-1} \equiv (n', n)$, a $\bar{\Gamma}$ je inverzni graf grafa Γ dobijen inverzijom svih linkova grafa Γ.

[4] *Pena* je uređena trojka $\mathcal{C} = (V_\mathcal{C}, E_\mathcal{C}, F_\mathcal{C})$, gde je $V_\mathcal{C}$ konačni skup verteksa, $E_\mathcal{C}$ set uređenih parova verteksa, tj. ivica $e = (v, v')$, i $F_\mathcal{C}$ konačni skup strana. Strana je konačan niz ivica $f = (e_1, \ldots, e_{n_f})$, gde je $\partial^+(e_n) = \partial^-(e_{n+1})$ i važi $\partial^+(e_{n_f}) = \partial^-(e_1)$. Primetimo da bilo koji podskup F skupa strana $F_\mathcal{C}$ prirodno definiše podkompleks pene $\mathcal{C}$, sačinjen od verteksa, ivica i strana koje se pojavljuju u F. Pena dobijena od $\mathcal{C}$ inverzijom svih njenih ivica i strana označava se sa $\bar{\mathcal{C}}$.

[5] Ivice $E_\mathcal{C}$ koje se pojavljuju tačno jednom i pripadaju samo jednoj strani pene zovemo njenim *linkovima*, dok su ostale ivice *unutrašnje ivice*. Analogno, vertekse $V_\mathcal{C}$ koji se pojavljuju samo jednom u unutrašnjim ivicama nazivamo *nodovima*, dok preostale nazivamo *unutrašnji verteksi*. Skupovi nodova i linkova pene $\mathcal{C}$ u opštem slučaju ne formiraju graf, ali kada to čine i kada se orijentacija svakog linka poklapa sa onom indukovanom jedinstvenom stranom koja prolazi kroz nju, takvu penu nazivamo *prava pena*. U tom slučaju definišemo *granicu pene* $\partial\mathcal{C}$ kao podkompleks pene $\mathcal{C}$ kom pripadaju sve strane $\mathcal{C}$ koje sadrže najmanje jedan link. Graf koji odgovara $\partial\mathcal{C}$ je *granični graf* $\mathcal{C}$.

[6] Definiše se *kompozicija dve prave pene* $\mathcal{C}_1$ i $\mathcal{C}_2$ duž orijentisanog grafa Γ, koju obeležavamo $\mathcal{C}_1 \cup_\Gamma \mathcal{C}_2$, ako je Γ povezana komponenta graničnih grafova $\mathcal{C}_1$ i $\bar{\mathcal{C}}_2$. Kompozicija se dobija uklanjanjem Γ i spajanjem, za svaki nod, jedinstvenih ivica $e_1 \in \mathcal{C}_1$ i $e_2 \in \mathcal{C}_2$ koje sadrže taj nod u jednu ivicu, a za svaki link l jedinstvenih strana $f_1 \in \mathcal{C}_1$ i $f_2 \in \mathcal{C}_2$ u jedinstvenu stranu.

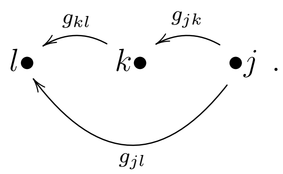
U kvantnoj elektrodinamici, perturbativnoj teoriji, bolja aproksimacija se postiže sumiranjem Fajnmanovih dijagrama višeg reda, dok se u kvantnoj hromodinamici, teoriji definisanoj na prostorvremenskoj rešetci, bolja aproksimacija dobija usitnjavanjem rešetke. Može se pokazati da su u kvantnoj gravitaciji na petnjama ova dva pristupa ista stvar, što je intuitivno jasno kada uzmemo u obzir da su tačke rešetke upravo kvanti prostora.

U narednim odeljcima fokusiraćemo se na konstrukcije topoloških BF suma po stanjima u slučaju trodimenzionalne i četvorodimenzionalne mnogostrukosti uobičajenom kvantizacionom procedurom spinske pene. U trodimenzionalnom slučaju, dobijena suma po stanjima predstavljena u odeljku 7.2.1 daje kvantnu teoriju trodimenzionalne gravitacije – *Ponzano-Redže model*, što je posledica činjenice da na klasičnom nivou odgovarajuća teorija nema lokalne propagirajuće stepene slobode. Kao što znamo, to nije rezultat u realnom četvorodimenzionalnom slučaju, pa kvantnu teoriju gravitacije moramo dobiti modifikacijom amplituda topološke sume po stanjima *Ouguri modela* predstavljene u odeljku 7.2.2. Ipak, ova konstrukcija je van okvira naše diskusije, pogledati [2] za pedagoški pristup kovarijantnoj kvantizacionoj proceduri formiranja sume po stanjima koja opisuje teoriju gravitacije u četiri dimenzije.

7.1 Gejdž invarijantni objekti

Klasične jednačine kretanja BF teorije nameću uslov da je gejdž koneksija ravna, tj. na jeziku holonomija, da svaka nul-homotopna kriva odgovara identitetu gejdž grupe. U Lemi 12 razmatrana je granična kriva trougla i uslov ravnosti gejdž koneksije formulisan je za ovaj element triangulacije mnogostrukosti.

Lema 12 *Posmatrajmo trougao $(jk\ell)$. Ivice trougla $(jk)\,, j<k$ su obeležene grupnim elementima $g_{jk} \in G$. Razmotrimo dijagram (7.6).*



(7.6)

Kriva $\gamma_1 = g_{k\ell} g_{jk}$ je jednaka krivoj $\gamma_2 = g_{j\ell}$, tj. važi identitet:

$$g_{j\ell} = g_{k\ell} g_{jk} \,. \tag{7.7}$$

7.2 Kvantizacija topološkog BF dejstva

U ovom odeljku predstavljen je postupak kvantovanja BF dejstva uobičajenom heurističkom kvantizacionom procedurom spinske pene. Najpre, konfiguracioni integral topološke sume po stanjima dat je izrazom:

$$Z = \int \mathcal{D}\alpha\, \mathcal{D}B \exp\left(i \int_{M_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} \right) . \tag{7.8}$$

Formalnom integracijom po Lagraževom množitelju B dobijamo izraz:

$$Z = \mathcal{N} \int \mathcal{D}\alpha\, \delta(\mathcal{F}) \,. \tag{7.9}$$

Zatim, 1-forma koneksije $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$ se diskretizuje bojenjem ivica triangulacije $\epsilon = (jk) \in \Lambda_1$ grupnim elementima $g_\epsilon \in G$. Meru konfiguracionog integrala (7.8) diskretizujemo smenom:

$$\int \mathcal{D}\alpha \qquad \mapsto \qquad \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \,, \tag{7.10}$$

gde dg_{jk} označava integraciju sa Harovom merom na grupi G.

Uslov nestajanja krivine diskretizuje na svakom trouglu $(jkl) \in \Lambda_2$ δ-funkcija $\delta(\mathcal{F})$. Prilikom prelaza sa glatke mnogostrukosti na njenu triangulaciju, δ-funkcija definiše se na skupu trouglova triangulacije,

$$\delta(\mathcal{F}) = \prod_{(jk\ell)\in\Lambda_2} \delta_G(g_{jk\ell})\,, \tag{7.11}$$

gde je za svaki trougao $(jkl) \in \Lambda_2$ odgovarajuća δ-funkcija $\delta_G(g_{jkl})$ data izrazom:

$$\delta_G(g_{jk\ell}) = \delta_G\big(g_{k\ell}\, g_{jk}\, g_{j\ell}^{-1}\big)\,. \tag{7.12}$$

Identitet (7.12) je posledica jednačine (7.7) iz Leme 12.

Zamenom prethodno definisane diskretizovane mere (7.10) i δ-funkcije (7.11) u jednačinu (7.9) dobija se suma po stanjima[7]:

$$Z = \mathcal{N} \prod_{(jk)\in\Lambda_1} \int_G dg_{jk} \Bigg(\prod_{(jk\ell)\in\Lambda_2} \delta_G\big(g_{jk\ell}\big) \Bigg)\,. \tag{7.18}$$

Zatim, zamenom izraza (7.12) u izraz (7.18), dobijamo eksplicitni izraz za sumu po stanjima date triangulacije prostorvremenske mnogostrukosti $\mathcal{M}$. Odgovarajućim izborom konstante ispred integrala $\mathcal{N}$, dobijene nakon integracije po Lagranževom množitelju B, ova suma postaje nezavisna od integracije, tj. invarijantna na Pahnerove poteze. Upravo zahtevanjem ove invarijantnosti, za sve Pahnerove poteze, dobijamo odgovarajući izbor konstatne $\mathcal{N}$, dat definicijom 7.2.1.

[7] Sličan postupak možemo sprovesti i u dualnoj triangulaciji. Najpre, diskretizujmo dejstvo:

$$S_{BF}[B,\alpha] \to S_{BF}^{disc}[B,\alpha] \equiv \sum_{f\in\Lambda_2^*} \mathrm{tr}\,\big[B_f g_f\big], \tag{7.13}$$

gde smo integral tenzora krivine zamenili sumom holonomija g_f po svim poligonima f dualnim sklopkama triangulacije, u trodimenzionalnom slučaju ivicama $(jk) \in \Lambda_1$, na kojima je krivina različita od nule, dok smo množilac B zamenili njegovom vrednošću B_f na svakom poligonu f. Holonomija g_f na poligonu f se može napisati kao proizvod holonomija g_l redom po svim ivicama $l \in \Lambda_2^*$ datog poligona $f \in \Lambda_2^*$ [38]:

$$g_f = \prod_{l\in f} g_l. \tag{7.14}$$

Zatim, integracijom po svim množiteljima i holonomijama

$$\begin{aligned} Z \to Z^{disc} &= \int \Big(\prod_{f\in\Lambda_2^*} \mathcal{D}B_f \Big) \int \Big(\prod_{l\in\Lambda_1^*} \mathcal{D}g_l \Big) \exp\Big(i \sum_{f\in\Lambda_2^*} \mathrm{tr}\,\big[B_f \prod_{l\in f} g_l\big] \Big) \\ &= \int \Big(\prod_{l\in\Lambda_1^*} \mathcal{D}g_l \Big) \prod_{f\in\Lambda_2^*} \Big(\int \mathcal{D}B_f \exp\Big(i\, \mathrm{tr}\,\big[B_f \prod_{l\in f} g_l\big] \Big) \Big)\,, \end{aligned} \tag{7.15}$$

gde prepoznajemo da je

$$\int \mathcal{D}B_f \exp\Big(i\, \mathrm{tr}\,\big[B_f \prod_{l\in f} g_l\big] \Big) = \mathcal{N}'\delta(\prod_{l\in f} g_l) = \mathcal{N}'\delta(g_f)\,, \tag{7.16}$$

dobija se suma po stanjima:

$$Z^{disc} = \mathcal{N} \Big(\prod_{l\in\Lambda_1^*} \int \mathcal{D}g_l \Big) \Big(\prod_{f\in\Lambda_2^*} \delta(g_f) \Big)\,. \tag{7.17}$$

Dobijena suma po stanjima ekvivalentna je jednačini (7.18) ali je izražena preko elemenata dualne triangulacije.

Definicija 7.2.1 *Neka je $\mathcal{M}_d$ kompaktna i orijentisana kombinatorna d-mnogostrukost, $d \in \{3,4\}$. Suma po stanjima* topološke gejdž teorije *je definisana kao:*

$$Z = |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}\left(\prod_{(jk)\in\Lambda_1}\int_G dg_{jk}\right)\left(\prod_{(jk\ell)\in\Lambda_2}\delta_G\big(g_{k\ell}\,g_{jk}\,g_{j\ell}^{-1}\big)\right) \tag{7.19}$$

U prethodnoj definiciji integracija se vrši po elementima $g_{jk} \in G$ za svaku ivicu $(jk) \in \Lambda_1$. Podintegralna δ-funkcija nameće sledeći uslov.

- Za svaki trougao $(jk\ell) \in \Lambda_2$, uslov $g_{k\ell}\,g_{jk} = g_{j\ell}$ (videti Lemu 12).

Teorema 19 *Neka je $\mathcal{M}_d$ zatvorena i orijentisana kombinatorna d-mnogostrukost za $d \in \{3,4\}$. Suma po stanjima (7.19) je invarijantna na Pahnerove poteze.*

Za sada Teoremu 19 ostavićemo bez dokaza. U poglavljima 8 i 9 razmatrane su generalizacije ove teoreme u okviru teorije kategorija i njihovi dokazi, pa se dokaz prethodne teoreme može jednostavno dobiti pojednostavljivanjem ovih opštijih slučajeva.

7.2.1 $d = 3$: Ponzano-Redže model

Trodimenzionalna kvantna gravitacija može biti definisana na više načina, a prvi uspešan pristup je *Ponzano-Redže model* kvantne gravitacije na diskretizovanoj trodimenzionalnoj mnogostrukosti formiranjem BF sume po stanjima za izbor gejdž grupe $G = SU(2)$.

Topološka suma po stanjima dobijena u prethodnom odeljku se posle izbora konkretne gejdž grupe G dalje transformiše korišćenjem Piter-Vejlove ili Planšarelove teoreme[8]. Prirodan izbor grupe G je grupa izometrija datog prostora, što je u trodimenzionalnom slučaju grupa $SO(3)$, odnosno $SU(2)$. Stoga za grupu $SU(2)$ možemo pisati:

$$\delta(g_f) = \sum_{j_f}\Big(\dim j_f\Big)\,\mathrm{tr}\Big[D^{(j_f)}(g_f)\Big]\,, \qquad j_f \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \dots\}\,. \tag{7.20}$$

Zamenom prethodnog izraza za $\delta(g_f)$ u izraz za sumu po stanjima u dualnoj triangulaciji (7.17) dobijamo:

$$\begin{aligned} Z^{disc} &= \mathcal{N}\int\Big(\prod_{l\in\Lambda_1^*}\mathcal{D}g_l\Big)\prod_{f\in\Lambda_2^*}\Bigg(\sum_{j_f\in\mathcal{N}_0/2}\Big(2j_f+1\Big)\,\mathrm{tr}\Big[\prod_{l\in f}D^{(j_f)}(g_l)\Big]\Bigg) \\ &= \mathcal{N}\sum_{\{j_f\}}\Big(\prod_{f\in\Lambda_2^*}(2j_f+1)\Big)\int\Big(\prod_{l\in\Lambda_1^*}\mathcal{D}g_l\Big)\Big(\dots D^{(j_f)}{}^{a}{}_{\alpha}(g_l)\dots D^{(j_{f^*})}{}^{a^*}{}_{\alpha^*}(g_{l^*})\dots\Big), \end{aligned} \tag{7.21}$$

[8] Piter-Vejlova, odnosno Planšarelova teorema, obezbeđuju dekompoziciju funkcija na grupi u sumu po odgovarajućim ireducibilnim reprezentacijama grupe. Teorema nosi ime *Piter-Vejl* za slučaj kada je grupa G kompaktna, dok je Planšarel dokazao analognu teoremu u slučaju nekompaktne grupe G.

Teorema 20 (Piter-Vejl, Planšarel teorema) *Za δ-funkciju čiji je argument element grupe $g_f \in G$ važi,*

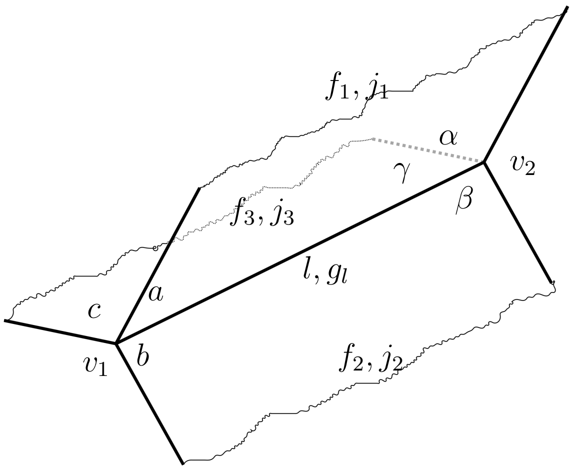
$$\delta(g_f) = \sum_{\Lambda_f}\dim(\Lambda_f)\,\chi(g_f,\Lambda_f)\,,$$

gde su Λ_f unitarne ireducibilne reprezentacije Lijeve grupe G, $\chi(g_f,\Lambda_f)$ je trag elementa g_f u reprezentaciji Λ_f i $\dim(\Lambda_f)$ je dimenzija reprezentacije.

gde su u poslednjem redu svi indeksi kontrakovani. Preuređivanjem elemenata proizvoda u zagradi u prethodnom izrazu tako da grupišemo reprezentacije elementa g_l dobijamo:

$$Z^{disc} = \mathcal{N} \sum_{\{j_f\}} \Big(\prod_{f\in\Lambda_2^*} (2j_f+1) \Big) \operatorname{tr} \Bigg[\prod_{l\in\Lambda_1^*} \Big(\int \mathcal{D}g_l D^{(j_1)}{}^a_\alpha(g_l) D^{(j_2)}{}^b_\beta(g_l) D^{(j_3)}{}^c_\gamma(g_l) \Big) \Bigg]. \tag{7.22}$$

Očigledno je da postoje tri takve reprezentacije, odnosno tri stranice f kojima odgovaraju brojevi j_f, koje dele ivicu l, kao što je prikazano na Slici 7.1. Razlog za to je jer ivici l dualne rešetke odgovara trougao, koji ima tri ivice, kojima odgovaraju stranice f. Za grupu $SU(2)$



Slika 7.1: Jedna ivica l dualne rešetke i stranice f_1, f_2 i f_3 kojima je zajednička.

i broj reprezentacija $n = 3$ nekog elementa grupe, postoji samo jedan intertvajner[9] i on je $\{3j\}$-simbol za koji važi

$$\int \mathcal{D}g_l D^{(j_1)}{}^a_\alpha(g_l) D^{(j_2)}{}^b_\beta(g_l) D^{(j_3)}{}^c_\gamma(g_l) = i^{abc} i_{\alpha\beta\gamma} = \begin{pmatrix} j_1 & j_2 & j_3 \\ a & b & c \end{pmatrix}_{SU(2)} \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha & \beta & \gamma \end{pmatrix}_{SU(2)}. \tag{7.24}$$

Korišćenjem (7.24) jednačina (7.22) postaje:

$$Z^{disc} = \mathcal{N} \sum_{\{j_f\}} \Big(\prod_{f\in\Lambda_2^*} (2j_f+1) \Big) \operatorname{tr} \Bigg[\prod_{l\in\Lambda_1^*} \begin{pmatrix} j_1 & j_2 & j_3 \\ a & b & c \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ \alpha & \beta & \gamma \end{pmatrix} \Bigg]. \tag{7.25}$$

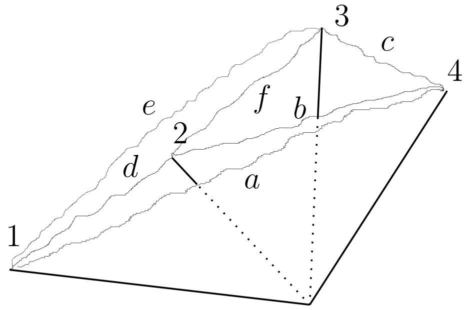
Daljim grupisanjem intertvajnera po zajedničkom verteksu, na način prikazan na Slici 7.2, dobijamo:

$$Z^{disc} = \mathcal{N} \sum_{\{j_f\}} \Big(\prod_{f\in\Lambda_2^*} (2j_f+1) \Big) \prod_{v\in\Lambda_0^*} \left(\begin{pmatrix} j_1 & j_4 & j_5 \\ a & d & e \end{pmatrix} \begin{pmatrix} j_2 & j_4 & j_6 \\ b & d & f \end{pmatrix} \begin{pmatrix} j_3 & j_5 & j_6 \\ c & e & f \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ a & b & c \end{pmatrix} \right). \tag{7.26}$$

[9] Za svaki element grupe $g \in G$ i njene reprezentacije Λ_i postoji jedan ili više objekata koje zovemo *intertvajneri* [1] koji zadovoljavaju jednakost:

$$D^{(\Lambda_1)}{}^a_\alpha(g) D^{(\Lambda_2)}{}^b_\beta(g) D^{(\Lambda_3)}{}^c_\gamma(g) \cdots D^{(\Lambda_n)}{}^n_\nu(g) i^{(\Lambda_1\Lambda_2\cdots\Lambda_n)}_{abc\cdots n} = i^{(\Lambda_1\Lambda_2\cdots\Lambda_n)}_{\alpha\beta\gamma\cdots\nu}. \tag{7.23}$$

Intertvajneri zavise od n, grupe G i njene reprezentacije, ali ne i od elementa grupe $g \in G$.



Slika 7.2: Četiri ivica i šest stranica koje se sastaju u jednom verteksu dualne triangulacije $3D$ mnogostrukosti.

Jednom verteksu odgovara šest[10] stranica i četiri ivica u dualnoj triangulaciji. Četiri $\{3j\}$-simbola grupisana u zagradi čine $\{6j\}$-simbol, funkciju šest brojeva $f(j_1, \cdots, j_6)$[11], pa možemo pisati:

$$Z^{disc} = \mathcal{N} \sum_{j_1} \cdots \sum_{j_{|\Lambda_2^*|}} \left(\prod_{f=1}^{|\Lambda_2^*|} (2j_f + 1) \right) \left(\prod_{v=1}^{|\Lambda_0^*|} \{6j\}^{v}_{SU(2)} \right) . \tag{7.27}$$

Zapisano preko elemenata triangulacije suma po stanjima *Ponzano-Redže modela* je:

$$Z^{disc} = \mathcal{N} \sum_{j_1} \cdots \sum_{j_{|\Lambda_1|}} \left(\prod_{\epsilon=1}^{|\Lambda_1|} (2j_\epsilon + 1) \right) \left(\prod_{\tau=1}^{|\Lambda_3|} \{6j\}^{\tau}_{SU(2)} \right) . \tag{7.28}$$

Konstruisana BF suma po stanjima daje kvantnu teoriju gravitacije u slučaju trodimenzionalne prostorvremenske mnogostrukosti. Stoga, klasična teorija mora biti dobijena u klasičnom limitu ove kvantne teorije. U kvantnoj mehanici, klasičan limit se dobija u limitu velikih kvantnih brojeva, gde kvantna diskretnost postaje zanemarljiva. Ponzano i Redže su ukazali, a Roberts je formalno dokazao, da u limitu velikih kvantnih brojeva j važi

$$\{6j\} \underset{j\to\infty}{\sim} \frac{1}{\sqrt{12\pi V}} \cos(S + \frac{\pi}{4}) , \tag{7.29}$$

gde je V zapremina tetraedra, a S klasično Redže dejstvo tetraedra. Raspisivanjem kosinusa i korišćenjem adicionih formula za kosinus dobijamo:

$$\{6j\} \underset{j\to\infty}{\sim} \frac{1}{2\sqrt{-12i\pi V}} \exp(iS) + \frac{1}{2\sqrt{12i\pi V}} \exp(-iS) . \tag{7.30}$$

U limitu velikih kvantnih brojeva, zbir po spinovima u jednačini (7.28) možemo aproksimirati integralom po dužinama u Redže geometriji. Na osnovu jednačine (7.30) vidimo da integrand ima oblik eksponenta dejstva, pa se ispostavlja da se Ajnštajn–Hilbertovo dejstvo krije u $\{6j\}$ simbolu. Stoga, u klasičnom limitu dobijamo[12] integral po putanjama za Ajnštajn–Hilbertovo dejstvo:

$$Z \sim \int \mathcal{D}g \, e^{\frac{i}{\hbar} \int \sqrt{-g} R} . \tag{7.31}$$

[10] Verteks dualne triangulacije odgovara tetraedru obične triangulacije $3D$ mnogostrukosti, četiri ivice koje se sastaju u verteksu dualne triangulacije odgovaraju trouglovima $\mathcal{T}(\mathcal{M}_3)$, šest stranica dualne triangulacije odgovara šest ivica $\mathcal{T}(\mathcal{M}_3)$, itd.

[11] U jednom verteksu sastaju se četiri ivice, gde za svake dve postoji stranica koja ih obe sadrži. Intertvajneri oblika i^{pqr} nose informaciju o tri stranice p, q i r kojima je zajednička ivica kojoj odgovara taj intertvajner, odnosno tri broja j_f koje stranice nose.

[12] Primetimo da u izrazu (7.30) postoje dva člana sa suprotnim predznacima dejstva u eksponentima, pa je dobijeni izraz malo komplikovaniji od jednačine (7.31).

Četvorodimenzionalni slučaj se ispostavlja neznatno komplikovanijim, što je i očekivano jer u četiri dimenzije gravitacija nije topološka teorija bez lokalnih propagirajućih stepeni slobode. Konstrukcija BF sume po stanjima za slučaj četvorodimenzionalne mnogostrukosti je predstavljena u narednom odeljku.

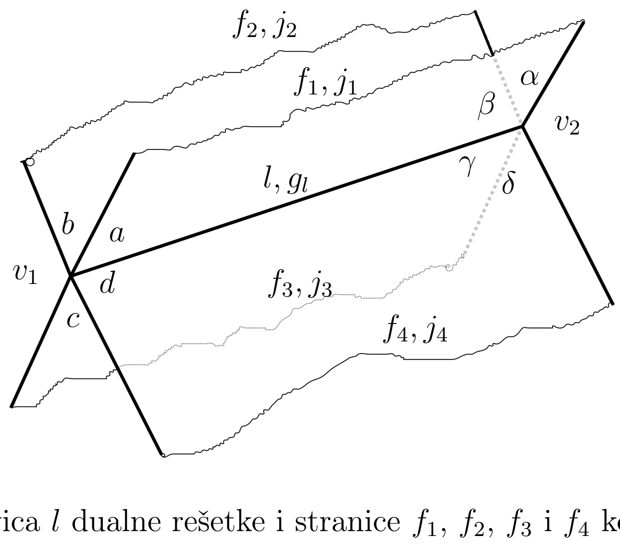
7.2.2 $d = 4$: Ouguri model

U slučaju četvorodimenzionalnog prostorvremena BF topološko dejstvo je zadato na isti način kao i u trodimenzionalnom slučaju,

$$S_{BF}[B,\omega] = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab}, \tag{7.32}$$

pr čemu su sada krivina $R_{ab} = \mathrm{d}\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b$ i Lagranžev množitelj B^{ab} 2-forme. Ponavljajući postupak iz odeljka 7.2.1 za BF dejstvo definisano za trodimenzionalnu prostorvremensku mnogostrukost, dolazimo do jednačine (7.17). Primenom Teoreme 20 za grupu $G = SO(4)$ dobijamo da suma po stanjima ima oblik (7.21), pri čemu indeks j_f sada označava ireducibilne reprezentacije grupe $SO(4)$.

Zatim, proizvod u zagradi sume po stanjima (7.17) se preuređuje tako da grupišemo reprezentacije elementa grupe g_l, pri čemu imamo u vidu da sada postoje četiri takve reprezentacije, tj. četiri broja j_f. Na Slici 7.3 je demonstrirano da u $4D$ postoje četiri stranice f koje dele ivicu l, iz razloga što ivici l dualne rešetke odgovara tetraedar, koji ima četiri trougla, kojima odgovaraju četiri stranice f.



Slika 7.3: Jedna ivica l dualne rešetke i stranice f_1, f_2, f_3 i f_4 kojima je zajednička.

Korišćenjem definicije intertvajnera za grupu $SO(4)$, može se dokazati sledeća teorema:

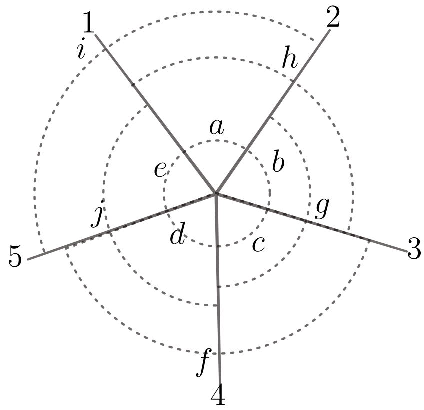
$$\int \mathcal{D}g_l D^{(j_1)a}{}_{\alpha}(g_l) D^{(j_2)b}{}_{\beta}(g_l) D^{(j_3)c}{}_{\gamma}(g_l) D^{(j_4)d}{}_{\delta}(g_l) = \sum_{I_l} i^{abcd}_{I_l} i^{I_l}_{\alpha\beta\gamma\delta}. \tag{7.33}$$

Primećujemo da za razliku od grupe $SU(2)$, koja ima samo jedan intertvajner $\{3j\}$ za dati skup reprezentacija, u slučaju grupe $SO(4)$ postoji više intertvajnera koji zadovoljavaju definicionu jednačinu (7.23). Ti intertvajneri se u jednačini (7.33) prebrojavaju nekim dodanim indeksom I_l.

Suma po stanjima koju dobijamo iz (7.21) odgovarajućim preuređivanjem i primenom jednačine (7.33) je:

$$\begin{aligned} Z^{disc} &= \sum_{\{j_f\}} \left(\prod_{f\in\Lambda_2^*} \dim j_f \right) \mathrm{tr} \left[\prod_{l\in\Lambda_1^*} \left(\int \mathcal{D}g_l D^{(j_1)}{}^a{}_\alpha(g_l) D^{(j_2)}{}^b{}_\beta(g_l) D^{(j_3)}{}^c{}_\gamma(g_l) D^{(j_4)}{}^d{}_\delta(g_l) \right) \right] \\ &= \sum_{\{j_f\}} \left(\prod_{f\in\Lambda_2^*} \dim j_f \right) \mathrm{tr} \left[\prod_{l\in\Lambda_1^*} \sum_{I_l} i^{abcd}_{I_l} i^{I_l}_{\alpha\beta\gamma\delta} \right] \\ &= \sum_{\{j_f\}} \sum_{\{I_l\}} \left(\prod_{f\in\Lambda_2^*} \dim j_f \right) \mathrm{tr} \left[\prod_{l\in\Lambda_1^*} i^{a_l b_l c_l d_l}_{I_l} i^{I_l}_{\alpha_l\beta_l\gamma_l\delta_l} \right] \\ &= \sum_{\{j_f\}} \sum_{\{I_l\}} \left(\prod_{f\in\Lambda_2^*} \dim j_f \right) \prod_{v\in\Lambda_0^*} \left(i^{efgb}_{I_l} i^{ehic}_{I_2} i^{fhja}_{I_3} i^{gjid}_{I_4} i^{abcd}_{I_5} \right). \end{aligned} \tag{7.34}$$

U poslednjem koraku smo grupisali intertvajnere po zajedničkom verteksu kao što je prikazano na Slici 7.4. Jednom verteksu odgovara deset[13] stranica i pet ivica u dualnoj triangulaciji. Pet intertvajnera grupisanih u zagradi predstavljaju $\{15j\}_{SO(4)}$-simbol, funkciju petnaest brojeva $f(I_1, \cdots, I_5, j_1, \cdots, j_{10})$[14]. Konačno, dobijamo da suma po stanjima topološke BF teorije za



Slika 7.4: Pet ivica i deset stranica koje se sastaju u jednom verteksu dualne triangulacije $4D$ mnogostrukosti.

četvorodimenzionalnu prostorvremensku mnogostrukost ima oblik:

$$Z^{disc} = \sum_{j_1} \cdots \sum_{j_{|\Lambda_2^*|}} \sum_{I_1} \cdots \sum_{I_{|\Lambda_1^*|}} \left(\prod_{f=1}^{|\Lambda_2^*|} \dim j_f \right) \left(\prod_{v=1}^{|\Lambda_0^*|} \{15j\}^v_{SO(4)} \right), \tag{7.35}$$

odnosno zapisano preko elemenata triangulacije:

$$Z^{disc} = \sum_{j_1} \cdots \sum_{j_{|\Lambda_2|}} \sum_{I_1} \cdots \sum_{I_{|\Lambda_3|}} \left(\prod_{\Delta=1}^{|\Lambda_2|} \dim j_\Delta \right) \left(\prod_{\sigma=1}^{|\Lambda_4|} \{15j\}^\sigma_{SO(4)} \right). \tag{7.36}$$

Dobijena suma po stanjima predstavlja *Ouguri model* [4].

[13]Verteks dualne triangulacije odgovara 4-simpleksu obične triangulacije $4D$ mnogostrukosti, pet ivica koje se sastaju u verteksu dualne triangulacije odgovaraju tetraedrima $\mathcal{T}(\mathcal{M}_4)$, deset stranica dualne triangulacije odgovaraju deset trouglova $\mathcal{T}(\mathcal{M}_4)$, itd.

[14]Intertvajneri oblika $i^{pqrs}_{I_l}$ nose informaciju o ivici l i o četiri stranice kojima je ona zajednička p, q, r i s, tj. o broju I_l koji nosi ivica i brojevima j_f koje nose stranice.

Za razliku od trodimenzionalnog slučaja gde je opšta teorija relativnosti teorija bez lokalnih propagirajućih stepeni slobode, u četiri dimenzije BF teorija nije ekvivalentna opštoj teoriji relativnosti. Da bi smo dobili teoriju koja opisuje OTR, neophodna je modifikacija BF dejstva dodavanjem odgovarajućih veza, tj. formulacija Plebanski dejstva (4.74) na nivou klasične teorije, ili deformacija topološke sume po stanjima (7.36) u netopološku sumu po stanjima na nivou kvantne teorije. Deformacija sume po stanjima se postiže izborom drugačijih amplituda u modelu, procedurom opisanom u [9][10]. Ovaj postupak rezultuje *EPRL/FK modelom spinske pene*, koji predstavlja jednu moguću kvantizaciju *Opšte teorije relativnosti.*

Kvantovanje BF teorije sa vezama i konstrukcija kvantne teorije gravitacije prevazilazi okvire naše diskusije, pa zainteresovanog čitaoca upućujemo na literaturu [9][10].

Glava 8

Formiranje topološke sume po stanjima: $2BF$ teorija

U ovom poglavlju fokusiraćemo se na drugi korak kovarijantne kvantizacione procedure spinske pene za $2BF$ teoriju. Demonstriraćemo kako se konstruiše suma po stanjima Z koja je nezavisna od triangulacije, na osnovu klasičnog $2BF$ dejstva za opštu striktnu 2-grupu i bilo koju triangulaciju bilo koje glatke d-dimenzionalne prostorvremenske mnogostrukosti, za slučajeve $d \in \{3, 4\}$. Za $d = 3$, kontruisana suma po stanjima je upravo Jeterov model, dok se za $d = 4$ poklapa sa Porterovom TKTP za $d = 4$ i $n = 2$.

Da bismo proverili da je konstruisana suma po stanjima topološka, analiziramo njeno ponašanje pri Pahnerovim potezima, lokalnim promenama triangulacije koje čuvaju topologiju, tako da su bilo koje dve triangulacije iste mnogostrukosti povezane konačnim brojem Pahnerovih poteza. U trodimenzionalnom slučaju postoji četiri Pahnerova poteza — potezi $1 \leftrightarrow 4$ i $2 \leftrightarrow 3$ i njihovi inverzi, dok u 4 dimenzije postoji pet različitih Pahnerovih poteza — potezi $3 \leftrightarrow 3$, $4 \leftrightarrow 2$ i $5 \leftrightarrow 1$ i njihovi inverzi. Postavka analize ponašanja konstruisane sume po stanjima pri ovim Pahnerovim potezima predstavljena je u odeljku 8.3, dok su detalji proračuna dati u Dodatku E.1. Dobijeno je da suma po stanjima nepromenjena pri ovim transformacijama mnogostrukosti, što dokazuje da je *topološka invarijanta* mnogostrukosti. Kako je nezavisna od triangulacije, suma po stanjima je nepromenjena pri proizvoljnom usitnjavanju triangulacije i stoga definiše teoriju kontinuuma na glatkoj mnogostrukosti.

Pogledati rad Žirelija, Pfajfera i Popeskua za više informacija [13].

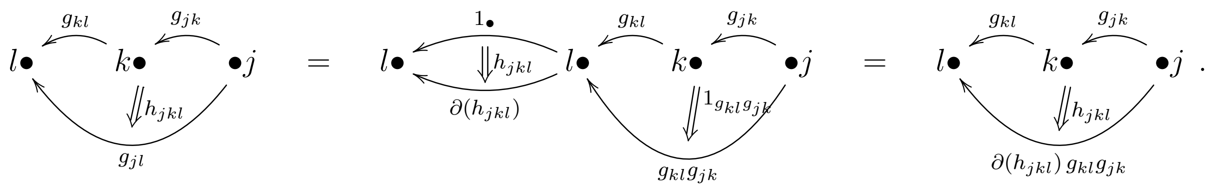
8.1 Gejdž invarijantni objekti

Dejstvo klasične BF teorije izabrano je tako da bude nezavisno od bilo kakve pozadinske metrike, tj. da zavisi samo od prostorvremenske mnogostrukosti. Klasične jednačine kretanja nameću uslov da je gejdž koneksija ravna, tj. na jeziku holonomija, da svaka nul-homotopna kriva odgovara identitetu gejdž grupe. U okviru viših gejdž teorija, konkretno 2-gejdž teorije i odgovarajuće $2BF$ teorije, ovaj uslov se generalizuje zahtevom da površinska holonomija granične 2-sfere svake 3-lopte bude trivijalna.

U odeljku 2.2 uveli smo niz operacija pomoću kojih na jeziku 2-gejdž teorije definišu kompozicije proizvoljnih puteva i površina, sve do proizvoljno velikih. U ovom odeljku ćemo koristiti ove kompozicije kako bismo konstruisali gejdž invarijantne veličine koje odgovaraju zatvorenim putevima i površinama. U Lemama 13 i 14, ovaj se postupak koristi za graničnu putanju trougla i graničnu površinu tetraedra, kao što je izvedeno u radu [13].

Lema 13 *Posmatrajmo trougao* $(jk\ell)$. *Ivice trougla* $(jk)\,, j < k$ *su obeležene grupnim elementima* $g_{jk} \in G$, *a trougao* $(jk\ell)\,,\ j < k < \ell$ *je obeležen elementom* $h_{jk\ell} \in H$. *Razmotrimo dijagram*

(8.1).



(8.1)

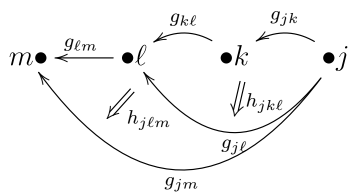
Kriva $\gamma_1 = g_{k\ell}g_{jk}$ *je izvor* 2*-morfizma, a kriva* $\gamma_2 = g_{j\ell}$ *je meta površinskog* 2*-morfizma* $\Sigma : \gamma_1 \to \gamma_2$ *obeleženog sa* $h_{jk\ell}$*,*

$$g_{j\ell} = \partial(h_{jk\ell})g_{k\ell}g_{jk}\,. \tag{8.2}$$

Lema 14 *Posmatrajmo tetraedar* $(jk\ell m)$*. Ivice* $(jk)\,, j < k$ *su obeležene grupnim elementima* $g_{jk} \in G$*, trouglovi* $(jk\ell)\,,\, j < k < \ell$ *grupnim elementima* $h_{jk\ell} \in H$*. Orijentisali smo trouglove* $(jk\ell)$ *tako da je kriva* $g_{k\ell}g_{jk}$ *izvor* 2*-morfizma, a kriva* $g_{j\ell}$ *meta, tj. zadovoljen je uslov* $g_{j\ell} = \partial(h_{jk\ell})g_{k\ell}g_{jk}$ *.*

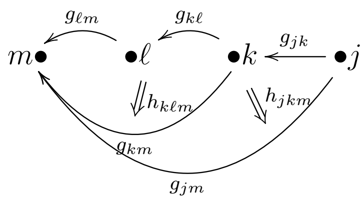
Najpre, presečemo površinu tetraedra po granici (jm)*. To određuje redosled vertikalne kompozicije sastavnih površina. Moramo biti sigurni da su sve površine kompozibilne, tj. da imaju odgovarajuće referentne točke i ispravnu orijentaciju za vertikalnu kompoziciju.*

Analizirajmo prvo kompoziciju prikazanu na dijagramu (8.3). Prvo pomeramo krivu od $g_{k\ell}g_{jk}$ *do krive* $g_{j\ell}$*. U ovom stadijumu dobijeni rezultat nije vertikalno kompozibilan sa trouglom* $(j\ell m)$*, prvo mu moramo dodati* $g_{\ell m}$ *sa leve strane. Sada su ova dva morfizma vertikalno kompozibilna, a dobijeni* 2*-morfizam prevlači krivu do* g_{jm}*. Dobijeni* 2*-morhizam je:*



$$= (g_{\ell m}g_{j\ell}, h_{j\ell m})\#_2\big(g_{\ell m}\#_1(g_{k\ell}g_{jk}, h_{jk\ell})\big) = \big(g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell m}(g_{\ell m} \rhd h_{jk\ell})\big)\,. \tag{8.3}$$

Razmotrimo dijagram (8.4). Prvo prevlačimo krivu $g_{\ell m}g_{k\ell}$ *do krive* g_{km}*. Da bi vertikalna kompozicija dobijenog rezultata sa trouglom* (jkm) *bila moguća, najpre mu dodamo krivu* g_{jk} *sa desne strane. Sada su dva morfizma vertikalno kompozibilna i dobijeni* 2*-morfizam prevlači krivu na* g_{jm}*. Dobijen je sledeći* 2*-morfizam:*



$$= (g_{km}g_{jk}, h_{jkm})\#_2\big((g_{\ell m}g_{k\ell}, h_{k\ell m})\#_1 g_{jk}\big) = (g_{\ell m}g_{k\ell}g_{jk}, h_{jkm}h_{k\ell m})\,. \tag{8.4}$$

Dve površine $\Sigma_1 : g_{\ell m}g_{k\ell}g_{jk} \to g_{jm}$ *i* $\Sigma_2 : g_{\ell m}g_{k\ell}g_{jk} \to g_{jm}$ *opisuju isti* 2*-morfizam, odnosno*

$$(g_{\ell m}g_{k\ell}g_{jk}, h_{jkm}h_{k\ell m}) = \big(g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell m}(g_{\ell m} \rhd h_{jk\ell})\big)\,, \tag{8.5}$$

na osnovu čega dobijamo relaciju:

$$h_{jkm}h_{k\ell m} = h_{j\ell m}(g_{\ell m} \rhd h_{jk\ell})\,. \tag{8.6}$$

8.2 Kvantizacija topološkog $2BF$ dejstva

U ovom odeljku predstavićemo kombinatorni opis konstrukcije sume po stanjima za $2BF$ teoriju formiranu za triangulaciju mnogostrukosti dimenzije $d = \{3, 4\}$.

Kvantovanje $2BF$ klasičnog dejstva, datog jednačinom (8) radi se na isti način kao i u slučaju BF teorija, uobičajenom heurističkom kvantizacionom procedurom spinske pene. Najpre, konfiguracioni integral topološke sume po stanjima dat je izrazom:

$$Z = \int \mathcal{D}\alpha \, \mathcal{D}\beta \, \mathcal{D}B \, \mathcal{D}C \exp\left(i \int_{M_d} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} \right) . \tag{8.7}$$

Formalnom integracijom po Lagraževim množiteljima B i C dobijamo izraz:

$$Z = \mathcal{N} \int \mathcal{D}\alpha \, \mathcal{D}\beta \, \delta(\mathcal{F}) \delta(\mathcal{G}) . \tag{8.8}$$

Slično kao i u običnoj gejdž teoriji 1-forma koneksije $\alpha \in \mathcal{A}^1(\mathcal{M}_d, \mathfrak{g})$ se diskretizuje bojenjem ivica triangulacije $\epsilon = (jk) \in \Lambda_1$ grupnim elementima $g_\epsilon \in G$, dok se 2-forma koneksije $\beta \in \mathcal{A}^2(\mathcal{M}_d \, , \mathfrak{h})$ diskretizuje bojenjem trouglova $\Delta = (jk\ell) \in \Lambda_2$ elementima grupe $h_\Delta \in H$. Mere konfiguracionog integrala (8.7) diskretizujemo smenom:

$$\int \mathcal{D}\alpha \quad \mapsto \quad \prod_{(jk) \in \Lambda_1} \int_G \mathrm{d}g_{jk} \, , \tag{8.9}$$

$$\int \mathcal{D}\beta \quad \mapsto \quad \prod_{(jk\ell) \in \Lambda_2} \int_H \mathrm{d}h_{jk\ell} \, , \tag{8.10}$$

gde dg_{jk} i $dh_{jk\ell}$ označavaju integraciju sa Harovom merom na grupama G i H. Uslov nestajanja lažne krivine zadaje se na svakom trouglu $(jkl) \in \Lambda_2$ dikretizacijom $\delta(\mathcal{F})$. Prilikom prelaza sa glatke mnogostrukosti na njenu triangulaciju, δ distribucija definiše se na skupu elemenata triangulacije,

$$\delta(\mathcal{F}) = \prod_{(jk\ell) \in \Lambda_2} \delta_G(g_{jk\ell}) \, , \tag{8.11}$$

gde je za svaki trougao $(jkl) \in \Lambda_2$ odgovarajuća δ-funkcija $\delta_G(g_{jkl})$ data izrazom (videti jednačinu (8.2) u Lemi 13):

$$\delta_G(g_{jk\ell}) = \delta_G\big(\partial(h_{jk\ell}) \, g_{k\ell} \, g_{jk} \, g_{j\ell}^{-1}\big) . \tag{8.12}$$

Uslov da je površinska holonomija granične 2-sfere svake 3-lopte trivijalna $\delta(\mathcal{G})$, diskretizovan na elemente triagulacije mnogostrukosti postaje

$$\delta(\mathcal{G}) = \prod_{(jk\ell m) \in \Lambda_3} \delta_H(h_{jk\ell m}) \, , \tag{8.13}$$

gde za svaki tetraedar $(jk\ell m) \in \Lambda_3$ važi:

$$\delta_H(h_{jk\ell m}) = \delta_H\big(h_{j\ell m} \, (g_{\ell m} \triangleright h_{jk\ell}) \, h_{k\ell m}^{-1} \, h_{jkm}^{-1}\big) . \tag{8.14}$$

Izraz (8.14) je posledica jednačine (8.6) izvedene u Lemi 14.

Zamenom prethodno definisanih diskretizovanih mera (8.9) i (8.10), δ-funkcija (8.11) i (8.13) u jednačinu (8.8) dobija se suma po stanjima:

$$Z = \ \mathcal{N} \prod_{(jk) \in \Lambda_1} \int_G \mathrm{d}g_{jk} \prod_{(jk\ell) \in \Lambda_2} \int_H \mathrm{d}h_{jk\ell} \Bigg(\prod_{(jk\ell m) \in \Lambda_3} \delta_G\big(g_{jk\ell}\big) \Bigg) \Bigg(\prod_{(jk\ell m) \in \Lambda_3} \delta_H\big(h_{jk\ell m}\big) \Bigg) . \tag{8.15}$$

Zatim, zamenom izraza (8.12) i (8.14) u izraz (8.15), dobijamo eksplicitni izraz za sumu po stanjima date triangulacije mnogostrukosti $\mathcal{M}_d$. Odgovarajućim izborom konstante ispred integrala $\mathcal{N}$, dobijene nakon integracije po Lagranževim množiteljima B i C, ova suma postaje nezavisna od triangulacije, tj. invarijantna na Pahnerove poteze[1]. Upravo zahtevanjem ove invarijantnosti, za sve Pahnerova poteza, dobijamo odgovarajući izbor konstatne $\mathcal{N}$, dat definicijom 8.2.1.

Definicija 8.2.1 *Neka je $\mathcal{M}_d$ kompaktna i orijentisana kombinatorna d-mnogostrukost za $d = \{3,4\}$ i neka je $(H \stackrel{\partial}{\to} G\,, \rhd\,)$ ukršten modul. Suma po stanjima topološke više gejdž teorije je definisana kao:*

$$\begin{aligned} Z \;=\; & |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|} \left(\prod_{(jk)\in\Lambda_1} \int_G \mathrm{d}g_{jk}\right) \left(\prod_{(jk\ell)\in\Lambda_2} \int_H \mathrm{d}h_{jk\ell}\right) \\ & \times \left(\prod_{(jk\ell)\in\Lambda_2} \delta_G\big(\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk}\, g_{j\ell}^{-1}\big)\right) \left(\prod_{(jk\ell m)\in\Lambda_3} \delta_H\big(h_{j\ell m}\,(g_{\ell m} \rhd h_{jk\ell})\, h_{k\ell m}^{-1}\, h_{jkm}^{-1}\big)\right) \end{aligned} \tag{8.16}$$

U prethodnoj definiciji integracija se vrši po elementima $g_{jk} \in G$ za svaku ivicu $(jk) \in \Lambda_1$ i elementima $h_{jk\ell} \in H$ za svaki trougao $(jk\ell) \in \Lambda_2$. Podintegralne δ-funkcije nameću sledeće uslove.

1. Za svaki trougao $(jk\ell) \in \Lambda_2$, obojen elementom grupe $h_{jk\ell}$, uslov $\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk} = g_{j\ell}$ zahteva da svaki trougao $h_{jk\ell}$ ima odgovarajući izvor i metu (videti Lemu 13);

2. Za svaki tetraedar $(jk\ell m) \in \Lambda_3$ uslov da je površinska holonomija tetraedra trivijantna svodi se na uslov da je $h_{j\ell m}\,(g_{\ell m} \rhd h_{jk\ell}) h_{k\ell m}^{-1} h_{jkm}^{-1}$ jednako neutralnom elementu grupe H (videti Lemu 14).

Teorema 22 *Neka je $\mathcal{M}_d$ zatvorena i orijentisana kombinatorna d-mnogostrukost za $d = \{3,4\}$ i $(H \stackrel{\partial}{\to} G\,, \rhd\,)$ uskršteni modul. Suma po stanjima (8.16) je invarijantna na Pahnerove poteze.*

8.3 Pahnerovi potezi

8.3.1 $d = 3$

U trodimenzionalnom slučaju, da bi se proverila invarijantnost sume po stanjima (8.16) dovoljno je pokazati da se ona ne menja pri četiri Pahnerova poteza, $1 \leftrightarrow 4$ i $2 \leftrightarrow 3$ i njihovim inverzima. Postavka dokaza invarijantnosti sume po stanjima (8.16) na Pahnerove poteze data je u ovom odeljku, dok su detalji računa prikazani u Dodatku E.1 [13].

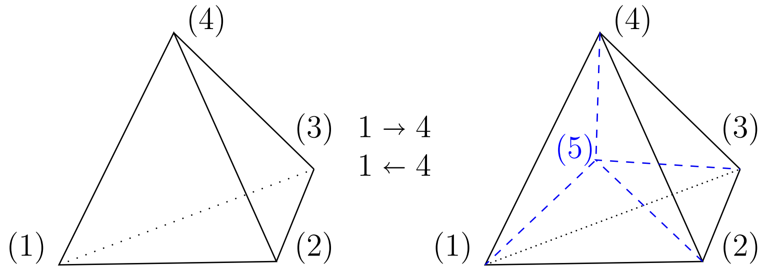
Pahnerov potez $1 \leftrightarrow 4$

Obeležimo vertekse sa leve strane $1 \leftrightarrow 4$ Pahnerovog poteza sa (1234). Dodavanjem verteksa (5) sa desne strane Pahnerovog poteza dobijamo četiri tetraedra:

$$M_3 = \{(2345), (1235), (1345), (1245)\}\,. \tag{8.17}$$

[1] Po Pahnerovoj teoremi, da bismo dokazali da je suma po stanjima topološka invarijantna, tj, da je invarijantna na promenu triangulacije, dovoljno je pokazati invarijantnost na Pahnerove poteze, koje je definisao Udo Pahner 1991. godine [44]. U trodimenzionalnom prostoru jedini Pahnerovi potezi su $2 \to 3$, $2 \leftarrow 3$, $1 \to 4$ i $1 \leftarrow 4$.

Teorema 21 (Pahnerova teorema) *Za datu deo-po-deo glatku mnogostrukost $\mathcal{M}_D$ svaka triangulacija te mnogostrukosti $T_1(\mathcal{M}_D)$ povezana je sa nekom triangulacijom homeomorfne (topološki izomorfne) mnogostrukosti $T_2(\mathcal{M}_D)$ konačnim brojem Pahnerovih poteza.*



Sa desne strane su prisutni dodatni trouglovi,

$$M_2 = \{(125), (135), (145), (235), (245), (345)\}\,, \tag{8.18}$$

odnosno dodatne ivice:

$$M_1 = \{(15), (25), (35), (45)\}\,. \tag{8.19}$$

Sa leve strane imamo sumu po stanjima,

$$\mathcal{Z}^{1\leftrightarrow 4}_{\text{levo}} = |G|^{-2}|H|^{1}\delta_H(h_{1234})\mathcal{Z}_{\text{ostatak}}, \tag{8.20}$$

dok sa desne strane imamo sumu po stanjima:

$$\begin{aligned}\mathcal{Z}^{1\leftrightarrow 4}_{\text{desno}} = \;& |G|^{-5}|H|^{1}\int_{G^4} \mathrm{d}g_{15}\mathrm{d}g_{25}\mathrm{d}g_{35}\mathrm{d}g_{45}\int_{H^6}\mathrm{d}h_{125}\mathrm{d}h_{135}\mathrm{d}h_{145}\mathrm{d}h_{235}\mathrm{d}h_{245}\mathrm{d}h_{345} \\ & \left(\prod_{(jk\ell)\in M_2}\delta_G(g_{jk\ell})\right)\left(\prod_{(jk\ell m)\in M_3}\delta_H(h_{jk\ell m})\right)\mathcal{Z}_{\text{ostatak}}.\end{aligned} \tag{8.21}$$

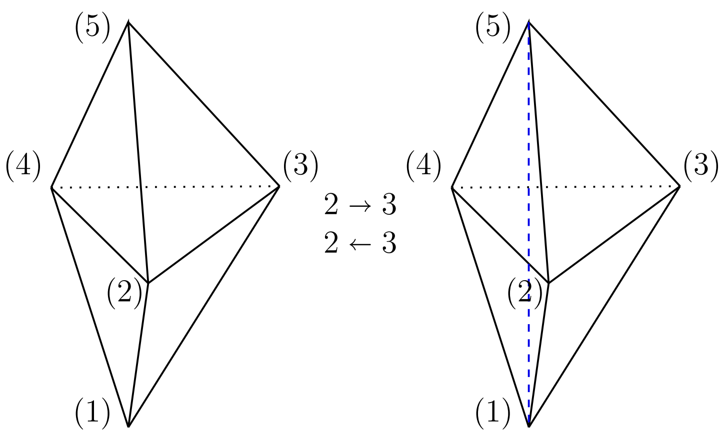
gde su M_2 i M_3 dati u izrazima (8.17) i (8.18). Broj k-simpleksa sa obe strane $1 \leftrightarrow 4$ poteza (pri čemu ne brojimo ostatak triangulacije) dat je u Tabeli 8.1.

	$\|\Lambda_0\|$	$\|\Lambda_1\|$	$\|\Lambda_2\|$	$\|\Lambda_3\|$
l.s.	4	6	4	1
d.s.	5	10	10	4

Tabela 8.1: Broj verteksa $|\Lambda_0|$, ivica $|\Lambda_1|$, trouglova $|\Lambda_2|$ i tetraedra $|\Lambda_3|$ sa leve i desne strane $1 \leftrightarrow 4$ Pahnerovog poteza.

Dokaz invarijantnosti svodi se na dokaz da su izrazi (8.20) i (8.21) jednaki, pri čemu činilac $\mathcal{Z}_{\text{ostatak}}$ označava deo sume koji ostaje nepromenjen po definiciji poteza.

Pahnerov potez 2 ↔ 3



Obeležimo tetraedre na levoj strani poteza $M_3^{\text{levo}} = \{(1234), (2345)\}$, koji dele trougao $M_2^{\text{levo}} = \{(234)\}$, dok sa desne strane imamo tetraedre $M_3^{\text{desno}} = \{(1235), (1245), (1345)\}$, koji dele ivicu $M_1^{\text{desno}} = \{(15)\}$, a svaka dva od njih dele jedan od trouglova $M_2^{\text{desno}} = \{(125), (135), (145)\}$.

Sa leve strane $2 \leftrightarrow 3$ poteza imamo sumu po stanjima

$$\mathcal{Z}_{\text{levo}}^{2\leftrightarrow 3} = |G|^{-3}|H|^{1} \int_H \mathrm{d}h_{234}\delta_G(g_{234})\,\delta_H(h_{1234})\delta_H(h_{2345})\mathcal{Z}_{\text{ostatak}}, \tag{8.22}$$

dok sa desne strane imamo sumu po stanjima:

$$\begin{aligned} \mathcal{Z}_{\text{desno}}^{2\leftrightarrow 3} = \; & |G|^{-4}|H|^{1} \int_G \mathrm{d}g_{15} \int_{H^3} \mathrm{d}h_{125}\mathrm{d}h_{135}\mathrm{d}h_{145} \\ & \delta_G(g_{125})\delta_G(g_{135})\delta_G(g_{145})\,\delta_H(h_{1235})\delta_H(h_{1245})\delta_H(h_{1345})\mathcal{Z}_{\text{ostatak}}\,. \end{aligned} \tag{8.23}$$

Broj k-simpleksa sa obe strane $2 \leftrightarrow 3$ poteza prikazana je u Tabeli 8.2.

	$\lvert\Lambda_0\rvert$	$\lvert\Lambda_1\rvert$	$\lvert\Lambda_2\rvert$	$\lvert\Lambda_3\rvert$
l.s.	5	9	7	2
d.s.	5	10	9	3

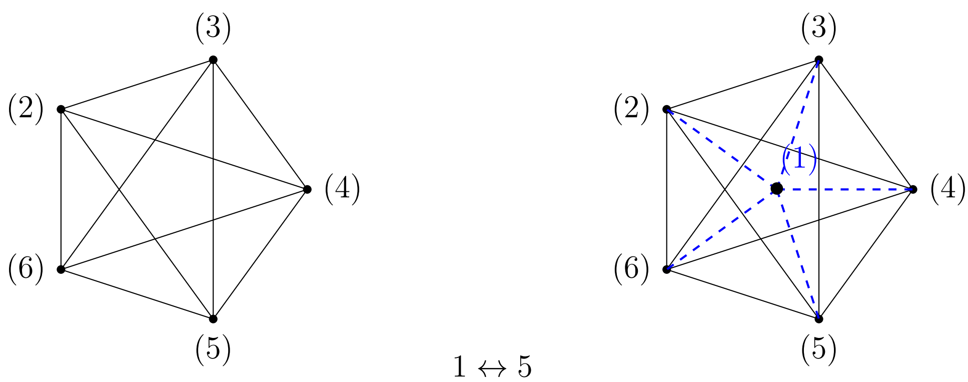
Tabela 8.2: Broj verteksa $|\Lambda_0|$, ivica $|\Lambda_1|$, trouglova $|\Lambda_2|$ i tetraedra $|\Lambda_3|$ sa leve i desne strane $2 \leftrightarrow 3$ Pahnerovog poteza.

Dokaz invarijantnosti svodi se na dokaz da su izrazi (8.22) i (8.23) jednaki, pri čemu činilac $\mathcal{Z}_{\text{ostatak}}$ označava deo sume koji ostaje nepromenjen po definiciji poteza.

8.3.2 $d = 4$

U četvorodimenzionalnom slučaju, da bi se proverila invarijantnost sume po stanjima (8.16) dovoljno je pokazati da se ona ne menja pri pet Pahnerovih poteza, $1 \leftrightarrow 5$, $2 \leftrightarrow 4$ i $3 \leftrightarrow 3$ Pahnerovim potezima i njihovim inverzima. Postavka dokaza invarijantnosti sume po stanjima (8.16) na Pahnerove poteze data je u ovom odeljku, dok su detalji računa prikazani u Dodatku E.1 [13].

Pahnerov potez $1 \leftrightarrow 5$



$1 \leftrightarrow 5$

Obeležimo vertekse 4-simpleksa na levoj strani $1 \leftrightarrow 5$ Pahnerovog poteza sa (23456). Dodavanjem verteksa (1) sa desne strane Pahnerovog poteza dobijamo pet 4-simpleksa

$$M_4 = \{(13456), (12456), (12356), (12346), (12345)\}\,.$$

Sa desne strane su prisutni dodatni tetraedri

$$M_3 = \{(1234),(1235),(1236),(1245),(1246),(1256),(1345),(1346),(1356),(1456)\}\,,$$

dodatni trouglovi

$$(jk\ell) \in M_2 = \{(123),(124),(125),(126),(134),(135),(136),(145),(146),(156)\}\,,$$

dodatne ivice $(jk) \in M_1 = \{(12),(13),(14),(15),(16)\}$ i dodatni verteks $(j) \in M_0 = \{(1)\}$. Svi ostali simpleksi su prisutni sa obe strane poteza.

