Научном већу Института за физику у Београду

Предмет: Покретање поступка за избор у звање научни сарадник

Молим Научно веће Института за физику у Београду да покрене поступак за мој избор у звање научни сарадник.

У прилогу достављам:

1. мишљење руководиоца лабораторије са предлогом комисије за избор у звање;
2. биографске и стручне податке;
3. преглед научне активности;
4. елементе за квалитативну оцену научног доприноса;
5. елементе за квантитативну оцену научног доприноса;
6. списак објављених радова;
7. податке о цитираности;
8. копије објављених радова;
9. копију докторске дисертације;
10. доказ о признању дипломе докторских студија;
11. докази о преосталим елементима оцена научног доприноса.

С поштовањем,

Петар Тадић

Петар Тадић

Научном већу Института за физику у Београду

Предмет: Мишљење руководиоца лабораторије о избору др Петра Тадића у звање научни сарадник

Др Петар Тадић је заинтересован за сарадњу са Лабораторијом за примену рачунара у науци, у оквиру Националног центра изузетних вредности за изучавање комплексних система Института за физику у Београду. У истраживачком раду бави се темама везаним за примену метода квантне теорије поља у физици кондензоване материје. С обзиром да испуњава све предвиђене услове у складу са Правилником о стицању истраживачких и научних звања Министарства НИТРА, сагласан сам са покретањем поступка за избор др Петра Тадића у звање научни сарадник.

За састав комисије за избор др Петра Тадића у звање научни сарадник предлажем:

(1) др Милица Миловановић, научни саветник, Институт за физику у Београду
(2) др Михаило Чубровић, научни сарадник, Институт за физику у Београду
(3) др Марија Димитријевић Ћирић, редовни професор Физичког факултета Универзитета у Београду

др Антун Балаж
научни саветник
Руководилац Лабораторије за примену рачунара у науци

2. БИОГРАФСКИ И СТРУЧНИ ПОДАЦИ О КАНДИДАТУ

Петар Тадић је рођен 29.7.1994. у Никшићу, у Црној Гори, где је завршио Основну школу „Милева Лајовић-Лалатовић“ и Гимназију „Стојан Церовић“. Основне академске студије на Физичком факултету Универзитета у Београду, смер Теоријска и експериментална физика, завршио је 2017. године са просечном оценом 10,00. Мастер академске студије је завршио 2018. године на Универзитету у Оксфорду. Мастер рад на тему „*Quasinormal modes of black branes*“ („Квазинормалне моде црних брана“) урадио је под менторством др Андреја Старинетса. По завршетку мастер академских студија, уписао је докторске студије на Математичком факултету Тринити колеџа у Даблину, у Ирској. Докторску дисертацију на тему „*Conformal bootstrap and thermalization in holographic CFTs*“ („Конформални бутстрап и термализација у холографским конформалним теоријама поља“) урадио је под менторством др Мануеле Кулаксизи, а одбранио је августу 2021. године. Диплома докторских студија призната је од стране Агенције за квалификације Републике Србије 12.1.2023. године, решење број 612-03-27/2023-03.

Током основних студија, током лета 2015., 2016. и 2017. године радио је стручну праксу на Инсититуту за физику у Београду у Лабораторији за примену компјутера у науци и Групи за гравитацију, честице и поља. Током докторских студија, током 2019. и 2020. године радио је као асистент-демонстратор на курсевима Статистичка физика 1 и 2, на основним академским студијама Тринити колеџа у Даблину. По завршетку докторских студија, од септембра 2021. године запослен је као научни сарадник на постдокторском усавршавању на Јејл Универзитету у Сједињеним Америчким Државама под руководством др Дејвида Поланда. На овој позицији изучава примене конформалног бутстрапа у решавању конформалних теорија поља.

Током средње школе учествовао је на три Међународне физичке олимпијаде за ученике средњих школа (IPhO) 2011., 2012. и 2013. године где је два пута освојио бронзану медаљу. Године 2012. добио је награду Министарства науке Црне Горе за најбољег младог научника. Током основних студија био је добитник стипендије „Проф. др Ђорђе Живановић“ за изузетне резултате, додељене од стране Физичког факултета и Института за физику у Београду, као и стипендије „Србија за Србе из региона“, додељене од стране Министарства просвете, науке и технолошког развоја Републике Србије. Током мастер студија био је добитник стипендије Дулвертон фондације намењене најбољим студентима из Источне и Централне Европе за студије на Универзитету у Оксфорду. Током докторских студија био је стипендиста Ушер фондације.

Петар је до сада објавио осам научних радова. Радови су до сада цитирани 163 пута, не рачунајући аутоцитате.

3. ПРЕГЛЕД НАУЧНЕ АКТИВНОСТИ

У досадашњем научном раду кандидат се примарно бавио истраживачким темама из области конформалних теорија поља, квантне гравитације и релативистичке хидродинамике. Методолошки приступ кандидата су теоријска изучавања и нумерички прорачуни.

Током докторских студија кандидат се бавио изучавањем холографских конформалних теорија поља које за финални циљ има разумевање квантне гравитације. На основу холографске хипотезе (која се још зове и „АдС-ЦФТ дуализам“) овакве теорије су еквивалентне гравитационим теоријама у Анти-де Ситер простору, без да експлицитно поседују гравитационе степене слободе. Дате теорије су дефинисане као конформалне теорије поља са централним наелектрисањем много већим од један где сви примарни оператори спина већег од два имају конформалне димензије много веће од један. Карактеристика холографског дуализма је да када је гравитациона теорија слабо куплована, њена дуална холографска конформална теорија поља је јако куплована и обратно. Да би било могуће проучавати ефекте квантне гравитације кроз дуалне конформалне теорије поља неопходно је прво разумети детаље овог дуализма на семикласичном нивоу, тј. у ситуацији када је гравитација на довољно ниским енергијама, самим тим и слабо куплована и могуће ју је описати Ајнштајновом општом теоријом релативности. Тада је дуална конформална теорија поља јако куплована и није могуће применити уобичајену пертурбативану анализу. Идући корак у опису ефеката квантне гравитације је увођење квантних корекција у гравитационе теорије и дуалне конформалне теорија поља, или други речима, корекција које су пропорционалне инверзном централном наелектрисању.

Кандидат се бавио аналитичким решавањем јако куплованих холографских конформалних теорија поља користећи непертурбативне услове конзистентности и анализом датих решења из перспективе дуалних гравитационих теорија. Резултати истраживања објављени су у четири студије:

- Robin Karlsson, Manuela Kulaxizi, Andrei Parnachev, and **Petar Tadić**, Black holes and conformal Regge bootstrap. J. High Energ. Phys. 2019, 46 (2019). (врхунски међународни часопис - категорија M21)
- Robin Karlsson, Manuela Kulaxizi, Andrei Parnachev, and **Petar Tadić**, Leading multi-stress tensors and conformal bootstrap. J. High Energ. Phys. 2020, 76 (2020). (врхунски међународни часопис - категорија M21)
- Robin Karlsson, Manuela Kulaxizi, Andrei Parnachev, and **Petar Tadić**, Stress tensor sector of conformal correlators. J. High Energ. Phys. 2020, 19 (2020). (врхунски међународни часопис - категорија M21)
- Robin Karlsson, Andrei Parnachev, and **Petar Tadić**, Thermalization in large-N CFTs. J. High Energ. Phys. 2021, 205 (2021). (врхунски међународни часопис - категорија M21)

У првој студији изучаван је Реге лимит корелатора два пара идентичних скаларних оператора у холографским конформалним теоријама поља, од којих један пар има конформалну димензију много већу од један, реда централног наелектрисања дате теорије. Овај лимит се у гравитационој теорији може интерпретирати као расејање високо-енергетске честице на црној рупи са фиксним параметром судара. Показано је да једнакост између фазног помака расејане честице у гравитационој теорији и аномалне димензије оператора дуплог твиста у холографској конформалној теорији поља важи у вишим редовима развоја аномалне димензије по степенима инверзног централног наелектрисања.

У другој студији изучаван је исти корелатор у холографским теоријама поља у простор-времену које има више од две димензије, међутим сада у лимиту када се један од четири скаларна оператора приближава светлосном конусу другог скаларног оператора. У овом лимиту, доминантну контрибуцију корелатору дају размењени примарни оператори минималног твиста, који је дефинисан као разлика конформалне димензије и спина примарног оператора. Битну класу оператора минималног твиста чине очуване струје, на првом месту, стрес-тензор. Производи више стрес-тензора такође дају примарне операторе минималног твиста за дату конформалну димензију, под условом да се индекси исправно симетризују и да се не контрахују. У овом раду су изучаване контрибуције свих оператора минималног твиста добијених од стрес-тензора и нађен је егзактан аналитички израз који сумира бесконачан број ових оператора у сваком реду у развоју по степенима инверзног централног наелектрисања. Показано је да су коефицијенти развоја операторског производа између мулти-стрес-тензора минималног твиста и два идентична скаларна оператора одређени условима конзистенције холографске конформалне теорије поља и самим тим не зависе од избора конкретне теорије, тј. експлицитно је потврђена њихова универзалност. У трећој студији разматрање је проширено на све операторе који се могу написати као производи стрес-тензора. Нађен је аналитички облик збира свих оператора фиксног твиста и показано је да једини коефицијенти развоја операторског производа који нису фиксирани условима конзистенције они за мулти-стрес-тензоре спина нула и два. Закључено је да ови коефицијенти параметризују холографске конформалне теорије поља.

У четвртој студији разматрана је термализација мулти-стрес-тензора произвољног спина у конформалним теоријама поља са централним наелектрисањем много већим од један. Показано је да је термализација ових оператора еквивалентна универзалности коефицијената развоја операторских производа датих оператора са два идентична скаларна оператора конформалне димензије реда централног наелектрисања. Експлицитно је показано да је ова универзалност задовољена упоређујући коефицијенте израчунате у слободној и холографској теорији. Такође је показано да мулти-стрес-тензори задовољавају дијагонални део хипотезе термализације својствених вредности.

У првој и другој студији кандидат је значајно допринео иницијалној теориској анализи проблема, произвео све резултате студије, значајно учествовао у анализи резултата и писању рада за објављивање. У трећој студији кандидат је такође произвео све резултате студије и водио анализу резултата и писање објављеног рада. У четвртој студији кандидат је водио избор метода студије, аналитичке прорачуне и анализу резултата, произвео све резултате студије и значајно је учествовао у писању објављеног рада.

У оквиру исте научне области кандидат се бавио и генерализацијом метода дводимензионалних конформалних теорија поља на стрес-тензоре минималног твиста у вишедимензионалним конформалним теоријама, будући да су њихови коефицијенти операторског развоја универзални на исти начин као и коефицијенти оператора у Вирасоровом блоку јединичног оператора у дводимензионалној конформалној теорији поља. Испоставља се да је Вирасоров блок јединичног оператора одређен Каталановим бројевима у одређеном лимиту. Користећи рекурзивну релацију која дефинише Каталанове бројеве могуће је написати диференцијалну једначину чије је решење Вирасоров блок јединичког оператора. У вишедимензионалном случају, показано је да се уместо Каталанових бројева јавља њихова генерализација у виду броја линеарних екстензија одређеног типа парцијално уређених сетова. Тренутно није позната рекурзивна релација коју ови бројеви задовољавају па није могуће написати диференцијалну једначину коју би задовољавала контрибуција свих мулти-стрес-тензора минималног твиста у више од две димензије. Такође су изучаване и генерализације Вирасорове алгебре у две димензије, којима се кандидат посебно бавио у оквиру ове студије. Кандидат је изучавао конформални блок јединичног оператора W_3 алгебре и његове везе са контрибуцијама мулти-стрес-тензора минималног твиста у холографској конформалној теорији поља у четвородимензионалном простор-времену:

- Robin Karlsson, Manuela Kulaxizi, Gim Seng Ng, Andrei Parnachev, and **Petar Tadić**, CFT correlators, W-algebras and generalized Catalan numbers. J. High Energ. Phys. 2022, 162 (2022). (врхунски међународни часопис - категорија M21)

Кандидат се бавио и релативистичком хидродинамиком јако куплованих система користећи методе АдС-ЦФТ дуализма. У овом приступу идеја је да се користи семикласични гравитациони дуал јако кbusпловане теорије поља како би се изучавала динамика јако купловане теорије без гравитационих степени слободе. У оквиру ове области кандидат је објавио три студије:

- Sašo Grozdanov, Pavel K. Kovtun, Andrei O. Starinets, and **Petar Tadić**, Convergence of the Gradient Expansion in Hydrodynamics. Phys. Rev. Lett. 122, 251601 (2019). (међународни часопис изузетних вредности - категорија M21a)
- Sašo Grozdanov, Pavel K. Kovtun, Andrei O. Starinets, and **Petar Tadić**, The complex life of hydrodynamic modes. J. High Energ. Phys. 2019, 97 (2019). (врхунски међународни часопис - категорија M21)
- Sašo Grozdanov, Andrei O. Starinets, and **Petar Tadić**, Hydrodynamic dispersion relations at finite coupling. J. High Energ. Phys. 2021, 180 (2021). (врхунски међународни часопис - категорија M21)

У првој студији показано је да је радијус конвергенције хидродинамичке дисперзионе релације одређен позицијом критичних тачака придружене спектралне криве у комплексној равни импулса које се поклапају са позицијом судара хидродинамичких мода са не-хидродинамичким модама. Нумерички су израчунати радијуси конвергенције хидродинамичких мода у Н=4 супер-Јанг-Милс теорији у четири просторно-временске димензије. У другој студији је ова анализа проширена на гравитационе теорије у простор-временима са другим бројем димензија. У тећој студији разматрана је зависност радијуса конвергенције хидродинамичких мода од инверзне константе куповања теорије и показано је ова зависност није монотона функција у случају Н=4 супер-Јанг-Милс теорије.

У првој и другој студији кандидат је урадио нумеричке прорачуне радијуса конвергенције и учествовао у анализи резултата. У трећој студији кандидат је водио избор метода истраживања и иницијалну теоријску анализу, произвео све резултате студије и учествовао у писању текста за објављивање.

4. ЕЛЕМЕНТИ ЗА КВАЛИТАТИВНУ ОЦЕНУ НАУЧНОГ ДОПРИНОСА КАНДИДАТА

4.1. Квалитет научних резултата

4.1.1. Научни ниво и значај резултата, утицај научних радова

Кандидат је у досадашњој каријери објавио 8 научних радова, од чега 1 рад категорије M21a и 7 радова категорије M21. Своја истраживања је представио и на 1 конференцији, тј. остварио 1 допринос категорије M34.

До сада најутицајнији рад кандидата из теме доктората је:

- Robin Karlsson, Manuela Kulaxizi, Andrei Parnachev, and **Petar Tadić**, Stress tensor sector of conformal correlators. J. High Energ. Phys. 2020, 19 (2020).
  DOI: 10.1007/JHEP07(2020)019

Тема овог рада су све контрибуције стрес-тензор сектора корелатору са четири скаларна оператора од којих два имају конформалне димензије реда централног наелектрисања холографске конформалне теорије поља. Нађен је аналитички облик за суме контрибуција мулти-стрес-тензора фиксног твиста и показано је да сви коефицијенти развоја операторског производа могу бити израчунати из услова конзистенције конформалне теорије осим оних за мулти-стрес-тензоре спина нула и два. Такође је показано како коефицијенти развоја операторског производа мулти-стрес-тензора спина два могу бити израчунати у дуалној гравитационој теорији разматрајући фазни помак расејане честице на црној рупи. Кандидат је произвео све резултате објављене у овом раду и водио интерпретацију и дискусију резултата као и писање рада.

Уочена је сличност аналитичког облика збира контрибуција мулти-стрес тензора збиру контрибуција Вирасорових наследника јединичног оператора корелатору у дводимензионалним конформалним теоријама поља. Поставило се питање постојања алгебре Вирасоровог типа у вишедимензионалним конформалним теоријама која се манифестује у лимиту светлосног конуса. Један од кандидата за дату алгебру је W_3 алгебра која је изучавана од стране кандидата у каснијем раду. Питање постојања алгебре Вирасоровиг типа у вишедимензионалним конформалним теоријама поља је такође касније изучавано у литератури од стране више аутора и за сада није пронађен пример алгебре која би задовољила све услове и дала аналитички облик стрес-тензорских контрибуција непертурбативно по инверзном централном наелектрисању.

4.1.2. Цитираност научних радова кандидата

Према подацима о цитираности аутора изведених из базе *Web of Science* 01.2.2023., радови чији је кандидат ко-аутор цитирани су 174 пута, од чега 163 пута без аутоцитата, а Хиршов фактор је 6.

4.1.3. Параметри квалитета радова и часописа

Кандидат је објавио 8 радова у часописима:

- 7 радова у часопису *Journal of High Energy Physics* (ISSN: 1029-8479), категорија M21, IF(2021)=5.62, SNIP(2021)=1.27;
- 1 рад у часопису *Physical Review Letters* (ISSN: 0031-9007), категорија M21a, IF(2019)=8.57, SNIP(2019)=2.50.

Додатни библиометријски показатељи квалитета часописа у којима је кандидат објављивао радове приказани су у табели:

	ИФ	М	СНИП
Укупно	47.91	66	11.39
Усредњено по чланку	5.99	8.25	1.42
Усредњено по аутору	12.63	17.43	3.00

4.1.4. Степен самосталности и степен учешћа у реализацији радова у научним центрима у земљи и иностранству

Кандидат је водећи аутор на четири објављена рада, којима је дао кључан допринос у погледу дефинисања проблема, иницијалне теоријске анализе, нумеричких и аналитичких прорачуна, интерпретације резултата и писања рада. Наведени радови су:

- Robin Karlsson, Manuela Kulaxizi, Andrei Parnachev, and **Petar Tadić**, Stress tensor sector of conformal correlators. J. High Energ. Phys. 2020, 19 (2020).
- Robin Karlsson, Andrei Parnachev, and **Petar Tadić**, Thermalization in large-N CFTs. J. High Energ. Phys. 2021, 205 (2021).
- Robin Karlsson, Manuela Kulaxizi, Gim Seng Ng, Andrei Parnachev, and **Petar Tadić**, CFT correlators, W-algebras and generalized Catalan numbers. J. High Energ. Phys. 2022, 162 (2022).
- Sašo Grozdanov, Andrei O. Starinets, and **Petar Tadić**, Hydrodynamic dispersion relations at finite coupling. J. High Energ. Phys. 2021, 180 (2021).

Редослед аутора на радовима је одређен алфабетски, не према доприносу аутора. На осталим радовима кандидат је значајно допринео иницијалној теоријској анализи, нумеричким и аналитичким прорачунима и интерпретацији резултата.

Кандидатк је већину досадашњих научних активности обављао на Тринити колеџу у Даблину, у Ирској и Јејл Универзитету у Сједињеним Америчким Државама. Кроз наведене доприносе остварио је сарадњу, између осталог, са истраживачима са Универзитета у Оксфорду и Универзитета у Единбургу у Уједињеном Краљевству, Универзитета у Љубљани у Словенији и Универзитета у Викторији у Канади.

4.2. Нормирање броја коауторских радова, патената и техничких решења

Од 8 радова, 3 рада су нумерички прорачуни од којих један има три аутора, а два имају по 4 аутора, тако да се сва три рада рачунају са пуним бројем бодова. Осталих 5 радова су теоријски радови, један од њих има три аутора, па се рачуна са пуним бројем бодова, а на осталим радовима бодови се нормирају. Укупан број М бодова је 66, а нормирани број је 60.

4.3. Активност у научним и научно-стручним друштвима

Кандидат је рецезент у два научна часописа: *Journal of High Energy Physics* и *Nuclear Physics, Section B.*

4.4. Утицај научних резултата

Утицај научних резултата огледа се у подацима о цитираности, наведеним у секцији 4.1.2.

Кандидат је одржао и 7 семинара по позиву на којима је представио своје научне резултате, на Тринити колеџу у Даблину 2019. године, Универзитету у Љубљани 2020. године, Holotube Junior конференцији 2020, Инсититуту за физику у Београду и Јејл Универзитету у САД 2021. године, Универзитету у Оксфорду и Универзитету у Бостону 2022 године.

4.5. Конкретан допринос кандидата у реализацији радова у научним центрима у земљи и иностранству

Кандидат је водећи аутор на 4 објављена рада, којима је дао кључан допринос у погледу дефинисања проблема, иницијалне теоријске анализе, нумеричких и аналитичких прорачуна, интерпретације резултата и писања рада.

Кандидат је већину досадашњих научних активности обављао на Тринити колеџу у Даблину у Ирској, као студент докторских студија и Јејл Универзитету у Сједињеним Америчким Државама, на постдокторском усавршавању.

5. ЕЛЕМЕНТИ ЗА КВАНТИТАТИВНУ ОЦЕНУ НАУЧНОГ ДОПРИНОСА КАНДИДАТА

Остварени резултати у периоду након одлуке Научног већа о предлогу за стицање претходног научног звања **:**

Категорија	**М бодова по раду**	**Број радова**	**Укупно М бодова**	**Нормирани број М бодова**
M21a	10	1	10	10
M21	8	7	56	49.7
M34	0.5	1	0.5	0.5
M70	6	1	6	6

Поређење са минималним квантитативним условима за избор у звање научни сарадник:

Минимални број М бодова	Неопходно	Остварено, број М бодова без нормирања	**Остварено, нормирани број М бодова**
Укупно	16	72.5	**66.2**
M10+M20+M31+M32+M33+M41+M42+M90	10	66	**59.7**
M11+M12+M21+M22+M23	6	66	**59.7**

6. СПИСАК ОБЈАВЉЕНИХ РАДОВА

6.1. Радови у међународним часописима изузетних вредности (M21a):

- Sašo Grozdanov, Pavel K. Kovtun, Andrei O. Starinets, and **Petar Tadić**, Convergence of the Gradient Expansion in Hydrodynamics. Phys. Rev. Lett. 122, 251601 (2019).
  DOI: 10.1103/PhysRevLett.122.251601
  M21a, IF(2019)=8.57, SNIP(2019)=2.50.

6.2. Радови у врхунским међународним часописима (M21):

- Robin Karlsson, Manuela Kulaxizi, Andrei Parnachev, and **Petar Tadić**, Black holes and conformal Regge bootstrap. J. High Energ. Phys. 2019, 46 (2019).
  DOI: 10.1007/JHEP10(2019)046
  M21, IF(2019)=5.47, SNIP(2019)=1.30.

- Sašo Grozdanov, Pavel K. Kovtun, Andrei O. Starinets, and **Petar Tadić**, The complex life of hydrodynamic modes. J. High Energ. Phys. 2019, 97 (2019).
  DOI: 10.1007/JHEP11(2019)097
  M21, IF(2019)=5.47, SNIP(2019)=1.30.

- Robin Karlsson, Manuela Kulaxizi, Andrei Parnachev, and **Petar Tadić**, Leading multi-stress tensors and conformal bootstrap. J. High Energ. Phys. 2020, 76 (2020).
  DOI: 10.1007/JHEP01(2020)076
  M21, IF(2020)=5.32, SNIP(2020)=1.23.

- Robin Karlsson, Manuela Kulaxizi, Andrei Parnachev, and **Petar Tadić**, Stress tensor sector of conformal correlators. J. High Energ. Phys. 2020, 19 (2020).
  DOI: 10.1007/JHEP07(2020)019
  M21, IF(2020)=5.32, SNIP(2020)=1.23.

- Robin Karlsson, Andrei Parnachev, and **Petar Tadić**, Thermalization in large-N CFTs. J. High Energ. Phys. 2021, 205 (2021).
  DOI: 10.1007/JHEP09(2021)205
  M21, IF(2021)=5.62, SNIP(2021)=1.27.

- Sašo Grozdanov, Andrei O. Starinets, and **Petar Tadić**, Hydrodynamic dispersion relations at finite coupling. J. High Energ. Phys. 2021, 180 (2021).
  DOI: 10.1007/JHEP06(2021)180
  M21, IF(2021)=5.62, SNIP(2021)=1.27.

- Robin Karlsson, Manuela Kulaxizi, Gim Seng Ng, Andrei Parnachev, and **Petar Tadić**, CFT correlators, W-algebras and generalized Catalan numbers. J. High Energ. Phys. 2022, 162 (2022).
  DOI: 10.1007/JHEP06(2022)162
  M21, IF(2021)=5.62, SNIP(2021)=1.27.

6.3. Саопштења са међународног скупа штампана у изводу (М34):

- Robin Karlsson, Andrei Parnachev, and **Petar Tadić**,
  Thermalization of stress tensor sector,
  XVI Avogadro Meeting, Galileo Galilei Institute for Theoretical Physics, 21-22 December 2020.

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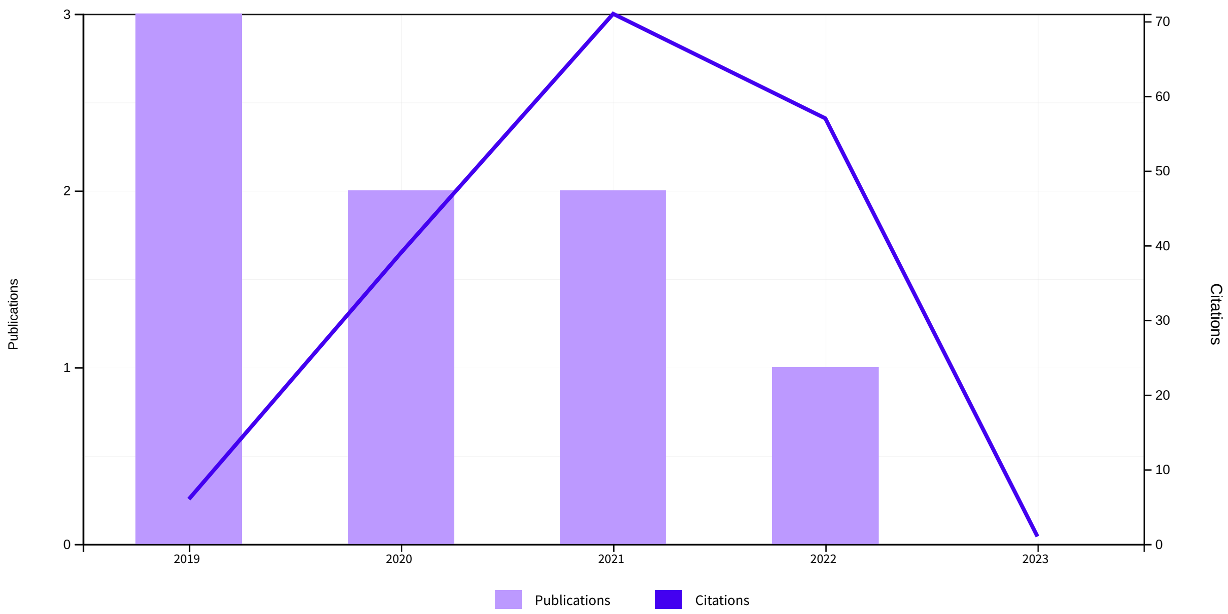

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			< Previous year Next year >					Average per year	Total
			2019	2020	2021	2022	2023		
		Total	6	39	71	57	1	34.8	174
1	The complex life of hydrodynamic modes Grozdanov, S; Kovtun, PK; (...); Tadic, P		1	12	30	18	0	12.2	61

	Nov 18 2019 | JOURNAL OF HIGH ENERGY PHYSICS (11)							
2	Convergence of the Gradient Expansion in Hydrodynamics Grozdanov, S; Kovtun, PK; (...); Tadic, P Jun 28 2019 | PHYSICAL REVIEW LETTERS 122 (25)	4	13	20	15	1	10.6	53
3	Black holes and conformal Regge bootstrap Karlsson, R; Kulaxizi, M; (...); Tadic, P Oct 4 2019 | JOURNAL OF HIGH ENERGY PHYSICS (10)	1	8	6	4	0	3.8	19
4	Leading multi-stress tensors and conformal bootstrap Karlsson, R; Kulaxizi, M; (...); Tadic, P Jan 14 2020 | JOURNAL OF HIGH ENERGY PHYSICS (1)	0	5	9	3	0	4.25	17
5	Stress tensor sector of conformal correlators operators in the Regge limit Karlsson, R; Kulaxizi, M; (...); Tadic, P Jul 3 2020 | JOURNAL OF HIGH ENERGY PHYSICS (7)	0	1	5	4	0	2.5	10
6	Hydrodynamic dispersion relations at finite coupling Grozdanov, S; Starinets, AO and Tadic, P Jun 30 2021 | JOURNAL OF HIGH ENERGY PHYSICS (6) Enriched Cited References	0	0	1	8	0	3	9
7	Thermalization in large-N CFTs Karlsson, R; Parnachev, A and Tadic, P Sep 29 2021 | JOURNAL OF HIGH ENERGY PHYSICS (9)	0	0	0	3	0	1	3
8	CFT correlators, W-algebras and generalized Catalan numbers Karlsson, R; Kulaxizi, M; (...); Tadic, P Jun 29 2022 | JOURNAL OF HIGH ENERGY PHYSICS (6) Enriched Cited References	0	0	0	2	0	1	2

Citation Report Publications Table

Published for SISSA by Springer

Received: June 5, 2019
Revised: August 19, 2019
Accepted: September 17, 2019
Published: October 4, 2019

Black holes and conformal Regge bootstrap

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Abstract: Highly energetic particles traveling in the background of an asymptotically AdS black hole experience a Shapiro time delay and an angle deflection. These quantities are related to the Regge limit of a heavy-heavy-light-light four-point function of scalar operators in the dual CFT. The Schwarzschild radius of the black hole in AdS units is proportional to the ratio of the conformal dimension of the heavy operator and the central charge. This ratio serves as a useful expansion parameter; its power counts the number of stress tensors in the multi-stress tensor operators which contribute to the four-point function. In the cross-channel the four-point function is determined by the OPE coefficients and anomalous dimensions of the heavy-light double-trace operators. We explain how this data can be obtained and explicitly compute the first and second order terms in the expansion of the anomalous dimensions. We observe perfect agreement with known results in the lightcone limit, which were obtained by computing perturbative corrections to the energy eigenstates in AdS spacetimes.

Keywords: AdS-CFT Correspondence, Conformal Field Theory

ArXiv ePrint: 1904.00060

https://doi.org/10.1007/JHEP10(2019)046

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1 Introduction and summary

1.1 Introduction

The AdS/CFT correspondence provides a non-perturbative definition of quantum gravity in negatively curved spacetimes [1–3]. The correspondence in principle provides an opportunity to study generic properties of quantum gravity, possibly probing regimes unattainable by low-energy effective theories. Recent years have seen a development in conformal bootstrap techniques following [4–7], leading to many results for CFTs with holographic duals (see e.g. [8–30]). CFT methods have therefore become a powerful tool in the study of quantum gravity.

Crossing symmetry in CFTs imposes highly non-trivial constraints on the theory. The idea of conformal bootstrap is to use these constraints to put restrictions on the CFT data or, if possible, even solve the theory. One way to make use of the crossing symmetry is to isolate a small number of contributing operators in one channel, e.g. by going to a certain kinematical regime. This typically has to be reproduced by the exchange of an infinite number of operators in another channel. One such example is the lightcone limit where the separation between two operators in a four-point function is close to being null. One can then infer [31, 32] the existence of double-twist operators at large spin in any CFT in dimensions $d > 2$. The Regge limit provides another opportunity to isolate the contribution of a class of operators, those of highest spin.

Holographic CFTs satisfy the following defining properties: (1) large central charge $C_T \sim N^2$ and large-N factorization of correlation functions and (2) a parametrically large gap in the spectrum of single trace operators above spin-2. As argued in [8], they are dual to theories of quantum gravity in asymptotically AdS spacetimes with local physics below the AdS scale. In holographic CFTs the Regge limit of a four-point function, extensively studied in [33–38],[1] is dominated by operators of spin two — the stress tensor and the double-trace operators (this is a consequence of the gap in the spectrum). In gravity, it reproduces a Witten diagram with graviton exchange (see e.g. [20]). The Regge limit corresponds to special kinematics, which on the gravity side is described by the scattering of highly energetic particles whose trajectories in the bulk are approximately null.

Such scattering can be described in the eikonal approximation where particles follow classical trajectories but their wavefunctions acquire a phase shift $\delta(S, L)$. The phase shift is a function of the total energy S and the impact parameter L. In the CFT language, this phase shift can be extracted from the Fourier transform of the four-point function. In [34] the phase shift extracted from the four-point function of the type $\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_2 \rangle$ was shown to be equal (up to a factor of $-\pi$) to the anomalous dimension of the double-trace operators $[\mathcal{O}_1 \mathcal{O}_2]_{n,l}$ at leading order in $1/N^2$. The Regge limit implies that the calculation is valid for $n, l \gg 1$. These anomalous dimensions have been subsequently verified in [26, 62–70].

So far both operators $\mathcal{O}_1$ and $\mathcal{O}_2$ were assumed to have conformal dimensions of order one. In the following, we will refer to them as "light" operators and denote them by $\mathcal{O}_L$. In [71] one pair of operators (which we denote by $\mathcal{O}_H$) was taken to be "heavy", with

[1] See also [22, 39–61] for other recent applications of Regge limit in CFTs.

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conformal dimension Δ_H scaling as the central charge. The ratio $\mu \sim \Delta_H/C_T$ is a useful expansion parameter; its power k corresponds to the number of stress tensors in the multi-stress tensor operators exchanged in the T-channel $(\mathcal{O}_H \times \mathcal{O}_H \to (T_{\mu\nu})^k \to \mathcal{O}_L \times \mathcal{O}_L)$.[2] As explained in [71], one can define the phase shift as a Fourier transform of the $\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle$ four-point function. It is related to the time delay and angle deflection of a highly energetic particle traveling along a null geodesic in the background of an asymptotically AdS black hole. The black hole corresponds to the insertion of the heavy operator $\mathcal{O}_H$; its mass in the units of AdS radius is proportional to μ.