Invarijantnost sume po stanjima (8.16) na Pahnerov potez $1 \leftrightarrow 5$ znači da je integral sa desne strane,

$$\begin{aligned} Z^{1\leftrightarrow 5}_{\text{desno}} = |G|^{-11}|H|^{-4} \int_{G^5} \prod_{(jk)\in M_1} \mathrm{d}g_{jk} \int_{H^{10}} \prod_{(jk\ell)\in M_2} \mathrm{d}h_{jk\ell} \\ \cdot \left(\prod_{(jk\ell)\in M_2} \delta_G(g_{jk\ell}) \right) \left(\prod_{(jk\ell m)\in M_3} \delta_H(h_{jk\ell m}) \right) Z_{\text{ostatak}}\,, \end{aligned} \tag{8.24}$$

jednak sumi po stanjima prisutnoj na levoj strani,

$$Z^{1\leftrightarrow 5}_{\text{levo}} = |G|^{-5}|H|^{0} Z_{\text{ostatak}}\,. \tag{8.25}$$

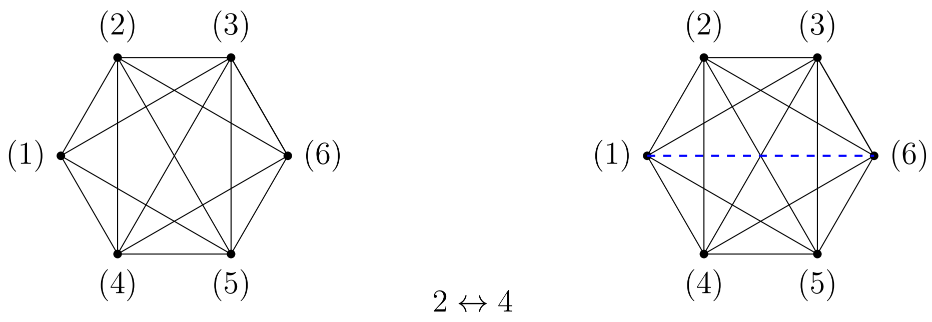
Faktore ispred integrala sume po stanjima, prisutne sa leve i desne strane poteza, izračunavamo na osnovu jednačine (8.16), odnosno koristimo $|G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}$ i $|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}$, gde su $|\Lambda_0|$, $|\Lambda_1|$, $|\Lambda_2|$, $|\Lambda_3|$ redom brojevi verteksa, ivica, trouglova i tetraedra u triangulaciji. Na osnovu podataka prikazanih u Tabeli 8.3 sa desne strane se dobija faktor $|G|^{-11}|H|^{-4}$, dok je faktor sa leve strane jednak $|G|^{-5}|H|^{0}$.

	$\lvert\Lambda_0\rvert$	$\lvert\Lambda_1\rvert$	$\lvert\Lambda_2\rvert$	$\lvert\Lambda_3\rvert$	$\lvert\Lambda_4\rvert$
l.s.	5	10	10	5	1
d.s.	6	15	20	15	5

Tabela 8.3: Broj verteksa $|\Lambda_0|$, ivica $|\Lambda_1|$, trouglova $|\Lambda_2|$, tetraedra $|\Lambda_3|$ i 4-simpleksa $|\Lambda_4|$ sa leve i desne strane $1 \leftrightarrow 5$ Pahnerovog poteza.

Dokaz invarijantnosti svodi se na dokaz da su izrazi (8.24) i (8.25) jednaki, pri čemu činilac Z_{ostatak} označava deo sume koji ostaje nepromenjen po definiciji poteza.

Pahnerov potez 2 $\leftrightarrow$ 4



Kako bi proverili invarijantnost sume po stanjima (8.16) pri $2 \leftrightarrow 4$ Pahnerovom potezu, poređajmo vertekse tako da na levoj strani poteza imamo dva 4-simpleksa

$$M_4^{\text{levo}} = \{(23456),(12345)\}$$

a na desnoj strani četiri 4-simpleksa

$$M_4^{\text{desno}} = \{(12346), (12356), (12456), (13456)\} .$$

Onda, na levoj strani imamo jedan tetraedar

$$M_3^{\text{levo}} = \{(2345)\} ,$$

dok na desnoj strani imamo šest tetraedra

$$M_3^{\text{desno}} = \{(1236), (1246), (1256), (1346), (1356), (1456)\} .$$

Svi ostali tetraedri su prisutni na obe strane poteza. Takođe, na desnoj strani su prisutni trouglovi $M_2^{\text{desno}} = \{(126), (136), (146), (156)\}$ i jedna ivica $M_1^{\text{desno}} = \{(16)\}$, dok su svi preostali trouglovi i ivice prisutni sa obe strane poteza. Takođe, svi verteksi su prisutni sa obe strane poteza.

Na levoj strani poteza imamo integral,

$$Z_{\text{levo}}^{2\leftrightarrow 4} = |G|^{-8}|H|^{-1}\delta_H(h_{2345}) Z_{\text{ostatak}} , \tag{8.26}$$

dok je sa desne strane integral

$$\begin{aligned} Z_{\text{desno}}^{2\leftrightarrow 4} = |G|^{-11}|H|^{-3} \int_G \mathrm{d}g_{16} \int_{H^4} \mathrm{d}h_{126}\mathrm{d}h_{136}\mathrm{d}h_{146}\mathrm{d}h_{156} \\ \left(\prod_{(jk\ell)\in M_2^{\text{desno}}} \delta_G(g_{jk\ell}) \right) \left(\prod_{(jk\ell m)\in M_3^{\text{desno}}} \delta_H(h_{jk\ell m}) \right) Z_{\text{ostatak}} . \end{aligned} \tag{8.27}$$

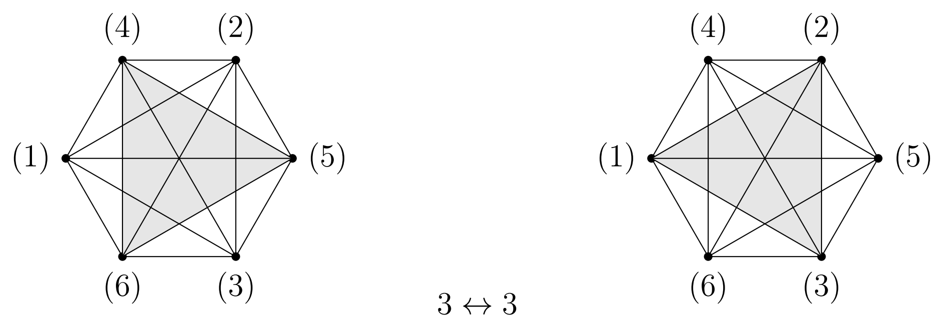
Prebrojavanjem k-simpleksa sa obe strane $2 \leftrightarrow 4$ poteza (vidi Tabelu 8.4) dobijamo koeficijente ispred integrala – $|G|^{-8}|H|^{-1}$ sa leve strane poteza i $|G|^{-11}|H|^{-3}$ sa desne strane poteza.

	$\lvert\Lambda_0\rvert$	$\lvert\Lambda_1\rvert$	$\lvert\Lambda_2\rvert$	$\lvert\Lambda_3\rvert$	$\lvert\Lambda_4\rvert$
l.s.	6	14	16	9	2
d.s.	6	15	20	14	4

Tabela 8.4: Broj verteksa $|\Lambda_0|$, ivica $|\Lambda_1|$, trouglova $|\Lambda_2|$, tetraedra $|\Lambda_3|$ i 4-simpleksa $|\Lambda_4|$ sa obe strane $2 \leftrightarrow 4$ poteza.

Dokaz invarijantnosti svodi se na dokaz da su izrazi (8.26) i (8.27) jednaki, pri čemu činilac Z_{ostatak} označava deo sume koji ostaje nepromenjen po definiciji poteza.

Pahnerov potez 3 ↔ 3



Obeležimo vertekse tako da sa leve strane $3 \leftrightarrow 3$ Pahnerovog poteza, imamo tri 4-simpleksa

$$M_4^{\text{levo}} = \{(23456), (13456), (12456)\} ,$$

a sa desne strane imamo 4-simplekse

$$M_4^{\text{desno}} = \{(12356), (12346), (12345)\} .$$

Sa leve strane su prisutni tetraedri $M_3^{\text{levo}} = \{(1456), (2456), (3456)\}$, dok su sa desne strane prisutni $M_3^{\text{desno}} = \{(1234), (1235), (1236)\}$. Dve strane poteza dele šest tetraedara, dok se sa svake strane nalazi tri tetraedra koje dele dva 4-simpleksa. Dalje, sa leve strane imamo trougao $M_2^{\text{levo}} = \{(456)\}$, a sa desne strane poteza trougao $M_2^{\text{desno}} = \{(123)\}$. Svi ostali trouglovi, ivice i verteksi se pojavljuju sa obe strane poteza.

Dakle, na levoj strani poteza imamo integral,

$$Z_{\text{levo}}^{3\leftrightarrow 3} = \int_H \mathrm{d}h_{456} \delta_G(g_{456}) \delta_H(h_{3456}) \delta_H(h_{2456}) \delta_H(h_{1456}) Z_{\text{ostatak}} , \tag{8.28}$$

dok sa desne strane imamo integral:

$$Z_{\text{desno}}^{3\leftrightarrow 3} = \int_H \mathrm{d}h_{123} \delta_G(g_{123}) \delta_H(h_{1234}) \delta_H(h_{1235}) \delta_H(h_{1236}) Z_{\text{ostatak}} . \tag{8.29}$$

Dokaz invarijantnosti svodi se na dokaz da su izrazi (8.28) i (8.29) jednaki, pri čemu činilac Z_{ostatak} označava deo sume koji ostaje nepromenjen po definiciji poteza.

Glava 9

Formiranje topološke sume po stanjima: $3BF$ teorija

U ovom poglavlju fokusiraćemo se na drugi korak kovarijantne kvantizacione procedure spinske pene za $3BF$ teoriju. Analogno postupku iz prethodnog poglavlja u slučaju sume po stanjima za $2BF$ teoriju, demonstriraćemo kako se konstruiše suma po stanjima Z koja je nezavisna od triangulacije, na osnovu klasičnog $3BF$ dejstva za opštu semistriktnu 3-grupu i bilo koju triangulaciju bilo koje glatke 4-dimenzionalne prostorvremenske mnogostrukosti, kso što je to urađeno u [27]. Moguće je formulisati $3BF$ teoriju samo u slučaju kada je dimenzija prostorvremenske mnogostrukosti $d \geqslant 4$, pa stoga ne razmatramo trodimenzinalni slučaj. Konstruisana suma po stanjima je generalizacija rada Žirelija, Pfajfera i Popeskua za $2BF$ sumu po stanjima predstavljenu u prethodnom poglavlju, tj. generalizacija Jeterovog modela, a poklapa sa Porterovom TKTP za $d = 4$ i $n = 3$.

Slično kao i u slučaju $2BF$ sume po stanjima, da bismo proverili da je konstruisana suma po stanjima topološka, analiziramo njeno ponašanje pri Pahnerovim potezima. Analiziramo samo četvorodimenzionalni slučaj, tj. invarijantnost pri Pahnerovim potezima $3 \leftrightarrow 3$, $4 \leftrightarrow 2$ i $5 \leftrightarrow 1$ i njihovim inverzima. Postavka analize ponašanja konstruisane sume po stanjima pri ovim Pahnerovim potezima predstavljena je u odeljku 9.3, dok su detalji proračuna dati u Dodatku E.2. Dobijeno je da suma po stanjima invarijantna na Pahnerove poteze, što dokazuje da je *topološka invarijanta* mnogostrukosti [27]. Zaključujemo da je suma po stanjima nepromenjena pri proizvoljnom usitnjavanju triangulacije i stoga definiše teoriju kontinuuma na glatkoj mnogostrukosti.

Međutim, da bi završili drugi korak kovarijantne kvantizacione procedure spinske pene, neophodne su generalizacije Peter-Vejl i Planšarel teorema za slučajeve 2-grupe i 3-grupe, matematički rezultati koji za sada predstavljaju otvorene probleme. Naime, ove teoreme treba da obezbede dekompoziciju funkcija na 3-grupi u sumu po odgovarajućim ireducibilnim reprezentacijama 3-grupe. Na ovaj način se određuje spektar oznaka simpleksa triangulacije, tj. domen vrednosti polja koja žive na simpleksima triangulacije, kao što je to urađeno u slučaju BF sume po stanjima. Trenutni pokušaji privođenja drugog koraka kvantizacije uopštenih BF teorija u okviru viših gejdž teorija, se svode na pogađanje ireducibilnih reprezentacija 2-grupa, kao što je urađeno na primer u slučaju spinkub modela kvantne gravitacije [15], ili drugim tehnikama, videti na primer [45]–[47].

Svakako, ovaj rezultat otvara put ka trećem i finalnom koraku kovarijantne kvantizacione procedure i formulaciji kvantne teorije gravitacije i materije iz Standardnog Modela nametanjem odgovarajućih ograničenja na varijable modela modifikacijom amplituda sume po stanjima.

9.1 Gejdž invarijantni objekti

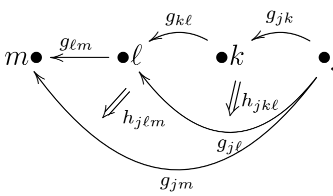
Uz prethodne uslove koji važe i u slučaju $2BF$ teorije, u $3BF$ teoriji jednačine kretanja nameću viši uslov ravnosti na 3-krivinu $(\mathcal{F}, \mathcal{G}, \mathcal{H})$, pa se dodatno zahteva da zapreminska holonomija oko granične 3-sfere bilo koje 4-kugle bude trivijalna.

U odeljku 2.3 uveli smo niz operacija pomoću kojih na jeziku 3-gejdž teorije definišu kompozicije puteva, površina i zapremina. Ta pravila se mogu koristiti za izračunavanje kompozicija elementarnih puteva, površina i zapremina, sve do proizvoljno velikih. U ovom odeljku ćemo koristiti ove kompozicije kako bismo konstruisali gejdž invarijantne veličine koje odgovaraju zatvorenim putevima, površinama i zapreminama. U Lemama 13, 15 i 16, ovaj se postupak koristi za graničnu putanju trougla, graničnu površinu tetraedra i graničnu zapreminu 4-simpleksa. Rezultat Leme 13 je izveden za slučaj 2-grupa i ostaje nepromenjen u 3-gejdž teoriji, vidi [13]. Zahtev ravnosti granične površine tetraedra izveden u Lemi 14, uopštavamo za slučaj 3-grupa u Lemi 15. Jedan od glavnih rezultata je Lema 16 u kojoj smo izveli uslov ravnosti granične zapremine 4-simpleksa.

Lema 15 *Posmatrajmo tetraedar $(jk\ell m)$. Ivice $(jk)\,, j<k$ su obeležene grupnim elementima $g_{jk}\in G$, trouglovi $(jk\ell)\,, j<k<\ell$ grupnim elementima $h_{jk\ell}\in H$, a tetraedri mnogostrukosti $(jk\ell m)\,, j<k<\ell<m$ grupnim elementima $l_{jk\ell m}\in L$. Orijentisali smo trouglove $(jk\ell)$ tako da je kriva $g_{k\ell}g_{jk}$ izvor 2-morfizma, a kriva $g_{j\ell}$ meta, tj. zadovoljen je uslov $g_{j\ell}=\partial(h_{jk\ell})g_{k\ell}g_{jk}$.*

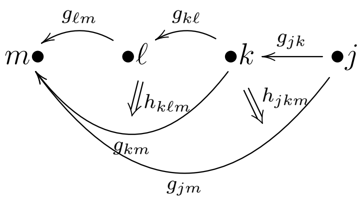
Prvo presečemo površinu tetraedra po granici (jm). To određuje redosled vertikalne kompozicije sastavnih površina. Moramo biti sigurni da su sve površine kompozibilne, tj. da imaju odgovarajuće referentne tačke i ispravnu orijentaciju za vertikalnu kompoziciju.

Analizirajmo prvo kompoziciju prikazanu na dijagramu (9.1). Prvo pomeramo krivu od $g_{k\ell}g_{jk}$ do krive $g_{j\ell}$. U ovom stadijumu dobijeni rezultat nije vertikalno kompozibilan sa trouglom $(j\ell m)$, prvo mu moramo dodati $g_{\ell m}$ sa leve strane. Sada su ova dva morfizma vertikalno kompozibilna, a dobijeni 2-morfizam prevlači krivu do g_{jm}. Dobijeni 2-morphizam je:



$$= (g_{\ell m}g_{j\ell}, h_{j\ell m})\#_2\big(g_{\ell m}\#_1(g_{k\ell}g_{jk}, h_{jk\ell})\big) = \big(g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell m}(g_{\ell m}\rhd h_{jk\ell})\big)\,. \tag{9.1}$$

Razmotrimo dijagram (9.2). Prvo prevlačimo krivu s $g_{\ell m}g_{k\ell}$ do krive g_{km}. Da bi vertikalna kompozicija dobijenog rezultata sa trouglom (jkm) bila moguća, najpre mu dodamo krivu g_{jk} sa desne strane. Sada su dva morfizma vertikalno kompozibilna i dobijeni 2-morfizam prevlači krivu na g_{jm}. Dobijen je sledeći 2-morfizam:



$$= (g_{km}g_{jk}, h_{jkm})\#_2\big((g_{\ell m}g_{k\ell}, h_{k\ell m})\#_1 g_{jk}\big) = (g_{\ell m}g_{k\ell}g_{jk}, h_{jkm}h_{k\ell m})\,. \tag{9.2}$$

Dve površine $\Sigma_1 : g_{\ell m}g_{k\ell}g_{jk}\to g_{jm}$ i $\Sigma_2 : g_{\ell m}g_{k\ell}g_{jk}\to g_{jm}$ imaju isti izvor i metu. Prevlačenje povšine prikazane na dijagramu (9.1) do površine prikazane na dijagramu (9.1) dato je zapreminskim morfizmom $\mathcal{V}:\Sigma_1\to\Sigma_2$ obeleženim sa grupnim elementom $l_{jk\ell m}$, tj.

$$(g_{\ell m}g_{k\ell}g_{jk}, h_{jkm}h_{k\ell m}) = \big(g_{\ell m}g_{k\ell}g_{jk}, \delta(l_{jk\ell m})h_{j\ell m}(g_{\ell m}\rhd h_{jk\ell})\big)\,, \tag{9.3}$$

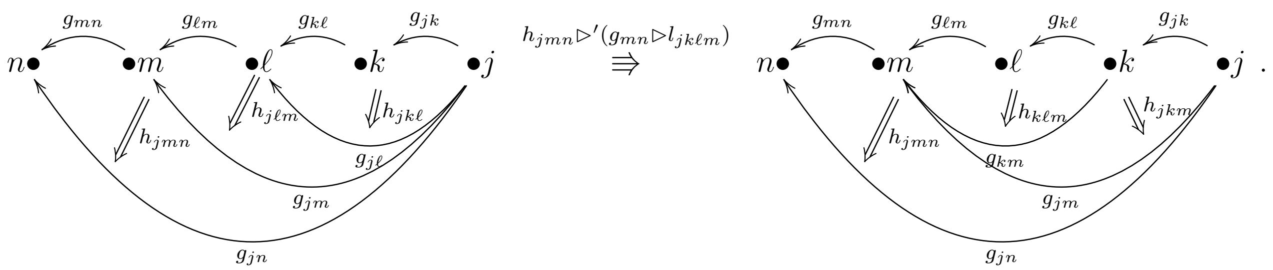
na osnovu čega dobijamo relaciju:

$$h_{jkm}h_{k\ell m} = \delta(l_{jk\ell m})h_{j\ell m}(g_{\ell m}\rhd h_{jk\ell})\,. \tag{9.4}$$

Lema 16 *Razmotrimo 4-simpleks,* $(jk\ell mn)$*. Ivice* $(jk)\,, j<k$*, su obeležene elementima grupe* $g_{jk}\in G$*, trouglovi* $(jk\ell)\,, j<k<\ell$ *su obeleženi elementima grupe* $h_{jk\ell}\in H$*, a tetraedri* $(jk\ell m)\,, j<k<\ell<m$ *elemetima grupe* $l_{jk\ell m}\in L$*. Trouglovi* $(jk\ell)$ *su orijentisani tako da je izvor* 2*-morfizma kriva* $g_{k\ell}g_{jk}$*, a njegova meta kriva* $g_{j\ell}$*, tj.* $g_{j\ell}=\partial(h_{jk\ell})g_{k\ell}g_{jk}$ *, dok su tetraedri* $(jk\ell m)$ *orijentisani tako da je izvor* 3*-morfizma površina* $h_{j\ell m}(g_{\ell m}\rhd h_{jk\ell})$ *a meta površina* $h_{jkm}h_{k\ell m}$*, tj.* $h_{jkm}h_{k\ell m}=\delta(l_{jk\ell m})h_{j\ell m}(g_{\ell m}\rhd h_{jk\ell})$*.*

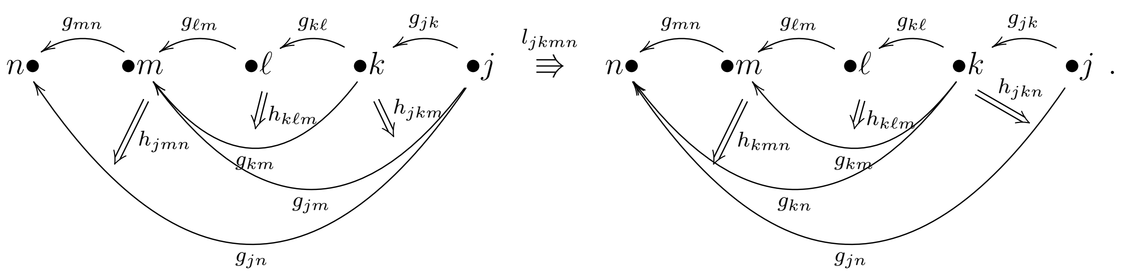
Najpre isecimo zapreminu 4*-simpleksa duž površine* $h_{jmn}g_{mn}\rhd(h_{j\ell m}g_{\ell m}\rhd h_{jk\ell})$*. To određuje redosled kompozicije* 3*-morfizama prema gore. Pritom se moramo pobrinuti da su sve zapremine kompozibilne, tj. da imaju odgovarajuće referentne površine i ispravnu orijentaciju kako bi njihove kompozicije bile definisane.*

Prvo, razmotrimo dijagram (9.5). Površinu $h_{j\ell m}g_{\ell m}\rhd h_{jk\ell}$ *prevlačimo do površine* $h_{jkm}h_{k\ell m}$ *uz pomoć* 3*-morfizma* $l_{jk\ell m}$*. Da bi kompozicija dobijenog* 3*-morfizma i* 2*-morfizma* h_{jmn} *bila definisana moramo najpre* 3*-morfizmu dodati krivu* g_{mn} *sa leve strane. Dobijeni* 3*-morfizam* $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{mn}\rhd(h_{j\ell m}g_{\ell m}\rhd h_{jk\ell}), g_{mn}\rhd l_{jk\ell m})$ *možemo vertikalno složiti sa* 2*-morfizmom* $(g_{mn}g_{jm}, h_{jmn})$ *sa donje strane, tako da je rezultujući* 3*-morfizam* $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn}\rhd(h_{j\ell m}g_{\ell m}\rhd h_{jk\ell}), h_{jmn}\rhd'(g_{mn}\rhd l_{jk\ell m}))$*, čiji je izvor površina* $h_{jmn}g_{mn}\rhd(h_{j\ell m}g_{\ell m}\rhd h_{jk\ell})$*,a meta površina* $h_{jmn}g_{mn}\rhd(h_{jkm}h_{k\ell m})$*,*



(9.5)

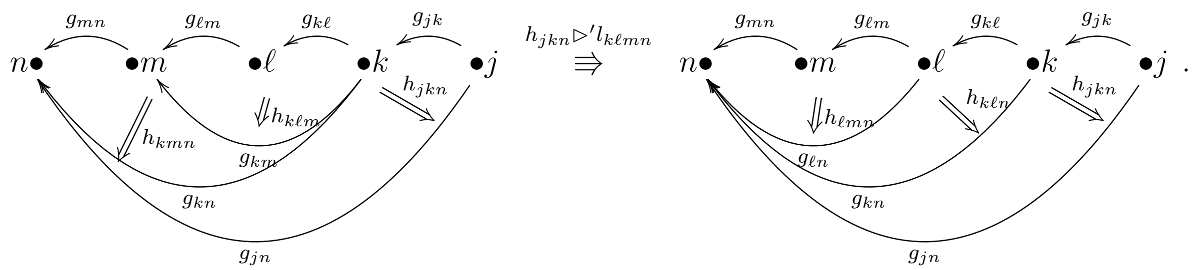
Sada ćemo prevući površinu do površine $h_{jkn}h_{kmn}g_{m\ell}\rhd h_{k\ell m}$*, kao što je prikazano na dijagramu (9.6). Najpre, razmotrimo* 3*-morfizam* $(g_{mn}g_{km}g_{jk}, h_{jmn}g_{mn}\rhd h_{jkm}, l_{jkmn})$ *čiji je izvor površina* $h_{jmn}g_{mn}\rhd h_{jkm}$*, a meta površina* $h_{jkn}h_{kmn}$*. Ovaj* 3*-morfizam možemo proširiti odozgo sa* 2*-morfizmom* $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{mn}\rhd h_{k\ell m})$*, što rezultuje* 3*-morfizmom* $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn}\rhd(h_{jkm}h_{k\ell m}), l_{jkmn})$*, čiji je izvor površina* $h_{jmn}g_{mn}\rhd(h_{jkm}h_{k\ell m})$*, a meta površina* $h_{jkn}h_{kmn}g_{mn}\rhd h_{k\ell m}$*,*



(9.6)

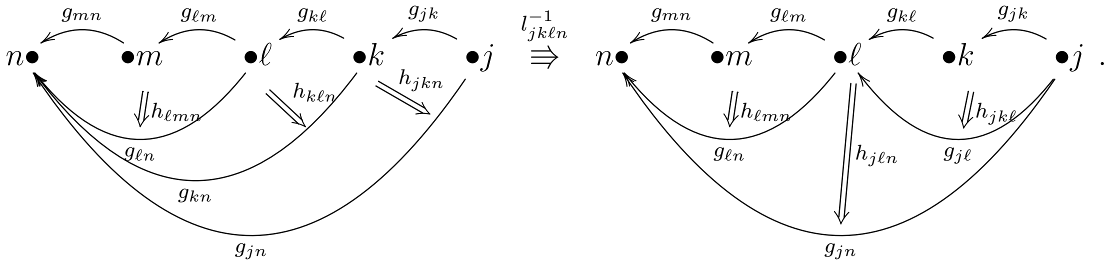
Sledeći korak je pomeranje površine od $h_{jkn}h_{kmn}g_{mn}\rhd h_{k\ell m}$ *do površine* $h_{jkn}h_{k\ell n}h_{\ell mn}$*, dijagram (9.7). Prvo, proširimo the* 3*-morfizam* $(g_{mn}g_{\ell m}g_{k\ell}, h_{kmn}g_{mn}\rhd h_{k\ell m}, l_{k\ell mn})$*, čiji je izvod površina* $h_{kmn}g_{mn}\rhd h_{k\ell m}$ *i meta površina* $h_{k\ell n}h_{\ell mn}$*, sa krivom* g_{jk} *sa desne strane, što rezultuje* 3*-morfizmom* $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{kmn}g_{mn}\rhd h_{k\ell m}, l_{k\ell mn})$*. Zatim, dobijeni* 3*-morfizam proširimo sa* 2*-morfizmom* $(g_{kn}g_{jk}, h_{jkn})$ *odozdo, posle čega se dobija* 3*-morfizam* $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{kmn}g_{mn}\rhd$

$h_{k\ell m}, h_{jkn} \rhd' l_{k\ell mn})$,



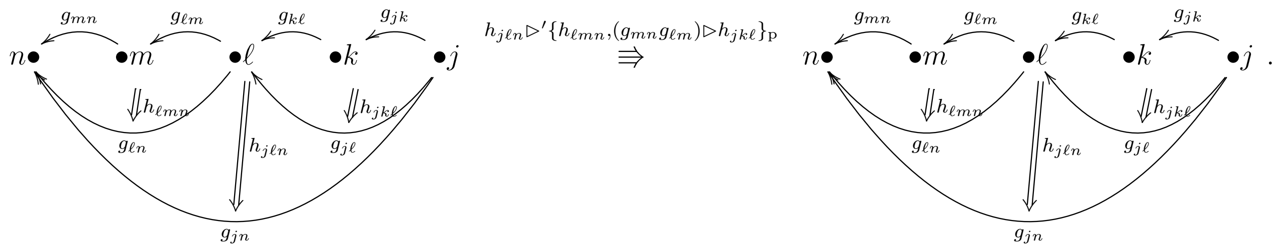
(9.7)

Pomeranje površine $h_{jkn}h_{k\ell n}h_{\ell mn}$ do površine $h_{j\ell n}g_{\ell n} \rhd h_{jk\ell}h_{\ell mn}$ prikazano je na dijagramu (9.8), a 3-morfizam sa odgovarajućim izvorom i metom dobijen je proširenjem 3-morfizma $(g_{\ell n}g_{k\ell}g_{jk}\,, h_{jkn}h_{k\ell n}, l^{-1}_{jk\ell n})$ sa 2-morfizmom $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{\ell mn})$ sa gornje strane. Tako dobijeni 3-morfizam je $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}h_{\ell mn}, l^{-1}_{jk\ell n})$,



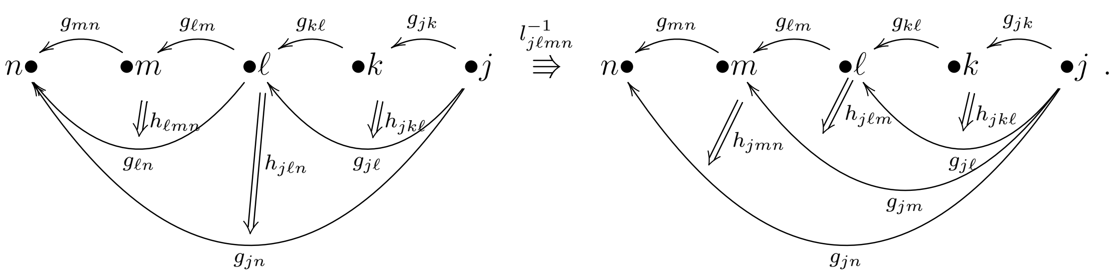
(9.8)

Zatim, želimo da mapiramo površinu $h_{j\ell n}g_{\ell n} \rhd h_{jk\ell}h_{\ell mn}$ u površinu $h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \rhd h_{jk\ell}$, videti dijagram (9.9). To je postignutno inverznom razmenjujućom 2-morfizam kompozicijom, koja preslikava $g_{\ell n} \rhd h_{jk\ell}h_{\ell mn}$ u površinu $h_{\ell mn}(g_{mn}g_{\ell m}) \rhd h_{jk\ell}$, tj. 3-morfizmom $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}$, $g_{\ell n} \rhd h_{jk\ell}h_{\ell mn}, \{h_{\ell mn}, (g_{mn}g_{\ell m}) \rhd h_{jk\ell}\}_{\rm pf})$. Zatim, dobijeni 3-morfizam proširujemo sa 2-morfizmom $(g_{\ell n}g_{j\ell}, h_{j\ell n})$ odozdo. Dobijeni 3-morfizam sa odgovarajućim izvorom i metom je $(g_{mn}g_{\ell m}g_{k\ell}g_{jk},\ h_{j\ell n}g_{\ell n} \rhd h_{jk\ell}h_{\ell mn}, h_{j\ell n} \rhd' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \rhd h_{jk\ell}\}_{\rm pf})$,



(9.9)

Najzad, postupak završavamo konstrukcijom 3-morfizma koji preslikava površinu $h_{j\ell n}h_{\ell mn}$ $(g_{mn}g_{\ell m}) \rhd h_{jk\ell}$ u početnu površinu $h_{jmn}g_{mn} \rhd (h_{j\ell m}g_{\ell m} \rhd h_{jk\ell})$. Prvo pomeramo površinu $h_{j\ell n}h_{\ell mn}$ u površinu $h_{jmn}g_{mn} \rhd h_{j\ell m}$ sa 3-morfizmom $(g_{mn}g_{\ell m}g_{j\ell}, h_{j\ell n}h_{\ell mn}, l^{-1}_{j\ell mn})$. Zatim, proširimo 3-morfizam $(g_{mn}g_{\ell m}g_{j\ell}, h_{j\ell n}h_{\ell mn}, l^{-1}_{j\ell mn})$ sa 2-morfizmom $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, (g_{mn}g_{\ell m}) \rhd h_{jk\ell})$ odozgo. Tako dobijeni 3-morfizam $(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \rhd h_{jk\ell}, l^{-1}_{j\ell mn})$ prevlači površinu u površinu od koje smo krenuli, kao što je prikazano na dijagramu (9.10),



(9.10)

Sada formiramo kompoziciju 3-morfizama predstavljenih dijagramima (9.5)-(9.10) prema gore, vodeći računa o redosledu. Dobijeni 3-morfizam je:

$$
\begin{aligned}
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{j\ell n}h_{\ell mn}(g_{mn}g_{\ell m}) \rhd h_{jk\ell}, l^{-1}_{j\ell mn})\#_3 \\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, g_{\ell n} \rhd h_{jk\ell}h_{\ell mn}, h_{j\ell n} \rhd' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \rhd h_{jk\ell}\}_{\mathrm{p}})\#_3 \\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{k\ell n}h_{\ell mn}, l^{-1}_{jk\ell n})\#_3 \\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jkn}h_{kmn}g_{m\ell} \rhd h_{k\ell m}, h_{jkn} \rhd' l_{jkmn})\#_3 \\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \rhd (h_{jkm}h_{k\ell m}), l_{jkmn})\#_3 \\
&(g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \rhd (h_{j\ell m}g_{\ell m} \rhd h_{jk\ell}), h_{jmn} \rhd' (g_{mn} \rhd l_{jk\ell m})) \\
&= (g_{mn}g_{\ell m}g_{k\ell}g_{jk}, h_{jmn}g_{mn} \rhd (h_{j\ell m}g_{\ell m} \rhd h_{jk\ell}), l^{-1}_{j\ell mn}\, h_{j\ell n} \rhd' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \rhd h_{jk\ell}\}_{\mathrm{pf}} \\
&\qquad l^{-1}_{jk\ell n}(h_{jkn} \rhd' l_{k\ell mn})l_{jkmn}h_{jmn} \rhd' (g_{mn} \rhd l_{jk\ell m}))\,.
\end{aligned}
\tag{9.11}
$$

Dobijeni 3-morfizam je jedinični 3-morfizam, odnosno njegov izvor i meta su površina $\mathcal{V}_1 = \mathcal{V}_2 = h_{jmn}g_{mn} \rhd (h_{j\ell m}g_{\ell m} \rhd h_{jk\ell})$, tj. važi identiet

$$
l^{-1}_{j\ell mn}\, h_{j\ell n} \rhd' \{h_{\ell mn}, (g_{mn}g_{\ell m}) \rhd h_{jk\ell}\}_{\mathrm{pf}}\, l^{-1}_{jk\ell n}(h_{jkn} \rhd' l_{k\ell mn})l_{jkmn}h_{jmn} \rhd' (g_{mn} \rhd l_{jk\ell m}) = e\,. \tag{9.12}
$$

9.2 Kvantizacija topološkog $3BF$ dejstva

U ovom odeljku predstavićemo kombinatorni opis konstrukcije $3BF$ sume po stanjima za bilo koju triangulaciju mnogostrukosti dimenzije $d = 4$. Model je definisan za bilo koju zatvorenu i orijentisanu kombinatornu mnogostrukost $\mathcal{M}_4$ dimenzije $d = 4$. Ovaj model se podudara sa Porterovim TKTP [26] za $d = 4$ i $n = 3$.

Najpre ćemo demonstrirati kako se formira suma po stanjima koja odgovara klasičnom $3BF$ dejstvu (6.1) uobičajenim postupkom diskretizacije. Stoga razmatramo sumu po stanjima:

$$
Z = \int \mathcal{D}\alpha\, \mathcal{D}\beta\, \mathcal{D}\gamma\, \mathcal{D}B\, \mathcal{D}C\, \mathcal{D}D \exp\left(\int_{M_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \right)\,. \tag{9.13}
$$

Formalna integracija po Lagranževim množiteljima B, C i D daje rezultat:

$$
Z = \int \mathcal{D}\alpha\, \mathcal{D}\beta\, \mathcal{D}\gamma\ \delta(\mathcal{F})\delta(\mathcal{G})\delta(\mathcal{H})\,. \tag{9.14}
$$

Slično kao i u običnoj gejdž teoriji, 1-forma koneksije $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$ je diskretizovana bojenjem ivica $\epsilon = (jk) \in \Lambda_1$ triangulacije grupnim elementima $g_\epsilon \in G$. Analogno, 2-forma koneksije $\beta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{h})$ je reprezentovana bojenjem trouglova triangulacije $\Delta = (jk\ell) \in \Lambda_2$ elementima $h_\Delta \in H$, a 3-forma koneksije $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,, \mathfrak{l})$ je reprezentovana bojenjem tetraedara $\tau = (jk\ell m) \in \Lambda_3$ elementima grupe $l_\tau \in L$.

Meru sume po stanjima (9.13) diskretizujemo smenama

$$
\int \mathcal{D}\alpha \quad \mapsto \quad \prod_{(jk)\in\Lambda_1} \int_G \mathrm{d}g_{jk}\,, \tag{9.15}
$$

$$
\int \mathcal{D}\beta \quad \mapsto \quad \prod_{(jk\ell)\in\Lambda_2} \int_H \mathrm{d}h_{jk\ell}, \tag{9.16}
$$

$$
\int \mathcal{D}\gamma \quad \mapsto \quad \prod_{(jk\ell m)\in\Lambda_3} \int_L \mathrm{d}l_{jk\ell m}\,, \tag{9.17}
$$

gde dg_{jk}, $dh_{jk\ell}$ i $dl_{jk\ell m}$ redom označavaju integracije na grupama G, H i L.

Uslov da lažna krivina mora nestati, dat jednačinom (8.2) u Lemi 13, diskretizovan je na svakom trouglu $(jkl) \in \Lambda_2$ zamenom $\delta(F)$ sa

$$\delta_G(g_{jk\ell}) = \delta_G\big(\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk}\, g_{j\ell}^{-1}\big)\,, \tag{9.18}$$

uslov o trivijalnosti 3-forme krivine $\delta(G)$ za svaki tetraedar $(jk\ell m) \in \Lambda_3$ na diskretizovanoj mnogostrukosti, dat jednačinom (9.4) u Lemi 15, pretvara se u uslov

$$\delta_H(h_{jk\ell m}) = \delta_H\big(\delta(l_{jk\ell m}) h_{j\ell m}\,(g_{\ell m} \rhd h_{jk\ell})\, h_{k\ell m}^{-1}\, h_{jkm}^{-1}\big)\,, \tag{9.19}$$

dok uslov o trivialnosti 4-forme krivine $\delta(H)$ za svaki 4-simpleks $(jk\ell mn) \in \Lambda_4$ postaje identitet (9.12) u Lemi 16:

$$\delta_L(l_{jk\ell mn}){=}\delta_L\big(l_{jk\ell n}^{-1}(h_{jkn}{\rhd'}l_{k\ell mn})l_{jkmn}(h_{jmn}{\rhd'}(g_{mn}{\rhd}l_{jk\ell m}))h_{j\ell mn}^{-1}h_{j\ell n}{\rhd'}\{h_{\ell mn},(g_{mn}g_{\ell m}){\rhd}h_{jk\ell}\}_{\mathrm{p}}\big). \tag{9.20}$$

Na osnovu prethodnog, integral po putanjama se može napisati kao suma po stanjima u sledećem obliku:

$$\begin{aligned} Z = & \prod_{(jk)\in\Lambda_1} \int_G \mathrm{d}g_{jk} \prod_{(jk\ell)\in\Lambda_2} \int_H \mathrm{d}h_{jk\ell} \prod_{(jk\ell m)\in\Lambda_3} \int_L \mathrm{d}l_{jk\ell m} \\ & \Bigg(\prod_{(jk\ell)\in\Lambda_2} \delta_G\big(g_{jk\ell}\big)\Bigg)\Bigg(\prod_{(jk\ell m)\in\Lambda_3} \delta_H\big(h_{jk\ell m}\big)\Bigg)\Bigg(\prod_{(jk\ell mn)\in\Lambda_4} \delta_L\big(l_{jk\ell mn}\big)\Bigg). \end{aligned} \tag{9.21}$$

Zamenom jednačina (9.18), (9.19) i (9.20) u sumu po stanjima (9.21), dobijamo izraz proporcionalan sumi (9.22). Da bi suma data izrazom (9.21) bila nezavisna od triangulacije mnogostrukosti moramo je pomnožiti sa odgovarajućim faktorom koji zavisi od broja verteksa, ivica, trouglova, tetraedara i 4-simpleksa trangulacije, što rezultuje u sumi po stanjima u jednačini (9.22).

Definicija 9.2.1 *Neka je $\mathcal{M}_d$ kompaktna orijentisana kombinatorna d-mnogostrukost, $d = 4$, i neka je $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \rhd\,, \{\_\,,\_\}_{\mathrm{pf}})$ jedan 2-ukršteni modul. Suma po stanjima topološke 3-gejdž teorije je definisana sledećim izrazom:*

$$\begin{aligned} Z = \; & |G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}\,|L|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|+|\Lambda_3|-|\Lambda_4|}\Bigg(\prod_{(jk)\in\Lambda_1}\int_G \mathrm{d}g_{jk}\Bigg)\Bigg(\prod_{(jk\ell)\in\Lambda_2}\int_H \mathrm{d}h_{jk\ell}\Bigg) \\ & \Bigg(\prod_{(jk\ell m)\in\Lambda_3}\int_L \mathrm{d}l_{jk\ell m}\Bigg)\Bigg(\prod_{(jk\ell)\in\Lambda_2}\delta_G\big(\partial(h_{jk\ell})\,g_{k\ell}\,g_{jk}\,g_{j\ell}^{-1}\big)\Bigg)\Bigg(\prod_{(jk\ell m)\in\Lambda_3}\delta_H\big(\delta(l_{jk\ell m})h_{j\ell m}\,(g_{\ell m}\rhd h_{jk\ell})\,h_{k\ell m}^{-1}\,h_{jkm}^{-1}\big)\Bigg) \\ & \Bigg(\prod_{(jk\ell mn)\in\Lambda_4}\delta_L\big(l_{j\ell mn}^{-1}\,h_{j\ell n}\rhd'\{h_{\ell mn},(g_{mn}g_{\ell m})\rhd h_{jk\ell}\}_{\mathrm{p}}\,l_{jk\ell n}^{-1}(h_{jkn}\rhd' l_{k\ell mn})l_{jkmn}h_{jmn}\rhd'(g_{mn}\rhd l_{jk\ell m})\big)\Bigg). \end{aligned} \tag{9.22}$$

U prethodnom izrazu integralimo po elementima $g_{jk} \in G$ za svaku ivicu $(jk) \in \Lambda_1$, po elementima $h_{jk\ell} \in H$ za svaki trougao $(jk\ell) \in \Lambda_2$ i po elementima l_{jklm} za svaki tetraedar mnogostrukosti $(jk\ell m) \in \Lambda_3$, dok δ-distribucije u podintegralnom izrazu nameću sledeće uslove na ove vrednosti. Uslovi koje elementi trangulacije moraju da zadovoljavaju izvedeni su u Lemama 13, 15 i 16.

1. Uslov da svaki trougao $(jk\ell) \in \Lambda_2$ koji nosi oznaku $h_{jk\ell}$ ima odgovarajući izvor i metu je $\partial(h_{jk\ell})\, g_{k\ell}\, g_{jk} = g_{j\ell}$, kao što je prikazano u Lemi 13.

2. Zatim, uslov $h_{jkm}\, h_{k\ell m} = \delta(l_{jk\ell m}) h_{j\ell m}\,(g_{\ell m} \rhd h_{jk\ell})$ važi za svaki tetraedar $(jk\ell m) \in \Lambda_3$, tj. svaki tetraedar koji nosi oznaku $l_{jk\ell m}$ ima dobro definisan izvor i metu, vidi Lemu 15.

3. Najzad, zapreminska holonomija oko svakog 4-simpleksa $(jk\ell mn) \in \Lambda_4$ je trivijalna, tj. $l^{-1}_{j\ell mn}\, h_{j\ell n} \rhd' \{h_{\ell mn}, (g_{mn} g_{\ell m}) \rhd h_{jk\ell}\}_{\mathrm{p}}\, l^{-1}_{jk\ell n} (h_{jkn} \rhd' l_{k\ell mn}) l_{jkmn} h_{jmn} \rhd' (g_{mn} \rhd l_{jk\ell m})$ je neutralni element grupe L za svaki 4-simpleks $(jk\ell mn) \in \Lambda_4$, kao što je dokazano u Lemi 16.

Teorema 23 *Neka je $\mathcal{M}_4$ zatvorena i orijentisana kombinatorna d-mnogostrukost, $d = 4$ i $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \rhd\,, \{\_\,,\_\}_{\mathrm{pf}})$ jedan 2-ukršteni modul. Suma po stanjima (9.22) je invarijantna na Pahnerove poteze.*

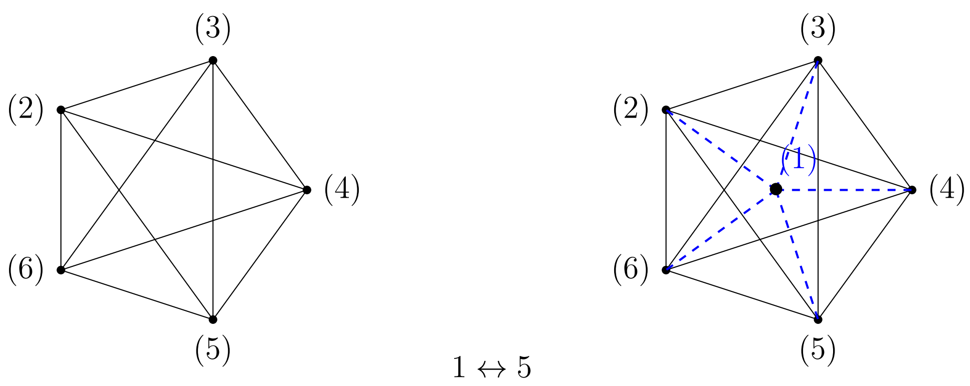
U Dodatku E skiciraćemo dokaz invarijantnosti sume po stanjima na Pahnerove poteze [44]. U sumi po stanjima (9.22), označavanje ivica elementima $g_{jk} \in G$, trouglova s elementima $h_{jk\ell} \in H$ i tetraedra s elementima $l_{jk\ell m} \in L$ naziva se *bojenje mnogostrukosti.*

9.3 Pahnerovi potezi

9.3.1 $d = 4$

U četvorodimenzionalnom slučaju, da bi se proverila invarijantnost sume po stanjima (9.22) pri lokalnim promenama triangulacije četvorodimenzione mnogostrukosti dovoljno je pokazati da se ona ne menja pri pet Pahnerovih poteza, $1 \leftrightarrow 5$, $2 \leftrightarrow 4$ i $3 \leftrightarrow 3$ Pahnerovim potezima i njihovim inverzima. Postavka dokaza invarijantnosti sume po stanjima (9.22) na Pahnerove poteze data je u ovom odeljku, dok su detalji računa prikazani u Dodatku E.2.

Pahnerov potez $1 \leftrightarrow 5$



Budući da suma po stanjima (9.22) ne zavisi od načina na koji su verteksi triangulacije obeleženi, kao ni od njihovog redoleda, invarijantnost je dovoljno utvrditi samo u jednom slučaju. Obeležimo vertekse 4-simpleksa na levoj strani $1 \leftrightarrow 5$ Pahnerovog pokreta sa (23456). Dodavanjem verteksa (1) sa desne strane Pahnerovog poteza dobijamo pet novih 4-simpleksa:

$$M_4 := \{(13456), (12456), (12356), (12346), (12345)\}\,.$$

Sa desne strane su prisutni dodatni tetraedri

$$M_3 := \{(1234), (1235), (1236), (1245), (1246), (1256), (1345), (1346), (1356), (1456)\}\,,$$

dodatni trouglovi

$$(jk\ell) \in M_2 := \{(123), (124), (125), (126), (134), (135), (136), (145), (146), (156)\}\,,$$

dodatne ivice $(jk) \in M_1 := \{(12),(13),(14),(15),(16)\}$ i dodatni verteksi $(j) \in M_0 := \{(1)\}$. Svi ostali simpleksi su prisutni sa obe strane poteza.

Invarijantnost sume po stanjima (9.22) na Pahnerov potez $1 \leftrightarrow 5$ znači da je integral sa desne strane,

$$
\begin{aligned}
Z^{1\leftrightarrow 5}_{\text{desno}} = |G|^{-11}|H|^{-4}|L|^{-1} \int_{G^5} \prod_{(jk)\in M_1} \mathrm{d}g_{jk} \int_{H^{10}} \prod_{(jk\ell)\in M_2} \mathrm{d}h_{jk\ell} \int_{L^{10}} \prod_{(jklm)\in M_3} \mathrm{d}l_{jklm} \\
\cdot \left(\prod_{(jk\ell)\in M_2} \delta_G(g_{jk\ell}) \right) \left(\prod_{(jk\ell m)\in M_3} \delta_H(h_{jk\ell m}) \right) \left(\prod_{(jk\ell mn)\in M_4} \delta_L(l_{jk\ell mn}) \right) Z_{\text{ostatak}}\,,
\end{aligned}
\tag{9.23}
$$

jednak δ-funkciji prisutnoj na levoj strani,

$$
Z^{1\leftrightarrow 5}_{\text{levo}} = |G|^{-5}|H|^{0}|L|^{-1}\delta_L(l_{23456})Z_{\text{ostatak}}\,. \tag{9.24}
$$

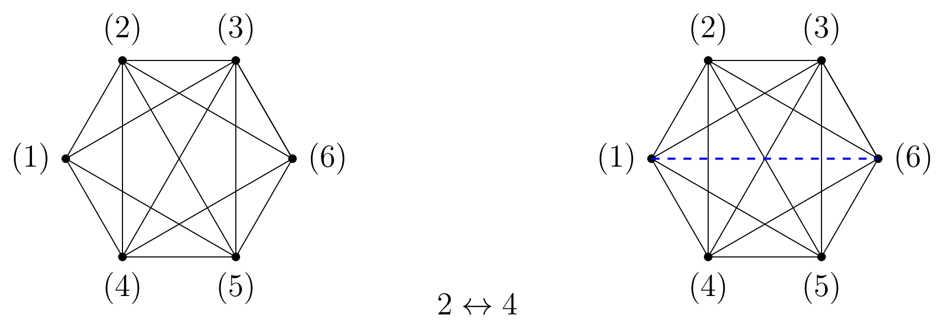
Faktore ispred integrala sume po stanjima, prisutne sa leve i desne strane pokreta, izračunavamo na osnovu jednačine (9.22), odnosno koristimo $|G|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|}$, $|H|^{|\Lambda_0|-|\Lambda_1|+|\Lambda_2|-|\Lambda_3|}$ i $|L|^{-|\Lambda_0|+|\Lambda_1|-|\Lambda_2|+|\Lambda_3|-|\Lambda_4|}$, gde su $|\Lambda_0|$, $|\Lambda_1|$, $|\Lambda_2|$, $|\Lambda_3|$ i $|\Lambda_4|$ redom brojevi verteksa, ivica, trouglova, tetraedra i 4-simpleksa u triangulaciji. Na osnovu podataka prikazanih u tabeli 9.1 sa desne strane se dobija faktor $|G|^{-11}|H|^{-4}|L|^{-1}$, dok je faktor sa leve strane jednak $|G|^{-5}|H|^{0}|L|^{-1}$.

	$\lvert\Lambda_0\rvert$	$\lvert\Lambda_1\rvert$	$\lvert\Lambda_2\rvert$	$\lvert\Lambda_3\rvert$	$\lvert\Lambda_4\rvert$
l.s.	5	10	10	5	1
d.s.	6	15	20	15	5

Tabela 9.1: Broj verteksa $|\Lambda_0|$, ivica $|\Lambda_1|$, trouglova $|\Lambda_2|$, tetraedra $|\Lambda_3|$ i 4-simpleksa $|\Lambda_4|$ sa leve i desne strane $1 \leftrightarrow 5$ Pahnerovog poteza.

Dokaz invarijantnosti sume po stanjima (9.22) pri $1 \leftrightarrow 5$ Pahnerovom potezu svodi se na dokaz da su izrazi (9.23) i (9.24) jednaki, pri čemu činilac Z_{ostatak} označava deo sume koji ostaje nepromenjen po definiciji poteza. Dokaz da je $Z^{1\leftrightarrow 5}_{\text{desno}} = Z^{1\leftrightarrow 5}_{\text{levo}}$ dat je u Dodatku E.

Pahnerov potez $2 \leftrightarrow 4$



Kako bi proverili invarijantnost sume po stanjima (9.22) pri $2 \leftrightarrow 4$ Pahnerovom potezu, poređajmo vertekse tako da na levoj strani poteza imamo dva 4-simpleksa

$$
M_4^{\text{levo}} = \{(23456),(12345)\}
$$

a na desnoj strani četiri 4-simpleksa

$$
M_4^{\text{desno}} = \{(12346),(12356),(12456),(13456)\}\,.
$$

Onda, na levoj strani imamo jedan tetraedar

$$M_3^{\text{levo}} = \{(2345)\}\,,$$

dok na desnoj strani imamo šest tetraedra

$$M_3^{\text{desno}} = \{(1236),(1246),(1256),(1346),(1356),(1456)\}\,.$$

Svi ostali tetraedri su prisutni na obe strane poteza. Takođe, na desnoj strani su prisutni trouglovi $M_2^{\text{desno}} = \{(126),(136),(146),(156)\}$ i jedna ivica $M_1^{\text{desno}} = \{(16)\}$, dok su svi preostali trouglovi i ivice prisutni sa obe strane poteza. Takođe, svi verteksi su prisutni sa obe strane poteza.

Dakle, koristeći izraz za definiciju sume po stanjima (9.22), na levoj strani poteza imamo integral,

$$Z_{\text{levo}}^{2\leftrightarrow 4} = |G|^{-8}|H|^{-1}|L|^{-1}\int_L \mathrm{d}l_{2345}\delta_H(h_{2345})\left(\prod_{(jk\ell mn)\in M_4^{\text{levo}}}\delta_L(l_{jk\ell mn})\right)Z_{\text{ostatak}}\,, \tag{9.25}$$

dok je sa desne strane integral:

$$Z_{\text{desno}}^{2\leftrightarrow 4} = |G|^{-11}|H|^{-3}|L|^{-1}\int_G \mathrm{d}g_{16}\int_{H^4}\mathrm{d}h_{126}\mathrm{d}h_{136}\mathrm{d}h_{146}\mathrm{d}h_{156}\int_L \mathrm{d}l_{1236}\mathrm{d}l_{1246}\mathrm{d}l_{1256}\mathrm{d}l_{1346}\mathrm{d}l_{1356}\mathrm{d}l_{1456}$$
$$\left(\prod_{(jk\ell)\in M_2^{\text{desno}}}\delta_G(g_{jk\ell})\right)\left(\prod_{(jk\ell m)\in M_3^{\text{desno}}}\delta_H(h_{jk\ell m})\right)\left(\prod_{(jk\ell mn)\in M_4^{\text{desno}}}\delta_L(l_{jk\ell mn})\right)Z_{\text{ostatak}}\,. \tag{9.26}$$

Prebrojavanjem k-simpleksa sa obe strane $2\leftrightarrow 4$ poteza (vidi Tabelu 9.2) dobijamo koeficijente ispred integrala, $|G|^{-8}|H|^{-1}|L|^{-1}$ sa leve strane poteza i $|G|^{-11}|H|^{-3}|L|^{-1}$ sa desne strane poteza.

	$\lvert\Lambda_0\rvert$	$\lvert\Lambda_1\rvert$	$\lvert\Lambda_2\rvert$	$\lvert\Lambda_3\rvert$	$\lvert\Lambda_4\rvert$
l.s.	6	14	16	9	2
d.s.	6	15	20	14	4

Tabela 9.2: Broj verteksa $|\Lambda_0|$, ivica $|\Lambda_1|$, trouglova $|\Lambda_2|$, tetraedra $|\Lambda_3|$ i 4-simpleksa $|\Lambda_4|$ sa obe strane $2\leftrightarrow 4$ poteza.

Dokaz invarijantnosti sume po stanjima (9.22) pri $2\leftrightarrow 4$ Pahnerovom potezu svodi se na dokaz da su izrazi (9.25) i (9.26) jednaki, pri čemu činilac Z_{ostatak} označava deo sume koji ostaje nepromenjen po definiciji poteza. Dokaz da je $Z_{\text{desno}}^{2\leftrightarrow 4} = Z_{\text{levo}}^{2\leftrightarrow 4}$ dat je u Dodatku E.

Pahnerov potez $3\leftrightarrow 3$

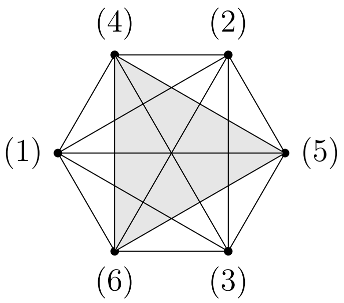
Obeležimo vertekse tako da sa leve strane $3\leftrightarrow 3$ Pahnerovog poteza, imamo tri 4-simpleksa

$$M_4^{\text{levo}} = \{(23456),(13456),(12456)\}\,,$$

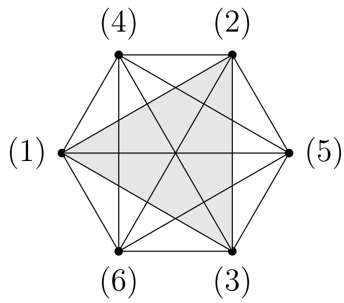
a sa desne strane imamo 4-simplekse

$$M_4^{\text{desno}} = \{(12356),(12346),(12345)\}\,.$$

Sa leve strane su prisutni tetraedri $M_3^{\text{levo}} = \{(1456),(2456),(3456)\}$, dok su sa desne strane prisutni $M_3^{\text{desno}} = \{(1234),(1235),(1236)\}$. Dve strane poteza dele šest tetraedara, dok se sa



$3 \leftrightarrow 3$



svake strane nalazi tri tetraedra koje dele dva 4-simpleksa. Dalje, sa leve strane imamo trougao $M_2^{\text{levo}} = \{(456)\}$, a sa desne strane poteza trougao $M_2^{\text{desno}} = \{(123)\}$. Svi ostali trouglovi, ivice i verteksi se pojavljuju sa obe strane poteza.

Dakle, na levoj strani poteza imamo integral,

$$
\begin{aligned}
Z_{\text{levo}}^{3\leftrightarrow 3} = & \int_H \mathrm{d}h_{456} \int_{L^3} \mathrm{d}l_{1456}\mathrm{d}l_{2456}\mathrm{d}l_{3456}\delta_G(g_{456}) \\
& \delta_H(h_{3456})\delta_H(h_{2456})\delta_H(h_{1456})\delta_L(l_{23456})\delta_L(l_{13456})\delta_L(l_{12456})Z_{\text{ostatak}}\,,
\end{aligned}
\tag{9.27}
$$

dok sa desne strane imamo integral:

$$
\begin{aligned}
Z_{\text{desno}}^{3\leftrightarrow 3} = & \int_H \mathrm{d}h_{123} \int_{L^3} \mathrm{d}l_{1234}\mathrm{d}l_{1235}\mathrm{d}l_{1236}\delta_G(g_{123}) \\
& \delta_H(h_{1234})\delta_H(h_{1235})\delta_H(h_{1236})\delta_L(l_{12356})\delta_L(l_{12346})\delta_L(l_{12345})Z_{\text{ostatak}}\,.
\end{aligned}
\tag{9.28}
$$

Dokaz invarijantnosti sume po stanjima (9.22) pri $3 \leftrightarrow 3$ Pahnerovom potezu svodi se na dokaz da su izrazi (9.27) i (9.28) jednaki, pri čemu činilac Z_{ostatak} označava deo sume koji ostaje nepromenjen po definiciji poteza.