The phase shift $\delta(S, L)$ was computed in gravity in [71] as an infinite series expansion in μ, i.e.,

$$\delta(S, L) = \sum_{k=1}^{\infty} \delta^{(k)} \mu^k \,, \tag{1.1}$$

with terms subleading in $1/N^2$ suppressed. The anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ admit a similar expansion

$$\gamma(n, l) = \sum_{k=1}^{\infty} \gamma^{(k)} \mu^k. \tag{1.2}$$

In [71] it was also proven that

$$\gamma^{(1)} = -\frac{\delta^{(1)}}{\pi} \tag{1.3}$$

where the following identifications are implied:

$$h = n + l, \qquad \bar{h} = n, \qquad S = 4h\bar{h}, \qquad e^{-2L} = \frac{\bar{h}}{h}. \tag{1.4}$$

However, it was observed that this relation does not hold for higher order terms, i.e. in general $\gamma^{(k)}$ is not proportional to $\delta^{(k)}$. One of the aims of this paper is to explain how higher order anomalous dimensions are related to higher order terms in the phase shift.

1.2 Summary of the results

In this paper we explain how to compute the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ order by order in μ, using the phase shift result of [71]. In particular, we show that the $\mathcal{O}(\mu^2)$ anomalous dimensions in any d are given by

$$\gamma^{(2)} = -\frac{\delta^{(2)}}{\pi} + \frac{\gamma^{(1)}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}, \qquad \Delta_H \gg l, n \gg 1. \tag{1.5}$$

Using known results for $\delta^{(1)}$ and $\delta^{(2)}$ from [71], we find an explicit expression for $\gamma^{(2)}$ and compare it with the known results in the lightcone limit ($\Delta_H \gg l \gg n \gg 1$). We find perfect agreement.

The rest of the paper is organized as follows. In the next section, we review the 4-point function with two heavy scalar operators $\mathcal{O}_H$ and two light scalar operators $\mathcal{O}_L$. This is the main object studied in this paper and we refer to it as a heavy-heavy-light-light correlator.

[2] Recently a similar limit was studied in [72].

We consider holographic CFTs, where the T-channel exchange (where $\mathcal{O}_H \to \mathcal{O}_H$ and $\mathcal{O}_L \to \mathcal{O}_L$) is dominated by multi-trace operators made out of the stress tensor. We relate this to corrections to the CFT data in the S-channel (OPE coefficients and anomalous dimensions).

In section 3 we focus on four-dimensional holographic CFTs. At $\mathcal{O}(\mu)$, we use the crossing equation between the S- and T-channel to solve for the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$. The result is eq. (1.3), valid for $l, n \gg 1$. We then introduce the impact parameter representation which allows us to rewrite the S-channel expansion as a Fourier transform. We use this to relate the phase shift to the anomalous dimensions of $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ at $\mathcal{O}(\mu^2)$, thereby deriving (1.5). Using a known result for the phase shift $\delta^{(2)}$, we write down an explicit expression for $\gamma^{(2)}$. In the subsequent $l \gg n$ limit it reduces to the result which has been obtained in [71] in a completely different way (by computing corrections to the energies of excited states in the AdS-Schwarzschild background).

In section 4 we generalize these results to any d ($d = 2$ is treated separately in appendix D). By solving the Casimir equation in the limit $\Delta_H \gg \Delta_L, l, n$, we obtain the conformal blocks for heavy-light double-trace operators in the S-channel. Using the explicit expression for the blocks together with the mean field theory OPE coefficients, we derive an impact parameter representation valid in general dimensions. Just as in the $d = 4$ case, this allows us to write the S-channel sum as a Fourier transform. Hence, we show that (1.5) holds for any d. We compute $\gamma^{(2)}$ in the lightcone limit and find perfect agreement with the results quoted in [71]. In addition, we find an expression for the $\mathcal{O}(\mu^2)$ corrections to the OPE coefficients.

Section 5 discusses various observations and mentions some open problems. Appendices contain additional technical details. The conformal bootstrap calculations are summarized in appendix A, the proof of the impact parameter representation in $d = 4$ in appendix B and the proof in general dimension d in appendix C. The special case of $d = 2$ is treated in appendix D. Appendix E discusses the fate of some boundary terms. Appendices F and G contain some identities which are used in section 5.

2 Heavy-heavy-light-light correlator in holographic CFTs

In this section, crossing relations for a heavy-heavy-light-light correlator of pairwise identical scalars are reviewed. We consider large N CFTs, with $N^2 \sim C_T$ and C_T the central charge, with a parametrically large gap Δ_{gap} in the spectrum of single trace operators with spin $J > 2$. The object that we study is a four-point correlation function between two light scalar operators $\mathcal{O}_L$, with scaling dimension of order one, and two heavy scalar operators $\mathcal{O}_H$, with scaling dimension Δ_H of $\mathcal{O}(C_T)$. Explicitly, we expand the CFT data in the parameter μ defined in [71] as

$$\mu = \frac{4\Gamma(d+2)}{(d-1)^2\Gamma(d/2)^2}\frac{\Delta_H}{C_T}, \tag{2.1}$$

which is kept fixed as $C_T \to \infty$. Our conventions mostly follow those of [71].

The four-point function is fixed by conformal symmetry up to a function $\mathcal{A}(u,v)$ of the cross-ratios as

$$\langle \mathcal{O}_H(x_4)\mathcal{O}_L(x_3)\mathcal{O}_L(x_2)\mathcal{O}_H(x_1)\rangle = \frac{\mathcal{A}(u,v)}{x_{14}^{2\Delta_H} x_{23}^{2\Delta_L}}, \tag{2.2}$$

where u, v are cross-ratios

$$\begin{aligned} u &= z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \\ v &= (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \end{aligned} \tag{2.3}$$

and $x_{ij} = x_i - x_j$. Using conformal symmetry we can fix $x_1 = 0$, $x_3 = 1$ and $x_4 \to \infty$, with the last operator confined to a plane parameterized by $(z, \bar{z})$. It will also be convenient to introduce the following coordinates after the analytic continuation ($z \to ze^{-2i\pi}$):

$$\begin{aligned} 1 - z &= \sigma e^{\rho} \\ 1 - \bar{z} &= \sigma e^{-\rho}. \end{aligned} \tag{2.4}$$

The Regge limit then corresponds to $\sigma \to 0$ with ρ kept fixed.

The main object of study is an appropriately rescaled version of (2.2)

$$G(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H} \langle \mathcal{O}_H(x_4)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle. \tag{2.5}$$

This can be expanded in the S-channel $\mathcal{O}_L(z,\bar{z}) \to \mathcal{O}_H(0)$ as

$$G(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{\mathcal{O}} \left(-\frac{1}{2}\right)^J \lambda_{\mathcal{O}_H\mathcal{O}_L\mathcal{O}} \lambda_{\mathcal{O}_L\mathcal{O}_H\mathcal{O}} g_{\mathcal{O}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}), \tag{2.6}$$

where $\Delta_{HL} = \Delta_H - \Delta_L$, λ_{ijk} are OPE coefficients and the sum runs over primaries $\mathcal{O}$ with spin J and corresponding conformal blocks $g_{\mathcal{O}}$. The correlator can likewise be expanded in the T-channel $\mathcal{O}_L(z,\bar{z}) \to \mathcal{O}_L(1)$ as

$$G(z,\bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\mathcal{O}'} \left(-\frac{1}{2}\right)^{J'} \lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}'} \lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}'} g_{\mathcal{O}'}^{0,0}(1-z,1-\bar{z}), \tag{2.7}$$

where we again sum over primaries $\mathcal{O}'$ with spin J'. The equality of (2.6) and (2.7) constitutes an example of a crossing relation, in both channels we sum over an infinite set of conformal blocks $g_{\mathcal{O}}^{\Delta_1,\Delta_2}(z,\bar{z})$. These contain the contribution from a primary $\mathcal{O}$ and all its descendants. (For recent reviews on conformal bootstrap see [73–75].) Here we have distinguished between operators $\mathcal{O}$ and $\mathcal{O}'$, in the S- and T-channel, respectively, in order to stress that generically different operators are relevant in different channels. As an example of this, in the lightcone limit in $d > 2$ one finds [31, 32] that the T-channel is dominated by the identity operator, while in the S-channel an infinite number of operators contribute. These are the so-called double-twist operators that exist at large spin in any $\mathrm{CFT}_{d>2}$.

We will assume the following scaling for a non-trivial single trace operator $\mathcal{O}$ (not including the stress tensor)

$$\langle \mathcal{O}_{H,L}\mathcal{O}_{H,L}\mathcal{O}\rangle \sim \frac{1}{\sqrt{C_T}}. \tag{2.8}$$

The conformal Ward identity fixes the following 3-pt function for the stress tensor

$$\langle \mathcal{O}_{H,L}\mathcal{O}_{H,L}T_{\mu\nu}\rangle \sim \Delta_{H,L}, \tag{2.9}$$

which implies the following scaling for the exchange of the stress tensor in the T-channel

$$\frac{\langle \mathcal{O}_H\mathcal{O}_H T_{\mu\nu}\rangle\langle T_{\mu\nu}\mathcal{O}_L\mathcal{O}_L\rangle}{C_T} \sim \frac{\Delta_H\Delta_L}{C_T} \sim \mu. \tag{2.10}$$

Keeping μ small, it follows that the leading contribution in the T-channel is given by the disconnected correlator $\langle \mathcal{O}_H\mathcal{O}_H\rangle\langle \mathcal{O}_L\mathcal{O}_L\rangle$, i.e., the exchange of the identity operator. Decomposing the disconnected correlator in the S-channel, we will infer the existence of the "double-trace operators" $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ for all integers n, l, with scaling dimension $\Delta = \Delta_H + \Delta_L + 2n + l + \gamma(n,l)$ and spin l. Moreover, the OPE coefficients scale as (the explicit expression is given below)

$$\langle \mathcal{O}_H\mathcal{O}_L[\mathcal{O}_H\mathcal{O}_L]_{n,l}\rangle \sim 1. \tag{2.11}$$

Keeping $\mu \sim \Delta_H/C_T$ fixed as $C_T \to \infty$, (2.10) implies that the CFT data of double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ receives perturbative corrections in μ. We therefore expand the anomalous dimensions of these double-trace operators, as well as the OPE coefficients

$$P_{n,l} \equiv \left(-\frac{1}{2}\right)^l \lambda_{\mathcal{O}_H\mathcal{O}_L[\mathcal{O}_H\mathcal{O}_L]_{n,l}}\lambda_{\mathcal{O}_L\mathcal{O}_H[\mathcal{O}_H\mathcal{O}_L]_{n,l}}\,, \tag{2.12}$$

in μ as

$$\begin{aligned}\gamma(n,l) &= \mu\gamma^{(1)} + \mu^2\gamma^{(2)} + \ldots \\ P_{n,l} &= P^{(0)}(1 + \mu P^{(1)} + \mu^2 P^{(2)}\ldots),\end{aligned} \tag{2.13}$$

with $\ldots$ denoting higher order terms.

To reach the Regge limit we analytically continue $z \to \mathrm{e}^{-2\pi i}z$, under which the blocks in the S-channel transform as (see e.g. [21, 50])

$$g_{\Delta,J}(z,\bar{z}) \to \mathrm{e}^{-i\pi(\Delta-J)}g_{\Delta,J}(z,\bar{z}). \tag{2.14}$$

In particular, for double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ with scaling dimension $\Delta = \Delta_H + \Delta_L + 2n + l + \gamma(n,l)$, the blocks transform as

$$g^{\Delta_{HL},-\Delta_{HL}}_{[\mathcal{O}_H\mathcal{O}_L]_{n,l}}(z,\bar{z}) \to e^{-i\pi(\Delta_H+\Delta_L)}e^{-i\pi\gamma(n,l)}g^{\Delta_{HL},-\Delta_{HL}}_{[\mathcal{O}_H\mathcal{O}_L]_{n,l}}(z,\bar{z}). \tag{2.15}$$

In what follows it will be convienent to do a change of variables to $h = n+l$ and $\bar{h} = n$ and to denote the block due to a heavy-light double-trace operator $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ as $g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}$.

Substituting the μ expansion (2.13) in the S-channel (2.6) and performing the usual analytic continuation to $\mathcal{O}(\mu)$ leads to

$$\begin{aligned}
G(z,\bar{z})|_{\mu^0} &= (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(0)} g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) \\
G(z,\bar{z})|_{\mu^1} &= (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(0)} \left(P^{(1)} + \gamma^{(1)} \left(\frac{1}{2}\left(\partial_h + \partial_{\bar{h}}\right) - i\pi\right)\right) \\
&\qquad \times g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}).
\end{aligned} \tag{2.16}$$

The new single trace operators that can possibly appear here would be subleading in $1/N^2$. Continuing to $\mathcal{O}(\mu^2)$, the imaginary part of the S-channel is given by

$$\begin{aligned}
\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} &= -i\pi(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \\
&\times \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(0)} \left(\gamma^{(2)} + \gamma^{(1)}P^{(1)} + \frac{(\gamma^{(1)})^2}{2}(\partial_h + \partial_{\bar{h}})\right) g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}).
\end{aligned} \tag{2.17}$$

Moreover, the real part of the correlator at the same order is given by

$$\begin{aligned}
\mathrm{Re}(G(z,\bar{z}))|_{\mu^2} &= (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(0)} \Bigg(P^{(2)} - \frac{1}{2}\pi^2(\gamma^{(1)})^2 \\
&+ \frac{1}{2}(\gamma^{(2)} + P^{(1)}\gamma^{(1)})(\partial_h + \partial_{\bar{h}}) + \frac{1}{8}(\gamma^{(1)})^2(\partial_h + \partial_{\bar{h}})^2\Bigg) g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z})\,.
\end{aligned} \tag{2.18}$$

The product of OPE coefficients $P^{(0)}$ is fixed by the correlator at $\mathcal{O}(\mu^0)$ in (2.16) and can be found in [15]:

$$\begin{aligned}
P^{(0)} = &\frac{(\Delta_H + 1 - d/2)_{\bar{h}}(\Delta_L + 1 - d/2)_{\bar{h}}(\Delta_H)_h(\Delta_L)_h}{\bar{h}!(h-\bar{h})!(\Delta_H + \Delta_L + \bar{h} + 1 - d)_{\bar{h}}(\Delta_H + \Delta_L + h + \bar{h} - 1)_{h-\bar{h}}} \\
&\times \frac{1}{(h - \bar{h} + d/2)_{\bar{h}}(\Delta_H + \Delta_L + h - d/2)_{\bar{h}}},
\end{aligned} \tag{2.19}$$

where $(a)_b$ is the Pochhammer symbol. In the limit $\Delta_H \gg \Delta_L, h, \bar{h}$, (2.19) simplifies

$$P^{(0)} \approx C_{\Delta_L} \frac{\Gamma(\Delta_L + \bar{h} - d/2 + 1)\Gamma(\Delta_L + h)}{\bar{h}!(h-\bar{h})!(h - \bar{h} + d/2)_{\bar{h}}}, \tag{2.20}$$

where $C_{\Delta_L} = (\Gamma(\Delta_L - d/2 + 1)\Gamma(\Delta_L))^{-1}$. As we will see below, in the Regge limit the dominant contribution in the S-channel comes from double-trace operators with $h, \bar{h} \gg 1$. In this limit the OPE coefficients are given by

$$P^{(0)} \approx C_{\Delta_L}(h\bar{h})^{\Delta_L - \frac{d}{2}}(h - \bar{h})^{\frac{d}{2}-1}. \tag{2.21}$$

We will further need $\lambda_{\mathcal{O}_L\mathcal{O}_L T}\lambda_{\mathcal{O}_H\mathcal{O}_H T}$ in (2.7), these are fixed by Ward Identities to be

$$\frac{\lambda_{\mathcal{O}_L\mathcal{O}_L T}\lambda_{\mathcal{O}_H\mathcal{O}_H T}}{\Delta_L} = \left(\frac{d}{d-1}\right)^2 \frac{\Delta_H}{C_T} = \frac{\mu\Gamma\left(\frac{d}{2}+1\right)^2}{\Gamma\left(d+2\right)}. \tag{2.22}$$

Note that as explained in [71], an expansion in μ corresponds in the bulk to an expansion in the black hole Schwarzschild radius in AdS units.

3 Anomalous dimensions of heavy-light double-trace operators in $d = 4$

In this section we investigate the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ in $d = 4$ using conformal bootstrap. Moreover, using a four-dimensional impact parameter representation we relate the anomalous dimensions to the bulk phase shift to $\mathcal{O}(\mu^2)$. This procedure can be repeated order by order in μ to obtain the OPE data (anomalous dimensions and OPE coefficients — see also section 4) to the desired order.

3.1 Anomalous dimensions in the Regge limit using bootstrap

The conformal blocks in $d = 4$ are given by [76]

$$g_{\Delta,J}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}\left(k_{\Delta+J}(z)k_{\Delta-J-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right) \tag{3.1}$$

where

$$k_\beta(z) = z^{\beta/2}{}_2F_1\left(\frac{\beta-\Delta_{12}}{2},\frac{\beta+\Delta_{34}}{2},\beta,z\right). \tag{3.2}$$

In the limit $\Delta_H \sim C_T \gg 1$ the hypergeometric functions in (3.1) can be substituted by the identity up to $1/\Delta_H$ corrections. Explicitly, the conformal blocks of $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ in the heavy limit are given by

$$g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = \frac{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L)}(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1})}{z-\bar{z}}. \tag{3.3}$$

Inserting this form of the conformal blocks in (2.16) together with the OPE coefficients in the Regge limit (2.21), we approximate the sums by integrals and find the following expression at $\mathcal{O}(\mu^0)$ in the S-channel

$$G(z,\bar{z})|_{\mu^0} = \frac{C_{\Delta_L}}{z-\bar{z}}\int_0^\infty dh\int_0^h d\bar{h}(h\bar{h})^{\Delta_L-2}(h-\bar{h})\left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right). \tag{3.4}$$

The integrals are explicitly computed in appendix A; the result is the disconnected correlator in the T-channel $[(1-z)(1-\bar{z})]^{-\Delta_L}$ in the Regge limit $\sigma \to 0$.

At $\mathcal{O}(\mu)$ in holographic CFTs the leading corrections in the T-channel come from the exchanges of the stress tensor and double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ ($[\mathcal{O}_H\mathcal{O}_H]_{n,l=2}$ are heavy and therefore decouple). The conformal block for the T-channel exchange of the stress tensor is found after $z \to e^{-2\pi i}z$ to be given by

$$g_{T_{\mu\nu}} = \frac{360 i\pi e^{-\rho}}{\sigma(e^{2\rho}-1)} + \ldots, \tag{3.5}$$

where $\ldots$ denotes non-singular terms. The contribution from the stress tensor exchange in the T-channel is thus imaginary for real values of σ and ρ. The only imaginary term at order μ in the S-channel expansion (2.16) comes from the term proportional to $-i\pi\gamma$; it must reproduce (3.5).

In the Regge limit, we approximate the sum in the S-channel by an integral and insert the OPE coefficients from (2.21); the imaginary part at $\mathcal{O}(\mu)$ in the S-channel is thus given by

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu^1} = \frac{-i\pi C_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L - 2}(h-\bar{h})\gamma^{(1)}(h,\bar{h}) \left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right). \tag{3.6}$$

With the ansatz $\gamma^{(1)}(h,\bar{h}) = c_1 h^a \bar{h}^b/(h-\bar{h})$ the integrals in (3.6) can be computed (for more details see appendix A). In order to reproduce the exchange of the stress tensor, the anomalous dimensions at $\mathcal{O}(\mu)$ must be equal to

$$\begin{aligned}\gamma^{(1)} &= -\frac{90\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}\lambda_{\mathcal{O}_L\mathcal{O}_L T_{\mu\nu}}}{\mu\Delta_L}\frac{\bar{h}^2}{h-\bar{h}} \\ &= -\frac{3\bar{h}^2}{h-\bar{h}},\end{aligned} \tag{3.7}$$

where in the second line we inserted the OPE coefficients from (2.22). With the form (3.7) not only the stress tensor exchange is reproduced, but also an infinite sum of spin-2 double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ with scaling dimension $\Delta_n = 2\Delta_L + 2 + 2n$. This is similar to what happens in the light-light case [50].

To determine the second order corrections to the anomalous dimensions we use the derivative relationship:

$$P^{(0)}P^{(1)} = \frac{1}{2}(\partial_h + \partial_{\bar{h}})\left(P^{(0)}\gamma^{(1)}\right). \tag{3.8}$$

We will prove below (see section 4.3) that this relationship is true in the limit $h,\bar{h} \gg 1$. The imaginary part at $\mathcal{O}(\mu^2)$ in the S-channel from (2.16) is then given by

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = -i\pi \int_0^\infty dh \int_0^h d\bar{h} P^{(0)} \left(\gamma^{(2)} + \gamma^{(1)}P^{(1)} + \frac{(\gamma^{(1)})^2}{2}(\partial_h + \partial_{\bar{h}})\right) g_{h,\bar{h}}. \tag{3.9}$$

With the help of (3.8), one can write (3.9) as

$$\begin{aligned}\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = -i\pi \int_0^\infty dh \int_0^h d\bar{h} P^{(0)} \left(\gamma^{(2)} - \frac{\gamma^{(1)}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}\right) g_{h,\bar{h}} \\ + \text{total derivative},\end{aligned} \tag{3.10}$$

where the total derivate term does not contribute (see appendix E for details). In order to fix $\gamma^{(2)}$ completely from crossing symmetry, we would need to consider the exchange of infinitely many double-trace operators made out of the stress tensor in the T-channel. Instead, we will use an impact parameter representation to relate $\gamma^{(2)}$ to the bulk phase shift calculated from the gravity dual in [71].

3.2 4*d* impact parameter representation and relation to bulk phase shift

In [34] the anomalous dimensions of light-light double-trace operators in the limit $h,\bar{h} \gg 1$ were shown to be related to the bulk phase shift. An impact parameter representation for

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the case when one of the operators is heavy was introduced in [71], where it was also shown that the bulk phase shift and the anomalous dimensions are equal at $\mathcal{O}(\mu)$. The goal of this section is to see explicitly how the bulk phase shift and the anomalous dimensions are related to $\mathcal{O}(\mu^2)$.

The correlator (2.5) can be written in an impact parameter representation as

$$G(z,\bar{z}) = \int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}} f(h,\bar{h}), \tag{3.11}$$

with $\mathcal{I}_{h,\bar{h}}$ given by

$$\mathcal{I}_{h,\bar{h}} = (z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(0)} g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) \tag{3.12}$$

and $f(h,\bar{h})$ some function that generically depends on the anomalous dimension and corrections to the OPE coefficients. In particular, for $f(h,\bar{h}) = 1$, (3.11) is equal to the disconnected correlator. In appendix B it is shown that $\mathcal{I}_{h,\bar{h}}$ can be equivalently written as

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta_L) \int_{M^+} \frac{d^4p}{(2\pi)^4} (-p^2)^{\Delta_L-2} e^{-ipx} (h-\bar{h}) \delta(p\cdot\bar{e}+h+\bar{h})\, \delta\left(\frac{p^2}{4}+h\bar{h}\right) \tag{3.13}$$

where M^+ is the upper Milne wedge with $\{p^2 \leq 0,\ p^0 \geq 0\}$, $C(\Delta_L)$ given by (with $d=4$)

$$C(\Delta) \equiv \frac{2^{d+1-2\Delta}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta)\Gamma(\Delta-\frac{d}{2}+1)} \tag{3.14}$$

and $\bar{e} = (1,0,0,0)$. Moreover, following [71], we will set $z = e^{ix^+}$ and $\bar{z} = e^{ix^-}$, with $x^+ = t+r$ and $x^- = t-r$ in spherical coordinates.

Using the identity

$$\delta(p\cdot\bar{e}+h+\bar{h})\, \delta\left(\frac{p^2}{4}+h\bar{h}\right) = \frac{1}{|h-\bar{h}|}\left(\delta\left(\frac{p^+}{2}-h\right)\delta\left(\frac{p^-}{2}-\bar{h}\right) + (h \leftrightarrow \bar{h})\right), \tag{3.15}$$

with $p^+ = p^t + p^r$, $p^- = p^t - p^r$, the integrals over $h, \bar{h}$ in (3.11) are easily computed. With the identification $h = \frac{p^+}{2}$ and $\bar{h} = \frac{p^-}{2}$ it follows that a generic term like (3.11) can be written as a Fourier transform

$$\int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}} f(h,\bar{h}) = C(\Delta_L) \int_{M^+} \frac{d^4p}{(2\pi)^4} (-p^2)^{\Delta_L-2} e^{-ipx} f\left(\frac{p^+}{2},\frac{p^-}{2}\right). \tag{3.16}$$

We thus see that the impact parameter representation allows us to rewrite the S-channel expression as a Fourier transform.

The phase shift $\delta(p)$ for a pair of operators $\mathcal{O}_H$ and $\mathcal{O}_L$, with scaling dimensions $\Delta_H/C_T \propto \mu$ and $\Delta_L/C_T \ll 1$, respectively, was defined in [71] by

$$\mathcal{B}(p) \equiv \int d^4x e^{ipx} G(x) = \mathcal{B}_0(p) e^{i\delta(p)}, \tag{3.17}$$

where $G(x)$ is given in (2.5) and $\mathcal{B}_0(p)$ denotes the Fourier transform of the disconnected correlator. As the OPE data, the phase shift admits an expansion in μ:

$$\delta(p) = \mu\delta^{(1)}(p) + \mu^2\delta^{(2)}(p) + \ldots, \tag{3.18}$$

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where … denotes higher order terms in the expansion. Expanding the exponential in (3.17) in μ we get

$$\mathcal{B}(p) = \mathcal{B}_0(p)\Big(1 + i\mu\delta^{(1)} + \mu^2(-\frac{(\delta^{(1)})^2}{2} + i\delta^{(2)}) + \dots\Big). \tag{3.19}$$

With (3.19) the relationship between the anomalous dimensions and the bulk phase shift to $\mathcal{O}(\mu^2)$ can be established using (2.16), (2.17) and (3.16):

$$\begin{aligned}\gamma^{(1)} &= -\frac{\delta^{(1)}}{\pi}\\ \gamma^{(2)} &= -\frac{\delta^{(2)}}{\pi} + \frac{\gamma^{(1)}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}(h,\bar{h}).\end{aligned} \tag{3.20}$$

The phase shift was calculated in closed form to all orders in μ for the four-dimensional case [71], with the first and second order terms given by

$$\begin{aligned}\delta^{(1)} &= \frac{3\pi}{2}\sqrt{-p^2}\frac{e^{-L}}{e^{2L}-1}\\ \delta^{(2)} &= \frac{35\pi}{8}\sqrt{-p^2}\frac{2e^{L}-e^{-L}}{(e^{2L}-1)^3},\end{aligned} \tag{3.21}$$

where

$$-p^2 = p^+p^-, \qquad \cosh L = \frac{p^+ + p^-}{2\sqrt{-p^2}}. \tag{3.22}$$

Using (3.21) and (3.22), the $\mathcal{O}(\mu)$ corrections to the anomalous dimensions are given by $\gamma^{(1)} = -3n^2/l$, which agrees with (3.7). From (3.21) and (3.20), we deduce the anomalous dimensions at $\mathcal{O}(\mu^2)$:

$$\gamma^{(2)} = -\frac{35}{4}\frac{(2l+n)n^3}{l^3} + 9\frac{n^3}{l^2}. \tag{3.23}$$

Taking the lightcone limit ($l \gg n \gg 1$) in (3.23) we find

$$\gamma^{(2)}_{\text{l.c.}} = -\frac{17}{2}\frac{n^3}{l^2}. \tag{3.24}$$

The anomalous dimensions in the lightcone limit (3.24) agree with eq. (6.40) in [71], which was obtained independently by considering corrections to the energy levels in the AdS-Schwarzschild background.

4 OPE data of heavy-light double-trace operators in generic d

In this section we will write the general form of conformal blocks for heavy-light double-trace operators in the limit $\Delta_H \sim C_T \gg 1$ and general $d > 2$. These blocks will be used to confirm the validity of the impact parameter representation in appendix C. Using the impact parameter representation the OPE data will be related to the bulk phase shift. In particular, we show that (3.20) remains valid in any number of dimensions and find explicit expressions for the corrections to the OPE coefficients up to $\mathcal{O}(\mu^2)$.

4.1 Conformal blocks in the heavy limit

In order to find conformal blocks in general spacetime dimension d in the limit $\Delta_H \gg \Delta_L, h, \bar{h}$, we write them in the following form:

$$g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = (z\bar{z})^{\frac{\Delta_H+\Delta_L}{2}} F(z,\bar{z}), \tag{4.1}$$

where the function $F(z,\bar{z})$ does not depend on Δ_H and is symmetric with respect to the exchange $z \leftrightarrow \bar{z}$. Let us now insert the expression (4.1) into the Casimir equation and consider the leading $\mathcal{O}(\Delta_H)$ term:

$$z\frac{\partial}{\partial z}F(z,\bar{z}) + \bar{z}\frac{\partial}{\partial \bar{z}}F(z,\bar{z}) - (h+\bar{h})F(z,\bar{z}) = 0. \tag{4.2}$$

The most general solution to eq. (4.2) is:

$$F(z,\bar{z}) = z^{h+\bar{h}} f\left(\frac{\bar{z}}{z}\right), \tag{4.3}$$

where f is an arbitrary function that satisfies $f(\frac{1}{x}) = x^{-h-\bar{h}}f(x)$, since conformal blocks must be symmetric with respect to the exchange $z \leftrightarrow \bar{z}$.

The behaviour of the conformal blocks as $z, \bar{z} \to 0$ and $z/\bar{z}$ fixed is given by [76, 77]

$$g_{\Delta,l}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) \to \frac{l!}{(\frac{d}{2}-1)_l}(z\bar{z})^{\frac{\Delta}{2}} C_l^{(\frac{d}{2}-1)}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right), \tag{4.4}$$

where $\Delta = \Delta_1 + \Delta_2 + 2n + l$ and $C_q^{(p)}(x)$ are the Gegenbauer polynomials. Using (4.4), we can completely determine the function f:

$$f\left(\frac{\bar{z}}{z}\right) = \frac{(h-\bar{h})!}{(\frac{d}{2}-1)_{h-\bar{h}}}\left(\frac{\bar{z}}{z}\right)^{\frac{h+\bar{h}}{2}} C_{h-\bar{h}}^{(\frac{d}{2}-1)}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right). \tag{4.5}$$

That is, the conformal blocks in the limit of large Δ_H are given by

$$g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = \frac{(h-\bar{h})!}{(\frac{d}{2}-1)_{h-\bar{h}}}(z\bar{z})^{\frac{\Delta_H+\Delta_L+h+\bar{h}}{2}} C_{h-\bar{h}}^{(\frac{d}{2}-1)}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right). \tag{4.6}$$

It is easy to explicitly check that this form of the conformal blocks agrees with the one we used in $d=4$ in the previous section.

4.2 Anomalous dimensions

In appendix C we prove the validity of the impact parameter representation in any d. This means that the derivation of (3.20) goes through for arbitrary d. Using known results for the bulk phase shift from [71], we thus find

$$\gamma^{(1)} = -\frac{\bar{h}^{\frac{d}{2}}}{h^{\frac{d}{2}-1}}\frac{\Gamma(d)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)}\,{}_2F_1\left(\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}\right). \tag{4.7}$$

In the lightcone limit ($h = l \gg \bar{h} = n$) this reduces to

$$\gamma_{l.c.}^{(1)} = -\frac{\bar{h}^{\frac{d}{2}}}{h^{\frac{d}{2}-1}}\frac{\Gamma(d)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)}. \tag{4.8}$$

Similarly, using (3.20) together with eq. (2.29) and eq. (A.5) from [71], we find the $\mathcal{O}(\mu^2)$ corrections to the anomalous dimensions in the limit $h, \bar{h} \gg 1$:

$$
\begin{aligned}
\gamma^{(2)} &= -\frac{\delta^{(2)}}{\pi} + \frac{1}{2}\gamma^{(1)}\left\{\frac{2}{h+\bar{h}}\gamma^{(1)} - \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)^2}\bar{h}^{\frac{d}{2}-1}h^{\frac{d}{2}-1}\frac{(h-\bar{h})^{3-d}}{h+\bar{h}}\right\} \\
&= -\left(\frac{\bar{h}^{d-1}}{h^{d-2}}\right)\frac{2^{2d-4}\Gamma\left(d+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(d)}\,{}_2F_1\left[2d-3, d-2, d, \frac{\bar{h}}{h}\right] \\
&\quad + \frac{\bar{h}^d h^{2-d}}{(h+\bar{h})}\,\frac{4\Gamma^2(d)}{d^2\,\Gamma^4\left(\frac{d}{2}\right)}\left({}_2F_1\left[\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}\right]\right)^2 \\
&\quad + \frac{\bar{h}^{d-1}(h-\bar{h})^{3-d}}{h+\bar{h}}\,\frac{\Gamma^2(d)}{d\,\Gamma^4\left(\frac{d}{2}\right)}\,{}_2F_1\left[\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}\right]
\end{aligned} \tag{4.9}
$$

Taking further the lightcone limit ($h \gg \bar{h}$) we find that

$$
\gamma^{(2)}_{l.c.} = \frac{\bar{h}^{d-1}}{h^{d-2}}\frac{2^{2d-4}}{\pi}\left(\frac{d\Gamma\left(\frac{d+1}{2}\right)^2}{\Gamma\left(\frac{d+2}{2}\right)^2} - \frac{\sqrt{\pi}\Gamma\left(d+\frac{1}{2}\right)}{\Gamma(d)}\right). \tag{4.10}
$$

The result (4.10) agrees with eq. (6.42) in [71] which was obtained independently using perturbation theory in the bulk. In order to see this explicitly, one should notice the following expression for the hypergeometric function:

$$
{}_3F_2\left(1, -\frac{d}{2}, -\frac{d}{2}; 1+\frac{d}{2}, 1+\frac{d}{2}; 1\right) = \frac{1}{2}\left(1 + \frac{\Gamma^4(1+\frac{d}{2})\Gamma(2d+1)}{\Gamma^4(d+1)}\right). \tag{4.11}
$$

4.3 Corrections to the OPE coefficients

So far, we have only considered the imaginary part of the S-channel. The real part at $\mathcal{O}(\mu)$ is given by the following expression:

$$
\mathrm{Re}(G(z,\bar{z}))|_\mu = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dh\int_0^h d\bar{h}P^{(0)}\left(P^{(1)} + \frac{1}{2}\gamma^{(1)}(\partial_h + \partial_{\bar{h}})\right)g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}), \tag{4.12}
$$

which can be rewritten as:

$$
\begin{aligned}
\mathrm{Re}(G(z,\bar{z}))|_\mu = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}&\int_0^{+\infty}dh\int_0^h d\bar{h}g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}} \\
&\times\left(P^{(0)}P^{(1)} - \frac{1}{2}(\partial_h + \partial_{\bar{h}})(P^{(0)}\gamma^{(1)})\right) + \text{total derivative}.
\end{aligned} \tag{4.13}
$$

The total derivative term in (4.13) can be shown to vanish as explained in appendix E.

To derive a relation between the corrections to the OPE coefficients and the anomalous dimensions at $\mathcal{O}(\mu)$, let us consider the limit $h, \bar{h} \gg 1$ and substitute $\bar{h}$ by h everywhere. Using (4.7), one can deduce $\gamma^{(1)} \propto h$. Then, it follows that $(\partial_h + \partial_{\bar{h}})(P^{(0)}\gamma^{(1)}) \propto P^{(0)}$ and hence the second term on the right hand side of (4.13) behaves as:

$$
(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dh\int_0^h d\bar{h}\left(-\frac{1}{2}g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(\partial_h + \partial_{\bar{h}})(P^{(0)}\gamma^{(1)})\right) \propto \frac{1}{\sigma^{2\Delta_L}}. \tag{4.14}
$$

On the other hand, we know that in the Regge limit the leading contribution in the T-channel at $\mathcal{O}(\mu)$ comes from the exchange of the stress tensor. The real part of its conformal block is proportional to σ^d, so the T-channel result behaves as $\frac{1}{\sigma^{2\Delta_L-d}}$. This is way less singular than (4.14). Hence (4.14) must be canceled by the first term on the right hand side of (4.13), at least in the limit $h,\bar{h}\gg 1$. That is:

$$P^{(0)}P^{(1)}=\frac{1}{2}(\partial_h+\partial_{\bar{h}})(P^{(0)}\gamma^{(1)}). \tag{4.15}$$

A similar relation holds for the OPE coefficients of light-light double-trace operators, e.g. see [8, 15, 30]. In that case it was observed in [26] that the relation is not exact in $(h,\bar{h})$. We expect the same to be true here. Furthermore, the real part at $\mathcal{O}(\mu^2)$ was given in (2.18) as:

$$\begin{aligned}\mathrm{Re}(G(z,\bar{z}))|_{\mu^2}=(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\sum_{h\geq\bar{h}\geq 0}^{\infty}P^{(0)}\bigg(P^{(2)}-\frac{1}{2}(\pi\gamma^{(1)})^2\\ +\frac{1}{2}(\gamma^{(2)}+P^{(1)}\gamma^{(1)})(\partial_h+\partial_{\bar{h}})+\frac{1}{8}(\gamma^{(1)})^2(\partial_h+\partial_{\bar{h}})^2\bigg)g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}.\end{aligned} \tag{4.16}$$

Using the impact parameter representation this can be expressed as:

$$\begin{aligned}\mathrm{Re}(G(z,\bar{z}))|_{\mu^2}=\int_0^\infty dh\int_0^h d\bar{h}\mathcal{I}_{h,\bar{h}}\bigg(P^{(2)}-\frac{\pi^2}{2}(\gamma^{(1)})^2\\ -\frac{1}{2P^{(0)}}(\partial_h+\partial_{\bar{h}})(P^{(0)}(\gamma^{(2)}+P^{(1)}\gamma^{(1)}))+\frac{1}{8P^{(0)}}(\partial_h+\partial_{\bar{h}})^2(P^{(0)}(\gamma^{(1)})^2)\bigg),\end{aligned} \tag{4.17}$$

where we repeatedly integrated by parts. It follows from (3.19) and (3.16), together with $\pi\gamma^{(1)}=-\delta^{(1)}$, that the corrections to the OPE coefficients at $\mathcal{O}(\mu^2)$ satisfy the following relationship:

$$P^{(0)}P^{(2)}=\frac{1}{2}(\partial_h+\partial_{\bar{h}})(P^{(0)}(\gamma^{(2)}+P^{(1)}\gamma^{(1)}))-\frac{1}{8}(\partial_h+\partial_{\bar{h}})^2(P^{(0)}(\gamma^{(1)})^2). \tag{4.18}$$

The arguments above are similar to the ones used in [34, 50].

4.4 Flat space limit

In the flat space limit the relation between the scattering phase shift and the anomalous dimensions has been previously discussed in [78]. Hence, it is interesting to consider the flat space limit of eq. (1.5). This limit is achieved by taking the apparent impact parameter to be much smaller than the AdS radius. This corresponds to the small L regime or, equivalently, using $e^{-2L}=\bar{h}/h$ to the $1\ll l\ll n\ll\Delta_H$ limit.

In this limit, according to (4.7), the behavior of $\gamma^{(1)}$ is given by

$$\gamma^{(1)}\propto n\left(\frac{n}{l}\right)^{d-3}. \tag{4.19}$$

Hence, the $\gamma^{(1)}(\partial_h+\partial_{\bar{h}})\gamma^{(1)}$ term in eq. (1.5) behaves as

$$\gamma^{(1)}\partial_n\gamma^{(1)}\propto n\left(\frac{n}{l}\right)^{2d-6}. \tag{4.20}$$

Similarly, using equation (A.5) from [71], one finds that $\delta^{(2)}$ behaves as

$$\delta^{(2)} \propto n \left(\frac{n}{l}\right)^{2d-5} . \tag{4.21}$$

Since (4.20) is subleading to (4.21), in the flat space limit the anomalous dimensions are proportional to the phase shift,

$$\gamma^{(2)} \approx -\frac{\delta^{(2)}}{\pi} \tag{4.22}$$

5 Discussion

In this paper we studied a four-point function of pairwise identical scalar operators, $\mathcal{O}_H$ and $\mathcal{O}_L$, in holographic CFTs of any dimensionality. Scaling Δ_H with the central charge, the CFT data admits an expansion in the ratio $\mu \sim \Delta_H/C_T$ which we keep fixed. Using crossing symmetry and the bulk phase shift calculated in [71], we studied $\mathcal{O}(\mu^2)$ corrections to the OPE data of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ for large l and n. In particular, the relationship between the bulk phase shift and the OPE data of heavy-light double-trace operators is found using an impact parameter representation. Furthermore, this allows us in principle to determine the OPE data of $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$, for $l, n \gg 1$ to all orders in μ, i.e., to all orders in an expansion in the dual black hole Schwarzschild radius.

Scaling Δ_H with the central charge enhances the effect of stress tensor exchanges compared to the $1/C_T$ corrections due to the exchange of generic operators. At $\mathcal{O}(\mu^2)$ and higher, we therefore expect multi-stress tensor operators to contribute. The OPE coefficients for such exchanges are not known in general. They would be needed to determine corrections to the OPE data of heavy-light double-trace operators using purely CFT methods. In a recent paper [72] some of these OPE coefficients have been computed. In particular, the OPE coefficients with the multi-stress tensor operators of lowest twist have been argued to be universal (independent of the higher derivative couplings in the bulk gravitational lagrangian). It would be interesting to connect these results to the ones discussed in this paper.

It is a curious fact that each term in the μ-expansion of the bulk phase shift as computed in gravity in [71] can be expressed as an infinite sum of "Regge conformal blocks" corresponding to operators of dimension $\Delta = k(d-2)+2n+2$ and spin $J=2$. Explicitly,

$$i\,\delta^{(k)}(S,L) = f(k) \sum_{n=0}^{\infty} \lambda_k(n)\;\; g^R_{k(d-2)+2n+2,\,2}(S,L)\,, \tag{5.1}$$

where the coefficients $(f(k), \lambda_k(n))$ are listed in appendix F and we set $S \equiv \sqrt{-p^2}$ compared to [71]. Here $g^R_{\Delta,J}(S,L)$ denotes a "Regge conformal block", and is equal to the leading behaviour of the analytically continued T-channel conformal block in the Regge limit [49, 79]

$$g^R_{\Delta,J}(S,L) = i\, c_{\Delta,J}\; S^{J-1}\; \Pi_{\Delta-1,d-1}(L) \tag{5.2}$$

defined in terms of

$$1-z = \frac{e^L}{S}, \quad 1-\bar{z} = \frac{e^{-L}}{S} \tag{5.3}$$

as $S \to \infty$ and L fixed. Here $c_{\Delta,J}$ are known coefficients which can be found in appendix F and $\Pi_{\Delta-1,d-1}(L)$ denotes the $(d-1)$-dimensional hyperbolic space propagator for a massive scalar of mass square $m^2 = (\Delta - 1)$.

To understand the implications of (5.1) let us focus on $k = 2$ and consider large impact parameters, a.k.a. the lightcone limit. In this case, one expects that the dominant contribution to the bulk phase shift comes from the infinite sum of the minimal twist double-trace operators built from the stress tensor, schematically denoted by $T_{\mu\nu}\partial_{\mu_1} \cdots \partial_{\mu_\ell} T_{\rho\sigma}$. (5.1) implies that this infinite sum gives rise to a contribution which can be interpreted as coming from a *single* conformal block of an "effective" operator of the same twist $\tau = 2(d-2)$, but spin $J = 2$. At finite impact parameter, one would then need to add the contributions of an infinite tower of such effective operators of twist $\tau = 2(d-2) + 2n$ and spin $J = 2$, as expressed by the infinite sum in (5.1). From this point of view, the coefficients λ_n in (5.1) can be interpreted as ratios of sums of OPE coefficients of double-trace operators. It is clear that this picture appears to hold to all orders in $\left(\frac{\Delta_H}{C_T}\right)$ or equivalently, the Schwarzschild radius of the black hole.

It would be interesting to investigate whether Rindler positivity constrains the Regge behaviour of the bulk phase shift to grow at most linearly with the energy S, similarly to section 5.2 in [49]. If this were true, one would perhaps only need to understand the origin of the λ_n to compute the bulk phase shift to arbitrary order in $\left(\frac{\Delta_H}{C_T}\right)$ purely from CFT techniques.

Acknowledgments

We thank S. Frolov, G.S. Ng, R. Pereira and P. Di Vecchia for discussions. The work of R.K. and A.P. is supported in part by the Irish Research Council Laureate Award. The work of P.T. is supported in part by an Ussher Fellowship Award. This work was also supported in part by the National Science Foundation under Grant No. NSF PHY-1748958. M.K. and A.P. thank KITP Santa Barbara for hospitality during the completion of this work.

A Details on the conformal bootstrap

Below we review some of the details of the confomal bootstrap calculations. Explicitly, we will show that exchanges of heavy-light double-trace operators in the S-channel reproduce the disconnected correlator at $\mathcal{O}(\mu^0)$ and the stress tensor exchange at $\mathcal{O}(\mu)$.

A.1 Solving the crossing equation to $\mathcal{O}(\mu)$ in $d = 4$

We start with the leading $\mathcal{O}(\mu^0)$ term in the S-channel that should reproduce the disconnected propagator in the T-channel. This is given in $d = 4$ by

$$G(z,\bar{z})|_{\mu^0} = \frac{C_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L - 2} (h - \bar{h}) (z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}). \tag{A.1}$$

Let us look at the following piece of (A.1):

$$-\int_0^\infty dh \int_0^h d\bar{h}(h\bar{h})^{\Delta_L-2}(h-\bar{h})z^{\bar{h}}\bar{z}^{h+1} = -\int_0^\infty d\bar{h}\int_{\bar{h}}^\infty dh(h\bar{h})^{\Delta_L-2}(h-\bar{h})z^{\bar{h}}\bar{z}^{h+1}$$
$$= \frac{\bar{z}}{z}\int_0^\infty dh \int_h^\infty d\bar{h}(h\bar{h})^{\Delta_L-2}(h-\bar{h})z^{h+1}\bar{z}^{\bar{h}}. \tag{A.2}$$

Setting $\bar{z}/z = 1$ to leading order in the Regge limit, we find that the S-channel expression reproduces the disconnected correlator:

$$G(z,\bar{z})|_{\mu^0} = \frac{zC_{\Delta_L}}{z-\bar{z}}\int_0^\infty dh \int_0^\infty d\bar{h}(h\bar{h})^{\Delta_L-2}(h-\bar{h})z^h z^{\bar{h}}$$
$$= \frac{zC_{\Delta_L}}{z-\bar{z}}\frac{(\log\bar{z}-\log z)}{(\log z\log\bar{z})^{\Delta_L}}\Gamma(\Delta_L)\Gamma(\Delta_L-1) \simeq \frac{1}{(1-z)^{\Delta_L}(1-\bar{z})^{\Delta_L}}. \tag{A.3}$$

Notice that to arrive in the last equality we expanded $(z,\bar{z})$ around unity and substituted $C_{\Delta_L} = (\Gamma(\Delta_L)\Gamma(\Delta_L-1))^{-1}$.

Consider now the imaginary part at $\mathcal{O}(\mu)$ in the S-channel. For convenience we define

$$I^{(d=4)} \equiv \mathrm{Im}(G(z,\bar{z}))|_\mu, \tag{A.4}$$

which is then equal to:

$$I^{(d=4)} = \frac{-i\pi C_{\Delta_L}}{\sigma(e^{-\rho}-e^\rho)}$$
$$\times\int_0^\infty dh\int_0^h d\bar{h}(h\bar{h})^{\Delta_L-2}(h-\bar{h})\gamma(h,\bar{h})\Big((1-\sigma e^\rho)^{h+1}(1-\sigma e^{-\rho})^{\bar{h}} - (h\leftrightarrow\bar{h})\Big). \tag{A.5}$$

Notice that we used the variables (σ,ρ) defined as $z = 1-\sigma e^\rho$ and $\bar{z} = 1-\sigma e^{-\rho}$.

Consider the following ansatz for $\gamma = \frac{ch^a\bar{h}^b}{h-\bar{h}}$, where (a,b,c) are numbers tobe determined by the crossing equation. Substituting into (A.5) and collecting the leading singularity σ^{-k} as $\sigma\to 0$ with $k = 2\Delta_L+a+b-1$ leads to

$$I^{(d=4)}|_{\sigma^{-k}} = \frac{-ic\pi C_{\Delta_L}}{(e^{-\rho}-e^\rho)}\Big(\Gamma(\Delta_L+a-1)\Gamma(\Delta_L+b-1)(e^{(b-a)\rho}-e^{(a-b)\rho})+$$
$$+\frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+a-1}e^{-(2\Delta_L+a+b-2)\rho}{}_2F_1(\Delta_L+a-1,2\Delta_L+a+b-2,\Delta_L+a,-e^{-2\rho})$$
$$-\frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+a-1}e^{(2\Delta_L+a+b-2)\rho}{}_2F_1(\Delta_L+a-1,2\Delta_L+a+b-2,\Delta_L+a,-e^{2\rho})\Big). \tag{A.6}$$

Note that in order to do these integrals we need $\Delta_L+a>1$ and $\Delta_L+b>1$. Using the following identity of the hypergeometric function

$${}_2F_1(a,b,c,x) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}{}_2F_1\left(a,a-c+1,a-b+1,\frac{1}{x}\right)$$
$$+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}{}_2F_1\left(b,b-c+1,-a+b+1,\frac{1}{x}\right), \tag{A.7}$$

the third line in (A.6) can be simplified and we are left with

$$I^{(d=4)}|_{\sigma^{-k}} = \frac{ic\pi C_{\Delta_L}}{(e^{2\rho}-1)}\Bigg(-\Gamma(\Delta_L+a-1)\Gamma(\Delta_L+b-1)e^{(a-b+1)\rho}$$
$$+\frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+a-1}e^{-(2\Delta_L+a+b-3)\rho}{}_2F_1(\Delta_L+a-1,2\Delta_L+a+b-2,\Delta_L+a,-e^{-2\rho})$$
$$+\frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+b-1}e^{-(2\Delta_L+a+b-3)\rho}{}_2F_1(\Delta_L+b-1,2\Delta_L+a+b-2,\Delta_L+b,-e^{-2\rho})\Bigg). \tag{A.8}$$

On the other hand, the Regge limit in the T-channel is dominated by operators of maximal spin. In a holographic CFT, we have $J = 2$. If we further take the lightcone limit, $\rho \gg 1$, the dominant contribution is due to the stress tensor exchange and behaves as $\sigma^{-1}e^{-(d-1)\rho}$. To reproduce this behavior from the S-channel, we must set $a = 0$ and $b = 2$ and make an appropriate choice for the overall constant c. Substituting the designated values of (a, b, c) revals that the first term in (A.8) precisely matches the T-channel stress tensor contribution, which in the Regge limit (after analytic continuation) behaves like:

$$g_{\Delta,J} \propto \frac{1}{\sigma^{J-1}}\frac{e^{-(\Delta-3)\rho}}{(e^{2\rho}-1)} + \dots, \tag{A.9}$$

with $\Delta = d$ and $J = 2$. Furthermore, the remaining terms correspond to the exchange of operators with spin 2 and dimension $2\Delta_L + 2 + 2n$; these are the double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$.

A.2 Integrating the S-channel result at $\mathcal{O}(\mu^2)$ in $d = 4$

Below we describe how to use the results for the anomalous dimensions at $\mathcal{O}(\mu^2)$ in order to recover the imaginary part of the correlator to the same order. Using the obtained expressions for the anomalous dimensions (3.7) and (3.23), we note that the integrand in (3.10) can be written as

$$\begin{aligned} P^{(0)}\left(\gamma^{(2)} - \frac{\gamma^{(1)}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}\right) &= -\frac{35\bar{h}^3(2h-\bar{h})}{4(h-\bar{h})^3}P^{(0)} \\ &= -\frac{35h^{\Delta_L-3}\bar{h}^{\Delta_L+1}}{2\Gamma(\Delta_L-1)\Gamma(\Delta_L)}\sum_{n=0}^{\infty}\left(\frac{\bar{h}}{h}\right)^n\left(1+\frac{n}{2}\right). \end{aligned} \tag{A.10}$$

Therefore we see that (3.10) can be written as an infinite sum of integrals of the same form that appeared at $\mathcal{O}(\mu)$ in (A.5). It then follows that the full S-channel result can be integrated in order to obtain the correlator in position space. Especially, the lightcone result is obtained by setting $k = 0$ in (A.10) and taking $\rho \to \infty$ which gives

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = \frac{i35\pi\Delta_L(\Delta_L+1)}{2(\Delta_L-2)}\frac{e^{-3\rho}}{\sigma^{2\Delta_L+1}(e^{2\rho}-1)} + \dots, \tag{A.11}$$

with $\dots$ denoting terms that are subleading in the lightcone limit. The result (A.11) has a form consistent with the contribution of an operator with spin-2 and $\Delta = 6$. The full result (beyond the lightcone limit) further contains an infinite number of operators with spin-2 of dimension $\Delta = 6 + 2n$ and $\Delta = 2\Delta_L + 2n + 2$.

A.3 Solving the crossing equation to $\mathcal{O}(\mu)$ in $d = 2$

Here we review the calculations needed for the $d = 2$ case explained in appendix D. To $\mathcal{O}(\mu^0)$ the S-channel (2.16) is given by

$$G(z,\bar{z})|_{\mu^0} = \frac{1}{\Gamma(\Delta_L)^2}\int_0^\infty \int_0^h d\bar{h}(h\bar{h})^{\Delta_L-1}(z^h\bar{z}^{\bar{h}} + (z\leftrightarrow\bar{z})). \tag{A.12}$$

The integrand in (A.12) is symmetric w.r.t. $h \leftrightarrow \bar{h}$ and can thus be rewritten as

$$G(z,\bar{z})|_{\mu^0} = \frac{1}{\Gamma(\Delta_L)^2}\int_0^\infty \int_0^\infty d\bar{h}(h\bar{h})^{\Delta_L-1}z^h\bar{z}^{\bar{h}}, \tag{A.13}$$

which can easily be seen to reproduce the disconnected correlator $[(1-z)(1-\bar{z})]^{-\Delta_L}$ in the Regge limit.

As in the previous subsection we proceed to consider the imaginary part of the correlator in the S-channel expansion to $\mathcal{O}(\mu)$. Using a similar notation,

$$I^{(d=2)} \equiv \mathrm{Im}(G(z,\bar{z}))|_\mu\,, \tag{A.14}$$

combined with the ansatz $\gamma_1(h,\bar{h}) = c\,h^a\bar{h}^b$, allows us to write:

$$I^{(d=2)} = -\tfrac{ic\,\pi}{\Gamma(\Delta_L)^2}\textstyle\int_0^\infty \int_0^h d\bar{h}(h\bar{h})^{\Delta_L-1}h^a\bar{h}^b(z^h\bar{z}^{\bar{h}} + (z\leftrightarrow\bar{z})). \tag{A.15}$$

The integrals in (A.15) can be easily performed given that $a + \Delta_L > 0$ and $b + \Delta_L > 0$. Changing variables to $z = 1 - \sigma e^\rho$, $\bar{z} = 1 - \sigma e^{-\rho}$ and collecting the most singular term σ^{-k}, with $k = 2\Delta_L + a + b$, leads to

$$\begin{aligned}I^{(d=2)}|_{\sigma^{-k}} = \frac{ic\pi}{\Gamma(\Delta_L)^2}\bigg(&\Gamma(a+\Delta_L)\Gamma(b+\Delta_L)(-e^{\rho(b-a)} - e^{\rho(a-b)})\\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{-\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L, a+b+2\Delta_L, 1+a+\Delta_L, -e^{-2\rho})\\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L, a+b+2\Delta_L, 1+a+\Delta_L, -e^{2\rho})\bigg).\end{aligned} \tag{A.16}$$

Using again (A.7) we express (A.16) as follows

$$\begin{aligned}I^{(d=2)}|_{\sigma^{-k}} = \frac{ic\pi}{\Gamma(\Delta_L)^2}\bigg(&-\Gamma(a+\Delta_L)\Gamma(b+\Delta_L)e^{\rho(a-b)}\\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{-\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L, a+b+2\Delta_L, 1+a+\Delta_L, -e^{-2\rho})\\ &- \frac{\Gamma(a+b+2\Delta_L)e^{-(a+b+2\Delta_L)\rho}}{b+\Delta_L}{}_2F_1(b+\Delta_L, a+b+2\Delta_L, 1+b+\Delta_L, -e^{-2\rho})\bigg).\end{aligned} \tag{A.17}$$

In matching (A.17) with the T-channel expansion, following the same logic as in the previous subsection we deduce that $a = 0$ and $b = 1$ and fix c. The first line in (A.17) then reproduces the exchange of the stress tensor in the T-channel. The other two lines match the contribution of double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ with dimension $\Delta = 2\Delta_L + 2n + 2$ and spin 2 in the T-channel expansion.

B Details on the impact parameter representation in $d = 4$

Here we will see how the impact parameter representation in four dimensions leads to the expression for the disconnected correlator in the Regge limit, in terms of the integral over $h, \bar{h}$.

The objective of this section is to explicitly see that the disconnected contribution of the correlator in the Regge limit

$$\frac{1}{[(1-z)(1-\bar{z})]^{\Delta}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta-2} \frac{h-\bar{h}}{z-\bar{z}} \left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right), \tag{B.1}$$

can be equivalently written as

$$\int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}}\,, \tag{B.2}$$

with

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta) \int_{M^+} \frac{d^4p}{(2\pi)^4} (-p^2)^{\Delta-2} e^{-ipx} (h-\bar{h}) \delta(p \cdot \bar{e} + h + \bar{h})\, \delta\left(\frac{p^2}{4} + h\bar{h}\right). \tag{B.3}$$

where M^+ is the upper Milne wedge with $\{p^2 \leq 0,\ p^0 \geq 0\}$ and

$$C(\Delta) \equiv \frac{2^{d+1-2\Delta}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta)\Gamma(\Delta - \frac{d}{2} + 1)}, \tag{B.4}$$

with d the dimensionality of the spacetime, here $d = 4$.

In practice, we need to perform the integral over p in (B.3). To do so, we will use spherical polar coordinates and write:

$$\begin{aligned}\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3} \int_{-\infty}^{\infty} dp^0 \int_0^\infty dp^r\, (p^r)^2 \int_{-1}^{1} d(\cos\theta)\ (-p^2)^{\Delta-2}\, \theta(p^0)\theta(-p^2) \\ \times e^{ip^0x^0}\, e^{-irp^r\cos\theta} \left[\delta\left(\frac{p^0+p^r}{2} - h\right) \delta\left(\frac{p^0-p^r}{2} - \bar{h}\right) + h \leftrightarrow \bar{h}\right].\end{aligned} \tag{B.5}$$

The overall factor of (2π) is simply the result of the integration with respect to the angular variable ϕ. Next we perform the integral over $\cos\theta$:

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3} \int_{-\infty}^{\infty} dp^0 \int_0^\infty dp^r\, (p^r)^2\, (-p^2)^{\Delta-2}\, e^{ip^0x^0} \left(\frac{e^{-irp^r} - e^{irp^r}}{-irp^r}\right) \theta(p^0)\theta(-p^2)\, (\delta\,\delta), \tag{B.6}$$

where we set

$$(\delta\,\delta) \equiv \delta\left(\frac{p^0+p^r}{2} - h\right) \delta\left(\frac{p^0-p^r}{2} - \bar{h}\right) + h \leftrightarrow \bar{h}\,. \tag{B.7}$$

Notice that

$$\begin{aligned}\int_0^\infty dp^r\, \frac{p^r}{ir}\, (-p^2)^{\Delta-2}\, e^{irp^r} (\delta\,\delta) - \int_0^\infty dp^r\, \frac{p^r}{ir}\, (-p^2)^{\Delta-2}\, e^{-irp^r} (\delta\,\delta) \\ = \int_{-\infty}^{\infty} dp^r\, \frac{p^r}{ir}\, (-p^2)^{\Delta-2}\, e^{irp^r} (\delta\,\delta)\,.\end{aligned} \tag{B.8}$$

Hence we can write (B.6) as follows

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp^+\, dp^-}{2} \frac{p^+ - p^-}{i(x^+ - x^-)} (-p^2)^{\Delta-2}\, e^{\frac{i}{2}(p^+x^- + p^-x^+)}\, \theta(p^+)\theta(p^-)\, (\delta\, \delta)\,. \quad \text{(B.9)}$$

Performing the last two integrations is trivial due to the delta-functions. The result is

$$\mathcal{I}_{h,\bar{h}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)} \frac{h-\bar{h}}{i(x^+ - x^-)} (h\bar{h})^{\Delta-2} \left(e^{ihx^+} e^{i\bar{h}x^-} - e^{i\bar{h}x^+} e^{ihx^-}\right), \quad \text{(B.10)}$$

which allows us to write (B.2) as follows:

$$\int_0^{\infty} dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)} \int_0^{\infty} dh \int_0^h d\bar{h} \frac{h-\bar{h}}{i(x^+ - x^-)} (h\bar{h})^{\Delta-2} \left(z^h \bar{z}^{\bar{h}} - z^{\bar{h}} \bar{z}^h\right). \quad \text{(B.11)}$$

Here we also used the identification $(z = e^{ix^+}, \bar{z} = e^{ix^-})$.

Observe that (B.11) is equal to (B.1) in the Regge limit, where

$$\frac{z}{z-\bar{z}} \simeq \frac{1}{i(x^+ - x^-)}, \qquad \frac{\bar{z}}{z-\bar{z}} \simeq \frac{1}{i(x^+ - x^-)}\,. \quad \text{(B.12)}$$

However, when considering next order corrections in (x^+, x^-) the impact parameter represention may require corrections. Below we show that these are irrelevant for the questions we are interested in.

B.1 Exact Fourier transform

Here we will compute the Fourier transform for the S-channel expression with the identification $(z = e^{ix^+}, \bar{z} = e^{ix^-})$ and show that the leading order results in the Regge limit given in the previous section do not miss any important contributions.

The generic term in the S-channel which we would like to Fourier transform looks like:

$$\int dh\, d\bar{h}\, g(x^+, x^-) \tilde{f}(h, \bar{h})\,, \quad \text{(B.13)}$$

where

$$g(x^+, x^-) = \frac{e^{i(1+h)x^+} e^{i\bar{h}x^-} - e^{i\bar{h}x^+} e^{i(h+1)x^-}}{(e^{ix^+} - e^{ix^-})}\,, \quad \text{(B.14)}$$

and

$$\tilde{f}(h, \bar{h}) = i\pi (h\bar{h})^{\Delta-2} (h - \bar{h}) f(h, \bar{h})\,, \quad \text{(B.15)}$$

where $f(h, \bar{h})$ stands for all the contributions in the S-channel to a given order.

The Fourier transform is:

$$\int d^4x\, e^{ipx} \int dh\, d\bar{h}\, g(x^+, x^-) \tilde{f}(h, \bar{h}) = \int dh\, d\bar{h} \tilde{f}(h, \bar{h}) \int d^4x\, e^{ipx} g(x^+, x^-)\,, \quad \text{(B.16)}$$

where we simply reversed the order of integration. Our focus in what follows will be the integral:

$$I \equiv \int d^4x\, e^{ipx} g(x^+, x^-)\,. \quad \text{(B.17)}$$

Since $x^+ = t + r$ and $x^- = t - r$, it is convenient to use spherical polar coordinates to perform the integration. The angular integration over ϕ gives us an overall factor of (2π) as the integrand is independent of ϕ. Next we perform the integration over the other angular variable. Similar to what was discussed in the previous section,

$$\int_{-1}^{1} d(\cos\theta)\, e^{ip^r r\cos\theta} = \frac{e^{irp^r} - e^{-irp^r}}{irp^r}\,. \tag{B.18}$$

Combining the above we can write:

$$I = 2\pi \int_{-\infty}^{\infty} dte^{-itp^t} \int_0^{\infty} drr\, \frac{e^{irp^r} - e^{-irp^r}}{ip^r}\, g(t,r)\,. \tag{B.19}$$

It is easy to see that $g(t,r) = g(t,-r)$ and as a result:

$$\int_0^{\infty} dr\, re^{-irp^r}\, g(t,r) = -\int_{-\infty}^{0} dr\, re^{irp^r}\, g(t,r)\,, \tag{B.20}$$

which allows us to write the integral as:

$$I = 2\pi \int_{-\infty}^{\infty} \frac{dx^+ dx^-}{2} e^{ip\cdot x} \frac{x^+ - x^-}{i(p^+ - p^-)} g(x^+, x^-)\,. \tag{B.21}$$

Here $e^{ip\cdot x} = e^{-\frac{i}{2}(p^+x^- + p^-x^+)}$ and the above integral can be thought of as a two-dimensional Fourier transform.

To proceed we need the explicit form of $g(x^+, x^-)$ which we write as

$$g(x^+, x^-) = \frac{e^{ihx^+} e^{i\bar{h}x^-}}{1 - e^{-i(x^+ - x^-)}} + (x^+ \leftrightarrow x^-) \tag{B.22}$$

and then expand the denominator in the Regge limit

$$\frac{1}{1 - e^{-i(x^+ - x^-)}} = \frac{1}{i(x^+ - x^-)} \left[1 - \frac{i}{2}(x^+ - x^-) + \cdots \right]. \tag{B.23}$$

Substituting into (B.21) leads to:

$$I = 2\pi \frac{1}{(-p^+ + p^-)} \int \frac{dx^+ dx^-}{2} e^{ip\cdot x} \left\{ e^{ihx^+} e^{i\bar{h}x^-} \left[1 - \frac{i}{2}(x^+ - x^-) + \cdots \right] + (x^+ \leftrightarrow x^-) \right\}. \tag{B.24}$$

Let us compute the integral term by term. The leading term in the Regge limit yields the standard delta functions:

$$\begin{aligned} I_0 &= 2^2\pi^3 \frac{1}{p^- - p^+} \delta\left(\frac{p^+}{2} - \bar{h}\right) \delta\left(\frac{p^-}{2} - h\right) + (p^+ \leftrightarrow p^-) \\ &= 2\pi^3 \frac{1}{h - \bar{h}} \left\{ \delta\left(\frac{p^+}{2} - \bar{h}\right) \delta\left(\frac{p^-}{2} - h\right) + (p^+ \leftrightarrow p^-) \right\} \\ &= 2\pi^3 \frac{1}{h - \bar{h}} \delta(p\cdot\bar{e} + h + \bar{h}) \delta\left(\frac{p^2}{4} + h\bar{h}\right). \end{aligned} \tag{B.25}$$

The subleading terms on the other hand produce the same result except that the delta functions are replaced with derivatives of themselves with respect to $p^r = \frac{p^+ - p^-}{2}$.