Dobijamo da suma po stanjima definisana u (9.22) ostaje nepromenjena pri svih pet Pahnerovih poteza, te stoga zaključujemo da je suma po stanjima nezavisna od triangulacije 4-dimenzionalne mnogostrukosti $\mathcal{M}_4$ (pogledati Dodatak E za dokaz).

Glava 10

Zaključak

Rezime

U ovoj disertaciji smo se upoznali sa osnovama modela kvantne gravitacije i materije u okviru $2BF$, odnosno $3BF$ teorije. Najpre, u poglavlju 2 je prikazan kratak pregled više kategorijske generalizacije gejdž teorija - *viših gejdž teorija*, naime, 2-gejdž teorija kod kojih je simetrija teorije data nekom 2-grupom, odnosno ukrštenim modulom $(H \stackrel{\partial}{\to} G, \triangleright)$ i 3-gejdž teorija kod kojih je simetrija teorije data nekom 3-grupom, odnosno 2-ukrštenim modulom $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G, \triangleright, \{\_, \_\}_{\mathrm{pf}})$. Takođe, uvedeni su neophodni matematički objekti za formiranje $3BF$ teorije — 3-koneksija (α, β, γ), gde su diferencijalne forme elementi algebri $\alpha \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{h})$ i $\gamma \in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{l})$ i lažna 3-krivina $(\mathcal{F}, \mathcal{G}, \mathcal{H})$, gde su $\mathcal{F} \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$, $\mathcal{G} \in \mathcal{A}^3(\mathcal{M}_4, \mathfrak{h})$ i $\mathcal{H} \in \mathcal{A}^4(\mathcal{M}_4, \mathfrak{l})$. Zatim, u poglavlju 3 je prikazan kratak pregled *Hamiltonove analize sistema sa vezama* sa definicijama kanonskog i totalnog Hamiltonijana, primarnih i sekundarnih veza, veza prve i druge klase, kao i načinom izračunavanja broja stepeni slobode u teoriji i *Kastelanijevom procedurom* za izračunavanje generatora gejdž simetrija.

Nakon toga, u poglavlju 4 je razmatrana BF teorija i urađena je kompletna Hamiltonova analiza za topološko BF dejstvo. Dobijeno je da, očekivano, BF dejstvo opisuje teoriju bez lokalnih propagirajućih stepeni slobode. Kastelanijevom procedurom je izračunat generator gejdž simetrija u BF teoriji i izračunate su varijacije formi za sve varijable u teoriji i njihove konjugovane impulse. Na osnovu ovih rezultata, dobijena su dva tipa gejdž transformacija simetrija u BF teoriji — G-gejdž i M-gejdž transformacije, koje su već poznate u literaturi, kao i komutacione relacije ukupne grupe gejdž simetrija BF dejstva $\mathcal{G}_{BF} = G \ltimes \tilde{M}$. Ovde je G podgrupa ukupne grupe simetrija koju formiraju G-gejdž transformacije, a $\tilde{M}$ invarijantna podgrupa ukupne grupe simetrija koju formiraju M-gejdž transformacije. Varijacije formi koje odgovaraju difeomorfizmima prikazane su kao zbir varijacija formi varijabli pri gejdž transformacijama i varijacija formi pri HT transformacijama za konkretan izbor parametara, pa je time demonstrirano da je BF teorija invarijantna na difeomorfizme. Zatim, razmatrana su dva za fiziku relevantna modela koji poseduju lokalne propagirajuće stepene slobode, dobijena dodavanjem odgovarajućih članova, *veza*, u BF dejstvo. Prvi razmatrani primer takvog dejstva je *Jang-Milsova teorija* za $SU(N)$ grupu u prostoru Minkovskog, a drugi je *Plebanski model* za Opštu relativnost.

Prateći istu liniju izlaganja, u poglavlju 5 je razmatrana viša kategorijska generalizacija BF teorije — $2BF$ teorija, u literaturi poznata i kao $BFCG$ teorija. Sprovedena je kompletna Hamiltonova analiza za topološko $2BF$ dejstvo. Kao i u slučaju BF teorije, dobijeno je da je $2BF$ topološka teorija bez lokalnih propagirajućih stepeni slobode. Nakon izračunatog generatora i varijacija formi varijabli i njihovih konjugovanih impulsa, dobijene su konačne transformacije simetrija za $2BF$ dejstvo: G-gejdž, H-gejdž, M-gejdž i N-gejdž transformacije. Ukupna gejdž grupa simetrija dobijena je kao $\mathcal{G}_{2BF} = G \ltimes (\tilde{H} \ltimes (\tilde{N} \times \tilde{M}))$, gde su grupe G

i $\tilde{M}$ definisane kao i u slučaju BF teorije, $\tilde{N}$ je grupa N-gejdž transformacija, a grupa $\tilde{H}$ je grupa H-gejdž transformacija. Slično kao i slučaju BF teorije, dobijeni su konkretni izbori gejdž parametara i HT parametara koji daju difeomorfizam transformacije. Pokazano je kako Opštu relativnost možemo prikazati kao $2BF$ teoriju sa vezama za konkretan izbor 2-grupe simetrija. Na kraju, poslednji odeljak poglavlja 5 posvećen je diskusiji *Ajnštajn-Jang-Milsove teorije*, odnosno teoriji gravitacije i gejdž polja formulisanoj kao $2BF$ teorija sa vezama. Prednost ove formulacije Opšte relativosti u odnosu na formulaciju preko BF teorije leži u tome što struktura 2-grupe uvodi tetradna polja u topološko dejstvo, što otvara mogućnost kuplovanja materije sa gravitacijom na pravolinijski način. Ipak, polja materije ne mogu biti prirodno izražena u okviru algebarske strukture 2-grupe, tj. sektor materije u dejstvu ne može biti napisan kao zbir topološkog člana i veze. Kako bi to bilo postignuto, neophodan je još jedan korak više kategorijske generalizacije BF teorije — tzv. $3BF$ teorija.

Konačno, poslednje poglavlje u prvom delu disertacije, poglavlje 6, posvećeno je klasičnoj $3BF$ teoriji. Nakon Hamiltonove analize teorije i postupka analognog onom u slučaju BF i $2BF$ teorije, dobijeno je da je $3BF$ teorija invarijantna na pet vrsta gejdž transformacija — G-gejdž, H-gejdž, L-gejdž, M-gejdž i N-gejdž transformacije. Analiza transformacija simetrija, tj. izračunavanje komutatora generatora ovih transformacija, ukazala je na jednu bitnu razliku u odnosu na $2BF$ teoriju, a to je da u $3BF$-teoriji H-gejdž transformacije ne čine grupu. Dobijena je ukupna gejdž grupa simetrije $\mathcal{G}_{3BF} = G \ltimes (\tilde{H}_L \ltimes (\tilde{N} \times \tilde{M}))$, gde je $\tilde{H}_L$ grupa koju formiraju H-gejdž i L-gejdž transformacije, dok su ostale grupe definisane kao u slučaju $2BF$ teorije. Zatim su diskutovane $3BF$ teorije sa vezama koje opisuju modele sa netrivijalnom dinamikom, dobijene modifikacijom topološkog $3BF$ dejstva dodavanjem odgovarajućih veza. Razmatrane su teorije koje opisuju *Klajn-Gordonovo* i *Dirakovo polje* u zakrivljenom prostoru, formulisane kao $3BF$ dejstvo sa vezama. Radi kompletnosti, analizirano je i *Vajlovo* i *Majorana polje* u interakciji sa Ajnštajn-Kartanovom gravitacijom. Ovi rezultati su zatim primenjeni za konstrukciju $3BF$ dejstva sa vezama koje opisuje svu materiju prisutnu u Standardnom Modelu kuplovanu sa gravitacionim poljem. Na kraju ovog poglavlja, predstavljen je jednostavan model koji opisuje skalarnu elektrodinamiku kao $3BF$ teoriju sa vezama.

Drugi deo disertacije posvećen je kvantnoj teoriji. U poglavlju 7 razmatrana je konstrukcija topološke BF sume po stanjima u slučaju trodimenzionalne i četvorodimenzionalne mnogostrukosti uobičajenom kvantizacionom procedurom spinske pene. U trodimenzionalnom slučaju, dobijena suma po stanjima daje kvantnu teoriju $3D$ gravitacije — *Ponzano-Redže model*, što je posledica činjenice da na klasičnom nivou odgovarajuća teorija nema lokalne propagirajuće stepene slobode. Zatim je predstavljena konstrukcija BF topološke sume po stanjima u realnom slučaju četvorodimenzionalne prostorvremenske mnogostrukosti — tzv. *Ouguri model*. Međutim, kako u $4D$ teorija gravitacije nije topološka teorija, dobijena suma po stanjima ne predstavlja fizičku teoriju, a kvantna teorija gravitacije može se dobiti modifikacijom amplituda topološke sume po stanjima. Poslednji, treći korak kvantizacione procedure spinske pene, tj. nametanje veza na varijable prisutne u topološkom sektoru dejstva modifikacijom amplituda topološke sume po stanjima, izlazi iz okvira ove disertacije.

U poglavlju 8 sproveden je drugi korak kovarijantne kvantizacione procedure spinske pene za $2BF$ teoriju. Demonstrirano je kako se konstruiše suma po stanjima Z koja je nezavisna od triangulacije, na osnovu klasičnog $2BF$ dejstva za opštu striktnu 2-grupu i bilo koju triangulaciju bilo koje glatke d-dimenzionalne prostorvremenske mnogostrukosti, za slučaje $d \in \{3, 4\}$. Za $d = 3$, kontruisana suma po stanjima je upravo Jeterov model, dok se za $d = 4$ poklapa sa Porterovom TKTP za $d = 4$ i $n = 2$. Analizirano je ponašanje konstruisane sume po stanjima pri Pahnerovim potezima, lokalnim promenama triangulacije koje čuvaju topologiju, tako da su bilo koje dve triangulacije iste mnogostrukosti povezane konačnim brojem Pahnerovih poteza. U trodimenzionalnom slučaju postoje četiri Pahnerova poteza — potezi $1 \leftrightarrow 4$ i $2 \leftrightarrow 3$ i njihovi inverzi, dok u 4 dimenzije postoji pet različitih Pahnerovih poteza — potezi $3 \leftrightarrow 3$, $4 \leftrightarrow 2$ i

$5 \leftrightarrow 1$ i njihovi inverzi. Postavka analize ponašanja konstruisane sume po stanjima pri ovim Pahnerovim potezima predstavljena je u odeljku 8.3, dok su detalji računa dati u Dodatku E.1. Dobijeno je da suma po stanjima ostaje nepromenjena pri ovim transformacijama triangulacije, što dokazuje da je *topološka invarijanta* mnogostrukosti. Kako je nezavisna od triangulacije, suma po stanjima je invarijantna na proizvoljno usitnjavanje triangulacije i stoga definiše teoriju kontinuuma na glatkoj mnogostrukosti.

Konačno, u poslednjem poglavlju, poglavlju 9, fokusirali smo se na drugi korak kovarijantne kvantizacione procedure spinske pene za $3BF$ teoriju. Analogno postupku iz prethodnog poglavlja u slučaju sume po stanjima za $2BF$ teoriju, demonstrirano je kako se konstruiše suma po stanjima Z koja je nezavisna od triangulacije, na osnovu klasičnog $3BF$ dejstva za opštu semistriktnu 3-grupu i bilo koju triangulaciju bilo koje 4-dimenzionalne prostorvremenske mnogostrukosti. Kako je moguće formulisati $3BF$ teoriju samo u slučaju kada je dimenzija prostorvremenske mnogostrukosti $d \geqslant 4$, razmatran je samo slučaj $d = 4$. Konstruisana suma po stanjima je generalizacija rada Žirelija, Pfajfera i Popeskua za $2BF$ sumu po stanjima predstavljenu u prethodnom poglavlju, tj. generalizacija Jeterovog modela, a poklapa sa Porterovom TKTP za $d = 4$ i $n = 3$. Slično kao i u slučaju $2BF$ sume po stanjima, kako bismo proverili da je konstruisana suma po stanjima topološka, analizirano je njeno ponašanje pri Pahnerovim potezima u slučaju četvorodimenzionalne mnogostrukosti, tj. invarijantnost pri potezima $3 \leftrightarrow 3$, $4 \leftrightarrow 2$ i $5 \leftrightarrow 1$ i njihovim inverzima. Analiza ponašanja konstruisane sume po stanjima pri Pahnerovim potezima predstavljena je u odeljku 9.3, sa detaljima u Dodatku E.2. Ponovo je dobijeno da je suma po stanjima invarijantna na Pahnerove poteze, tj. da je *topološka invarijanta* mnogostrukosti.

U dodacima su prikazani računski detalji koji prate osnovni tekst.

Diskusija i budući pravci istraživanja

Kategorijska generalizacija BF dejstva na $3BF$ dejstvo pružila je značajan uvid u sadržaj sektora materije. Pokazano je kako su polja materije u teoriji određena izborom gejdž grupe L, elementom 2-ukrštenog modula $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \triangleright, \{\_, \_\}_{\mathrm{pf}})$. Grupa L je potpuno nova struktura koja nije prisutna u Standardnom Modelu i u potpunosti proizilazi iz više kategorijske strukture teorije. Tako su formulisane gejdž grupe koje odgovaraju Klajn-Gordonovom, Dirakovom, Vajlovom i Majorana polju i konstruisana odgovarajuća $3BF$ dejstva koja opisuju dinamiku ovih polja kuplovanih sa gravitacijom na standardan način. Zatim, jednostavnim izborom grupe L kao direktnog proizvoda grupa koje odgovaraju pojedinačnim relevantnim poljima u Standardnom Modelu, tj. kao $L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G})$, dobijena je grupa koja opisuje materiju kompletnog Standardnog Modela i formulisana je teorija gravitacije i polja materije kao $3BF$ teorija sa vezama. Ovakav izbor 3-grupe je trivijalan, u smislu da je grupa L izabrana kao direktni proizvod, a Pajferovo podizanje i preslikavanja ∂ i δ kao trivijalna. Različitim izborom 3-grupe mogu se konstruisani različiti modeli gravitacije i materije, slično kao što je to rađeno u okviru *teorija velikog ujedinjenja*[1], gde su konstruisani razni modeli vektorskog polja različitim izborima Jang-Milsove gejdž grupe. Postavlja se pitanje da li postoji neki bolji izbor 3-grupe koji bi odgovarao teoriji gravitacije sa materijom, a koji bi pružio odgovor na neke od otvorenih pitanja Standardnog Modela. Mogućnost netrivijalnog ujedinjenja polja u teoriji je interesantan budući pravac istraživanja u kome leži najveći potencijal ovog pristupa.

U prvom delu teksta urađena je Hamiltonova analiza topološkog BF, $2BF$ i $3BF$ dejstva. Međutim, kao topološka dejstva, ona ne opisuju realistične fizičke teorije koje sadrže lokalne propagirajuće stepene slobode. Uvođenje fizičkih stepeni slobode postiže se nametanjem veza na topološko dejstvo. Formulisana $2BF$, odnosno $3BF$ dejstva sa vezama za netopološke teorije,

[1]eng. *Grand Unified Theory (GUT).*

konkretno $2BF$ dejstva sa vezama koja opisuju Jang-Milsovo polje i Ajnštajn-Kartanovu gravitaciju, kao i $3BF$ dejstva sa vezama koja opisuju Klajn-Gordonovo, Dirakovo, Vejlovo i Majorana polje kuplovana sa gravitacijom na standardni način napisana su u obliku sume topološkog člana i člana sa vezama. Prirodan sledeći korak ovog pravca istraživanja bila bi Hamiltonova analiza svih takvih $2BF$, odnosno $3BF$ modela gravitacije kuplovanih sa različitim vektorskim poljima i poljima materije i proučavanje njihovih simetrija.

Dobijena je grupa simetrija $\mathcal{G}_{3BF}$ koja opisuje gejdž simetrije topološkog $3BF$ dejstva. Međutim, fizičke teorije opisane su modifikovanim $3BF$ dejstvima, dobijenim dodavanjem veza, čije grupe simetrija predstavljaju neku podgrupu ukupne grupe simetrija topološkog dejstva. Ovako eksplicitno narušena grupa simetrija, dalje se može spontano narušiti Higsovim mehanizmom, što predstavlja jedan od zanimljivih budućih pravaca istraživanja.

Jedan važan rezultat je veza između 2-ukrštenog modula i grupe simetrija $3BF$ dejstva, tzv. *dualnost*. Iz izračunatih komutacionih relacija Lijeve algebre grupe simetrije $\mathcal{G}_{3BF}$ vidi se da strukturne konstante zavise od izbora grupa G, H i L 2-ukrštenog modula ($L \xrightarrow{\delta} H \xrightarrow{\partial} G$, $\triangleright$, $\{\_,\ \_\}_{\mathrm{pf}}$), dejstva $\triangleright$ i simetričnog dela Pajferovog podizanja $\{\_,\ \_\}_{\mathrm{pf}}$. Međutim, grupa $\mathcal{G}_{3BF}$ ne zavisi od antisimetričnog dela Pajferovog podizanja, niti od homomorfizama ∂ i δ. Na osnovu ovog rezultata sledi da može postojati nekoliko različitih 2-ukrštenih modula *dualnih* istoj grupi simetrija $\mathcal{G}_{3BF}$, odnosno ne postoji korespondencija jedan-na-jedan između 2-ukrštenog modula i grupe simetrija odgovarajućeg $3BF$ dejstva. Ovaj rezultat može imati praktičnu primenu u konstrukciji $3BF$ modela, gde bismo dakle prvo definisali izbor grupe $\mathcal{G}_{3BF}$ koji odgovara željenoj simetriji modela, a koji automatski fiksira izbor grupa G, H i L, dejstva $\triangleright$ i simetričnog dela Pajferovog podizanja, a zatim definisali preostale elemente 2-ukrštenog modula tako da su zadovoljene sve aksiome definicije 2-ukrštenog modula.

U drugom delu teze konstruisana je topološka suma po stanjima Z za opštu semistriktnu 3-grupu i $4D$ prostorvremensku mnogostrukost $\mathcal{M}_4$ i dokazano je da je ona topološka invarijanta te mnogostrukosti. Konstruisana suma po stanjima predstavlja kombinatornu konstrukciju topološke kvantne teorije polja (TKTP) u smislu Atijinih aksioma, što se može eksplicitno proveriti. Dokaz da suma po stanjima zadovoljava Atijine aksiome prevazilazi okvire ove teze i ostavljen je za dalji rad.

Ipak, da bi uspešno završili drugi korak kovarijantne kvantizacione procedure spinske pene, neophodne su generalizacije Peter-Vejlove i Planšarelove teorema za slučajeve 2-grupe i 3-grupe, matematički rezultati koji za sada predstavljaju otvorene probleme. Naime, ove teoreme treba da obezbede dekompoziciju funkcija na 3-grupi u sumu po odgovarajućim ireducibilnim reprezentacijama 3-grupe. Na ovaj način se određuje spektar oznaka simpleksa triangulacije, tj. domen vrednosti polja koja žive na simpleksima triangulacije, kao što je to urađeno u slučaju BF sume po stanjima. Trenutni pokušaji privođenja kraju drugog koraka kvantizacije uopštenih BF teorija u okviru viših gejdž teorija se svode na pogađanje ireducibilnih reprezentacija 2-grupa, kao što je urađeno na primer u slučaju spinkub modela kvantne gravitacije.

Ovaj rezultat otvara put ka trećem i finalnom koraku kovarijantne kvantizacione procedure spinske pene. Kako je cilj da opišemo realnu fizičku teoriju, tj. teoriju koja sadrži lokalne propagirajuće stepene slobode, potrebno je konstruisati netopološku sumu po stanjima koja opisuje teoriju sa netrivijalnom dinamikom. Poslednji, treći korak kvantizacione procedure spinske pene podrazumeva nametanje veza koje deformišu topološku teoriju u fizičku teoriju. Nastavak ovog istraživanja ima za cilj formulaciju sume po stanjima koja opisuje kvantnu teoriju gravitacije kuplovanu sa materijom. Izgradnju ovog modela ostavljamo za budući rad.

Ova lista nije kompletna, pa pored navedenih tema postoje i mnogobrojni drugi pravci istraživanja u okviru viših kategorijskih generalizacija BF teorija, kako u fizici tako i u matematici.

Dodatak A

Konstrukcija dejstva invarijantnog na gejdž transformacije

A.1 Konstrukcija $2BF$ dejstva

Simetrične bilinearne invarijantne forme za algebre $\mathfrak{h}$ i $\mathfrak{g}$ obeležavamo kao:

$$\langle t_a \,, t_b \rangle_{\mathfrak{h}} = g_{ab} \,, \qquad \langle \tau_\alpha \,, \tau_\beta \rangle_{\mathfrak{g}} = g_{\alpha\beta} \,. \tag{A.1}$$

Bilinearne forme imaju osobine:

- $\langle \_ \,, \_ \rangle_{\mathfrak{g}}$ je G-invarijantna:

$$\langle g\tau_\alpha g^{-1} \,, g\tau_\beta g^{-1} \rangle_{\mathfrak{g}} = \langle \tau_\alpha \,, \tau_\beta \rangle_{\mathfrak{g}} \,, \quad \forall g \in G \,; \tag{A.2}$$

- $\langle \_ \,, \_ \rangle_{\mathfrak{h}}$ je G-invarijantna

$$\langle g \triangleright t_a \,, g \triangleright t_b \rangle_{\mathfrak{h}} = \langle t_a \,, t_b \rangle_{\mathfrak{h}} \,, \quad \forall g \in G \,, \tag{A.3}$$

  a takođe i H-invarijantna:

$$\langle h t_a h^{-1} \,, h t_b h^{-1} \rangle_{\mathfrak{h}} = \langle \partial(h) \triangleright t_a \,, \partial(h) \triangleright t_b \rangle_{\mathfrak{h}} = \langle t_a \,, t_b \rangle_{\mathfrak{h}} \,, \quad \forall h \in H \,. \tag{A.4}$$

A.1.1 2-Gejdž transformacije 2-krivine

Grupe G i H ukrštenog modula $(H \stackrel{\partial}{\to} G, \triangleright)$ generišu dva tipa 2-gejdž transformacija 2-koneksije, definisanih izrazima (2.47) i (2.48).

Teorema 24 *Kompozicija G-gejdž i H-gejdž transformacija dovodi do transformacije 2-koneksije:*

$$\begin{aligned} \alpha'' &= g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta) \,, \\ \beta'' &= g^{-1} \triangleright \beta + \mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta - \eta \wedge \eta \,, \end{aligned} \tag{A.5}$$

gde su $g : \mathcal{M}_4 \to G$ i $\eta \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ redom parametri G- i H-gejdž transformacija.

Dokaz. Pri G-gejdž transformacijama 2-koneksija (α, β) se transformiše po pravilu (2.47):

$$\begin{aligned} \alpha' &= g^{-1}\alpha g + g^{-1}\mathrm{d}g \,, \\ \beta' &= g^{-1} \triangleright \beta \,. \end{aligned}$$

Daljom transformacijom 2-koneksije H-gejdž transformacijama po pravilu (2.48) dobija se:

$$\alpha'' = \alpha' + \partial(\eta) = g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta)\,,$$
$$\beta'' = \beta' + \mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta - \eta\wedge\eta = g^{-1}\triangleright\beta + \mathrm{d}\eta + \big(g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta)\big)\wedge^{\triangleright}\eta - \eta\wedge\eta\,.$$

Time smo dokazali Teoremu 24. ■

Na osnovu definicije (2.36) i transformacionih pravila za 2-koneksiju (2.47) i (2.48), dobijaju se transformacije 2-krivine $(\mathcal{F},\mathcal{G})$ pri 2-gejdž transformacijama.

Teorema 25 *Pri G-gejdž transformacijama 2-krivina $(\mathcal{F},\mathcal{H})$ se transformiše na sledeći način*

$$\mathcal{F}\to g^{-1}\mathcal{F}g\,,\qquad \mathcal{G}\to g^{-1}\triangleright\mathcal{G}\,, \tag{A.6}$$

gde je $g:\mathcal{M}_4\to G$ parametar G–gejdž transformacija.

Dokaz. Primenom definicije 2-krivine, dobijamo da se pri G-gejdž transformacijama krivina $\mathcal{F}$ transformiše kao

$$\begin{aligned}
\mathcal{F}' &= \mathrm{d}\alpha' + \alpha'\wedge\alpha' - \partial\beta' \\
&= \mathrm{d}(g^{-1}\alpha g + g^{-1}\mathrm{d}g) + (g^{-1}\alpha g + g^{-1}\mathrm{d}g)\wedge(g^{-1}\alpha g + g^{-1}\mathrm{d}g) - \partial(g^{-1}\triangleright\beta) \\
&= \mathrm{d}g^{-1}\wedge\alpha g + g^{-1}\mathrm{d}\alpha g - g^{-1}\alpha\wedge\mathrm{d}g + \mathrm{d}g^{-1}\wedge\mathrm{d}g \\
&\quad + g^{-1}\alpha g\wedge g^{-1}\alpha g + g^{-1}\alpha g\wedge g^{-1}\mathrm{d}g + g^{-1}\mathrm{d}g\wedge g^{-1}\alpha g + g^{-1}\mathrm{d}g\wedge g^{-1}\mathrm{d}g - g^{-1}\triangleright\partial(\beta) \\
&= \mathrm{d}g^{-1}\wedge\alpha g + g^{-1}\mathrm{d}\alpha g + \mathrm{d}g^{-1}\wedge\mathrm{d}g \\
&\quad + g^{-1}\alpha\wedge\alpha g - \mathrm{d}g^{-1}\wedge\alpha g - \mathrm{d}g^{-1}\wedge\mathrm{d}g - g^{-1}\triangleright\partial(\beta) \\
&= g^{-1}\mathrm{d}\alpha g + g^{-1}\alpha\wedge\alpha g - g^{-1}\triangleright\partial(\beta) \\
&= g^{-1}\mathcal{F}g\,,
\end{aligned} \tag{A.7}$$

gde smo koristili $g^{-1}dg = -dg^{-1}g$. Pri G-gejdž transformacijama krivina $\mathcal{G}$ se transformiše na sledeći način

$$\begin{aligned}
\mathcal{G}' &= \mathrm{d}\beta' + \alpha'\wedge^{\triangleright}\beta' \\
&= \mathrm{d}(g^{-1}\triangleright\beta) + (g^{-1}\alpha g + g^{-1}\mathrm{d}g)\wedge^{\triangleright}(g^{-1}\triangleright\beta) \\
&= \mathrm{d}g^{-1}\wedge^{\triangleright}\beta + g^{-1}\triangleright\mathrm{d}\beta + (g^{-1}\alpha g)\wedge^{\triangleright}(g^{-1}\triangleright\beta) - \mathrm{d}g^{-1}\wedge^{\triangleright}\beta \\
&= g^{-1}\triangleright\mathrm{d}\beta + g^{-1}\triangleright(\alpha\wedge^{\triangleright}\beta) \\
&= g^{-1}\triangleright\mathcal{G}\,.
\end{aligned} \tag{A.8}$$

Time smo dokazali tvrđenje Teoreme 25. ■

Teorema 26 *Pri H-gejdž transformacijama 2-krivina se transformiše po zakonu transformacije*

$$\mathcal{F}\to\mathcal{F}\,,\qquad \mathcal{G}\to\mathcal{G} + \mathcal{F}\wedge^{\triangleright}\eta\,, \tag{A.9}$$

gde je $\eta\in\mathcal{A}^1(\mathcal{M}_4,\mathfrak{h})$ parametar H-gejdž transformacija.

Dokaz. Primenom definicije 3-krivine, dobijamo da je krivina $\mathcal{F}$ invarijantna na H-gejdž transformacije

$$
\begin{aligned}
\mathcal{F}'' &= \mathrm{d}\alpha'' + \alpha'' \wedge \alpha'' - \partial\beta'' \\
&= \mathrm{d}(\alpha + \partial\eta) + (\alpha + \partial\eta) \wedge (\alpha + \partial\eta) - \partial(\beta + \mathrm{d}\eta + (\alpha + \partial\eta) \wedge^{\triangleright} \eta - \eta \wedge \eta) \\
&= \mathrm{d}\alpha + \alpha \wedge \alpha + \alpha'' \wedge \partial\eta + \partial\eta \wedge \alpha'' - \partial\beta - \partial(\alpha'' \wedge^{\triangleright} \eta) \\
&= \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta \\
&= \mathcal{F}\,,
\end{aligned}
\tag{A.10}
$$

gde smo primenili identitete:

$$
\begin{gathered}
\mathrm{d}(\partial\eta) = \partial(\mathrm{d}\eta)\,, \\
\partial(\eta \wedge \eta) = \partial\eta \wedge \partial\eta\,, \\
\alpha'' \wedge \partial\eta + \partial\eta \wedge \alpha'' = \partial(\alpha'' \wedge^{\triangleright} \eta)\,.
\end{gathered}
$$

Pri H-gejdž transformacijama krivina $\mathcal{G}$ se transformiše na sledeći način:

$$
\begin{aligned}
\mathcal{G}'' &= \mathrm{d}\beta'' + \alpha'' \wedge^{\triangleright} \beta'' \\
&= \mathrm{d}(\beta + \mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta - \eta \wedge \eta) + (\alpha + \partial\eta) \wedge^{\triangleright} (\beta + \mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta - \eta \wedge \eta) \\
&= \mathrm{d}\beta + \mathrm{d}\alpha'' \wedge^{\triangleright} \eta - \alpha'' \wedge^{\triangleright} \mathrm{d}\eta - \mathrm{d}\eta \wedge \eta + \eta \wedge \mathrm{d}\eta \\
&\quad + \alpha \wedge^{\triangleright} (\beta + \mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta - \eta \wedge \eta) + \partial\eta \wedge^{\triangleright} (\beta + \mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta - \eta \wedge \eta) \\
&= \mathrm{d}\beta + \mathrm{d}\alpha \wedge^{\triangleright} \eta + \mathrm{d}(\partial\eta) \wedge^{\triangleright} \eta - \partial\eta \wedge^{\triangleright} \mathrm{d}\eta - \mathrm{d}\eta \wedge \eta + \eta \wedge \mathrm{d}\eta \\
&\quad + \alpha \wedge^{\triangleright} \beta + (\alpha \wedge \alpha) \wedge^{\triangleright} \eta + \alpha \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) - \alpha \wedge^{\triangleright} (\eta \wedge \eta) \\
&\quad - \partial\beta \wedge^{\triangleright} \eta + \partial\eta \wedge^{\triangleright} \mathrm{d}\eta + \partial\eta \wedge^{\triangleright} (\alpha \wedge^{\triangleright} \eta) + \partial\eta \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) - \partial\eta \wedge^{\triangleright} (\eta \wedge \eta) \\
&= \mathrm{d}\beta + \mathrm{d}\alpha \wedge^{\triangleright} \eta + \alpha \wedge^{\triangleright} \beta + (\alpha \wedge \alpha) \wedge^{\triangleright} \eta - \partial\beta \wedge^{\triangleright} \eta \\
&= \mathcal{G} + \mathcal{F} \wedge \eta\,.
\end{aligned}
\tag{A.11}
$$

Ovde smo koristili sledeće identitete:

$$
\begin{gathered}
\mathrm{d}\eta \wedge \eta - \eta \wedge \mathrm{d}\eta = \mathrm{d}(\partial\eta) \wedge^{\triangleright} \eta\,, \\
\alpha \wedge^{\triangleright} (\alpha \wedge^{\triangleright} \eta) = (\alpha \wedge \alpha) \wedge^{\triangleright} \eta\,, \\
\partial\eta \wedge^{\triangleright} \beta = -\partial\beta \wedge^{\triangleright} \eta\,, \\
\alpha \wedge^{\triangleright} (\eta \wedge \eta) = \alpha \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) + \partial\eta \wedge^{\triangleright} (\alpha \wedge^{\triangleright} \eta)\,, \\
\partial\eta \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) - \partial\eta \wedge^{\triangleright} (\eta \wedge \eta) = 0\,.
\end{gathered}
$$

Time smo dokazali tvrđenje Teoreme 26. ■

Teorema 27 *Pri 2-gejdž transformacijama 2-krivina $(\mathcal{F}, \mathcal{G})$ se transformiše na sledeći način:*

$$
\begin{gathered}
\mathcal{F} \to g^{-1} \triangleright \mathcal{F}\,, \\
\mathcal{G} \to g^{-1} \triangleright \mathcal{G} + (g^{-1} \triangleright \mathcal{F}) \wedge^{\triangleright} \eta\,,
\end{gathered}
\tag{A.12}
$$

gde su $g : \mathcal{M}_4 \to G$ i $\eta \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ redom parametri G- i H-gejdž transformacija.

Dokaz. Uzatopnom transformacijom 2-krivine najpre G-gejdž transformacijom definisanom u Teoremi 25, a zatim H-gejdž transformacijom definisanom u Teoremi 26 dobijamo rezultat Teoreme 27. ■

A.1.2 2-Gejdž transformacije Lagranževih množitelja

Korišćenjem G-invarijntne simetrične nedegenerisane bilinearne forme u $\mathfrak{g}$ i $\mathfrak{h}$, može se definisati bilinearno antisimetrično preslikavanje $\mathcal{T} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ na sledeći način [17]:

$$\langle \mathcal{T}(\underline{h}_1,\, \underline{h}_2)\,, \underline{g} \rangle_{\mathfrak{g}} = -\langle \underline{h}_1,\, \underline{g} \rhd \underline{h}_2 \rangle_{\mathfrak{h}}, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{g} \in \mathfrak{g}\,. \tag{A.13}$$

U prethodnom izrazu podvučeni simboli obeležavaaju elemente algebri, a nepodvučeni elemente grupa, što je notacija koju podrazumevamo i u daljem tekstu.

Za svako $g \in G$ i svake $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ važi

$$\langle \underline{h}_1, g \rhd \underline{h}_2 \rangle_{\mathfrak{h}} = -\langle g \rhd \underline{h}_1, \underline{h}_2 \rangle_{\mathfrak{h}} = -\langle \underline{h}_2, g \rhd \underline{h}_1 \rangle_{\mathfrak{h}}\,.$$

Može se pokazati da za sve $g \in G$ i $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$, važi:

$$\mathcal{T}(g \rhd \underline{h}_1, g \rhd \underline{h}_2) = g\mathcal{T}(\underline{h}_1, \underline{h}_2)g^{-1}\,. \tag{A.14}$$

Pokažimo to. Za neko $\underline{g}_0 \in \mathfrak{g}$, primenom osobina G-invarijantnosti $\langle \_, \_ \rangle_{\mathfrak{g}}$ i $\langle \_, \_ \rangle_{\mathfrak{h}}$ dobijamo:

$$\begin{aligned} \langle g^{-1}\mathcal{T}(g \rhd \underline{h}_1, g \rhd \underline{h}_2)g, \underline{g}_0 \rangle_{\mathfrak{g}} &= \langle \mathcal{T}(g \rhd \underline{h}_1, g \rhd \underline{h}_2), g\underline{g}_0 g^{-1} \rangle_{\mathfrak{g}} \\ &= -\langle g \rhd \underline{h}_1, (g\underline{g}_0 g^{-1}) \rhd g \rhd \underline{h}_2 \rangle_{\mathfrak{h}} \\ &= -\langle \underline{h}_1, \underline{g}_0 \rhd \underline{h}_2 \rangle_{\mathfrak{h}} \\ &= \langle \underline{g}_0, \mathcal{T}(\underline{h}_1, \underline{h}_2) \rangle_{\mathfrak{g}} \\ &= \langle \mathcal{T}(\underline{h}_1, \underline{h}_2), \underline{g}_0 \rangle_{\mathfrak{g}}\,. \end{aligned}$$

Zadovoljen je sledeći identitet:

$$\mathcal{T}(\underline{g}_0 \rhd \underline{h}_1, \underline{h}_2) + \mathcal{T}(\underline{h}_1, \underline{g}_0 \rhd \underline{h}_2) = [\underline{g}_0, \mathcal{T}(\underline{h}_1, \underline{h}_2)]\,.$$

Nakon fiksiranja bazisa, možemo definisati koeficijent bilinearnog antisimetričnog preslikavanja $\mathcal{T} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ na sledeći način

$$\mathcal{T}(t_a, t_b) = \mathcal{T}_{ab}{}^{\alpha}\tau_\alpha\,, \tag{A.15}$$

pa definiciju preslikavanja $\mathcal{T}$ možemo izraziti koristeći ovaj koeficijent:

$$\mathcal{T}_{ab}{}^{\alpha} g_{\alpha\beta} = - \rhd_{\alpha[b}{}^{c} g_{a]c}\,. \tag{A.16}$$

Bilinearno antisimetrično preslikavanje $\mathcal{T}$ dve diferencijalne forme elemenata algebre $\mathfrak{h}$, η i ω, definiše diferencijalnu formu element algebre $\mathfrak{g}$:

$$\omega \wedge^{\mathcal{T}} \eta = \omega^a \wedge \eta^b \mathcal{T}_{ab}{}^{\alpha}\tau_\alpha\,.$$

Preslikavanje $\mathcal{T}$ igra ključnu ulogu u konstrukciji $2BF$ dejstva invarijantnog na 2-gejdž transformacije, tj. definiciji zakona transformacije Langranževih množitelja pri 2-gejdž tranformacijama.

Kako bi dejstvo (5.1) bilo gejdž invarijantno pri transformacijama krivina (2.47) i (2.48), Lagranževi množitelji $B \in \mathcal{A}^2(\mathcal{M}, \mathfrak{g})$ i $C \in \mathcal{A}^1(\mathcal{M}, \mathfrak{h})$ moraju se pri G-gejdž transformacijama transformisati na sledeći način

$$B \to B' = g^{-1}Bg\,, \qquad C \to C' = g^{-1} \rhd C\,, \tag{A.17}$$

a pri H-gejdž transformacijama

$$B \to B'' = B + C'' \wedge^{\mathcal{T}} \eta\,, \qquad C \to C'' = C\,, \tag{A.18}$$

gde je preslikavanje $\mathcal{T}$ definisano jednačinom (A.13).

A.2 Konstrukcija 3BF dejstva

Simetrične bilinearne invarijantne forme za algebre $\mathfrak{l}, \mathfrak{h}$ i $\mathfrak{g}$ obeležavamo kao:

$$\langle T_A \,, T_B \rangle_{\mathfrak{l}} = g_{AB} \,, \qquad \langle t_a \,, t_b \rangle_{\mathfrak{h}} = g_{ab} \,, \qquad \langle \tau_\alpha \,, \tau_\beta \rangle_{\mathfrak{g}} = g_{\alpha\beta} \,. \tag{A.19}$$

Bilinearne forme imaju osobine:

- $\langle \_ \,, \_ \rangle_{\mathfrak{g}}$ je G-invarijantna:

$$\langle g\tau_\alpha g^{-1} \,, g\tau_\beta g^{-1} \rangle_{\mathfrak{g}} = \langle \tau_\alpha \,, \tau_\beta \rangle_{\mathfrak{g}} \,, \quad \forall g \in G \,; \tag{A.20}$$

- $\langle \_ \,, \_ \rangle_{\mathfrak{h}}$ je G-invarijantna:

$$\langle g \rhd t_a \,, g \rhd t_b \rangle_{\mathfrak{h}} = \langle t_a \,, t_b \rangle_{\mathfrak{h}} \,, \quad \forall g \in G \,, \tag{A.21}$$

  a kada je $(H \xrightarrow{\partial} G \,, \rhd)$ ukršteni modul, takođe i H-invarijantna:

$$\langle h t_a h^{-1} \,, h t_b h^{-1} \rangle_{\mathfrak{h}} = \langle \partial(h) \rhd t_a \,, \partial(h) \rhd t_b \rangle_{\mathfrak{h}} = \langle t_a \,, t_b \rangle_{\mathfrak{h}} \,, \quad \forall h \in H \,; \tag{A.22}$$

- $\langle \_ \,, \_ \rangle_{\mathfrak{l}}$ je G-invarijantna:

$$\langle g \rhd T_A \,, g \rhd T_B \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall g \in G \,, \tag{A.23}$$

  a u specijalnom slučaju kada je Pajferovo podizanje ili preslikavanje δ trivijalno takođe i H-invarijantna:

$$\langle h \rhd' T_A \,, h \rhd' T_B \rangle_{\mathfrak{l}} = \langle T_A - \{\delta(T_A),\, h\} \,, T_B - \{\delta(T_B),\, h\} \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall h \in H \,. \tag{A.24}$$

  Iz H-invarijantnosti $\langle \_ \,, \_ \rangle_{\mathfrak{l}}$ i osobina ukrštenog modula $(L \xrightarrow{\delta} H \,, \rhd')$ sledi L-invarijantnost bilinearne forme:

$$\langle l T_A l^{-1} \,, l T_B l^{-1} \rangle_{\mathfrak{l}} = \langle \delta(l) \rhd' T_A \,, \delta(l) \rhd' T_B \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall l \in L \,. \tag{A.25}$$

A.2.1 3-Gejdž transformacije 3-krivine

Struktura 3-grupe, tj. 2-ukrštenog modula $(L \xrightarrow{\delta} H \xrightarrow{\partial} G, \rhd, \{\_ , \_\}_{\mathrm{pf}})$, generiše tri tipa gejdž transformacija, G-, H- i L-gejdž transformacije 3-koneksije definisane izrazima (2.128), (2.129) i (2.130).

Teorema 28 *Kompozicija G-gejdž, H-gejdž i L-gejdž transformacija dovodi do transformacije 3-koneksije:*

$$\begin{aligned} \tilde{\alpha} &= g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta) \,, \\ \tilde{\beta} &= g^{-1} \rhd \beta + \mathrm{d}\eta + \tilde{\alpha} \wedge^{\rhd} \eta - \eta \wedge \eta - \delta(\theta) \,, \\ \tilde{\gamma} &= g^{-1} \rhd \gamma - \mathrm{d}\theta - \tilde{\alpha} \wedge \theta - \tilde{\beta} \wedge^{\{,\}} \eta - \eta \wedge^{\{,\}} (g^{-1} \rhd \beta) + \eta \wedge^{\rhd'} \theta \,, \end{aligned} \tag{A.26}$$

gde su $g : \mathcal{M}_4 \to G$, $\eta \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ i $\theta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{l})$ redom parametri G-, H- i L-gejdž transformacija.

Dokaz. Pri G-gejdž transformacijama 3-koneksija se transformiše po pravilu (2.128):

$$\begin{aligned}\alpha' &= g^{-1}\alpha g + g^{-1}\mathrm{d}g\,,\\ \beta' &= g^{-1}\rhd\beta\,,\\ \gamma' &= g^{-1}\rhd\gamma\,.\end{aligned}$$

Pri H-gejdž transformacijama transformiše se po pravilu (2.129):

$$\begin{aligned}\alpha'' &= \alpha' + \partial(\eta) = g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta)\,,\\ \beta'' &= \beta' + \mathrm{d}\eta + \alpha''\wedge^{\rhd}\eta - \eta\wedge\eta = g^{-1}\rhd\beta + \mathrm{d}\eta + \big(g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta)\big)\wedge^{\rhd}\eta - \eta\wedge\eta\,,\\ \gamma'' &= \gamma' - \beta''\wedge^{\{,\}}\eta - \eta\wedge^{\{,\}}\beta'\\ &= g^{-1}\rhd\gamma - \big(g^{-1}\rhd\beta + \mathrm{d}\eta + \big(g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta)\big)\wedge^{\rhd}\eta - \eta\wedge\eta\big)\wedge^{\{,\}}\eta - \eta\wedge^{\{,\}}(g^{-1}\rhd\beta)\,.\end{aligned}$$

Pri L-gejdž transformacijama koneksija se transformiše po pravilu (2.130):

$$\begin{aligned}\tilde\alpha &= \alpha'' = g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta)\,,\\ \tilde\beta &= \beta'' - \delta(\theta) = g^{-1}\rhd\beta + \mathrm{d}\eta + \big(g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta)\big)\wedge^{\rhd}\eta - \eta\wedge\eta - \delta(\theta)\\ &= g^{-1}\rhd\beta + \mathrm{d}\eta + \tilde\alpha\wedge^{\rhd}\eta - \eta\wedge\eta - \delta(\theta)\,,\\ \tilde\gamma &= \gamma'' - \mathrm{d}\theta - \tilde\alpha\wedge^{\rhd}\theta\\ &= g^{-1}\rhd\gamma - \big(g^{-1}\rhd\beta + \mathrm{d}\eta + \big(g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta)\big)\wedge^{\rhd}\eta - \eta\wedge\eta\big)\wedge^{\{,\}}\eta\\ &\quad - \eta\wedge^{\{,\}}(g^{-1}\rhd\beta) - \mathrm{d}\theta - \big(g^{-1}\alpha g + g^{-1}\mathrm{d}g + \partial(\eta)\big)\wedge^{\rhd}\theta\\ &= g^{-1}\rhd\gamma - \tilde\beta\wedge^{\{,\}}\eta - \eta\wedge^{\{,\}}(g^{-1}\rhd\beta) - \mathrm{d}\theta - \tilde\alpha\wedge^{\rhd}\theta - \eta\wedge^{\rhd'}\theta\,,\end{aligned}$$

gde smo koristili identitet $\delta(\theta)\wedge^{\{,\}}\eta = -\eta\wedge^{\rhd'}\theta$. ■

Na osnovu definicije (2.118) i transformacionih pravila za 3-koneksiju, dobija se transformacija 3-krivine $(\mathcal{F},\mathcal{G},\mathcal{H})$ pri 3-gejdž transformacijama.

Teorema 29 *Pri G-gejdž transformacijama 3-krivina $(\mathcal{F},\mathcal{G},\mathcal{H})$ se transformiše na sledeći način*

$$\mathcal{F}\to\mathcal{F}' = g^{-1}\mathcal{F}g\,,\qquad \mathcal{G}\to\mathcal{G}' = g^{-1}\rhd\mathcal{G}\,,\qquad \mathcal{H}\to\mathcal{H}' = g^{-1}\rhd\mathcal{H}\,,\tag{A.27}$$

gde je $g:\mathcal{M}_4\to G$ parametar G-gejdž transformacija.

Dokaz.

Primenom definicije 3-krivine i transformacionog pravila 3-koneksije pri G-gejdž transformacijama, dobijamo da se pri G-gejdž transformacijama krivina $\mathcal{F}$ transformiše na isti način kao u slučaju $2BF$ teorije.

Pri G-gejdž transformacijama krivina $\mathcal{G}$ se transformiše na sledeći način

$$\begin{aligned}\mathcal{G}' &= \mathrm{d}\beta' + \alpha'\wedge^{\rhd}\beta' - \delta(\gamma')\\ &= \mathrm{d}(g^{-1}\rhd\beta) + (g^{-1}\alpha g + g^{-1}\mathrm{d}g)\wedge^{\rhd}(g^{-1}\rhd\beta) - \delta(g^{-1}\rhd\gamma)\\ &= g^{-1}\rhd G - g^{-1}\rhd\delta(\gamma)\\ &= g^{-1}\rhd\mathcal{G}\,,\end{aligned}\tag{A.28}$$

gde smo treći red dobili na sličan način kao u slučaju $2BF$ teorije i primenom osobine preslikavanja δ. Pri G-gejdž transformacijama krivina $\mathcal{H}$ se transformiše na sledeći način

$$\begin{aligned}
\mathcal{H}' &= \mathrm{d}\gamma' + \alpha' \wedge^{\triangleright} \gamma' + \beta' \wedge^{\{,\}} \beta' \\
&= \mathrm{d}(g^{-1} \triangleright \gamma) + (g^{-1}\alpha g + g^{-1}\mathrm{d}g) \wedge^{\triangleright} (g^{-1} \triangleright \gamma) + g^{-1} \triangleright \beta \wedge^{\{,\}} g^{-1} \triangleright \beta \\
&= \mathrm{d}g^{-1} \wedge^{\triangleright} \gamma + g^{-1} \triangleright \mathrm{d}\gamma + (g^{-1}\alpha g) \wedge^{\triangleright} (g^{-1} \triangleright \gamma) - \mathrm{d}g^{-1} \wedge^{\triangleright} \gamma + g^{-1} \triangleright \beta \wedge^{\{,\}} g^{-1} \triangleright \beta \\
&= g^{-1} \triangleright \gamma + g^{-1} \triangleright (\alpha \wedge \gamma) + g^{-1} \triangleright (\beta \wedge^{\{,\}} \beta) \\
&= g \triangleright \mathcal{H}\,,
\end{aligned} \tag{A.29}$$

gde smo u poslednjem redu iskoristili osobinu G-ekvivarijantnosti Pajferovog podizanja. Time smo dokazali tvrđenje Teoreme 29. ■

Teorema 30 *Pri H-gejdž transformacijama 3-krivina $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ se transformiše na sledeći način*

$$\mathcal{F} \to \mathcal{F}'' = \mathcal{F}\,, \qquad \mathcal{G} \to \mathcal{G}'' = \mathcal{G} + \mathcal{F} \wedge^{\triangleright} \eta\,, \qquad \mathcal{H} \to \mathcal{H}'' = \mathcal{H} - \mathcal{G}'' \wedge^{\{,\}} \eta + \eta \wedge^{\{,\}} \mathcal{G}\,, \tag{A.30}$$

gde je $\eta \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ parametar H–gejdž transformacija.

Dokaz. Primenom definicije 3-krivine i transformacionog pravila 3-koneksije pri H-gejdž transformacijama, dobijamo da se krivina $\mathcal{F}$ transformiše na isti način kao u slučaju $2BF$ teorije. Pri H-gejdž transformacijama krivina $\mathcal{G}$ se transformiše

$$\begin{aligned}
\mathcal{G}'' &= \mathrm{d}\beta'' + \alpha'' \wedge^{\triangleright} \beta'' - \delta(\gamma'') \\
&= \mathrm{d}\beta + \mathrm{d}\alpha \wedge^{\triangleright} \eta + \mathrm{d}(\partial\eta) \wedge^{\triangleright} \eta - \partial\eta \wedge^{\triangleright} \mathrm{d}\eta - \mathrm{d}\eta \wedge \eta + \eta \wedge \mathrm{d}\eta \\
&\quad + \alpha \wedge^{\triangleright} \beta + (\alpha \wedge \alpha) \wedge^{\triangleright} \eta + \alpha \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) - \alpha \wedge^{\triangleright} (\eta \wedge \eta) \\
&\quad + \partial\eta \wedge^{\triangleright} \beta + \partial\eta \wedge^{\triangleright} \mathrm{d}\eta + \partial\eta \wedge^{\triangleright} (\alpha \wedge^{\triangleright} \eta) + \partial\eta \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) - \partial\eta \wedge^{\triangleright} (\eta \wedge \eta) \\
&\quad - \delta(\gamma - (\beta + \mathrm{d}\eta + (\alpha + \partial\eta) \wedge^{\triangleright} \eta - \eta \wedge \eta) \wedge^{\{,\}} \eta - \eta \wedge^{\{,\}} \beta) \\
&= \underline{\mathrm{d}\beta} + \underline{\mathrm{d}\alpha \wedge^{\triangleright} \eta} + \mathrm{d}(\partial\eta) \wedge^{\triangleright} \eta - \partial\eta \wedge^{\triangleright} \mathrm{d}\eta - \mathrm{d}\eta \wedge \eta + \eta \wedge \mathrm{d}\eta \\
&\quad + \underline{\alpha \wedge^{\triangleright} \beta} + \underline{(\alpha \wedge \alpha) \wedge^{\triangleright} \eta} + \alpha \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) - \alpha \wedge^{\triangleright} (\eta \wedge \eta) \\
&\quad + \partial\eta \wedge^{\triangleright} \beta + \partial\eta \wedge^{\triangleright} \mathrm{d}\eta + \partial\eta \wedge^{\triangleright} (\alpha \wedge^{\triangleright} \eta) + \partial\eta \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) - \partial\eta \wedge^{\triangleright} (\eta \wedge \eta) \\
&\quad - \underline{\delta(\gamma)} + \beta \wedge^{[,]} \eta - \underline{\partial\beta \wedge^{\triangleright} \eta} + \mathrm{d}\eta \wedge^{[,]} \eta - \partial(\mathrm{d}\eta) \wedge^{\triangleright} \eta + (\alpha \wedge^{\triangleright} \eta) \wedge^{[,]} \eta - \partial(\alpha \wedge^{\triangleright} \eta) \wedge^{\triangleright} \eta \\
&\quad + (\partial\eta \wedge^{\triangleright} \eta) \wedge^{[,]} \eta - \partial(\partial\eta \wedge^{\triangleright} \eta) \wedge^{\triangleright} \eta - (\eta \wedge \eta) \wedge^{[,]} \eta + \partial(\eta \wedge \eta) \wedge^{\triangleright} \eta \\
&\quad + \eta \wedge^{[,]} \beta - \partial\eta \wedge^{\triangleright} \beta\,,
\end{aligned} \tag{A.31}$$

gde prepoznajući da podvučeni članovi daju $\mathcal{G} + \mathcal{F} \wedge^{\triangleright} \eta$ i skraćivanjem određenih članova dobijamo:

$$
\begin{aligned}
\mathcal{G}'' \;=\;& \mathcal{G} + \mathcal{F} \wedge^{\triangleright} \eta \\
&+\alpha \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) - \alpha \wedge^{\triangleright} (\eta \wedge \eta) \\
&+\partial\eta \wedge^{\triangleright} (\alpha \wedge^{\triangleright} \eta) + \partial\eta \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) - \partial\eta \wedge^{\triangleright} (\eta \wedge \eta) \\
&+(\alpha \wedge^{\triangleright} \eta) \wedge^{[,]} \eta - \partial(\alpha \wedge^{\triangleright} \eta) \wedge^{\triangleright} \eta \\
&+(\partial\eta \wedge^{\triangleright} \eta) \wedge^{[,]} \eta - \partial(\partial\eta \wedge^{\triangleright} \eta) \wedge^{\triangleright} \eta - (\eta \wedge \eta) \wedge^{[,]} \eta + \partial(\eta \wedge \eta) \wedge^{\triangleright} \eta \,.
\end{aligned}
\tag{A.32}
$$

Primenili smo da je $\mathrm{d}\partial\eta = \partial\mathrm{d}\eta$, $\mathrm{d}\eta \wedge^{[,]} \eta = \mathrm{d}\eta \wedge \eta - \eta \wedge \mathrm{d}\eta$ i $\beta \wedge^{[,]} \eta = -\eta \wedge^{[,]} \beta$ po definiciji. Zatim, koristeći identitete

$$
\begin{aligned}
(\alpha \wedge^{\triangleright} \eta) \wedge^{[,]} \eta &= \alpha \wedge^{\triangleright} (\eta \wedge \eta) \,, \\
(\partial\eta \wedge^{\triangleright} \eta) \wedge^{[,]} \eta &= \partial\eta \wedge^{\triangleright} (\eta \wedge \eta) \,, \\
(\eta \wedge \eta) \wedge^{[,]} \eta &= 0 \,,
\end{aligned}
$$

dobijamo:

$$
\begin{aligned}
\mathcal{G}'' \;=\;& \mathcal{G} + \mathcal{F} \wedge^{\triangleright} \eta + \alpha \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) + \partial\eta \wedge^{\triangleright} (\alpha \wedge^{\triangleright} \eta) + \partial\eta \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) \\
&-\partial(\alpha \wedge^{\triangleright} \eta) \wedge^{\triangleright} \eta - \partial(\partial\eta \wedge^{\triangleright} \eta) \wedge^{\triangleright} \eta + \partial(\eta \wedge \eta) \wedge^{\triangleright} \eta \,.
\end{aligned}
\tag{A.33}
$$

Na kraju, koristeći da je

$$
\begin{aligned}
\partial(\alpha \wedge^{\triangleright} \eta) \wedge^{\triangleright} \eta &= \alpha \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) + \partial\eta \wedge^{\triangleright} (\alpha \wedge^{\triangleright} \eta) \,, \\
\partial(\partial\eta \wedge^{\triangleright} \eta) \wedge^{\triangleright} \eta &= \partial(\eta \wedge \eta) \wedge^{\triangleright} \eta + \partial\eta \wedge^{\triangleright} (\partial\eta \wedge^{\triangleright} \eta) \,,
\end{aligned}
$$

konačno sledi:

$$
\mathcal{G}'' \;=\; \mathcal{G} + \mathcal{F} \wedge^{\triangleright} \eta \,. \tag{A.34}
$$

Najzad, primenom transformacionih pravila za 3-koneksiju dobijamo da je transformacija krivine $\mathcal{H}$ pri H-gejdž transformacijama

$$
\begin{aligned}
\mathcal{H}'' \;=\;& \mathrm{d}\gamma'' + \alpha'' \wedge^{\triangleright} \gamma'' + \beta'' \wedge^{\{,\}} \beta'' \\
=\;& \mathrm{d}\gamma + \alpha'' \wedge^{\triangleright} \gamma - \mathrm{d}(\beta'' \wedge^{\{,\}} \eta) - \mathrm{d}(\eta \wedge^{\{,\}} \beta) \\
&-\alpha'' \wedge^{\triangleright} (\beta'' \wedge^{\{,\}} \eta) - \alpha'' \wedge^{\triangleright} (\eta \wedge^{\{,\}} \beta) + \beta'' \wedge^{\{,\}} \beta'' \\
=\;& \mathrm{d}\gamma + \alpha'' \wedge^{\triangleright} \gamma - \mathrm{d}\beta'' \wedge^{\{,\}} \eta - \beta'' \wedge^{\{,\}} \mathrm{d}\eta - \mathrm{d}\eta \wedge^{\{,\}} \beta + \eta \wedge^{\{,\}} \mathrm{d}\beta \\
&-(\alpha'' \wedge^{\triangleright} \beta'') \wedge^{\{,\}} \eta - \beta'' \wedge^{\{,\}} (\alpha'' \wedge^{\triangleright} \eta) - (\alpha'' \wedge^{\triangleright} \eta) \wedge^{\{,\}} \beta + \eta \wedge^{\{,\}} (\alpha'' \wedge^{\triangleright} \beta) \\
&+\beta'' \wedge^{\{,\}} \beta'' \\
=\;& \mathrm{d}\gamma + \alpha'' \wedge^{\triangleright} \gamma - (\mathrm{d}\beta'' + \alpha'' \wedge^{\triangleright} \beta'') \wedge^{\{,\}} \eta - \beta'' \wedge^{\{,\}} (\mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta) \\
&-(\mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta) \wedge^{\{,\}} \beta + \eta \wedge^{\{,\}} (\mathrm{d}\beta + \alpha'' \wedge^{\triangleright} \beta) \\
&+\beta'' \wedge^{\{,\}} \beta'' \,,
\end{aligned}
\tag{A.35}
$$

gde je:

$$\beta'' \wedge^{\{,\}} \beta'' = \beta \wedge^{\{,\}} \beta + (\mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta) \wedge^{\{,\}} \beta - (\eta \wedge \eta) \wedge^{\{,\}} \beta + \beta'' \wedge^{\{,\}} (\mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta) - \beta'' \wedge^{\{,\}} (\eta \wedge \eta)\,.$$

Dobijamo da je:

$$\begin{aligned} \mathcal{H}'' \;=\;& \mathcal{H} + \partial\eta \wedge^{\triangleright} \gamma - G'' \wedge^{\{,\}} \eta + \eta \wedge^{\{,\}} G + \eta \wedge^{\{,\}} (\partial\eta \wedge^{\triangleright} \beta) \\ & - \beta'' \wedge^{\{,\}} (\mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta) - (\mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta) \wedge^{\{,\}} \beta + (\mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta) \wedge^{\{,\}} \beta \\ & - (\eta \wedge \eta) \wedge^{\{,\}} \beta + \beta'' \wedge^{\{,\}} (\mathrm{d}\eta + \alpha'' \wedge^{\triangleright} \eta) - \beta'' \wedge^{\{,\}} (\eta \wedge \eta)\,. \end{aligned} \tag{A.36}$$

Određeni članovi se skraćuju, a nakon primene identiteta

$$\beta'' \wedge^{\{,\}} (\eta \wedge \eta) = (\beta'' \wedge^{\langle,\rangle} \eta) \wedge^{\{,\}} \eta\,,$$

$$\begin{aligned} \partial\eta \wedge^{\triangleright} \gamma =& - \eta \wedge^{\{,\}} \delta\gamma + \delta\gamma \wedge^{\{,\}} \eta \\ =& - \eta \wedge^{\{,\}} \delta\gamma + \delta\gamma'' \wedge^{\{,\}} \eta + (\beta'' \wedge^{\langle,\rangle} \eta) \wedge^{\{,\}} \eta + (\eta \wedge^{\langle,\rangle} \beta) \wedge^{\{,\}} \eta\,, \end{aligned}$$

sledi:

$$\begin{aligned} \mathcal{H}'' \;=\;& \mathcal{H} - \mathcal{G}'' \wedge^{\{,\}} \eta + \eta \wedge^{\{,\}} \mathcal{G} \\ & + \eta \wedge^{\{,\}} (\partial\eta \wedge^{\triangleright} \beta) - (\eta \wedge \eta) \wedge^{\{,\}} \beta + (\eta \wedge^{\langle,\rangle} \beta) \wedge^{\{,\}} \eta \\ =\;& \mathcal{H} - \mathcal{G}'' \wedge^{\{,\}} \eta + \eta \wedge^{\{,\}} \mathcal{G}\,. \end{aligned} \tag{A.37}$$

U poslednjem redu primenili smo identitet:

$$\eta \wedge^{\{,\}} (\partial\eta \wedge^{\triangleright} \beta) - (\eta \wedge \eta) \wedge^{\{,\}} \beta + (\eta \wedge^{\langle,\rangle} \beta) \wedge^{\{,\}} \eta = 0\,.$$

Time smo dokazali tvrđenje Teoreme 30. ■

Teorema 31 *Pri L-gejdž transformacijama 3-krivina $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ se transformiše na sledeći način*

$$\mathcal{F} \to \tilde{\mathcal{F}} = \mathcal{F}\,, \qquad \mathcal{G} = \tilde{\mathcal{G}} \to \mathcal{G}\,, \qquad \mathcal{H} \to \tilde{\mathcal{H}} = \mathcal{H} - \mathcal{F} \wedge^{\triangleright} \theta\,, \tag{A.38}$$

gde je $\theta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{l})$ parametar L-gejdž transformacija.