Let us now consider the full result which up to an overall numerical coefficient can be written as:

$$\int dh\, d\bar{h}\, \tilde{f}(h,\bar{h}) \left(1 - \frac{\partial}{\partial p^r} + \cdots\right) \delta(p\cdot \bar{e} + h + \bar{h})\delta\left(\frac{p^2}{4} + h\bar{h}\right). \tag{B.26}$$

To evaluate the terms with derivatives of the delta function we need to integrate by parts. Now recall that we are interested in the imaginary piece of the S-channel whose leading behaviour is $\sim \sqrt{-p^2}$ (this dependence is hidden in what we called $\tilde{f}$). It is obvious that the derivatives will produce subleading terms which we are not interested in.

What about the other pieces in the S-channel which are not imaginary? To $\mathcal{O}(\mu^2)$ in this case, we know that the leading behaviour grows like $\sim (\sqrt{-p^2})^2$, so by differentiation, a term of the order $\sqrt{-p^2}$ may be produced. However, it is clear that this term will never contribute to the *imaginary* term of the S-channel (note that the coefficient in the first term in the parenthesis in (B.26) is real). We thus deduce that the subleading terms in (B.24) are irrelevant for our study.

C Impact parameter representation in general spacetime dimension d

Here we want to prove the following equation for general spacetime dimension d:

$$\mathcal{I}_{h,\bar{h}} = (z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(0)} g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}), \tag{C.1}$$

using the form of conformal blocks given in (4.6). We start with the definition of $\mathcal{I}_{h,\bar{h}}$ that is given as:

$$\mathcal{I}_{h,\bar{h}} = C(\Delta_L) \int_{M^+} \frac{d^dp}{(2\pi)^d} (-p^2)^{\Delta_L - \frac{d}{2}} e^{-ipx} (h-\bar{h}) \delta(p\cdot\bar{e} + h + \bar{h}) \delta\left(\frac{p^2}{4} + h\bar{h}\right), \tag{C.2}$$

where:

$$C(\Delta_L) \equiv \frac{2^{d+1-2\Delta_L}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta_L)\Gamma(\Delta_L - \frac{d}{2} + 1)}. \tag{C.3}$$

Using spherical coordinates we write (C.2) as:

$$\begin{aligned}\mathcal{I}_{h,\bar{h}} = C(\Delta_L) \int_{-\infty}^{\infty} dp^t e^{ip^t t} \int_0^{\infty} dp^r (p^r)^{d-2} \int_{S_{d-2}} \sin^{d-3}\phi_1 d\phi_1\, d\Omega_{d-3} \\ \times e^{-ip^r r\cos\phi_1} (-p^2)^{\Delta_L - \frac{d}{2}} \theta(-p^2)\theta(p^t) \left\{\delta\left(\frac{p^t + p^r}{2} - h\right)\delta\left(\frac{p^t - p^r}{2} - \bar{h}\right) + (h \leftrightarrow \bar{h})\right\},\end{aligned} \tag{C.4}$$

where $\Omega_{d-3} = \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d-2}{2})}$ denotes the area of the unit $(d-3)$-dimensional hypersphere.

Notice now that

$$\int_0^{\pi} \sin^{d-3}\phi_1 e^{-ip^r r\cos\phi_1} d\phi_1 = \sqrt{\pi}\Gamma\left(\frac{d}{2} - 1\right) {}_0F_1\left(\frac{d-1}{2}; -\frac{1}{4}(p^r)^2 r^2\right). \tag{C.5}$$

Substituting (C.5) back in to (C.4), one is left with integrals with respect to p^t and p^r only. These integrals are trivial due to the presence of delta functions.[3] When these integrations are done, the expression for $\mathcal{I}_{h,\bar{h}}$ is given as:

$$\mathcal{I}_{h,\bar{h}} = \frac{2^{3-d}\sqrt{\pi}}{\Gamma(\Delta_L)\Gamma(\Delta_L - \frac{d}{2}+1)} e^{it(h+\bar{h})}(h-\bar{h})^{d-2}(h\bar{h})^{\Delta_L-\frac{d}{2}} {}_0F_{1R}\left(\frac{d-1}{2}; -\frac{1}{4}(h-\bar{h})^2 r^2\right), \quad \text{(C.6)}$$

where ${}_0F_{1R}(a,x) = \Gamma(a)^{-1} {}_0F_1(a,x)$. Relations between coordinates t and r with x^+ and x^- are given as: $x^+ = t + r$ and $x^- = t - r$.

On the other hand, using the explicit form for conformal blocks (4.6) and OPE coefficients in the Regge limit (2.21) one finds that:

$$\begin{aligned}&(z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(0)} g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) \\ &= \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(\Delta_L)\Gamma(\Delta_L - \frac{d}{2}+1)} (h\bar{h})^{\Delta_L+\frac{d}{2}}(h-\bar{h})(z\bar{z})^{\frac{h+\bar{h}}{2}} C_{h-\bar{h}}^{(\frac{d}{2}-1)}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right).\end{aligned} \quad \text{(C.7)}$$

Using the relations between coordinates r, t and $z, \bar{z}$ it is easy to see that $(z\bar{z})^{\frac{h+\bar{h}}{2}} = e^{it(h+\bar{h})}$. Next, one can use the relation between Gegenbauer polynomials and hypergeometric functions:

$$C_n^{(\alpha)}(z) = \frac{(2\alpha)_n}{n!} {}_2F_1\left(-n, 2\alpha+n, \alpha+\frac{1}{2}; \frac{1-z}{2}\right), \quad \text{(C.8)}$$

which for $h - \bar{h} = l \gg 1$ gives:

$$C_l^{(\frac{d}{2}-1)}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right) = \frac{l^{d-3}}{\Gamma(d-2)} {}_2F_1\left(-l, l+d-2, \frac{d-1}{2}; \frac{1}{2} - \frac{1}{2}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right)\right). \quad \text{(C.9)}$$

With the help of the following properties of hypergeometric functions:

$$\begin{aligned}{}_2F_1(a,b,c;z) &= (1-z)^{-b} {}_2F_1\left(c-a, b, c; \frac{z}{z-1}\right), \\ \lim_{m,n\to\infty} {}_2F_1\left(m,n,b;\frac{z}{mn}\right) &= {}_0F_1(b;z).\end{aligned} \quad \text{(C.10)}$$

Using these, together with the assumption that in the Regge limit the values of $x^+ l$ and $x^- l$ are fixed constants: $x^+ l = a_1$ and $x^- l = a_2$ while $l \to \infty$, one can easily see[4] that (C.6) reproduces (C.1). This confirms the validity of the impact parameter representation.

D Anomalous dimensions of heavy-light double-trace operators in $d = 2$

The OPE data of the heavy-light double trace operators in $d = 2$ dimensions can be directly obtained from the heavy-light Virasoro vacuum block [17, 80]. For completeness, in this

[3] One only needs to remember that $h \geq \bar{h} \geq 0$.

[4] By noting that:

$$\Gamma\left(x - \frac{1}{2}\right) = 2^{2-2x}\sqrt{\pi}\frac{\Gamma(2x-1)}{\Gamma(x)}. \quad \text{(C.11)}$$

appendix we investigate the anomalous dimension of $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ in $d=2$ following the discussion in section 3. As in $d=4$, we introduce an impact parameter representation following [71]. We calculate the anomalous dimensions to $\mathcal{O}(\mu)$ by solving the crossing equation and then use the impact parameter representation to relate them to the bulk phase shift. We find a precise agreement between the two. Using the bulk phase shift we furthermore determine the anomalous dimension to second order in μ. Much of the discussion follows closely the four-dimensional case and will be briefer.

D.1 Anomalous dimensions in the Regge limit using bootstrap

The conformal blocks in two dimension are given by [73, 76]

$$g^{\Delta_{12},\Delta_{34}}_{\Delta,J}(z,\bar{z}) = k_{\Delta+J}(z)k_{\Delta-J}(\bar{z}) + (z\leftrightarrow\bar{z})\,, \tag{D.1}$$

where $k_\beta(z)$ was defined in (3.2). Similar to the four dimensional case, the blocks for heavy-light double-trace operators simplify in the heavy limit ($\Delta_H \sim C_T$)

$$g^{\Delta_{HL},-\Delta_{HL}}_{[\mathcal{O}_H\mathcal{O}_L]_{h,\bar{h}}}(z,\bar{z}) = (z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L)}(z^h\bar{z}^{\bar{h}} + (z\leftrightarrow\bar{z}))\,. \tag{D.2}$$

Inserting this form of the conformal blocks in (2.16) together with the OPE coefficients in the Regge limit (2.21) and approximating the sums with integrals, one can due to symmetry extend the region of integration and it is easily found that the disconnected correlator in the T-channel is reproduced.

Similar to the four-dimensional case the stress tensor dominates at order μ in the T-channel. The block of the stress tensor after analytic continuation in the Regge limit is given by

$$g_{T_{\mu\nu}} = \frac{24i\pi e^{-\rho}}{\sigma} + \dots\,, \tag{D.3}$$

where $\dots$ denote non-singular terms. As in the four-dimensional case, this has to be reproduced in the S-channel by the term in (2.16) proportional to $-i\pi\gamma$.

With the conformal blocks (D.2), the imaginary part in the S-channel to $\mathcal{O}(\mu)$ is given by

$$\mathrm{Im}(G(z,\bar{z}))|_\mu = -i\pi C_{\Delta_L}\int_0^\infty dh\int_0^h d\bar{h}(h\bar{h})^{\Delta_L-1}\gamma^{(1)}(h,\bar{h})\left(z^h\bar{z}^{\bar{h}} + z^{\bar{h}}\bar{z}^h\right). \tag{D.4}$$

Using the ansatz $\gamma^{(1)}(h,\bar{h}) = c_1 h^a\bar{h}^b$ we find that the T-channel contribution is reproduced for $a=0$ and $b=1$ (see appendix A.2 for details). We thus find using (2.22)

$$\gamma^{(1)} = -\frac{6\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}\lambda_{\mathcal{O}_L\mathcal{O}_L T_{\mu\nu}}}{\mu\Delta_L}\bar{h} = -\bar{h}. \tag{D.5}$$

To $\mathcal{O}(\mu^2)$ we can use (3.10) to find the following contribution to the purely imaginary terms in the S-channel

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = -i\pi C_{\Delta_L}\int_0^\infty dh\int_0^h d\bar{h}(h\bar{h})^{\Delta_L-1}\left(\gamma^{(2)} - \frac{c_1^2\bar{h}}{2}\right)(z^h\bar{z}^{\bar{h}} + z^{\bar{h}}\bar{z}^h). \tag{D.6}$$

D.2 2d impact parameter representation and relation to bulk phase shift

Similar to the four-dimensional case we introduce an impact parameter representation in order to relate the anomalous dimension with the bulk phase shift. The impact parameter representation in $d = 2$ is given by

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta_L) \int_{M^+} d^2p(-p^2)^{\Delta-1} e^{-ipx}(h-\bar{h})\delta(p\cdot\bar{e}+h+\bar{h})\,\delta\left(\frac{p^2}{4}+h\bar{h}\right), \tag{D.7}$$

with straightforward generalization of the $d = 4$ case explained above. This is again chosen such that when the impact parameter represetation is integrated over $h, \bar{h}$ the disconnected correlator is reproduced:

$$\int_0^\infty dh \int_0^h \mathcal{I}_{h,\bar{h}} = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}}. \tag{D.8}$$

The discussion of the phase shift is completely analogous to the four-dimensional case, as in (3.20) we find the following relation between the bulk phase shift and the anomalous dimension to second order in μ

$$\begin{aligned} \gamma^{(1)} &= -\frac{\delta^{(1)}}{\pi} \\ \tilde{\gamma}^{(2)} - \frac{c_1^2 p^-}{4} &= -\frac{\delta^{(2)}}{\pi}. \end{aligned} \tag{D.9}$$

In [71] the phase shift in $d = 2$ was found to be

$$\begin{aligned} \delta^{(1)} &= \frac{\pi}{2}\sqrt{-p^2}e^{-L} \\ \delta^{(2)} &= \frac{3\pi}{8}\sqrt{-p^2}e^{-L}. \end{aligned} \tag{D.10}$$

Using the identification $p^+ = 2h$ and $p^- = 2\bar{h}$ together with (3.22) we find for the anomalous dimension in the Regge limit

$$\begin{aligned} \gamma^{(1)} &= -\bar{h} \\ \gamma^{(2)} &= -\frac{1}{4}\bar{h}. \end{aligned} \tag{D.11}$$

We thus see that the first order result agrees with that obtained from bootstrap (D.5). Furthermore, the second order correction agrees also in $d = 2$ with the result (6.40) in [71].

E Discussion of the boundary term integrals

There are a few integrals containing total derivative terms that we have ignored throughout this paper and we analyze more carefully here. Let us start with a total derivative term which shows up in the real part of the correlator at $\mathcal{O}(\mu)$. It is given by:[5]

$$I_1 = \frac{1}{2}(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \int_0^{+\infty} dl \left[P^{(0)}\gamma^{(1)} g_{n+l,n}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z})\right]_{n=0}^{n\to\infty}. \tag{E.1}$$

[5] We are again using variables n and l, one can notice that $n = \bar{h}$ and $l = h - \bar{h}$. It is trivial to prove that $\partial_n = \partial_h + \partial_{\bar{h}}$.

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Let us focus on the integrand: $\left[P^{(0)}\gamma^{(1)}g^{\Delta_{HL},-\Delta_{HL}}_{n+l,n}(z,\bar{z})\right]^{n\to\infty}_{n=0}$. When $n=0$, the expression within the brackets trivially vanishes. On the other hand, when $n\to\infty$, it takes the form $n^{2\Delta_L-2}(z\bar{z})^n\times f(l)$, where f is some function of l only. We are instructed here to take the limit $n\to\infty$ independently of all other limits (recall that the Regge limit is taken after the integration). For generic values $0<(z,\bar{z})<1$ it is clear that $\lim_{n\to\infty}\left[P^{(0)}\gamma^{(1)}g^{\Delta_{HL},-\Delta_{HL}}_{n+l,n}(z,\bar{z})\right]=\lim_{n\to\infty}n^{2\Delta_L-2}(z\bar{z})^n\times f(l)\to 0$. In other words, the expression $\left[P^{(0)}\gamma^{(1)}g^{\Delta_{HL},-\Delta_{HL}}_{n+l,n}(z,\bar{z})\right]^{n\to\infty}_{n=0}\to 0$, and we conclude that the integral (E.1) does not contribute to the S-channel expansion of the correlator.

There are a few more integrals of similar kind that appear at $\mathcal{O}(\mu^2)$. We will analyse one of them here:

$$I_2=\frac{-i\pi}{2}(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dl\left[P^{(0)}(\gamma^{(1)})^2g^{\Delta_{HL},-\Delta_{HL}}_{n+l,n}(z,\bar{z})\right]^{n\to\infty}_{n=0}. \tag{E.2}$$

The same logic can be applied here. Again, the value of the expression in brackets at $n=0$ is trivially zero, while for large n it behaves like: $n^{2\Delta_L+d-4}(z\bar{z})^n\tilde{f}(l)$. As long as $(z,\bar{z})<1$, this vanishes exponentially in the limit $n\to\infty$. One concludes therefore that the integral (E.2) vanishes. The same logic is valid for all other integrals of similar total derivative terms that appear at $\mathcal{O}(\mu^2)$.

F An identity for the bulk phase shift

The aim is to elaborate on the results of [71] for the bulk phase shift in a black hole background as computed in gravity. Firstly, let us note the following identity involving hypergeometric functions:

$$\begin{aligned}\sum_{n=0}^{\infty}a(n)x^n\,{}_2F_1\left[\tau_0+2n+1,\frac{d}{2}-1,\tau_0+2n-\frac{d}{2}+3,x\right]&={}_2F_1\left[\tau_0+1,\frac{\tau_0}{2},\frac{\tau_0}{2}+2,x\right]\\ a(n)=\frac{2^{2n}}{n!}\frac{\tau_0+2}{\tau_0+2+2n}\frac{(\frac{\tau_0}{2}+1-\frac{d}{2})_n\left(\frac{\tau_0+1}{2}\right)_n}{(\tau_0+n+2-\frac{d}{2})_n},&\quad\tau_0\neq 0\,.\end{aligned}\tag{F.1}$$

Given that both sides of the equality can be expressed as an infinite series expansion around $x=0$, one simply needs to show that the expansion coefficients match to all orders in x. This is proven in appendix G.

Consider now the case $\tau_0=k(d-2)$ where $k\in N^{\star}$. Setting $x\equiv e^{-2L}$ and multiplying both sides with $e^{-[k(d-2)+1]L}$ yields:

$$\begin{aligned}\Pi_{k(d-2)+1,k(d-2)+1}(L)&=\sum_{n=0}^{\infty}\beta_n\Pi_{k(d-2)+2n+1,d-1}(L)\\ \beta(n)&\equiv\pi^{\frac{(1-k)(d-2)}{2}}\frac{a(n)}{(k(d-2)+1)_n}\frac{\Gamma\left[k(d-2)-\frac{d}{2}+2n+3\right]}{\Gamma\left[\frac{k(d-2)}{2}+2\right]}\,.\end{aligned}\tag{F.2}$$

The left hand side represents the hyperbolic space propagator for a scalar field of squared mass equal to $k(d-2)+1$ in a hyperbolic space of dimensionality $k(d-2)+1$ and is proportional to the k-th order expression for the bulk phase shift computed in gravity in [71], where

$$\delta^{(k)}(S,L) = \frac{1}{k!}\frac{2\Gamma\left(\frac{dk+1}{2}\right)}{\Gamma\left(\frac{k(d-2)+1}{2}\right)}\frac{\pi^{1+\frac{k(d-2)}{2}}}{\Gamma\left(\frac{k(d-2)}{2}+1\right)} S\,\Pi_{k(d-2)+1,k(d-2)+1}(L)\,. \tag{F.3}$$

On the other hand, the right-hand side of (F.2) expresses the k-th order term of the bulk phase shift as an infinite sum of $(d-1)$-dimensional hyperbolic space propagators for fields with mass-squared equal to $m^2 = k(d-2)+1+2n$.

It can be shown [49, 79] that the analytically continued T-channel scalar conformal block in the Regge limit behaves like:

$$g_{\Delta,J}(\sigma,\rho) = i\,c_{\Delta,J}\frac{\Pi_{\Delta-1,d-1}(\rho)}{\sigma^{J-1}}\,, \tag{F.4}$$

where

$$c_{\Delta,J} = \frac{4^{\Delta+J-1}\Gamma\left(\frac{\Delta+J-1}{2}\right)\Gamma\left(\frac{\Delta+J+1}{2}\right)}{\Gamma(\frac{\Delta+J}{2})^2}\frac{2\Gamma\left(\Delta-\frac{d}{2}+1\right)}{\pi^{1-\frac{d}{2}}\Gamma\left(\Delta-1\right)}\,. \tag{F.5}$$

Here $\Pi_{\Delta-1,d-1}$ denotes as usual the $(d-1)$-dimensional hyperbolic space propagator for a massive scalar of mass-squared $m^2 = (\Delta-1)$.

It follows that the k-th order term in the μ-expansion of the bulk phase shift in a black hole background can be expressed as an infinite sum of conformal blocks corresponding to operators of twist $\tau = \tau_0(k)+2n = k(d-2)+2n$ and spin $J=2$ in the Regge limit. In other words, we can write:

$$\begin{aligned} i\,\delta^{(k)}(S,L) &= f(k)\sum_{n=0}^{\infty}\lambda_k(n)\, g^R_{\tau_0(k)+2n+2,2}(S,L) \\ \lambda_k(n) &= a(n)\frac{2^{-4n}\left[\left(\frac{\tau_0(k)+4}{2}\right)_n\right]^2}{\left(\frac{\tau_0(k)+3}{2}\right)_n\left(\frac{\tau_0(k)+5}{2}\right)_n}, \qquad \tau_0(k) = k(d-2) \end{aligned} \tag{F.6}$$

where

$$f(k) \equiv \frac{\sqrt{\pi}}{64}\frac{1}{2^{k(d-2)}\,k!}\frac{\Gamma\left(\frac{kd+1}{2}\right)\Gamma\left(\frac{k(d-2)+4}{2}\right)}{\Gamma\left(\frac{k(d-2)+5}{2}\right)\Gamma\left(\frac{k(d-2)+3}{2}\right)}\,, \tag{F.7}$$

and

$$g^R_{\Delta,J}(S,L) = ic_{\Delta,J}\,S^{J-1}\,\Pi_{\Delta-1,d-1}(L)\,. \tag{F.8}$$

G An identity for hypergeometric functions

Here we will show that for $q \neq 0$,

$$\begin{aligned} &\sum_{n=0}^{\infty} a(n)x^n\,{}_2F_1\left[q+2n+1,\frac{d}{2}-1,q+2n-\frac{d}{2}+3,x\right] = {}_2F_1\left[q+1,\frac{q}{2},\frac{q}{2}+2,x\right] \\ &a(n) = \frac{2^{2n}}{n!}\frac{q+2}{q+2+2n}\frac{(\frac{q}{2}+1-\frac{d}{2})_n\left(\frac{q+1}{2}\right)_n}{(q+n+2-\frac{d}{2})_n}, \quad q\neq 0\,. \end{aligned} \tag{G.1}$$

Given that both sides of the equality can be expressed as an infinite series expansion around $x = 0$, one simply needs to show that the expansion coefficients match to all orders in x. Let us first set:

$$b(n,m) \equiv \frac{1}{m!}\frac{(q+1+2n)_m\left(\frac{d}{2}-1\right)_m}{\left(q-\frac{d}{2}+2n+3\right)_m}$$
$$c(\ell) \equiv \frac{1}{\ell!}\frac{(q+1)_\ell\left(\frac{q}{2}\right)_\ell}{\left(\frac{q}{2}+2\right)_\ell} = \frac{(q+1)_\ell}{\ell!}\frac{q(q+2)}{(q+2\ell)(q+2\ell+2)}\,, \tag{G.2}$$

such that:

$${}_2F_1[q+2n+1,\frac{d}{2}-1,q+2n-\frac{d}{2}+3,x] = \sum_{m=0}^{\infty} b(n,m)x^m,$$
$${}_2F_1[q+1,\frac{q}{2},\frac{q}{2}+2,x] = \sum_{\ell=0}^{\infty} c(\ell)x^\ell. \tag{G.3}$$

It is easy to check that the coefficients of the first few powers of x precisely match. Indeed, e.g.,

$$a(0)b(0,0) - c(0) = 0$$
$$a(1)b(1,0) + a(0)b(0,1) - c(1) = 0 \tag{G.4}$$
$$a(2)b(2,0) + a(1)b(1,1) + a(0)b(0,2) - c(2) = 0.$$

To show that the above identity is true for all powers of x we must show that:

$$\sum_{k=0}^{\ell} a(k)b(k,\ell-k) = c(\ell)\,, \tag{G.5}$$

for all $\ell \in N$. The left-hand side of (G.5) can be easily summed to yield:

$$\sum_{k=0}^{\ell} a(k)b(k,\ell-k) = \frac{1}{\ell!}\frac{\Gamma[q+1+\ell]}{\Gamma[q]}\frac{(q+2)}{(q+2\ell)(2+2\ell+q)}\,, \tag{G.6}$$

which can be trivially shown to be equal to $c(\ell)$.

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JHEP10(2019)046

Convergence of the Gradient Expansion in Hydrodynamics

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(Received 6 May 2019; published 28 June 2019)

Hydrodynamic excitations corresponding to sound and shear modes in fluids are characterized by gapless dispersion relations. In the hydrodynamic gradient expansion, their frequencies are represented by power series in spatial momenta. We investigate the analytic structure and convergence properties of the hydrodynamic series by studying the associated spectral curve in the space of complexified frequency and complexified spatial momentum. For the strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills plasma, we use the holographic duality methods to demonstrate that the derivative expansions have finite nonzero radii of convergence. Obstruction to the convergence of hydrodynamic series arises from level crossings in the quasinormal spectrum at complex momenta.

DOI: 10.1103/PhysRevLett.122.251601

Introduction.—Hydrodynamics is an established universal language for describing near-equilibrium phenomena in fluids [1]. The equations of hydrodynamics are the local conservation laws which can be written as

$$\partial_t \rho_a + \nabla \cdot \mathbf{J}_a = 0, \tag{1}$$

where ρ_a are the densities of locally conserved charges (energy, momentum, particle number, etc.) and $\mathbf{J}_a$ are the corresponding fluxes. The conservation equations (1) can be solved once the fluxes $\mathbf{J}_a$ are expressed in terms of the densities ρ_a through the so-called constitutive relations, $\mathbf{J}_a = \mathbf{J}_a(\rho)$. Conventionally, one works with the quantities ϕ_a such as temperature, fluid velocity, and the chemical potential which are conjugate to ρ_a in the grand canonical ensemble. The constitutive relations $\rho_a = \rho_a(\phi)$, $\mathbf{J}_a = \mathbf{J}_a(\phi)$ are then used in the conservation laws (1) in order to determine the macroscopic space-time evolution of the fluid [1].

There are two basic physics principles that constrain possible forms of the constitutive relations: symmetry and the derivative expansion. Symmetry is what distinguishes different types of fluids. The derivative expansion is a reflection of the fact that hydrodynamics is only an effective description on length scales much larger than the microscopic scale (such as the mean free path). Thus, the constitutive relations are schematically written as

$$\rho_a = O(\phi) + O(\nabla\phi) + O(\nabla^2\phi) + \cdots, \tag{2a}$$

$$\mathbf{J}_a = O(\phi) + O(\nabla\phi) + O(\nabla^2\phi) + \cdots, \tag{2b}$$

which is a derivative (gradient) expansion. For normal fluids, truncating the expansions (2) at $O(\phi)$ [i.e., neglecting the terms $O(\nabla\phi)$ and higher] gives rise to perfect fluids and Euler equations of hydrodynamics. Truncating the expansions at $O(\nabla\phi)$ gives rise to viscous fluids and Navier-Stokes equations. Truncating at $O(\nabla^2\phi)$ gives rise to second-order hydrodynamics and Burnett equations, and so on.

The naive expectation is that going to higher orders in the derivative expansion improves the hydrodynamic description of the fluid, similar to how the Navier-Stokes equations improve the perfect-fluid approximation by including the viscous effects. The purpose of this Letter is thus to address the following foundational question: viewed as an expansion in small gradients, does the hydrodynamic derivative expansion in fact converge?

In order to make this question precise, we will choose a specific physical quantity whose exact value can be compared with the prediction of the derivative expansion. For fluids, the characteristic feature of the hydrodynamic description is the existence of gapless modes: small near-equilibrium fluctuations of the fluid whose frequencies $\omega_i(q)$ are such that $\omega_i(q) \to 0$ as the magnitude of the wave vector $q \to 0$. The well-known example is the sound wave whose dispersion relation is $\omega_{\text{sound}}(q) = \pm v_s q + O(q^2)$, where v_s is the speed of sound. More generally, the hydrodynamic prediction is that

$$\omega_i(q) = \sum_{n=1}^{\infty} b_n^{(i)} q^n, \tag{3}$$

where, in principle, all orders in the derivative expansion (2) contribute to the dispersion relation of the ith mode. In this Letter, we shall investigate whether the infinite power series expansions (3) converge, and if so, what determines their radii of convergence. Recently, the convergence of the shear-diffusion mode series in $d = 2 + 1$ was investigated in a holographic model with nonzero chemical potential [2]. Using the coefficients of the series (3) and Padé approximants, Ref. [2] found an obstruction to the convergence in the form of a branch point at purely imaginary momentum and identified this singularity as the collision of two gapped quasinormal modes. Here, we show how branch point singularities generically arise from the spectral curves in classical hydrodynamics.

Besides the general physics interest in the foundations of hydrodynamics, our motivation comes from the success of the relativistic hydrodynamic framework to describe the quark-gluon plasma produced in the collisions of heavy nuclei [3]. Similarly, in the examples of strongly interacting quantum field theories whose nonequilibrium evolution can be determined from first principles using holographic methods, the hydrodynamic description appears to be unexpectedly robust, even when the gradients are large [4,5]. If the expansions (3) indeed converge, this convergence would be a step towards understanding the “unreasonable effectiveness” of the hydrodynamic description of the quark-gluon plasma and of similar strongly interacting holographic fluids [6–9].

An important comment has to be made before we proceed. The expansion (3) is a prediction of classical hydrodynamics, which neglects the effects of statistical fluctuations. As is well known [10], fluctuations lead to infinitely many fractional powers of q appearing in $\omega_i(q)$, thereby rendering the expansion (3) insufficient. While one may rightly question the applicability of classical second- and higher-order hydrodynamics to the quark-gluon plasma on these grounds [11], one should keep in mind that the complete effective description of the fluid involves both the classical hydrodynamics and the fluctuation effects. Our focus here is on the classical hydrodynamics part, with the understanding that the fluctuation effects are to be included later. In holographic hydrodynamics, these effects are suppressed in the large-N limit of the corresponding models [12].

Hydrodynamic modes from complex curves.—In order to understand the origin of the series (3), consider the constitutive relations (2) truncated at a finite order k in the derivative expansion. Suppose there are m hydrodynamic variables ϕ_a, with $a = 1, \ldots, m$. The hydrodynamic equations (1) linearized near equilibrium can be written as $\mathcal{L}_a[\delta\phi] = 0$, where $\mathcal{L}_a$ are linear differential operators of order at most $k+1$. Upon Fourier transforming the linearized fluctuations, $\delta\phi \propto \exp(-i\omega t + i\mathbf{q}\cdot\mathbf{x})$, the frequencies are determined by the eigenvalue equation $\det L_{ab}(\omega, \mathbf{q}) = 0$, where L_{ab} is an $m \times m$ matrix whose elements are polynomials in ω and $\mathbf{q}$. Assuming a rotation-invariant equilibrium state, the eigenvalue equation can only depend on $\mathbf{q}^2$ and ω, and can be written as $P_k(\mathbf{q}^2, \omega) = 0$, where P_k is a polynomial in $\mathbf{q}^2$ and ω. Continuing the gradient expansion, we set $P(\mathbf{q}^2, \omega) = \lim_{k\to\infty} P_k(\mathbf{q}^2, \omega)$. The all-order eigenvalue equation is then

$$P(\mathbf{q}^2, \omega) = 0. \tag{4}$$

It is useful to treat $z \equiv \mathbf{q}^2$ and ω as complex variables. The eigenvalue equation $P(z, \omega) = 0$ then defines a complex spectral curve in $\mathbb{C}^2$. Regular points of the curve satisfy the condition of the analytic implicit function theorem [13],

$$P(z, \omega) = 0, \qquad \frac{\partial P(z, \omega)}{\partial \omega} \neq 0, \tag{5}$$

which guarantees analyticity and uniqueness of the branch $\omega = \omega(z)$ in the vicinity of a regular point. Of particular interest are the so-called critical points, i.e., the points (z_c, ω_c) where, in addition to $P(z_c, \omega_c) = 0$, the first $(p-1)$ derivatives with respect to ω vanish,

$$\frac{\partial P(z_c, \omega_c)}{\partial \omega} = 0, \ldots, \frac{\partial^p P(z_c, \omega_c)}{\partial \omega^p} \neq 0. \tag{6}$$

Assuming the analyticity of $P(z, \omega)$ at (z_c, ω_c), the generalization of the implicit function theorem guarantees the existence of p branches $\omega_j = \omega_j(z)$, $j = 1, \ldots, p$, represented by Puiseux series [series in fractional powers of $(z - z_c)$] converging in the vicinity of the branch point $z = z_c$ [14]. The hydrodynamic (gapless) dispersion relations are defined implicitly by Eq. (4). They arise as m functions $\omega_i = \omega_i(z)$ satisfying $\omega_i(z \to 0) = 0$, $i = 1, \ldots, m$, and $P(z, \omega(z)) = 0$.

Relativistic hydrodynamics.—In what follows, we focus on relativistic hydrodynamics for concreteness [15], and set $\hbar = c = 1$. In first-order hydrodynamics of an uncharged fluid in $3+1$ dimensions, one finds

$$P_1(\mathbf{q}^2, \omega) = (\omega + iD\mathbf{q}^2)^2(\omega^2 + i\Gamma\omega\mathbf{q}^2 - v_s^2\mathbf{q}^2) = 0, \tag{7}$$

where $v_s = (\partial p/\partial\epsilon)^{1/2}$ is the speed of sound, p and ϵ are the equilibrium pressure and energy density, $D = \eta/(\epsilon + p)$ is the diffusion coefficient of the transverse velocity, $\Gamma = [(4/3)\eta + \zeta]/(\epsilon + p)$ is the damping coefficient of sound waves, η is the shear viscosity, and ζ is the bulk viscosity. More generally, one can show that the eigenvalue equation (4) factorizes as $P(\mathbf{q}^2, \omega) = F_{\text{shear}}^2 F_{\text{sound}}$, with

$$F_{\text{shear}} \equiv \omega + i\mathbf{q}^2\gamma_\eta(\mathbf{q}^2, \omega) = 0, \tag{8}$$

$$F_{\text{sound}} \equiv \omega^2 + i\omega\mathbf{q}^2\gamma_s(\mathbf{q}^2, \omega) - \mathbf{q}^2 H(\mathbf{q}^2, \omega) = 0. \tag{9}$$

This factorization is a consequence of rotation invariance. In the derivative expansion, the functions γ_η, γ_s, and H are given by power series around $(z,\omega)=(0,0)$.