Dokaz. Primenom definicije 3-krivine, dobijamo da je krivina $\mathcal{F}$ invarijantna na L-gejdž transf-o
–rmacije

$$\begin{aligned} \tilde{\mathcal{F}} \;=\;& \mathrm{d}\tilde{\alpha} + \tilde{\alpha} \wedge \tilde{\alpha} - \partial\tilde{\beta} \\ =\;& \mathrm{d}\alpha + \alpha \wedge \alpha - \partial(\beta - \delta(\theta)) \\ =\;& \mathcal{F}\,, \end{aligned} \tag{A.39}$$

gde smo primenili identitet $\partial\delta = 0$. Pri L-gejdž transformacijama krivina $\mathcal{G}$ se transformiše na sledeći način

$$\begin{aligned} \tilde{\mathcal{G}} \;=\;& \mathrm{d}\tilde{\beta} + \tilde{\alpha} \wedge^{\triangleright} \tilde{\beta} - \delta(\tilde{\gamma}) \\ =\;& \mathrm{d}(\beta - \delta(\theta)) + \alpha \wedge^{\triangleright} (\beta - \delta(\theta)) - \delta(\gamma - \mathrm{d}\theta - \alpha \wedge^{\triangleright} \theta) \\ =\;& \mathcal{G} - \mathrm{d}(\delta(\theta)) - \alpha \wedge^{\triangleright} \delta(\theta) + \delta(\mathrm{d}\theta) + \delta(\alpha \wedge^{\triangleright} \theta) \\ =\;& \mathcal{G}\,, \end{aligned} \tag{A.40}$$

gde smo koristili da je $\mathrm{d}(\delta\theta) = \delta(\mathrm{d}\theta)$. Najzad, transformacija krivine $\mathcal{H}$ pri L-gejdž transformacijama je:

$$
\begin{aligned}
\tilde{\mathcal{H}} &= \mathrm{d}\tilde{\gamma} + \tilde{\alpha} \wedge^{\rhd} \tilde{\gamma} + \tilde{\beta} \wedge^{\{,\}} \tilde{\beta} \\
&= \mathrm{d}(\gamma - \mathrm{d}\theta - \alpha \wedge^{\rhd} \theta) + \alpha \wedge^{\rhd} (\gamma - \mathrm{d}\theta - \alpha \wedge^{\rhd} \theta) + (\beta - \delta(\theta)) \wedge^{\{,\}} (\beta - \delta(\theta)) \\
&= \mathcal{H} - \mathrm{d}\alpha \wedge^{\rhd} \theta + \alpha \wedge^{\rhd} \mathrm{d}\theta - \alpha \wedge^{\rhd} \mathrm{d}\theta - \alpha \wedge^{\rhd} (\alpha \wedge^{\rhd} \theta) \\
&\quad -\beta \wedge^{\{,\}} \delta(\theta) - \delta(\theta) \wedge^{\{,\}} \beta + \delta(\theta) \wedge^{\{,\}} \delta(\theta) \\
&= \mathcal{H} - \mathrm{d}\alpha \wedge^{\rhd} \theta - (\alpha \wedge \alpha) \wedge^{\rhd} \theta + \partial(\beta) \wedge^{\rhd} \theta \\
&= \mathcal{H} - \mathcal{F} \wedge^{\rhd} \theta \,,
\end{aligned}
\tag{A.41}
$$

gde smo primenili identitete:

$$
\delta(\theta) \wedge^{\{,\}} \delta(\theta) = 0 \,,
$$
$$
-\beta \wedge^{\{,\}} \delta(\theta) - \delta(\theta) \wedge^{\{,\}} \beta = \partial(\beta) \wedge^{\rhd} \theta \,.
$$

Tvrđenje Teoreme 31 je dokazano. ■

Teorema 32 *Pri* 3*-gejdž transformacijama* 3*-krivina* $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ *se transformiše na sledeći način:*

$$
\begin{aligned}
&\mathcal{F} \to g^{-1} \rhd \mathcal{F} \,, \\
&\mathcal{G} \to g^{-1} \rhd \mathcal{G} + (g^{-1} \rhd \mathcal{F}) \wedge^{\rhd} \eta \,, \\
&\mathcal{H} \to g^{-1} \rhd \mathcal{H} - (g^{-1} \rhd \mathcal{G} + (g^{-1} \rhd \mathcal{F}) \wedge^{\rhd} \eta) \wedge^{\{,\}} \eta - \eta^{\{,\}}(g^{-1} \rhd \mathcal{G}) - (g^{-1} \rhd \mathcal{F}) \wedge^{\rhd} \theta \,,
\end{aligned}
\tag{A.42}
$$

gde su $g : \mathcal{M}_4 \to G$, $\eta \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ *i* $\theta \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{l})$ *redom parametri* G*-,* H*- i* L*-gejdž transformacija.*

Dokaz. Uzatopnom transformacijom 2-krivine najpre G-gejdž transformacijom definisanom u Teoremi 29, zatim H-gejdž transformacijom definisanom u Teoremi 30 i L-gejdž transformacijom definisanom u Teoremi 31 dobijamo rezultat Teoreme 32. ■

A.2.2 3-Gejdž transformacije Lagranževih množitelja

Bilinearno antisimetrično preslikavanje $\mathcal{T} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ se definiše na isti način kao u slučaju $2BF$ dejstva (A.13).

Da bi se definisalo $3BF$ gejdž invarijantno topološko dejstvo potrebno je definisati bilinearno antisimetrično preslikavanje $\mathcal{S} : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{g}$ na sledeći način:

$$
\langle \mathcal{S}(\underline{l}_1, \underline{l}_2), \underline{g} \rangle_{\mathfrak{g}} = -\langle \underline{l}_1, \underline{g} \rhd \underline{l}_2 \rangle_{\mathfrak{l}} \,, \qquad \forall \underline{l}_1, \forall \underline{l}_2 \in \mathfrak{l}, \quad \forall \underline{g} \in \mathfrak{g} \,. \tag{A.43}
$$

Primetimo na osnovu simetričnosti i nedegenerisanosti bilinearne forme $\langle \_ \,, \_ \rangle_{\mathfrak{g}}$ važi:

$$
\langle \underline{l}_1, \underline{g} \rhd \underline{l}_2 \rangle_{\mathfrak{l}} = -\langle \underline{g} \rhd \underline{l}_1, \underline{l}_2 \rangle_{\mathfrak{l}} = -\langle \underline{l}_2, \underline{g} \rhd \underline{l}_1 \rangle_{\mathfrak{l}} \,, \qquad \forall \underline{g} \in \mathfrak{g}, \quad \forall \underline{l}_1, \underline{l}_2 \in \mathfrak{l} \,.
$$

Takođe, za svako $g \in G$ i $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$ važi identitet:

$$
\mathcal{S}(g \rhd \underline{l}_1, g \rhd \underline{l}_2) = g \, \mathcal{S}(\underline{l}_1, \underline{l}_2) \, g^{-1} \,, \tag{A.44}
$$

što se lako može pokazati:

$$\begin{aligned}\langle \underline{g},\, g^{-1}\mathcal{S}(g\rhd \underline{l}_1\,, g\rhd \underline{l}_2)g\rangle_{\mathfrak{g}} &= \langle g\underline{g}g^{-1},\, \mathcal{S}(g\rhd \underline{l}_1,\, g\rhd \underline{l}_2)\rangle_{\mathfrak{g}} \\ &= -\langle (g\,\underline{g}\,g^{-1})\rhd g\rhd \underline{l}_1,\, g\rhd \underline{l}_2\rangle_{\mathfrak{l}} \\ &= -\langle \underline{g}\rhd \underline{l}_1\,, \underline{l}_2\rangle_{\mathfrak{l}} = \langle \underline{g}\,, \mathcal{S}(\underline{l}_1,\, \underline{l}_2)\rangle_{\mathfrak{g}}\,,\end{aligned}$$

Ovde je korišćen identitet:

$$g\rhd(\underline{g}\rhd \underline{l}) = (g\,\underline{g}\,g^{-1})\rhd g\rhd \underline{l}\,.$$

Zadovoljen je sledeći identitet:

$$\mathcal{S}(\underline{g}\rhd \underline{l}_1,\, \underline{l}_2) + \mathcal{S}(\underline{l}_1,\, \underline{g}\rhd \underline{l}_2) = [\underline{g},\, \mathcal{S}(\underline{l}_1,\, \underline{l}_2)]\,.$$

Kada fiksiramo bazis, možemo definisati koeficijent bilinearnog antisimetričnog preslikavanja $\mathcal{S}:\mathfrak{l}\times\mathfrak{l}\to\mathfrak{g}$,

$$\mathcal{S}(T_A, T_B) = \mathcal{S}_{AB}{}^{\alpha}\tau_\alpha\,, \tag{A.45}$$

pa definiciju preslikavanja $\mathcal{S}$ možemo izraziti koristeći ovaj koeficijent na sledeći način:

$$\mathcal{S}_{AB}{}^{\alpha} g_{\alpha\beta} = -\rhd_{\alpha[B}{}^{C} g_{A]C}\,. \tag{A.46}$$

Bilinearno antisimetrično preslikavanje $\mathcal{S}$ dve diferencijalne forme elemenata algebre $\mathfrak{l}$, η i ω, definiše diferencijalnu formu element algebre $\mathfrak{g}$:

$$\omega\wedge^{\mathcal{S}}\eta = \omega^A\wedge\eta^B\mathcal{S}_{AB}{}^{\alpha}\tau_\alpha\,.$$

Sada možemo definisati zakone transformacije Lagranževih množitelja pri L-gejdž transformacijama (A.58).

Transformacije Lagranževih množitelja pri H-gejdž transformacijama definišemo koristeći bilinearno preslikavanje $\mathcal{X}_1:\mathfrak{l}\times\mathfrak{h}\to\mathfrak{h}$ definisano sledećim identitetom

$$\langle \mathcal{X}_1(\underline{l},\, \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}} = -\langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2\in\mathfrak{h}\,,\quad \forall \underline{l}\in\mathfrak{l}\,, \tag{A.47}$$

i bilinearno preslikavanje $\mathcal{X}_2:\mathfrak{l}\times\mathfrak{h}\to\mathfrak{h}$ definisano pravilom:

$$\langle \mathcal{X}_2(\underline{l},\, \underline{h}_2),\, \underline{h}_1\rangle_{\mathfrak{h}} = -\langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2\in\mathfrak{h}\,,\quad \forall \underline{l}\in\mathfrak{l}\,. \tag{A.48}$$

Fiksiranjem bazisa možemo definisati koeficijente bilinearnih preslikavanja $\mathcal{X}_1$ i $\mathcal{X}_2$:

$$\mathcal{X}_1(T_A, t_a) = \mathcal{X}_{1Aa}{}^{b}\, t_b\,, \qquad \mathcal{X}_2(T_A, t_a) = \mathcal{X}_{2Aa}{}^{b}\, t_b\,. \tag{A.49}$$

Definicije preslikavanja $\mathcal{X}_1$ i $\mathcal{X}_2$ zapisane u bazisu postaju:

$$\mathcal{X}_{1Ab}{}^{c} g_{ac} = -X_{ba}{}^{B} g_{AB}\,, \qquad \mathcal{X}_{2Ab}{}^{c} g_{ac} = -X_{ab}{}^{B} g_{AB}\,. \tag{A.50}$$

Za dve diferencijalne forme, element algebre $\mathfrak{l}$ formu ω i element algebre $\mathfrak{h}$ formu η, definiše se diferencijalna forma element algebre $\mathfrak{h}$ na sledeći način:

$$\omega\wedge^{\mathcal{X}_1}\eta = \omega^A\wedge\eta^a\mathcal{X}_{1Aa}{}^{b}t_b\,, \qquad \omega\wedge^{\mathcal{X}_2}\eta = \omega^A\wedge\eta^a\mathcal{X}_{2Aa}{}^{b}t_b\,.$$

Za sve $g\in G$, $\underline{l}\in\mathfrak{l}$ i $\underline{h}\in\mathfrak{h}$, važi:

$$\mathcal{X}_1(g\rhd\underline{l},\, g^{-1}\rhd\underline{h}) = g\rhd\mathcal{X}_1(\underline{l},\, \underline{h})\,, \qquad \mathcal{X}_2(g\rhd\underline{l},\, g\rhd\underline{h}) = g^{-1}\rhd\mathcal{X}_2(\underline{l},\, \underline{h})\,,$$

što sledi na osnovu pravila da za svako $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ i $\underline{l} \in \mathfrak{l}$ važi:

$$
\begin{aligned}
\langle \underline{h}_2,\, g^{-1} \rhd \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{h}_1)\rangle_{\mathfrak{h}} &= \langle g \rhd \underline{h}_2,\, \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{h}_1)\rangle_{\mathfrak{h}} \\
&= \langle g \rhd \underline{l},\, \{g \rhd \underline{h}_1,\, g \rhd \underline{h}_2\}\rangle_{\mathfrak{l}} \\
&= \langle g \rhd \underline{l},\, g \rhd \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}} \\
&= \langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}} \\
&= \langle \underline{h}_2,\, \mathcal{X}_1(l,\, \underline{h}_1)\rangle_{\mathfrak{h}}\,,
\end{aligned}
$$

i slično za $\mathcal{X}_2$.

Najzad, potrebno je definisati trilinearno preslikavanje $\mathcal{D} : \mathfrak{h} \times \mathfrak{h} \times \mathfrak{l} \to \mathfrak{g}$ sledećim pravilom:

$$
\langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l}),\, \underline{g}\rangle_{\mathfrak{g}} = -\langle \underline{l},\, \{\underline{g} \rhd \underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}, \quad \forall \underline{g} \in \mathfrak{g}\,, \tag{A.51}
$$

Koeficijenti trilinearnog preslikavanja se definišu kao:

$$
\mathcal{D}(t_a,\, t_b,\, T_A) = \mathcal{D}_{abA}{}^{\alpha}\tau_\alpha\,, \tag{A.52}
$$

pa iz definicije preslikavanja $\mathcal{D}$ sledi da ga možemo zapisati i preko koeficijenata:

$$
\mathcal{D}_{abA}{}^{\beta} g_{\alpha\beta} = -\rhd_{\alpha a}{}^{c} X_{cb}{}^{B} g_{AB}\,. \tag{A.53}
$$

Za dve diferencijalne forme, elemente algebre $\mathfrak{h}$, ω i η, i diferencijalnu formu ξ element algebre $\mathfrak{l}$, definiše se diferencijalna forma element algebre g:

$$
\omega \wedge^{\mathcal{D}} \eta \wedge^{\mathcal{D}} \xi = \omega^a \wedge \eta^b \wedge \xi^A \mathcal{D}_{abA}{}^{\beta}\tau_\beta\,.
$$

Važe relacije kompatibilsnosti između preslikavanja $\mathcal{X}_1$ i $\mathcal{D}$:

$$
\langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l}),\, \underline{g}\rangle_{\mathfrak{g}} = \langle \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}, \quad \forall \underline{g} \in \mathfrak{g}\,. \tag{A.54}
$$

Takođe, može se pokazati da za svako $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$ i $g \in G$ važi:

$$
\mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l}) = g\, \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})\, g^{-1}\,, \tag{A.55}
$$

što je posledica toga da za svako $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$, $\underline{g} \in \mathfrak{g}$ i $g \in G$ možemo pisati:

$$
\begin{aligned}
\langle g^{-1}\mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l})g,\, \underline{g}\rangle_{\mathfrak{g}} &= \langle \mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l}),\, g\underline{g}g^{-1}\rangle_{\mathfrak{g}} \\
&= \langle \mathcal{X}_1(g \rhd \underline{l},\, g\underline{g}g^{-1} \rhd g \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{g} \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle g \rhd \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})\,,\underline{g}\rangle_{\mathfrak{g}}\,,
\end{aligned}
$$

gde su korišćene relacije (A.2.2) i relacije kompatibilnosti (A.54). Sledi da za sve $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$ i $\underline{g} \in \mathfrak{g}$ važi sledeći identitet:

$$
\mathcal{D}(\underline{g} \rhd \underline{h}_1,\, \underline{h}_2,\, \underline{l}) + \mathcal{D}(\underline{h}_1,\, \underline{g} \rhd \underline{h}_2,\, \underline{l}) + \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{g} \rhd \underline{l}) = [\underline{g},\, \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})]\,.
$$

Ovim su definisana sva preslikavanja neophodna da se definišu zakoni transformacije Langranževih množitelja pri H-gejdž tranformacijama.

Kako bi dejstvo (6.1) bilo gejdž invarijantno pri transformacijama krivina (2.128), (2.129) i (2.130), Lagranževi množitelji B, C i D moraju se pri G-gejdž transformacijama transformisati na sledeći način

$$B \to g^{-1} B g \,, \qquad C \to g^{-1} \rhd C \,, \qquad D \to g^{-1} \rhd D \,, \tag{A.56}$$

pri H-gejdž transformacijama

$$B \to B + C' \wedge^{\mathcal{T}} \eta - \eta \wedge^{\mathcal{D}} \eta \wedge^{\mathcal{D}} D \,, \qquad C \to C + D \wedge^{\mathcal{X}_1} \eta + D \wedge^{\mathcal{X}_2} \eta \,, \qquad D \to D \,, \tag{A.57}$$

i najzad pri L-gejdž transformacijama na sledeći način:

$$B \to B - D \wedge^{\mathcal{S}} \theta \,, \qquad C \to C \,, \qquad D \to D \,. \tag{A.58}$$

Preslikavanja $\mathcal{T}$, $\mathcal{D}$, $\mathcal{X}_1$, $\mathcal{X}_2$ i $\mathcal{S}$ definisana su jednačinama (A.13), (A.43), (A.47), (A.48) i (A.51).

Dodatak B

Jednačine kretanja za $3BF$ dejstvo sa vezama za Vajlovo i Majorana polje kuplovano sa Ajnštajn-Kartanovom gravitacijom

Dejstvo za Vajlovo spinorsko polje kuplovano sa Ajnštajn-Kartanovom gravitacijom dato je izrazom (6.151). Variranjem ovog dejstva redom po varijablama B_{ab}, λ^{ab}, γ_α, $\bar{\gamma}^{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, e^a, β^a i ω^{ab} dobijaju se jednačine kretanja:

$$R^{ab} - \lambda^{ab} = 0\,,$$

$$B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd}e^c \wedge e^d = 0\,,$$

$$\nabla\psi_\alpha + \lambda_\alpha = 0\,,$$

$$\nabla\bar{\psi}^{\dot{\alpha}} + \bar{\lambda}^{\dot{\alpha}} = 0\,,$$

$$-\gamma_\alpha + \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}} = 0\,,$$

$$-\bar{\gamma}^{\dot{\alpha}} + \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta = 0\,,$$

$$\nabla\gamma_\alpha - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}}\bar{\lambda}_L{}^{\dot{\beta}} = 0\,,$$

$$\nabla\bar{\gamma}^{\dot{\alpha}} - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta}\lambda_{L\beta} = 0\,,$$

$$\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{i}{2}\varepsilon_{abcd}e^b \wedge e^c \wedge (\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta + \lambda^\alpha\sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}) - 8\pi i l_p^2\varepsilon_{abcd}e^b\beta^c\big(\psi^\alpha(\sigma^d)_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}\big) = 0\,,$$

$$\nabla e_a - 4\pi l_p^2\varepsilon_{abcd}e^b \wedge e^c \wedge (\bar{\psi}_{\dot{\alpha}}\bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta) = 0\,,$$

$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} - \frac{1}{2}\gamma\sigma^{ab}{}_\alpha{}^\beta\psi_\beta - \frac{1}{2}\bar{\gamma}_{\dot{\alpha}}\bar{\sigma}^{ab\dot{\alpha}}{}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}} = 0\,.$$

U slučaju $3BF$ dejstva sa vezama koje odgovara teoriji Majorana spinorskog polja kuplovanog sa Ajnštajn-Kartanovom gravitacijom, dodajemo maseni član (6.154) dejstvu (6.151). Variranjem ovako dobijenog dejstva redom po varijablama to B_{ab}, ψ^{ab}, γ^α, $\bar{\gamma}_{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}_I$, e^a, β^a i ω_{ab}

dobijamo jednačine kretanja:

$$
\begin{aligned}
&R^{ab} - \lambda^{ab} = 0\,,\\
&B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd}e^c \wedge e^d = 0\,,\\
&-\nabla\psi_\alpha + \lambda_\alpha = 0\,,\\
&-\nabla\bar{\psi}^{\dot{\alpha}} + \lambda^{\dot{\alpha}} = 0\,,\\
&\gamma^\alpha - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \bar{\psi}_{\dot{\beta}}(\bar{\sigma}^d)^{\dot{\beta}\alpha} = 0\,,\\
&\bar{\gamma}_{\dot{\alpha}} - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \psi^\beta(\sigma^d)_{\beta\dot{\alpha}} = 0\,,\\
&\nabla\gamma^\alpha + \frac{i}{6}\varepsilon_{abcd}\lambda^{\dot{\beta}} \wedge e^a \wedge e^b \wedge e^c (\sigma^d)^\alpha{}_{\dot{\beta}} - \frac{1}{6}\, m\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d \psi^\alpha\\
&\qquad - 4i\pi l_p^2 \varepsilon_{abcd}e^a \wedge e^b \wedge \beta^c \bar{\psi}_{\dot{\beta}}(\bar{\sigma}^d)^{\dot{\beta}\alpha} = 0\,,\\
&\nabla\bar{\gamma}_{\dot{\alpha}} + \frac{i}{6}\varepsilon_{abcd}\lambda_\beta \wedge e^a \wedge e^b \wedge e^c (\bar{\sigma}^d)_{\dot{\alpha}}{}^\beta - \frac{1}{6}m\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d \psi_{\dot{\alpha}}\\
&\qquad - 4i\pi l_p^2 \varepsilon_{abcd}e^a \wedge e^b \wedge \beta^c \psi^\beta(\sigma^d)_{\beta\dot{\alpha}} = 0\,,\\
&\nabla\beta^a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{i}{2}\varepsilon_{abcd}\lambda_\alpha \wedge e^b \wedge e^c \bar{\psi}_{\dot{\beta}}(\bar{\sigma}^d)^{\dot{\beta}\alpha} + \frac{i}{2}\varepsilon_{abcd}\lambda^{\dot{\alpha}} \wedge e^b \wedge e^c \psi^\beta(\sigma^d)_{\beta\dot{\alpha}}\\
&\qquad - \frac{1}{3}m\varepsilon_{abcd}e^b \wedge e^c \wedge e^d(\psi^\alpha\psi_\alpha + \bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}) - 8\pi i l_p^2 \varepsilon_{abcd}e^b\beta^c\big(\psi^\alpha(\sigma^d)_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}\big) = 0\,,\\
&\nabla e_a - 4i\pi l_p^2 \varepsilon_{abcd}e^b \wedge e^c\big(\psi^\alpha(\sigma^d)_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}\big) = 0\,,\\
&\nabla B_{ab} - e_{[a} \wedge \beta_{b]} - \frac{1}{2}\psi^\alpha(\sigma^{ab})_\alpha{}^\beta\gamma_\beta - \frac{1}{2}\bar{\psi}_{\dot{\alpha}}(\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\gamma}^{\dot{\beta}} = 0\,.
\end{aligned}
$$

Dodatak C

Hamiltonova analiza skalarne elektrodinamike

U ovom poglavlju urađena je kompletna Hamiltonova analiza za dejstvo (6.174).

Pretpostavljajući da je prostorvremenska mnogostrukost globalno hiperbolička, $\mathcal{M}_4 = \mathbb{R} \times \Sigma_3$, Lagranžijan za dejstvo (6.174) ima oblik:

$$L_{3BF} = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \epsilon^{\mu\nu\rho\sigma} \big(\frac{1}{4} B^{ab}{}_{\mu\nu}\, R^{cd}{}_{\rho\sigma}\, g_{ab,\,cd} + \frac{1}{4} B_{\mu\nu} F_{\rho\sigma} + \frac{1}{3!}\, e^a{}_\mu\, \mathcal{G}^b{}_{\nu\rho\sigma}\, g_{ab} + \frac{1}{4!}\, \phi^A \mathcal{H}^B{}_{\mu\nu\rho\sigma}\, g_{AB} \big) \,. \tag{C.1}$$

Kanonski impulsi varijabli $B^{ab}{}_{\mu\nu}$, $\omega^{ab}{}_\mu$, $B_{\mu\nu}$, A_μ, $e^a{}_\mu$, $\beta^a{}_{\mu\nu}$, ϕ^A and $\gamma^A{}_{\mu\nu\rho}$ su:

$$\begin{array}{llllllll}
\pi(B)_{ab}{}^{\mu\nu} & = & \dfrac{\delta L}{\delta \partial_0 B^{ab}{}_{\mu\nu}} & = & 0\,, & \pi(\omega)_{ab}{}^{\mu} & = & \dfrac{\delta L}{\delta \partial_0 \omega^{ab}{}_{\mu}} & = & \epsilon^{0\mu\nu\rho} B_{ab\nu\rho}\,, \\
\pi(B)^{\mu\nu} & = & \dfrac{\delta L}{\delta \partial_0 B_{\mu\nu}} & = & 0\,, & \pi(A)^{\mu} & = & \dfrac{\delta L}{\delta \partial_0 A_{\mu}} & = & \dfrac{1}{2}\epsilon^{0\mu\nu\rho} B_{\nu\rho}\,, \\
\pi(e)_{a}{}^{\mu} & = & \dfrac{\delta L}{\delta \partial_0 e^{a}{}_{\mu}} & = & 0\,, & \pi(\beta)_{a}{}^{\mu\nu} & = & \dfrac{\delta L}{\delta \partial_0 \beta^{a}{}_{\mu\nu}} & = & -\epsilon^{0\mu\nu\rho} e_{a\rho}\,, \\
\pi(\phi)_{A} & = & \dfrac{\delta L}{\delta \partial_0 \phi^{A}} & = & 0\,, & \pi(\gamma)_{A}{}^{\mu\nu\rho} & = & \dfrac{\delta L}{\delta \partial_0 \gamma^{A}{}_{\mu\nu\rho}} & = & \epsilon^{0\mu\nu\rho} \phi_{A}\,.
\end{array} \tag{C.2}$$

Primarne veze u teoriji su:

$$\begin{array}{llllllllll}
P(B)_{ab}{}^{\mu\nu} & \equiv & \pi(B)_{ab}{}^{\mu\nu} & \approx & 0\,, & P(\omega)_{ab}{}^{\mu} & \equiv & \pi(\omega)_{ab}{}^{\mu} - \epsilon^{0\mu\nu\rho} B_{ab\nu\rho} & \approx & 0\,, \\
P(B)^{\mu\nu} & \equiv & \pi(B)^{\mu\nu} & \approx & 0\,, & P(A)^{\mu} & = & \pi(A)^{\mu} - \dfrac{1}{2}\epsilon^{0\mu\nu\rho} B_{\nu\rho} & \approx & 0\,, \\
P(e)_{a}{}^{\mu} & \equiv & \pi(e)_{a}{}^{\mu} & \approx & 0\,, & P(\beta)_{a}{}^{\mu\nu} & \equiv & \pi(\beta)_{a}{}^{\mu\nu} + \epsilon^{0\mu\nu\rho} e_{a\rho} & \approx & 0\,, \\
P(\phi)_{A} & \equiv & \pi(\phi)_{A} & \approx & 0\,, & P(\gamma)_{A}{}^{\mu\nu\rho} & \equiv & \pi(\gamma)_{A}{}^{\mu\nu\rho} - \epsilon^{0\mu\nu\rho} \phi_{A} & \approx & 0\,.
\end{array} \tag{C.3}$$

Fundamentalne Poasonove zagrade definišemo na sledeći način:

$$
\begin{aligned}
\{ B^{ab}{}_{\mu\nu}(x) \,,\, \pi(B)_{cd}{}^{\rho\sigma}(y) \} &= 4\delta^a{}_{[c}\delta^b{}_{d]}\delta^\rho{}_{[\mu}\delta^\sigma{}_{\nu]}\, \delta^{(3)}(\vec{x}-\vec{y}) \,, \\
\{ \omega^{ab}{}_{\mu}(x) \,,\, \pi(\omega)_{cd}{}^{\nu}(y) \} &= 2\delta^a{}_{[c}\delta^b{}_{d]}\delta^\nu{}_{\mu}\, \delta^{(3)}(\vec{x}-\vec{y}) \,, \\
\{ B_{\mu\nu}(x) \,,\, \pi(B)^{\rho\sigma}(y) \} &= 2\delta^\rho{}_{[\mu}\delta^\sigma{}_{\nu]}\, \delta^{(3)}(\vec{x}-\vec{y}) \,, \\
\{ A_{\mu}(x) \,,\, \pi(A)^{\nu}(y) \} &= \delta^\nu{}_{\mu}\, \delta^{(3)}(\vec{x}-\vec{y}) \,, \\
\{ e^{a}{}_{\mu}(x) \,,\, \pi(e)_{b}{}^{\nu}(y) \} &= \delta^a{}_{b}\delta^\nu{}_{\mu}\, \delta^{(3)}(\vec{x}-\vec{y}) \,, \\
\{ \beta^{a}{}_{\mu\nu}(x) \,,\, \pi(\beta)_{b}{}^{\rho\sigma}(y) \} &= 2\delta^a{}_{b}\, \delta^\rho{}_{[\mu}\delta^\sigma{}_{\nu]}\, \delta^{(3)}(\vec{x}-\vec{y}) \,, \\
\{ \phi^{A}(x) \,,\, \pi(\phi)_{B}(y) \} &= \delta^A{}_{B}\, \delta^{(3)}(\vec{x}-\vec{y}) \,, \\
\{ \gamma^{A}{}_{\mu\nu\rho}(x) \,,\, \pi(\gamma)_{B}{}^{\alpha\beta\gamma}(y) \} &= 3!\delta^A{}_{B}\, \delta^\alpha{}_{[\mu}\delta^\beta{}_{\nu}\delta^\gamma{}_{\rho]}\, \delta^{(3)}(\vec{x}-\vec{y}) \,.
\end{aligned}
\tag{C.4}
$$

Koristeći fundamentalne Poasonove zagrade, nalazimo *algebru primarnih veza*,

$$
\begin{aligned}
\{ P(B)^{abjk}(x) \,,\, P(\omega)_{cd}{}^{i}(y) \} &= 4\epsilon^{0ijk}\, \delta^a{}_{[c}\delta^b{}_{d]}\, \delta^{(3)}(\vec{x}-\vec{y}) \,, \\
\{ P(B)^{jk}(x) \,,\, P(A)^{i}(y) \} &= \epsilon^{0ijk}\, \delta^{(3)}(\vec{x}-\vec{y}) \,, \\
\{ P(e)^{ak} \,,\, P(\beta)_{b}{}^{ij}(y) \} &= -\epsilon^{0ijk}\, \delta^a{}_{b}(x)\, \delta^{(3)}(\vec{x}-\vec{y}) \,, \\
\{ P(\phi)^{A}(x) \,,\, P(\gamma)_{B}{}^{ijk}(y) \} &= \epsilon^{0ijk}\, \delta^A{}_{B}\, \delta^{(3)}(\vec{x}-\vec{y}) \,,
\end{aligned}
\tag{C.5}
$$

dok su sve ostale Poasonove zagrade identički jednake nuli. Kanonski *on-shell* Hamiltonijan ima oblik:

$$
\begin{aligned}
H_c = \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big[& \frac{1}{4}\pi(B)_{ab}{}^{\mu\nu}\, \partial_0 B^{ab}{}_{\mu\nu} + \frac{1}{2}\pi(\omega)_{ab}{}^{\mu}\, \partial_0\omega^{ab}{}_{\mu} + \frac{1}{2}\pi(B)^{\mu\nu}\, \partial_0 B_{\mu\nu} + \pi(A)^{\mu}\, \partial_0 A_{\mu} \\
& + \pi(e)_{a}{}^{\mu}\, \partial_0 e^{a}{}_{\mu} + \frac{1}{2}\pi(\beta)_{a}{}^{\mu\nu}\, \partial_0\beta^{a}{}_{\mu\nu} + \pi(\phi)_{A}\, \partial_0 D^{A} + \frac{1}{3!}\pi(\gamma)_{A}{}^{\mu\nu\rho}\, \partial_0\gamma^{A}{}_{\mu\nu\rho} \Big] - L \,.
\end{aligned}
\tag{C.6}
$$

Prepisivanjem (C.6) tako da su brzine pomnožene sa vezama prve klase, koje su *on-shell* jednake nuli, dobijamo:

$$
\begin{aligned}
H_c \approx -\int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \epsilon^{0ijk} \Big[& \frac{1}{2}B_{ab0i}\, R^{ab}{}_{jk} + \frac{1}{2}B_{0i}F_{jk} + \frac{1}{6}e_{a0}\, \mathcal{G}^{a}{}_{ijk} + \beta^{a}{}_{0i}\nabla_j e_{ak} \\
& + \frac{1}{2}\omega^{ab}{}_{0}\Big(\nabla_i B_{ab\,jk} - e_{[a|i}\,\beta_{|b]jk} \Big) + \frac{1}{2}A_0\Big(\partial_i B_{jk} + \frac{1}{3}\,\phi_A\, \rhd_B{}^{A}\, \gamma^{B}{}_{ijk} \Big) + \frac{1}{2}\gamma^{A}{}_{0ij}\nabla_k\phi_A \Big] \,.
\end{aligned}
\tag{C.7}
$$

Dodavanjem Lagranževog množitelja λ za svaku primarnu vezu dobijamo totalni *off-shell* Hamiltonijan:

$$
\begin{aligned}
H_T \;=\; H_c + \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big[& \tfrac{1}{4}\lambda(B)^{ab}{}_{\mu\nu}P(B)_{ab}{}^{\mu\nu} + \tfrac{1}{2}\lambda(\omega)^{ab}{}_{\mu}P(\omega)_{ab}{}^{\mu} + \tfrac{1}{2}\lambda(B)_{\mu\nu}P(B)^{\mu\nu} + \lambda(A)_{\mu}P(A)^{\mu} \\
& + \lambda(e)^{a}{}_{\mu}P(e)_{a}{}^{\mu} + \tfrac{1}{2}\lambda(\beta)^{a}{}_{\mu\nu}P(\beta)_{a}{}^{\mu\nu} + \lambda(\phi)^{A}P(\phi)_{A} + \tfrac{1}{3!}\lambda(\gamma)^{A}{}_{\mu\nu\rho}P(\gamma)_{A}{}^{\mu\nu\rho} \Big] \,.
\end{aligned}
\tag{C.8}
$$

Uslov konzistentnosti primarnih veza (3.26) za primarne veze $P(B)_{ab}{}^{0i}$, $P(\omega)_{ab}{}^{0}$, $P(B)^{0i}$, $P(A)^0$, $P(e)_a{}^0$, $P(\beta)_a{}^{0i}$ i $P(\gamma)_A{}^{0ij}$,

$$\begin{array}{cccc} \dot{P}(B)_{ab}{}^{0i} \approx 0\,, & \dot{P}(\omega)_{ab}{}^{0} \approx 0\,, & \dot{P}(B)^{0i} \approx 0\,, & \dot{P}(A)^0 \approx 0\,, \\ \dot{P}(e)_a{}^0 \approx 0\,, & \dot{P}(\beta)_a{}^{0i} \approx 0\,, & \dot{P}(\gamma)_A{}^{0ij} \approx 0\,, & \end{array} \tag{C.9}$$

dovodi do pojave *sekundarnih veza* $\mathcal{S}$ u teoriji,

$$\begin{array}{lcllll} \mathcal{S}(R)_{ab}{}^{i} & \equiv & \epsilon^{0ijk} R_{abjk} & \approx & 0\,, \\ \mathcal{S}(\nabla B)_{ab} & \equiv & \epsilon^{0ijk}\big(\nabla_i B_{ab\,jk} - e_{[a|i}\,\beta_{|b]\,jk}\big) & \approx & 0\,, \\ \mathcal{S}(F)^{i} & \equiv & \dfrac{1}{2}\epsilon^{0ijk} F_{jk} & \approx & 0\,, \\ \mathcal{S}(\nabla B) & \equiv & \dfrac{1}{2}\epsilon^{0ijk}\big(\partial_i B_{jk} + \dfrac{1}{3}\,\phi_A \rhd_B{}^A \gamma^B{}_{ijk}\big) & \approx & 0\,, \\ \mathcal{S}(\mathcal{G})_a & \equiv & \dfrac{1}{6}\epsilon^{0ijk}\mathcal{G}_{aijk} & \approx & 0\,, \\ \mathcal{S}(\nabla e)_a{}^{i} & \equiv & \epsilon^{0ijk}\nabla_j e_{ak} & \approx & 0\,, \\ \mathcal{S}(\nabla \phi)_A{}^{ij} & \equiv & \epsilon^{0ijk}\nabla_k \phi_A & \approx & 0\,, \end{array} \tag{C.10}$$

dok u slučaju primarnih veza $P(B)_{ab}{}^{jk}$, $P(\omega)_{ab}{}^{k}$, $P(B)^{jk}$, $P(A)^k$, $P(e)_a{}^k$, $P(\beta)_a{}^{jk}$, $P(\phi)_A$ i $P(\gamma)_A{}^{ijk}$ uslovi konzistentnosti,

$$\begin{array}{cccc} \dot{P}(B)_{ab}{}^{jk} \approx 0\,, & \dot{P}(\omega)_{ab}{}^{k} \approx 0\,, & \dot{P}(B)^{jk} \approx 0\,, & \dot{P}(A)^k \approx 0\,, \\ \dot{P}(e)_a{}^k \approx 0\,, & \dot{P}(\beta)_a{}^{jk} \approx 0\,, & \dot{P}(\phi)_A \approx 0\,, & \dot{P}(\gamma)_A{}^{ijk} \approx 0\,, \end{array} \tag{C.11}$$

određuju Lagranževe množitelje:

$$\begin{array}{lcl} \lambda(\omega)_{ab}{}^{i} & \approx & \nabla^i\,\omega_{ab\,0}\,, \\ \lambda(B)_{ab}{}^{ij} & \approx & 2\nabla^{[i|}B_{ab}{}^{0|j]} + e_{[a|\,0}\beta_{|b]}{}^{ij} - 2e_{[a|}{}^{[i|}\beta_{|b]}{}^{0|j]} + 2\omega_{[a|}{}^{c}B_{|b]}{}^{c\,ij}\,, \\ \lambda(A)^{i} & \approx & \partial^i\,A_0\,, \\ \lambda(B)^{ij} & \approx & 2\partial^{[i|}\,B^{0|j]} + \gamma_A{}^{0ij} \rhd_B{}^{A}\,\phi^B\,, \\ \lambda(\beta)_a{}^{ij} & \approx & 2\nabla^{[i|}\,\beta_a{}^{0|j]} - \omega_{ab}{}^{0}\,\beta^{b\,ij}\,, \\ \lambda(e)_a{}^{i} & \approx & \nabla^i\,e_a{}^0 - \omega_a{}^{b\,0}\,e_b{}^i\,, \\ \lambda(\gamma)_A{}^{ijk} & \approx & -A^0 \rhd_A{}^B\,\gamma_B{}^{ijk} + \nabla^i\gamma_A{}^{0jk} - \nabla^j\gamma_A{}^{0ik} + \nabla^k\gamma_A{}^{0ij}\,. \\ \lambda(\phi)^{A} & \approx & A^0 \rhd^A{}_B\,\phi^B\,. \end{array} \tag{C.12}$$

Primetimo da Lagranževi množitelji

$$\lambda(B)^{ab}{}_{0i}\,, \qquad \lambda(\omega)^{ab}{}_{0}\,, \qquad \lambda(B)_{0i}\,, \qquad \lambda(A)_0\,, \qquad \lambda(e)^a{}_0\,, \qquad \lambda(\beta)^a{}_{0i}\,, \qquad \lambda(\gamma)^A{}_{0ij} \tag{C.13}$$

ostaju neodređeni. Uslovi konzistentnosti sekundarnih veza ne dovode do pojave novih veza u teoriji tj. može se pokazati da važi:

$$
\begin{aligned}
\dot{\mathcal{S}}(R)^{abi} &= \{\mathcal{S}(R)^{abi}\,,\,H_T\} &&= \omega^{[a|}{}_{c0}\,\mathcal{S}(R)^{c|b]i}\,,\\
\dot{\mathcal{S}}(\nabla B)_{ab} &= \{\mathcal{S}(\nabla B)_{ab}\,,\,H_T\} &&= \mathcal{S}(R)_{[a|c}{}^{k}\,B^{c}{}_{|b]0k} + \omega_{[a|}{}^{c}{}_{0}\mathcal{S}(\nabla B)_{|b]c}\\
&&& \quad -\beta_{[a|0k}\,\mathcal{S}(\nabla e)_{|b]}{}^{k} + e_{[a|0}\,\mathcal{S}(\mathcal{G})_{|b]}\,,\\
\dot{\mathcal{S}}(F)^{i} &= \{\mathcal{S}(F)^{i}\,,\,H_T\} &&= 0\,,\\
\dot{\mathcal{S}}(\nabla B) &= \{\mathcal{S}(\nabla B),\,H_T\} &&= -\rhd_{B}{}^{A}\,\gamma^{B}{}_{0ij}\,\mathcal{S}(\nabla\phi)_{A}{}^{ij}\,,\\
\dot{\mathcal{S}}(\mathcal{G})^{a} &= \{\mathcal{S}(\mathcal{G})^{a}\,,\,H_T\} &&= \beta_{b0k}\,\mathcal{S}(R)^{abk} - \omega^{ab}{}_{0}\,\mathcal{S}(\mathcal{G})_{b}\,,\\
\dot{\mathcal{S}}(\nabla e)_{a}{}^{i} &= \{\mathcal{S}(\nabla e)_{a}{}^{i}\,,\,H_T\} &&= e^{b}{}_{0}\,\mathcal{S}(R)_{ab}{}^{i} - \omega_{a}{}^{b}{}_{0}\,\mathcal{S}(\nabla e)_{b}{}^{i}\,,\\
\dot{\mathcal{S}}(\nabla\phi)_{A}{}^{ij} &= \{\mathcal{S}(\nabla\phi)_{A}{}^{ij}\,,\,H_T\} &&= A_0\,\rhd_{A}{}^{B}\mathcal{S}(\nabla\phi)_{B}{}^{ij}\,.
\end{aligned}
\tag{C.14}
$$

Totalni Hamiltonijan se može napisati u sledećem obliku:

$$
\begin{aligned}
H_T = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\,\Big[& \frac{1}{2}\lambda(B)_{ab}{}^{0i}\,\Phi(B)^{ab}{}_{i} + \frac{1}{2}\lambda(\omega)_{ab}{}^{0}\,\Phi(\omega)^{ab} + \lambda(B)^{0i}\,\Phi(B)_{i} + \lambda(A)^{0}\,\Phi(A)\\
& + \lambda(e)_{a}{}^{0}\,\Phi(e)^{a} + \lambda(\beta)_{a}{}^{0i}\,\Phi(\beta)^{a}{}_{i} + \frac{1}{2}\lambda(\gamma)_{A}{}^{0ij}\Phi(\gamma)^{A}{}_{ij}\\
& - \frac{1}{2}B_{ab0i}\,\Phi(R)^{abi} - \frac{1}{2}\omega_{ab0}\,\Phi(\nabla B)^{ab} - B_{0i}\,\Phi(F)^{i} - A_0\,\Phi(\nabla B)\\
& - e_{a0}\,\Phi(\mathcal{G})^{a} - \beta_{a0i}\,\Phi(\nabla e)^{ai} - \frac{1}{2}\gamma_{A0ij}\,\Phi(\nabla\phi)^{Aij}\Big]\,,
\end{aligned}
\tag{C.15}
$$

gde su

$$
\begin{aligned}
\Phi(B)^{ab}{}_{i} &= P(B)^{ab}{}_{0i}\,, \\
\Phi(\omega)^{ab} &= P(\omega)^{ab}{}_{0}\,, \\
\Phi(B)_{i} &= P(B)_{0i}\,, \\
\Phi(A) &= P(A)_{0}\,, \\
\Phi(e)^{a} &= P(e)^{a}{}_{0}\,, \\
\Phi(\beta)^{a}{}_{i} &= P(\beta)^{a}{}_{0i}\,, \\
\Phi(\gamma)^{A}{}_{ij} &= P(\gamma)^{A}{}_{0ij}\,, \\
\Phi(R)^{abi} &= \mathcal{S}(R)^{abi} - \nabla_{j}P(B)^{ab\,ij}\,, \\
\Phi(\nabla B)^{ab} &= \mathcal{S}(\nabla B)^{ab} + \nabla_{i}P(\omega)^{abi} + B^{[a|}{}_{c\,ij}\,P(B)^{c|b]\,ij} - 2e^{[a|}{}_{i}\,P(e)^{|b]\,i} - \beta^{[a|}{}_{ij}\,P(\beta)^{|b]\,ij}\,, \\
\Phi(F)^{i} &= \mathcal{S}(F)^{i} - \partial_{j}P(B)^{ij}\,, \\
\Phi(\nabla B) &= \mathcal{S}(\nabla B) + \partial_{i}P(A)^{i} + \frac{1}{3!}\,\gamma^{A}{}_{ijk}\,\rhd_{A}{}^{B}\,P(\gamma)_{B}{}^{ijk} - \phi_{A}\,\rhd_{B}{}^{A}\,P(\phi)^{B} \\
\Phi(\mathcal{G})^{a} &= \mathcal{S}(\mathcal{G})^{a} + \nabla_{i}P(e)^{a\,i} - \frac{1}{4}\,\beta_{b\,ij}\,P(B)^{ab\,ij}\,, \\
\Phi(\nabla e)^{a\,i} &= \mathcal{S}(\nabla e)^{a\,i} - \nabla_{j}P(\beta)^{a\,ij} + \frac{1}{2}\,e_{b\,j}\,P(B)^{ab\,ij} \\
\Phi(\nabla\phi)^{A\,ij} &= \mathcal{S}(\nabla\phi)^{A\,ij} + \nabla_{k}P(\gamma)^{A\,ijk} - \rhd_{B}{}^{A}\,\phi^{B}\,P(B)^{ij}\,,
\end{aligned}
\tag{C.16}
$$

veze prve klase, dok su *veze druge klase*:

$$
\begin{gathered}
\chi(B)_{ab}{}^{jk} = P(B)_{ab}{}^{jk}\,, \qquad \chi(B)^{jk} = P(B)^{jk}\,, \qquad \chi(e)_{a}{}^{i} = P(e)_{a}{}^{i}\,, \qquad \chi(\phi)_{A} = P(\phi)_{A}\,, \\
\chi(\omega)_{ab}{}^{i} = P(\omega)_{ab}{}^{i}\,, \qquad \chi(A)^{i} = P(A)^{i}\,, \qquad \chi(\beta)_{a}{}^{ij} = P(\beta)_{a}{}^{ij}\,, \qquad \chi(\gamma)_{A}{}^{ijk} = P(\gamma)_{A}{}^{ijk}\,.
\end{gathered}
\tag{C.17}
$$

Poasonova algebra veza prve klase je:

$$
\begin{aligned}
\{\,\Phi(\mathcal{G})^{a}(x)\,,\,\Phi(\nabla e)_{b}{}^{i}(y)\,\} &= -\,\Phi(R)^{a}{}_{b}{}^{i}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\mathcal{G})^{a}(x)\,,\,\Phi(\nabla B)_{bc}(y)\,\} &= 2\delta^{a}{}_{[b|}\,\Phi(\mathcal{G})_{|c]}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla e)^{a}{}_{i}(x)\,,\,\Phi(\nabla B)_{bc}(y)\,\} &= 2\delta^{a}{}_{[b|}\Phi(\nabla e)_{|c]i}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(R)^{abi}(x)\,,\,\Phi(\nabla B)_{cd}(y)\,\} &= -4\delta^{[a|}{}_{[c}\,\Phi(R)^{|b]}{}_{d]}{}^{i}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\Phi(\nabla B)_{cd}(y)\,\} &= -4\delta^{[a|}{}_{[c|}\,\Phi(\nabla B)^{|b]}{}_{|d]}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla B)(x)\,,\,\Phi(\nabla\phi)_{A}{}^{ij}(y)\,\} &= -2\rhd^{B}{}_{A}\,\Phi(\nabla\phi)_{B}{}^{ij}(x)\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{C.18}
$$

Poasonova algebra veza prve klase i veza druge klase je:

$$
\begin{aligned}
\{\, \Phi(R)^{abi}(x)\,,\, \chi(\omega)_{cd}{}^{j}(y)\,\} &= 4\,\delta^{[a|}{}_{[c|}\,\chi(B)^{|b|}{}_{|d]}{}^{ij}(x)\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \Phi(\mathcal{G})^{a}(x)\,,\, \chi(\omega)_{cd}{}^{i}(y)\,\} &= 2\,\delta^{a}{}_{[c|}\,\chi(e)_{|d]}{}^{i}(x)\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \Phi(\mathcal{G})^{a}(x)\,,\, \chi(\beta)_{c}{}^{jk}(y)\,\} &= -\frac{1}{2}\,\chi(B)^{a}{}_{c}{}^{jk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \Phi(\nabla e)^{ai}(x)\,,\, \chi(\omega)_{cd}{}^{j}(y)\,\} &= -2\,\delta^{a}{}_{[c|}\,\chi(\beta)_{|d]}{}^{ij}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \Phi(\nabla e)^{ai}(x)\,,\, \chi(e)_{b}{}^{j}(y)\,\} &= \frac{1}{2}\,\chi(B)^{a}{}_{b}{}^{ij}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \Phi(\nabla B)^{ab}(x)\,,\, \chi(\omega)_{cd}{}^{i}(y)\,\} &= 4\,\delta^{[a|}{}_{[c|}\,\chi(\omega)_{|d]}{}^{|b]i}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \Phi(\nabla B)(x)\,,\, \chi(A)^{i}(y)\,\} &= 2\,\chi(A)^{i}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \Phi(\nabla B)^{ab}(x)\,,\, \chi(\beta)_{c}{}^{jk}(y)\,\} &= -2\delta^{[a|}{}_{c}\,\chi(\beta)^{|b]jk}\,\delta^{(3)}(x-y)\,,\\
\{\, \Phi(\nabla B)(x)\,,\, \chi(\gamma)_{A}{}^{ijk}(y)\,\} &= \rhd_{A}{}^{B}\,\chi(\gamma)_{B}{}^{ijk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \Phi(\nabla B)^{ab}(x)\,,\, \chi(B)_{cd}{}^{jk}(y)\,\} &= 4\,\delta^{[a|}{}_{[c}\,\chi(B)_{d]}{}^{|b]jk}\,\delta^{(3)}(\vec{x}-\vec{y})\,.\\
\{\, \Phi(\nabla B)^{ab}(x)\,,\, \chi(e)_{a}{}^{i}(y)\,\} &= -2\delta^{[a|}{}_{c}\,\chi(e)^{|b]i}\,\delta^{(3)}(\vec{x}-\vec{y})\,.\\
\{\, \Phi(\nabla B)(x)\,,\, \chi(\phi)_{A}(y)\,\} &= -\rhd^{B}{}_{A}\,\chi(\phi)_{B}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \Phi(\nabla\phi)^{Aij}(x)\,,\, \chi(A)^{k}\,\} &= -\rhd_{B}{}^{A}\,\chi(\gamma)^{Bijk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \Phi(\nabla\phi)^{Aij}(x)\,,\, \chi(\phi)_{B}\,\} &= -\rhd_{B}{}^{A}\,\chi(B)^{ij}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{C.19}
$$

Poasonova zagrada veza druge klase je izračunata u (C.5).

C.1 Bijankijevi identiteti

Kako bi se izračunao broj stepeni slobode u teoriji, potrebno je koristiti *Bijankijeve identitete* (BI), kao i dodatne, *generalizovane Bijankijeve identitete* (GBI) koji predstavljaju analog običnim BI za dodatna polja prisutna u teoriji.

Konkretno, u teoriji postoje BI za 1-forme polja ω^{ab} i e^a, kao i GBI za 1-formu A. Odgovarajuća 2-forma krivine za koneksiju ω,

$$
R^{ab} = \mathrm{d}\omega^{ab} + \omega^{a}{}_{c}\wedge\omega^{cb}\,, \qquad T^{a} = \mathrm{d}e^{a} + \omega^{a}{}_{b}\wedge e^{b}\,, \qquad F = \mathrm{d}A\,, \tag{C.20}
$$

zadovoljava identitetete:

$$
\epsilon^{\lambda\mu\nu\rho}\,\nabla_{\mu}R^{ab}{}_{\nu\rho} = 0\,, \tag{C.21}
$$

$$
\epsilon^{\lambda\mu\nu\rho}\left(\nabla_{\mu}T^{a}{}_{\nu\rho} - R^{ab}{}_{\mu\nu}\,e_{b\rho}\right) = 0\,, \tag{C.22}
$$

$$
\epsilon^{\lambda\mu\nu\rho}\,\nabla_{\mu}F_{\nu\rho} = 0\,. \tag{C.23}
$$

Birajući slobodan indeks da bude vremenska koordinata $\lambda = 0$, ovi identiteti, kao vremenski nezavisni delovi Bijankijevih identiteta, postaju *"off-shell"* ograničenja u smislu Hamiltonove analize. Sa druge strane, odabirom slobodnog indeksa da bude prostorna koordinata, dobijaju se vremenski zavisni delovi Bjankijevih identiteta, koji ne nameću nikakva ograničenja, već mogu biti izvedeni kao posledica Hamiltonovih jednačina kretanja.

Pored toga, u teoriji postoje GBI asocirani 2-formama polja B^{ab}, B i β^a. Odgovarajuće 3-forme krivina su:

$$S^{ab} = \mathrm{d}B^{ab} + 2\omega^{[a|}{}_c \wedge B^{c|b]}\,, \qquad P = \mathrm{d}B\,, \qquad G^a = \mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b\,. \tag{C.24}$$

Diferenciranjem ovih izraza dobijaju se sledeći GBI:

$$\epsilon^{\lambda\mu\nu\rho}\left(\frac{1}{3}\nabla_\lambda\, S^{ab}{}_{\mu\nu\rho} - R^{[a|\,c}{}_{\lambda\mu}\, B_c{}^{|b]}{}_{\nu\rho}\right) = 0\,, \tag{C.25}$$

$$\epsilon^{\lambda\mu\nu\rho}\partial_\lambda\, P_{\mu\nu\rho} = 0\,, \tag{C.26}$$

$$\epsilon^{\lambda\mu\nu\rho}\left(\frac{2}{3}\nabla_\lambda\, G^a{}_{\mu\nu\rho} - R^{ab}{}_{\lambda\mu}\,\beta_{b\,\nu\rho}\right) = 0\,. \tag{C.27}$$

Međutim, u četvorodimenzionalnom prostorvremenu, svaki od ovih identiteta je jedna jednačina, bez slobodnih prostorvremenskih indeksa, pa stoga obavezno sadrže vremenske izvode polja. Sledi da ovi identiteti ne nameću nikakva off-shell ograničenja na kanonske promenljive.

Konačno, postoji GBI za 0-formu ϕ. Odgovarajuća 1-forma krivine je,

$$Q^A = \mathrm{d}\phi^A + \rhd_B{}^A\, A \wedge \phi^B\,, \tag{C.28}$$

tako da je GBI:

$$\epsilon^{\lambda\mu\nu\rho}\left(\nabla_\nu Q^A{}_\rho - \frac{1}{2}\,\rhd_B{}^A\, F_{\nu\rho}\phi^B\right) = 0. \tag{C.29}$$

Ovaj GBI čini 12 jednačina, koje se dobijaju za šest izbora antisimetrizovanog para prostorvremenskih indeksa $\lambda\mu$ i dva izbora slobodnog grupnog indeksa A. Međutim, ovih 12 identiteta nisu nezavisni, što vidimo diferenciranjem jednačine (C.29), pri čemu se dobija osam jednačina

$$\rhd_B{}^A\,\epsilon^{\lambda\mu\nu\rho}\,\nabla_\mu\, F_{\nu\rho}\,\phi^B = 0\,, \tag{C.30}$$

koje su automatski zadovoljene primenom GBI (C.23). Sledi da su u teoriji samo četiri nezavisna identiteta (C.29). Fiksiranjem indeksa $\lambda = 0$, dobijamo vremenski nezavisne komponente jednačina (C.29) i (C.30),

$$\epsilon^{0ijk}\left(\nabla_j Q^A{}_k - \frac{1}{2}\,\rhd_B{}^A\, F_{jk}\phi^B\right) = 0\,, \tag{C.31}$$

$$\rhd_B{}^A\,\epsilon^{0ijk}\,\partial_i\, F_{jk}\,\phi^B = 0\,. \tag{C.32}$$

Analizom slobodnih indeksa ovih jednačina vidimo da od šest jednačina (C.31), zbog dve jednačine (C.32), preostaju četiri nezavisna GBI relevantna za Hamiltonovu analizu.

C.2 Broj stepeni slobode

U ovom odeljku ćemo pokazati da iz strukture veza u teoriji sledi da ne postoje lokalni stepeni slobode (DOF) u topološkom sektoru $3BF$ teorije skalarne elektrodinamike. U opštem slučaju, broj lokalnih stepeni slobode neke teorije dat je jednačinom (3.41).

U našem slučaju, inicijalan broj polja u teoriji N određujemo prebrojavanjem komponenti polja prikazanih u Tabeli C.1, na osnovu čega dobijamo da je $N = 120$. Zatim, broj nezavisnih

$\omega^{ab}{}_{\mu}$	A_μ	$\beta^a{}_{\mu\nu}$	$\gamma^A{}_{\mu\nu\rho}$	$B^{ab}{}_{\mu\nu}$	$B_{\mu\nu}$	$e^a{}_\mu$	ϕ^A
24	4	24	8	36	6	16	2

Tabela C.1: Broj inicijalnih polja u $3BF$ teoriji skalarne elektrodinamike.

komponenti veza druge klase $S = 70$ prikazan je u Tabeli C.2.

$\chi(B)_{ab}{}^{jk}$	$\chi(B)^{jk}$	$\chi(e)_a{}^i$	$\chi(\phi)_A$	$\chi(\omega)_{ab}{}^i$	$\chi(A)^i$	$\chi(\beta)_a{}^{ij}$	$\chi(\gamma)_A{}^{ijk}$
18	3	12	2	18	3	12	2

Tabela C.2: Veze druge klase u $3BF$ teoriji skalarne elektrodinamike.

Veze prve klase nisu nezavisne zbog prisustva BI i GBI u teoriji. Diferenciranjem veze $\Phi(R)^{abi}$ dobijamo:

$$\nabla_i \Phi(R)^{abi} = \varepsilon^{0ijk} \nabla_i R^{ab}{}_{jk} + \frac{1}{2} R^{c[a|}{}_{ij} P(B)_c{}^{|b]ij} \,. \tag{C.33}$$

Prvi član na desnoj strani izraza je jednak nuli off-shell, $\epsilon^{ijk} \nabla_i R^{ab}{}_{jk} = 0$, kao $\lambda = 0$ komponenta Bijankijevog identiteta(C.21). Drugi član je takođe nula off-shell, kao proizvod dve veze,

$$R^{c[a|}{}_{ij} \, P(B)_c{}^{|b]ij} \equiv \frac{1}{2} \epsilon_{0ijk} \mathcal{S}(R)^{c[a|k} \, P(B)_c{}^{|b]ij} = 0 \,. \tag{C.34}$$

Iz prethodnog sledi da imamo off-shell identitet

$$\nabla_i \Phi(R)^{abi} = 0 \,, \tag{C.35}$$

tj. postoji dvanaest nezavisnih komponenti $\Phi(R)^{abi}$. Slično, diferenciranjem veze $\Phi(F)^i$, dobijamo

$$\nabla_i \Phi(F)^i = \varepsilon^{0ijk} \, \nabla_i F_{jk} + \frac{1}{2} \, F_{ij} \, P(B)^{ij} \,. \tag{C.36}$$

Prvi član na desnoj strani izraza je jednak nuli $\epsilon^{ijk} \, \nabla_i F_{jk} = 0$, kao $\lambda = 0$ komponenta GBI (C.21). Za drugi član opet dobijamo da je jednak nuli kao proizvod dve veze,

$$F_{ij} \, P(B)^{ij} \equiv \frac{1}{2} \epsilon_{0ijk} \, \mathcal{S}(F)^k \, P(B)^{ij} = 0 \,. \tag{C.37}$$

Sledi da je zadovoljen off-shell identitet

$$\nabla_i \Phi(F)^i = 0 \,, \tag{C.38}$$

tj. postoji samo dve nezavisne komponente veze $\Phi(F)^i$. Slično, možemo pokazati da je:

$$\nabla_i \Phi(\nabla e)_a{}^i - \frac{1}{2} \, \Phi(R)_{ab}{}^i \, e^b{}_i + \frac{1}{4} \epsilon^{0ijk} \mathcal{S}(R)_{abk} \, P(\beta)^b{}_{ij} = \frac{1}{2} \epsilon^{0ijk} \left(\nabla_i T_{ajk} - R_{ab\,ij} \, e^b{}_k \right) \,. \tag{C.39}$$

Desna strana izraza (C.39) je $\lambda = 0$ komponenta BI (C.22), tako da jednačina (C.39) postaje:

$$\nabla_i \Phi(\nabla e)_a{}^i - \frac{1}{2}\,\Phi(R)_{ab}{}^i\, e^b{}_i = 0\,. \tag{C.40}$$

U prethodnoj relaciji smo iskoristili da je treći član na levoj strani izraza (C.39) jednak nuli kao proizvod dve veze. Iz jednačine (C.40) sledi da četiri komponente veza $\Phi(\nabla e)_a{}^i$ i $\Phi(R)_{ab}{}^i$ možemo izraziti kao linearnu kombinaciju preostalih. Konačno, može se pokazati da važi relacija

$$\begin{aligned}\nabla_i \Phi(\nabla\phi)_A{}^{ij} - \frac{1}{2}\epsilon_{0ikl} \rhd_A\, \mathcal{S}(F)^l\, \chi(\gamma)_B{}^{ijk} + \rhd^B{}_A\, \phi_B\, \Phi(F)^j \\ + \frac{1}{2}\epsilon_{0ilm} \rhd^B{}_A\, P(B)^{ij}\, \mathcal{S}(\nabla\phi)_B{}^{lm} = \epsilon^{0ijk}\left(\nabla_i Q_{Ak} + \frac{1}{2} \rhd^B{}_A\, F_{ik}\, \phi_B\right)\,,\end{aligned} \tag{C.41}$$

iz koje sledi:

$$\nabla_i \Phi(\nabla\phi)_A{}^{ij} + \rhd^B{}_A\, \phi_B\, \Phi(F)^j = 0\,. \tag{C.42}$$

Desna strana izraza (C.41) je $\lambda = 0$ komponenta GBI (C.29), a članovi koji su proizvod dve veze su nula off-shell. Na osnovu identiteta (C.42) sledi da se šest komponenti veza prve klase $\Phi(\nabla\phi)_A{}^{ij}$ i $\Phi(F)^j$ mogu prikazati kao neka linearna kombinacija ostalih. Međutim, u prethodnom odeljku smo analizom Bijankijevih identiteta zaključili da su samo četiri od ovih šesti identiteta linearno nezavisni, pa sledi da preostaju četiri identiteta (C.42).

Uzimajući u obzir dobijene identitete (C.35), (C.38), (C.40) i (C.42), konačno možemo da izračunamo ukupan broj nezavisnih veza prve klase. Iz Tabele C.3 vidimo da ukupan broj komponenti veza prve klase $F^* = 100$. Identiteti (C.35), (C.38), (C.40) i (C.42) smanjuju broj nezavisnih komponenti ovih veza, što je eksplicitno naznačeno u tabeli. Dakle, umanjujući ovaj broj za šest identiteta (C.35), identitet (C.38), četiri identiteta (C.40) i četiri identiteta (C.42), dobijamo da je broj nezavisnih komponenti veza prve klase $F = 85$.

$\Phi(B)_{ab}{}^i$	$\Phi(B)^i$	$\Phi(e)_a$	$\Phi(\omega)_{ab}$	$\Phi(A)$	$\Phi(\beta)_a{}^i$	$\Phi(\gamma)_A{}^{ij}$	$\Phi(R)_{ab}{}^i$	$\Phi(F)^i$	$\Phi(\mathcal{G})_a$	$\Phi(\nabla e)_a{}^i$	$\Phi(\nabla B)_{ab}$	$\Phi(\nabla B)$	$\Phi(\nabla\phi)_A{}^{ij}$
18	3	4	6	1	12	6	$18-6$	$3-1$	4	$12-4$	6	1	$6-4$

Tabela C.3: Veze prve klase u $3BF$ teoriji skalarne elektrodinamike.

Stoga, zamenom svih dobijenih rezultata u jednačini (3.41), dobijamo

$$n = 120 - 85 - \frac{70}{2} = 0, \tag{C.43}$$

tj. da $3BF$ teorija data dejstvom (6.174) ne poseduje lokalne propagirajuće stepene slobode.

C.3 Generator gejdž simetrije

Na osnovu rezultata Hamiltonove analize dejstva (6.174), možemo da izračunamo generator gejdž simetrije i varijacije formi varijabli u $3BF$ topološkoj teoriji skalarne elektrodinamike. Rezultati prikazani u ovom odeljku su generalizacija računa u [48] za generator i varijacije formi varijabli u $2BF$ topološkom dejstvu za Poenkareovu 2-grupu, a specijalan slučaj rezultata za $3BF$ teoriju za generalnu semistriktnu 2-grupu datih u Glavi 6.

Kastelanijevom procedurom dobijamo generator:

$$
\begin{aligned}
G \;=\; \int_{\Sigma_3} \mathrm{d}^3\vec{x}\Big(& \frac{1}{2}(\nabla_0\epsilon^{ab}{}_i)\Phi(B)_{ab}{}^i - \frac{1}{2}\epsilon^{ab}{}_i\Phi(R)_{ab}{}^i + \frac{1}{2}(\nabla_0\epsilon^{ab})\Phi(\omega)_{ab} - \frac{1}{2}\epsilon^{ab}\Phi(\nabla B)_{ab} \\
& + (\partial_0\epsilon_i)\Phi(B)^i - \epsilon_i\Phi(F)^i + (\partial_0\epsilon)\Phi(A) - \epsilon\Phi(\nabla B) \\
& + (\nabla_0\epsilon^a)\Phi(e)_a - \epsilon^a\Phi(\mathcal{G})_a + (\nabla_0\epsilon^a{}_i)\Phi(\beta)_a{}^i - \epsilon^a{}_i\Phi(\nabla e)_a{}^i \\
& + \frac{1}{2}(\nabla_0\epsilon^A{}_{ij})\Phi(\gamma)_A{}^{ij} - \frac{1}{2}\epsilon^A{}_{ij}\Phi(\nabla\phi)_A{}^{ij} \\
& + \epsilon^{ab}\left(\beta_{[a|0i}P(\beta)_{|b]}{}^i + e_{[a|0}P(e)_{|b]} + B_{[a|c0i}P(B)^c{}_{|b]}{}^i\right) - \epsilon\,\gamma_{A0ij} \rhd_B{}^A P(\gamma)^{Bij} \\
& + \epsilon^a\beta_{b0i}P(B)^{abi} + \epsilon^a{}_i\, e_{b0}P(B)_a{}^{bi}\Big)\,,
\end{aligned}
\tag{C.44}
$$

gde su $\epsilon^{ab}{}_i$, ϵ^{ab}, ϵ_i, ϵ, ϵ^a, $\epsilon^a{}_i$ i $\epsilon^A{}_{ij}$ nezavisni parametri gejdž transformacija.

Dalje, možemo izračunati varijacije formi varijabli i konjugovanih impulsa varijabli u teoriji:

$$
\begin{aligned}
\delta_0\omega^{ab}{}_0 &= \nabla_0\epsilon^{ab}\,, & \delta_0\pi(\omega)_{ab}{}^0 &= -2\epsilon_{[a|}{}^c{}_i\pi(B)_{c|b]}{}^{0i} - 2\epsilon_{[a|}{}^c\pi(\omega)_{c|b]}{}^0\,, \\
&& &\quad +2\epsilon_{[a|}\pi(e)_{|b]}{}^0 + 2\epsilon_{[a|i}\pi(\beta)_{|b]}{}^{0i}\,, \\
\delta_0\omega^{ab}{}_i &= \nabla_i\epsilon^{ab}\,, & \delta_0\pi(\omega)_{ab}{}^i &= -2\epsilon_{[a|}{}^c{}_j\,\pi(B)_{c|b]}{}^{ij} - 2\epsilon_{[a|}{}^c{}_i\,\pi(\omega)_{|b]c}{}^i \\
&& &\quad +2\epsilon_{[a|}\,\pi(e)_{|b]i} + 2\epsilon_{[a|j}\pi(\beta)_{|b]}{}^{ij} \\
&& &\quad +2\epsilon^{0ijk}\,\nabla_{[j|}\epsilon_{ab|k]} + \epsilon^{0ijk}\epsilon_{[a|}\beta_{|b]jk}\,, \\
\delta_0 B^{ab}{}_{0i} &= \nabla_0\epsilon^{ab}{}_i + \epsilon^{[a|}{}_i e^{|b]}{}_0 & \delta_0\pi(B)_{ab}{}^{0i} &= 2\epsilon_{[a|c}\,\pi(B)_{|b]}{}^{ci}\,, \\
&\quad +2\epsilon^{[a|c}B^{|b]}{}_{c0i} + \epsilon^{[a|}\beta^{|b]}{}_{0i}\,, && \\
\delta_0 B^{ab}{}_{ij} &= 2\nabla_{[i|}\epsilon^{ab}{}_{|j]} + 2\epsilon^{[a|c}B^{|b]}{}_{cij} & \delta_0\pi(B)_{ab}{}^{ij} &= 2\epsilon_{[a|c}\,\pi(B)_{|b]}{}^{cij}\,, \\
&\quad +2\epsilon^{[a|}{}_{[i}e^{|b]}{}_{j]} + \epsilon^{[a|}\beta^{|b]}{}_{ij}\,, && \\
\delta_0 A_0 &= \partial_0\epsilon\,, & \delta_0\pi(A)^0 &= -\frac{1}{2}\epsilon^A{}_{ij} \rhd^B{}_A\,\pi(\gamma)_B{}^{0ij}\,, \\
\delta_0 A_i &= \partial_i\epsilon\,, & \delta_0\pi(A)^i &= \epsilon^{0ijk}\partial_j\epsilon_k - \frac{1}{2}\epsilon^A{}_{jk} \rhd^B{}_A\,\pi(\gamma)_B{}^{ijk}\,, \\
\delta_0 B_{0i} &= \partial_0\epsilon_i\,, & \delta_0\pi(B)^{0i} &= 0\,,
\end{aligned}
\tag{C.45}
$$

$$
\begin{aligned}
\delta_0 B_{ij} &= 2\,\partial_{[i|}\epsilon_{|j]} + \epsilon^A{}_{ij} \rhd^B{}_A \phi_B\,, & \delta_0\pi(B)^{ij} &= -\epsilon^{0ijk}\partial_k\epsilon\,,\\
\delta_0\beta^a{}_{0i} &= \nabla_0\epsilon^a{}_i - \epsilon^{ab}\beta_{b0i}\,, & \delta_0\pi(\beta)_a{}^{0i} &= -\epsilon_{ab}\pi(\beta)^{b0i} + \frac{1}{2}\epsilon^b\pi(B)_{ab}{}^{0i}\,,\\
\delta_0\beta^a{}_{ij} &= 2\nabla_{[i|}\epsilon^a{}_{|j]} - \epsilon^{ab}\,\beta_{bij}\,, & \delta_0\pi(\beta)_a{}^{ij} &= -\epsilon_{ab}\,\pi(\beta)^{bij} + \frac{1}{2}\epsilon^b\,\pi(B)_{ab}{}^{ij}\\
& & & \quad -\epsilon^{0ijk}\,\nabla_k\epsilon^a\,,\\
\delta_0 e^a{}_0 &= \nabla_0\epsilon^a - \epsilon^{ab}\,e_{b0}\,, & \delta_0\pi(e)_a{}^0 &= -\epsilon_{ab}\,\pi(e)^{b0} + \frac{1}{2}\epsilon^b{}_i\,\pi(B)_{ab}{}^{0i}\,,\\
\delta_0 e^a{}_i &= \nabla_i\epsilon^a - \epsilon^{ab}\,e_{bi}\,, & \delta_0\pi(e)_a{}^i &= -\epsilon_{ab}\,\pi(e)^{bi} + \epsilon^{0ijk}\Big(\nabla_{[j|}\epsilon_{a|k]} + \epsilon_{ab}\beta^{bjk}\Big)\\
& & & \quad +\frac{1}{2}\epsilon^b{}_j\,\pi(B)_{ab}{}^{ij}\,,\\
\delta_0\gamma^A{}_{0ij} &= \nabla_0\epsilon^A{}_{ij} - \epsilon\,\gamma^B{}_{0ij} \rhd^A{}_B\,, & \delta_0\pi(\gamma)_A{}^{0ij} &= \epsilon \rhd^B{}_A\,\pi(\gamma)_B{}^{0ij}\,,\\
\delta_0\gamma^A{}_{ijk} &= -\,\epsilon\,\gamma^B{}_{ijk} \rhd_B{}^A + \nabla_i\epsilon^A{}_{jk} & \delta_0\pi(\gamma)_A{}^{ijk} &= \epsilon \rhd_A{}^B\left(\pi(\gamma)_B{}^{ijk} + \epsilon^{0ijk}\,\phi_B\right)\,,\\
&\quad -\nabla_j\epsilon^A{}_{ik} + \nabla_k\epsilon^A{}_{ij}\,, & &\\
\delta_0\phi^A &= \epsilon\,\phi^B \rhd^A{}_B\,, & \delta_0\pi(\phi)_A &= -\epsilon \rhd^B{}_A\,\pi(\phi)_B + \frac{1}{3!}\,\epsilon\,\epsilon^{0ijk} \rhd^B{}_A\,\gamma_{Bijk}\\
& & & \quad -\frac{1}{2}\rhd_{A\ B}\,\epsilon^B{}_{ij}\,\pi(B)^{ij} - \frac{1}{2}\epsilon^{0ijk}\,\nabla_i\epsilon^A{}_{jk}\,,
\end{aligned}
\tag{C.46}
$$

Dodatak D

Ukupna grupa gejdž simetrija

D.1 Gejdž transformacije u BF topološkoj teoriji

Generator gejdž transformacija u BF teoriji je

$$G = \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big((\nabla_0 \epsilon_{\mathfrak{g}}{}^{\alpha})\,(\tilde{G}_1)_{\alpha} + \epsilon_{\mathfrak{g}}{}^{\alpha}\,(\tilde{G}_0)_{\alpha} + (\nabla_0 \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i)\,(\tilde{M}_1)_{\alpha}{}^i + \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i\,(\tilde{M}_0)_{\alpha}{}^i \Big)\,, \tag{D.1}$$

gde je:

$$\begin{aligned} (\tilde{G}_1)_{\alpha} &= -\Phi(\alpha)_{\alpha}\,, \\ (\tilde{G}_0)_{\alpha} &= -\big(f_{\alpha\gamma}{}^{\beta} B_{\beta 0 i} \Phi(B)^{\gamma i} + C_{a0} \rhd_{\alpha b}{}^{a} \Phi(C)^{b0} + \beta_{a0i} \rhd_{\alpha b}{}^{a} \Phi(\beta)^{b0i} - \Phi(\nabla B)_{\alpha} \big)\,, \\ (\tilde{M}_1)_{\alpha}{}^i &= -\Phi(B)_{\alpha}{}^i\,, \\ (\tilde{M}_0)_{\alpha}{}^i &= \Phi(\mathcal{F})_{\alpha}{}^i\,, \end{aligned} \tag{D.2}$$

gde su $\epsilon_{\mathfrak{g}}{}^{\alpha}$ i $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_i$ nezavisni parametri gejdž transformacija.

D.1.1 Gejdž grupa simetrije prostora BF dejstva

Algebra koju čine generatori grupe simetrija $(M_0)_{\alpha}{}^i$, $(M_1)_{\alpha}{}^i$, $(G_0)_{\alpha}$ i $(G_1)_{\alpha}$ definisani u Dodatku D.1 je:

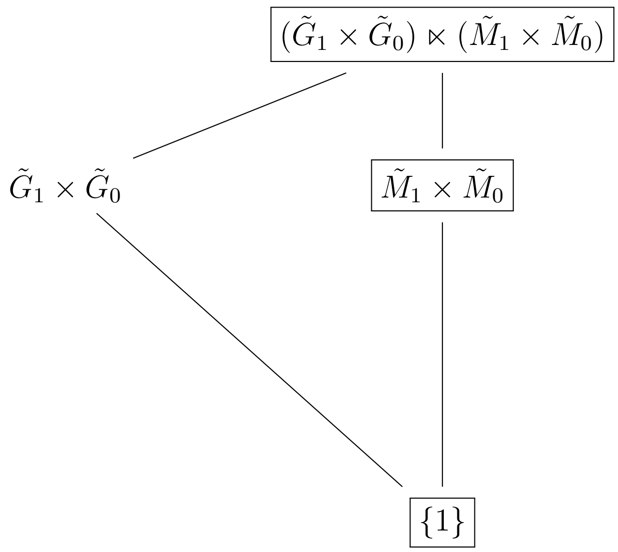
$$\begin{aligned} \{(\tilde{G}_0)_{\alpha}(\vec{x})\,, (\tilde{G}_0)_{\beta}(\vec{y})\} &= f_{\alpha\beta}{}^{\gamma} (\tilde{G}_0)_{\gamma}\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{(\tilde{G}_0)_{\alpha}(\vec{x})\,, (\tilde{M}_0)_{\beta}{}^i(\vec{y})\} &= f_{\alpha\beta}{}^{\gamma} (\tilde{M}_0)_{\gamma}{}^i\, \delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{(\tilde{G}_0)_{\alpha}(\vec{x})\,, (\tilde{M}_1)_{\beta}{}^i(\vec{y})\} &= f_{\alpha\beta}{}^{\gamma} (\tilde{M}_1)_{\gamma}{}^i\, \delta^{(3)}(\vec{x}-\vec{y})\,, \end{aligned} \tag{D.3}$$

Na osnovu ovih komutacionih relacija zaključujemo da je gejdž grupa simetrije ima sledeću strukturu. Najpre, generatori $(\tilde{M}_1)_{\alpha}{}^i$ i $(\tilde{M}_1)_{\alpha}{}^i$ formiraju algebru $\mathfrak{a}_1$,

$$\mathfrak{a}_1 = \mathrm{span}\{(\tilde{M}_1)_{\alpha}{}^i\} \oplus \mathrm{span}\{(\tilde{M}_0)_{\alpha}{}^i\}\,,$$

koja generiše podgrupu $\tilde{M}_1 \times \tilde{M}_0$ ukupne grupe simetrije $\tilde{\mathcal{G}}_{\Sigma_3}$. Osim toga, podgrupa $\tilde{M}_1 \times \tilde{M}_0$ je invarijantna podgrupa grupe $\tilde{\mathcal{G}}_{\Sigma_3}$. Može se primetiti da $\tilde{G}_1 \times \tilde{G}_0$ takođe čini podgrupu ukupne grupe simetrije. Sada se ove dve podgrupe, od kojih je jedna invarijantna podruga, a druga ne, mogu pomnožiti semidirektnim proizvodom, pri čemu dobijamo da je ukupna grupa simetrije $\tilde{\mathcal{G}}_{\Sigma_3}$:

$$\tilde{\mathcal{G}}_{\Sigma_3} = (\tilde{G}_1 \times \tilde{G}_0) \ltimes (\tilde{M}_1 \times \tilde{M}_0)\,.$$



Slika D.1: Grupa simetrije $\mathcal{G}_{\Sigma_3}$ u faznom prostoru. Invarijantne grupe su uokvirene.

D.1.2 Konstrukcija generatora simetrija BF teorije

Kada zamenimo generatore (D.2) u jednačinu (D.1), dobijamo generator gejdž simetrija u BF teoriji sledećeg oblika

$$G = -\int_{\Sigma_3} \mathrm{d}^3\vec{x}\Big((\nabla_0 \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i)\Phi(B)_{\alpha}{}^i - \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i \Phi(\mathcal{F})_{\alpha}{}^i + (\nabla_0 \epsilon_{\mathfrak{g}}{}^{\alpha})\Phi(\alpha)_{\alpha} + \epsilon_{\mathfrak{g}}{}^{\alpha}\big(f_{\alpha\gamma}{}^{\beta} B_{\beta 0 i}\Phi(B)^{\gamma i} - \Phi(\nabla B)_{\alpha}\big)\Big)\,, \tag{D.4}$$

gde su $\epsilon_{\mathfrak{g}}{}^{\alpha}$ i $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_i$ nezavisni parametri gejdž transformacija.

Generator gejdž transformacija u BF topološkoj teoriji (4.31) dobijamo *Kastelanijevom procedurom* (3.53) pri čemu su za svaki par generatora G_0 i G_1 zadovoljene relacije (3.55).

Pretpostavimo najpre da generator ima oblik:

$$G = \int \dot{\epsilon}^{\alpha}{}_i (G_1)_{\alpha}{}^i + \epsilon^{\alpha}{}_i (G_0)_{\alpha}{}^i + \dot{\epsilon}^{\alpha}(G_1)_{\alpha} + \epsilon^{\alpha}(G_0)_{\alpha}\,. \tag{D.5}$$

Izborom $(G_1)_{\alpha}{}^i = C_{PFC}$ i $(G_1)_{\alpha} = C_{PFC}$, gde je C_{PFC} oznaka za neku primarnu vezu prve klase, prirodan izbor je:

$$(G_1)_{\alpha}{}^i = \Phi(B)_{\alpha}{}^i\,, \qquad (G_1)_{\alpha} = \Phi(\alpha)_{\alpha}\,. \tag{D.6}$$

Ostaje da se utvrde dva generatora G_0. Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{m}\alpha}{}^i$ daje

$$\begin{aligned} (G_0)_{\mathfrak{m}\alpha}{}^i - \{\Phi(B)_{\alpha}{}^i, H_T\} &= (C_{PFC})_{\alpha}{}^i\,, \\ (G_0)_{\mathfrak{m}\alpha}{}^i - \Phi(\mathcal{F})_{\alpha}{}^i &= (C_{PFC})_{\alpha}{}^i\,, \end{aligned} \tag{D.7}$$

gde je $(G_0)_{\mathfrak{m}\alpha}{}^i = (C_{PFC})_{\alpha}{}^i + \Phi(\mathcal{F})_{\alpha}{}^i$. Zatim, iz Kastelanijevog trećeg uslova sledi

$$\begin{aligned} &\{(G_0)_{\mathfrak{m}\alpha}{}^i, H_T\} = (C_{PFC1})_{\alpha}{}^i\,, \\ &\{(C_{PFC})_{\alpha}{}^i + \Phi(\mathcal{F})_{\alpha}{}^i, H_T\} = (C_{PFC1})_{\alpha}{}^i\,, \\ &\{(C_{PFC})_{\alpha}{}^i, H_T\} - f_{\beta\gamma\alpha}\alpha^{\beta}{}_0\Phi(\mathcal{F})^{\gamma i} = (C_{PFC1})_{\alpha}{}^i\,, \end{aligned} \tag{D.8}$$

što daje jednačinu

$$(C_{PFC})_{\alpha}{}^i = f_{\beta\gamma\alpha}\alpha^{\beta}{}_0\Phi(B)^{\gamma i}\,.$$

Iz toga sledi da je generator:

$$(G_0)_{\mathfrak{m}\alpha}{}^i = f_{\beta\gamma\alpha}\alpha^\beta{}_0\Phi(B)^{\gamma i} + \Phi(\mathcal{F})_\alpha{}^i\,. \tag{D.9}$$

Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{g}\alpha}$ daje

$$\begin{aligned} &(G_0)_{\mathfrak{g}\alpha} - \{\Phi(\alpha)_\alpha\,, H_T\} = (C_{PFC})_\alpha\,,\\ &(G_0)_{\mathfrak{g}\alpha} - \Phi(\nabla B)_\alpha = (C_{PFC})_\alpha\,, \end{aligned} \tag{D.10}$$

tj. dobija se da je $(G_0)_{\mathfrak{g}\alpha} = (C_{PFC})_\alpha + \Phi(\nabla B)_\alpha$. Nakon toga, iz trećeg Kastelanijevog uslova sledi

$$\begin{aligned} &\{(G_0)_{\mathfrak{g}\alpha}\,, H_T\} = (C_{PFC1})_\alpha\,,\\ &\{(C_{PFC})_\alpha + \Phi(\nabla B)_\alpha\,, H_T\} = (C_{PFC1})_\alpha\,,\\ &\{(C_{PFC})_\alpha\,, H_T\} + B_{\beta 0 i} f_{\alpha\gamma}{}^\beta\Phi(\mathcal{F})^{\gamma i} - \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma\Phi(\nabla B)_\gamma = (C_{PFC1})_\alpha\,, \end{aligned} \tag{D.11}$$

tj.

$$(C_{PFC})_\alpha = -B_{\beta 0 i} f_{\alpha\gamma}{}^\beta\Phi(B)^{\gamma i} + \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma\Phi(\alpha)_\gamma\,.$$

Sledi da je generator:

$$(G_0)_{\mathfrak{g}\alpha} = -B_{\beta 0 i} f_{\alpha\gamma}{}^\beta\Phi(B)^{\gamma i} + \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma\Phi(\alpha)_\gamma + \Phi(\nabla B)_\alpha\,. \tag{D.12}$$

U ovom trenutku, korisno je rezimirati rezultate i uvesti novu notaciju:

$$\begin{aligned} \dot\epsilon_{\mathfrak{m}}{}^\alpha{}_i(G_1)_{\mathfrak{m}\alpha}{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i(G_0)_{\mathfrak{m}\alpha}{}^i &= -\nabla_0\epsilon_{\mathfrak{m}}{}^\alpha{}_i\Phi(B)_\alpha{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i\Phi(\mathcal{F})_\alpha{}^i\\ &= \nabla_0\epsilon_{\mathfrak{m}}{}^\alpha{}_i(\tilde M_1)_\alpha{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i(\tilde M_0)_\alpha{}^i\,. \end{aligned} \tag{D.13}$$

Primetimo da se vremenski izvod parametra kombinuje sa nekim drugim članovima u kovarijantni izvod u vremenskom pravcu.

Za drugi deo ukupnog generatora dobijamo:

$$\begin{aligned} \dot\epsilon_{\mathfrak{g}}{}^\alpha(G_1)_{\mathfrak{g}\alpha} + \epsilon_{\mathfrak{g}}{}^\alpha(G_0)_{\mathfrak{g}\alpha} &= -\dot\epsilon_{\mathfrak{g}}{}^\alpha\Phi(\alpha)_\alpha - \epsilon_{\mathfrak{g}}{}^\alpha\big(B_{\beta 0 i} f_{\alpha\gamma}{}^\beta\Phi(B)^{\gamma i} - \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma\Phi(\alpha)_\gamma - \Phi(\nabla B)_\alpha\big)\\ &= -\nabla_0\epsilon_{\mathfrak{g}}{}^\alpha\Phi(\alpha)_\alpha - \epsilon_{\mathfrak{g}}{}^\alpha\big(B_{\beta 0 i} f_{\alpha\gamma}{}^\beta\Phi(B)^{\gamma i} - \Phi(\nabla B)_\alpha\big)\\ &= \nabla_0\epsilon_{\mathfrak{g}}{}^\alpha(\tilde G_1)_\alpha + \epsilon_{\mathfrak{g}}{}^\alpha(\tilde G_0)_\alpha\,. \end{aligned} \tag{D.14}$$

D.2 Gejdž transformacije u $2BF$ topološkoj teoriji

Generator gejdž transformacija u $2BF$ teoriji je

$$
\begin{aligned}
G \;=\; \int_{\Sigma_3} d^3\vec{x} \Big(& (\nabla_0 \epsilon_{\mathfrak{g}}{}^{\alpha})\,(\tilde{G}_1)_\alpha + \epsilon_{\mathfrak{g}}{}^{\alpha}\,(\tilde{G}_0)_\alpha + (\nabla_0 \epsilon_{\mathfrak{h}}{}^{a}{}_i)\,(\tilde{H}_1)_a{}^i + \epsilon_{\mathfrak{h}}{}^{a}{}_i\,(\tilde{H}_0)_a{}^i \\
& + (\nabla_0 \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i)\,(\tilde{M}_1)_\alpha{}^i + \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i\,(\tilde{M}_0)_\alpha{}^i + (\nabla_0 \epsilon_{\mathfrak{n}}{}^{a})\,(\tilde{N}_1)_a + \epsilon_{\mathfrak{n}}{}^{a}(\tilde{N}_0)_a \Big)\,,
\end{aligned}
\tag{D.15}
$$

gde je:

$$
\begin{aligned}
(\tilde{G}_1)_\alpha &= -\Phi(\alpha)_\alpha\,, \\
(\tilde{G}_0)_\alpha &= -\big(f_{\alpha\gamma}{}^{\beta} B_{\beta 0 i} \Phi(B)^{\gamma i} + C_{a0} \rhd_{\alpha b}{}^{a} \Phi(C)^{b0} + \beta_{a0i} \rhd_{\alpha b}{}^{a} \Phi(\beta)^{b0i} - \Phi(\nabla B)_\alpha \big)\,, \\
(\tilde{H}_1)_a{}^i &= -\Phi(\beta)_a{}^i\,, \\
(\tilde{H}_0)_a{}^i &= C_{b0} \rhd_{\alpha a}{}^{b} \Phi(B)^{\alpha i} + \Phi(\nabla C)_a{}^i\,, \\
(\tilde{M}_1)_\alpha{}^i &= -\Phi(B)_\alpha{}^i\,, \\
(\tilde{M}_0)_\alpha{}^i &= \Phi(\mathcal{F})_\alpha{}^i\,, \\
(\tilde{N}_1)_a &= -\Phi(C)_a\,, \\
(\tilde{N}_0)_a &= \beta_{b0i} \rhd_{\alpha a}{}^{b} \Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a\,,
\end{aligned}
\tag{D.16}
$$

gde su $\epsilon_{\mathfrak{g}}{}^{\alpha}$, $\epsilon_{\mathfrak{h}}{}^{a}{}_i$, $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_i$ i $\epsilon_{\mathfrak{n}}{}^{a}$ nezavisni parametri gejdž transformacija.

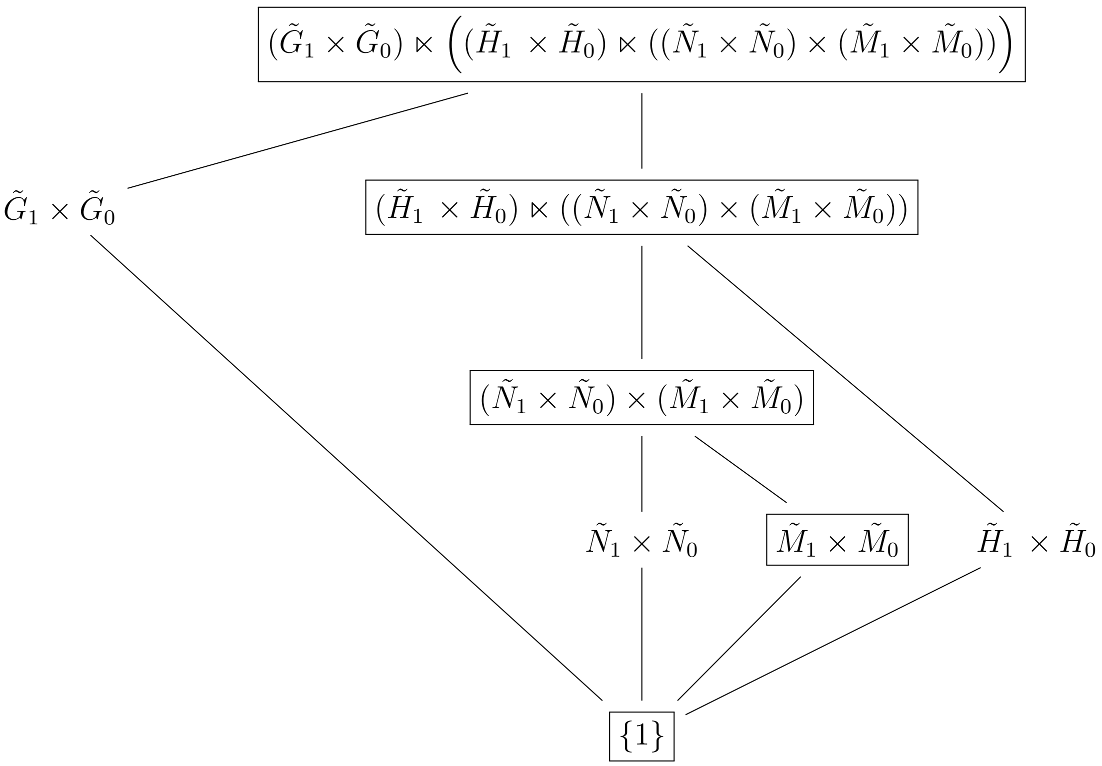
D.2.1 Gejdž grupa simetrije $2BF$ dejstva

Algebra koju čine generatori grupe simetrija $(M_0)_\alpha{}^i$, $(M_1)_\alpha{}^i$, $(G_0)_\alpha$, $(G_1)_\alpha$, $(H_0)_a{}^i$, $(H_1)_a{}^i$, $(N_0)_a$ i $(N_1)_a$ definisani u prethodnom delu D.2 je:

$$
\{(\tilde{G}_0)_\alpha(\vec{x})\,,(\tilde{G}_0)_\beta(\vec{y})\} = f_{\alpha\beta}{}^{\gamma}(\tilde{G}_0)_\gamma\,\delta^{(3)}(\vec{x}-\vec{y})\,,
\tag{D.17}
$$

$$
\begin{aligned}
\{(\tilde{H}_0)_a{}^i(\vec{x})\,,(\tilde{N}_0)^b(\vec{y})\} &= \rhd_{\alpha a}{}^{b}(\tilde{M}_0)^{\alpha i}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{(\tilde{H}_1)_a{}^i(\vec{x})\,,(\tilde{N}_0)^b(\vec{y})\} &= \rhd_{\alpha a}{}^{b}(\tilde{M}_1)^{\alpha i}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{(\tilde{H}_0)_a(\vec{x})\,,(\tilde{N}_1)^{bi}(\vec{y})\} &= \rhd_{\alpha a}{}^{b}(\tilde{M}_1)^{\alpha i}\,\delta^{(3)}(\vec{x}-\vec{y})\,,
\end{aligned}
\tag{D.18}
$$

$$
\begin{aligned}
\{(\tilde{G}_0)_\alpha(\vec{x})\,,(\tilde{M}_0)_\beta{}^i(\vec{y})\} &= f_{\alpha\beta}{}^{\gamma}(\tilde{M}_0)_\gamma{}^i\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{(\tilde{G}_0)_\alpha(\vec{x})\,,(\tilde{M}_1)_\beta{}^i(\vec{y})\} &= f_{\alpha\beta}{}^{\gamma}(\tilde{M}_1)_\gamma{}^i\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,(\tilde{G}_0)_\alpha(\vec{x})\,,\,(\tilde{H}_1)_a{}^i(\vec{y})\,\} &= \rhd_{\alpha a}{}^{b}\,(\tilde{H}_1)_b{}^i(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,(\tilde{G}_0)_\alpha(\vec{x})\,,\,(\tilde{H}_0)_a{}^i(\vec{y})\,\} &= \rhd_{\alpha a}{}^{b}\,(\tilde{H}_0)_b{}^i(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,(\tilde{G}_0)_\alpha(\vec{x})\,,\,(\tilde{N}_1)_a(\vec{y})\,\} &= \rhd_{\alpha a}{}^{b}\,(\tilde{N}_1)_b(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,(\tilde{G}_0)_\alpha(\vec{x})\,,\,(\tilde{N}_0)_a(\vec{y})\,\} &= \rhd_{\alpha a}{}^{b}\,(\tilde{N}_0)_b(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,
\end{aligned}
\tag{D.19}
$$



Slika D.2: Grupa simetrije $\mathcal{G}_{\Sigma_3}$ u faznom prostoru. Invarijantne grupe su okvirene.

Grupa gejdž simetrije ima sledeću strukturu. Prvo, grupe $\tilde{M}_1 \times \tilde{M}_0$, $\tilde{N}_1 \times \tilde{N}_0$ i $\tilde{H}_1 \times \tilde{H}_0$ sa odgovarajućim algebrama $\mathfrak{a}_1$, $\mathfrak{a}_2$ i $\mathfrak{a}_3$, gde je

$$\begin{gathered}\mathfrak{a}_1 = \operatorname{span}\{(\tilde{M}_1)_\alpha{}^i\} \oplus \operatorname{span}\{(\tilde{M}_0)_\alpha{}^i\}\,, \quad \mathfrak{a}_2 = \operatorname{span}\{(\tilde{N}_1)_a\} \oplus \operatorname{span}\{(\tilde{N}_0)_a\}\,, \\ \mathfrak{a}_3 = \operatorname{span}\{(\tilde{H}_1)_a{}^i\} \oplus \operatorname{span}\{(\tilde{H}_0)_a{}^i\}\end{gathered} \tag{D.20}$$

su podgrupe ukupne grupe simetrije $\tilde{\mathcal{G}}_{\Sigma_3}$. Pritom, podgrupa $\tilde{M}_1 \times \tilde{M}_0$ je invarijantna podgrupa ukupne grupe simetrije. Grupe $\tilde{N}_1 \times \tilde{N}_0$ i $\tilde{H}_1 \times \tilde{H}_0$ nisu invarijantne podgrupe ukupne grupe simetrije, što vidimo na osnovu Poasonovih zagrada $\{(\tilde{H}_0)_a{}^i(\vec{x})\,,(\tilde{N}_0)^b(\vec{y})\}$ i $\{(\tilde{H}_1)_a{}^i(\vec{x})\,,(\tilde{N}_0)^b(\vec{y})\}$ koje su jednake nekim linearnim kombinacijama generatora grupa $\tilde{M}_1$, tj. $\tilde{M}_0$. Može se formirati direktan proizvod $(\tilde{N}_1 \times \tilde{N}_0) \times (\tilde{M}_1 \times \tilde{M}_0)$, kako generatori ovih grupa međusobno komutiraju, a dobijena grupa je invarijantna podgrupa ukupne grupe simetrija.

Dobijenu grupu možemo pomnožiti sa grupom $\tilde{H}_1 \times \tilde{H}_0$, pri čemu je $(\tilde{N}_1 \times \tilde{N}_0) \times (\tilde{M}_1 \times \tilde{M}_0)$ invarijantna, a $\tilde{H}_1 \times \tilde{H}_0$ ne, koristeći semidirektan proizvod, čime se dobija invarijantna podgrupa $(\tilde{H}_1 \times \tilde{H}_0) \ltimes ((N_1 \times N_0) \times (M_1 \times M_0))$, kojoj odgovara algebra $\mathfrak{a}_4$:

$$\mathfrak{a}_4 = \operatorname{span}\{(\tilde{M}_0)_\alpha{}^i, (\tilde{M}_1)_\alpha{}^i, (\tilde{H}_0)_a{}^i, (\tilde{H}_1)_a{}^i, (\tilde{N}_0)_a, (\tilde{N}_1)_a\}\,.$$

Na kraju, dobijenu grupu možemo pomnožiti sa grupom $\tilde{G}_1 \times \tilde{G}_0$ koristeći semidirektan proizvod, pri čemu dobijamo ukupnu grupu simetrija $\tilde{\mathcal{G}}_{\Sigma_3}$ koja je jednaka:

$$\tilde{\mathcal{G}}_{\Sigma_3} = (\tilde{G}_1 \times \tilde{G}_0) \ltimes \Big((\tilde{H}_1 \times \tilde{H}_0) \ltimes \Big((\tilde{N}_1 \times \tilde{N}_0) \times (\tilde{M}_1 \times \tilde{M}_0)\Big)\Big)\,.$$

Kompletna struktura ukupne grupe simetrija prikazana je na Slici D.2. Ovde su invarijantne podgrupe ukupne grupe simetrija uokvirene.

D.2.2 Konstrukcija generatora simetrija $2BF$ teorije

Kada zamenimo generatore (D.16) u jednačinu (D.15), dobijamo generator gejdž simetrija u $2BF$ teoriji sledećeg oblika

$$\begin{aligned} G \;=\; -\int_{\Sigma_3} \mathrm{d}^3\vec{x}\Big(& (\nabla_0\epsilon_{\mathfrak{m}}{}^{\alpha}{}_i)\Phi(B)_{\alpha}{}^i - \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i\Phi(\mathcal{F})_{\alpha}{}^i + (\nabla_0\epsilon_{\mathfrak{g}}{}^{\alpha})\Phi(\alpha)_{\alpha} \\ & + \epsilon_{\mathfrak{g}}{}^{\alpha}\big(f_{\alpha\gamma}{}^{\beta}B_{\beta 0i}\Phi(B)^{\gamma i} + C_{a0}\rhd_{\alpha b}{}^{a}\Phi(C)^{b0} + \beta_{a0i}\rhd_{\alpha b}{}^{a}\Phi(\beta)^{b0i} - \Phi(\nabla B)_{\alpha}\big) \\ & + (\nabla_0\epsilon_{\mathfrak{n}}{}^{a})\Phi(C)_a - \epsilon_{\mathfrak{n}}{}^{a}\left(\beta_{b0i}\rhd_{\alpha a}{}^{b}\Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a\right) \\ & + (\nabla_0\epsilon_{\mathfrak{h}}{}^{a}{}_i)\Phi(\beta)_a{}^i - \epsilon_{\mathfrak{h}}{}^{a}{}_i\left(C_{b0}\rhd_{\alpha a}{}^{b}\Phi(B)^{\alpha i} + \Phi(\nabla C)_a{}^i\right)\Big)\,, \end{aligned} \tag{D.21}$$

gde su $\epsilon_{\mathfrak{g}}{}^{\alpha}$, $\epsilon_{\mathfrak{h}}{}^{a}{}_i$, $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_i$ i $\epsilon_{\mathfrak{n}}{}^{a}$ nezavisni parametri gejdž transformacija.

Generator gejdž transformacija simetrije (D.15) u $2BF$ teoriji (5.1), dobija se Kastelanijevom procedurom. Pretpostavimo, najpre, da generator ima strukturu:

$$\begin{aligned} G \;=\; \int_{\Sigma_3} \mathrm{d}^3\vec{x}\,\Big(& \dot{\epsilon}_{\mathfrak{m}}{}^{\alpha}{}_i(G_1)_{\mathfrak{m}\alpha}{}^i + \epsilon_{\mathfrak{m}}{}^{\alpha}{}_i(G_0)_{\mathfrak{m}\alpha}{}^i + \dot{\epsilon}_{\mathfrak{g}}{}^{\alpha}(G_1)_{\mathfrak{g}\alpha} + \epsilon_{\mathfrak{g}}{}^{\alpha}(G_0)_{\mathfrak{g}\alpha} \\ & + \dot{\epsilon}_{\mathfrak{h}}{}^{a}{}_i(G_1)_{\mathfrak{h}a}{}^i + \epsilon_{\mathfrak{h}}{}^{a}{}_i(G_0)_{\mathfrak{h}a}{}^i + \dot{\epsilon}_{\mathfrak{n}}{}^{a}(G_1)_{\mathfrak{n}a} + \epsilon_{\mathfrak{n}}{}^{a}(G_0)_{\mathfrak{n}a}\Big)\,. \end{aligned} \tag{D.22}$$

Prvi korak Kastelanijeve procedure predstavlja nametanje uslova

$$(G_1)_{\mathfrak{m}\alpha}{}^i = C_{PFC}\,, \quad (G_1)_{\mathfrak{g}\alpha} = C_{PFC}\,, \quad (G_1)_{\mathfrak{h}a}{}^i = C_{PFC}\,, \quad (G_1)_{\mathfrak{n}a} = C_{PFC}\,, \tag{D.23}$$

prirodnim izborom:

$$(G_1)_{\mathfrak{m}\alpha}{}^i = -\Phi(B)_{\alpha}{}^i\,, \quad (G_1)_{\mathfrak{g}\alpha} = -\Phi(\alpha)_{\alpha}\,, \quad (G_1)_{\mathfrak{h}a}{}^i = -\Phi(C)_{\alpha}{}^i\,, \quad (G_1)_{\mathfrak{n}a} = -\Phi(\beta)_a\,. \tag{D.24}$$

Ostaje da se utvrdi kako glase četiri generatora G_0.

Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{m}\alpha}{}^i$ daje

$$\begin{aligned} (G_0)_{\mathfrak{m}\alpha}{}^i - \{\Phi(B)_{\alpha}{}^i\,, H_T\} &= (C_{PFC})_{\alpha}{}^i\,, \\ (G_0)_{\mathfrak{m}\alpha}{}^i - \Phi(\mathcal{F})_{\alpha}{}^i &= (C_{PFC})_{\alpha}{}^i\,, \end{aligned} \tag{D.25}$$

gde je $(G_0)_{\mathfrak{m}\alpha}{}^i = (C_{PFC})_{\alpha}{}^i + \Phi(\mathcal{F})_{\alpha}{}^i$. Zatim, iz Kastelanijevog trećeg uslova sledi

$$\begin{aligned} \{(G_0)_{\mathfrak{m}\alpha}{}^i\,, H_T\} &= (C_{PFC1})_{\alpha}{}^i\,, \\ \{(C_{PFC})_{\alpha}{}^i + \Phi(\mathcal{F})_{\alpha}{}^i\,, H_T\} &= (C_{PFC1})_{\alpha}{}^i\,, \\ \{(C_{PFC})_{\alpha}{}^i\,, H_T\} - f_{\beta\gamma\alpha}\alpha^{\beta}{}_0\Phi(\mathcal{F})^{\gamma i} &= (C_{PFC1})_{\alpha}{}^i\,, \end{aligned} \tag{D.26}$$

što daje jednačinu

$$(C_{PFC})_{\alpha}{}^i = f_{\beta\gamma\alpha}\alpha^{\beta}{}_0\Phi(B)^{\gamma i}\,.$$

Iz toga sledi da je generator:

$$(G_0)_{\mathfrak{m}\alpha}{}^i = f_{\beta\gamma\alpha}\alpha^{\beta}{}_0\Phi(B)^{\gamma i} + \Phi(\mathcal{F})_{\alpha}{}^i\,. \tag{D.27}$$

Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{g}\alpha}$ daje

$$\begin{aligned} (G_0)_{\mathfrak{g}\alpha} - \{\Phi(\alpha)_{\alpha}\,, H_T\} &= (C_{PFC})_{\alpha}\,, \\ (G_0)_{\mathfrak{g}\alpha} - \Phi(\nabla B)_{\alpha} &= (C_{PFC})_{\alpha}\,, \end{aligned} \tag{D.28}$$

tj. dobija se da je $(G_0)_{\mathfrak{g}\alpha} = (C_{PFC})_\alpha + \Phi(\nabla B)_\alpha$. Nakon toga, iz trećeg Kastelanijevog uslova sledi

$$\begin{aligned}
&\{(G_0)_{\mathfrak{g}\alpha}\,, H_T\} = (C_{PFC1})_\alpha\,,\\
&\{(C_{PFC})_\alpha + \Phi(\nabla B)_\alpha\,, H_T\} = (C_{PFC1})_\alpha\,,\\
&\{(C_{PFC})_\alpha\,, H_T\} + B_{\beta 0i} f_{\alpha\gamma}{}^\beta \Phi(\mathcal{F})^{\gamma i} - \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma \Phi(\nabla B)_\gamma + C_{a0} \rhd_{\alpha b}{}^a \Phi(\mathcal{G})^b + \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\nabla C)^{bi} = (C_{PFC1})_\alpha\,,
\end{aligned} \tag{D.29}$$

tj.

$$(C_{PFC})_\alpha = -B_{\beta 0i} f_{\alpha\gamma}{}^\beta \Phi(B)^{\gamma i} + \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma \Phi(\alpha)_\gamma - C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^b - \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)^{bi}\,.$$

Sledi da je generator:

$$(G_0)_{\mathfrak{g}\alpha} \;=\; -B_{\beta 0i} f_{\alpha\gamma}{}^\beta \Phi(B)^{\gamma i} + \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma \Phi(\alpha)_\gamma - C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^b - \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)^{bi} + \Phi(\nabla B)_\alpha\,. \tag{D.30}$$

Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{n}a}$ daje

$$\begin{aligned}
&(G_0)_{\mathfrak{n}a} - \{\Phi(C)_a\,, H_T\} = (C_{PFC})_a\,,\\
&(G_0)_{\mathfrak{n}a} - \Phi(\mathcal{G})_a = (C_{PFC})_a\,,
\end{aligned} \tag{D.31}$$

gde je $(G_0)_{\mathfrak{n}a} = (C_{PFC})_a + \Phi(\mathcal{G})_a$.

Zatim, iz trećeg Kastelanijevog uslova dobijamo

$$\begin{aligned}
&\{(G_0)_{\mathfrak{n}a}\,, H_T\} = (C_{PFC1})_a\,,\\
&\{(C_{PFC})_a + \Phi(\mathcal{G})_a\,, H_T\} = (C_{PFC1})_a\,,\\
&\{(C_{PFC})_a, H_T\} + \alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(\mathcal{G})_b - \beta_{b0i} \rhd_{\alpha a}{}^b \Phi(\mathcal{F})^{\alpha i} = (C_{PFC1})_a\,,
\end{aligned} \tag{D.32}$$

što daje:

$$(C_{PFC})_a = -\alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(C)_b + \beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i}\,.$$

Iz toga sledi da je generator:

$$(G_0)_{\mathfrak{n}a} = -\alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(C)_b + \beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a\,.$$

Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{h}a}{}^i$ daje

$$\begin{aligned}
&(G_0)_{\mathfrak{h}a}{}^i - \{\Phi(\beta)_a{}^i\,, H_T\} = (C_{PFC})_a{}^i\,,\\
&(G_0)_{\mathfrak{h}a}{}^i - \Phi(\nabla C)_a{}^i = (C_{PFC})_a{}^i\,,
\end{aligned} \tag{D.33}$$

tj. dobija se $(G_0)_{\mathfrak{h}a}{}^i = (C_{PFC})_a{}^i + \Phi(\nabla C)_a{}^i$. Nakon toga, iz trećeg Kastelanijevog uslova sledi

$$\begin{aligned}
&\{(G_0)_{\mathfrak{h}a}{}^i\,, H_T\} = (C_{PFC1})_a{}^i\,,\\
&\{(C_{PFC})_a{}^i + \Phi(\nabla C)_a{}^i\,, H_T\} = (C_{PFC1})_a{}^i\,,\\
&\{(C_{PFC})_a{}^i, H_T\} + \alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(\nabla C)_b{}^i - C_{b0} \rhd_{\alpha a}{}^b \Phi(\mathcal{F})^{\alpha i} = (C_{PFC1})_a{}^i\,,
\end{aligned}$$

što daje rezultat:

$$(C_{PFC})_a{}^i = -\alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(\beta)_b{}^i + C_{b0} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} .$$

Iz toga sledi da je generator:

$$(G_0)_{\mathfrak{h}a}{}^i = -\alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(\beta)_b{}^i + C_{b0} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\nabla C)_a{}^i .$$

U ovom trenutku, korisno je rezimirati rezultate i uvesti novu notaciju:

$$\begin{aligned} \dot{\epsilon}_{\mathfrak{m}}{}^\alpha{}_i (G_1)_{\mathfrak{m}\alpha}{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i (G_0)_{\mathfrak{m}\alpha}{}^i &= -\nabla_0 \epsilon_{\mathfrak{m}}{}^\alpha{}_i \Phi(B)_\alpha{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i \Phi(\mathcal{F})_\alpha{}^i \\ &= \nabla_0 \epsilon_{\mathfrak{m}}{}^\alpha{}_i (\tilde{M}_1)_\alpha{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i (\tilde{M}_0)_\alpha{}^i . \end{aligned} \tag{D.34}$$

Primetimo da se vremenski izvod parametra kombinuje sa nekim drugim članovima u kovarijantni izvod u vremenskom pravcu.

Za drugi deo ukupnog generatora dobijamo:

$$\begin{aligned} \dot{\epsilon}_{\mathfrak{g}}{}^\alpha (G_1)_{\mathfrak{g}\alpha} + \epsilon_{\mathfrak{g}}{}^\alpha (G_0)_{\mathfrak{g}\alpha} &= -\dot{\epsilon}_{\mathfrak{g}}{}^\alpha \Phi(\alpha)_\alpha - \epsilon_{\mathfrak{g}}{}^\alpha \big(B_{\beta 0 i} f_{\alpha\gamma}{}^\beta \Phi(B)^{\gamma i} - \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma \Phi(\alpha)_\gamma \\ &\quad + C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^b + \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)_b{}^i - \Phi(\nabla B)_\alpha \big) \\ &= -\nabla_0 \epsilon_{\mathfrak{g}}{}^\alpha \Phi(\alpha)_\alpha - \epsilon_{\mathfrak{g}}{}^\alpha \big(B_{\beta 0 i} f_{\alpha\gamma}{}^\beta \Phi(B)^{\gamma i} \\ &\quad + C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^b + \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)_b{}^i - \Phi(\nabla B)_\alpha \big) \\ &= \nabla_0 \epsilon_{\mathfrak{g}}{}^\alpha (\tilde{G}_1)_\alpha + \epsilon_{\mathfrak{g}}{}^\alpha (\tilde{G}_0)_\alpha . \end{aligned} \tag{D.35}$$

Osim toga, sledi:

$$\begin{aligned} \dot{\epsilon}_{\mathfrak{h}}{}^a{}_i (G_1)_{\mathfrak{h}a}{}^i + \epsilon_{\mathfrak{h}}{}^a{}_i (G_0)_{\mathfrak{h}a}{}^i &= -\nabla_0 \epsilon_{\mathfrak{h}}{}^a{}_i \Phi(\beta)_\alpha{}^i + \epsilon_{\mathfrak{h}}{}^a{}_i \big(C_{b0} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\nabla C)_a{}^i \big) \\ &= \nabla_0 \epsilon_{\mathfrak{h}}{}^a{}_i (\tilde{H}_1)_a{}^i + \epsilon_{\mathfrak{h}}{}^a{}_i (\tilde{H}_0)_a{}^i , \end{aligned} \tag{D.36}$$

$$\begin{aligned} \dot{\epsilon}_{\mathfrak{n}}{}^a (G_1)_{\mathfrak{n}a} + \epsilon_{\mathfrak{n}}{}^a (G_0)_{\mathfrak{n}a} &= -\nabla_0 \epsilon_{\mathfrak{n}}{}^a \Phi(C)_a + \epsilon_{\mathfrak{n}}{}^a (\beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a) \\ &= \nabla_0 \epsilon_{\mathfrak{n}}{}^a (\tilde{N}_1)_a + \epsilon_{\mathfrak{n}}{}^a (\tilde{N}_0)_a . \end{aligned} \tag{D.37}$$

D.2.3 Izračunavanje algebre simetrija 2*BF* dejstva

Da bi se dobila struktura grupe simetrija $2BF$ dejstva, kao što je predstavljeno u podsekciji 5.1.2, moramo najpre izračunati komutatore između generatora G-, H-, M- i N-gejdž simetrija. Ovaj proces je opisan u odeljku 5.1.2, dok su detalji izračunavanja dati u ovom odeljku.