For example, assuming analyticity of the function γ_η at $(z,\omega)=(0,0)$, the origin (0,0) is a regular point of the complex curve (8). The implicit function theorem then implies that the shear dispersion relation is given by a series in powers of $\mathbf{q}^2$ converging in the vicinity of $\mathbf{q}^2=0$,

$$\mathfrak{w}_{\rm shear} = -i\sum_{n=1}^{\infty} c_n \mathfrak{q}^{2n} = -ic_1\mathfrak{q}^2+\cdots, \qquad (10)$$

where we defined $\mathfrak{w}\equiv\omega/2\pi T$, $\mathfrak{q}\equiv|\mathbf{q}|/2\pi T$, and $c_1=2\pi TD$. On the other hand, for the curve (9) we have $\partial F_{\rm sound}/\partial\omega=0$ but $\partial^2F_{\rm sound}/\partial\omega^2\neq 0$ at (0,0)—again, assuming analyticity of the functions γ_s and H at the origin. In this case, the two branches $\omega^\pm=\omega^\pm(\mathbf{q}^2)$ are given by Puiseux series in powers of $q\equiv(\mathbf{q}^2)^{\frac{1}{2}}$,

$$\mathfrak{w}^{\pm}_{\rm sound} = -i\sum_{n=1}^{\infty} a_n e^{\pm(i\pi n/2)}\mathfrak{q}^n = \pm a_1\mathfrak{q}+ia_2\mathfrak{q}^2+\cdots, \qquad (11)$$

converging in the vicinity of $\mathbf{q}^2=0$. The dimensionless coefficients are $a_1=v_s$, $a_2=-\Gamma\pi T$. We expect the radius of convergence of the hydrodynamic dispersion relation series to be determined by the distance from the origin to the nearest critical point (6).

The main example.—An example of a quantum field theory in which the dispersion relations $\omega_i(q)$ can be analyzed to all orders in q is the $\mathcal{N}=4$ supersymmetric $SU(N)$ Yang-Mills (SYM) theory at infinitely large ‘t Hooft coupling and infinite N. This theory has been extensively studied through the use of holographic methods in various contexts [16,17], including as a model of collective properties of quantum chromodynamics in the deconfined phase [18]. The theory is conformal, and the only dimensionful scale is the equilibrium temperature T that sets the finite correlation length. In any conformal theory, we have $a_1=1/\sqrt{3}$, $a_2=-(2/3)c_1$, and the bulk viscosity $\zeta=0$. In $\mathcal{N}=4$ SYM theory, we further have $c_1=1/2$ due to the universal holographic relation $\eta/s=1/4\pi$ (where s is the density of entropy) [19].

Holography.—To compute the coefficients c_n and a_n in $\mathcal{N}=4$ SYM theory, we use the holographic duality to map the quantum field-theoretic problem into a calculation in classical general relativity. The duality implies that the hydrodynamic modes (3) coincide with the gapless quasinormal modes of black branes in one higher dimension [20,21]. The relevant gravitational perturbations of the black brane are described by two functions: $Z_1(u)$ (the shear mode) and $Z_2(u)$ (the sound mode), where u is the radial coordinate ranging from $u=0$ (asymptotic boundary) to $u=1$ (event horizon) [21]. The shear mode equation is

$$Z_1'' - \frac{(\mathfrak{w}^2-\mathfrak{q}^2f)f-u\mathfrak{w}^2f'}{uf(\mathfrak{w}^2-\mathfrak{q}^2f)}Z_1' + \frac{\mathfrak{w}^2-\mathfrak{q}^2f}{uf^2}Z_1 = 0, \qquad (12)$$

where $f(u)=1-u^2$. The sound mode equation is

$$Z_2'' - \frac{3\mathfrak{w}^2(1+u^2)+\mathfrak{q}^2(2u^2-3u^4-3)}{uf[3\mathfrak{w}^2+\mathfrak{q}^2(u^2-3)]}Z_2' + \frac{3\mathfrak{w}^4+\mathfrak{q}^4(3-4u^2+u^4)+\mathfrak{q}^2(4u^5-4u^3+4u^2\mathfrak{w}^2-6\mathfrak{w}^2)}{uf^2[3\mathfrak{w}^2+\mathfrak{q}^2(u^2-3)]}Z_2 = 0. \qquad (13)$$

Both equations have to be solved with the boundary condition $Z_i(u)\sim(1-u)^{-i\mathfrak{w}/2}$ as $u\to 1$, corresponding to the infalling wave at the horizon. Near the boundary $u=0$, the two independent solutions have exponents 0 and 2. Hence, the solution satisfying the infalling condition at the horizon can be written near the boundary as $Z_i(u)\sim\mathcal{A}_i(1+\cdots)+\mathcal{B}_iu^2+\cdots$, where the dots denote higher powers of u, and $\mathcal{A}_i$, $\mathcal{B}_i$ are the two integration constants which depend on $\mathfrak{w}$ and $\mathfrak{q}^2$. The Dirichlet condition [the holographic analogue of Eqs. (8) and (9)],

$$\mathcal{A}_i(\mathfrak{q}^2,\mathfrak{w}) = 0, \qquad (14)$$

relates $\mathfrak{w}$ to $\mathfrak{q}^2$ and gives the dispersion relations of the hydrodynamic and other (gapped) modes [21]. We note that the analyticity of the coefficient $\mathcal{A}_i(\mathfrak{q}^2,\mathfrak{w})$ at the origin is not *a priori* guaranteed, and thus it is not obvious that the hydrodynamic series in holography have a nonzero radius of convergence. In order to find the coefficients c_n, a_n in the expansions (10), (11), we must solve Eqs. (12), (13). We do this by constructing the Frobenius series solution at $u=1$, and truncating the series at a sufficiently high order [21].

The results are shown in Fig. 1. The plots indicate that the coefficients c_n and a_n decrease exponentially with n, and therefore the convergence radii are nonzero for both the shear and the sound modes. Finite radii of convergence of the series (10) and (11) imply the existence of singularities in the complex $\mathfrak{q}$-plane obstructing analyticity. The absolute value of the critical $\mathfrak{q}_i^{\rm c}$ that sets the finite radius of convergence is determined by the slope, and the argument of $\mathfrak{q}_i^{\rm c}$ is determined by the period of the oscillations in Fig. 1. While the critical values $\mathfrak{q}_i^{\rm c}$ can be extracted from these data by fitting the coefficients to exponential functions of n with complex exponents, a more precise way to

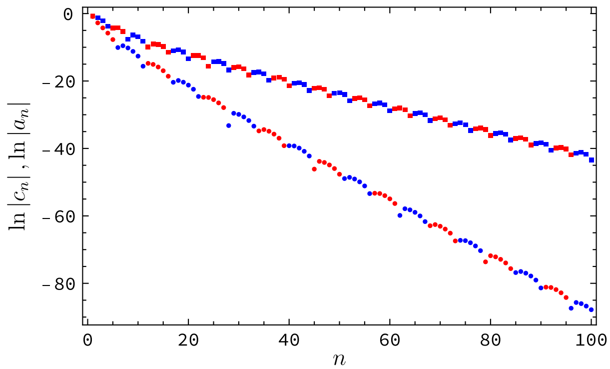

FIG. 1. Coefficients of the expansions (10) and (11) in $\mathcal{N} = 4$ SYM theory. The circles are $\ln |c_n|$ (shear mode), the squares are $\ln |a_n|$ (sound mode). Red (blue) indicate positive (negative) values of c_n or a_n.

find $\mathfrak{q}_i^c$ is by solving the set of equations [the holographic analogue of Eq. (6)]

$$\mathcal{A}_i(\mathfrak{q}^2, \mathfrak{w}) = 0, \qquad \frac{\partial \mathcal{A}_i(\mathfrak{q}^2, \mathfrak{w})}{\partial \mathfrak{w}} = 0, \tag{15}$$

which determines the critical points. The Frobenius expansion around the horizon gives $\mathcal{A}_i(\mathfrak{q}^2, \mathfrak{w})$ as explicit algebraic functions of $\mathfrak{w}$ and $\mathfrak{q}^2$, and Eqs. (15) can be solved numerically. For the shear mode, we find two pairs $(\mathfrak{q}_c^2, \mathfrak{w}_c)$ with

$$\mathfrak{q}_c^2 \approx 1.8906469 \pm 1.1711505i, \tag{16a}$$

$$\mathfrak{w}_c \approx \pm 1.4436414 - 1.0692250i, \tag{16b}$$

corresponding to the convergence radius of the shear mode dispersion relation $|\mathfrak{q}^c_{\text{shear}}| \approx 1.49131$. For the sound mode, we find, similarly,

$$\mathfrak{q}_c^2 = \pm 2i, \qquad \mathfrak{w}_c = \pm 1 - i, \tag{17}$$

within the limits of our numerical accuracy. One can check that the values (17) indeed satisfy Eq. (14), with a simple analytic solution for $Z_2(u)$. This corresponds to the convergence radius of the sound mode dispersion relation $|\mathfrak{q}^c_{\text{sound}}| = \sqrt{2} \approx 1.41421$. The values of $\mathfrak{q}_c$ for the shear and sound modes are not equal, but are quite close. Thus the slopes of the two lines in Fig. 1 differ by approximately a factor of 2, as the shear mode frequency is expanded in $\mathfrak{q}^2$, while the sound mode frequency is expanded in $(\mathfrak{q}^2)^{1/2}$.

Quasinormal spectrum level crossing.—The origin of the critical values (16) and (17) can be understood if we

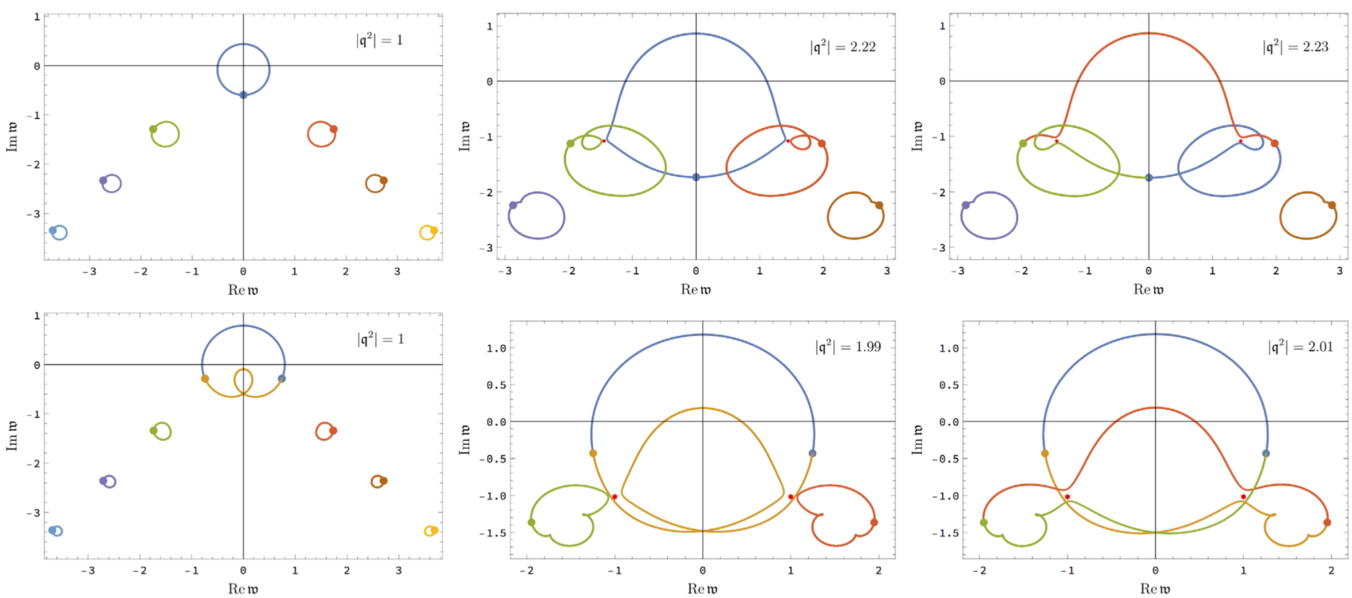

FIG. 2. Poles of the retarded two-point function of the energy-momentum tensor in the complex $\mathfrak{w}$ plane, at various values of the complexified momentum $\mathfrak{q}^2 = |\mathfrak{q}^2| e^{i\theta}$. Top row is the shear channel, bottom row is the sound channel. Large dots correspond to the location of the poles for real $\mathfrak{q}^2$ ($\theta = 0$) [21]. The hydrodynamic shear and sound poles are the poles closest to the real axis in the top left and bottom left panels, correspondingly. As θ increases from 0 to 2π, each pole moves counterclockwise, following the trajectory of its color. In the shear channel (top row), at $|\mathfrak{q}^2| = 1$, each pole follows a closed orbit (top left). At $|\mathfrak{q}^2| = 2.22$ (top center), the hydrodynamic pole almost collides with the two gapped poles closest to the real axis. The actual collision would happen at the critical momentum (16), $|\mathfrak{q}_c^2| \approx 2.224$, with the corresponding frequencies marked by red asterisks in the figure. At $|\mathfrak{q}^2| = 2.23$ (top right), the orbits of the three uppermost poles are no longer closed: the hydrodynamic pole and the two gapped poles exchange their positions cyclically as the phase θ increases from 0 to 2π. Similar behavior is observed for the sound mode (bottom row). The dispersion relations $\mathfrak{w}_i(\mathfrak{q})$ thus have branch cuts starting at $\mathfrak{q}_c$.

consider the singularities (poles) of the retarded two-point correlation functions of the energy-momentum tensor for complex $\mathfrak{w}$, $\mathfrak{q}$. The poles, or quasinormal frequencies, determined by Eq. (14), contain the hydrodynamic modes $\mathfrak{w}_i(\mathfrak{q})$ as well as an infinite tower of gapped modes $\mathfrak{w}_n^{\text{gapped}}(\mathfrak{q})$, such that $\mathfrak{w}_n^{\text{gapped}}(\mathfrak{q} \to 0) \neq 0$ [21]. Consider now the locations of the poles $\mathfrak{w}(\mathfrak{q})$ in the complex $\mathfrak{w}$ plane as the phase θ of the complex momentum $\mathfrak{q}^2 = |\mathfrak{q}^2| e^{i\theta}$ changes from 0 to 2π, as illustrated in Fig. 2. At small $|\mathfrak{q}^2|$, the poles (whose original location at $\theta = 0$ is indicated by large dots) move along simple curves, as shown in the left-most panels of Fig. 2. With $|\mathfrak{q}^2|$ increasing, the trajectories of the poles exhibit more complicated behavior. The poles effectively "interact" with each other, and at special values of $\mathfrak{q} = \mathfrak{q}_c$, the hydrodynamic poles collide with one of the gapped-mode poles. This is illustrated in Fig. 2, where we show the trajectories of the poles just before and just after the collision, with the collision points marked by asterisks. The figures clearly show that the collision of poles happens at the critical values given by Eqs. (16), (17) when the hydrodynamic poles transform into one of the former gapped poles. With $|\mathfrak{q}^2|$ further increasing, other poles from the infinite tower of gapped quasinormal modes become involved. By analogy with quantum mechanics, we call this phenomenon the quasinormal spectrum level crossing. Thus, the radius of convergence of the hydrodynamic series $|\mathfrak{q}_c|$ can be viewed as the absolute value of (complex) $\mathfrak{q}$ with the smallest possible $|\mathfrak{q}|$ at which the hydrodynamic pole collides with a gapped pole.

Discussion.—We have shown that the gradient expansions for the hydrodynamic shear and sound frequencies in the strongly coupled $\mathcal{N} = 4$ SYM theory have finite radii of convergence given by $q^c_{\text{sound}} = \sqrt{2}(2\pi T)$ for the sound mode, and by $q^c_{\text{shear}} \approx 1.49(2\pi T)$ for the shear mode. In general, all-order hydrodynamics gives rise to convergent dispersion relations (3), provided the analyticity of the corresponding spectral curves at the origin is established independently. While the radius of convergence could, in principle, be infinite, in the example of $\mathcal{N} = 4$ SYM theory, it was limited by the collision of the poles of the two-point correlation function of the energy-momentum tensor at complex q. This obstruction to convergence would be invisible had we only considered real values of q.

Returning to the question of the unreasonable effectiveness of hydrodynamics, we note that the derivative expansion in relativistic hydrodynamics has been previously argued to diverge [22]. This is based on assuming that the fluid undergoes a one-dimensional expansion, such that all quantities only depend on proper time τ. One assumes that an expansion of the fluid energy density in powers of $\tau^{-2/3}$ can be performed, identifying this as a gradient expansion. This large-τ expansion is divergent in holographic models [22], and in the Müller-Israel-Stewart (MIS) extension of hydrodynamics [23]. The gradient expansion we are considering here is different and thus our results do not contradict [22,23]. Our interest is in the near-equilibrium spatial gradient expansion, rather than in the boost-invariant flow. In the same MIS theory where the large-τ expansion diverges, the small-q expansions of Eq. (3) converge [24]. Here we have shown that in the same strongly coupled $\mathcal{N} = 4$ SYM theory where the large-τ expansion diverges, the small-q expansions of Eq. (3), again, converge. Therefore one must be careful not to identify the divergence of the large-τ expansion with the failure of the hydrodynamic gradient expansion. The common conclusion of our results and those of Ref. [22] is that the presence of the nonhydrodynamic degrees of freedom (the gapped quasinormal modes) is what sets the limit on the applicability of hydrodynamics.

Finally, we comment on the dependence of the radii of convergence on coupling, limiting the discussion to the sound mode in first-order hydrodynamics, where the first nontrivial critical point occurs at $|\mathfrak{q}^c_{\text{sound}}| = v_s/\pi T\Gamma$. For conformal theories $v_s = 1/\sqrt{d}$, and at infinitely strong coupling, dual gravity approximation gives $|\mathfrak{q}^c_{\text{sound}}| = 2\sqrt{d}/(d-1) = \sqrt{3}$ in $d = 3$, not too far from the correct value in Eq. (17). For $\mathcal{N} = 4$ SYM theory, taking into account the leading order correction to the shear viscosity-entropy density ratio at large but finite 't Hooft coupling λ [25], we find $|\mathfrak{q}^c_{\text{sound}}| = \sqrt{3}[1 - 15\zeta(3)\lambda^{-3/2} + \cdots]$. Therefore, it appears that the radius of convergence is smaller at weaker coupling, in line with the earlier observations regarding validity of hydrodynamics at finite coupling [9,26].

S. G. was supported by the U.S. DOE Grant No. DE-SC0011090. P. K. was supported in part by NSERC of Canada. The work of P. T. is supported by an Ussher Fellowship from Trinity College Dublin.

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Published for SISSA by Springer

Received: *June 28, 2019*
Revised: *October 5, 2019*
Accepted: *November 3, 2019*
Published: *November 18, 2019*

The complex life of hydrodynamic modes

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Abstract: We study analytic properties of the dispersion relations in classical hydrodynamics by treating them as Puiseux series in complex momentum. The radii of convergence of the series are determined by the critical points of the associated complex spectral curves. For theories that admit a dual gravitational description through holography, the critical points correspond to level-crossings in the quasinormal spectrum of the dual black hole. We illustrate these methods in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 3+1 dimensions, in a holographic model with broken translation symmetry in 2+1 dimensions, and in conformal field theory in 1+1 dimensions. We comment on the pole-skipping phenomenon in thermal correlation functions, and show that it is not specific to energy density correlations.

Keywords: Black Holes in String Theory, Effective Field Theories, Gauge-gravity correspondence, Holography and quark-gluon plasmas

ArXiv ePrint: 1904.12862

https://doi.org/10.1007/JHEP11(2019)097

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1 Introduction

Hydrodynamics is an effective theory of fluids valid at sufficiently long times and sufficiently large distances. Classical hydrodynamics is formulated by combining conservation laws for energy, momentum, and other charges (such as the particle number in non-relativistic systems) with the constitutive relations expressing the conserved fluxes in terms of the hydrodynamic variables: local temperature, fluid velocity, and charge density [1]. The constitutive relations are written down on the basis of symmetry, using the derivative expansion. The constitutive relations which contain no derivatives of the hydrodynamic variables are said to describe "perfect fluids", corresponding to "zeroth-order" hydrodynamics. The constitutive relations which contain up to one derivative of the hydrodynamic variables are commonly said to describe "viscous fluids", corresponding to "first-order" or "Navier-Stokes" hydrodynamics. One can proceed by adding terms with more derivatives of the hydrodynamic variables to the constitutive relations, hoping to improve the hydrodynamic description of the actual physical fluids. The constitutive relations which contain up to n derivatives of the hydrodynamic variables then give rise to n^{th}-order hydrodynamics. In this paper, we will use simple facts from the theory of complex curves in order to study some aspects of convergence of the above derivative expansion. Our focus will be on relativistic fluids, and we shall illustrate general results with the examples of strongly interacting quantum field theories possessing a dual holographic description.

A neutral homogeneous and isotropic relativistic fluid whose energy-momentum tensor is conserved supports collective excitations in the form of shear and sound hydrodynamic modes [2]. The collective modes arise from the analysis of linearised fluctuations of energy and momentum densities around the equilibrium state at temperature T. The mode's frequency ω is related to the wave vector $\mathbf{q}$ by the dispersion relation $\omega = \omega(\mathbf{q})$. In hydrodynamics, the dispersion relations are given by

$$\text{Shear mode:} \qquad \omega_{\text{shear}}(\mathbf{q}) = -iD\mathbf{q}^2 + \cdots \,, \tag{1.1}$$

$$\text{Sound mode:} \qquad \omega^{\pm}_{\text{sound}}(\mathbf{q}) = \pm v_s|\mathbf{q}| - i\frac{\Gamma_s}{2}\,\mathbf{q}^2 + \cdots \,, \tag{1.2}$$

where v_s is the speed of sound, and D and Γ_s can be expressed through relevant transport coefficients. In d_s spatial dimensions, we have

$$D = \frac{\eta}{\epsilon + p}\,, \tag{1.3}$$

$$\Gamma_s = \frac{1}{\epsilon + p}\left[\zeta + \frac{2d_s - 2}{d_s}\,\eta\right], \tag{1.4}$$

where ϵ and p are the equilibrium energy density and pressure, and η and ζ are the shear and bulk viscosities. The shear mode (1.1) describes diffusion of the transverse (to the wave vector) velocity fluctuations which are damped by the shear viscosity. The sound mode (1.2) describes propagation of the longitudinal velocity fluctuations together with the fluctuations in the energy density and pressure. The terms written down in (1.1) and (1.2) represent the contributions from first-order hydrodynamics, while the ellipses denote terms with higher powers of $\mathbf{q}$, which can be matched to transport coefficients in second- and higher-order hydrodynamics [3–5]. To all orders in the hydrodynamic derivative expansion, the dispersion relations (1.1), (1.2) are represented by infinite series in $\mathbf{q}$.

Are these hydrodynamic series convergent? If so, what are their radii of convergence and what determines them? If the series are only asymptotic, can they be resummed? The shear mode (1.1) *appears* to be an expansion in powers of $\mathbf{q}^2$, whereas the sound

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mode (1.2) seems to contain odd and even powers of $|\mathbf{q}| = \sqrt{\mathbf{q}^2}$ in its real and imaginary part, respectively. Can we prove that no other power of $\mathbf{q}^2$ or non-analytic terms appears in the hydrodynamic expansions at *any* order?[1] These questions were considered in our recent short paper [8], and we shall provide more details here.

In general, proving convergence of a perturbative series and finding the corresponding radius of convergence may constitute rather difficult (and rather different) problems. For example, the convergence of the so called $1/Z$ series representing the lowest energy eigenvalue of the two-electron atom with the nucleus of charge Z was rigorously proven by Kato in 1951 [9] but reliably computing the actual value of the radius of convergence from the series coefficients and their Padé extensions remained a controversial problem for many decades [10, 11]. Yet another example is the series solution to Kepler's equation whose "mysterious" convergence properties were discussed by the giants such as Laplace and Cauchy and were finally understood as arising from the critical points of the associated curve in the complex eccentricity plane (see appendix C).

The problems involving re-summing all-order hydrodynamic expansions and finding the radius of convergence of hydrodynamic series have been previously addressed in the literature. All-order linearised hydrodynamics was investigated by Bu and Lublinsky [12, 13] using fluid-gravity correspondence. Re-summations of the hydrodynamic derivative expansion have been also considered in the framework of kinetic theory [14] and in cosmological models [15]. By far the largest body of work on the subject has been done in the setting of the boost-invariant flow, where the gradient expansion in the position space typically produces asymptotic rather than convergent series, and the Borel-Padé and "resurgence" methods [16] can be used to re-sum the series and extract information about gapped modes in the spectrum from the hydrodynamic series [17–26]. In ref. [27], Withers studied the convergence properties and analytic continuation of a dispersion relation for the shear-diffusion mode in a holographic model involving a dual Reissner-Nordstrom-AdS$_4$ black brane. There, a finite radius of convergence resulted from a branch point singularity at a certain value of the purely imaginary momentum. From the point of view of the quasinormal spectrum, this point corresponds to the collision of the modes or level-crossing, very similar to the discussion in ref. [8] and in the present paper.

The prediction of classical hydrodynamics is that the above frequencies $\omega(\mathbf{q})$ appear as poles of the retarded two-point functions[2] of the energy-momentum tensor in thermal equilibrium, as $\mathbf{q} \to 0$ [2, 28]. Assuming that the prediction of classical hydrodynamics is correct and that the actual response functions computed from quantum field theory (viewed as functions of ω) indeed have isolated poles at $\omega = \omega(\mathbf{q})$ with $\omega(\mathbf{q} \to 0) \to 0$, we can define the function $\omega(\mathbf{q})$ as the location of the relevant pole. In what follows, we shall discuss several models where the poles of the full response functions can be readily analysed, both analytically and numerically, for generic values of $\mathbf{q}$, either real or complex,

[1] In this paper, we consider classical hydrodynamics only, ignoring the effects of statistical fluctuations such as those discussed for example in refs. [6, 7]. Such fluctuation effects are suppressed in the holographic models we study below.

[2] We shall often call the retarded two-point functions "response functions", as they form the basis of the linear response theory.

small or large. This will allow us to study the analytic properties of the derivative expansion in hydrodynamics by studying the dispersion relations $\omega(\mathbf{q})$ obtained from the poles of the relevant retarded functions in thermal equilibrium.

In general, the hydrodynamic dispersion relations arise as solutions to

$$P(\mathbf{q}^2, \omega) = 0\,, \tag{1.5}$$

where P is proportional to the inverse of the corresponding retarded two-point function.[3] The hydrodynamic dispersion relations $\omega(\mathbf{q})$ are solutions to (1.5) obeying $\omega(\mathbf{q} \to 0) \to 0$, where $\mathbf{q}^2$ and ω are treated as complex variables. We shall refer to $P(\mathbf{q}^2, \omega)$ as the hydrodynamic spectral curve. In order to obtain $P(\mathbf{q}^2, \omega)$ from classical hydrodynamics, we choose a set of hydrodynamic variables φ_a (such as the fluid velocity and temperature), and linearise the hydrodynamic equations about the equilibrium state, $\varphi_a = \varphi_a^{(\mathrm{eq})} + \delta\varphi_a$. In the absence of external sources, the hydrodynamic equations are homogeneous and, upon Fourier transforming, can be written as a set of linear algebraic equations, $K_{ab}\delta\varphi_b = 0$. The hydrodynamic spectral curve is then simply $P = \det K$.

In n^{th}-order (classical) hydrodynamics, $P(\mathbf{q}^2, \omega)$ is a polynomial of a finite order, and eq. (1.5) defines a complex algebraic curve. The theorems of analysis such as the implicit function theorem and the theorem of Puiseux then determine the structure and properties of $\omega(\mathbf{q})$. In particular, these theorems guarantee that for any finite order of the derivative expansion, the dispersion relations $\omega(\mathbf{q})$ are given by series converging in some vicinity of the origin $(\mathbf{q}^2, \omega) = (0, 0)$, and the same is true as $n \to \infty$, provided $P(\mathbf{q}^2, \omega)$ is an analytic function at $(0, 0)$. We discuss the spectral curve and the associated dispersion relations of the hydrodynamic modes in section 2 of the paper.

In addition to the spectral curve, we shall also study the retarded functions of the energy-momentum tensor. As a simple example, the prediction of 1^{st}-order hydrodynamics for the retarded function of the transverse momentum density is [2]

$$G^R_{\perp,\perp}(\mathbf{q}^2, \omega) = \frac{(\epsilon + p)D\mathbf{q}^2}{i\omega - D\mathbf{q}^2}\,, \tag{1.6}$$

where $D = \eta/(\epsilon + p)$, as before. When viewed as a function of ω, there is a simple pole given by the shear mode dispersion relation (1.1). When viewed as a function of two variables $\mathbf{q}^2$ and ω, one can immediately see that the point $(\mathbf{q}^2_*, \omega_*) = (0, 0)$ is a singular point of $G^R(\mathbf{q}^2, \omega)$, such that the value of the response function at $(\mathbf{q}^2_*, \omega_*)$ is undefined. This is commonly expressed by saying that the limits $\omega \to 0$ and $\mathbf{q} \to 0$ do not commute. Such indeterminacy-type singularities in physical response functions can also exist at finite non-zero $(\mathbf{q}^2_*, \omega_*)$. This was recently explored for the sound mode (retarded function of the energy density) in refs. [29–31], where the phenomenon of the indeterminacy-type singularities at non-zero $(\mathbf{q}^2_*, \omega_*)$ was called "pole-skipping". Put differently, pole-skipping happens when $P(\mathbf{q}^2_*, \omega_*) = 0$, and the residue of the corresponding pole of $G^R(\mathbf{q}^2, \omega)$ vanishes at $(\mathbf{q}^2_*, \omega_*)$. In the example of eq. (1.6), the "skipping" of the shear pole happens at the origin.

[3] Here, the proportionality is assumed to be trivial in the sense that the equations $G_R^{-1} = 0$ and (1.5) are equivalent.

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A study of pole-skipping at non-zero $(\mathbf{q}_*^2, \omega_*)$ will be another focus of our paper.[4] In fact, the original motivation for our work was to see whether the pole-skipping singularities in response functions and the non-zero radius of convergence of the hydrodynamic derivative expansion might be related to each other.

To illustrate our approach in a simple exactly solvable example, in section 3 we discuss the holographic bottom-up model studied, in particular, by Davison and Goutéraux in ref. [33]. The advantage of the model is that the effects of translation symmetry breaking on hydrodynamics can be studied in a controlled manner, and that the hydrodynamic and non-hydrodynamic degrees of freedom can be easily separated. The explicit breaking of the translation symmetry means that momentum is no longer conserved, and the sound mode is absent from the spectrum as $\mathbf{q} \to 0$. Thus, the only modes with $\omega(\mathbf{q} \to 0) = 0$ are the diffusive modes of the energy and charge densities. For a certain special value of the translation symmetry breaking parameter in the model, the response functions of the energy and charge densities can be found exactly for all (not just small) momenta [33]. One then finds the following dispersion relation for the diffusive modes:

$$\mathfrak{w}(\mathfrak{q}) = -\frac{i}{2}\left(1 - \sqrt{1 - 4\mathfrak{q}^2}\right) = -i\mathfrak{q}^2 - i\mathfrak{q}^4 + \cdots . \tag{1.7}$$

It is clear that the corresponding hydrodynamic series converges in the circle $|\mathfrak{q}| < |\mathfrak{q}_c| = 1/2$ due to the branch point singularities at $\mathfrak{q}_c = \pm 1/2$. We shall study the exact and approximate spectral curves and obstruction to convergence in this model in section 3.

Our main example, considered in section 4, is the $\mathcal{N} = 4$ supersymmetric $SU(N_c)$ Yang-Mills theory at infinite N_c and infinite 't Hooft coupling, which we will abbreviate as "strongly coupled $\mathcal{N} = 4$ SYM theory". In this theory, real-time equilibrium correlation functions can be calculated by the methods of holographic duality [34, 35] (see for example refs. [36–39] for an introduction to the holographic duality and applications). The dispersion relations of the shear and sound modes in the strongly coupled $\mathcal{N} = 4$ SYM theory obtained by holographic methods are shown in figure 1. Using the units $\hbar = c = 1$, we plot the dispersion relations in terms of dimensionless variables $\mathfrak{w} \equiv \omega/2\pi T$ and $\mathfrak{q} \equiv |\mathbf{q}|/2\pi T$. The function $\mathfrak{w}(\mathfrak{q})$ for the shear mode is purely imaginary for real $\mathfrak{q}$, while $\mathfrak{w}(\mathfrak{q})$ for the sound mode has both real and imaginary parts for real $\mathfrak{q}$. The functions $\mathfrak{w}(\mathfrak{q})$ in figure 1 appear to be smooth and generally unremarkable functions of real positive $\mathfrak{q}$. Their behaviour at $\mathfrak{q} \ll 1$ has a clear hydrodynamic interpretation [40, 41] fully compatible with eqs. (1.1), (1.2), and their asymptotics as $\mathfrak{q} \to \infty$ were studied in refs. [42, 43]. Thus, at least in the case of the $\mathcal{N} = 4$ SYM theory, if the series (1.1), (1.2) have finite radii of convergence, the obstacle to convergence must lie either at negative $\mathfrak{q} = \sqrt{\mathfrak{q}^2}$, or more generally, at complex momenta. By studying complex $\mathfrak{q}$, one indeed finds that the functions $\mathfrak{w}(\mathfrak{q})$ in the $\mathcal{N} = 4$ SYM theory have finite non-zero radii of convergence: $|\mathfrak{q}_c^{\text{sound}}| = \sqrt{2}$, and $|\mathfrak{q}_c^{\text{shear}}| \approx 1.49$ [8]. In section 4, we show in detail how the finite radius of convergence arises from the quasinormal mode level-crossing in the shear and sound channels,

[4] The connection between the pole-skipping values $(\mathbf{q}_*^2, \omega_*)$ and the growth of the out-of-time-ordered four-point correlation functions (OTOC) of local operators in quantum field theory has been studied in refs. [29–32].

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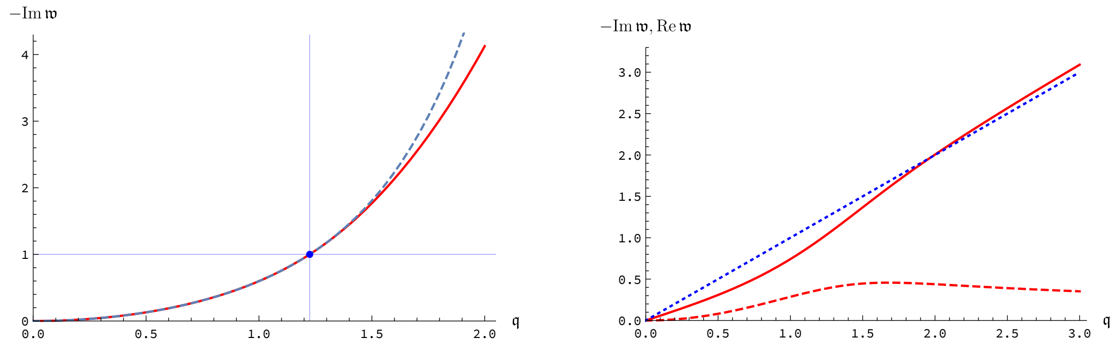

Figure 1. Dispersion relations of the hydrodynamic modes in the strongly coupled $\mathcal{N} = 4$ SYM theory, obtained using the dual holographic description. The dispersion relations are plotted in terms of dimensionless $\mathfrak{w} \equiv \omega/2\pi T$ and $\mathfrak{q} \equiv |\mathbf{q}|/2\pi T$, with complex $\mathfrak{w}$ as functions of real positive $\mathfrak{q}$. The left panel shows $\mathfrak{w}_{\text{shear}}(\mathfrak{q})$ for the shear mode, the right panel shows $\mathfrak{w}^{+}_{\text{sound}}(\mathfrak{q})$ for one of the two sound modes. In the left panel, the actual $-\mathrm{Im}\,\mathfrak{w}_{\text{shear}}(\mathfrak{q})$ for the shear mode is shown by the solid red curve, and the analytic hydrodynamic approximation to $O(\mathfrak{q}^8)$ (computed in section 4.1) is shown by the dashed blue curve. The blue dot indicates the pole-skipping point at $\mathfrak{q}_* = \sqrt{3/2}$, $\mathfrak{w}_* = -i$, discussed in section 4.7. The right panel shows $\mathrm{Re}\,\mathfrak{w}^{+}_{\text{sound}}(\mathfrak{q})$ (solid red curve) and $-\mathrm{Im}\,\mathfrak{w}^{+}_{\text{sound}}(\mathfrak{q})$ (dashed red curve) for the "+" sound mode. The straight dotted line indicates the light cone $\mathrm{Re}\,\mathfrak{w} = \mathfrak{q}$.

and demonstrate that the level-crossing phenomenon is also observed in the scalar channel of the correlators.

One of our main results concerns the response functions of the energy-momentum tensor in the strongly coupled $\mathcal{N} = 4$ SYM theory. It was shown in refs. [29, 31] that there is a pole-skipping singularity in the retarded two-point function of the energy density at $(\mathfrak{q}_*^2, \mathfrak{w}_*) = (-3/2, i)$, at which point the sound pole is "skipped". The sound dispersion curves pass through the point $(\mathfrak{q}_*^2, \mathfrak{w}_*)$, as illustrated in figure 2. We observe that in close analogy with the sound mode, the shear mode dispersion relation (1.1) passes through the point $\mathfrak{q}_*^2 = 3/2$, $\mathfrak{w}_* = -i$ (see figure 1). It turns out that this is not an accident: we will show that the pole-skipping phenomenon in the strongly coupled $\mathcal{N} = 4$ SYM theory is exhibited not only by the response functions which give rise to the sound mode (energy density correlations), but also by the response functions which give rise to the shear mode (transverse momentum density correlations). Moreover, we find that the pole-skipping at non-zero complex momentum also occurs in response functions of those components of the energy-momentum tensor that are not related to hydrodynamic modes. For example, for $\mathbf{q}$ along the z direction, the fluctuations of T^{xy} are gapped, and the response function of T^{xy} has no hydrodynamic singularities. Nevertheless, the gapped singularities of the retarded function of T^{xy} pass through $(\mathfrak{q}_*^2, \mathfrak{w}_*) = (-3/2, -i)$, at which point one of the gapped poles is "skipped". This is illustrated in figure 3. In other words, all retarded functions of $T^{\mu\nu}$ in strongly coupled $\mathcal{N} = 4$ SYM theory exhibit pole-skipping at $|\mathfrak{q}_*| = \sqrt{3/2}$ and $|\mathfrak{w}_*| = 1$. The retarded functions of the energy-momentum tensor and the convergence of the derivative expansion in the $\mathcal{N} = 4$ SYM theory are discussed in section 4.

A natural question to ask is whether pole-skipping happens within the domain of validity of the hydrodynamic approximation, as far as the convergence of the hydrodynamic

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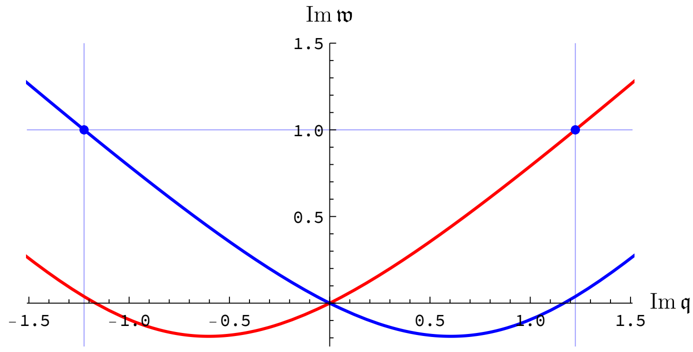

Figure 2. The analytically continued sound mode frequencies in the strongly coupled $\mathcal{N}=4$ SYM theory, obtained using the dual holographic description. The dimensionless frequencies $\mathfrak{w}^{\pm}_{\text{sound}}$ of the two sound modes are plotted for purely imaginary dimensionless spatial momentum $\mathfrak{q}$, with the "+" branch in red and the "−" branch in blue. The frequencies $\mathfrak{w}^{\pm}_{\text{sound}}$ are purely imaginary at imaginary $\mathfrak{q}$. At small momenta, the curves are linear with slopes $\pm v_s$, with $v_s = 1/\sqrt{3}$. The curves pass through pole-skipping points $(\mathfrak{q}_*, \mathfrak{w}_*) = (\pm i\sqrt{3/2}, i)$ indicated by the blue dots.

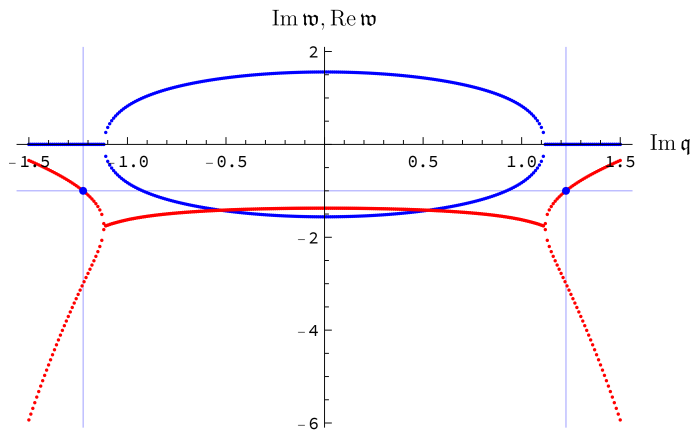

Figure 3. The first two (closest to the origin) poles of the retarded function of T^{xy} in the strongly coupled $\mathcal{N}=4$ SYM theory, obtained using the dual holographic description. The locations of the poles are plotted as functions of the dimensionless wave vector for $\mathfrak{q}$ purely imaginary, with $\text{Re}\,\mathfrak{w}$ shown in blue, and $\text{Im}\,\mathfrak{w}$ shown in red. The dots indicate the points $(\mathfrak{q}_*, \mathfrak{w}_*) = (\pm i\sqrt{3/2}, -i)$, where the response function of T^{xy} exhibits pole-skipping.

derivative series is concerned. In other words, if the hydrodynamic dispersion relation $\mathfrak{w}_i(\mathfrak{q})$ has a finite radius of convergence $|\mathfrak{q}^i_{\text{c}}|$ and pole-skipping in the corresponding response function happens at $|\mathfrak{q}^i_*|$, how does $|\mathfrak{q}^i_{\text{c}}|$ compare with $|\mathfrak{q}^i_*|$? In the strongly coupled $\mathcal{N}=4$ SYM theory we have $|\mathfrak{q}^{\text{sound}}_{\text{c}}| = \sqrt{2}$, $|\mathfrak{q}^{\text{shear}}_{\text{c}}| \approx 1.49$ [8], and $|\mathfrak{q}_*| = \sqrt{3/2}$. Thus $|\mathfrak{q}_*| < |\mathfrak{q}_{\text{c}}|$, and therefore pole-skipping in correlation functions takes place within the convergence domain of the hydrodynamic derivative expansion. On the other hand, in the model of ref. [33], we have $|\mathfrak{q}_{\text{c}}| = 1/2$, $|\mathfrak{q}_*| = \sqrt{2}$, hence $|\mathfrak{q}_*| > |\mathfrak{q}_{\text{c}}|$, so pole-skipping occurs outside the convergence domain of the hydrodynamic derivative expansion. This indicates that the "skipping" of hydrodynamic poles in retarded functions of energy and momentum

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densities is not directly related to the convergence radius of the derivative expansion in hydrodynamics.

More generally, pole-skipping singularities in real-time response functions at non-zero momentum do not have to have any relation to hydrodynamics at all. As an example, we consider response functions of spin-zero operators in 1+1 dimensional conformal field theory (CFT). For a primary operator, the Euclidean correlation function on $\mathbb{R}^2$ is fixed by conformal symmetry. Performing a conformal transformation to the cylinder $\mathbb{R} \times \mathbb{S}^1$ gives Euclidean thermal correlations functions [44], which can be Fourier transformed and analytically continued to produce exact real-time retarded functions $G^R(\mathfrak{w}, \mathfrak{q})$ [34]. These functions have no hydrodynamic poles, yet we will see that there is an infinite number of pole-skippings at non-zero values of $(\mathfrak{q}_*, \mathfrak{w}_*)$. We discuss this in detail in section 5. Our conclusions and discussion of the issues raised in the paper appear in section 6.

2 Hydrodynamic dispersion relations as Puiseux series

2.1 Hydrodynamic spectral curves

We start with a brief review of how the hydrodynamic dispersion relations are derived. Consider hydrodynamics of a neutral homogeneous and isotropic relativistic fluid in flat space in d_s spatial dimensions. We are interested in linearised fluctuations in a homogeneous and isotropic equilibrium state, $T^{\mu\nu} = T^{\mu\nu}_{\text{eq.}} + \delta T^{\mu\nu}$, where $T^{\mu\nu}$ denotes the expectation value of the symmetric energy-momentum tensor operator, and the equilibrium state is characterised by $T^{00}_{\text{eq.}} = \epsilon$, $T^{ij}_{\text{eq.}} = p\delta^{ij}$, $T^{0i}_{\text{eq.}} = 0$, where ϵ and p are the equilibrium energy density and pressure. The equations of hydrodynamics follow from the conservation of the energy-momentum tensor, $\partial_\mu T^{\mu\nu} = 0$. Translation invariance of the equilibrium state implies that we can Fourier transform the fluctuations and take all variables to be proportional to $\exp(-i\omega t + i\mathbf{q}\cdot\mathbf{x})$. Furthermore, rotation invariance allows us to choose the direction of the z axis along $\mathbf{q}$. We then have the following system of conservation equations for the linearised fluctuations:

$$-\omega\, \delta T^{0a} + q_z\, \delta T^{za} = 0\,, \tag{2.1a}$$
$$-\omega\, \delta T^{00} + q_z\, \delta T^{z0} = 0\,, \tag{2.1b}$$
$$-\omega\, \delta T^{0z} + q_z\, \delta T^{zz} = 0\,, \tag{2.1c}$$

where we use the index a and subsequent Latin indices to denote any of the $d_s - 1$ spatial directions orthogonal to z.

The above conservation equations need to be supplemented by the constitutive relations which express $\delta T^{\mu\nu}$ in terms of the hydrodynamic degrees of freedom. For linearised hydrodynamics, a convenient choice of the degrees of freedom is the energy density δT^{00} and momentum density δT^{0i}. This choice implies that we only need the constitutive relations for the spatial stress, $\delta T^{ij} = \delta T^{ij}(\delta T^{00}, \delta T^{0k})$. The constitutive relations will contain derivatives of δT^{00} and δT^{0k}, as is needed for example to describe the viscosity of fluids. We will organise the constitutive relations according to the number of derivatives of the hydrodynamic variables. The hydrodynamics of k-th order is determined by the constitutive relations in which δT^{ij} contains up to k derivatives of δT^{00} and δT^{0i}. It is then straightforward to write down the linearised constitutive relations at any order, by noting that under the spatial SO(d_s), the stress fluctuation δT^{ij} is a rank-two tensor, momentum density δT^{0i} is a vector, and the energy density δT^{00} is a scalar. For example, in the first-order hydrodynamics of ref. [1], we have

$$\begin{aligned}\delta T^{ij} &= \delta^{ij}\frac{\partial p}{\partial \epsilon}\,\delta T^{00}\\ &\quad -\frac{1}{\epsilon+p}\left[\eta\left(\partial_i\,\delta T^{0j}+\partial_j\,\delta T^{0i}-\frac{2}{d_s}\delta^{ij}\partial_k\delta T^{0k}\right)+\zeta\delta^{ij}\partial_k\,\delta T^{0k}\right]+\cdots\,,\end{aligned} \tag{2.2}$$

where η is the shear viscosity, ζ is the bulk viscosity, and the ellipses denote terms with more than one derivative of δT^{00}, δT^{0i}. Combining the constitutive relations (2.2) with the conservation equations (2.1) gives a system of linear equations for the fluctuations δT^{00} and δT^{0i}. The equations have non-trivial solutions provided the corresponding determinant vanishes:

$$P_1(\mathbf{q}^2,\omega)\equiv\left(\omega+iD\mathbf{q}^2\right)^{d_s-1}\left(\omega^2+i\Gamma_s\omega\mathbf{q}^2-v_s^2\mathbf{q}^2\right)=0\,, \tag{2.3}$$

where $v_s^2=\partial p/\partial\epsilon$ is the speed of sound squared, and D, Γ_s are defined by eqs. (1.3), (1.4).

In fact, rotation invariance implies that the most general linearised constitutive relations in momentum space take the following form:

$$\begin{aligned}\delta T^{nm} =& -iA\left(q^n\delta T^{0m}+q^m\delta T^{0n}\right)+\delta T^{00}\left(Bq^nq^m+C\delta^{nm}\right)\\ &+iq_l\delta T^{0l}\left(Dq^nq^m+E\delta^{nm}\right)\,,\end{aligned} \tag{2.4}$$

where A, B, C, D, E are scalar functions of ω and $\mathbf{q}^2$. Substituting the constitutive relations (2.4) into the conservation equations (2.1), we find a system of d_s+1 linear equations for d_s+1 hydrodynamic variables. This system has non-trivial solutions provided the determinant of the corresponding matrix vanishes. The vanishing of the determinant is equivalent to the vanishing of

$$P(\mathbf{q}^2,\omega)\equiv F_{\text{shear}}^{d_s-1}F_{\text{sound}}\,, \tag{2.5}$$

where

$$F_{\text{shear}}(\mathbf{q}^2,\omega)\equiv\omega+i\mathbf{q}^2\gamma_\eta(\mathbf{q}^2,\omega)=0\,, \tag{2.6}$$

$$F_{\text{sound}}(\mathbf{q}^2,\omega)\equiv\omega^2+i\omega\mathbf{q}^2\gamma_s(\mathbf{q}^2,\omega)-\mathbf{q}^2H(\mathbf{q}^2,\omega)=0\,. \tag{2.7}$$

Here the coefficients are $\gamma_\eta\equiv A$, $\gamma_s\equiv 2A-E-D\mathbf{q}^2$, $H=B\mathbf{q}^2+C$. Thus, the shear and the sound modes decouple as a consequence of rotation invariance.[5]

[5]See refs. [12, 13] for a study of "resummed" holographic hydrodynamics to all orders in the derivative expansion. In our language, this amounts to studying the functions $\gamma_\eta(\mathbf{q}^2,\omega)$, $\gamma_s(\mathbf{q}^2,\omega)$, $H(\mathbf{q}^2,\omega)$ in a holographic model.

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If the constitutive relations (2.4) are given by a local derivative expansion, then the functions $\gamma_\eta(\mathbf{q}^2,\omega)$, $\gamma_s(\mathbf{q}^2,\omega)$ and $H(\mathbf{q}^2,\omega)$ are power series in ω and $\mathbf{q}^2$ with finite values at $\omega=0$, $\mathbf{q}^2=0$:

$$\gamma_\eta(0,0)=D\,,\qquad \gamma_s(0,0)=\Gamma_s\,,\qquad H(0,0)=v_s^2\,, \tag{2.8}$$

with v_s, D and Γ_s as above. Truncating the derivative expansion at order k then gives a sequence of algebraic equations defined by finite polynomials in $\mathbf{q}^2$ and ω,

$$F^{(k)}_{\text{shear}}(\mathbf{q}^2,\omega)=0\,, \tag{2.9}$$

$$F^{(k)}_{\text{sound}}(\mathbf{q}^2,\omega)=0\,. \tag{2.10}$$

For general complex ω and $\mathbf{q}^2$, eqs. (2.6), (2.7), or (2.9), (2.10) define complex algebraic curves[6,7] which we will call hydrodynamic spectral curves. The complete dispersion relations of the i-th mode, $\omega_i=\omega_i(\mathbf{q}^2)$, can be obtained by solving eqs. (2.6), (2.7) for ω, while the corresponding approximate expressions arising in k-th order hydrodynamics are found from eqs. (2.9), (2.10). The hydrodynamic dispersion relations are the solutions which satisfy $\omega_i(\mathbf{q}^2\to 0)=0$. They correspond to infinite relaxation times for infinite-wavelength perturbations of conserved densities, i.e. to the conservation of energy and momentum.

Note that the polynomials $F^{(k)}_{\text{shear}}(\mathbf{q}^2,\omega)$, $F^{(k)}_{\text{sound}}(\mathbf{q}^2,\omega)$ are not defined uniquely because of the freedom to organise the derivative expansion in hydrodynamics, such as the choice of "frame" and the use of on-shell conditions [2, 46]. As an example, the conservation equations (2.1) imply that the factors of ω in the constitutive relations (2.4) can be eliminated at each order in the derivative expansion. Thus one can organise the derivative expansion in such a way that the functions $\gamma_\eta(\mathbf{q}^2,\omega)$, $\gamma_s(\mathbf{q}^2,\omega)$ and $H(\mathbf{q}^2,\omega)$ are all ω-independent at each given order in the expansion. Then eqs. (2.9), (2.10) give simple explicit expressions for $\omega_{\text{shear}}(\mathbf{q}^2)$ and $\omega_{\text{sound}}(\mathbf{q}^2)$ in terms of three scalar functions $\gamma_\eta(\mathbf{q}^2)$, $\gamma_s(\mathbf{q}^2)$ and $H(\mathbf{q}^2)$, whose small-$\mathbf{q}$ limits are given by eq. (2.8). In this way of implementing the derivative expansion, the hydrodynamic dispersion relations are the only solutions to (2.9), (2.10). Of course, other choices of organising the derivative expansion are possible where the ω-dependence in $\gamma_\eta(\mathbf{q}^2,\omega)$, $\gamma_s(\mathbf{q}^2,\omega)$ and $H(\mathbf{q}^2,\omega)$ is retained, and non-hydrodynamic (gapped) modes appear in addition to the hydrodynamic modes.

The above discussion was in the context of classical hydrodynamics. A similar factorisation of the shear and sound modes happens in the full response functions of the energy-momentum tensor, without any hydrodynamic assumptions [41]. For example, for the wave vector along z, the shear mode is described by fluctuations of δT^{0a}, where the direction a is orthogonal to z. The condition that the inverse of the equilibrium response function of the T^{0a} operator vanishes can be written as $P_{\text{shear}}(\mathbf{q}^2,\omega)=0$. In general, $P_{\text{shear}}(\mathbf{q}^2,\omega)$ is a complicated function which describes both hydrodynamic (long-distance, long-time) and non-hydrodynamic (short-distance, short-time) physics. For small (appropriately defined) $\mathbf{q}^2$ and ω, the exact $P_{\text{shear}}(\mathbf{q}^2,\omega)$ will reduce to the above $F_{\text{shear}}(\mathbf{q}^2,\omega)$,

[6] Eq. (2.5) is an example of a reduced curve $f(x,y)=\prod_i g_i(x,y)$, where each g_i can be considered independently [45].

[7] The equations $F^{d_s-1}_{\text{shear}}=0$ and $F_{\text{shear}}=0$ define the same curve. To avoid any confusion, by the "shear curve", we shall always mean the definition $F_{\text{shear}}=0$.

provided hydrodynamics is a valid effective description of the system. The same applies to the sound mode: the condition that the inverse of the equilibrium response function of the T^{00} operator vanishes can be written as $P_{\text{sound}}(\mathbf{q}^2,\omega) = 0$. For small (relative to the appropriately defined scale) $\mathbf{q}^2$ and ω, the exact $P_{\text{sound}}(\mathbf{q}^2,\omega)$ will reduce to the above $F_{\text{sound}}(\mathbf{q}^2,\omega)$, provided hydrodynamics is a valid effective description of the system. In this paper, we will only be studying physical systems in which the near-equilibrium physics governed by the conserved densities can be described by classical hydrodynamics. In other words, we will assume that the functions $\gamma_\eta(\mathbf{q}^2,\omega)$, $\gamma_s(\mathbf{q}^2,\omega)$, $H(\mathbf{q}^2,\omega)$ are defined by the exact response functions of the $T^{\mu\nu}$ operator.

2.2 Small-momentum expansions

For a given spectral curve, the small-$\mathbf{q}^2$ expansion of the hydrodynamic dispersion relation $\omega_i(\mathbf{q}^2)$ can be found using the theorem of Puiseux (see appendix A and refs. [45, 47, 48]).

Starting with the shear mode, let us assume that $\gamma_\eta(\mathbf{q}^2,\omega)$ is analytic at $(\mathbf{q}^2,\omega)$=(0,0) and thus eq. (2.6) defines an analytic curve for complex $(\mathbf{q}^2,\omega) \in \mathbb{C}^2$. This is of course not guaranteed *a priori* and should be established by independent methods, e.g. by finding the exact response function.[8] The analyticity of $\gamma_\eta(\mathbf{q}^2,\omega)$ implies that $F_{\text{shear}}(\mathbf{q}^2,\omega)$ is analytic at the origin as well, and, since $\partial F_{\text{shear}}/\partial\omega = 1 \neq 0$ at $(0,0)$, the origin is a regular point of the analytic curve (2.6). Then the analytic implicit function theorem (see appendix A) guarantees that for sufficiently small $\mathbf{q}^2$ and ω, there exists a unique solution of eq. (2.6) of the form

$$\omega_{\text{shear}}(\mathbf{q}^2) = -i\sum_{n=1}^{\infty} c_n \mathbf{q}^{2n} = -ic_1\mathbf{q}^2 + O(\mathbf{q}^4)\,, \tag{2.11}$$

convergent in a neighbourhood of $\mathbf{q}^2 = 0$. The radius of convergence is determined by the location of the nearest to the origin critical point of the curve (2.6).

Continuing with the sound mode, let us again assume that $\gamma_s(\mathbf{q}^2,\omega)$ and $H(\mathbf{q}^2,\omega)$ are analytic functions at $(0,0)$. The function $F_{\text{sound}}(\mathbf{q}^2,\omega)$ is then analytic at the origin as well. Now we have $\partial F_{\text{sound}}/\partial\omega = 0$ at $(0,0)$, and thus the origin is a critical point of the spectral curve. On the other hand, $\partial^2 F_{\text{sound}}/\partial\omega^2 = 2 \neq 0$ at $(0,0)$, thus we expect the sound dispersion relation to have $p = 2$ branches. The Puiseux series expansions are then given by eqs. (A.6), (A.7), in other words $\omega^{(j)}_{\text{sound}}(\mathbf{q}^2)$ can be represented by series in non-negative powers of $(\mathbf{q}^2)^{1/m_j}$, where m_j are positive integers, and $j = 1, 2$ labels the two branches corresponding to the two sound modes. Following the general analysis of algebraic curves, the integers m_j may be found using the Newton's polygon method (see refs. [45, 47, 48] for details). For analytic γ_s and H, we have the expansions

$$\gamma_s(\mathbf{q}^2,\omega) = \sum_{n,m=0}^{\infty} \gamma^s_{nm}\omega^n\mathbf{q}^{2m}\,, \tag{2.12}$$

$$H(\mathbf{q}^2,\omega) = \sum_{n,m=0}^{\infty} H_{nm}\omega^n\mathbf{q}^{2m}\,. \tag{2.13}$$

[8] Moreover, the analyticity fails when the statistical fluctuation effects are taken into account, see footnote 1 and ref. [49].

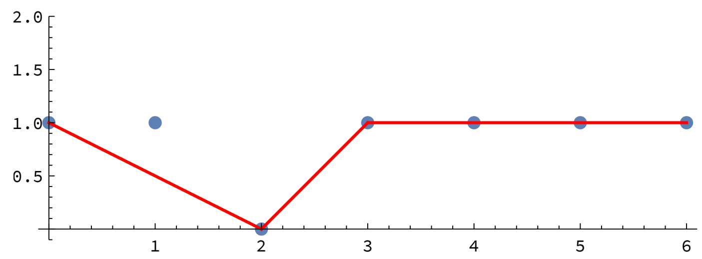

Figure 4. The Newton polygon for the sound mode.

The coefficients in front of various powers of ω in the expression (2.7) are then given by

$$\omega^0 : \qquad -\sum_{k=0}^{\infty} H_{0k}\mathbf{q}^{2k+2}\,, \tag{2.14}$$

$$\omega^1 : \qquad i\mathbf{q}^2\gamma^s_{00} + \sum_{k=1}^{\infty}\left(i\gamma^s_{0k} - H_{1k}\right)\mathbf{q}^{2k+2}\,, \tag{2.15}$$

$$\omega^2 : \qquad 1 + \sum_{k=0}^{\infty}\left(i\gamma^s_{1k} - H_{2k}\right)\mathbf{q}^{2k+2}\,, \tag{2.16}$$

$$\omega^3 : \qquad \sum_{k=0}^{\infty}\left(i\gamma^s_{2k} - H_{3k}\right)\mathbf{q}^{2k+2}\,, \tag{2.17}$$

$$\vdots \tag{2.18}$$

$$\omega^n : \qquad \sum_{k=0}^{\infty}\left(i\gamma^s_{n-1k} - H_{nk}\right)\mathbf{q}^{2k+2}\,. \tag{2.19}$$

The vertices of the Newton polygon are thus given by $(0, 1+k_0)$, $(1, 1+k_1)$, $(2, 0)$, $(3, 1+k_3)$, $(4, 1+k_4)$, $\ldots$, where $k_0, k_1, \ldots$ are the smallest indices such that $H_{0k_0} \neq 0$ as well as $H_{0k_1} \neq 0$ or/and $\gamma^s_{0k_1} \neq 0$, etc. The Newton polygon for the sound mode is shown in figure 4, where it is assumed that $H_{00} \neq 0$ and that either $H_{n,0} \neq 0$ or $\gamma^s_{n-1,0} \neq 0$ (or both) are non-zero for $n = 3, 4, \ldots$. The exponents of $x \equiv \mathbf{q}^2$ in the Puiseux series are given by the negative slopes of the polygon's lines, i.e. by $1/2$ for $H_{00} \neq 0$. Thus $m_j = 2$, and the lowest order term in the two branches is

$$\omega = \pm\sqrt{H_{00}}\,(\mathbf{q}^2)^{\frac{1}{2}} + \cdots\,. \tag{2.20}$$

From the Newton polygon, it is clear that $H_{00} \neq 0$ is the necessary and sufficient condition for the fractional powers of $\mathbf{q}^2$ to appear in the dispersion relation. In the hydrodynamic derivative expansion, $H_{00} = \partial p/\partial\epsilon = v_s^2$ is the speed of sound squared. One may have $H_{00} = 0$ at the point of a phase transition (see e.g. ref. [50]) in which case the dispersion relation contains only positive powers of $\mathbf{q}^2$.

Generically, for $H_{00} = v_s^2 \neq 0$, the sound mode dispersion relation will be given by the two branches of Puiseux series in $(\mathbf{q}^2)^{1/2}$ converging in some neighbourhood of the point $\mathbf{q}^2 = 0$,

$$\omega^{\pm}_{\text{sound}}(\mathbf{q}^2) = -i\sum_{n=1}^{\infty} a_n e^{\pm\frac{i\pi n}{2}}(\mathbf{q}^2)^{\frac{n}{2}} = \pm a_1(\mathbf{q}^2)^{\frac{1}{2}} + ia_2\mathbf{q}^2 \mp a_3(\mathbf{q}^2)^{\frac{3}{2}} + O(\mathbf{q}^4)\,, \tag{2.21}$$

where $a_n \in \mathbb{R}$ and $a_1 = c_s$ is the speed of sound. In particular, for $\mathbf{q}^2 = e^{\pm i\pi}|\mathbf{q}^2|$, the functions $\omega^{\pm}_{\rm sound}(\mathbf{q}^2)$ are purely imaginary as anticipated in ref. [29].

2.3 Convergence of the hydrodynamic series

The shear and sound dispersion relation series (2.11) and (2.21) have non-zero radii of convergence as long as the corresponding spectral curves (2.6) and (2.7) are given by the functions of $\mathbf{q}^2$ and ω analytic at the origin $(0,0)$. One way to find the radius of convergence of the series is to analyse the behaviour of the coefficients a_n, c_n at large n. This behaviour will of course depend on the microscopic details of the particular physical system, and may be difficult to study in practice (see, however, ref. [27]). Instead, here we use the spectral curves to determine the radii of convergence.

The Puiseux analysis implies that the domain of convergence of Puiseux series centred e.g. at the origin is the circle whose radius is set by the distance from the origin to the nearest critical point of the associated spectral curve. Critical points of the spectral curve $F(\mathbf{q}^2,\omega) = 0$, where $\mathbf{q}^2, \omega \in \mathbb{C}$, are determined by the conditions

$$F(\mathbf{q}^2_{\rm c},\omega_{\rm c}) = 0\,, \qquad \frac{\partial F}{\partial \omega}(\mathbf{q}^2_{\rm c},\omega_{\rm c}) = 0\,. \tag{2.22}$$

There are $p > 1$ branches of the solution $\omega = \omega(\mathbf{q}^2)$ in the vicinity of the critical point, provided that

$$F(\mathbf{q}^2_{\rm c},\omega_{\rm c}) = 0\,, \qquad \frac{\partial F}{\partial \omega}(\mathbf{q}^2_{\rm c},\omega_{\rm c}) = 0\,, \;\ldots,\; \frac{\partial^p F}{\partial \omega^p}(\mathbf{q}^2_{\rm c},\omega_{\rm c}) \neq 0\,. \tag{2.23}$$

For example, the origin $(0,0)$ is the critical point (with $p=2$) of the sound hydrodynamic spectral curve (2.7), as discussed in section 2.2. If the spectral curves happen to be known exactly or approximately (as will be the case in the holographic models we study below), eqs. (2.22) provide an efficient method to find the radii of convergence, without performing the large-n analysis of the expansion coefficients.

When the function $F(\mathbf{q}^2,\omega)$ is a polynomial, the condition (2.22) means that the equation $F(\mathbf{q}^2,\omega) = 0$, where $F(\mathbf{q}^2,\omega)$ is considered as a polynomial in ω with $\mathbf{q}^2$-dependent coefficients, has multiple roots at $\omega = \omega_{\rm c}$. This is equivalent to the condition that the discriminant of the polynomial $F(\mathbf{q}^2,\omega)$ vanishes. As an example, consider the first-order hydrodynamics of [1], where the spectral curves following from eq. (2.3) are[9]

$$F^{(1)}_{\rm shear}(\mathbf{q}^2,\omega) = \omega + iD\mathbf{q}^2 = 0\,, \tag{2.24}$$

$$F^{(1)}_{\rm sound}(\mathbf{q}^2,\omega) = \omega^2 + i\Gamma_s\omega\mathbf{q}^2 - v_s^2\mathbf{q}^2 = 0\,. \tag{2.25}$$

[9]As mentioned in section 2.1, the expressions for the truncated spectral curves $F^{(k)}$ at each order k of the hydrodynamic derivative expansion are not unique due to the freedom allowed by the "frame" choice. Correspondingly, the critical points determined by the approximate spectral curves $F^{(k)}$ depend on the "frame" choice. This dependence becomes less and less pronounced with k increasing and disappears in the limit $k \to \infty$. Thus, although the critical points of the exact spectral curve are "frame"-independent, the rate of convergence of the approximate location of the critical points to the exact values can be affected by the "frame" choice.