Komutator $[H, H]$

Sada ćemo izračunati komutator generatora H-gejdž transformacija, tj. jednačinu (5.51). Nakon transformacije promenljivih pri H-gejdž transformacijama za parametar $\epsilon_{\mathfrak{h}1}$ dobija se

$$\alpha' = \alpha - \partial\epsilon_{\mathfrak{h}1}\,, \tag{D.38}$$

$$\beta' = \beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1}\,, \tag{D.39}$$

$$B' = B - C\wedge^{\mathcal{T}}\epsilon_{\mathfrak{h}1}\,, \tag{D.40}$$

$$C' = C\,, \tag{D.41}$$

Zatim, daljom transformacijom varijabli H-gejdž transformacijama sa parametrom $\epsilon_{\mathfrak{h}2}$ dobija se:

$$\begin{aligned} \alpha'' &= \alpha - \partial\epsilon_{\mathfrak{h}1} - \partial\epsilon_{\mathfrak{h}2}\,, \\ \beta'' &= \beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1} - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}-\partial\epsilon_{\mathfrak{h}2}}{\nabla}\epsilon_{\mathfrak{h}2} - \epsilon_{\mathfrak{h}2}\wedge\epsilon_{\mathfrak{h}2}\,, \\ B'' &= B - C\wedge^{\mathcal{T}}\epsilon_{\mathfrak{h}1} - C\wedge^{\mathcal{T}}\epsilon_{\mathfrak{h}2} \\ C'' &= C\,, \end{aligned} \tag{D.42}$$

Vidimo da za promenljive $\alpha^{\alpha}{}_{\mu}$, $B^{\alpha}{}_{\mu\nu}$ i $C^{a}{}_{\mu}$ dobijamo da transformacije komutiraju:

$$\begin{aligned} \mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\alpha^{\alpha}{}_{\mu} &= \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\alpha^{\alpha}{}_{\mu}\,, \\ \mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}B^{\alpha}{}_{\mu\nu} &= \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}B^{\alpha}{}_{\mu\nu}\,, \\ \mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}C^{a}{}_{\mu} &= \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}C^{a}{}_{\mu}\,, \end{aligned} \tag{D.43}$$

Za preostale promenljivu $\beta^{a}{}_{\mu\nu}$ razlika jednačine (D.42) i analogne jednačine gde $\epsilon_{\mathfrak{h}1}\leftrightarrow\epsilon_{\mathfrak{h}2}$ je:

$$\begin{aligned} \left(\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H} - \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\right)\frac{1}{2}\beta^{a}{}_{\mu\nu} &= \partial_{b}{}^{\alpha}\epsilon_{\mathfrak{h}2}{}^{b}{}_{[\mu|}\epsilon_{\mathfrak{h}1}{}^{c}{}_{|\nu]}\rhd_{\alpha c}{}^{a} - \partial_{b}{}^{\alpha}\epsilon_{\mathfrak{h}1}{}^{b}{}_{[\mu|}\epsilon_{\mathfrak{h}2}{}^{c}{}_{|\nu]}\rhd_{\alpha c}{}^{a} \\ &= 0\,. \end{aligned} \tag{D.44}$$

Uzimajući u obzir rezultate (D.43) i (D.44) zaključujemo da H-gejdž transformacije komutiraju:

$$\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H} - \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H} = 0\,. \tag{D.45}$$

Komutator $[H, N]$

Izračunajmo komutator između generatora H-gejdž i N-gejdž transformacija, tj. izvedimo jednačinu (5.62). Ovo se radi izračunavanjem izraza

$$\left(\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N} - \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\right)A\,, \tag{D.46}$$

za sve varijable A prisutne u teoriji. Primećujemo da je za varijable $\alpha^{\alpha}{}_{\mu}$, $\beta^{a}{}_{\mu\nu}$ i $C^{a}{}_{\mu}$ dobijeno da transformacije komutiraju:

$$\begin{aligned} \mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\alpha^{\alpha}{}_{\mu} &= \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\alpha^{\alpha}{}_{\mu}\,, \\ \mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\beta^{a}{}_{\mu\nu} &= \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\beta^{a}{}_{\mu\nu}\,, \\ \mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}C^{a}{}_{\mu} &= \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}C^{a}{}_{\mu}\,. \end{aligned} \tag{D.47}$$

Preostala varijabla $B^{\alpha}{}_{\mu\nu}$ se pri H-gejdž transformacijama transformiše na sledeći način:

$$\begin{aligned} B' &= B - C \wedge^{\tau} \epsilon_{\mathfrak{h}}\,, \\ C' &= C\,. \end{aligned} \tag{D.48}$$

Daljom transformacijom varijable N-gejdž transformacijom:

$$\begin{aligned} B'' &= B' - \beta' \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}} \\ &= B - C \wedge^{\tau} \epsilon_{\mathfrak{h}} - (\beta - \overset{\{\alpha^{\alpha} - \partial_a{}^{\alpha}\epsilon_{\mathfrak{h}}{}^{a}\}}{\nabla\epsilon_{\mathfrak{h}}} - \epsilon_{\mathfrak{h}} \wedge \epsilon_{\mathfrak{h}}) \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}}\,, \\ C'' &= C' - \overset{\{\alpha^{\alpha} - \partial_a{}^{\alpha}\epsilon_{\mathfrak{h}}{}^{a}\}}{\nabla} \epsilon_{\mathfrak{n}} \\ &= C - \overset{\{\alpha^{\alpha} - \partial_a{}^{\alpha}\epsilon_{\mathfrak{h}}{}^{a}\}}{\nabla} \epsilon_{\mathfrak{n}}\,. \end{aligned} \tag{D.49}$$

Zatim, izmenimo redosled transformacija. Najpre, transformacija varijabli pri N-gejdž transformacijama je

$$\begin{aligned} B^{\cdot} &= B - \beta \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}}\,, \\ C^{\cdot} &= C - \nabla\epsilon_{\mathfrak{n}}\,, \end{aligned} \tag{D.50}$$

dok dalja H-gejdž transformacija daje:

$$\begin{aligned} B^{\cdot\cdot} &= B^{\cdot} - C^{\cdot} \wedge^{\tau} \epsilon_{\mathfrak{h}} \\ &= B - \beta \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}} - (C - \nabla\epsilon_{\mathfrak{n}}) \wedge^{\tau} \epsilon_{\mathfrak{h}}\,, \\ C^{\cdot\cdot} &= C^{\cdot} \\ &= C - \nabla\epsilon_{\mathfrak{n}}\,. \end{aligned} \tag{D.51}$$

Razlika jednačina (D.49) i (D.51) je:

$$\begin{aligned} \left(e^{\epsilon_{\mathfrak{h}}\cdot H} e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N} e^{\epsilon_{\mathfrak{h}}\cdot H}\right) B^{\alpha} &= \nabla\epsilon_{\mathfrak{n}}{}^{a} \wedge \epsilon_{\mathfrak{h}}{}^{b} \mathcal{T}_{ab}{}^{\alpha} - \nabla\epsilon_{\mathfrak{h}}{}^{a} \wedge \epsilon_{\mathfrak{n}}{}^{b} \mathcal{T}_{ab}{}^{\alpha} + \partial_a{}^{\beta} \rhd_{\beta c}{}^{b} \epsilon_{\mathfrak{h}}{}^{a} \wedge \epsilon_{\mathfrak{h}}{}^{c} \wedge \epsilon_{\mathfrak{n}}{}^{d} \mathcal{T}_{bd}{}^{\alpha} - \epsilon_{\mathfrak{h}}{}^{a} \wedge \epsilon_{\mathfrak{h}}{}^{b} f_{ab}{}^{c} \epsilon_{\mathfrak{n}}{}^{d} \mathcal{T}_{cd}{}^{\alpha}\,, \\ \left(e^{\epsilon_{\mathfrak{h}}\cdot H} e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N} e^{\epsilon_{\mathfrak{h}}\cdot H}\right) C^{c} &= -\partial_a{}^{\beta} \epsilon_{\mathfrak{h}}{}^{a} \rhd_{\beta b}{}^{c} \epsilon_{\mathfrak{n}}{}^{b}\,, \end{aligned} \tag{D.52}$$

Primenom definicija preslikavanja $\mathcal{T}$ prethodne jednačine se svode na

$$\begin{aligned} \left(e^{\epsilon_{\mathfrak{h}}\cdot H} e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N} e^{\epsilon_{\mathfrak{h}}\cdot H}\right) B^{\alpha} &= \nabla\epsilon_{\mathfrak{n}}{}^{a} \wedge \epsilon_{\mathfrak{h}}{}^{b} \mathcal{T}_{ab}{}^{\alpha} - \nabla\epsilon_{\mathfrak{h}}{}^{a} \wedge \epsilon_{\mathfrak{n}}{}^{b} \mathcal{T}_{ab}{}^{\alpha} = \nabla(\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}})^{\alpha}\,, \\ \left(e^{\epsilon_{\mathfrak{h}}\cdot H} e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N} e^{\epsilon_{\mathfrak{h}}\cdot H}\right) C^{c} &= \partial^{c}{}_{\alpha}(\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}})^{\alpha}\,, \end{aligned} \tag{D.53}$$

Upoređivanjem jednačina (D.47) i (D.53) sa jednačinom (6.58) dobijamo konačan rezultat za komutator

$$\left(e^{\epsilon_{\mathfrak{h}}\cdot H} e^{\epsilon_{\mathfrak{n}}\cdot N} - e^{\epsilon_{\mathfrak{n}}\cdot N} e^{\epsilon_{\mathfrak{h}}\cdot H}\right) = -(\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}}) \cdot M\,. \tag{D.54}$$

D.3 Gejdž transformacije u $3BF$ topološkoj teoriji

Generator gejdž transformacija u $3BF$ teoriji je:

$$
\begin{aligned}
G \;=\; \int_{\Sigma_3} \mathrm{d}^3\vec{x}\Big(& (\nabla_0 \epsilon_{\mathfrak{g}}{}^{\alpha})\,(\tilde{G}_1)_{\alpha} + \epsilon_{\mathfrak{g}}{}^{\alpha}\,(\tilde{G}_0)_{\alpha} + (\nabla_0 \epsilon_{\mathfrak{h}}{}^{a}{}_{i})\,(\tilde{H}_1)_{a}{}^{i} + \epsilon_{\mathfrak{h}}{}^{a}{}_{i}\,(\tilde{H}_0)_{a}{}^{i} \\
& + \frac{1}{2}(\nabla_0 \epsilon_{\mathfrak{l}}{}^{A}{}_{ij})\,(\tilde{L}_1)_{A}{}^{ij} + \frac{1}{2}\epsilon_{\mathfrak{l}}{}^{A}{}_{ij}\,(\tilde{L}_0)_{A}{}^{ij} \\
& + (\nabla_0 \epsilon_{\mathfrak{m}}{}^{\alpha}{}_{i})\,(\tilde{M}_1)_{\alpha}{}^{i} + \epsilon_{\mathfrak{m}}{}^{\alpha}{}_{i}\,(\tilde{M}_0)_{\alpha}{}^{i} + (\nabla_0 \epsilon_{\mathfrak{n}}{}^{a})\,(\tilde{N}_1)_{a} + \epsilon_{\mathfrak{n}}{}^{a}(\tilde{N}_0)_{a}\Big)\,,
\end{aligned}
\tag{D.55}
$$

gde je

$$
\begin{aligned}
(\tilde{G}_1)_{\alpha} &= -\Phi(\alpha)_{\alpha}\,, \\
(\tilde{G}_0)_{\alpha} &= -\big(f_{\alpha\gamma}{}^{\beta}B_{\beta 0i}\Phi(B)^{\gamma i} + C_{a0}\rhd_{\alpha b}{}^{a}\Phi(C)^{b0} + \beta_{a0i}\rhd_{\alpha b}{}^{a}\Phi(\beta)^{b0i} \\
&\quad -\frac{1}{2}\gamma^{A}{}_{0ij}\rhd_{\alpha A}{}^{B}\Phi(\gamma)_{B}{}^{ij} - \Phi(\nabla B)_{\alpha}\big)\,, \\
(\tilde{H}_1)_{a}{}^{i} &= -\Phi(\beta)_{a}{}^{i}\,, \\
(\tilde{H}_0)_{a}{}^{i} &= C_{b0}\rhd_{\alpha a}{}^{b}\Phi(B)^{\alpha i} - 2\beta^{b}{}_{0j}X_{(ab)}{}^{A}\Phi(\gamma)_{A}{}^{ij} + \Phi(\nabla C)_{a}{}^{i}\,, \\
(\tilde{L}_1)_{a}{}^{ij} &= \Phi(\gamma)_{A}{}^{ij}\,, \\
(\tilde{L}_0)_{a}{}^{ij} &= -\Phi(\nabla D)_{A}{}^{ij}\,, \\
(\tilde{M}_1)_{\alpha}{}^{i} &= -\Phi(B)_{\alpha}{}^{i}\,, \\
(\tilde{M}_0)_{\alpha}{}^{i} &= \Phi(\mathcal{F})_{\alpha}{}^{i}\,, \\
(\tilde{N}_1)_{a} &= -\Phi(C)_{a}\,, \\
(\tilde{N}_0)_{a} &= \beta_{b0i}\rhd_{\alpha a}{}^{b}\Phi(B)^{\alpha i} + \Phi(\mathcal{G})_{a}\,,
\end{aligned}
\tag{D.56}
$$

gde su $\epsilon_{\mathfrak{g}}{}^{\alpha}$, $\epsilon_{\mathfrak{h}}{}^{a}{}_{i}$, $\epsilon_{\mathfrak{l}}{}^{A}{}_{ij}$, $\epsilon_{\mathfrak{m}}{}^{\alpha}{}_{i}$, i $\epsilon_{\mathfrak{n}}{}^{a}$ nezavisni parametri gejdž transformacija.

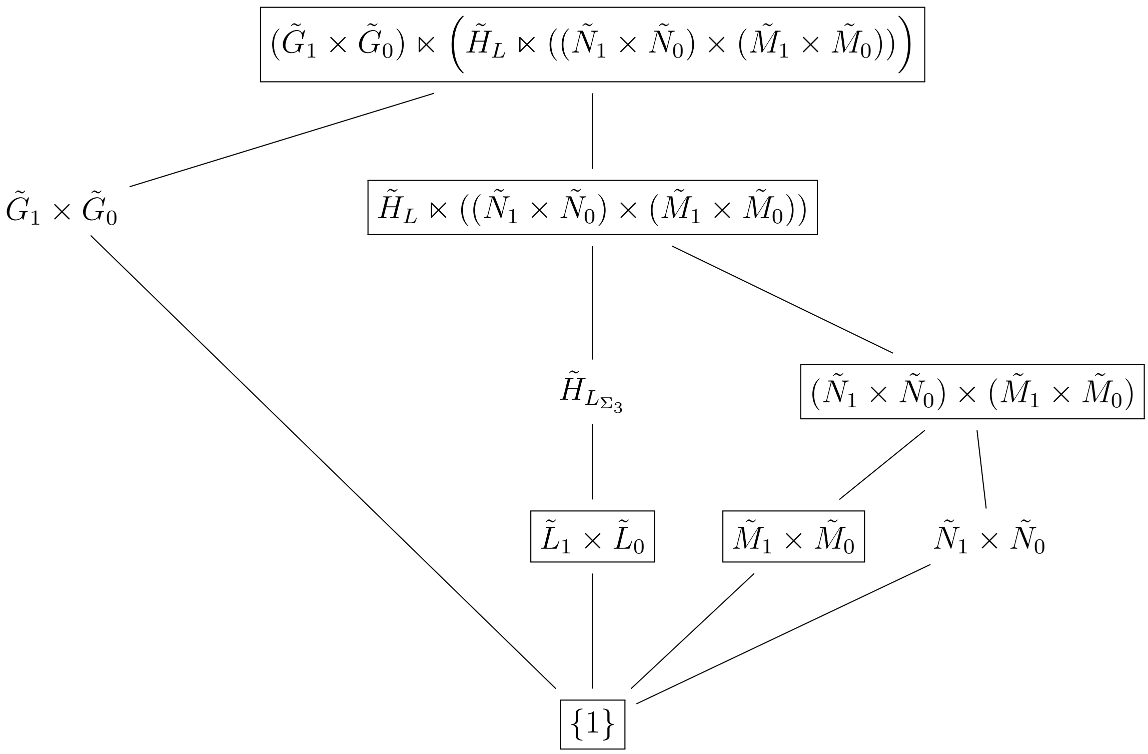
D.3.1 Gejdž grupa simetrije $3BF$ dejstva

Algebra koju čine generatori grupe simetrija $(\tilde{M}_0)_{\alpha}{}^{i}$, $(\tilde{M}_1)_{\alpha}{}^{i}$, $(\tilde{G}_0)_{\alpha}$, $(\tilde{G}_1)_{\alpha}$, $(\tilde{H}_0)_{a}{}^{i}$, $(\tilde{H}_1)_{a}{}^{i}$, $(\tilde{N}_0)_{a}$, $(\tilde{N}_1)_{a}$, $(\tilde{L}_0)_{A}{}^{ij}$ i $(\tilde{L}_1)_{A}{}^{ij}$ definisani u Dodatku D.3 je:

$$
\{(\tilde{G}_0)_{\alpha}(\vec{x})\,,(\tilde{G}_0)_{\beta}(\vec{y})\} = f_{\alpha\beta}{}^{\gamma}(\tilde{G}_0)_{\gamma}\,\delta^{(3)}(\vec{x}-\vec{y})\,,
\tag{D.57}
$$

$$
\begin{aligned}
\{(\tilde{H}_0)_{a}{}^{i}(\vec{x})\,,(\tilde{H}_0)_{b}{}^{j}(\vec{y})\} &= 2X_{(ab)}{}^{A}(\tilde{L}_0)_{A}{}^{ij}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{(\tilde{H}_0)_{a}{}^{i}(\vec{x})\,,(\tilde{H}_1)_{b}{}^{j}(\vec{y})\} &= 2X_{(ab)}{}^{A}(\tilde{L}_1)_{A}{}^{ij}\,\delta^{(3)}(\vec{x}-\vec{y})\,,
\end{aligned}
\tag{D.58}
$$

$$
\begin{aligned}
\{(\tilde{H}_0)_{a}{}^{i}(\vec{x})\,,(\tilde{N}_0)^{b}(\vec{y})\} &= \rhd_{\alpha a}{}^{b}(\tilde{M}_0)^{\alpha i}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{(\tilde{H}_1)_{a}{}^{i}(\vec{x})\,,(\tilde{N}_0)^{b}(\vec{y})\} &= \rhd_{\alpha a}{}^{b}(\tilde{M}_1)^{\alpha i}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{(\tilde{H}_0)_{a}(\vec{x})\,,(\tilde{N}_1)^{bi}(\vec{y})\} &= \rhd_{\alpha a}{}^{b}(\tilde{M}_1)^{\alpha i}\,\delta^{(3)}(\vec{x}-\vec{y})\,,
\end{aligned}
\tag{D.59}
$$



Slika D.3: Grupa simetrije $\mathcal{G}_{\Sigma_3}$ u faznom prostoru. Invarijantne grupe su okvirene.

$$
\begin{aligned}
\{(\tilde{G}_0)_\alpha(\vec{x})\,,(\tilde{M}_0)_\beta{}^i(\vec{y})\} &= f_{\alpha\beta}{}^\gamma(\tilde{M}_0)_\gamma{}^i\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{(\tilde{G}_0)_\alpha(\vec{x})\,,(\tilde{M}_1)_\beta{}^i(\vec{y})\} &= f_{\alpha\beta}{}^\gamma(\tilde{M}_1)_\gamma{}^i\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,(\tilde{G}_0)_\alpha(\vec{x})\,,\,(\tilde{H}_1)_a{}^i(\vec{y})\,\} &= \rhd_{\alpha a}{}^b\,(\tilde{H}_1)_b{}^i(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,(\tilde{G}_0)_\alpha(\vec{x})\,,\,(\tilde{H}_0)_a{}^i(\vec{y})\,\} &= \rhd_{\alpha a}{}^b\,(\tilde{H}_0)_b{}^i(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,(\tilde{G}_0)_\alpha(\vec{x})\,,\,(\tilde{N}_1)_a(\vec{y})\,\} &= \rhd_{\alpha a}{}^b\,(\tilde{N}_1)_b(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,(\tilde{G}_0)_\alpha(\vec{x})\,,\,(\tilde{N}_0)_a(\vec{y})\,\} &= \rhd_{\alpha a}{}^b\,(\tilde{N}_0)_b(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,(\tilde{G}_0)_\alpha(\vec{x})\,,\,(\tilde{L}_0)_A{}^{ij}(\vec{y})\,\} &= \rhd_{\alpha A}{}^B(\tilde{L}_0)_B{}^{ij}(\vec{x})\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{D.60}
$$

Grupa gejdž simetrije ima sledeću strukturu. Prvo, grupe $\tilde{M}_1 \times \tilde{M}_0$, $\tilde{N}_1 \times \tilde{N}_0$ i $\tilde{L}_1 \times \tilde{L}_0$ sa odgovarajućim algebrama $\mathfrak{a}_1$, $\mathfrak{a}_2$ i $\mathfrak{a}_3$, gde je

$$
\begin{gathered}
\mathfrak{a}_1 = \operatorname{span}\{(\tilde{M}_1)_\alpha{}^i\} \oplus \operatorname{span}\{(\tilde{M}_0)_\alpha{}^i\}\,, \quad \mathfrak{a}_2 = \operatorname{span}\{(\tilde{N}_1)_a\} \oplus \operatorname{span}\{(\tilde{N}_0)_a\}\,,\\
\mathfrak{a}_3 = \operatorname{span}\{(\tilde{L}_1)_A{}^{ij}\} \oplus \operatorname{span}\{(\tilde{L}_0)_A{}^{ij}\}\,,
\end{gathered}
\tag{D.61}
$$

su podgrupe ukupne grupe simetrije $\tilde{\mathcal{G}}_{\Sigma_3}$. Pored toga, podgrupe $\tilde{L}_1 \times \tilde{L}_0$ i $\tilde{M}_1 \times \tilde{M}_0$ su invarijantne podgrupe ukupne grupe simetrije. Grupa $\tilde{N}_1 \times \tilde{N}_0$ nije invarijantna podgrupa ukupne grupe simetrije, što vidimo na osnovu Poasonovih zagrada $\{(\tilde{H}_0)_a{}^i(\vec{x})\,,(\tilde{N}_0)^b(\vec{y})\}$ i $\{(\tilde{H}_1)_a{}^i(\vec{x})\,,(\tilde{N}_0)^b(\vec{y})\}$ koje su jednake nekim linearnim kombinacijama generatora $\tilde{M}_1 \times \tilde{M}_0$. Može se formirati direktan proizvod $(\tilde{N}_1 \times \tilde{N}_0) \times (\tilde{M}_1 \times \tilde{M}_0)$, kako generatori ovih grupa međusobno komutiraju, a dobijena grupa je invarijantna podgrupa ukupne grupe simetrija.

Zatim, razmotrimo podgrupu $\tilde{H}_{L_{\Sigma_3}}$ određenu algebrom definisanu generatorima $(\tilde{L}_1)_A{}^{ij}$, $(\tilde{L}_0)_A{}^{ij}$, $(\tilde{H}_1)_a{}^i$ i $(\tilde{H}_0)_a{}^i$. Ova grupa nije invarijantna pogrupa ukupne grupe simetrija, zbog

Poasonovih zagrada $\{(\tilde{H}_0)_a{}^i(\vec{x})\,,(\tilde{N}_0)^b(\vec{y})\}$ i $\{(\tilde{H}_1)_a{}^i(\vec{x})\,,(\tilde{N}_0)^b(\vec{y})\}$, iz istog razloga kao i pre. Sada se mogu pomnožiti ove dve podgrupe, od kojih je jedna invarijantna, a druga ne, koristeći semidirektan proizvod, čime se dobija invarijantna podgrupa $H_L \ltimes ((N_1\times N_0)\times(M_1\times M_0))$, kojoj odgovara algebra $\mathfrak{a}_4$:

$$\mathfrak{a}_4 = \operatorname{span}\{(\tilde{M}_0)_\alpha{}^i,(\tilde{M}_1)_\alpha{}^i,(\tilde{H}_0)_a{}^i,(\tilde{H}_1)_a{}^i,(\tilde{N}_0)_a,(\tilde{N}_1)_a,(\tilde{L}_0)_A{}^{ij},(\tilde{L}_1)_A{}^{ij}\}\,.$$

Na kraju, prateći istu liniju rezonovanja, razmatranjem grupe $\tilde{G}_1\times\tilde{G}_0$ dobijamo ukupnu grupu simetrija $\tilde{\mathcal{G}}_{\Sigma_3}$ koja je jednaka:

$$\tilde{\mathcal{G}}_{\Sigma_3} = (\tilde{G}_1\times\tilde{G}_0)\ltimes\Big(\tilde{H}_L\ltimes((\tilde{N}_1\times\tilde{N}_0)\times(\tilde{M}_1\times\tilde{M}_0))\Big)\,.$$

Kompletna struktura ukupne grupe simetrija prikazana je na Slici D.3. Ovde su invarijantne podgrupe ukupne grupe simetrija uokvirene.

D.3.2 Konstrukcija generatora simetrija $3BF$ teorije

Kada zamenimo generatore (D.56) u jednačinu (6.30), dobijamo generator gejdž simetrija u $3BF$ teoriji sledećeg oblika

$$\begin{aligned}G \;=\; -\int_{\Sigma_3} \mathrm{d}^3\vec{x}\Big(& (\nabla_0\epsilon_{\mathfrak{m}}{}^\alpha{}_i)\Phi(B)_\alpha{}^i - \epsilon_{\mathfrak{m}}{}^\alpha{}_i\Phi(\mathcal{F})_\alpha{}^i + (\nabla_0\epsilon_{\mathfrak{g}}{}^\alpha)\Phi(\alpha)_\alpha + \epsilon_{\mathfrak{g}}{}^\alpha\big(f_{\alpha\gamma}{}^\beta B_{\beta 0i}\Phi(B)^{\gamma i}\\ & + C_{a0}\rhd_{\alpha b}{}^a\Phi(C)^{b0} + \beta_{a0i}\rhd_{\alpha b}{}^a\Phi(\beta)^{b0i} - \frac{1}{2}\gamma^A{}_{0ij}\rhd_{\alpha A}{}^B\Phi(\gamma)_B{}^{ij} - \Phi(\nabla B)_\alpha\big)\\ & + (\nabla_0\epsilon_{\mathfrak{n}}{}^a)\Phi(C)_a - \epsilon_{\mathfrak{n}}{}^a\left(\beta_{b0i}\rhd_{\alpha a}{}^b\Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a\right)\\ & + (\nabla_0\epsilon_{\mathfrak{h}}{}^a{}_i)\Phi(\beta)_a{}^i - \epsilon_{\mathfrak{h}}{}^a{}_i\left(C_{b0}\rhd_{\alpha a}{}^b\Phi(B)^{\alpha i} - 2\beta^b{}_{0j}X_{(ab)}{}^A\Phi(\gamma)_A{}^{ij} + \Phi(\nabla C)_a{}^i\right)\\ & - \frac{1}{2}(\nabla_0\epsilon_{\mathfrak{l}}{}^A{}_{ij})\Phi(\gamma)_A{}^{ij} + \frac{1}{2}\epsilon_{\mathfrak{l}}{}^A{}_{ij}\Phi(\nabla D)_A{}^{ij}\Big)\,,\end{aligned} \tag{D.62}$$

gde su $\epsilon_{\mathfrak{g}}{}^\alpha$, $\epsilon_{\mathfrak{h}}{}^a{}_i$, $\epsilon_{\mathfrak{l}}{}^A{}_{ij}$, $\epsilon_{\mathfrak{m}}{}^\alpha{}_i$ i $\epsilon_{\mathfrak{n}}{}^a$ nezavisni parametri gejdž transformacija.

Generator gejdž transformacija simetrije (D.55) u $3BF$ teoriji (6.1), dobija se Kastelanijevom procedurom, pri čemu su za svaki par generatora G_0 i G_1 zadovoljene relacije

$$G_1 \;=\; C_{PFC}\,, \tag{D.63}$$
$$G_0 + \{G_1, H_T\} \;=\; C_{PFC}\,, \tag{D.64}$$
$$\{G_0, H_T\} \;=\; C_{PFC}\,, \tag{D.65}$$

gde C_{PFC} predstavlja neku vezu prve klase. Pretpostavimo, najpre, da generator ima strukturu:

$$\begin{aligned}G \;=\; \int_{\Sigma_3}\mathrm{d}^3\,\vec{x}\,\Big(&\dot{\epsilon}_{\mathfrak{m}}{}^\alpha{}_i(G_1)_{\mathfrak{m}\alpha}{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i(G_0)_{\mathfrak{m}\alpha}{}^i + \dot{\epsilon}_{\mathfrak{g}}{}^\alpha(G_1)_{\mathfrak{g}\alpha} + \epsilon_{\mathfrak{g}}{}^\alpha(G_0)_{\mathfrak{g}\alpha}\\ & + \dot{\epsilon}_{\mathfrak{h}}{}^a{}_i(G_1)_{\mathfrak{h}a}{}^i + \epsilon_{\mathfrak{h}}{}^a{}_i(G_0)_{\mathfrak{h}a}{}^i + \dot{\epsilon}_{\mathfrak{n}}{}^a(G_1)_{\mathfrak{n}a} + \epsilon_{\mathfrak{n}}{}^a(G_0)_{\mathfrak{n}a}\\ & + \frac{1}{2}\dot{\epsilon}_{\mathfrak{l}}{}^A{}_{ij}(G_1)_{\mathfrak{l}A}{}^{ij} + \frac{1}{2}\epsilon_{\mathfrak{l}}{}^A{}_{ij}(G_0)_{\mathfrak{l}A}{}^{ij}\Big)\,.\end{aligned} \tag{D.66}$$

Prvi korak Kastelanijeve procedure predstavlja nametanje uslova

$$\begin{gathered}(G_1)_{\mathfrak{m}\alpha}{}^i = C_{PFC}\,,\quad (G_1)_{\mathfrak{g}\alpha} = C_{PFC}\,,\quad (G_1)_{\mathfrak{h}a}{}^i = C_{PFC}\,,\\ (G_1)_{\mathfrak{n}a} = C_{PFC}\,,\quad (G_1)_{\mathfrak{l}A}{}^{ij} = C_{PFC}\,,\end{gathered} \tag{D.67}$$

prirodnim izborom:

$$\begin{gathered}(G_1)_{\mathfrak{m}\alpha}{}^i = -\Phi(B)_\alpha{}^i\,,\quad (G_1)_{\mathfrak{g}\alpha} = -\Phi(\alpha)_\alpha\,,\quad (G_1)_{\mathfrak{h}a}{}^i = -\Phi(C)_\alpha{}^i\,,\\ (G_1)_{\mathfrak{n}a} = -\Phi(\beta)_a\,,\quad (G_1)_{\mathfrak{l}A}{}^{ij} = \Phi(\gamma)_A{}^{ij}\,.\end{gathered} \tag{D.68}$$

Ostaje da se utvrdi pet generatora G_0.

Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{m}\alpha}{}^i$ daje

$$\begin{aligned}(G_0)_{\mathfrak{m}\alpha}{}^i - \{\Phi(B)_\alpha{}^i\,, H_T\} &= (C_{PFC})_\alpha{}^i\,,\\ (G_0)_{\mathfrak{m}\alpha}{}^i - \Phi(\mathcal{F})_\alpha{}^i &= (C_{PFC})_\alpha{}^i\,,\end{aligned} \tag{D.69}$$

gde je $(G_0)_{\mathfrak{m}\alpha}{}^i = (C_{PFC})_\alpha{}^i + \Phi(\mathcal{F})_\alpha{}^i$. Zatim, iz Kastelanijevog trećeg uslova sledi

$$\begin{aligned}\{(G_0)_{\mathfrak{m}\alpha}{}^i\,, H_T\} &= (C_{PFC1})_\alpha{}^i\,,\\ \{(C_{PFC})_\alpha{}^i + \Phi(\mathcal{F})_\alpha{}^i\,, H_T\} &= (C_{PFC1})_\alpha{}^i\,,\\ \{(C_{PFC})_\alpha{}^i\,, H_T\} - f_{\beta\gamma\alpha}\alpha^\beta{}_0\Phi(\mathcal{F})^{\gamma i} &= (C_{PFC1})_\alpha{}^i\,,\end{aligned} \tag{D.70}$$

što daje jednačinu

$$(C_{PFC})_\alpha{}^i = f_{\beta\gamma\alpha}\alpha^\beta{}_0\Phi(B)^{\gamma i}\,.$$

Iz toga sledi da je generator:

$$(G_0)_{\mathfrak{m}\alpha}{}^i = f_{\beta\gamma\alpha}\alpha^\beta{}_0\Phi(B)^{\gamma i} + \Phi(\mathcal{F})_\alpha{}^i\,. \tag{D.71}$$

Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{g}\alpha}$ daje

$$\begin{aligned}(G_0)_{\mathfrak{g}\alpha} - \{\Phi(\alpha)_\alpha\,, H_T\} &= (C_{PFC})_\alpha\,,\\ (G_0)_{\mathfrak{g}\alpha} - \Phi(\nabla B)_\alpha &= (C_{PFC})_\alpha\,,\end{aligned} \tag{D.72}$$

tj. dobija se da je $(G_0)_{\mathfrak{g}\alpha} = (C_{PFC})_\alpha + \Phi(\nabla B)_\alpha$. Nakon toga, iz trećeg Kastelanijevog uslova sledi

$$\begin{aligned}&\{(G_0)_{\mathfrak{g}\alpha}\,, H_T\} = (C_{PFC1})_\alpha\,,\\ &\{(C_{PFC})_\alpha + \Phi(\nabla B)_\alpha\,, H_T\} = (C_{PFC1})_\alpha\,,\\ &\{(C_{PFC})_\alpha\,, H_T\} + B_{\beta 0i}f_{\alpha\gamma}{}^\beta\Phi(\mathcal{F})^{\gamma i} - \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma\Phi(\nabla B)_\gamma\\ &+C_{a0}\rhd_{\alpha b}{}^a\Phi(\mathcal{G})^b + \beta_{a0i}\rhd_{\alpha b}{}^a\Phi(\nabla C)^{bi} - \frac{1}{2}\gamma^A{}_{0ij}\rhd_{\alpha A}{}^B\Phi(\nabla D)_B{}^{ij} = (C_{PFC1})_\alpha\,,\end{aligned} \tag{D.73}$$

tj.

$$(C_{PFC})_\alpha = -B_{\beta 0i}f_{\alpha\gamma}{}^\beta\Phi(B)^{\gamma i} + \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma\Phi(\alpha)_\gamma - C_{a0}\rhd_{\alpha b}{}^a\Phi(C)^b - \beta_{a0i}\rhd_{\alpha b}{}^a\Phi(\beta)^{bi} + \tfrac{1}{2}\gamma^A{}_{0ij}\rhd_{\alpha A}{}^B\Phi(\gamma)_B{}^{ij}\,.$$

Sledi da je generator:

$$\begin{aligned}(G_0)_{\mathfrak{g}\alpha} \;=\; &-B_{\beta 0i}f_{\alpha\gamma}{}^\beta\Phi(B)^{\gamma i} + \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma\Phi(\alpha)_\gamma - C_{a0}\rhd_{\alpha b}{}^a\Phi(C)^b\\ &-\beta_{a0i}\rhd_{\alpha b}{}^a\Phi(\beta)^{bi} + \frac{1}{2}\gamma^A{}_{0ij}\rhd_{\alpha A}{}^B\Phi(\gamma)_B{}^{ij} + \Phi(\nabla B)_\alpha\,.\end{aligned} \tag{D.74}$$

Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{n}a}$ daje

$$\begin{aligned}(G_0)_{\mathfrak{n}a} - \{\Phi(C)_a\,, H_T\} &= (C_{PFC})_a\,,\\ (G_0)_{\mathfrak{n}a} - \Phi(\mathcal{G})_a &= (C_{PFC})_a\,,\end{aligned} \tag{D.75}$$

gde je $(G_0)_{\mathfrak{n}a} = (C_{PFC})_a + \Phi(\mathcal{G})_a$.

Zatim, iz trećeg Kastelanijevog uslova dobijamo

$$\begin{aligned}
&\{(G_0)_{\mathfrak{n}a}\,, H_T\} = (C_{PFC1})_a\,, \\
&\{(C_{PFC})_a + \Phi(\mathcal{G})_a\,, H_T\} = (C_{PFC1})_a\,, \\
&\{(C_{PFC})_a, H_T\} + \alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(\mathcal{G})_b - \beta_{b0i} \rhd_{\alpha a}{}^b \Phi(\mathcal{F})^{\alpha i} = (C_{PFC1})_a\,,
\end{aligned} \tag{D.76}$$

što daje

$$(C_{PFC})_a = -\alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(C)_b + \beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i}\,.$$

Iz toga sledi da je generator:

$$(G_0)_{\mathfrak{n}a} = -\alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(C)_b + \beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a\,.$$

Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{h}a}{}^i$ daje

$$\begin{aligned}
&(G_0)_{\mathfrak{h}a}{}^i - \{\Phi(\beta)_a{}^i\,, H_T\} = (C_{PFC})_a{}^i\,, \\
&(G_0)_{\mathfrak{h}a}{}^i - \Phi(\nabla C)_a{}^i = (C_{PFC})_a{}^i\,,
\end{aligned} \tag{D.77}$$

tj. dobija se $(G_0)_{\mathfrak{h}a}{}^i = (C_{PFC})_a{}^i + \Phi(\nabla C)_a{}^i$. Nakon toga, iz trećeg Kastelanijevog uslova sledi

$$\begin{aligned}
&\{(G_0)_{\mathfrak{h}a}{}^i\,, H_T\} = (C_{PFC1})_a{}^i\,, \\
&\{(C_{PFC})_a{}^i + \Phi(\nabla C)_a{}^i\,, H_T\} = (C_{PFC1})_a{}^i\,, \\
&\{(C_{PFC})_a{}^i, H_T\} + \alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(\nabla C)_b{}^i - C_{b0} \rhd_{\alpha a}{}^b \Phi(\mathcal{F})^{\alpha i} + 2\beta^b{}_{0j} X_{(ab)}{}^A \Phi(\nabla D)_A{}^{ij} = (C_{PFC1})_a{}^i\,,
\end{aligned}$$

što daje rezultat:

$$(C_{PFC})_a{}^i = -\alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(\beta)_b{}^i + C_{b0} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} - 2\beta^b{}_{0j} X_{(ab)}{}^A \Phi(\gamma)_A{}^{ij}\,.$$

Iz toga sledi da je generator:

$$(G_0)_{\mathfrak{h}a}{}^i = -\alpha^\alpha{}_0 \rhd_{\alpha a}{}^b \Phi(\beta)_b{}^i + C_{b0} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} - 2\beta^b{}_{0j} X_{(ab)}{}^A \Phi(\gamma)_A{}^{ij} + \Phi(\nabla C)_a{}^i\,.$$

Kastelanijev drugi uslov za generator $(G_0)_{\mathfrak{l}A}{}^{ij}$ daje:

$$\begin{aligned}
&(G_0)_{\mathfrak{l}A}{}^{ij} + \{\Phi(\gamma)_A{}^{ij}\,, H_T\} = (C_{PFC})_A{}^{ij}\,, \\
&(G_0)_{\mathfrak{l}A}{}^{ij} + \Phi(\nabla D)_A{}^{ij} = (C_{PFC})_A{}^{ij}\,,
\end{aligned} \tag{D.78}$$

tj. dobijamo $(G_0)_{\mathfrak{l}A}{}^{ij} = (C_{PFC})_A{}^{ij} - \Phi(\nabla D)_A{}^{ij}$. Zatim, iz trećeg Kastelanijevog uslova sledi

$$\begin{aligned}
&\{(G_0)_{\mathfrak{l}A}{}^{ij}\,, H_T\} = (C_{PFC1})_A{}^{ij}\,, \\
&\{(C_{PFC})_A{}^{ij} - \Phi(\nabla D)_A{}^{ij}, H_T\} = (C_{PFC1})_A{}^{ij}\,, \\
&\{(C_{PFC})_A{}^{ij}\,, H_T\} - \alpha^\alpha{}_0 \rhd_{\alpha A}{}^B \Phi(\nabla D)_B{}^{ij} = (C_{PFC1})_A{}^{ij}\,,
\end{aligned} \tag{D.79}$$

što daje rezultat

$$(C_{PFC})_A{}^{ij} = \alpha^\alpha{}_0 \rhd_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij} \,.$$

Dobija se da je generator:

$$(G_0)_{\mathfrak{l}A}{}^{ij} = \alpha^\alpha{}_0 \rhd_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij} - \Phi(\nabla D)_A{}^{ij} \,. \tag{D.80}$$

U ovom trenutku, korisno je rezimirati rezultate i uvesti novu notaciju:

$$\begin{aligned} \dot{\epsilon}_{\mathfrak{m}}{}^\alpha{}_i (G_1)_{\mathfrak{m}\alpha}{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i (G_0)_{\mathfrak{m}\alpha}{}^i &= -\nabla_0 \epsilon_{\mathfrak{m}}{}^\alpha{}_i \Phi(B)_\alpha{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i \Phi(\mathcal{F})_\alpha{}^i \\ &= \nabla_0 \epsilon_{\mathfrak{m}}{}^\alpha{}_i (\tilde{M}_1)_\alpha{}^i + \epsilon_{\mathfrak{m}}{}^\alpha{}_i (\tilde{M}_0)_\alpha{}^i \,. \end{aligned} \tag{D.81}$$

Primetimo da se vremenski izvod parametra kombinuje sa nekim drugim članovima u kovarijantni izvod u vremenskom pravcu.

Za drugi deo ukupnog generatora dobijamo:

$$\begin{aligned} \dot{\epsilon}_{\mathfrak{g}}{}^\alpha (G_1)_{\mathfrak{g}\alpha} + \epsilon_{\mathfrak{g}}{}^\alpha (G_0)_{\mathfrak{g}\alpha} &= -\dot{\epsilon}_{\mathfrak{g}}{}^\alpha \Phi(\alpha)_\alpha - \epsilon_{\mathfrak{g}}{}^\alpha \big(B_{\beta 0 i} f_{\alpha\gamma}{}^\beta \Phi(B)^{\gamma i} - \alpha^\beta{}_0 f_{\alpha\beta}{}^\gamma \Phi(\alpha)_\gamma \\ &\quad + C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^b + \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)_b{}^i - \frac{1}{2} \gamma^A{}_{0ij} \rhd_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij} - \Phi(\nabla B)_\alpha \big) \\ &= -\nabla_0 \epsilon_{\mathfrak{g}}{}^\alpha \Phi(\alpha)_\alpha - \epsilon_{\mathfrak{g}}{}^\alpha \big(B_{\beta 0 i} f_{\alpha\gamma}{}^\beta \Phi(B)^{\gamma i} \\ &\quad + C_{a0} \rhd_{\alpha b}{}^a \Phi(C)^b + \beta_{a0i} \rhd_{\alpha b}{}^a \Phi(\beta)_b{}^i - \frac{1}{2} \gamma^A{}_{0ij} \rhd_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij} - \Phi(\nabla B)_\alpha \big) \\ &= \nabla_0 \epsilon_{\mathfrak{g}}{}^\alpha (\tilde{G}_1)_\alpha + \epsilon_{\mathfrak{g}}{}^\alpha (\tilde{G}_0)_\alpha \,. \end{aligned} \tag{D.82}$$

Osim toga, sledi:

$$\begin{aligned} \dot{\epsilon}_{\mathfrak{h}}{}^a{}_i (G_1)_{\mathfrak{h}a}{}^i + \epsilon_{\mathfrak{h}}{}^a{}_i (G_0)_{\mathfrak{h}a}{}^i &= -\nabla_0 \epsilon_{\mathfrak{h}}{}^a{}_i \Phi(\beta)_\alpha{}^i + \epsilon_{\mathfrak{h}}{}^a{}_i \big(C_{b0} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} - 2\beta^b{}_{0j} X_{(ab)}{}^A \Phi(\gamma)_A{}^{ij} + \Phi(\nabla C)_a{}^i \big) \\ &= \nabla_0 \epsilon_{\mathfrak{h}}{}^a{}_i (\tilde{H}_1)_a{}^i + \epsilon_{\mathfrak{h}}{}^a{}_i (\tilde{H}_0)_a{}^i \,, \end{aligned} \tag{D.83}$$

$$\begin{aligned} \dot{\epsilon}_{\mathfrak{n}}{}^a (G_1)_{\mathfrak{n}a} + \epsilon_{\mathfrak{n}}{}^a (G_0)_{\mathfrak{n}a} &= -\nabla_0 \epsilon_{\mathfrak{n}}{}^a \Phi(C)_a + \epsilon_{\mathfrak{n}}{}^a (\beta_{b0i} \rhd_{\alpha a}{}^b \Phi(B)^{\alpha i} + \Phi(\mathcal{G})_a) \\ &= \nabla_0 \epsilon_{\mathfrak{n}}{}^a (\tilde{N}_1)_a + \epsilon_{\mathfrak{n}}{}^a (\tilde{N}_0)_a \,. \end{aligned} \tag{D.84}$$

Na kraju, dobija se

$$\begin{aligned} \frac{1}{2} \dot{\epsilon}_{\mathfrak{l}}{}^A{}_{ij} (G_1)_{\mathfrak{l}A}{}^{ij} + \frac{1}{2} \epsilon_{\mathfrak{l}}{}^A{}_{ij} (G_0)_{\mathfrak{l}A}{}^{ij} &= \frac{1}{2} \dot{\epsilon}_{\mathfrak{l}}{}^A{}_{ij} \Phi(\gamma)_A{}^{ij} + \frac{1}{2} \epsilon_{\mathfrak{l}}{}^A{}_{ij} \alpha^\alpha{}_0 \rhd_{\alpha A}{}^B \Phi(\gamma)_B{}^{ij} \\ &\quad - \frac{1}{2} \epsilon_{\mathfrak{l}}{}^A{}_{ij} \Phi(\nabla D)_A{}^{ij} \\ &= \frac{1}{2} \nabla_0 \epsilon_{\mathfrak{l}}{}^A{}_{ij} \Phi(\gamma)_A{}^{ij} - \frac{1}{2} \epsilon_{\mathfrak{l}}{}^A{}_{ij} \Phi(\nabla D)_A{}^{ij} \\ &= \frac{1}{2} \nabla_0 \epsilon_{\mathfrak{l}}{}^A{}_{ij} (\tilde{L}_1)_A{}^{ij} + \frac{1}{2} \epsilon_{\mathfrak{l}}{}^A{}_{ij} (\tilde{L}_0)_A{}^{ij} \,. \end{aligned} \tag{D.85}$$

D.3.3 Izračunavanje algebre simetrija 3*BF* dejstva

Da bi se dobila struktura grupe simetrija $3BF$ dejstva, kao što je predstavljeno u podsekciji 6.1.2, moramo najpre izračunati komutatore između generatora G-, H-, L-, M- i N-gejdž simetrija. Ovaj proces je opisan u odeljku 6.1.2, dok su detalji izračunavanja dati u ovom odeljku.

Komutator $[H, H]$

Sada ćemo izračunati komutator generatora H-gejdž transformacija, tj. jednačinu (6.49). Nakon transformacije promenljivih pri H-gejdž transformacijama za parametar $\epsilon_{\mathfrak{h}1}$ dobija se

$$\begin{aligned}
\alpha' &= \alpha - \partial\epsilon_{\mathfrak{h}1}\,, && \text{(D.86)}\\
\beta' &= \beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1}\,, && \text{(D.87)}\\
\gamma' &= \gamma + \{\beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1}, \epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}} + \{\epsilon_{\mathfrak{h}1}, \beta\}_{\mathrm{pf}}\,, && \text{(D.88)}\\
B' &= B - (C - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}1})\wedge^{\mathcal{T}}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge^{\mathcal{D}}\epsilon_{\mathfrak{h}1}\wedge^{\mathcal{D}} D\,, && \text{(D.89)}\\
C' &= C - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}1}\,, && \text{(D.90)}\\
D' &= D\,. && \text{(D.91)}
\end{aligned}$$

Zatim, daljom transformacijom varijabli H-gejdž transformacijama sa parametrom $\epsilon_{\mathfrak{h}2}$ dobija se:

$$\begin{aligned}
\alpha'' &= \alpha - \partial\epsilon_{\mathfrak{h}1} - \partial\epsilon_{\mathfrak{h}2}\,,\\
\beta'' &= \beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1} - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}-\partial\epsilon_{\mathfrak{h}2}}{\nabla}\epsilon_{\mathfrak{h}2} - \epsilon_{\mathfrak{h}2}\wedge\epsilon_{\mathfrak{h}2}\,,\\
\gamma'' &= \gamma + \{\beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1}, \epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}} + \{\epsilon_{\mathfrak{h}1}, \beta\}_{\mathrm{pf}}\\
&\quad + \{\beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1} - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}-\partial\epsilon_{\mathfrak{h}2}}{\nabla}\epsilon_{\mathfrak{h}2} - \epsilon_{\mathfrak{h}2}\wedge\epsilon_{\mathfrak{h}2}, \epsilon_{\mathfrak{h}2}\}_{\mathrm{pf}}\\
&\quad + \{\epsilon_{\mathfrak{h}2}, \beta - \overset{\alpha-\partial\epsilon_{\mathfrak{h}1}}{\nabla}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge\epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}}\,,\\
B'' &= B - (C - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}1})\wedge^{\mathcal{T}}\epsilon_{\mathfrak{h}1} - \epsilon_{\mathfrak{h}1}\wedge^{\mathcal{D}}\epsilon_{\mathfrak{h}1}\wedge^{\mathcal{D}} D\\
&\quad - (C - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}2} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}2})\wedge^{\mathcal{T}}\epsilon_{\mathfrak{h}2} - \epsilon_{\mathfrak{h}2}\wedge^{\mathcal{D}}\epsilon_{\mathfrak{h}2}\wedge^{\mathcal{D}} D\,,\\
C'' &= C - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}1} - D\wedge^{\mathcal{X}_1}\epsilon_{\mathfrak{h}2} - D\wedge^{\mathcal{X}_2}\epsilon_{\mathfrak{h}2}\,,\\
D'' &= D\,.
\end{aligned} \qquad \text{(D.92)}$$

Vidimo da za promenljive $\alpha^\alpha{}_\mu$, $C^a{}_\mu$ i D^A dobijamo da transformacije komutiraju:

$$\begin{aligned}
\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\alpha^\alpha{}_\mu &= \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\alpha^\alpha{}_\mu\,,\\
\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}C^a{}_\mu &= \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}C^a{}_\mu\,,\\
\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}D^A &= \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}D^A\,.
\end{aligned} \qquad \text{(D.93)}$$

Za preostale promenljive, $\beta^a{}_{\mu\nu}$, $\gamma^A{}_{\mu\nu\rho}$ i $B^\alpha{}_{\mu\nu}$, razlika jednačine (D.3.3) i analogne jednačine gde $\epsilon_{\mathfrak{h}1} \leftrightarrow \epsilon_{\mathfrak{h}2}$ je:

$$
\begin{aligned}
\left(\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H} - \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\right)\frac{1}{2}\beta^a{}_{\mu\nu} &= \partial_b{}^\alpha \epsilon_{\mathfrak{h}2}{}^b{}_{[\mu|}\epsilon_{\mathfrak{h}1}{}^c{}_{|\nu]} \rhd{}_{\alpha c}{}^a - \partial_b{}^\alpha \epsilon_{\mathfrak{h}1}{}^b{}_{[\mu|}\epsilon_{\mathfrak{h}2}{}^c{}_{|\nu]} \rhd{}_{\alpha c}{}^a \\
&= 2\delta_A{}^a\, X_{(bc)}{}^A \epsilon_{\mathfrak{h}1}{}^b{}_{[\mu|}\epsilon_{\mathfrak{h}2}{}^c{}_{|\nu]} \\
&= \delta_A{}^a(\{\epsilon_{\mathfrak{h}1} \wedge \epsilon_{\mathfrak{h}2}\}_{\mathrm{pf}} - \{\epsilon_{\mathfrak{h}2} \wedge \epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}})^A{}_{\mu\nu}\,, \\
\left(\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H} - \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\right)\frac{1}{3!}\gamma^A{}_{\mu\nu\rho} &= 2(\partial_{[\mu}\epsilon_{\mathfrak{h}_1}{}^a{}_\nu)\epsilon_{\mathfrak{h}_2}{}^b{}_{\rho]}X_{(ab)}{}^A + 2\epsilon_{\mathfrak{h}_1}{}^a{}_{[\nu}(\partial_\mu \epsilon_{\mathfrak{h}_2}{}^b{}_{\rho]})X_{(ab)}{}^A \\
&\quad +2\alpha^\alpha{}_{[\mu}\epsilon_{\mathfrak{h}_1}{}^a{}_\nu \epsilon_{\mathfrak{h}_2}{}^b{}_{\rho]}X_{(ab)}{}^B \rhd_{\alpha B}{}^A \\
&= \nabla_{[\mu}(\{\epsilon_{\mathfrak{h}1} \wedge \epsilon_{\mathfrak{h}2}\}_{\mathrm{pf}} - \{\epsilon_{\mathfrak{h}2} \wedge \epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}})^A{}_{\nu\rho]}\,, \\
\left(\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H} - \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\right)\frac{1}{2}B^\alpha{}_{\mu\nu} &= D^A \epsilon_{\mathfrak{h}2}{}^a{}_{[\mu|}\epsilon_{\mathfrak{h}1}{}^b{}_{|\nu]}(X_{1Aa}{}^c + X_{2Aa}{}^c)\mathcal{T}_{cb}{}^\alpha \\
&\quad -D^A \epsilon_{\mathfrak{h}1}{}^b{}_{[\mu|}\epsilon_{\mathfrak{h}2}{}^a{}_{|\nu]}(X_{1Ab}{}^c + X_{2Ab}{}^c)\mathcal{T}_{ca}{}^\alpha \\
&= -2D_A \epsilon_{\mathfrak{h}1}{}^a{}_{[\mu|}\epsilon_{\mathfrak{h}2}{}^b{}_{|\nu]}(X_{(ac)}{}^A \rhd_{\alpha b}{}^c + X_{(bc)}{}^A \rhd_{\alpha a}{}^c) \\
&= -2D_A \epsilon_{\mathfrak{h}1}{}^a{}_{[\mu|}\epsilon_{\mathfrak{h}2}{}^b{}_{|\nu]}X_{(ab)}{}^B \rhd_{\alpha B}{}^A \\
&= (D \wedge^{\mathcal{S}} (\{\epsilon_{\mathfrak{h}1} \wedge \epsilon_{\mathfrak{h}2}\}_{\mathrm{pf}} - \{\epsilon_{\mathfrak{h}2} \wedge \epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}})^\alpha{}_{\mu\nu}\,.
\end{aligned}
\tag{D.94}
$$

Upoređivanjem jednačina (D.93) i (D.94) sa jednačinama (6.45), zaključujemo da je komutator dve H-gejdž transformacije L-gejdž transformacija sa parametrom $\epsilon_{\mathfrak{l}}{}^A{}_{\mu\nu} = 4\epsilon_{\mathfrak{h}1}{}^a{}_{[\mu|}\epsilon_{\mathfrak{h}2}{}^b{}_{|\nu]}X_{(ac)}{}^A$:

$$
\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H} - \mathrm{e}^{\epsilon_{\mathfrak{h}2}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{h}1}\cdot H} = 2\left(\{\epsilon_{\mathfrak{h}1} \wedge \epsilon_{\mathfrak{h}2}\}_{\mathrm{pf}} - \{\epsilon_{\mathfrak{h}2} \wedge \epsilon_{\mathfrak{h}1}\}_{\mathrm{pf}}\right)\cdot \hat{L}\,. \tag{D.95}
$$

Komutator $[\boldsymbol{H}, \boldsymbol{N}]$

Izračunajmo komutator između generatora H-gejdž i N-gejdž transformacija, tj. izvedimo jednačinu (6.81). Ovo se radi izračunavanjem izraza

$$
\left(\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N} - \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\right)A\,, \tag{D.96}
$$

za sve varijable A prisutne u teoriji. Primećujemo da je za varijable $\alpha^\alpha{}_\mu$, $\beta^a{}_{\mu\nu}$, $\gamma^A{}_{\mu\nu\rho}$ i D^A dobijeno da transformacije komutiraju:

$$
\begin{aligned}
\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\alpha^\alpha{}_\mu &= \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\alpha^\alpha{}_\mu\,, \\
\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\beta^a{}_{\mu\nu} &= \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\beta^a{}_{\mu\nu}\,, \\
\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\gamma^A{}_{\mu\nu\rho} &= \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\gamma^A{}_{\mu\nu\rho}\,, \\
\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}\mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}D^A &= \mathrm{e}^{\epsilon_{\mathfrak{n}}\cdot N}\mathrm{e}^{\epsilon_{\mathfrak{h}}\cdot H}D^A\,.
\end{aligned}
\tag{D.97}
$$

Preostale varijable $B^\alpha{}_{\mu\nu}$ i $C^a{}_\mu$ se pri H-gejdž transformacijama transformišu na sledeći način:

$$
\begin{aligned}
B' &= B - (C - D \wedge^{\chi_1} \epsilon_{\mathfrak{h}} - D \wedge^{\chi_2} \epsilon_{\mathfrak{h}}) \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} D\,, \\
C' &= C - D \wedge^{\chi_1} \epsilon_{\mathfrak{h}} - D \wedge^{\chi_2} \epsilon_{\mathfrak{h}}\,.
\end{aligned}
\tag{D.98}
$$

Daljom transformacijom ovih varijabla N-gejdž transformacijama:

$$
\begin{aligned}
B'' &= B' - \beta' \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}} \\
&= B - (C - D \wedge^{\chi_1} \epsilon_{\mathfrak{h}} - D \wedge^{\chi_2} \epsilon_{\mathfrak{h}}) \wedge^{\tau} \epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} D - (\beta - \overset{\{\alpha^\alpha - \partial_a{}^\alpha \epsilon_{\mathfrak{h}}{}^a\}}{\nabla \epsilon_{\mathfrak{h}}} - \epsilon_{\mathfrak{h}} \wedge \epsilon_{\mathfrak{h}}) \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}} \,, \\
C'' &= C' - \overset{\{\alpha^\alpha - \partial_a{}^\alpha \epsilon_{\mathfrak{h}}{}^a\}}{\nabla} \epsilon_{\mathfrak{n}} \\
&= C - D \wedge^{\chi_1} \epsilon_{\mathfrak{h}} - D \wedge^{\chi_2} \epsilon_{\mathfrak{h}} - \overset{\{\alpha^\alpha - \partial_a{}^\alpha \epsilon_{\mathfrak{h}}{}^a\}}{\nabla} \epsilon_{\mathfrak{n}} \,.
\end{aligned}
\tag{D.99}
$$

Zatim, izmenimo redosled transformacija. Najpre, transformacija varijabli pri N-gejdž transformacijama je

$$
\begin{aligned}
B^{\cdot} &= B - \beta \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}} \,, \\
C^{\cdot} &= C - \nabla \epsilon_{\mathfrak{n}} \,,
\end{aligned}
\tag{D.100}
$$

dok dalja H-gejdž transformacija daje:

$$
\begin{aligned}
B^{\cdot\cdot} &= B^{\cdot} - (C^{\cdot} - D^{\cdot} \wedge^{\chi_1} \epsilon_{\mathfrak{h}} - D^{\cdot} \wedge^{\chi_2} \epsilon_{\mathfrak{h}}) \wedge^{\tau} \epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} D^{\cdot} \\
&= B - \beta \wedge^{\mathcal{T}} \epsilon_{\mathfrak{n}} - (C - \nabla \epsilon_{\mathfrak{n}} - (D + \delta \epsilon_{\mathfrak{n}}) \wedge^{\chi_1} \epsilon_{\mathfrak{h}} - (D + \delta \epsilon_{\mathfrak{n}}) \wedge^{\chi_2} \epsilon_{\mathfrak{h}}) \wedge^{\tau} \epsilon_{\mathfrak{h}} - \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} \epsilon_{\mathfrak{h}} \wedge^{\mathcal{D}} (D + \delta \epsilon_{\mathfrak{n}}) \,, \\
C^{\cdot\cdot} &= C^{\cdot} - D^{\cdot} \wedge^{\chi_1} \epsilon_{\mathfrak{h}} - D^{\cdot} \wedge^{\chi_2} \epsilon_{\mathfrak{h}} \\
&= C - \nabla \epsilon_{\mathfrak{n}} - (D + \delta \epsilon_{\mathfrak{n}}) \wedge^{\chi_1} \epsilon_{\mathfrak{h}} - (D + \delta \epsilon_{\mathfrak{n}}) \wedge^{\chi_2} \epsilon_{\mathfrak{h}} \,.
\end{aligned}
\tag{D.101}
$$

Razlika jednačina (D.99) i (D.101) je:

$$
\begin{aligned}
\left(e^{\epsilon_{\mathfrak{h}} \cdot H} e^{\epsilon_{\mathfrak{n}} \cdot N} - e^{\epsilon_{\mathfrak{n}} \cdot N} e^{\epsilon_{\mathfrak{h}} \cdot H}\right) B^\alpha \quad &= \nabla \epsilon_{\mathfrak{n}}{}^a \wedge \epsilon_{\mathfrak{h}}{}^b \mathcal{T}_{ab}{}^\alpha + \delta^A{}_a \epsilon_{\mathfrak{n}}{}^a \epsilon_{\mathfrak{h}}{}^b \wedge \epsilon_{\mathfrak{h}}{}^d X_{1Ab}{}^c \mathcal{T}_{cd}{}^\alpha \\
&\quad + \delta^A{}_a \epsilon_{\mathfrak{n}}{}^a \epsilon_{\mathfrak{h}}{}^b \wedge \epsilon_{\mathfrak{h}}{}^d X_{2Ab}{}^c \mathcal{T}_{cd}{}^\alpha - \epsilon_{\mathfrak{h}}{}^a \wedge \epsilon_{\mathfrak{h}}{}^b \delta_A{}^c \epsilon_{\mathfrak{n}}{}^c D_{Aab}{}^\alpha \,, \\
&\quad \nabla \epsilon_{\mathfrak{h}}{}^a \wedge \epsilon_{\mathfrak{n}}{}^b \mathcal{T}_{ab}{}^\alpha + \partial_a{}^\beta \epsilon_{\mathfrak{h}}{}^a \rhd_{\beta c}{}^b \epsilon_{\mathfrak{h}}{}^c \epsilon_{\mathfrak{n}}{}^d \mathcal{T}_{bd}{}^\alpha - \epsilon_{\mathfrak{h}}{}^a \wedge \epsilon_{\mathfrak{h}}{}^b f_{ab}{}^c \epsilon_{\mathfrak{n}}{}^d \mathcal{T}_{cd}{}^\alpha \,, \\
\left(e^{\epsilon_{\mathfrak{h}} \cdot H} e^{\epsilon_{\mathfrak{n}} \cdot N} - e^{\epsilon_{\mathfrak{n}} \cdot N} e^{\epsilon_{\mathfrak{h}} \cdot H}\right) C^c \quad &= -(\delta^A{}_a \epsilon_{\mathfrak{n}}^a) \wedge \epsilon_{\mathfrak{h}}{}^b X_{1Ab}{}^c - (\delta^A{}_a \epsilon_{\mathfrak{n}}^a) \wedge \epsilon_{\mathfrak{h}}{}^b X_{2Ab}{}^c - \partial_a{}^\beta \epsilon_{\mathfrak{h}}{}^a \rhd_{\beta b}{}^c \epsilon_{\mathfrak{n}}{}^b \,,
\end{aligned}
\tag{D.102}
$$

Primenom definicija preslikavanja $\mathcal{T}$, $\mathcal{D}$, χ_1 i χ_2 prethodne jednačine se svode na

$$
\begin{aligned}
\left(e^{\epsilon_{\mathfrak{h}} \cdot H} e^{\epsilon_{\mathfrak{n}} \cdot N} - e^{\epsilon_{\mathfrak{n}} \cdot N} e^{\epsilon_{\mathfrak{h}} \cdot H}\right) B^\alpha &= \nabla \epsilon_{\mathfrak{n}}{}^a \wedge \epsilon_{\mathfrak{h}}{}^b \mathcal{T}_{ab}{}^\alpha - \nabla \epsilon_{\mathfrak{h}}{}^a \wedge \epsilon_{\mathfrak{n}}{}^b \mathcal{T}_{ab}{}^\alpha = \nabla (\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}})^\alpha \,, \\
\left(e^{\epsilon_{\mathfrak{h}} \cdot H} e^{\epsilon_{\mathfrak{n}} \cdot N} - e^{\epsilon_{\mathfrak{n}} \cdot N} e^{\epsilon_{\mathfrak{h}} \cdot H}\right) C^c &= \partial^c{}_\alpha (\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}})^\alpha \,,
\end{aligned}
\tag{D.103}
$$

Upoređivanjem jednačina (D.97) i (D.103) sa jednačinom (6.58) dobijamo konačan rezultat za komutator

$$
\left(e^{\epsilon_{\mathfrak{h}} \cdot H} e^{\epsilon_{\mathfrak{n}} \cdot N} - e^{\epsilon_{\mathfrak{n}} \cdot N} e^{\epsilon_{\mathfrak{h}} \cdot H}\right) = -(\epsilon_{\mathfrak{n}} \wedge^{\mathcal{T}} \epsilon_{\mathfrak{h}}) \cdot M \,. \tag{D.104}
$$

Dodatak E

Invarijantnost sume po stanjima na Pahnerove poteze

E.1 Invarijantnost $2BF$ sume po stanjima na Pahnerove poteze

E.1.1 $n = 3$

Pahnerov potez $1 \leftrightarrow 4$

Leva strana Pahnerovog poteza $1 \leftrightarrow 4$ data je izrazom (8.20) i ne može se pojednostaviti:

$$l.s. = \delta_H\big(h_{134}\, g_{34} \rhd h_{123}\, h_{234}^{-1}\, h_{124}^{-1}\big)\,. \tag{E.1}$$

Analizirajmo čemu je jednaka *desna strana poteza* $1 \leftrightarrow 4$ data izrazom (8.21). Integralimo g_{15} koristeći $\delta_G(g_{125})$, g_{25} koristeći $\delta_G(g_{235})$ i g_{35} koristeći $\delta_G(g_{345})$:

$$\begin{aligned} g_{15} &= \partial(h_{125})\, g_{25}\, g_{12}\,, \\ g_{25} &= \partial(h_{235})\, g_{35}\, g_{23}\,, \\ g_{35} &= \partial(h_{345})\, g_{45}\, g_{34}\,. \end{aligned} \tag{E.2}$$

Zatim, integralimo h_{135} koristeći $\delta_H(h_{1345})$, h_{125} koristeći $\delta_H(h_{1245})$ i h_{235} koristeći $\delta_H(h_{2345})$:

$$\begin{aligned} h_{135} &= h_{145}\,(g_{45} \rhd h_{134})\, h_{345}^{-1}\,, \\ h_{125} &= h_{145}\,(g_{45} \rhd h_{124})\, h_{245}^{-1}\,, \\ h_{235} &= h_{245}\,(g_{45} \rhd h_{234})\, h_{345}^{-1}\,. \end{aligned} \tag{E.3}$$

Preostale δ-funkcije na grupi G svode se na $\delta_G(e)^3$. Pokažimo to. Korišćenjem jednačina (E.3) i identiteta (8.2) za trougao (234):

$$\begin{aligned} \delta_G(g_{245}) &= \delta_G\big(\partial(h_{245})\, g_{45}\, g_{24}\, g_{25}^{-1}\big) \\ &= \delta_G\big(\partial(h_{245})\, g_{45}\, g_{24}\, g_{23}^{-1}\, g_{35}^{-1}\, \partial(h_{235})^{-1}\big) \\ &= \delta_G\big(\partial(h_{245})\, g_{45}\, g_{24}\, g_{23}^{-1}\, g_{35}^{-1}\, \partial(h_{345})\, g_{45} \rhd \partial(h_{234})^{-1}\, \partial(h_{245})^{-1}\big) \\ &= \delta_G\big(g_{45}\, g_{24}\, g_{23}^{-1}\, g_{34}^{-1}\, g_{45}^{-1}\, \partial(h_{345})^{-1}\, \partial(h_{345})\, g_{45}\, g_{34}\, g_{23}\, g_{24}^{-1}\, g_{45}^{-1}\big) \\ &= \delta_G(e)\,. \end{aligned} \tag{E.4}$$

Zatim, za preostale dve δ-funkcije na grupi G dobijamo da su ekvivalentne prvoj, pa sledi

$$\begin{aligned}
\delta_G(g_{135}) &= \delta_G\big(\partial(h_{135})\, g_{35}\, g_{13}\, g_{15}^{-1}\big) \\
&= \delta_G\big(\partial(h_{145})\, g_{45} \rhd \partial(h_{134})\, \partial(h_{345})^{-1}\, \partial(h_{345})\, g_{45}\, g_{34}\, g_{13}\, g_{12}^{-1}\, g_{25}^{-1}\, \partial(h_{125})^{-1}\big) \\
&= \delta_G\big(\partial(h_{145})\, g_{45}\, \partial(h_{134}) g_{45}^{-1}\, g_{45}\, g_{34}\, g_{13}\, g_{12}^{-1}\, g_{25}^{-1}\, \partial(h_{245})\, g_{45} \rhd \partial(h_{124})^{-1}\, \partial(h_{145})^{-1}\big) \\
&= \delta_G\big(g_{45}\, g_{14}\, g_{13}^{-1}\, g_{34}^{-1}\, g_{34}\, g_{13}\, g_{12}^{-1}\, g_{25}^{-1}\, \partial(h_{245})\, g_{45}\, g_{24}\, g_{12}\, g_{14}^{-1}\, g_{45}^{-1}\big) \\
&= \delta_G\big(\, g_{25}^{-1}\, \partial(h_{245})\, g_{45}\, g_{24}\big) \\
&= \delta_G\big(g_{245}\big) \\
&= \delta_G(e)\,,
\end{aligned} \tag{E.5}$$

$$\begin{aligned}
\delta_G(g_{145}) &= \delta_G\big(\partial(h_{145})\, g_{45}\, g_{14}\, g_{15}^{-1}\big) \\
&= \delta_G\big(\partial(h_{145})\, g_{45}\, g_{14}\, g_{12}^{-1}\, g_{25}^{-1} \partial(h_{125})^{-1}\big) \\
&= \delta_G\big(\partial(h_{145})\, g_{45}\, g_{14}\, g_{12}^{-1}\, g_{25}^{-1}\, \partial(h_{245})\, g_{45} \rhd \partial(h_{124})^{-1}\, \partial(h_{145})^{-1}\big) \\
&= \delta_G\big(g_{45}\, g_{14}\, g_{12}^{-1}\, g_{25}^{-1}\, \partial(h_{245})\, g_{45} \partial(h_{124})^{-1}\, g_{45}^{-1}\big) \\
&= \delta_G\big(g_{14}\, g_{12}^{-1}\, g_{25}^{-1}\, \partial(h_{245})\, g_{45}\, g_{24}\, g_{12}\, g_{14}^{-1}\big) \\
&= \delta_G\big(g_{25}^{-1}\, \partial(h_{245})\, g_{45}\, g_{24}\big) \\
&= \delta_G\big(g_{245}\big) \\
&= \delta_G(e)\,.
\end{aligned} \tag{E.6}$$

Koristili smo jednačine (E.2) i (E.3), kao i identitet (8.2) za trouglove (134) i (124). Zatim, analiziranjem $\delta_H(h_{1235})$ dobijamo:

$$\begin{aligned}
\delta_H(h_{1235}) &= \delta_H\big(h_{135}\, (g_{35} \rhd h_{123})\, h_{235}^{-1}\, h_{125}^{-1}\big) \\
&= \delta_H\big(h_{145}\, (g_{45} \rhd h_{134})\, h_{345}^{-1}\, (g_{35} \rhd h_{123})\, h_{235}^{-1}\, h_{245}\, g_{45} \rhd h_{124}^{-1} h_{145}^{-1}\big) \\
&= \delta_H\big(g_{45} \rhd h_{134}\, h_{345}^{-1}\, (g_{35} \rhd h_{123})\, (h_{245}\, (g_{45} \rhd h_{234})\, h_{345}^{-1})^{-1}\, h_{245}\, g_{45} \rhd h_{124}^{-1}\big) \\
&= \delta_H\big(g_{45} \rhd h_{134}\, h_{345}^{-1}\, (g_{35} \rhd h_{123})\, h_{345}\, g_{45} \rhd h_{234}^{-1}\, g_{45} \rhd h_{124}^{-1}\big) \\
&= \delta_H\big(g_{45} \rhd h_{134}\, (\partial(h_{345})^{-1} g_{35}) \rhd h_{123}\, g_{45} \rhd h_{234}^{-1}\, g_{45} \rhd h_{124}^{-1}\big) \\
&= \delta_H\big(g_{45} \rhd h_{134}\, (g_{45} g_{34}) \rhd h_{123}\, g_{45} \rhd h_{234}^{-1}\, g_{45} \rhd h_{124}^{-1}\big) \\
&= \delta_H\big(h_{134}\, g_{34} \rhd h_{123}\, h_{234}^{-1}\, h_{124}^{-1}\big)\,.
\end{aligned} \tag{E.7}$$

Zaključujemo dakle da je *desna strana poteza*:

$$d.s. = \delta_G(e)^3\, \delta_H\big(h_{134}\, g_{34} \rhd h_{123}\, h_{234}^{-1}\, h_{124}^{-1}\big)\,. \tag{E.8}$$

Konstante ispred integrala su $|G|^{-2}|H|$ sa leve strane poteza, odnosno $|G|^{-5}|H|$ sa desne strane poteza, što kompenzuje razliku u faktorima $|G|$ u izrazima (E.1) i (E.8). Zaključujemo da je suma po stanjima (8.16) invarijantna na $1 \leftrightarrow 4$ Pahnerov potez.

Pahnerov potez 2 ↔ 3

Sa *leve strane* $2 \leftrightarrow 3$ Pahnerovog poteza imamo integral:

$$\int \mathrm{d}h_{234}\, \delta_G(h_{234})\, \delta_H(h_{1234})\, \delta_H(h_{2345})\,. \tag{E.9}$$

Najpre, integralimo h_{234} koristeći $\delta_H(h_{2345})$. Dobijamo da je

$$h_{234} = g_{45}^{-1} \rhd h_{245}^{-1}\, g_{45}^{-1} \rhd h_{235}\, g_{45}^{-1} \rhd h_{345}\,, \tag{E.10}$$

pa, zamenjujući ovaj rezultat u izraz za preostalu δ-funkciju na grupi G i korišćenjem identiteta (8.2) za trouglove (245), (235) i (345), sledi:

$$\begin{aligned} \delta_G(g_{234}) &= \delta_G\big(\partial(h_{234})\, g_{34}\, g_{23}\, g_{24}^{-1}\big) \\ &= \delta_G\big(g_{45}^{-1} \rhd (\partial(h_{245})^{-1}\, \partial(h_{235})\, \partial(h_{345}))\, g_{34}\, g_{23}\, g_{24}^{-1}\big) \\ &= \delta_G\big(g_{45}^{-1}\, g_{45}\, g_{24}\, g_{25}\, g_{25}^{-1}\, g_{23}^{-1}\, g_{35}^{-1}\;\; g_{35}^{-1}\, g_{34}^{-1}\, g_{45}^{-1}\, g_{45}\, g_{34}\, g_{23}\, g_{24}^{-1}\big) \\ &= \delta_G(e)\,. \end{aligned} \tag{E.11}$$

Preostala δ-funkcija na grupi H je:

$$\begin{aligned} \delta_H(h_{1234}) &= \delta_H\big(h_{134}\,(g_{34} \rhd h_{123})\, h_{234}^{-1}\, h_{124}^{-1}\big) \\ &= \delta_H\big(h_{134}\,(g_{34} \rhd h_{123})\, g_{45}^{-1} \rhd (h_{345}^{-1}\, h_{235}^{-1}\, h_{245})\, h_{124}^{-1}\big)\,. \end{aligned} \tag{E.12}$$

Zaključujemo da je *leva strana poteza* jednaka:

$$l.s. = |G|\, \delta_H\big(h_{134}\,(g_{34} \rhd h_{123})\, g_{45}^{-1} \rhd (h_{345}^{-1}\, h_{235}^{-1}\, h_{245})\, h_{124}^{-1}\big)\,. \tag{E.13}$$

Sa *desne strane* Pahnerovog poteza imamo integral:

$$\int \mathrm{d}g_{15}\, \mathrm{d}h_{125}\, \mathrm{d}h_{135}\, \mathrm{d}h_{145}\, \delta_G(g_{125})\delta_G(g_{135})\delta_G(g_{145})\delta_H(h_{1235})\delta_H(h_{1245})\delta_H(h_{1345})\,. \tag{E.14}$$

Najpre, integralimo g_{15} koristeći $\delta_G(g_{135})$,

$$g_{15} = \partial(h_{135})\, g_{35}\, g_{13}\,, \tag{E.15}$$

a zatim h_{125} koristeći $\delta_H(h_{1235})$ i h_{135} koristeći $\delta_H(h_{1345})$:

$$\begin{aligned} h_{125} &= h_{135}\,(g_{35} \rhd h_{123})\, h_{235}^{-1}\,, \\ h_{135} &= h_{145}\,(g_{45} \rhd h_{134})\, h_{345}^{-1}\,. \end{aligned} \tag{E.16}$$

Koristeći ove rezultate u izrazima za preostale dve δ-funkcije na grupi G dobijamo da se svode na $\delta_G(e)^2$:

$$
\begin{aligned}
\delta_G(g_{125}) &= \delta_G\big(\partial(h_{125})\, g_{25}\, g_{12}\, g_{15}^{-1}\big) \\
&= \delta_G\big(\partial(h_{135})\, g_{35} \triangleright \partial(h_{123})\, \partial(h_{235})^{-1}\, g_{25}\, g_{12}\, g_{13}^{-1}\, g_{35}^{-1}\, \partial(h_{135})^{-1}\big) \\
&= \delta_G\big(g_{35} \triangleright \partial(h_{123})\, \partial(h_{235})^{-1}\, g_{25}\, g_{12}\, g_{13}^{-1}\, g_{35}^{-1}\big) \\
&= \delta_G\big(g_{35}\, g_{13}\, g_{12}^{-1}\, g_{23}^{-1}\, g_{35}^{-1}\, g_{35}\, g_{23}\, g_{25}^{-1}\, g_{25}\, g_{12}\, g_{13}^{-1}\, g_{35}^{-1}\big) \\
&= \delta_G(e)\,,
\end{aligned}
\tag{E.17}
$$

$$
\begin{aligned}
\delta_G(g_{145}) &= \delta_G\big(\partial(h_{145})\, g_{45}\, g_{14}\, g_{15}^{-1}\big) \\
&= \delta_G\big(\partial(h_{145})\, g_{45}\, g_{14}\, g_{13}^{-1}\, g_{35}^{-1}\, \partial(h_{135})^{-1}\big) \\
&= \delta_G\big(\partial(h_{145})\, g_{45}\, g_{14}\, g_{13}^{-1}\, g_{35}^{-1}\, \partial(h_{345})\, g_{45} \triangleright \partial(h_{134})^{-1}\, \partial(h_{145})^{-1}\big) \\
&= \delta_G\big(g_{45}\, g_{14}\, g_{13}^{-1}\, g_{35}^{-1}\, g_{35}\, g_{34}^{-1}\, g_{45}^{-1}\, g_{45}\, g_{34}\, g_{13}\, g_{14}^{-1}\, g_{45}^{-1}\big) \\
&= \delta_G(e)\,.
\end{aligned}
\tag{E.18}
$$

Preostala δ-funkcija na grupi H je:

$$
\begin{aligned}
\delta_H(h_{1245}) &= \delta_H\big(h_{145}\, (g_{45} \triangleright h_{124})\, h_{245}^{-1}\, h_{125}^{-1}\big) \\
&= \delta_H\big(h_{145}\, (g_{45} \triangleright h_{124})\, h_{245}^{-1}\, h_{235}\, g_{35} \triangleright h_{123}^{-1}\, h_{135}^{-1}\big) \\
&= \delta_H\big(h_{145}\, g_{45} \triangleright h_{124}\, h_{245}^{-1}\, h_{235}\, g_{35} \triangleright h_{123}^{-1}\, h_{345}\, g_{45} \triangleright h_{134}^{-1}\, h_{145}^{-1}\big) \\
&= \delta_H\big(g_{45} \triangleright h_{124}\, h_{245}^{-1}\, h_{235}\, g_{35} \triangleright h_{123}^{-1}\, h_{345}\, g_{45} \triangleright h_{134}^{-1}\big) \\
&= \delta_H\big(g_{45} \triangleright h_{124}\, h_{245}^{-1}\, h_{235}\;\, h_{345}\, (\partial(h_{345})^{-1} g_{35}) \triangleright h_{123}^{-1}\, g_{45} \triangleright h_{134}^{-1}\big) \\
&= \delta_H\big(g_{45} \triangleright h_{124}\, h_{245}^{-1}\, h_{235}\;\, h_{345}\, (g_{45} g_{34}) \triangleright h_{123}^{-1}\, g_{45} \triangleright h_{134}^{-1}\big) \\
&= \delta_H\big(h_{124}\, g_{45}^{-1} \triangleright (h_{245}^{-1}\, h_{235}\, h_{345})\, g_{34} \triangleright h_{123}^{-1}\, h_{134}^{-1}\big) \\
&= \delta_H\big(h_{134}\, (g_{34} \triangleright h_{123})\, g_{45}^{-1} \triangleright (h_{345}^{-1}\, h_{235}^{-1}\, h_{245})\, h_{124}^{-1}\big)\,.
\end{aligned}
\tag{E.19}
$$

Vidimo da su izrazi (E.12) i (E.19) jednaki. Preostala integracija po elementu h_{145} je trivijalna, pa je *desna strana poteza*:

$$
\begin{aligned}
d.s. &= \delta_G(e)^2\, \delta_H\big(h_{134}\, (g_{34} \triangleright h_{123})\, g_{45}^{-1} \triangleright (h_{345}^{-1}\, h_{235}^{-1}\, h_{245})\, h_{124}^{-1}\big) \\
&= |G|^2\, \delta_H\big(h_{134}\, (g_{34} \triangleright h_{123})\, g_{45}^{-1} \triangleright (h_{345}^{-1}\, h_{235}^{-1}\, h_{245})\, h_{124}^{-1}\big)\,.
\end{aligned}
\tag{E.20}
$$

Razlika u faktorima $|G|$ u izrazima (E.13) i (E.20) kompenzovana je konstantama ispred integrala – faktorom $|G|^{-4}|H|^1$ sa desne strane i faktorom $|G|^{-3}|H|^1$ sa leve strane poteza. Zaključujemo da je suma (8.16) invarijantna na $2 \leftrightarrow 3$ Pahnerov potez.

E.1.2 $n = 4$

Pahnerov potez $1 \leftrightarrow 5$

Leva strana Pahnerovog poteza $1 \leftrightarrow 5$ data je izrazom (8.25) i ne može se pojednostaviti. Ispitajmo čemu je jednaka *desna strana poteza*, data jednačinom (8.24). Najpre integralimo po varijabli g_{12} iskoristivši pritom δ-funkciju $\delta_G(g_{123})$, zatim varijabli g_{13} koristeći $\delta_G(g_{134})$, g_{14} koristeći $\delta_G(g_{145})$ i varijabli g_{15} koristeći $\delta_G(g_{156})$:

$$\begin{aligned} g_{12} = &\ g_{23}^{-1}\,\partial(h_{123})^{-1}\,g_{13}\,, \\ g_{13} = &\ g_{34}^{-1}\,\partial(h_{134})^{-1}\,g_{14}\,, \\ g_{14} = &\ g_{45}^{-1}\,\partial(h_{145})^{-1}\,g_{15}\,, \\ g_{15} = &\ g_{56}^{-1}\,\partial(h_{156})^{-1}\,g_{16}\,. \end{aligned} \tag{E.21}$$

Zatim, integralimo varijablu h_{123} koristeći $\delta_H(h_{1234})$, h_{124} koristeći $\delta_H(h_{1245})$, h_{125} koristeći $\delta_H(h_{1256})$, h_{134} koristeći $\delta_H(h_{1345})$, h_{135} koristeći $\delta_H(h_{1356})$ i h_{145} koristeći $\delta_H(h_{1456})$:

$$\begin{aligned} h_{123} = &\ g_{34}^{-1} \rhd h_{134}^{-1}\,g_{34}^{-1} \rhd h_{124}\,g_{34}^{-1} \rhd h_{234}\,, \\ h_{124} = &\ g_{45}^{-1} \rhd h_{145}^{-1}\,g_{45}^{-1} \rhd h_{125}\,g_{45}^{-1} \rhd h_{245}\,, \\ h_{125} = &\ g_{56}^{-1} \rhd h_{156}^{-1}\,g_{56}^{-1} \rhd h_{126}\,g_{56}^{-1} \rhd h_{256}\,, \\ h_{134} = &\ g_{45}^{-1} \rhd h_{145}^{-1}\,g_{45}^{-1} \rhd h_{135}\,g_{45}^{-1} \rhd h_{345}\,, \\ h_{135} = &\ g_{56}^{-1} \rhd h_{156}^{-1}\,g_{56}^{-1} \rhd h_{136}\,g_{56}^{-1} \rhd h_{356}\,, \\ h_{145} = &\ g_{56}^{-1} \rhd h_{156}^{-1}\,g_{56}^{-1} \rhd h_{146}\,g_{56}^{-1} \rhd h_{456}\,. \end{aligned} \tag{E.22}$$

Nakon ovih integracija šest δ-funkcija na grupi G prisutnih sa desne strane poteza svode se na $\delta_G(e)^6$. Dobijamo:

$$\delta_G(g_{124}) = \delta_G(g_{125}) = \delta_G(g_{126}) = \delta_G(g_{135}) = \delta_G(g_{136}) = \delta_G(g_{146}) = \delta_G(e)\,.$$

Detalji računa su isti kao i u slučaju $1 \leftrightarrow 5$ Pahnerovog poteza za $3BF$ sumu po stanjima i dati su u narednom odeljku. Pokažimo da se sada preostale δ-funkcije na grupi H svedu na $\delta_H(e)^4$. Najpre, pravolinijskim računom dobijamo za $\delta_H(h_{1235})$:

$$\begin{aligned} \delta_H(h_{1235}) &= \delta_H\big(h_{135}\,(g_{35} \rhd h_{123})\,h_{235}^{-1}\,h_{125}^{-1}\big) \\ &= \delta_H\big(h_{135}\,(g_{35}g_{34}^{-1}) \rhd (h_{134}^{-1}\,h_{124}\,h_{234})\,h_{235}^{-1}\,h_{125}^{-1}\big) \\ &= \delta_H\big(h_{135}\,(g_{35}g_{34}^{-1}) \rhd \big((g_{45}^{-1} \rhd (h_{345}^{-1}\,h_{135}^{-1}\,h_{145}\,h_{145}^{-1}\,h_{125}\,h_{245})h_{234}\big)\ h_{235}^{-1}\,h_{125}^{-1}\big) \\ &= \delta_H\big(h_{135}\,h_{345}h_{345}^{-1}\,h_{135}^{-1}\,h_{125}\,h_{245}h_{345}^{-1}(g_{35}g_{34}^{-1}) \rhd h_{234}\,h_{235}^{-1}\,h_{125}^{-1}\big) \\ &= \delta_H\big(h_{245}h_{345}^{-1}(g_{35}g_{34}^{-1}) \rhd h_{234}\,h_{235}^{-1}\big) \\ &= \delta_H\big(h_{245}h_{345}^{-1}\,h_{345}\,(\partial(h_{345})^{-1}g_{35}g_{34}^{-1}) \rhd h_{234}\,h_{345}^{-1}\,h_{235}^{-1}\big) \\ &= \delta_H\big(h_{245}\,g_{45} \rhd h_{234}\,h_{345}^{-1}\,h_{235}^{-1}\big) \\ &= \delta_H(e)\,. \end{aligned} \tag{E.23}$$

U prethodnom smo koristili identitet $g_{45} = \partial(h_{345})^{-1}g_{35}g_{34}^{-1}$ za trougao (345), kao i izraz $h_{245}\,g_{45} \rhd h_{234}\,h_{345}^{-1}\,h_{235}^{-1} = e$ za tetraedar (2345). Analognim postupkom za vrednost δ-funkcije

$\delta_H(h_{1236})$ dobijamo:

$$\begin{aligned}
\delta_H(h_{1236}) &= \delta_H\big(h_{136}\,(g_{36}\triangleright h_{123})\,h_{236}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{136}\,h_{346}\,(g_{46}g_{34})\triangleright h_{123}\,h_{346}^{-1}\,h_{236}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{136}\,h_{346}\,g_{46}\triangleright(h_{134}^{-1}\,h_{124}\,h_{234})\,h_{346}^{-1}\,h_{236}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{136}\,h_{346}\,g_{46}\triangleright(h_{134}^{-1}\,h_{124}\,h_{234})\,g_{46}\triangleright h_{234}^{-1}\,h_{246}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{136}\,h_{346}\,(g_{46}g_{45}^{-1})\triangleright(h_{345}^{-1}\,h_{135}^{-1}\,h_{125}\,h_{245})\,h_{246}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{136}\,h_{346}\,(g_{46}g_{45}^{-1})\triangleright(h_{345}^{-1}\,g_{56}^{-1}\triangleright(h_{356}^{-1}\,h_{136}^{-1}\,h_{126}\,h_{256})h_{245})\,h_{246}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{136}\,h_{346}\,(g_{46}g_{45}^{-1})\triangleright h_{345}^{-1}\,h_{456}\,h_{356}^{-1}\,h_{136}^{-1}\,h_{126}\,h_{256}\,g_{56}\triangleright h_{245}\,h_{456}^{-1}\,h_{246}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{136}\,h_{346}\,h_{456}\,g_{56}\triangleright h_{345}^{-1}\,h_{356}^{-1}\,h_{136}^{-1}\,h_{126}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{136}\,h_{136}^{-1}\,h_{126}\,h_{126}^{-1}\big)\\
&= \delta_H(e)\,.
\end{aligned} \tag{E.24}$$

Koristili smo identitet (8.6) za tetraedre (2456) i (3456), kao i identitet (8.2) za trougao (456).

Sličnim postupkom dobijamo da su δ-funkcije $\delta_H(h_{1246})=\delta_H(h_{1346})=\delta_H(e)$. Dobijamo da je *desna strana* poteza jednaka:

$$d.s.=\delta_G(e)^6\delta_H(e)^4=|G|^6|H|^4\,. \tag{E.25}$$

Faktori u izrazu (E.25) kompenzovani su konstantama ispred integrala – faktorom $|G|^{-11}|H|^{-4}$ sa desne strane i faktorom $|G|^{-5}|H|^0$ sa leve strane poteza. Zaključujemo da je suma (8.16) invarijantna na $1\leftrightarrow 5$ Pahnerov potez.

Pahnerov potez 2 ↔ 4

Leva strana poteza jednaka je δ-funkciji

$$\delta_H(h_{2345})=h_{235}\,h_{345}\,(g_{45}\triangleright h_{234}^{-1})\,h_{245}^{-1}\,. \tag{E.26}$$

Da bismo videli čemu je jednaka leva strana poteza, koristićemo Lemu 17.

Lema 17 *Neka je za dati 4-simplex $(jk\ell mn)$ zadovoljen identitet (8.6) za četiri tetraedra $(k\ell mn)$, $(j\ell mn)$, $(jkmn)$ i $(jk\ell n)$ i identitet (8.2) za sve trouglove na njihovoj granici, sledi da je identitet (8.6) takođe zadovoljen za peti tetraedar $(jk\ell m)$.*

Na osnovu ovog opšteg rezultata, možemo pokazati da se leva strana poteza svodi na

$$\delta_H(h_{2345})=\delta_H(e)=|H|\,. \tag{E.27}$$

Ispitajmo sada čemu je jednaka *desna strana poteza*, tj. integral (8.27):

$$\int dg_{16}\int dh_{126}dh_{136}dh_{146}dh_{156}\delta_G(g_{126})\delta_G(g_{136})\delta_G(g_{146})\delta_G(g_{156})\delta_H(h_{1236})\delta_H(h_{1246})\delta_H(h_{1256})\delta_H(h_{1346})\delta_H(h_{1356})\delta_H(h_{1456}). \tag{E.28}$$

Prvo integralimo g_{16} koristeći $\delta_G(g_{126})$,

$$g_{16}=\partial(h_{126})\,g_{26}\,g_{12}\,. \tag{E.29}$$

Zatim, integralimo h_{126} koristeći $\delta_H(h_{1236})$, h_{136} koristeći $\delta_H(h_{1346})$ i h_{146} koristeći $\delta_H(h_{1456})$, na osnovu čega dobijamo

$$
\begin{aligned}
h_{126} = &\quad h_{136}\,(g_{36} \triangleright h_{123})\,h_{236}^{-1}\,, \\
h_{136} = &\quad h_{146}\,(g_{46} \triangleright h_{134})\,h_{346}^{-1}\,, \\
h_{146} = &\quad h_{156}\,(g_{56} \triangleright h_{145})\,h_{456}^{-1}\,.
\end{aligned} \tag{E.30}
$$

Preostale tri δ-funkcije na grupi G svode se na $\delta_G(e)^3$, tj. pokazuje se da je $\delta_G(g_{136}) = \delta_G(g_{146}) = \delta_G(g_{156}) = \delta_G(e)$. Dokaz je isti kao u slučaju $3BF$, za detalje pogledati sledeći odeljak.

Preostale tri δ-funkcija na grupi H svode se na $\delta_H(e)^3$, sličnim postupkom kao i u slučaju $1 \leftrightarrow 5$ Pahnerovog poteza, tj. dobijamo $\delta_H(h_{1356}) = \delta_H(h_{1246}) = \delta_H(h_{1256}) = \delta_H(e)$. Najpre, pokažimo da je $\delta_H(h_{1356}) = \delta_H(e)$:

$$
\begin{aligned}
\delta_H(h_{1356}) &= \delta_H\big(h_{156}\,(g_{56} \triangleright h_{135})\,h_{356}^{-1}\,h_{136}^{-1}\big) \\
&= \delta_H\big(h_{156}\,(g_{56} \triangleright h_{135})\,h_{356}^{-1}\,h_{346}\,g_{46} \triangleright h_{134}^{-1}\,h_{146}^{-1}\big) \\
&= \delta_H\big(g_{56} \triangleright h_{135}\,h_{356}^{-1}\,h_{346}\,g_{46} \triangleright h_{134}^{-1}\,h_{456}\,g_{56} \triangleright h_{145}^{-1}\big) \\
&= \delta_H\big(g_{56} \triangleright (g_{45} \triangleright h_{134}h_{345})\,h_{356}^{-1}\,h_{346}\,g_{46} \triangleright h_{134}^{-1}\,h_{456}\big) \\
&= \delta_H\big(g_{46} \triangleright h_{134}\,h_{456}\,g_{56} \triangleright h_{345}\,h_{356}^{-1}\,h_{346}\,g_{46} \triangleright h_{134}^{-1}\big) \\
&= \delta_H\big(h_{456}\,g_{56} \triangleright h_{345}\,h_{356}^{-1}\,h_{346}\big) \\
&= \delta_H(e)\,.
\end{aligned} \tag{E.31}
$$

Zatim, δ-funkcija $\delta_H(h_{1246})$ je:

$$
\begin{aligned}
\delta_H(h_{1246}) &= \delta_H\big(h_{146}\,(g_{46} \triangleright h_{124})\,h_{246}^{-1}\,h_{126}^{-1}\big) \\
&= \delta_H\big(h_{146}\,(g_{46} \triangleright h_{124})\,h_{246}^{-1}\,h_{236}\,g_{36} \triangleright h_{123}^{-1}\,h_{346}\,g_{46} \triangleright h_{134}^{-1}\,h_{146}^{-1}\big) \\
&= \delta_H\big(g_{46} \triangleright h_{124}\,h_{246}^{-1}\,h_{236}\,g_{36} \triangleright h_{123}^{-1}\,h_{346}\,g_{46} \triangleright h_{134}^{-1}\big) \\
&= \delta_H\big(g_{46} \triangleright h_{124}\,(\partial(h_{246})^{-1}\,\partial(h_{236})\,g_{36}) \triangleright h_{123}^{-1}\,h_{246}^{-1}\,h_{236}\,h_{346}\,g_{46} \triangleright h_{134}^{-1}\big) \\
&= \delta_H\big(g_{46} \triangleright h_{124}\,(g_{46}g_{24}g_{23}^{-1}) \triangleright h_{123}^{-1}\,g_{46} \triangleright h_{234}\,g_{46} \triangleright h_{134}^{-1}\big) \\
&= \delta_H\big(g_{46} \triangleright h_{124}\,g_{46} \triangleright h_{234}\,(g_{46}g_{34}) \triangleright h_{123}^{-1}\,g_{46} \triangleright h_{134}^{-1}\big) \\
&= \delta_H(e)\,.
\end{aligned} \tag{E.32}
$$

Preostala δ-funkcija $\delta_H(h_{1256})$ je jednaka:

$$
\begin{aligned}
\delta_H(h_{1256}) &= \delta_H\big(h_{156}\,(g_{56} \rhd h_{125})\,h_{256}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{156}\,(g_{56} \rhd h_{125})\,h_{256}^{-1}\,(h_{236}\,g_{36} \rhd h_{123}^{-1}\,h_{136}^{-1})\big)\\
&= \delta_H\big(h_{156}\,(g_{56} \rhd h_{125})\,h_{256}^{-1}\,h_{236}\,g_{36} \rhd h_{123}^{-1}\,h_{346}\,g_{46} \rhd h_{134}^{-1}\,h_{146}^{-1}\big)\\
&= \delta_H\big(h_{156}\,(g_{56} \rhd h_{125})\,h_{256}^{-1}\,h_{236}\,g_{36} \rhd h_{123}^{-1}\,h_{346}\,g_{46} \rhd h_{134}^{-1}\,h_{456}g_{56} \rhd h_{145}^{-1}h_{156}^{-1}\big)\\
&= \delta_H\big(g_{56} \rhd h_{125}\,h_{256}^{-1}\,h_{236}\,g_{36} \rhd h_{123}^{-1}\,h_{346}\,g_{46} \rhd h_{134}^{-1}\,h_{456}g_{56} \rhd h_{145}^{-1}\big)\\
&= \delta_H\big(h_{256}^{-1}\,h_{236}\,g_{36} \rhd h_{123}^{-1}\,h_{346}\,g_{46} \rhd h_{134}^{-1}\,h_{456}g_{56} \rhd (g_{45} \rhd h_{124}h_{245}^{-1})\big)\\
&= \delta_H\big(h_{236}\,g_{36} \rhd h_{123}^{-1}\,h_{346}\,g_{46} \rhd h_{134}^{-1}\,g_{46} \rhd h_{124}\,h_{246}^{-1}\big)\\
&= \delta_H\big(h_{236}\,h_{346}\,(g_{46}g_{34}^{-1}) \rhd h_{123}^{-1}\,g_{46} \rhd h_{134}^{-1}\,g_{46} \rhd h_{124}\,h_{246}^{-1}\big)\\
&= \delta_H\big(g_{46} \rhd h_{234}\,(g_{46}g_{34}^{-1}) \rhd h_{123}^{-1}\,g_{46} \rhd h_{134}^{-1}\,g_{46} \rhd h_{124}\big)\\
&= \delta_H(e)\,.
\end{aligned}
\tag{E.33}
$$

Preostala integracija – po elementu h_{156} grupe H je trivijalna, odnosno *desna strana* se konačno svodi na:

$$
d.s. = \delta_G(e)^3\,\delta_H(e)^3 = |G|^3\,|H|^3\,. \tag{E.34}
$$

Konstante ispred integrala su $|G|^{-8}|H|^{-1}$ sa leve strane poteza, odnosno $|G|^{-11}|H|^{-3}$ sa desne strane poteza, što kompenzuje razliku i izrazima (E.27) i (E.34), na osnovu čega zaključujemo da je suma po stanjima (8.16) invarijantna na $2 \leftrightarrow 4$ Pahnerov potez.

Pahnerov potez 3 ↔ 3

Analizirajmo najpre desnu stranu poteza, tj. integral:

$$
\int_H dh_{123}\delta_G(g_{123})\,\delta_H(h_{1234})\delta_H(h_{1235})\delta_H(h_{1236})\,. \tag{E.35}
$$

Integralimo h_{123} koristeći $\delta_H(h_{1234})$ i dobijamo:

$$
h_{123} = g_{34}^{-1} \rhd h_{134}^{-1}\,g_{34}^{-1} \rhd h_{124}\,g_{34}^{-1} \rhd h_{234}. \tag{E.36}
$$

Može se pokazati da je preostala δ-funkcija na grupi G, δ-funkcija $\delta_G(g_{123}) = \delta_G(e)$, videti dokaz invarijantnosti $3BF$ sume po stanjima na $3 \leftrightarrow 3$ Paherov potez.

Za δ-funkciju $\delta_H(h_{1235})$, primenom identiteta (8.2) za trougao (345) i identiteta (8.6) za tetraedre (1345), (1245) i (2345) dobijamo:

$$
\begin{aligned}
\delta_H(h_{1235}) &= \delta_H\big(h_{135}\,(g_{35}\triangleright h_{123})\,h_{235}^{-1}\,h_{125}^{-1}\big)\\
&= \delta_H\big(h_{135}\,(g_{35}g_{34}^{-1})\triangleright (h_{134}^{-1}\,h_{124}\,h_{234})\,h_{235}^{-1}\,h_{125}^{-1}\big)\\
&= \delta_H\big(h_{135}\,h_{345}g_{45}\triangleright (h_{134}^{-1}\,h_{124}\,h_{234})\,h_{345}^{-1}\,h_{235}^{-1}\,h_{125}^{-1}\big)\\
&= \delta_H\big(h_{145}\,g_{45}\triangleright (h_{124}\,h_{234})\,h_{345}^{-1}\,h_{235}^{-1}\,h_{125}^{-1}\big)\\
&= \delta_H\big(h_{245}\,g_{45}\triangleright h_{234}\,h_{345}^{-1}\,h_{235}^{-1}\big)\\
&= \delta_H(e)\,.
\end{aligned}
\tag{E.37}
$$

Za δ-funkciju $\delta_H(h_{1236})$, primenom identiteta (8.2) za trougao (346) i identiteta (8.6) za tetraedre (1346), (1246) i (2346) dobijamo:

$$
\begin{aligned}
\delta_H(h_{1236}) &= \delta_H\big(h_{136}\,(g_{36}\triangleright h_{123})\,h_{236}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{136}\,(g_{36}g_{34}^{-1})\triangleright (h_{134}^{-1}\,h_{124}\,h_{234})\,h_{236}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{136}\,h_{346}g_{46}\triangleright (h_{134}^{-1}\,h_{124}\,h_{234})\,h_{346}^{-1}\,h_{236}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{146}\,g_{46}\triangleright (h_{124}\,h_{234})\,h_{346}^{-1}\,h_{236}^{-1}\,h_{126}^{-1}\big)\\
&= \delta_H\big(h_{246}\,g_{46}\triangleright h_{234}\,h_{346}^{-1}\,h_{236}^{-1}\big)\\
&= \delta_H(e)\,.
\end{aligned}
\tag{E.38}
$$

Zaključujemo da se izraz sa desne strane Pahnerovog poteza svodi na

$$
d.s. = \delta_G(e)\delta_H(e)^2 = |G|\,|H|^2\,. \tag{E.39}
$$

Analizirajmo sada *levu stranu poteza*, tj. integral:

$$
\int_H dh_{456}\delta_G(g_{456})\,\delta_H(h_{3456})\delta_H(h_{2456})\delta_H(h_{1456})\,. \tag{E.40}
$$

Integralimo h_{456} koristeći δ-funkciju $\delta_H(h_{3456})$. Zamenjujući svuda dobijeni izraz za h_{456} dobijamo da se δ-funkcija $\delta_G(g_{456})$, primenom identiteta (8.2) za trouglove (346), (356) i (345), svodi na:

$$
\delta_G(g_{456}) = \delta_G\big(e\big)\,. \tag{E.41}
$$

Sličnim postupkom kao i sa desne strane poteza, dobijamo da su δ-funkcije $\delta_H(h_{1456})$ i $\delta_H(h_{2456})$ jednake $\delta_H(e)^2$. Konačno dobijamo da je leva strana poteza jednaka:

$$
l.s. = \delta_G(e)\delta_H(e)^2 = |G|\,|H|^2\,. \tag{E.42}
$$

Broj k-simpleksa sa obe strane $3\leftrightarrow 3$ poteza je isti za sve k, tj. koeficijenti ispred integrala su jednaki u ovom slučaju i prema tome nisu od značaja.

E.2 Invarijantnost $3BF$ sume po stanjima na Pahnerove poteze

U ovom dodatku prikazan je dokaz da je suma po stanjima (9.22) nezavisna od triangulacije mnogostrukosti, tj. invarijantna na Pahnerove poteze.

E.2.1 $n = 4$

Pahnerov potez 1 ↔ 5

Koristeći identitet (2.90) δ-funkcija na *levoj strani poteza* $\delta_L(l_{23456})$ je:

$$\delta_L(l_{23456}) = \delta_L\big(l_{2346}{}^{-1}(h_{236} \rhd' l_{3456})l_{2356}h_{256} \rhd' (g_{56} \rhd l_{2345})l_{2456}{}^{-1}h_{246} \rhd' \{h_{456},(g_{56}g_{45}) \rhd h_{234}\}_{\rm pf}\big). \tag{E.43}$$

Ispitajmo čemu je jednaka *desna strana poteza*, data jednačinom (9.23) posle integracije. Najpre integralimo po varijabli g_{12} iskoristivši pritom δ-funkciju $\delta_G(g_{123})$, zatim varijabli g_{13} koristeći $\delta_G(g_{134})$, g_{14} koristeći $\delta_G(g_{145})$ i varijabli g_{15} koristeći $\delta_G(g_{156})$:

$$\begin{aligned} g_{12} =& \;\; g_{23}^{-1}\,\partial(h_{123})^{-1}\,g_{13}\,, \\ g_{13} =& \;\; g_{34}^{-1}\,\partial(h_{134})^{-1}\,g_{14}\,, \\ g_{14} =& \;\; g_{45}^{-1}\,\partial(h_{145})^{-1}\,g_{15}\,, \\ g_{15} =& \;\; g_{56}^{-1}\,\partial(h_{156})^{-1}\,g_{16}\,. \end{aligned} \tag{E.44}$$

Zatim, integralimo varijablu h_{123} koristeći $\delta_H(h_{1234})$, h_{124} koristeći $\delta_H(h_{1245})$, h_{125} koristeći $\delta_H(h_{1256})$, h_{134} koristeći $\delta_H(h_{1345})$, h_{135} koristeći $\delta_H(h_{1356})$ i h_{145} koristeći $\delta_H(h_{1456})$:

$$\begin{aligned} h_{123} =& \;\; g_{34}^{-1} \rhd h_{134}^{-1}\, g_{34}^{-1} \rhd \delta(l_{1234})^{-1}\, g_{34}^{-1} \rhd h_{124}\, g_{34}^{-1} \rhd h_{234}\,, \\ h_{124} =& \;\; g_{45}^{-1} \rhd h_{145}^{-1}\, g_{45}^{-1} \rhd \delta(l_{1245})^{-1}\, g_{45}^{-1} \rhd h_{125}\, g_{45}^{-1} \rhd h_{245}\,, \\ h_{125} =& \;\; g_{56}^{-1} \rhd h_{156}^{-1}\, g_{56}^{-1} \rhd \delta(l_{1256})^{-1}\, g_{56}^{-1} \rhd h_{126}\, g_{56}^{-1} \rhd h_{256}\,, \\ h_{134} =& \;\; g_{45}^{-1} \rhd h_{145}^{-1}\, g_{45}^{-1} \rhd \delta(l_{1345})^{-1}\, g_{45}^{-1} \rhd h_{135}\, g_{45}^{-1} \rhd h_{345}\,, \\ h_{135} =& \;\; g_{56}^{-1} \rhd h_{156}^{-1}\, g_{56}^{-1} \rhd \delta(l_{1356})^{-1}\, g_{56}^{-1} \rhd h_{136}\, g_{56}^{-1} \rhd h_{356}\,, \\ h_{145} =& \;\; g_{56}^{-1} \rhd h_{156}^{-1}\, g_{56}^{-1} \rhd \delta(l_{1456})^{-1}\, g_{56}^{-1} \rhd h_{146}\, g_{56}^{-1} \rhd h_{456}\,. \end{aligned} \tag{E.45}$$

Nakon ovih integracija šest δ-funkcija na grupi G prisutnih sa desne strane poteza svode se na $\delta_G(e)^6$. Skicirajmo dokaz. Najpre,

$$\begin{aligned} \delta_G(g_{124}) &= \delta_G\big(\partial(h_{124})g_{24}g_{12}g_{14}^{-1}\big) \\ &= \delta_G\big(\partial(h_{124})g_{24}g_{23}^{-1}\partial(h_{123})^{-1}g_{13}g_{14}^{-1}\big) \\ &= \delta_G\big(\partial(h_{124})g_{24}g_{23}^{-1}g_{34}^{-1}\partial(h_{234})^{-1}\partial(h_{124})^{-1}\partial(h_{134})g_{34}g_{13}g_{14}^{-1}\big) \\ &= \delta_G\big(\partial(h_{124})g_{24}g_{23}^{-1}g_{34}^{-1}(g_{34}g_{23}^{-1}g_{24}^{-1})\partial(h_{124})^{-1}e\big) \\ &= \delta_G(e). \end{aligned} \tag{E.46}$$

Zatim, dobijamo

$$\begin{aligned} \delta_G(g_{125}) &= \delta_G\big(\partial(h_{125})g_{25}g_{12}g_{15}^{-1}\big) \\ &= \delta_G\big(\partial(h_{125})g_{25}g_{23}^{-1}\partial(h_{123})^{-1}g_{13}g_{15}^{-1}\big) \\ &= \delta_G\big(\partial(h_{125})g_{25}g_{23}^{-1}g_{34}^{-1}\partial(h_{234})^{-1}\partial(h_{124})^{-1}\partial(h_{134})g_{34}g_{13}g_{15}^{-1}\big) \\ &= \delta_G\big(\partial(h_{125})g_{25}g_{23}^{-1}g_{34}^{-1}\partial(h_{234})^{-1}g_{45}^{-1}(\partial(h_{245})^{-1}\partial(h_{125})^{-1}\partial(h_{145}))g_{45}g_{14}g_{15}^{-1}\big) \\ &= \delta_G\big(\partial(h_{125})g_{25}g_{23}^{-1}g_{34}^{-1}(g_{34}g_{23}^{-1}g_{24}^{-1})g_{45}^{-1}(g_{45}g_{24}^{-1}g_{25}^{-1})\partial(h_{125})^{-1}e\big) \\ &= \delta_G(e). \end{aligned} \tag{E.47}$$

Slično,

$$
\begin{aligned}
\delta_G(g_{126}) &= \delta_G\big(\partial(h_{126})g_{26}g_{12}g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{126})g_{26}g_{23}^{-1}\partial(h_{123})^{-1}g_{13}g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{126})g_{26}g_{23}^{-1}g_{34}^{-1}\partial(h_{234})^{-1}\partial(h_{124})^{-1}\partial(h_{134})g_{34}g_{13}g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{126})g_{26}g_{23}^{-1}g_{34}^{-1}\partial(h_{234})^{-1}g_{45}^{-1}\partial(h_{245})^{-1}\partial(h_{125})^{-1}\partial(h_{145})g_{45}\partial(h_{134})g_{34}g_{13}g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{126})g_{26}g_{23}^{-1}g_{34}^{-1}\partial(h_{234})^{-1}g_{45}^{-1}\partial(h_{245})^{-1}g_{56}^{-1}\partial(h_{256})^{-1}\partial(h_{126})^{-1}\partial(h_{156})g_{56} \\
&\quad \partial(h_{145})g_{45}g_{14}g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{126})g_{26}g_{23}^{-1}g_{34}^{-1}(g_{34}g_{23}^{-1}g_{24}^{-1})g_{45}^{-1}(g_{45}g_{24}^{-1}g_{25}^{-1})g_{56}^{-1}(g_{56}g_{25}^{-1}g_{26}^{-1})\partial(h_{126})^{-1} \\
&\quad (g_{16}g_{15}^{-1}g_{56}^{-1})g_{56}g_{15}g_{16}^{-1}\big) \\
&= \delta_G(e).
\end{aligned}
\tag{E.48}
$$

Za $\delta_G(g_{135})$ dobijamo takođe

$$
\begin{aligned}
\delta_G(g_{135}) &= \delta_G\big(\partial(h_{135})\, g_{35}\, g_{13}\, g_{15}^{-1}\big) \\
&= \delta_G\big(\partial(h_{135})\, g_{35}\, g_{34}^{-1}\, \partial(h_{134})^{-1}\, g_{14}\, g_{15}^{-1}\big) \\
&= \delta_G\big(\partial(h_{135})\, g_{35}\, g_{34}^{-1}\, g_{45}^{-1}\, \partial(h_{345})^{-1}\partial(h_{135})^{-1}\partial(h_{145})\, g_{45}\, g_{14}\, g_{15}^{-1}\big) \\
&= \delta_G\big(\partial(h_{135})\, g_{35}\, g_{34}^{-1}\, g_{45}^{-1}\, \partial(h_{345})^{-1}\partial(h_{135})^{-1}\, \partial(h_{145})\, g_{45}\, g_{45}^{-1}\, \partial(h_{145})^{-1}\, g_{15}\, g_{15}^{-1}\big) \\
&= \delta_G\big(\partial(h_{135})\, g_{35}\, g_{34}^{-1}\, g_{45}^{-1}\, (g_{45}\, g_{34}^{-1}\, g_{35}^{-1})\partial(h_{135})^{-1}\big) \\
&= \delta_G(e)\,,
\end{aligned}
\tag{E.49}
$$

kao i za $\delta_G(g_{136})$:

$$
\begin{aligned}
\delta_G(g_{136}) &= \delta_G\big(\partial(h_{136})g_{36}g_{13}g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{136})g_{36}g_{34}^{-1}\partial(h_{134})^{-1}g_{14}g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{136})g_{36}g_{34}^{-1}g_{45}^{-1}\partial(h_{345})^{-1}\partial(h_{135})^{-1}\partial(h_{145})g_{45}g_{14}g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{136})g_{36}g_{34}^{-1}g_{45}^{-1}\partial(h_{345})^{-1}g_{56}^{-1}\partial(h_{356})^{-1}\partial(h_{136})^{-1}\partial(h_{156})g_{56}\partial(h_{145})g_{45}g_{14}g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{136})g_{36}g_{34}^{-1}g_{45}^{-1}(g_{45}g_{34}^{-1}g_{35}^{-1})g_{56}^{-1}(g_{56}g_{35}^{-1}g_{36}^{-1})\partial(h_{136})^{-1}e\big) \\
&= \delta_H(e).
\end{aligned}
\tag{E.50}
$$

Najzad, preostala δ-funkcija $\delta_G(g_{146})$ na grupi G postaje

$$\begin{aligned}
\delta_G(g_{146}) &= \delta_G\big(\partial(h_{146})\, g_{46}\, g_{14}\, g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{146})\, g_{46}\, (g_{45}^{-1}\, \partial(h_{145})^{-1}\, g_{15})\, g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{146})\, g_{46}\, g_{45}^{-1}\, \partial(h_{145})^{-1}\, (g_{56}^{-1}\, \partial(h_{156})^{-1}\, g_{16})\, g_{16}^{-1}\big) \\
&= \delta_G\big(\partial(h_{146})\, g_{46}\, g_{45}^{-1}\, g_{56}^{-1} \partial(h_{456})^{-1} \partial(h_{146})^{-1} \partial(h_{156}) g_{56}\, (g_{56}^{-1}\, \partial(h_{156})^{-1}\, g_{16})\, g_{16}^{-1}\big) \\
&= \delta_G(e)\,.
\end{aligned} \tag{E.51}$$

Zatim, integralimo l_{1235} koristeći $\delta_L(l_{12345})$, l_{1236} koristeći $\delta_L(l_{12346})$, l_{1246} koristeći $\delta_L(l_{12456})$ i l_{1346} koristeći $\delta_L(l_{13456})$,

$$l_{1235} = (h_{125} \rhd' l_{2345}) l_{1245} h_{145} \rhd' (g_{45} \rhd l_{1234}) l_{1345}^{-1}\, h_{135} \rhd' \{h_{345}, (g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{pf}}\,, \tag{E.52}$$

$$l_{1236} = (h_{126} \rhd' l_{2346}) l_{1246} h_{146} \rhd' (g_{46} \rhd l_{1234}) l_{1346}^{-1}\, h_{136} \rhd' \{h_{346}, (g_{46} g_{34}) \rhd h_{123}\}_{\mathrm{pf}}\,, \tag{E.53}$$

$$l_{1246} = (h_{126} \rhd' l_{2456}) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1245}) {l_{1456}}^{-1}\, h_{146} \rhd' \{h_{456}, (g_{56} g_{45}) \rhd h_{124}\}_{\mathrm{pf}}\,, \tag{E.54}$$

$$l_{1346} = (h_{136} \rhd' l_{3456}) l_{1356} h_{156} \rhd' (g_{56} \rhd l_{1345}) {l_{1456}}^{-1}\, h_{146} \rhd' \{h_{456}, (g_{56} g_{45}) \rhd h_{134}\}_{\mathrm{pf}}\,. \tag{E.55}$$

Pokažimo da se sada preostale δ-funkcije na grupi H svedu na $\delta_H(e)^4$. Najpre, pravolinijskim računom dobijamo za $\delta_H(h_{1235})$:

$$\begin{aligned}
\delta_H(h_{1235}) &= \delta_H\big(\delta(l_{1235}) h_{135} (g_{35} \rhd h_{123}) h_{235}^{-1} h_{125}^{-1}\big) \\
&= \delta_H\Big(\delta\big((h_{125} \rhd' l_{2345}) l_{1245} h_{145} \rhd' (g_{45} \rhd l_{1234}) l_{1345}^{-1} h_{135} \rhd' \{h_{345}, (g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{pf}}\big) h_{135} \\
&\quad (g_{35} \rhd h_{123}) h_{235}^{-1} h_{125}^{-1}\Big) \\
&= \delta_H\Big(\big(h_{125} \delta(l_{2345}) h_{125}^{-1} \delta(l_{1245}) h_{145} (g_{45} \rhd \delta(l_{1234})) h_{145}^{-1} \delta(l_{1345})^{-1} h_{135} \delta(\{h_{345}, (g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{pf}}) h_{135}^{-1}\big) \\
&\quad h_{135} (g_{35} \rhd h_{123}) h_{235}^{-1} h_{125}^{-1}\Big) \\
&= \delta_H\Big(h_{235} h_{345} (g_{45} \rhd h_{234}^{-1}) h_{245}^{-1} h_{125}^{-1} h_{125} h_{245} (g_{45} \rhd h_{124}^{-1}) h_{145}^{-1} h_{145} (g_{45} \rhd (h_{124} h_{234} (g_{34} \rhd h_{123}^{-1}) h_{134}^{-1})) \\
&\quad h_{145}^{-1} (h_{145} (g_{45} \rhd h_{134}) h_{345}^{-1} h_{135}^{-1}) h_{135} \delta(\{h_{345}, (g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{pf}}) h_{135}^{-1} h_{135} (g_{35} \rhd h_{123}) h_{235}^{-1}\Big) \\
&= \delta_H(h_{345} \big((g_{45} g_{34}) \rhd h_{123}^{-1}\big) h_{345}^{-1} \delta(\{h_{345}, (g_{45} g_{34}) \rhd h_{123}\}_{\mathrm{pf}}) (g_{35} \rhd h_{123}).
\end{aligned} \tag{E.56}$$

Iskoristivši identitet

$$\delta\{h_1\,, h_2\}_{\mathrm{pf}} (\partial(h_1) \rhd h_2) h_1 h_2^{-1} h_1^{-1} = e\,, \tag{E.57}$$

i jednačinu $g_{35} = \partial(h_{345}) g_{45} g_{34}$, pri čemu za elemente h_1 i h_2 biramo $h_1 = h_{345}$ i $h_2 = (g_{45} g_{34}) \rhd h_{123}$, dobijamo

$$\delta_H(h_{1235}) = \delta_H(e). \tag{E.58}$$

Analognim postupkom za vrednost δ-funkcije $\delta_H(h_{1236})$ dobijamo:

$$\begin{aligned}\delta_H(h_{1236}) &= \delta_H\big(\delta(l_{1236})h_{136}(g_{36}\triangleright h_{123})h_{236}^{-1}h_{126}^{-1}\big)\\ &= \delta_H\Big(\delta\big((h_{126}\triangleright' l_{2346})l_{1246}h_{146}\triangleright'(g_{46}\triangleright l_{1236}l_{1346}^{-1}h_{136}\triangleright'\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{pf}}\big)h_{136}(g_{36}\triangleright h_{123})h_{236}^{-1}h_{126}^{-1}\Big)\\ &= \delta_H\Big(\big(h_{126}\delta(l_{2346})h_{126}^{-1}\delta(l_{1246})h_{146}(g_{46}\triangleright\delta(l_{1234}))h_{146}^{-1}\delta(l_{1346})^{-1}h_{136}\delta(\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{pf}})h_{136}^{-1}\big)\\ &\quad h_{136}(g_{36}\triangleright h_{123})h_{236}^{-1}h_{126}^{-1}\Big)\\ &= \delta_H\Big(h_{236}h_{346}(g_{46}\triangleright h_{234}^{-1})h_{246}^{-1}h_{126}^{-1}h_{126}h_{246}(g_{46}\triangleright h_{124}^{-1})h_{146}^{-1}h_{146}(g_{46}\triangleright(h_{124}h_{234}(g_{34}\triangleright h_{123}^{-1})h_{134}^{-1}))\\ &\quad h_{146}^{-1}(h_{146}(g_{46}\triangleright h_{134})h_{346}^{-1}h_{136}^{-1})h_{136}\delta(\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{pf}})h_{136}^{-1}h_{136}(g_{36}\triangleright h_{123})h_{236}^{-1}\Big)\\ &= \delta_H(h_{346}\big((g_{46}g_{34})\triangleright h_{123}^{-1}\big)h_{346}^{-1}\delta(\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{pf}})(g_{36}\triangleright h_{123}).\end{aligned} \tag{E.59}$$

Koristeći jednačinu $g_{36}=\partial(h_{346})g_{46}g_{34}$, i identitet (2.6) gde su $h_1=h_{346}$ i $h_2=(g_{46}g_{34})\triangleright h_{123}$, dobijamo:

$$\delta_H(h_{1236})=\delta_H(e)\,. \tag{E.60}$$

Sličnim postupkom dobijamo da su δ-funkcije $\delta_H(h_{1246})=\delta_H(h_{1346})=\delta_H(e)$. Preostala δ-funkcija na grupi L je $\delta_L(l_{12356})$,

$$\delta_L(l_{12356})=\ \delta_L\big({l_{1236}}^{-1}(h_{126}\triangleright' l_{2356})l_{1256}h_{156}\triangleright'(g_{56}\triangleright l_{1235}){l_{1356}}^{-1}h_{136}\triangleright'\{h_{356},(g_{56}g_{35})\triangleright h_{123}\}_{\mathrm{pf}}\big). \tag{E.61}$$

Koristeći jednačine (E.52), (E.53), (E.54) i (E.55), dobijamo:

$$\begin{aligned}\delta_L(l_{12356}) &= \delta_L\Big(h_{136}\triangleright'\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{pf}}^{-1}(h_{136}\triangleright' l_{3456})l_{1356}h_{156}\triangleright'(g_{56}\triangleright l_{1345}){l_{1456}}^{-1}\\ &\quad h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{134}\}_{\mathrm{pf}}h_{146}\triangleright'(g_{46}\triangleright l_{1234})^{-1}h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{124}\}_{\mathrm{pf}}^{-1}l_{1456}\\ &\quad h_{156}\triangleright'(g_{56}\triangleright l_{1245})^{-1}l_{1256}^{-1}(h_{126}\triangleright' l_{2456})^{-1}(h_{126}\triangleright'{l_{2346}}^{-1})(h_{126}\triangleright' l_{2356})l_{1256}\\ &\quad h_{156}\triangleright'(g_{56}\triangleright((h_{125}\triangleright' l_{2345})l_{1245}h_{145}\triangleright'(g_{45}\triangleright l_{1234})l_{1345}^{-1}h_{135}\triangleright'\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{pf}}))\\ &\quad {l_{1356}}^{-1}h_{136}\triangleright'\{h_{356},(g_{56}g_{35})\triangleright h_{123}\}_{\mathrm{pf}}\Big).\end{aligned} \tag{E.62}$$