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These equations are simple enough to be solved explicitly: eq. (2.24) is solved by $\omega = -iD\mathbf{q}^2$, whereas the solutions of eq. (2.25) are

$$\omega^{\pm}_{\text{sound}}(\mathbf{q}^2) = -\frac{i\Gamma_s}{2}\,\mathbf{q}^2 \pm \sqrt{v_s^2\mathbf{q}^2\left(1 - \frac{\Gamma_s^2\mathbf{q}^2}{4v_s^2}\right)} = \pm v_s q - \frac{i\Gamma_s}{2}\,q^2 + \cdots\,, \tag{2.26}$$

where in the expansion we only kept terms quadratic in $q \equiv \sqrt{\mathbf{q}^2}$, since we expect the coefficients in front of the higher powers of q to be modified by higher-derivative terms in the hydrodynamic expansion. The series in q on the right-hand side of eq. (2.26) has the radius of convergence

$$R^{(1)}_{\text{sound}} = \frac{2v_s}{\Gamma_s}\,, \tag{2.27}$$

determined by the branch points of the square root in eq. (2.26) or, equivalently, by the zeros of the discriminant[10] $(-\Gamma_s^2\mathbf{q}^4 + 4v_s^2\mathbf{q}^2)$ of the polynomial (2.25). Since first-order hydrodynamics can only be trusted for $|q| \ll 2v_s/\Gamma_s = R^{(1)}_{\text{sound}}$, the result (2.27) is only an approximation. Alternatively, applying the condition (2.22) to the spectral curve (2.25), we find

$$\mathbf{q}_{\text{c}}^2 = \frac{4v_s^2}{\Gamma_s^2}\,, \qquad \omega_{\text{c}}^{(\text{sound})} = -i\frac{2v_s}{\Gamma_s}\,, \tag{2.28}$$

which coincides with (2.27). In what follows, we will be studying models where the convergence radii of the small-$\mathbf{q}$ expansions can be determined from the exact response functions of the theory, without resorting to the derivative expansion of hydrodynamics.

3 A holographic model with translation symmetry breaking

To illustrate the methods discussed in section 2.3, we consider the holographic model with translation symmetry breaking [52], studied, in particular, in ref. [33]. The model is a bottom-up gravity construction in $4d$ describing a hypothetical dual 2+1-dimensional QFT with broken translational invariance. In the context of the present paper, the significance of the construction discussed in ref. [33] lies in the fact that it provides exact analytic formulae for the current and energy-momentum correlator two-point functions at a special, self-dual symmetry point (see section 4 of ref. [33] for details). In particular, among the poles of the correlation function, one finds a gapless excitation whose dispersion relation is known analytically, possibly a unique example in holography.

The bulk action of the model is given by [33, 52] (see also ref. [31])

$$S = \int d^4x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}\sum_{i=1}^{2}\partial^\mu\phi_i\partial_\mu\phi_i - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{3.1}$$

where $\Lambda = -3/L^2$ (we set $L = 1$ in the following). The background solution of interest involves the AdS-Schwarzschild black brane with translational invariance broken by the linear dilaton fields, and a vanishing Maxwell field.

[10] Similar methods have been used in spectroscopy [51].

First, we recast the setup of ref. [33] in the form more convenient for our purposes, having in mind the variables used in ref. [41]. The metric at the special symmetry point has the form (see section 4 of ref. [33])

$$ds^2 = -\frac{r_0^2}{u} f(u)\, dt^2 + \frac{r_0^2}{u}\left(dx^2 + dy^2\right) + \frac{du^2}{4fu^2}\,, \tag{3.2}$$

where $f = 1 - u$, and we have used the coordinate $u = r_0^2/r^2$. The horizon is located at $u = 1$ and the boundary at $u = 0$. The Hawking temperature of the background is $T = r_0/2\pi$. The dilaton fields are given by $\phi_1 = \sqrt{2} r_0 x$ and $\phi_2 = \sqrt{2} r_0 y$. They will not play any role in the following.

Considering the Maxwell field fluctuations in the background (3.2), coupled to the current operator on the boundary, one finds the equations of motion

$$a_t'' + \frac{1}{2u} a_t' - \frac{1}{4uf}\left(\mathfrak{q}^2 a_t + \mathfrak{w}\mathfrak{q}\, a_x\right) = 0\,, \tag{3.3}$$

$$a_x'' + \frac{1-3u}{2uf} a_x' + \frac{1}{4uf^2}\left(\mathfrak{q}\mathfrak{w} a_t + \mathfrak{w}^2 a_x\right) = 0\,, \tag{3.4}$$

$$\mathfrak{w} a_t' + \mathfrak{q} f a_x' = 0\,, \tag{3.5}$$

where $\mathfrak{w} = \omega/2\pi T = \omega/r_0$, $\mathfrak{q} = k/2\pi T = k/r_0$, and the momentum k is directed along x. For the gauge-invariant variable (the longitudinal component of the electric field) $E_x = \mathfrak{q} a_t + \mathfrak{w} a_x$ [41], the equation of motion reads

$$E_x'' + \frac{\mathfrak{w}^2(1-3u) - \mathfrak{q}^2 f^2}{2uf(\mathfrak{w}^2 - \mathfrak{q}^2 f)} E_x' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{4uf^2} E_x = 0\,. \tag{3.6}$$

The equation (3.6) has 4 regular singular points (located at $u = 0, 1, 1 - \mathfrak{w}^2/\mathfrak{q}^2, \infty$) and thus is of the Heun type. The indices of this equation are, respectively,

$$u = 0: \qquad 0, 1/2 \tag{3.7}$$

$$u = 1: \qquad \pm i\mathfrak{w}/2 \tag{3.8}$$

$$u = 1 - \frac{\mathfrak{w}^2}{\mathfrak{q}^2}: \qquad 0, 2 \tag{3.9}$$

$$u = \infty: \qquad \frac{1}{4}\left(1 \pm \sqrt{1 - 4\mathfrak{q}^2}\right). \tag{3.10}$$

Note that the local solution at $u = 0$ does not contain logarithms. The exact solution to eq. (3.6) can be written as

$$E_x(u) = \frac{4uf}{\mathfrak{q}} G'(u) + \frac{2f}{\mathfrak{q}} G(u)\,, \tag{3.11}$$

where $G \equiv a_t'$ is the solution of the hypergeometric equation[11]

$$G'' - \frac{5u-3}{2uf} G' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f - 2f}{4uf^2} G = 0\,, \tag{3.12}$$

[11]The very fact that an exact solution to the Heun equation can be found via the supplementary hypergeometric equation is rather curious and may warrant further reflection.

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obeying the incoming wave boundary condition at $u = 1$:

$$G = \frac{\mathfrak{q}}{2i\mathfrak{w}} x^{-i\mathfrak{w}/2} {}_2F_1\left(\frac{3-2i\mathfrak{w}-\sqrt{1-4\mathfrak{q}^2}}{4}, \frac{3-2i\mathfrak{w}+\sqrt{1-4\mathfrak{q}^2}}{4}; 1-i\mathfrak{w}; x\right), \tag{3.13}$$

where $x = 1 - u$ and the normalisation is chosen to ensure $E_x \to (1-u)^{-i\mathfrak{w}/2}(1+\cdots)$ at $u \to 1$.

3.1 The exact spectral curve

For the boundary value of the electric field, we find

$$E_x(u=0; \mathfrak{q}^2, \mathfrak{w}) \equiv F(\mathfrak{q}^2, \mathfrak{w}) = \frac{2\sqrt{\pi}\,\Gamma(-i\mathfrak{w})}{\Gamma\left[A_+(\mathfrak{q}^2, \mathfrak{w})\right]\Gamma\left[A_-(\mathfrak{q}^2, \mathfrak{w})\right]}, \tag{3.14}$$

where

$$A_\pm = \frac{1}{4}\left(1 \pm \sqrt{1-4\mathfrak{q}^2} - 2i\mathfrak{w}\right). \tag{3.15}$$

The condition $E_x(u=0) = 0$ determines the quasinormal modes and thus the poles of the corresponding current-current correlators [34, 41]. One can also write down the explicit analytic expressions for the correlators themselves [33], but this will not be necessary: the expression $F(\mathfrak{q}^2, \mathfrak{w}) = 0$, where $F(\mathfrak{q}^2, \mathfrak{w})$ is given by eq. (3.14), is the exact spectral curve containing all information about the poles of the two-point function. There are two sequences of quasinormal frequencies

$$\mathfrak{w}_n^\pm(\mathfrak{q}^2) = -i\left(2n_\pm + \frac{1}{2}\right) \pm \frac{i}{2}\sqrt{1-4\mathfrak{q}^2}, \qquad n_\pm = 0, 1, 2\ldots. \tag{3.16}$$

The solutions $E^\pm_{x,n_\pm}(u)$ themselves (the quasinormal modes) have the form

$$E^\pm_{x,n_\pm}(u) = \sqrt{u}\,(1-u)^{-n_\pm-\frac{1}{4}\pm\frac{1}{4}\sqrt{1-4\mathfrak{q}^2}}\,P^\pm_{n_\pm}(u, \mathfrak{q}^2), \tag{3.17}$$

where $P^\pm_{n_\pm}(u, \mathfrak{q}^2)$ are polynomials of degree $n_\pm$ in u with $\mathfrak{q}^2$−dependent coefficients. In particular,

$$E^\pm_{x,0}(u) = \sqrt{u}\,(1-u)^{-\frac{1}{4}\pm\frac{1}{4}\sqrt{1-4\mathfrak{q}^2}}. \tag{3.18}$$

Note that apart from the prefactors determined by the indices, the solutions (3.17) are polynomials, with $P_0^\pm = 1$. We are especially interested in the gapless mode

$$\begin{aligned}\mathfrak{w} = \mathfrak{w}_0^+ &= -\frac{i}{2}\left(1-\sqrt{1-4\mathfrak{q}^2}\right) = \frac{i}{2}\sum_{n=1}^{\infty}(-1)^n\binom{1/2}{n}\left(4\mathfrak{q}^2\right)^n \\ &= -i\mathfrak{q}^2 - i\mathfrak{q}^4 - 2i\mathfrak{q}^6 - 5i\mathfrak{q}^8 - 14i\mathfrak{q}^{10} + \cdots.\end{aligned} \tag{3.19}$$

The power series in the second line of eq. (3.19) converges in the circle $|\mathfrak{q}^2| < 1/4$, due to the branch point singularity of the function at $\mathfrak{q}^2 = 1/4$ evident from eq. (3.19). The same conclusion can be obtained by analysing critical points of the spectral curve (3.14).

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Indeed, the critical points are determined by the equations (2.22) whose solutions are $\mathfrak{w}^+_{n_+}(\mathfrak{q}^2) = \mathfrak{w}^-_{n_-}(\mathfrak{q}^2)$, i.e.

$$\mathfrak{q}_c^2 = \frac{1}{4} - (n_+ - n_-)^2 \,, \tag{3.20a}$$

$$\mathfrak{w}_c = -\frac{i}{2}\left[1 + 2\,(n_+ + n_-)\right]. \tag{3.20b}$$

It is also clear that $\partial^2_{\mathfrak{w}} F(\mathfrak{q}_c^2, \mathfrak{w}_c) \neq 0$ and thus there are two branches of the spectral curve emerging at each critical point. Put simply, the critical points occur when the two quasinormal frequencies in eq. (3.16) collide. This happens for real $\mathfrak{q}$ in case of $n_+ = n_-$, and for purely imaginary $\mathfrak{q}$ for $n_+ \neq n_-$. The critical point closest to the origin $\mathfrak{q}^2 = 0$ is at $\mathfrak{q}^2 = 1/4$ (it corresponds to the collision of the modes $\mathfrak{w}_0^+$ and $\mathfrak{w}_0^-$). This value determines the radius of convergence of the series in eq. (3.19).

Exact spectral curves are rare: in addition to (3.14), we are aware of only one example (involving the exact R-current two-point correlators in $\mathcal{N} = 4$ SYM theory in appropriate limit [53]) for a QFT in dimension higher than $1+1$, and even in that case it is only known exactly for $\mathfrak{q}^2 = 0$:

$$F_R(\mathfrak{q}^2 = 0, \mathfrak{w}) = 2^{-\frac{(1+i)\mathfrak{w}}{2}} \frac{\Gamma\left[1 - i\mathfrak{w}\right]}{\Gamma\left[1 - \frac{(1+i)\mathfrak{w}}{2}\right]\Gamma\left[1 + \frac{(1-i)\mathfrak{w}}{2}\right]} \,. \tag{3.21}$$

One can use the Weierstrass decomposition

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \,, \tag{3.22}$$

where γ is the Euler–Mascheroni constant, to write eqs. (3.14), (3.21) as infinite products: this illustrates explicitly why the critical points determined by the condition (2.22) correspond to multiple roots of (infinite order) polynomials.

3.2 The hydrodynamic approximation to the spectral curve

The expansion of the expression $\mathfrak{w}F(\mathfrak{q}^2, \mathfrak{w})$ for small $\mathfrak{w}$, $\mathfrak{q}$ (assuming the scaling $\mathfrak{w} \to \epsilon\mathfrak{w}$, $\mathfrak{q}^2 \to \epsilon\mathfrak{q}^2$ with $\epsilon \to 0$) truncated at order $\mathfrak{w}^k$, $\mathfrak{q}^{2k}$ is a polynomial $F_k(\mathfrak{q}^2, \mathfrak{w})$, with

$$F_1(\mathfrak{q}^2, \mathfrak{w}) = \mathfrak{w} + i\mathfrak{q}^2 \,, \tag{3.23}$$

$$F_2(\mathfrak{q}^2, \mathfrak{w}) = \mathfrak{w} + i\mathfrak{q}^2 - i\left[\mathfrak{w}^2 \ln 2 - \mathfrak{q}^4\,(1 - \ln 2)\right] \,, \tag{3.24}$$

$$\begin{aligned} F_3(\mathfrak{q}^2, \mathfrak{w}) &= \mathfrak{w} + i\mathfrak{q}^2 - i\left[\mathfrak{w}^2 \ln 2 - \mathfrak{q}^4\,(1 - \ln 2)\right] \\ &\quad - \frac{i}{12}\left[\mathfrak{q}^6\left(\pi^2 - 6(\ln 2 - 2)^2\right) - 6i\mathfrak{q}^4\mathfrak{w}\ln^2 2 - 6i\mathfrak{w}^3\ln^2 2 + \mathfrak{q}^2\mathfrak{w}^2(\pi^2 - 6\ln^2 2)\right] \,, \end{aligned} \tag{3.25}$$

and so on. Using this expansion to solve the equation $\mathfrak{w}F(\mathfrak{q}^2, \mathfrak{w}) = 0$ for $\mathfrak{w}$ *perturbatively* in $\mathfrak{q}^2$, one reproduces the series in eq. (3.19). The equation $F_k(\mathfrak{q}^2, \mathfrak{w}) = 0$ defines the hydrodynamic spectral curve of order k as discussed in section 2. At each order, the critical points are determined by the condition (2.22). In figure 5, we plot the corresponding value $|\mathfrak{q}_c^2|$ for each of the spectral curves $F_k(\mathfrak{q}^2, \mathfrak{w}) = 0$ for $k = 2, 3, \ldots, 13$. The resulting points converge rapidly to the exact value $|\mathfrak{q}_c^2| = 1/4$.

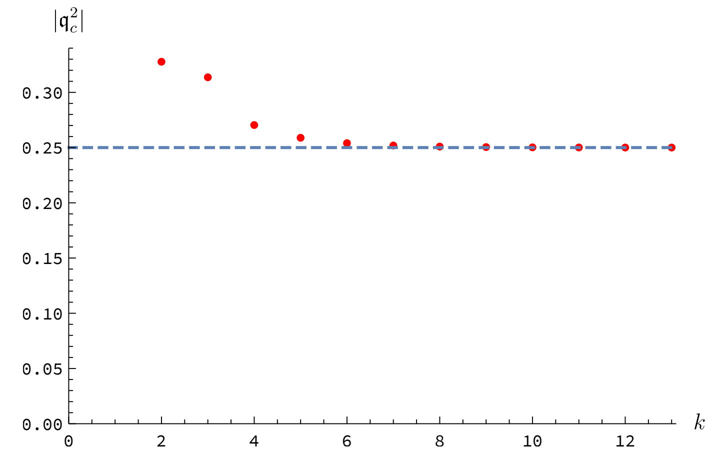

Figure 5. The approximations to the exact position of the critical point $|\mathfrak{q}_c^2| = 1/4$ in the holographic model with broken translation symmetry determined from the hydrodynamic algebraic curves $F_k(\mathfrak{q}^2, \mathfrak{w}) = 0$ as a function of k.

3.3 Pole-skipping in the full response functions

Finally, we comment on the relationship between the critical point defining the hydrodynamic series radius of convergence and the pole-skipping point in the holographic model with broken translation symmetry, which occurs at $|\mathfrak{q}_*^2| = 2 > |\mathfrak{q}_c^2|$ [31]. Because gapless excitations in the current and the energy density correlators have the same dispersion relation, we can directly use $\mathfrak{w}(\mathfrak{q})$ from eq. (3.19) to discuss both the charge and the energy sectors. At the pole-skipping point, the hydrodynamic series stated in the second line of eq. (3.19) diverges but can be resummed using the Borel transform[12]

$$\mathcal{B}\mathfrak{w}(\mathfrak{q}) = \frac{i}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \binom{1/2}{n} \left(4\mathfrak{q}^2\right)^n \tag{3.26}$$

$$= -\frac{i}{2} \left[1 - e^{2\mathfrak{q}^2} \left((1 - 4\mathfrak{q}^2) I_0(2\mathfrak{q}^2) + 4\mathfrak{q}^2 I_1(2\mathfrak{q}^2)\right)\right] , \tag{3.27}$$

where $I_n(x)$ is the modified Bessel function. Since this is a series in $\mathfrak{q}^2$ and not $\mathfrak{q}$, the corresponding Borel integral has the form

$$\boldsymbol{\Omega}(\mathfrak{q}) = \int_0^\infty dt\, e^{-t} \mathcal{B}\mathfrak{w}(\mathfrak{q}\sqrt{t}) = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 \,, \tag{3.28}$$

where

$$\mathcal{I}_1 = -\frac{i}{2} \int_0^\infty dt\, e^{-t} = -\frac{i}{2} \,, \tag{3.29}$$

$$\mathcal{I}_2 = \frac{i}{2} \int_0^\infty dt\, e^{-t} e^{2\mathfrak{q}^2 t} I_0(2\mathfrak{q}^2 t) = \frac{i}{2} \frac{1}{\sqrt{1 - 4\mathfrak{q}^2}} \,, \tag{3.30}$$

$$\mathcal{I}_3 = -i \int_0^\infty dt\, e^{-t} (2\mathfrak{q}^2 t) e^{2\mathfrak{q}^2 t} I_0(2\mathfrak{q}^2 t) = -2i\mathfrak{q}^2 \frac{(1 - 2\mathfrak{q}^2)}{(1 - 4\mathfrak{q}^2)^{3/2}} \,, \tag{3.31}$$

$$\mathcal{I}_4 = i \int_0^\infty dt\, e^{-t} (2\mathfrak{q}^2 t) e^{2\mathfrak{q}^2 t} I_1(2\mathfrak{q}^2 t) = 4i \frac{\mathfrak{q}^2 \sqrt{\mathfrak{q}^4}}{(1 - 4\mathfrak{q}^2)^{3/2}} \,, \tag{3.32}$$

[12] An alternative analytic continuation can be provided e.g. by the Mittag-Leffler summation. See e.g. ref. [54].

each with their respective region of $\mathfrak{q}$ for which the integral converges. Together, we find that the Borel integral representation (3.28) of the series converges for $\mathfrak{q} \in \mathbb{C}$ in the region defined by the function $\mathcal{C}(\mathfrak{q})$:

$$\mathcal{C}(\mathfrak{q}) \equiv \mathrm{Re}[\mathfrak{q}^2] < 1/4 \,. \tag{3.33}$$

This is a significant improvement over the convergence region of the hydrodynamic power series, $|\mathfrak{q}^2| < |\mathfrak{q}_c^2| = 1/4$. In particular, the Borel series is well suited for studying purely imaginary $\mathfrak{q}$. By writing $\mathfrak{q} = i\ell$, with $\ell \in \mathbb{R}$, we see that

$$\mathcal{C}(i\ell) = \mathrm{Re}[-\ell^2] = -\mathrm{Re}[\ell^2] \leq 0 < 1/4 \,. \tag{3.34}$$

Hence, the Borel representation of the series converges for all values of imaginary $\mathfrak{q}$, including $\mathfrak{w}(\mathfrak{q})$ at the chaos point $\mathfrak{w}_*(\mathfrak{q}_* = \sqrt{2}i) = i$. Finally, for $\mathfrak{q}$ such that $\mathcal{C}(\mathfrak{q}) < 1/4$, it is easy to check that the sum of four terms in (3.28) indeed reproduces the full dispersion relation (3.19).

4 Response functions in strongly coupled $\mathcal{N} = 4$ SYM theory

In this section, we use holographic duality to find the spectral curves, determine the radii of convergence of hydrodynamic series and analyse the pole-skipping phenomenon in the three channels of the response function of the energy-momentum tensor in the $\mathcal{N} = 4$ SU(N_c) SYM theory at infinite 't Hooft coupling and infinite N_c. The details of the duality are well known, and the necessary ingredients can be found e.g. in refs. [35, 41, 55]. In short, holography reduces the study of the response functions to the analysis of the fluctuations of the dual gravitational background involving a black hole with translationally invariant horizon — the AdS-Schwarzschild black brane.

The equations of motion describing fluctuations of the gravitational background dual to finite-temperature $\mathcal{N} = 4$ SYM theory are of the Heun type [56], and the exact analytic solution for the spectral curve similar to eq. (3.14) is not available. The equations can be solved perturbatively and analytically in $\mathfrak{w} \ll 1$, $\mathfrak{q} \ll 1$, as was done in refs. [35, 40], thereby giving a hydrodynamic approximation to the spectral curve, or numerically, for arbitrary $\mathfrak{w}$ and $\mathfrak{q}$, along the lines of ref. [41]. In this section, we consider and compare these two approaches.

4.1 Shear mode: hydrodynamic approximation to the spectral curve

We start with the analysis of the shear channel of the energy-momentum tensor response function [41]. In the hydrodynamic approximation, the spectral curve can be found analytically from the boundary value of the solution to the ODE obeyed by one of the components of the shear perturbation in dual gravity (see section 6.2 of ref. [35] for details):

$$G''(u) - \left(\frac{2u}{f} - \frac{i\mathfrak{w}}{1-u}\right) G'(u) + \frac{1}{f}\left(2 + \frac{i\mathfrak{w}}{2} - \frac{\mathfrak{q}^2}{u} + \frac{\mathfrak{w}^2[4 - u(1+u)^2]}{4uf}\right) G(u) = 0 \,, \tag{4.1}$$

where G is regular at $u = 1$. Rescaling $\mathfrak{w} \to \lambda^2 \mathfrak{w}$ and $\mathfrak{q}^2 \to \lambda^2 \mathfrak{q}^2$, assuming $\lambda \ll 1$, and looking for a perturbative solution of the form

$$G(u) = \sum_{n=0}^{\infty} \lambda^{2n} G_n(u)\,, \tag{4.2}$$

we find the following equation for coupled components G_n, G_{n-1}, G_{n-2}:

$$\begin{aligned} G_n'' - \frac{2u}{f} G_n' + \frac{2}{f} G_n & \\ + \frac{i\mathfrak{w}}{1-u} G_{n-1}' + \frac{i\mathfrak{w}}{2f} G_{n-1} - \frac{\mathfrak{q}^2}{uf} G_{n-1} & \\ + \frac{\mathfrak{w}^2 [4 - u(1+u)^2]}{4uf^2} G_{n-2} = 0\,, & \end{aligned} \tag{4.3}$$

where $G_n = 0$ for $n < 0$, and we can set $G_0(1) = 1$, $G_i(1) = 0$, $i \geq 1$, without loss of generality.[13] The explicit formulae for $G_0(u)$, $G_1(u)$ and $G_2(u)$ obeying the boundary conditions at $u = 1$ are written in appendix B. The solution of the homogeneous equation

$$g'' - \frac{2u}{f} g' + \frac{2}{f} g = 0 \tag{4.4}$$

is given by $g = C_1 \, g_1(u) + C_2 \, g_2(u)$, where

$$g_1 = u\,, \tag{4.5}$$

$$g_2 = \frac{u}{2} \ln \frac{1+u}{1-u} - 1\,. \tag{4.6}$$

Note that the Wronskian is $W(g_1, g_2) = 1/f$. Then one can write the following expression for G_n, $n \geq 2$:

$$\begin{aligned} G_n(u) = & \, g_1(u) \int_u^1 g_2(t) f(t) \mathcal{F}_n(t) dt - g_2(u) \int_u^1 g_1(t) f(t) \mathcal{F}_n(t) dt \\ & + C_1 \, g_1(u) + C_2 \, g_2(u)\,, \end{aligned} \tag{4.7}$$

where

$$\mathcal{F}_n(u) = -\frac{i\mathfrak{w}}{1-u} G_{n-1}' - \frac{i\mathfrak{w}}{2f} G_{n-1} + \frac{\mathfrak{q}^2}{uf} G_{n-1} - \frac{\mathfrak{w}^2 [4 - u(1+u)^2]}{4uf^2} G_{n-2}\,. \tag{4.8}$$

Boundary conditions at $u = 1$ (regularity of $G_n(u)$ at $u = 1$ and $G_n(1) = 0$ for $n > 0$) require $C_1 = 0$, $C_2 = 0$. Thus, we have the equation determining G_n from G_{n-1} and G_{n-2},

$$\begin{aligned} G_n(u) = & \, u \int_u^1 (1 - t^2) \left(\frac{t}{2} \ln \frac{1+t}{1-t} - 1 \right) \mathcal{F}_n(t) dt \\ & - \left(\frac{u}{2} \ln \frac{1+u}{1-u} - 1 \right) \int_u^1 t(1 - t^2) \mathcal{F}_n(t) dt\,, \end{aligned} \tag{4.9}$$

[13] See appendix C of ref. [57].

where $\mathcal{F}_n$ is given by eq. (4.8). This can be written as

$$G_n(u) = \frac{u}{2}\int_u^1 t(1-t^2)\ln\left[\frac{1+t-u-ut}{1-t+u-ut}\right]\mathcal{F}_n(t)\,dt \\ + \int_u^1 (t-u)(1-t^2)\mathcal{F}_n(t)dt\,, \tag{4.10}$$

or

$$G_n(u) = \int_u^1 \left\{\frac{ut}{2}\ln\left[\frac{1+t-u-ut}{1-t+u-ut}\right] + t - u\right\}(1-t^2)\,\mathcal{F}_n(t)\,dt\,. \tag{4.11}$$

Note that $\mathcal{F}_n(t) \sim 1/(1-t)$ at $t \to 1$, and so all integrals converge. In particular,

$$G_n(0) = \int_0^1 t(1-t^2)\mathcal{F}_n(t)dt\,. \tag{4.12}$$

The explicit expressions for $G_i(0)$, with $i = 1,2,3,4$, are written in appendix B. The results for $G_0(0)$, $G_1(0)$ and $G_2(0)$ coincide with those in ref. [3]. The results for $G_3(0)$ and $G_4(0)$ are new. The condition $G(0) = 0$ at this order in the hydrodynamic expansion defines the algebraic curve

$$\begin{aligned} F(\mathfrak{q}^2,\mathfrak{w}) =& -i\mathfrak{w} + \frac{\mathfrak{q}^2}{2} \\ &+ \frac{\mathfrak{q}^4}{4} - \frac{i\mathfrak{w}\mathfrak{q}^2\ln 2}{4} + \frac{\mathfrak{w}^2\ln 2}{2} \\ &+ i\mathfrak{w}^3\left(\frac{\pi^2}{24} + \ln 2 - \frac{3}{8}\ln^2 2\right) + \mathfrak{q}^6\left(\frac{\ln 2}{4} - \frac{1}{8}\right) + i\mathfrak{w}\mathfrak{q}^4\left(\frac{1}{4} - \frac{\ln 2}{8}\right) \\ &+ \mathfrak{q}^2\mathfrak{w}^2\left(\frac{\pi^2}{48} - \frac{\ln 2}{2} - \frac{\ln^2 2}{16}\right) + \mathfrak{q}^8\left(-\frac{1}{16} + \frac{\pi^2}{64} - \frac{\ln 2}{8}\right) \\ &- \mathfrak{q}^4\mathfrak{w}^2\left(\frac{\pi^2}{96} + (12 - 7\ln 2)\frac{\ln 8}{96}\right) - i\mathfrak{q}^6\mathfrak{w}\left(\frac{\pi^2}{96} + (-5 + \ln 4)\frac{\ln 64}{96}\right) \\ &+ \mathfrak{w}^4\left((24 - 5\ln 2)\frac{\ln^2 2}{48} + \frac{\pi^2}{48}(-4 + \ln 8) - \frac{1}{2}\zeta(3)\right) \\ &+ i\mathfrak{q}^2\mathfrak{w}^3\left(-\frac{\pi^2\ln 2}{96} + (-24 + \ln 2)\frac{\ln^2 2}{96} + \frac{3}{16}\zeta(3)\right) = 0\,. \end{aligned} \tag{4.13}$$

Here, $\zeta(z)$ is the Riemann zeta function. Solving eq. (4.13) perturbatively in $\mathfrak{q}^2$, one finds the dispersion relation for the shear mode

$$\begin{aligned} \mathfrak{w} =& -\frac{i}{2}\mathfrak{q}^2 - \frac{i(1-\ln 2)}{4}\mathfrak{q}^4 - \frac{i(24\ln^2 2 - \pi^2)}{96}\mathfrak{q}^6 \\ &- \frac{i(2\pi^2(5\ln 2 - 1) - 21\zeta(3) - 24\ln 2[1 + \ln 2(5\ln 2 - 3)])}{384}\mathfrak{q}^8 + O(\mathfrak{q}^{10})\,. \end{aligned} \tag{4.14}$$

The first two terms in (4.14) agree with the ones obtained in refs. [35] and [3], respectively.

Encouraged by the success of finding the critical point in the holographic model with broken translation symmetry in section 3, we can apply the condition (2.22) to the polynomial (4.13). Truncating (4.13) at linear (in $\mathfrak{w}$ and $\mathfrak{q}^2$), quadratic, cubic and quartic order,

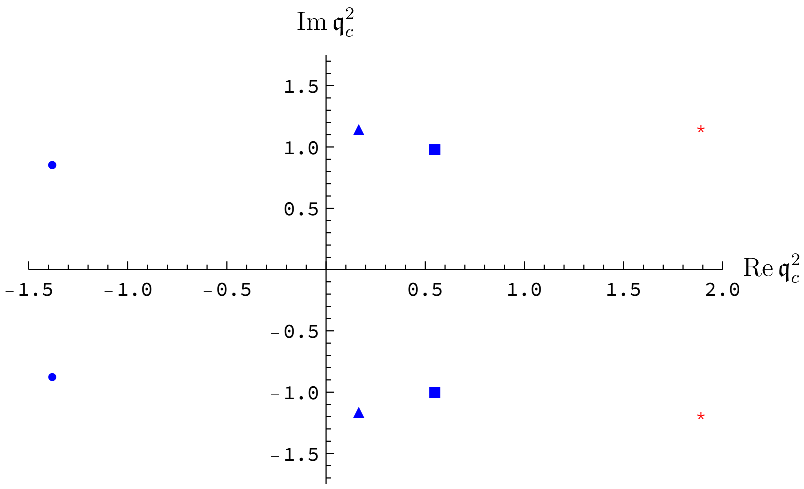

Figure 6. Approximations to the position of the critical point in the complex $\mathfrak{q}_c^2$ plane in the shear channel of the $\mathcal{N} = 4$ SYM theory arising from the order k hydrodynamic polynomials $F_k(\mathfrak{q}^2, \mathfrak{w}) = 0$ (circles for $k = 2$, triangles for $k = 3$, squares for $k = 4$). The "exact" position $\mathfrak{q}_c^2 \approx 1.8906469 \pm 1.1711505i$ (see section 4.2) is marked by red asterisks.

correspondingly, we obtain the algebraic curves $F_k(\mathfrak{q}^2, \mathfrak{w}) = 0$ for $k = 1, 2, 3, 4$. Using the condition (2.22), we find that there are no solutions at $k = 1$, whereas for $k = 2, 3, 4$, we have

$$k = 2: \quad \mathfrak{q}_c^2 \approx -1.380398 \pm 0.865925i\,, \quad \mathfrak{w}_c \approx \pm 0.216481 + 1.097596i\,, \tag{4.15}$$

$$k = 3: \quad \mathfrak{q}_c^2 \approx 0.164953 \pm 1.151910i\,, \quad \mathfrak{w}_c \approx \pm 0.735771 + 0.164407i\,, \tag{4.16}$$

$$k = 4: \quad \mathfrak{q}_c^2 \approx 0.548173 \pm 0.988705i\,, \quad \mathfrak{w}_c \approx \pm 0.706672 - 0.140043i\,. \tag{4.17}$$

In figure 6, the $k = 2, 3, 4$ approximations are shown in the complex plane of $\mathfrak{q}_c^2$ together with the "exact" value $\mathfrak{q}_c^2 \approx 1.8906469 \pm 1.1711505i$ obtained via the quasinormal level-crossing method (see section 4.2). In contrast with the holographic model with broken translation symmetry (see figure 5), the convergence is slow (admittedly, we only have 3 points in the present case). However, we learn an important lesson, namely, that the critical point can be located at a generic complex value of $\mathfrak{q}^2$. In the next section, we use a more efficient method of the quasinormal modes level-crossing to determine the critical points.