Koristeći identitet (2.62) δ-funkcija $\delta_L(l_{12356})$ se svede na:

$$\begin{aligned}\delta_L(l_{12356}) &= \delta_L\Big((h_{136}\triangleright' l_{3456})l_{1356}h_{156}\triangleright'(g_{56}\triangleright l_{1345}){l_{1456}}^{-1}\\ &\quad h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{134}\}_{\mathrm{pf}}h_{146}\triangleright'(g_{46}\triangleright l_{1234})^{-1}h_{146}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{124}\}_{\mathrm{pf}}^{-1}l_{1456}\\ &\quad \delta(h_{156}\triangleright'(g_{56}\triangleright l_{1245})^{-1})\triangleright'\Big(\big(\delta(l_{1256})^{-1}h_{126}\big)\triangleright'\big(l_{2456}^{-1}l_{2346}^{-1}l_{2356}\big)h_{156}\triangleright'(g_{56}\triangleright(h_{125}\triangleright' l_{2345}))\Big)\\ &\quad h_{156}\triangleright'(g_{56}\triangleright(h_{145}\triangleright'(g_{45}\triangleright l_{1234})l_{1345}^{-1})){l_{1356}}^{-1}(h_{136}h_{346})\triangleright'\{h_{346}^{-1}h_{356}g_{56}\triangleright h_{345},(g_{56}g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{pf}}\Big).\end{aligned} \tag{E.63}$$

Posle komutacije elemenata, δ-funkcija se svodi na

$$\begin{aligned}\delta_L(l_{12356}) &= \delta_L\Big((h_{156}\triangleright'(g_{56}\triangleright\delta(l_{1245})^{-1})\delta(l_{1256})^{-1}h_{126})\triangleright'\big(l_{2456}^{-1}l_{2346}^{-1}l_{2356}h_{256}\triangleright'(g_{56}\triangleright l_{2345})\big)\\ &\quad h_{156}\triangleright'\big(g_{56}\triangleright(h_{145}\triangleright'(g_{45}\triangleright l_{1234})l_{1345}^{-1})\big){l_{1356}}^{-1}(h_{136}h_{346})\triangleright'\{h_{346}^{-1}h_{356}g_{56}\triangleright h_{345},(g_{56}g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{pf}}\\ &\quad h_{136}\triangleright' l_{3456}h_{156}\triangleright'(g_{56}\triangleright l_{1345})(\delta(l_{1456})^{-1}h_{146})\triangleright'\big(\{h_{456},(g_{56}g_{45})\triangleright h_{134}\}_{\mathrm{pf}}\big)\\ &\quad (\delta(l_{1456})^{-1}h_{146})\triangleright'\big((g_{46}\triangleright l_{1234})^{-1}\big)(\delta(l_{1456})^{-1}h_{146})\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{124}\}_{\mathrm{pf}}^{-1}\Big).\end{aligned} \tag{E.64}$$

Tetraedar (3456) je deo podintegralne funkcije sa obe strane poteza, pa možemo koristiti identitet (9.19) za $\delta_H(h_{3456})$, odnosno jednakost $h_{346}^{-1}h_{356}g_{56} \rhd h_{345} = h_{346}^{-1} \rhd' \delta(l_{3456})^{-1}h_{456}$. Zatim, primenom identiteta (2.62) dobijamo

$$\begin{aligned}
\{h_{346}^{-1}h_{356}g_{56} \rhd h_{345}, (g_{56}g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{pf}} &= \{h_{346}^{-1} \rhd' \delta(l_{3456})^{-1}h_{456}, (g_{56}g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{pf}} \\
&= \left(h_{346}^{-1} \rhd' \delta(l_{3456})^{-1}\right) \rhd' \{h_{456}, (g_{56}g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{pf}} \\
&\quad \{h_{346}^{-1} \rhd' \delta(l_{3456})^{-1}, (g_{46}g_{34}) \rhd h_{123}\}_{\mathrm{pf}} \\
&= h_{346}^{-1} \rhd' l_{3456}^{-1}\{h_{456}, (g_{56}g_{45}g_{34}) \rhd h_{123}\}_{\mathrm{pf}} \\
&\quad \left((g_{46}g_{34}) \rhd h_{123}h_{346}^{-1}\right) \rhd' l_{3456}\,,
\end{aligned} \tag{E.65}$$

gde smo u zadnjem redu iskoristili definiciju dejstva $\rhd'$ grupe H na grupu L. Zamenom jednakosti (E.65) u jednačinu (E.64) dobijamo:

$$\begin{aligned}
\delta_L(l_{12356}) \;=\; & \delta_L\Big((h_{156}\rhd'(g_{56}\rhd\delta(l_{1245})^{-1})\delta(l_{1256})^{-1}h_{126}\delta(l_{2456})^{-1})\rhd'\Big(l_{2346}^{-1}l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})l_{2456}^{-1}\Big) \\
& h_{156}\rhd'(g_{56}\rhd(h_{145}\rhd'(g_{45}\rhd l_{1234})))(h_{156}\rhd'(g_{56}\rhd\delta(l_{1345})^{-1})\delta(l_{1356})^{-1}h_{136}\delta(l_{3456})^{-1}h_{346})\rhd' \\
& \big(\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{pf}}((g_{46}g_{34})\rhd h_{123})\rhd' l_{3456}\big)(\delta(l_{1456})^{-1}h_{146})\rhd'\big(\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{pf}}\big) \\
& (\delta(l_{1456})^{-1}h_{146})\rhd'\big((g_{46}\rhd l_{1234})^{-1}\big)(\delta(l_{1456})^{-1}h_{146})\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\mathrm{pf}}^{-1}\Big).
\end{aligned} \tag{E.66}$$

Komutiranjem elementa l_{3456} na kraj izraza, dobijamo

$$\begin{aligned}
\delta_L(l_{12356}) \;=\; & \delta_L\big((h_{156}\rhd'(g_{56}\rhd\delta(l_{1245})^{-1})\delta(l_{1256})^{-1}h_{126}\delta(l_{2456})^{-1})\rhd'(l_{2346}^{-1}l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})l_{2456}^{-1}) \\
& h_{156}\rhd'(g_{56}\rhd(h_{145}\rhd'(g_{45}\rhd l_{1234})))(h_{156}\rhd'(g_{56}\rhd\delta(l_{1345})^{-1})\delta(l_{1356})^{-1}h_{136}\delta(l_{3456})^{-1}h_{346})\rhd' \\
& \big(\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{pf}}\big)(\delta(l_{1456})^{-1}h_{146})\rhd'\big(\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{pf}}\big) \\
& (\delta(l_{1456})^{-1}h_{146})\rhd'\big((g_{46}\rhd l_{1234})^{-1}\big)(\delta(l_{1456})^{-1}h_{146})\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\mathrm{pf}}^{-1} \\
& (h_{156}g_{56}\rhd h_{145}h_{246}g_{46}\rhd h_{234}h_{346}^{-1})\rhd' l_{3456}\big)\big).
\end{aligned} \tag{E.67}$$

Delovanjem na ceo argument δ-funkcije sa $(h_{156} \rhd' (g_{56} \rhd \delta(l_{1245})^{-1})\delta(l_{1256})^{-1}h_{126}\delta(l_{2456})^{-1})^{-1}\rhd'$, dobijamo

$$\begin{aligned}
\delta_L(l_{12356}) \;=\; & \delta_L\big(l_{2346}^{-1}l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})l_{2456}^{-1}\big(h_{246}h_{456}(g_{56}g_{45})\rhd h_{124}^{-1}\big)\rhd' \\
& \Big((g_{56}g_{45})\rhd l_{1234}\big((g_{56}g_{45})\rhd h_{134}h_{456}^{-1}\big)\rhd'\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{pf}} \\
& h_{456}^{-1}\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{pf}}h_{456}^{-1}\rhd g_{46}\rhd l_{1234}^{-1}\big(h_{456}^{-1}g_{46}\rhd h_{124}\big)\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}^{-1}\}_{\mathrm{pf}}\Big) \\
& (h_{246}g_{46}\rhd h_{234}h_{346}^{-1})\rhd' l_{3456}.
\end{aligned} \tag{E.68}$$

Nakon što iskoristimo identitet (2.63) za $\{h_{456}, (g_{56}g_{45}) \rhd (h_{134}g_{34} \rhd h_{123})\}_{\mathrm{pf}}$, tj. jednakost

$$\{h_{456},(g_{56}g_{45})\rhd(h_{134}g_{34}\rhd h_{123})\}_{\mathrm{pf}}= \{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{pf}}(g_{46}\rhd h_{134})\rhd'\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\mathrm{pf}}, \tag{E.69}$$

izraz se svodi na:

$$\begin{aligned}\delta_L(l_{12356}) &= \delta_L\big(l_{2346}^{-1}l_{2356}h_{256}\triangleright'(g_{56}\triangleright l_{2345})l_{2456}^{-1}\\ &\quad h_{246}\triangleright'\Big(\big(h_{456}(g_{56}g_{45})\triangleright h_{124}^{-1}\big)\triangleright'\Big((g_{56}g_{45})\triangleright l_{1234}h_{456}^{-1}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright(h_{134}g_{34}\triangleright h_{123})\}_{\mathrm{pf}}\\ &\quad h_{456}^{-1}\triangleright g_{46}\triangleright l_{1234}^{-1}\Big)\{h_{456},(g_{56}g_{45})\triangleright h_{124}^{-1}\}_{\mathrm{pf}}\Big)(h_{246}g_{46}\triangleright h_{234}h_{346}^{-1})\triangleright' l_{3456}.\end{aligned} \tag{E.70}$$

Zatim, primenom identiteta (2.63) na član $\{h_{456},(g_{56}g_{45})\triangleright(h_{124}^{-1}\delta(l_{1234})h_{134}g_{34}\triangleright h_{123})\}_{\mathrm{pf}}$ vidimo da se članovi sa l_{1234} ponište, tj. dobijamo izraz,

$$\begin{aligned}\delta_L(l_{12356}) &= \delta_L\big(l_{2346}^{-1}l_{2356}h_{256}\triangleright'(g_{56}\triangleright l_{2345})l_{2456}^{-1}\\ &\quad h_{246}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright(h_{124}^{-1}\delta(l_{1234})h_{134}g_{34}\triangleright h_{123})\}_{\mathrm{pf}}(h_{246}g_{46}\triangleright h_{234}h_{346}^{-1})\triangleright' l_{3456}\\ &= \delta_L\big(l_{2346}{}^{-1}l_{2356}h_{256}\triangleright'(g_{56}\triangleright l_{2345})l_{2456}{}^{-1}h_{246}\triangleright'\{h_{456},(g_{56}g_{45})\triangleright h_{234}\}_{\mathrm{pf}}(\delta(l_{2346})^{-1}h_{236})\triangleright' l_{3456})\big)\\ &= \delta_L(l_{23456}).\end{aligned} \tag{E.71}$$

Zaključujemo da se preostala δ-funkcija $\delta_L(l_{12356})$ sa desne strane poteza svodi na delta funkciju $\delta_L(l_{23456})$ sa leve strane poteza. Integracije po elementima l_{1234}, l_{1245}, l_{1256}, l_{1345}, l_{1356} i l_{1456} su trivijalne i konačan izraz sa integral sa desne strane je:

$$d.s. = \delta_G(e)^6\delta_H(e)^4\delta_L(l_{23456}) = |G|^6|H|^4\delta_L(l_{23456})\,. \tag{E.72}$$

Razlika u faktorima u izrazima (E.43) i (E.72) kompenzovana je konstantama ispred integrala – faktorom $|G|^{-11}|H|^{-4}|L|^{-1}$ sa desne strane i faktorom $|G|^{-5}|H|^{0}|L|^{-1}$ sa leve strane. Zaključujemo da je suma (9.22) invarijantna na $1\leftrightarrow 5$ Pahnerov potez.

Pahnerov potez 2 $\leftrightarrow$ 4

Najpre analizirajmo *levu stranu poteza*, izraz

$$\int_L dl_{2345}\delta_H(h_{2345})\delta_L(l_{23456})\delta_L(l_{12345})\,. \tag{E.73}$$

Prvo integralimo l_{2345} koristeći δ-funkciju $\delta_L(l_{12345})$,

$$l_{2345} = h_{125}{}^{-1}\triangleright'\left(l_{1235}h_{135}\triangleright'\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{pf}}^{-1}l_{1345}h_{145}\triangleright'(g_{45}\triangleright l_{1234})^{-1}l_{1245}^{-1}\right). \tag{E.74}$$

Preostala δ-funkcija na grupi H $\delta_H(h_{2345})$ sada postaje,

$$\begin{aligned}\delta_H(h_{2345}) &= \delta_H\big(\delta(l_{2345})h_{245}(g_{45}\triangleright h_{234})h_{345}^{-1}h_{235}^{-1}\big)\\ &= \delta_H\Big(h_{125}{}^{-1}\delta(l_{1235})h_{135}\delta(\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{pf}}^{-1})h_{135}^{-1}\delta(l_{1345})h_{145}(g_{45}\triangleright\delta(l_{1234}))^{-1}h_{145}^{-1}\\ &\quad \delta(l_{1245})^{-1}h_{125}h_{245}(g_{45}\triangleright h_{234})h_{345}^{-1}h_{235}^{-1}\Big).\end{aligned} \tag{E.75}$$

Primenom identiteta (9.19) za tetraedre (1235), (1345), (1234) i (1245) izraz (E.75) se svodi na:

$$\begin{aligned}\delta_H(h_{2345}) &= \delta_H\Big(h_{125}{}^{-1}h_{125}h_{235}(g_{35}\triangleright h_{123}^{-1})h_{135}^{-1}h_{135}\delta(\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{pf}}^{-1})h_{135}^{-1}h_{135}h_{345}(g_{45}\triangleright h_{134}^{-1})\\ &\quad h_{145}^{-1}h_{145}g_{45}\triangleright(h_{134}(g_{34}\triangleright h_{123})h_{234}^{-1}h_{124}^{-1})h_{145}^{-1}h_{145}(g_{45}\triangleright h_{124})h_{245}^{-1}h_{125}^{-1}h_{125}h_{245}(g_{45}\triangleright h_{234})h_{345}^{-1}h_{235}^{-1}\big)\\ &= \delta_H\Big((g_{35}\triangleright h_{123}^{-1})\delta(\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{pf}}^{-1})h_{345}(g_{45}g_{34})\triangleright h_{123})h_{345}^{-1}\Big).\end{aligned} \tag{E.76}$$

Zatim, primenom identiteta (2.6) za $h_1 = h_{345}$ i $h_2 = (g_{45}g_{34}) \rhd h_{123}$ i identiteta $g_{35} = \partial(h_{345})g_{45}g_{34}$ dobijamo:

$$\delta_H(h_{2345}) = \delta_H(e)\,. \tag{E.77}$$

Preostala δ-funkcija na grupi L $\delta_L(l_{23456})$ je

$$\delta_L(l_{23456}) = \delta_L\big({l_{2346}}^{-1}(h_{236}\rhd' l_{3456})l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345}){l_{2456}}^{-1}h_{246}\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{234}\}_{\mathrm{pf}}\big). \tag{E.78}$$

Primenom jednačine (E.75) dobijamo:

$$\begin{aligned}\delta_L(l_{23456}) = \delta_L\Big(&{l_{2346}}^{-1}(h_{236}\rhd' l_{3456})l_{2356}h_{256}\rhd'\Big(g_{56}\rhd\big({h_{125}}^{-1}\rhd'\big(l_{1235}h_{135}\rhd'\{h_{345},(g_{45}g_{34})\rhd h_{123}\}^{-1}_{\mathrm{pf}}\\ &l_{1345}h_{145}\rhd'(g_{45}\rhd l_{1234})^{-1}l^{-1}_{1245}\big)\big)\Big){l_{2456}}^{-1}h_{246}\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{234}\}_{\mathrm{pf}}\Big).\end{aligned} \tag{E.79}$$

Komutiranjem elemenata dobijamo izraz:

$$\begin{aligned}\delta_L(l_{23456}) = \delta_L\Big(&{l_{2456}}^{-1}{l_{2346}}^{-1}l_{2356}(h_{256}g_{56}\rhd {h_{125}}^{-1})\rhd' g_{56}\rhd l_{1235}\big(h_{256}g_{56}\rhd {h_{125}}^{-1}g_{56}\rhd h_{135}\big)\rhd'\\ &\Big((g_{35}\rhd h_{123}h^{-1}_{356})\rhd' l_{3456})\{g_{56}\rhd h_{345},(g_{56}g_{45}g_{34})\rhd h_{123}\}^{-1}_{\mathrm{pf}}(g_{56}\rhd h_{345}(g_{56}g_{45})\rhd(h_{123}h^{-1}_{234})h^{-1}_{456})\rhd'\\ &\{h_{456},(g_{56}g_{45})\rhd h_{234}\}_{\mathrm{pf}}\Big)(h_{256}g_{56}\rhd {h_{125}}^{-1})\rhd' g_{56}\rhd l_{1345}\\ &(h_{256}g_{56}\rhd {h_{125}}^{-1}g_{56}\rhd h_{145})\rhd'((g_{56}g_{45})\rhd l_{1234})^{-1}(h_{256}g_{56}\rhd {h_{125}}^{-1})\rhd' g_{56}\rhd l^{-1}_{1245}\Big).\end{aligned} \tag{E.80}$$

Naposletku, leva strana poteza je:

$$l.s. = \delta_H(e)\delta_L(l_{23456}) = |H|\delta_L(l_{23456})\,. \tag{E.81}$$

Ispitajmo sada čemu je jednaka *desna strana poteza*, tj. integral (9.26). Prvo integralimo g_{16} koristeći $\delta_G(g_{126})$,

$$g_{16} = \partial(h_{126})\,g_{26}\,g_{12}\,. \tag{E.82}$$

Zatim, integralimo h_{126} koristeći $\delta_H(h_{1236})$, h_{136} koristeći $\delta_H(h_{1346})$ i h_{146} koristeći $\delta_H(h_{1456})$, na osnovu čega dobijamo

$$\begin{aligned} h_{126} &= \delta(l_{1236})h_{136}\,(g_{36}\rhd h_{123})\,h^{-1}_{236}\,,\\ h_{136} &= \delta(l_{1346})h_{146}\,(g_{46}\rhd h_{134})\,h^{-1}_{346}\,,\\ h_{146} &= \delta(l_{1456})h_{156}\,(g_{56}\rhd h_{145})\,h^{-1}_{456}\,.\end{aligned} \tag{E.83}$$

Preostale tri δ-funkcije na grupi G svode se na $\delta_G(e)^3$. Lako možemo pokazati da δ-funkcija $\delta_G(g_{136})$,

$$\delta_G(g_{136}) = \delta_G\big(\partial(h_{136})\,g_{36}\,g_{13}\,g^{-1}_{16}\big)\,, \tag{E.84}$$

posle zamene jednačine (E.82) postaje:

$$\delta_G(g_{136}) = \delta_G\big(\partial(h_{136})\,g_{36}\,g_{13}\,g^{-1}_{12}g^{-1}_{26}\partial(h_{126})^{-1}\big)\,. \tag{E.85}$$

Koristeći identitet (E.83) za elemente h_{126}, h_{136} i h_{146} i činjenicu da je $\partial(\delta l) = 0$ za svaki element $l \in L$, a nakon primene identiteta (9.18) za trouglove (156), (145), (456) (134), (346), (236) i (123), δ-funkcija $\delta_G(g_{136})$ postaje $\delta_G(e)$. Slično se pokazuje da važi $\delta_G(g_{146}) = \delta_G(g_{156}) = \delta_G(e)$. Zatim, integralimo l_{1236} koristeći $\delta_L(l_{12346})$ i dobijamo

$$l_{1236} = (h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234})l^{-1}_{1346}\,h_{136}\rhd'\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\mathrm{pf}}\,, \tag{E.86}$$

l_{1246} koristeći $\delta_L(l_{12456})$,

$$l_{1246} = (h_{126} \rhd' l_{2456}) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1245}) l_{1456}{}^{-1} h_{146} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{124}\}_{\text{pf}} \,, \tag{E.87}$$

i l_{1346} koristeći $\delta_L(l_{13456})$,

$$l_{1346} = (h_{136} \rhd' l_{3456}) l_{1356} h_{156} \rhd' (g_{56} \rhd l_{1345}) l_{1456}{}^{-1} h_{146} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{134}\}_{\text{pf}} \,. \tag{E.88}$$

Preostale δ-funkcije na grupi H svode se na $\delta_H(e)^3$, sličnim postupkom kao i u slučaju $1 \leftrightarrow 5$ Pahnerovog poteza, tj. dobijamo $\delta_H(h_{1256}) = \delta_H(h_{1356}) = \delta_H(h_{1456}) = \delta_H(e)$. Preostala δ-funkcija na grupi L, funkcija $\delta_L(l_{12356})$ glasi:

$$\delta_L(l_{12356}) = \delta_L\Big(l_{1236}{}^{-1}(h_{126} \rhd' l_{2356}) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1235}) l_{1356}{}^{-1} h_{136} \rhd' \{h_{356}, (g_{56}g_{35}) \rhd h_{123}\}_{\text{pf}}\Big). \tag{E.89}$$

Nakon zamene jednačina (E.86), (E.87) i (E.88), dobijamo

$$\begin{aligned}
\delta_L(l_{12356}) &= \delta_L\Big(h_{136} \rhd' \{h_{346}, (g_{46}g_{34}) \rhd h_{123}\}_{\text{pf}}{}^{-1} l_{1346} h_{146} \rhd' (g_{46} \rhd l_{1234})^{-1} l_{1246}^{-1} (h_{126} \rhd' l_{2346})^{-1} \\
&\quad (h_{126} \rhd' l_{2356}) l_{1256} h_{156} \rhd' (g_{56} \rhd l_{1235}) l_{1356}{}^{-1} h_{136} \rhd' \{h_{356}, (g_{56}g_{35}) \rhd h_{123}\}_{\text{pf}}\Big) \\
&= \delta_L\Big((h_{126} \rhd' l_{2456})^{-1} (h_{126} \rhd' l_{2346})^{-1} (h_{126} \rhd' l_{2356}) (h_{256} g_{56} \rhd h_{125}{}^{-1}) \rhd' l_{1235} \\
&\quad \delta(l_{1256}) \rhd' \Big(\delta(l_{1356})^{-1} \rhd' \big(h_{136} \rhd' \{h_{356}, (g_{56}g_{35}) \rhd h_{123}\}_{\text{pf}} (h_{136}h_{346}) \rhd' \{h_{346}^{-1}, g_{36} \rhd h_{123}\}_{\text{pf}} \\
&\quad (h_{136} \rhd' l_{3456})\big) h_{156} \rhd' (g_{56} \rhd l_{1345}) l_{1456}{}^{-1} h_{146} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{134}\}_{\text{pf}} h_{146} \rhd' (g_{46} \rhd l_{1234})^{-1} \\
&\quad h_{146} \rhd' \{h_{456}, (g_{56}g_{45}) \rhd h_{124}\}_{\text{pf}}^{-1} l_{1456} h_{156} \rhd' (g_{56} \rhd l_{1245})^{-1}\Big)\Big).
\end{aligned} \tag{E.90}$$

Komutiranjem elemenata kako bi se redosled elemenata slagao sa redosledom na levoj strani poteza, tj. sa δ-funkcijom (E.80), i primenom identiteta (2.62), tj.

$$\{h_{346}^{-1} h_{356}, (g_{56}g_{35}) \rhd h_{123}\}_{\text{pf}} = h_{346}^{-1} \rhd' \{h_{356}, (g_{56}g_{35}) \rhd h_{123}\}_{\text{pf}} \{h_{346}^{-1}, g_{36} \rhd h_{123}\}_{\text{pf}} \,, \tag{E.91}$$

dobijamo:

$$\begin{aligned}
\delta_L(l_{12356}) &= \delta_L\Big((h_{126} \rhd' l_{2456})^{-1} (h_{126} \rhd' l_{2346})^{-1} (h_{126} \rhd' l_{2356}) (h_{126} h_{256} g_{56} \rhd h_{125}{}^{-1}) \rhd' l_{1235} \\
&\quad \delta(l_{1256}) \rhd' \Big(\delta(l_{1356})^{-1} \rhd' \big((h_{136}h_{346}) \rhd' \{h_{346}^{-1} h_{356}, (g_{56}g_{35}) \rhd h_{123}\}_{\text{pf}} (h_{136} \rhd' l_{3456})\big) \\
&\quad h_{156} \rhd' (g_{56} \rhd l_{1345}) (\delta(l_{1456})^{-1} h_{146}) \rhd' \big(\{h_{456}, (g_{56}g_{45}) \rhd h_{134}\}_{\text{pf}} (g_{46} \rhd l_{1234})^{-1} \\
&\quad \{h_{456}, (g_{56}g_{45}) \rhd h_{124}\}_{\text{pf}}^{-1}\big) h_{156} \rhd' (g_{56} \rhd l_{1245})^{-1}\Big)\Big).
\end{aligned} \tag{E.92}$$

Ponovnom primenom identita (2.62),

$$\begin{aligned}
&(h_{136}h_{346}) \rhd' \{h_{346}^{-1} h_{356}, (g_{56}g_{35}) \rhd h_{123}\}_{\text{pf}} (h_{136} \rhd' l_{3456}) = \\
&(h_{136}h_{346}) \rhd' \{h_{346}^{-1} \rhd' \delta(l_{3456})^{-1} h_{456} g_{56} \rhd h_{345}^{-1}, (g_{56}g_{35}) \rhd h_{123}\}_{\text{pf}} (h_{136} \rhd' l_{3456}) = \\
&(h_{136} \rhd' \delta(l_{3456})^{-1} h_{136} h_{346}) \rhd' \big(\{h_{456} g_{56} \rhd h_{345}^{-1}, (g_{56}g_{35}) \rhd h_{123}\}_{\text{pf}} ((g_{46}g_{34}) \rhd h_{123} h_{346}^{-1}) \rhd' l_{3456}^{-1}\big) \,,
\end{aligned} \tag{E.93}$$

i zamenom ovog izraza u (E.92) dobijamo:

$$\begin{aligned}\delta_L(l_{12356}) &= \delta_L\Big((h_{126}\rhd' l_{2456})^{-1}(h_{126}\rhd' l_{2346})^{-1}(h_{126}\rhd' l_{2356})(h_{126}h_{256}g_{56}\rhd h_{125}{}^{-1})\rhd' l_{1235}\\ &\quad \delta(l_{1256})\rhd'\Big((\delta(l_{1356})^{-1}h_{136}\rhd'\delta(l_{3456})^{-1}h_{136}h_{346})\rhd'\\ &\quad (\{h_{456}g_{56}\rhd h_{345}^{-1},(g_{56}g_{35})\rhd h_{123}\}_{\rm pf}((g_{46}g_{34})\rhd h_{123}h_{346}^{-1})\rhd' l_{3456}))\\ &\quad (h_{156}g_{56}\rhd h_{135}g_{56}\rhd(h_{345}g_{45}\rhd h_{134}^{-1})h_{456}^{-1})\rhd'(\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\rm pf}(g_{46}\rhd l_{1234})^{-1}\\ &\quad \{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\rm pf}^{-1})\Big)(h_{126}h_{256}g_{56}\rhd h_{125}{}^{-1})\rhd'(h_{156}\rhd'(g_{56}\rhd l_{1345})(g_{56}\rhd l_{1245})^{-1})\Big).\end{aligned} \tag{E.94}$$

Komutiranjem elementa l_{3456} i $\{h_{456}g_{56}\rhd h_{345},(g_{56}g_{35})\rhd h_{123}\}_{\rm pf}$, a zatim korišćenjem identiteta (2.62), dobijamo:

$$\begin{aligned}\delta_L(l_{12356}) &= \delta_L\Big((h_{126}\rhd' l_{2456})^{-1}(h_{126}\rhd' l_{2346})^{-1}(h_{126}\rhd' l_{2356})(h_{126}h_{256}g_{56}\rhd h_{125}{}^{-1})\rhd' l_{1235}\\ &\quad (h_{126}h_{256}g_{56}\rhd h_{125}{}^{-1}h_{135}(g_{56}g_{35})\rhd h_{123}g_{56}\rhd h_{356}^{-1})\rhd' g_{56}\rhd l_{3456}\\ &\quad (h_{126}h_{256}g_{56}\rhd h_{125}^{-1}g_{56}\rhd h_{135}g_{56}\rhd h_{345})\rhd'\Big(\{g_{56}\rhd h_{345}^{-1},(g_{56}g_{35})\rhd h_{123}\}_{\rm pf}\\ &\quad h_{456}^{-1}\rhd'\{h_{456},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\rm pf}((g_{56}g_{45})\rhd h_{134}^{-1}h_{456}^{-1})\rhd'(\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\rm pf}(g_{46}\rhd l_{1234})^{-1}\\ &\quad \{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\rm pf}^{-1})\Big)(h_{126}h_{256}g_{56}\rhd h_{125}{}^{-1})\rhd'(h_{156}\rhd'(g_{56}\rhd l_{1345})(g_{56}\rhd l_{1245})^{-1})\Big).\end{aligned} \tag{E.95}$$

Nakon sličnih transformacija kao i u slučaju $1\leftrightarrow 5$ Pahnerovog poteza, komutiranjem elemenata l_{1234} kako bi poredak elemenata bio isti kao u (E.80), a zatim delovanjem na ceo izraz sa h_{126}^{-1}, dobijamo:

$$\begin{aligned}\delta_L(l_{12356}) &= \delta_L\Big(l_{2456}{}^{-1}l_{2346}{}^{-1}l_{2356}(h_{256}g_{56}\rhd h_{125}{}^{-1})\rhd' g_{56}\rhd l_{1235}(h_{256}g_{56}\rhd h_{125}{}^{-1}g_{56}\rhd h_{135})\rhd'\\ &\quad \Big((g_{35}\rhd h_{123}h_{356}^{-1})\rhd' l_{3456})\{g_{56}\rhd h_{345},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\rm pf}^{-1}(g_{56}\rhd h_{345}(g_{56}g_{45})\rhd(h_{123}h_{234}^{-1})h_{456}^{-1})\rhd'\\ &\quad \{h_{456},(g_{56}g_{45})\rhd h_{234}\}_{\rm pf}\Big)(h_{256}g_{56}\rhd h_{125}{}^{-1})\rhd' g_{56}\rhd l_{1345}\\ &\quad (h_{256}g_{56}\rhd h_{125}{}^{-1}g_{56}\rhd h_{145})\rhd'((g_{56}g_{45})\rhd l_{1234})^{-1}(h_{256}g_{56}\rhd h_{125}{}^{-1})\rhd' g_{56}\rhd l_{1245}^{-1}\Big).\end{aligned} \tag{E.96}$$

Ovaj izraz identičan je izrazu u jednačini (E.80). Preostale integracije – po elementu h_{156} grupe H i tri elementa grupe L, l_{1246}, l_{1256} i l_{1356}, su trivijalne, odnosno desna strana se konačno svodi na:

$$d.s. = \delta_G(e)^3\,\delta_H(e)^3\,\delta_L(l_{12356}) = |G|^3\,|H|^3\,\delta_L(l_{12356})\,. \tag{E.97}$$

Konstante ispred integrala su $|G|^{-8}|H|^{-1}|L|^{-1}$ sa leve strane poteza, odnosno $|G|^{-11}|H|^{-3}|L|^{-1}$ sa desne strane poteza, što kompenzuje razliku i izrazima (E.81) i (E.97), na osnovu čega zaključujemo da je suma po stanjima (9.22) invarijantna na $2\leftrightarrow 4$ Pahnerov potez.

Pahnerov potez $3\leftrightarrow 3$

Razmotrimo najpre desnu stranu poteza. Prvo integralimo l_{1235} koristeći $\delta_L(l_{12345})$

$$l_{1235} = (h_{125}\rhd' l_{2345})l_{1245}h_{145}\rhd'(g_{45}\rhd l_{1234})l_{1345}{}^{-1}\,h_{135}\rhd'\{h_{345},(g_{45}g_{34})\rhd h_{123}\}_{\rm pf} \tag{E.98}$$

i l_{1236} koristeći $\delta_L(l_{12356})$,

$$l_{1236} = (h_{126}\rhd' l_{2356})l_{1256}h_{156}\rhd'(g_{56}\rhd l_{1235})l_{1356}{}^{-1}h'_{136}\rhd\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\rm pf}\,. \tag{E.99}$$

Zatim, integralimo h_{123} koristeći $\delta_H(l_{1234})$ i dobijamo:

$$h_{123} = g_{34}^{-1} \triangleright h_{134}^{-1}\, g_{34}^{-1} \triangleright \delta(l_{1234})^{-1}\, g_{34}^{-1} \triangleright h_{124}\, g_{34}^{-1} \triangleright h_{234}. \tag{E.100}$$

Preostala δ-funkcija na grupi G, δ-funkcija $\delta_G(g_{123})$, nakon primene jednačine (E.100) postaje,

$$\delta_G(g_{123}) = \delta_G\big(g_{34}^{-1} \triangleright \partial(h_{134})^{-1}\, g_{34}^{-1} \triangleright \partial(\delta(l_{1234}))^{-1}\, g_{34}^{-1} \triangleright \partial(h_{124})\, g_{34}^{-1} \triangleright \partial(h_{234})\, g_{23}\, g_{12}\, g_{13}^{-1}\big)\,, \tag{E.101}$$

odnosno nakon primene identiteta $\partial\delta = 0$:

$$\delta_G(g_{123}) = \delta_G\big(\partial(h_{134})^{-1}\, \partial(h_{124})\, \partial(h_{234})\, g_{34}^{-1} g_{23}\, g_{12}\, g_{13}^{-1} g_{34}\big)\,. \tag{E.102}$$

Zatim, primenom identiteta (9.18) na trouglove (134), (124) i (234), dobijamo:

$$\delta_G(g_{123}) = \delta_G\big(e\big)\,. \tag{E.103}$$

Za δ-funkciju $\delta_H(h_{1235})$, nakon primene jednačine (E.98) dobijamo:

$$\begin{aligned}\delta_H(h_{1235}) \;=\;& \delta_H\Big((h_{125}\delta(l_{2345})h_{125}^{-1})\delta(l_{1245})(h_{145}(g_{45}\triangleright\delta(l_{1234}))h_{145}^{-1})\delta(l_{1345})^{-1}h_{135}\triangleright'\{h_{345},g_{35}\triangleright h_{123}\}_{\mathrm{pf}}h_{135}\\ & ((g_{35}g_{34}{}^{-1})\triangleright(h_{134}^{-1}\delta(l_{1234})^{-1}h_{124}h_{234}))h_{235}^{-1}h_{125}^{-1}\Big).\end{aligned} \tag{E.104}$$

Zatim, koristeći uslove (9.20) za δ-funkcije $\delta_L(h_{2345})$, $\delta_L(h_{1245})$ i$\delta_L(h_{1345})$, prisutne sa obe strane poteza,

$$\begin{aligned}\delta(l_{2345}) &= h_{235}\, h_{345}\, (g_{45} \triangleright h_{234}^{-1})\, h_{245}^{-1}\,,\\ \delta(l_{1245}) &= h_{125}\, h_{245}\, (g_{45} \triangleright h_{124}^{-1})\, h_{145}^{-1}\,,\\ \delta(l_{1345})^{-1} &= h_{145}\, (g_{45} \triangleright h_{134})\, h_{345}^{-1}\, h_{135}^{-1}\,,\end{aligned} \tag{E.105}$$

dobijamo:

$$\begin{aligned}\delta_H(h_{1235}) \;=\;& \delta_H\Big(h_{125}h_{235}h_{345}(g_{45}\triangleright h_{234}^{-1})h_{245}^{-1}h_{125}^{-1}h_{125}h_{245}(g_{45}\triangleright h_{124}^{-1})h_{145}^{-1}h_{145}(g_{45}\triangleright\delta(l_{1234}))h_{145}^{-1}\\ & h_{145}(g_{45}\triangleright h_{134})h_{345}^{-1}h_{135}^{-1}h_{135}\triangleright\delta(\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{pf}})\\ & h_{135}((g_{35}g_{34}{}^{-1})\triangleright(h_{134}^{-1}\delta(l_{1234})^{-1}h_{124}h_{234}))h_{235}^{-1}h_{125}^{-1}\Big)\\ =\;& \delta_H\Big(h_{345}(g_{45}g_{34})\triangleright h_{123}^{-1}h_{345}^{-1}\delta(\{h_{345},(g_{45}g_{34})\triangleright h_{123}\}_{\mathrm{pf}})(g_{35}\triangleright h_{123})\Big).\end{aligned} \tag{E.106}$$

Primenom identiteta (2.6) za $h_1 = h_{345}$ i $h_2 = (g_{45}g_{34}) \triangleright h_{123}$ i jednakosti $g_{35} = \partial(h_{345})g_{45}g_{34}$, dobijamo:

$$\delta_H(h_{1235}) = \delta_H(e). \tag{E.107}$$

Sličnim postupkom dobija se $\delta_H(h_{1236}) = \delta_H(e)$. Preostala δ-funkcija $\delta_H(l_{12346})$ je:

$$\delta_L(l_{12346}) = \;\delta_L\big(l_{1236}{}^{-1}(h_{126}\triangleright' l_{2346})l_{1246}h_{146}\triangleright'(g_{46}\triangleright l_{1234})l_{1346}{}^{-1}h_{136}\triangleright'\{h_{346},(g_{46}g_{34})\triangleright h_{123}\}_{\mathrm{pf}}\big). \tag{E.108}$$

Zamenom izraza (E.99), a zatim izraza (E.98), dobijamo:

$$\begin{aligned}\delta_L(l_{12346}) &= \delta_L\big(h_{136}\rhd'\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\text{pf}}^{-1}l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1235})^{-1}l_{1256}^{-1}h_{126}\rhd' l_{2356}^{-1} \\ &\quad (h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234})l_{1346}{}^{-1}h_{136}\rhd'\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\text{pf}}\big) \\ &= \delta_L\big(h_{136}\rhd'\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\text{pf}}^{-1}l_{1356} \\ &\quad h_{156}\rhd'(g_{56}\rhd((h_{125}\rhd' l_{2345})l_{1245}h_{145}\rhd'(g_{45}\rhd l_{1234})l_{1345}{}^{-1}h_{135}\rhd'\{h_{345},(g_{45}g_{34})\rhd h_{123}\}_{\text{pf}}))^{-1} \\ &\quad l_{1256}^{-1}h_{126}\rhd' l_{2356}^{-1}(h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234})l_{1346}{}^{-1}h_{136}\rhd'\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\text{pf}}\big).\end{aligned} \tag{E.109}$$

Komutiranjem elemenata, tj. korišćenjem Pajferovog identiteta za ukršteni modul $(L \xrightarrow{\delta} H, \rhd')$, dobijamo

$$\begin{aligned}\delta_L(l_{12346}) &= \delta_L\big(h_{136}\rhd'\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\text{pf}}^{-1} \\ &\quad (\delta(l_{1356})h_{156}g_{56}\rhd h_{135})\rhd' g_{56}\rhd\{h_{345},(g_{45}g_{34})\rhd h_{123}\}_{\text{pf}}^{-1}l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1345}) \\ &\quad (h_{156}g_{56}\rhd h_{145})\rhd'((g_{56}g_{45})\rhd l_{1234})^{-1}h_{156}\rhd'(g_{56}\rhd l_{1245})^{-1}(h_{156}g_{56}\rhd h_{125})\rhd'(g_{56}\rhd l_{2345}^{-1})l_{1256}^{-1} \\ &\quad h_{126}\rhd' l_{2356}^{-1}(h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234})l_{1346}{}^{-1}h_{136}\rhd'\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\text{pf}}\big) \\ &= \delta_L\big((\delta(l_{1346})^{-1}h_{136})\rhd'\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\text{pf}}(\delta(l_{1346})^{-1}h_{136})\rhd'\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\text{pf}}^{-1} \\ &\quad ((\delta(l_{1346})^{-1}\delta(l_{1356})h_{156}g_{56}\rhd h_{135})\rhd' g_{56}\rhd\{h_{345},(g_{45}g_{34})\rhd h_{123}\}_{\text{pf}}^{-1} \\ &\quad (\delta(l_{1346})^{-1}\delta(l_{1356})h_{156}\rhd'(g_{56}\rhd\delta(l_{1345}))h_{156}g_{56}\rhd h_{145})\rhd'((g_{56}g_{45})\rhd l_{1234})^{-1}l_{1346}^{-1}l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1345}) \\ &\quad h_{156}\rhd'(g_{56}\rhd l_{1245})^{-1}(h_{156}g_{56}\rhd h_{125})\rhd'(g_{56}\rhd l_{2345}^{-1})l_{1256}^{-1}h_{126}\rhd' l_{2356}^{-1}(h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234})\big).\end{aligned} \tag{E.110}$$

Primenom identiteta (2.65), tj. da je

$$\{h_{346},(g_{46}g_{34})\rhd h_{123}\}_{\text{pf}} = h_{346}\rhd'\{h_{346}^{-1}, g_{36}\rhd h_{123}\}_{\text{pf}}^{-1}, \tag{E.111}$$

a zatim varijante identiteta (2.62), tj. da je

$$\{h_1h_2h_3,h_4\}_{\text{pf}}^{-1} = \{h_1,\partial(h_2h_3)\rhd h_4\}_{\text{pf}}^{-1}h_1\rhd'\{h_2,\partial(h_2)\rhd h_4\}_{\text{pf}}^{-1}(h_1h_2)\rhd'\{h_3,h_4\}_{\text{pf}}^{-1}, \tag{E.112}$$

dobijamo:

$$\begin{aligned}\{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\text{pf}}^{-1} &= \{h_{346}^{-1},(g_{46}g_{34})\rhd h_{123}\}_{\text{pf}}^{-1}h_{346}^{-1}\rhd'\{h_{356},(g_{56}g_{35})\rhd h_{123}\}_{\text{pf}}^{-1} \\ &\quad (h_{346}^{-1}h_{356})\rhd'\{g_{56}\rhd h_{345},(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\text{pf}}^{-1}.\end{aligned} \tag{E.113}$$

Primenom ove jednakosti u izrazu (E.110) dobijamo:

$$\begin{aligned}\delta_L(l_{12346}) &= \delta_L\big((h_{146}g_{46}\rhd h_{134})\rhd'\{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\text{pf}}^{-1} \\ &\quad (\delta(l_{1346})^{-1}\delta(l_{1356})h_{156}\rhd'(g_{56}\rhd\delta(l_{1345}))h_{156}g_{56}\rhd h_{145})\rhd'((g_{56}g_{45})\rhd l_{1234})^{-1}l_{1346}^{-1}l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1345}) \\ &\quad h_{156}\rhd'(g_{56}\rhd l_{1245})^{-1}(h_{156}g_{56}\rhd h_{125})\rhd'(g_{56}\rhd l_{2345}^{-1})l_{1256}^{-1}h_{126}\rhd' l_{2356}^{-1}(h_{126}\rhd' l_{2346})l_{1246}h_{146}\rhd'(g_{46}\rhd l_{1234})\big).\end{aligned} \tag{E.114}$$

Primenom (E.100) i identiteta (2.63), dobijamo da važi jednakost,

$$\begin{aligned}\{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\rm pf}^{-1} &= \{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45})\rhd((h_{134}^{-1}\rhd'\delta(l_{1234})^{-1})h_{134}^{-1}h_{124}h_{234}\}_{\rm pf}^{-1}\\ &= (g_{46}\rhd(h_{134}^{-1}\rhd'\delta(l_{1234})^{-1}))\rhd'\\ &\quad \{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45})\rhd(h_{134}^{-1}h_{124}h_{234})\}_{\rm pf}^{-1}\\ &\quad \{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45})\rhd(h_{134}^{-1}\rhd'\delta(l_{1234})^{-1})\}_{\rm pf}^{-1}.\end{aligned} \tag{E.115}$$

Odnosno, kada raspišemo član

$$\begin{aligned}\{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45})\rhd(h_{134}^{-1}\rhd'\delta(l_{1234})^{-1})\}_{\rm pf}^{-1} &= (g_{46}\rhd(h_{134}^{-1}\rhd' l_{1234}^{-1}))\\ &\quad (h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}))\rhd'((g_{56}g_{45})\rhd(h_{134}^{-1}\rhd' l_{1234})).\end{aligned} \tag{E.116}$$

izraz (E.115) je jednak:

$$\begin{aligned}\{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45}g_{34})\rhd h_{123}\}_{\rm pf}^{-1} &= g_{46}\rhd(h_{134}^{-1}\rhd'\delta(l_{1234})^{-1})\\ &\quad \{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45})\rhd(h_{134}^{-1}h_{124}h_{234})\}_{\rm pf}^{-1}\\ &\quad (h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}))\rhd'((g_{56}g_{45})\rhd(h_{134}^{-1}\rhd' l_{1234})).\end{aligned} \tag{E.117}$$

Zamenom ovog rezultata u jednačinu (E.114) dobijamo da se članovi sa l_{1234} pokrate, pa sledi da je $\delta_L(l_{12346})$:

$$\begin{aligned}\delta_L(l_{12346}) &= \delta_L\big((h_{146}g_{46}\rhd h_{134})\rhd'\{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45})\rhd(h_{134}^{-1}h_{124}h_{234})\}_{\rm pf}^{-1}l_{1346}^{-1}l_{1356}\\ &\quad h_{156}\rhd'(g_{56}\rhd l_{1345})h_{156}\rhd'(g_{56}\rhd l_{1245})^{-1}(h_{156}g_{56}\rhd h_{125})\rhd'(g_{56}\rhd l_{2345}^{-1})l_{1256}^{-1}h_{126}\rhd' l_{2356}^{-1}(h_{126}\rhd' l_{2346})l_{1246}\big).\end{aligned} \tag{E.118}$$

Iz ovog izraza primećujemo da je integracija po l_{1234} trivijana, pa za izraz sa desne strane Pahnerovog poteza konačno dobijamo:

$$\begin{aligned}d.s. &= \delta_G(e)\delta_H(e)^2\delta_L\big(h_{156}\rhd'(g_{56}\rhd l_{1245})^{-1}h_{156}\rhd'(g_{56}\rhd(h_{125}\rhd' l_{2345}))^{-1}l_{1256}^{-1}h_{126}\rhd' l_{2356}^{-1}(h_{126}\rhd' l_{2346})\\ &\quad l_{1246}(h_{146}g_{46}\rhd h_{134})\rhd'\{h_{346}^{-1}h_{356}(g_{56}\rhd h_{345}),(g_{56}g_{45})\rhd(h_{134}^{-1}h_{124}h_{234})\}_{\rm pf}^{-1}l_{1346}^{-1}l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1345}).\end{aligned} \tag{E.119}$$

Analizirajmo sada levu stranu poteza, tj. integral:

$$\int_H dh_{456}\int_{L^3} dl_{1456}dl_{2456}dl_{3456}\delta_G(g_{456})\,\delta_H(h_{3456})\delta_H(h_{2456})\delta_H(h_{1456})\delta_L(l_{23456})\delta_L(l_{13456})\delta_L(l_{12456})\,. \tag{E.120}$$

Najpre, integralimo l_{1456} koristeći δ-funkciju $\delta_L(l_{13456})$ i dobijamo:

$$l_{1456} = h_{146}\rhd\{h_{456},(g_{56}g_{45})\rhd h_{134}\}{l_{1346}}^{-1}(h_{136}\rhd' l_{3456})l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1345})\,. \tag{E.121}$$

Zatim, integralimo l_{2456}, koristeći $\delta_L(l_{23456})$,

$$l_{2456} = h_{246}\rhd\{h_{456},(g_{56}g_{45})\rhd h_{234}\}{l_{2346}}^{-1}(h_{236}\rhd' l_{3456})l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})\,, \tag{E.122}$$

i h_{456} koristeći $\delta_H(h_{3456})$:

$$h_{456} = h_{346}^{-1}\,\delta(l_{3456})h_{356}\,(g_{56}\rhd h_{345})\,. \tag{E.123}$$

Koristeći jednačinu (E.123), dobijamo da se δ-funkcija $\delta_G(g_{456})$,

$$\delta_G(g_{456}) = \delta_G\big(\partial(h_{346})^{-1}\,\partial(h_{356})\,g_{56} \rhd \partial(h_{345})\,g_{56}\,g_{45}\,g_{46}^{-1}\big)\,, \tag{E.124}$$

primenom identiteta (9.18) za trouglove (346), (356) i (345), svodi na:

$$\delta_G(g_{456}) = \delta_G\big(e\big)\,. \tag{E.125}$$

Sličnim postupkom kao i sa desne strane poteza, dobijamo da su δ-funkcije $\delta_H(h_{1456})$ i $\delta_H(h_{2456})$, primenom jednačina (E.121) i (E.122), jednake $\delta_H(e)^2$. Preostala $\delta_L(l_{12456})$ je jednaka

$$\delta_L(l_{12456}) = \delta_L\big(l_{1246}{}^{-1}(h_{126} \rhd' l_{2456})l_{1256}h_{156} \rhd' (g_{56} \rhd l_{1245})l_{1456}{}^{-1}\, h_{146} \rhd \{h_{456}, (g_{56}g_{45}) \rhd h_{124}\}_{\mathrm{pf}}\big). \tag{E.126}$$

Zamenom jednačina (E.121) i (E.122) u izraz za δ-funkciju $\delta_L(l_{12456})$ dobijamo,

$$\begin{aligned}\delta_L(l_{12456}) \;=\;& \delta_L\big(l_{1246}{}^{-1}(h_{126}\rhd'(h_{246}\rhd\{h_{456},(g_{56}g_{45})\rhd h_{234}\}_{\mathrm{pf}}l_{2346}{}^{-1}(h_{236}\rhd' l_{3456})l_{2356}h_{256}\rhd'(g_{56}\rhd l_{2345})))\\ & l_{1256}h_{156}\rhd'(g_{56}\rhd l_{1245})h_{156}\rhd'(g_{56}\rhd l_{1345})^{-1}l_{1356}^{-1}(h_{136}\rhd' l_{3456})^{-1}l_{1346}h_{146}\rhd\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{pf}}^{-1}\\ & h_{146}\rhd\{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\mathrm{pf}}\big),\end{aligned} \tag{E.127}$$

odnosno posle komutacije elemenata:

$$\begin{aligned}\delta_L(l_{12456}) \;=\;& \delta_L\big((\delta(l_{1246})^{-1}h_{126}h_{246})\rhd\{h_{456},(g_{56}g_{45})\rhd h_{234}\}_{\mathrm{pf}}(\delta(l_{1246})^{-1}h_{126}\rhd\delta(l_{2346})^{-1}h_{126}h_{236})\rhd' l_{3456}\\ & l_{1246}^{-1}h_{126}\rhd' l_{2346}{}^{-1}h_{126}\rhd' l_{2356}(h_{126}h_{256})\rhd'(g_{56}\rhd l_{2345}))\\ & l_{1256}h_{156}\rhd'(g_{56}\rhd l_{1245})h_{156}\rhd'(g_{56}\rhd l_{1345})^{-1}l_{1356}^{-1}l_{1346}(\delta(l_{1346})^{-1}h_{136})\rhd' l_{3456}{}^{-1}\\ & h_{146}\rhd\{h_{456},(g_{56}g_{45})\rhd h_{134}\}_{\mathrm{pf}}^{-1}h_{146}\rhd\{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\mathrm{pf}}\big).\end{aligned} \tag{E.128}$$

Primenom identiteta (2.68) za inverz Pajferovog podizanja $\{h_{456}, (g_{56}g_{45}) \rhd h_{134}\}_{\mathrm{pf}}^{-1}$, a zatim identiteta (2.63) za tri elementa, tj.

$$\{h_1, h_2h_3h_4\}_{\mathrm{pf}} = \{h_1, h_2\}_{\mathrm{pf}}(\partial(h_1) \rhd h_2) \rhd' \{h_1, h_3\}_{\mathrm{pf}}(\partial(h_1) \rhd (h_2h_3)) \rhd' \{h_1, h_4\}_{\mathrm{pf}}\,, \tag{E.129}$$

dobijamo:

$$\begin{aligned}\{h_{456},(g_{56}g_{45})\rhd(h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{pf}} \;=\;& \{h_{456},(g_{56}g_{45})\rhd h_{134}^{-1}\}_{\mathrm{pf}}(g_{46}\rhd h_{134}^{-1})\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\mathrm{pf}}\\ & (g_{46}\rhd(h_{134}^{-1}h_{124}))\rhd'\{h_{456},(g_{56}g_{45})\rhd h_{124}\}_{\mathrm{pf}}.\end{aligned} \tag{E.130}$$

Ovaj identitet zatim možemo primeniti u jednačini (E.128), iz čega sledi jednakost:

$$\begin{aligned}\delta_L(l_{12456}) \;=\;& \delta_L\big((\delta(l_{1246})^{-1}h_{126} \rhd \delta(l_{2346})^{-1}h_{126}h_{236}) \rhd' l_{3456}\\ & l_{1246}^{-1}h_{126} \rhd' l_{2346}{}^{-1}h_{126} \rhd' l_{2356}(h_{126}h_{256}) \rhd' (g_{56} \rhd l_{2345})\,)\\ & l_{1256}h_{156} \rhd' (g_{56} \rhd l_{1245})h_{156} \rhd' (g_{56} \rhd l_{1345})^{-1}l_{1356}^{-1}l_{1346}(\delta(l_{1346})^{-1}h_{136}) \rhd' l_{3456}{}^{-1}\\ & (h_{146}g_{46} \rhd h_{134}) \rhd' \{h_{456}, (g_{56}g_{45}) \rhd (h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{pf}}\big)\,.\end{aligned} \tag{E.131}$$

Primenom jednačine (E.123) i identiteta (2.62), sličnim postupkom kao i kod desne strane poteza dobijamo da članovi sa l_{3456} pokrate, tj. δ-funkcija $\delta_L(l_{12456})$ glasi:

$$\begin{aligned}\delta_L(l_{12456}) &= \delta_L\big(l_{1246}^{-1} h_{126}\rhd' l_{2346}{}^{-1} h_{126}\rhd' l_{2356}(h_{126}h_{256})\rhd'(g_{56}\rhd l_{2345}))l_{1256}h_{156}\rhd'(g_{56}\rhd l_{1245})\\ &\quad h_{156}\rhd'(g_{56}\rhd l_{1345})^{-1} l_{1356}^{-1} l_{1346}(h_{146}g_{46}\rhd h_{134})\rhd'\{h_{456},(g_{56}g_{45})\rhd(h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{pf}}\big).\end{aligned} \tag{E.132}$$

Zaključujemo da je integral po l_{3456} trivijalan, pa konačno dobijamo da je leva strana poteza jednaka:

$$\begin{aligned} l.s. &= \delta_G(e)\delta_H(e)^2\delta_L\big(h_{126}\rhd' l_{2346}l_{1246}(h_{146}g_{46}\rhd h_{134})\rhd'\{h_{456},(g_{56}g_{45})\rhd(h_{134}^{-1}h_{124}h_{234})\}_{\mathrm{pf}}^{-1} l_{1346}^{-1}\\ &\quad l_{1356}h_{156}\rhd'(g_{56}\rhd l_{1345})h_{156}\rhd'(g_{56}\rhd l_{1245})^{-1}(h_{156}g_{56}\rhd h_{125})\rhd'(g_{56}\rhd l_{2345})^{-1} l_{1256}^{-1} h_{126}\rhd' l_{2356}^{-1}\big).\end{aligned} \tag{E.133}$$

Izrazi (E.119) i (E.126) su identični, na osnovu čega zaključujemo da je suma po stanjima (9.2.1) invarijantna na $3 \leftrightarrow 3$ Pahnerov potez. Broj k-simpleksa sa obe strane $3 \leftrightarrow 3$ poteza je isti za sve k, tj. koeficijenti ispred integrala su jednaki u ovom slučaju i prema tome nisu od značaja.

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Biografija

Tijana Radenković rođena je 21.3.1992. u Beogradu, gde je završila osnovnu školu "Miloš Crnjanski" i Matematičku gimnaziju 2011. godine. Upisala je Fizički fakultet Univerziteta u Beogradu 2011. godine, smer teorijska i eksperimentalna fizika, gde je diplomirala 30.9.2016. sa prosečnom ocenom 9,33. Godine 2016. upisuje master akademske studije na Fizičkom fakultetu Univerziteta u Beogradu, koje završava 27.9.2017. odbranivši master tezu naslovljenu

"Kvantna gravitacija na deo po deo ravnim mnogostrukostima",

sa prosečnom ocenom 9,33. Master teza je fokusirana na pregled poznatih rezultata u teoriji spin-kub modela. Spin-kub model je poznat model kvantne gravitacije koji je generalizacija modela spinske pene u kontekstu 2-kategorija. Međutim, u literaturi je nedostajao pregledan rad koji bi omogućio lakše upoznavanje sa ovim modelom i osnovama neophodne matematike na kojoj se on zasniva. U Tijaninom master radu urađen je pregled Redže računa i 2-grupa u teoriji kategorija, a zatim je pokazano formiranje sume po stanjima u okviru Ponzano-Redže topološkog modela $3D$ gravitavije i Ouguri modela, topološkog sektora $4D$ gravitacije, zasnovanih na BF dejstvu. Pokazano je da je opšta relativnost ekvivalentna $2BF$ dejstvu sa vezom i da je možemo posmatrati kao gradijentnu teoriju za Poenkareovu 2-grupu. Procedura kvantizacije za topološki deo teorije, tj. $BFCG$ dejstvo sprovedena je na sličan način kao u slučaju BF dejstva. Pokazano je da je problem prisutan u modelu spinske pene rešen njegovom kategorijskom generalizacijom: fundamentalne promenljive sada boje 3-kompleks dualne rešetke i dužine ivica u triangulaciji postaju osnovni stepeni slobode teorije. Teza je nagrađena od strane fonda "prof. Ljubomir Ćirković" kao najbolja master tezu odbranjena na Fizičkom fakultetu školske 2017/2018 godine.

Godine 2017. započinje doktorske studije na Fizičkom fakultetu Univerziteta u Beogardu, na studijskom programu *Polja, čestice i gravitacija*. Prosečna ocena je 9,5. Istraživanje na doktorskim studijama radi pod mentorstvom dr. Marka Vojinovića. Od aprila 2018. Tijana je zaposlena na Institutu za fiziku u Beogradu kao istraživač pripravnik. Tijana se u svom naučnom radu tokom doktorskih studija bavi problemima kvantne gravitacije i njenog ujedinjenja sa ostalim fundamentalnim poljima. Naime, formalizam 2-grupa, odnosno 3-grupa predstavlja odličnu platformu za ujedinjenje svih interakcija.

Od 2011. do 2019. Tijana je radila honorarno pri programima Fizika i TEH u Istraživačkoj stanici Petnica, gde je pomagala u organizaciji seminara, laboratorijskih vežbi tokom seminara i držala predavanja. Pored dugogodišnjeg učestvovanju u radu istraživačke stanice, Tijana je iskustvo u predavanju stekla i školske 2016/2017 kada je bila zaposlena kao nastavnica fizike u X Beogradskoj gimnaziji, i školske 2019/2020 kada je predavala akustiku u muzičkim školama "Stanković", "Vatroslav Lisinski" i "Josip Slavenski".

Izjava o istovetnosti štampane i elektronske verzije doktorskog rada

Ime i prezime autora: *Tijana Radenković*
Broj indeksa: **8009/2017**
Studijski program: *Kvantna polja, čestice i gravitacija*
Naslov rada: *Više gradijentne teorije i kvantna gravitacija*
Mentor: *dr Marko Vojinović*

Izjavljujem da je štampana verzija mog doktorskog rada istovetna elektronskoj verziji koju sam predala radi pohranjivanja u **Digitalnom repozitorijumu Univerziteta u Beogradu**.

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Potpis autora

U Beogradu, 28.6.2023.

Тијана Раденковић

Izjava o autorstvu

Ime i prezime autora: *Tijana Radenković*
Broj indeksa: **8009/2017**

Izjavljujem

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Više gradijentne teorije i kvantna gravitacija

- rezultat sopstvenog istraživačkog rada;
- da disertacija u celini ni u delovima nije bila predložena za sticanje druge diplome prema studijskim programima drugih visokoškolskih institucija;
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Potpis autora

U Beogradu, 28.6.2023.

Тијана Раденковић

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