4.2 Shear mode: full spectral curve

To compute the spectral curve numerically for all values of $\mathfrak{q}^2$ and $\mathfrak{w}$, it is more convenient to use the ODE obeyed by the gauge-invariant perturbations. The gauge-invariant shear mode gravitational perturbations of the AdS-Schwarzschild black brane are described by the function $Z_1(u)$, where u is the radial coordinate ranging from $u = 0$ (asymptotic boundary) to $u = 1$ (event horizon) [41]. The function $Z_1(u)$ obeys the equation

$$Z_1'' - \frac{(\mathfrak{w}^2 - \mathfrak{q}^2 f) f - u\mathfrak{w}^2 f'}{u f (\mathfrak{w}^2 - \mathfrak{q}^2 f)} Z_1' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{u f^2} Z_1 = 0\,, \tag{4.18}$$

where $f(u) = 1 - u^2$, and the solution must obey the incoming wave boundary condition at the horizon, $Z_1(u) \sim (1-u)^{-i\mathfrak{w}/2}$ as $u \to 1$ [34, 41]. The full shear spectral curve is then given by

$$F(\mathfrak{q}^2, \mathfrak{w}) = Z_1(u=0; \mathfrak{q}^2, \mathfrak{w}) = 0\,. \tag{4.19}$$

The spectral curve (4.19) can be determined numerically by e.g. using N terms of the Frobenius series solution at $u = 1$ [41]. The stability of the numerical procedure is ensured by checking that adding several more terms to the series does not appreciably change the result. Applying (numerically) the criterion (2.22) to (4.19), we find the critical point closest to the origin,

$$\mathfrak{q}_c^2 \approx 1.8906469 \pm 1.1711505i\,, \qquad \mathfrak{w}_c \approx \pm 1.4436414 - 1.0692250i\,, \tag{4.20}$$

implying the convergence radius of the shear mode dispersion relation $|\mathfrak{q}^c_{\text{shear}}| \approx 1.49131$. There are other critical points with $|\mathfrak{q}^2| > |\mathfrak{q}^c_{\text{shear}}|$. The first three of them are located at

$$\mathfrak{q}_{c,1}^2 \approx -2.37737\,, \qquad \mathfrak{w}_{c,1} \approx -1.64659i\,, \tag{4.21}$$

$$\mathfrak{q}_{c,2}^2 \approx -3.11051 \pm 0.8105i\,, \qquad \mathfrak{w}_{c,2} \approx \pm 1.41043 - 2.87086i\,, \tag{4.22}$$

$$\mathfrak{q}_{c,3}^2 \approx 2.90692 \pm 1.66606i\,, \qquad \mathfrak{w}_{c,3} \approx \pm 2.38819 - 2.13154i\,. \tag{4.23}$$

In figures 7 and 8, we show the corresponding quasinormal spectrum (solutions $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)$ of eq. (4.19)) in the complex plane of frequency $\mathfrak{w}$, parametrised by $\mathfrak{q}^2$. The difference with previous works is that now, we treat $\mathfrak{q}^2$ as a generic complex variable, $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\theta}$, and vary its phase, $\theta \in [0, 2\pi]$, at specific fixed values of the magnitude $|\mathfrak{q}^2|$. From eq. (4.18), it is clear that if $Z_1(u; \mathfrak{q}, \mathfrak{w})$ is a solution satisfying the incoming wave boundary condition at the horizon, then $Z_1(u; \mathfrak{q}, -\mathfrak{w})$ is also a solution. The spectrum in figures 7 and 8 therefore appears to be symmetric with respect to the imaginary axis. For real $\mathfrak{q}^2$, the spectrum coincides with the one originally found in ref. [58].

At small $|\mathfrak{q}^2|$, the poles follow closed orbits as the phase θ varies from 0 to 2π (figure 7, top panels). With the parameter $|\mathfrak{q}^2|$ increasing, the poles start feeling the presence of each other, and their orbits become more complicated. Finally, at $|\mathfrak{q}_c^2| \approx 2.224$, the hydrodynamic shear pole collides with one of the two closest non-hydrodynamic poles (see figure 7, bottom panels). For $|\mathfrak{q}^2| > |\mathfrak{q}_c^2|$, those three poles no longer have closed orbits: as the phase θ varies from 0 to 2π, they interchange their positions cyclically (figure 7, bottom right panel). For even larger $|\mathfrak{q}^2|$, other gapped poles become involved in this collective motion of quasinormal modes in a similar manner (figure 8).

The phenomenon observed in figures 7, 8 is the quasinormal spectrum level-crossing, reminiscent of the well-known effect in quantum mechanics. Indeed, for real $\mathfrak{q}^2$, the real and imaginary parts of the quasinormal frequencies do not exhibit level-crossing (see e.g. figures 13 and 14 in ref. [58]). For complex momentum $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\theta}$, the real and imaginary parts of the shear hydrodynamic mode and the nearest gapped mode do cross at the fixed phase $\theta \approx 0.338858\pi$ and $|\mathfrak{q}^2| = |\mathfrak{q}_c^2| \approx 2.224$, as shown in figure 9. A similar effect at a purely imaginary momentum was observed by Withers (see figure 3 in ref. [27]), and numerous instances of pole collisions (i.e. quasinormal level-crossings) at real momenta were reported earlier (see e.g. [33, 57, 59–68]).

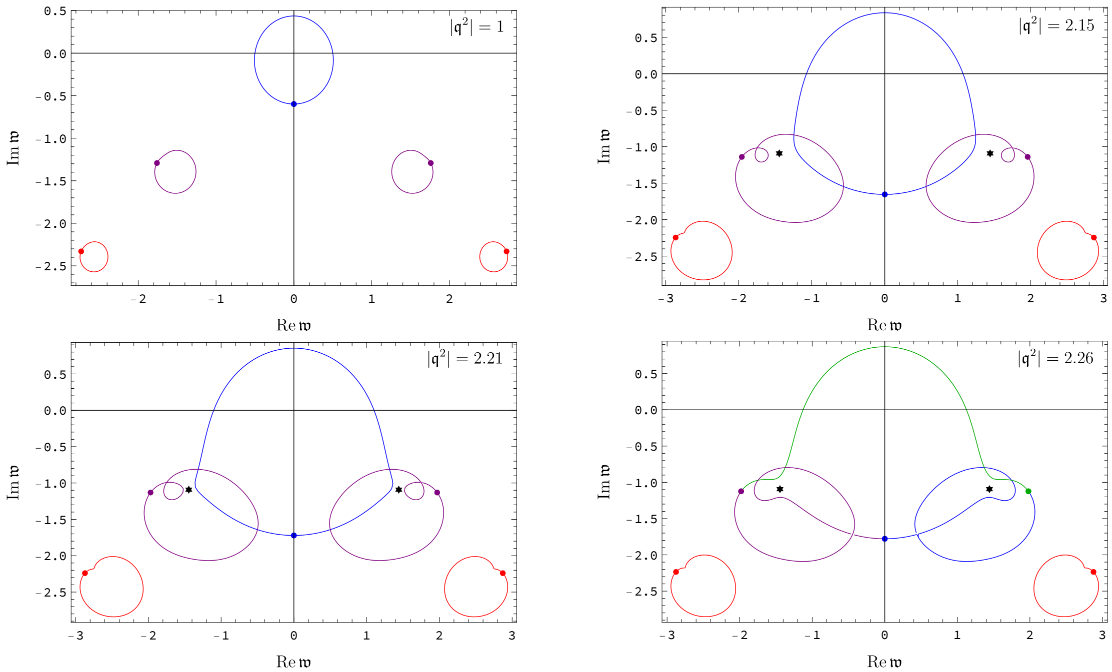

Figure 7. Quasinormal spectrum (poles of the retarded energy-momentum tensor two-point function in the $\mathcal{N} = 4$ SYM theory) in the shear channel plotted in the complex $\mathfrak{w}$-plane for different values of the complex momentum $\mathfrak{q}^2 = |\mathfrak{q}^2| e^{i\theta}$. Large dots in all four plots correspond to the location of the poles for purely real momentum, $\mathfrak{q}^2$ (i.e. at $\theta = 0$) [41]. The hydrodynamic shear pole is the blue pole closest to the real axis in the top left panel. As θ increases from 0 to 2π, each pole moves in a counter-clockwise direction, following the trajectory of its colour. At $|\mathfrak{q}^2| = 1$, all poles follow a closed orbit (top left panel). At $|\mathfrak{q}^2| = 2.15$ (top right panel), the trajectory of the hydrodynamic pole intersects the trajectories of the two nearest gapped poles. With $|\mathfrak{q}^2|$ further increasing to $|\mathfrak{q}^2| = 2.21$, the poles nearly collide at the positions marked by asterisks (bottom left panel). The actual collision occurs at the critical point with the momentum (4.20), $|\mathfrak{q}_c^2| \approx 2.224$. At $|\mathfrak{q}^2| = 2.26$ (bottom right panel), the orbits of the three poles closest to the origin ($\mathfrak{w} = 0$) are no longer closed: the hydrodynamic pole and the two gapped poles exchange their positions cyclically as the phase θ increases from 0 to 2π. This is a manifestation of the quasinormal mode level-crossing. The dispersion relation $\mathfrak{w}(\mathfrak{q}^2)$ therefore has branch point singularities in the complex momentum squared plane at $\mathfrak{q}_c^2$.

4.3 Sound mode: hydrodynamic approximation to the spectral curve

The gauge-invariant perturbation corresponding to the sound mode obeys the equation [41]

$$Z_2'' - \frac{3\mathfrak{w}^2(1+u^2) + \mathfrak{q}^2(2u^2 - 3u^4 - 3)}{uf(3\mathfrak{w}^2 + \mathfrak{q}^2(u^2-3))} Z_2' + \frac{3\mathfrak{w}^4 + \mathfrak{q}^4(3 - 4u^2 + u^4) + \mathfrak{q}^2(4u^5 - 4u^3 + 4u^2\mathfrak{w}^2 - 6\mathfrak{w}^2)}{uf^2(3\mathfrak{w}^2 + \mathfrak{q}^2(u^2-3))} Z_2 = 0 \,. \tag{4.24}$$

The full sound spectral curve is constructed from the solution $Z_2(u; \mathfrak{q}^2, \mathfrak{w})$ obeying the incoming wave boundary conditions at the horizon and is given by

$$F(\mathfrak{q}^2, \mathfrak{w}) = Z_2(u = 0; \mathfrak{q}^2, \mathfrak{w}) = 0 \,. \tag{4.25}$$

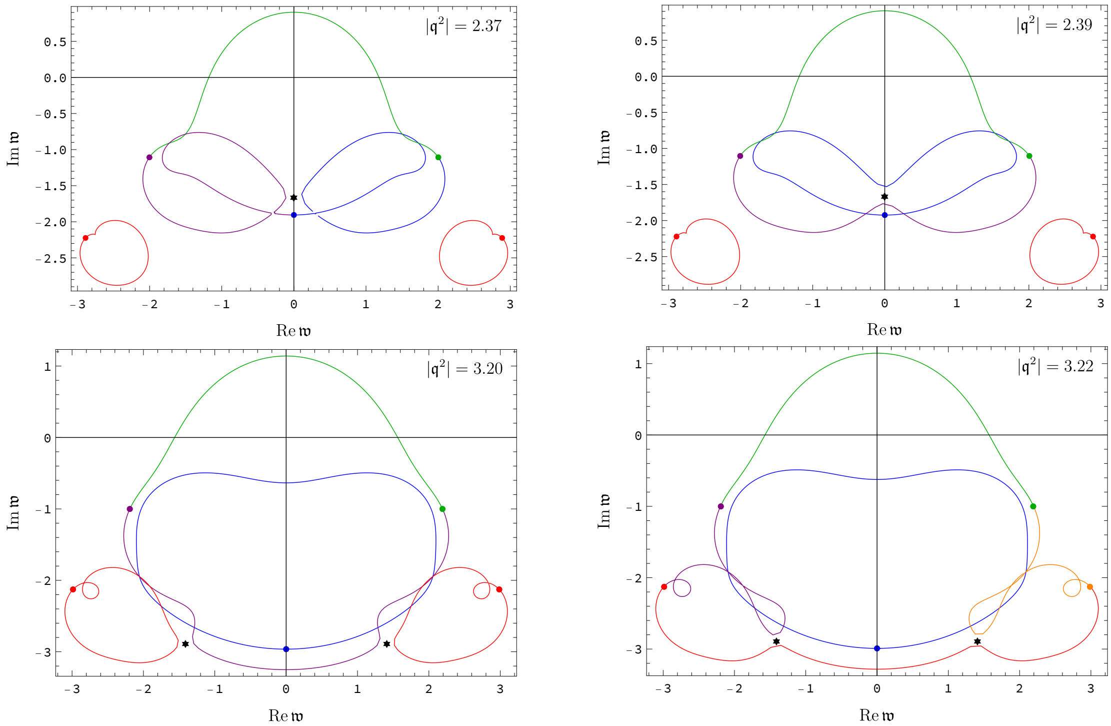

Figure 8. Quasinormal spectrum (poles of the retarded energy-momentum tensor two-point function in the $\mathcal{N} = 4$ SYM theory) in the shear channel plotted in the complex $\mathfrak{w}$-plane for different values of the complex momentum $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\theta}$. Large dots in all four plots correspond to the location of the poles for purely real momentum, $\mathfrak{q}^2$ (i.e. at $\theta = 0$) [41]. The second critical point (occuring at $|\mathfrak{q}^2_{c,1}| \approx 2.377$) is marked by asterisks in the figures showing the trajectories just before and just after the pole collision (top panels). With $|\mathfrak{q}^2|$ further increasing, the second pair of gapped modes becomes involved in the collisions leading to the third critical point at $|\mathfrak{q}^2_{c,2}| \approx 3.214$ (bottom panels).

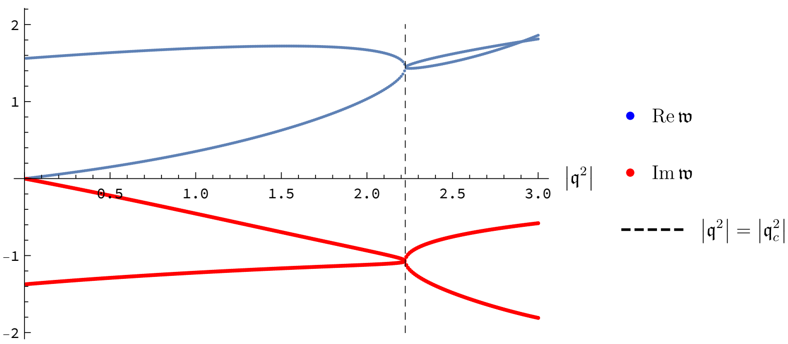

Figure 9. Quasinormal spectrum level-crossing: the real (blue curves) and the imaginary (red curves) parts of the hydrodynamic shear mode and the closest gapped quasinormal mode dispersion relations plotted as functions of $|\mathfrak{q}^2|$ at the fixed phase $\theta \approx 0.338858\pi$ of the complex momentum $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\theta}$. At $|\mathfrak{q}^2| = |\mathfrak{q}^2_c| \approx 2.224$, the level-crossing occurs.

Similarly to the shear mode case, one can first find a hydrodynamic approximation to eq. (4.25) analytically. For $\mathfrak{w} \ll 1$ and $\mathfrak{q} \ll 1$, eq. (4.24) can be solved perturbatively. Writing

$$Z_2(u) = \left(1-u^2\right)^{-\frac{i\mathfrak{w}}{2}} \sum_{n=0}^{\infty} \lambda^n z_n(u) \tag{4.26}$$

to enforce the boundary condition at the horizon, rescaling $\mathfrak{w} \to \lambda\mathfrak{w}$ and $\mathfrak{q} \to \lambda\mathfrak{q}$, and assuming $\lambda \ll 1$, we find the following recurrence relation for the functions $z_n(u)$:

$$\begin{aligned} z_n'' - &\frac{3\left(1+u^2\right)\mathfrak{w}^2 - \left(3-2u^2+3u^4\right)\mathfrak{q}^2}{u\left(1-u^2\right)\left[3\mathfrak{w}^2-\left(3-u^2\right)\mathfrak{q}^2\right]} z_n' - \frac{4u^2\mathfrak{q}^2}{\left(1-u^2\right)\left[3\mathfrak{w}^2-\left(3-u^2\right)\mathfrak{q}^2\right]} z_n \\ &+ \frac{2iu\mathfrak{w}}{1-u^2} z_{n-1}' - \frac{4iu^2\mathfrak{w}\mathfrak{q}^2}{\left(1-u^2\right)\left[3\mathfrak{w}^2-\left(3-u^2\right)\mathfrak{q}^2\right]} z_{n-1} \\ &+ \frac{\left(1+u+u^2\right)\mathfrak{w}^2 - \left(1+u\right)\mathfrak{q}^2}{u\left(1-u\right)\left(1+u\right)^2} z_{n-2} = 0\,, \end{aligned} \tag{4.27}$$

with $z_{-1} = z_{-2} = 0$. To fourth order in λ, we find the hydrodynamic algebraic curve to be (see also eq. (4.18) in ref. [3])

$$\begin{aligned} F(\mathfrak{q}^2,\mathfrak{w}) = \frac{\mathfrak{q}^2}{2} - \frac{3\mathfrak{w}^2}{2} - i\mathfrak{w}\mathfrak{q}^2 + \frac{\mathfrak{w}^4}{16}\left(\pi^2 - 12\ln^2 2 + 24\ln 2\right) - \frac{\mathfrak{q}^4}{12}\left(2\ln 2 - 8\right) \\ - \frac{\mathfrak{w}^2\mathfrak{q}^2}{48}\left(\pi^2 - 12\ln^2 2 + 48\ln 2\right) = 0\,. \end{aligned} \tag{4.28}$$

The form of $F(\mathfrak{q}^2,\mathfrak{w})$ for general $\mathfrak{w}$ and $\mathfrak{q}$ is at present not known to $O(\lambda^5)$. However, assuming that $\mathfrak{w}$ can be expanded in a series in powers of $\mathfrak{q}$, in ref. [5], $F(\mathfrak{q}^2)$ was computed to order $O(\lambda^5) = O(\mathfrak{q}^5)$, which was sufficient to find the coefficient in the sound mode dispersion relation multiplying $\mathfrak{q}^4$ (i.e. to third order in the gradient expansion):

$$\mathfrak{w} = \pm\frac{1}{\sqrt{3}}\mathfrak{q} - \frac{i}{3}\mathfrak{q}^2 \pm \frac{3-2\ln 2}{6\sqrt{3}}\mathfrak{q}^3 - \frac{i\left(\pi^2 - 24 + 24\ln 2 - 12\ln^2 2\right)}{108}\mathfrak{q}^4 + O(\mathfrak{q}^5)\,. \tag{4.29}$$

The first two terms in the dispersion relation (4.29) coincide with those obtained in ref. [40]. The third term was found in ref. [3] (see also ref. [69]). The fourth term was computed in ref. [5]. It appeared earlier in ref. [13] (with an incorrect coefficient in front of ln 2) and (correctly) in ref. [70] in the context of the fluid-gravity correspondence.

Here, we use the form of the algebraic curve (4.28) and apply the condition (2.22) to show that in the sound channel, the small $\mathfrak{w}$ and $\mathfrak{q}$ expansion of the spectral curve $F(\mathfrak{q}^2,\mathfrak{w})$ qualitatively correctly accounts for two sets of critical points:

$$\mathfrak{q}_c^2 = 0\,, \qquad \mathfrak{w}_c = 0\,, \tag{4.30}$$

$$\mathfrak{q}_c^2 \approx 0.34739 \pm 0.56763i\,, \qquad \mathfrak{w}_c \approx \pm 0.71165 + 0.39882i\,. \tag{4.31}$$

The critical point (4.30) is expected from hydrodynamics, as discussed in section 2.2. Having in mind the results of section 4.1, we may expect that the hydrodynamic approximation (4.31) to the position of the non-trivial critical point is not accurate. In section 4.4 we confirm this by finding the critical point exactly.

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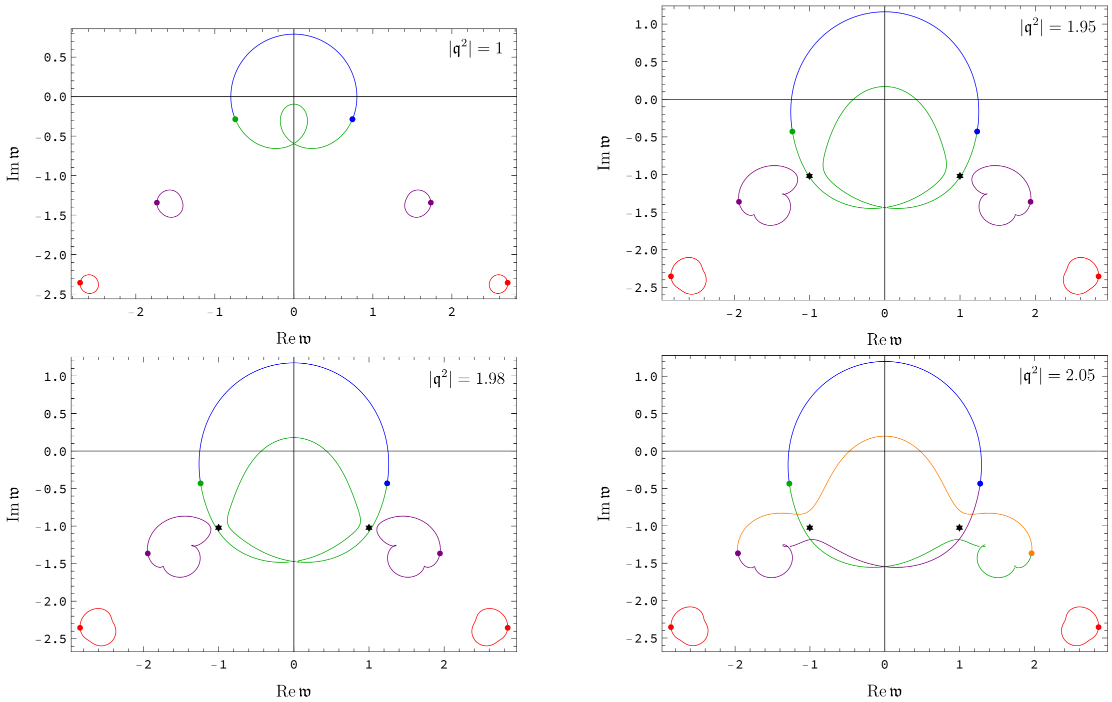

Figure 10. Quasinormal spectrum (poles of the retarded energy-momentum tensor two-point function in the $\mathcal{N} = 4$ SYM theory) in the sound channel plotted in the complex $\mathfrak{w}$-plane for different values of the complex momentum $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\theta}$. Large dots in all four plots correspond to the location of the poles for purely real momentum, $\mathfrak{q}^2$ (i.e. at $\theta = 0$) [41]. The hydrodynamic sound poles are the blue and the green poles closest to the real axis. As θ is tuned from 0 to 2π, each pole moves in a counter-clockwise direction and follows the trajectory of its colour. At $|\mathfrak{q}^2| = 1$ (top left panel), all poles follow a closed orbit. At $|\mathfrak{q}^2| = 1.95$ (top right panel), the trajectory of the two hydrodynamic sound poles comes close to the trajectories of the nearest gapped poles. With $|\mathfrak{q}^2|$ further increasing to $|\mathfrak{q}^2| = 1.98$, the poles nearly collide at the positions marked by asterisks (bottom left panel). The actual collision occurs at the critical value of the momentum (4.32), $|\mathfrak{q}_c^2| = 2$. At $|\mathfrak{q}^2| = 2.05$ (bottom right panel), the orbits of the four uppermost poles are no longer closed: the hydrodynamic poles and the two gapped poles exchange their positions cyclically as the phase θ increases from 0 to 2π — again, a manifestation of the quasinormal mode level-crossing. The dispersion relation $\mathfrak{w}(\mathfrak{q})$ therefore has branch cuts starting at $\mathfrak{q}_c$.

4.4 Sound mode: full spectral curve

As discussed in section 4.3, the origin (4.30) is a critical point of the sound mode dispersion relation $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)$, as predicted by hydrodynamics (see section 2.2). Proceeding as in section 4.2, we find the first set of critical points nearest to the origin at

$$\mathfrak{q}_c^2 = \pm 2i\,, \qquad \mathfrak{w}_c = \pm 1 - i\,, \tag{4.32}$$

within the limits of our numerical accuracy. Curiously, although eq. (4.24) looks rather complicated, one can check that with $\mathfrak{w}$ and $\mathfrak{q}^2$ given by (4.32), a simple analytic solution satisfying the correct boundary conditions at $u = 1$ and $u = 0$ is available. Explicitly, the

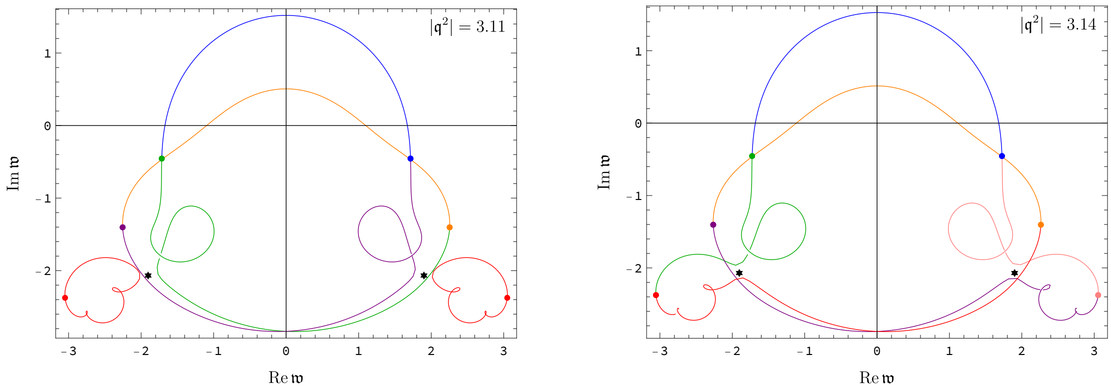

Figure 11. Quasinormal spectrum (poles of the retarded energy-momentum tensor two-point function in the $\mathcal{N} = 4$ SYM theory) in the sound channel plotted in the complex $\mathfrak{w}$-plane for different values of the complex momentum $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\theta}$. Large dots in all four plots correspond to the location of the poles for purely real momentum, $\mathfrak{q}^2$ (i.e. at $\theta = 0$) [41]. What is shown is the level-crossing phenomenon for the critical points in eq. (4.35).

two solutions corresponding to the pair of critical points in (4.32) are

$$(\mathfrak{q}_c^2 = 2i, \mathfrak{w}_c = 1 - i): \quad Z_2(u) = \mathcal{C}_2^+ (1-u)^{-\frac{i(1-i)}{2}} (1+u)^{-\frac{1-i}{2}} u^2 (u - 3i) \,, \tag{4.33}$$

$$(\mathfrak{q}_c^2 = -2i, \mathfrak{w}_c = -1 - i): \quad Z_2(u) = \mathcal{C}_2^- (1-u)^{-\frac{i(-1-i)}{2}} (1+u)^{\frac{-1-i}{2}} u^2 (u + 3i) \,, \tag{4.34}$$

where $\mathcal{C}_2^{\pm}$ are arbitrary constants. Although we were not able to show analytically that $\partial_{\mathfrak{w}} Z(u = 0) = 0$ at (4.32) as well, we have verified this numerically to high precision. The existence of the critical point (4.32) implies that the convergence radius of the sound mode dispersion relation is given by $|\mathfrak{q}^c_{\text{sound}}| = \sqrt{2} \approx 1.41421$. The next set of critical points is located at

$$\mathfrak{q}_c^2 \approx -0.01681 \pm 3.12967i \,, \qquad \mathfrak{w}_c \approx \pm 1.90134 - 2.04492i \,. \tag{4.35}$$

The behaviour of poles in the complex frequency plane is shown in figures 10 and 11, and the quasinormal level-crossing phenomenon is presented in figure 12.

4.5 Scalar mode: full spectral curve

In the scalar channel, the gauge-invariant metric perturbation $Z_3(u)$ obeys the equation [41]

$$Z_3'' - \frac{(1+u^2)}{uf} Z_3' + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{uf^2} Z_3 = 0 \,. \tag{4.36}$$

The full spectral curve is constructed from the solution $Z_3(u; \mathfrak{q}^2, \mathfrak{w})$ obeying the incoming wave boundary conditions at the horizon and is given by

$$F(\mathfrak{q}^2, \mathfrak{w}) = Z_3(u = 0; \mathfrak{q}^2, \mathfrak{w}) = 0 \,. \tag{4.37}$$

The quasinormal spectrum in the scalar channel has no hydrodynamic modes, but it exhibits the phenomenon of level-crossing for the gapped modes, as shown in figure 13. The

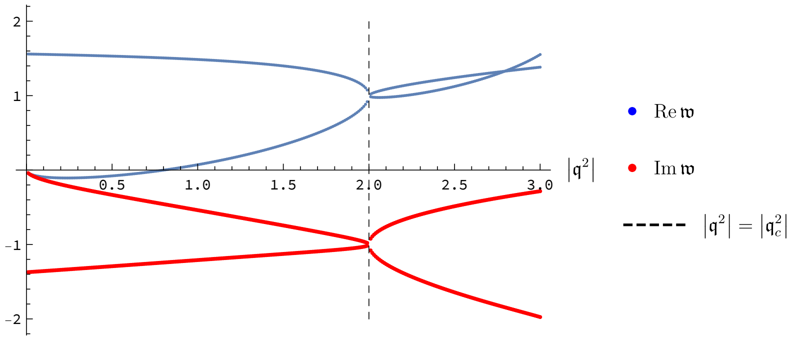

Figure 12. Quasinormal spectrum level-crossing in the sound channel: the real (blue curves) and the imaginary (red curves) parts of the hydrodynamic sound mode and the closest gapped quasinormal mode dispersion relations plotted as functions of $|\mathfrak{q}^2|$ at the fixed phase $\theta = \pi/2$ of the complex momentum $\mathfrak{q}^2 = |\mathfrak{q}^2| e^{i\theta}$. At $|\mathfrak{q}^2| = |\mathfrak{q}_c^2| = 2$, the level-crossing occurs.

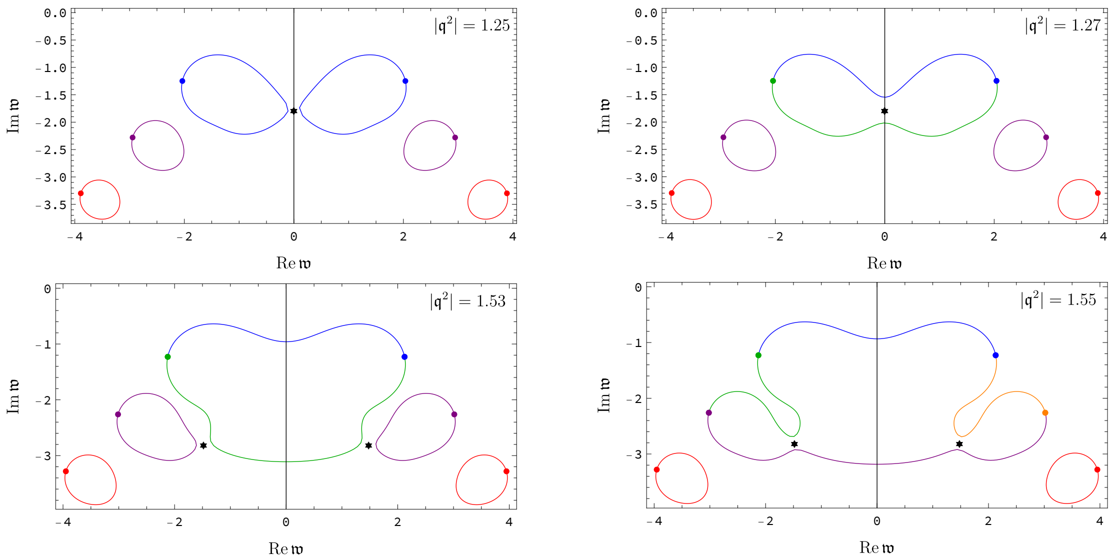

Figure 13. Quasinormal spectrum (poles of the retarded energy-momentum tensor two-point function in the $\mathcal{N} = 4$ SYM theory) in the scalar channel plotted in the complex $\mathfrak{w}$-plane for different values of the complex momentum $\mathfrak{q}^2 = |\mathfrak{q}^2| e^{i\theta}$. Large dots in all plots correspond to the location of the poles for purely real momentum, $\mathfrak{q}^2$ (i.e. at $\theta = 0$) [41]. There are no hydrodynamic modes in this channel, but the gapped modes exhibit the level-crossing phenomena at complex values of momenta given by eq. (4.38).

first two sets of critical points nearest to the origin are given by

$$\mathfrak{q}_c^2 \approx -1.25309\,, \qquad \mathfrak{w}_c \approx -1.76937i\,, \tag{4.38a}$$

$$\mathfrak{q}_c^2 \approx -1.49704 \pm 0.36674i\,, \qquad \mathfrak{w}_c \approx \mp 1.47977 - 2.79404i\,. \tag{4.38b}$$

These are branch points in the scalar (gapped) dispersion relation $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)$.

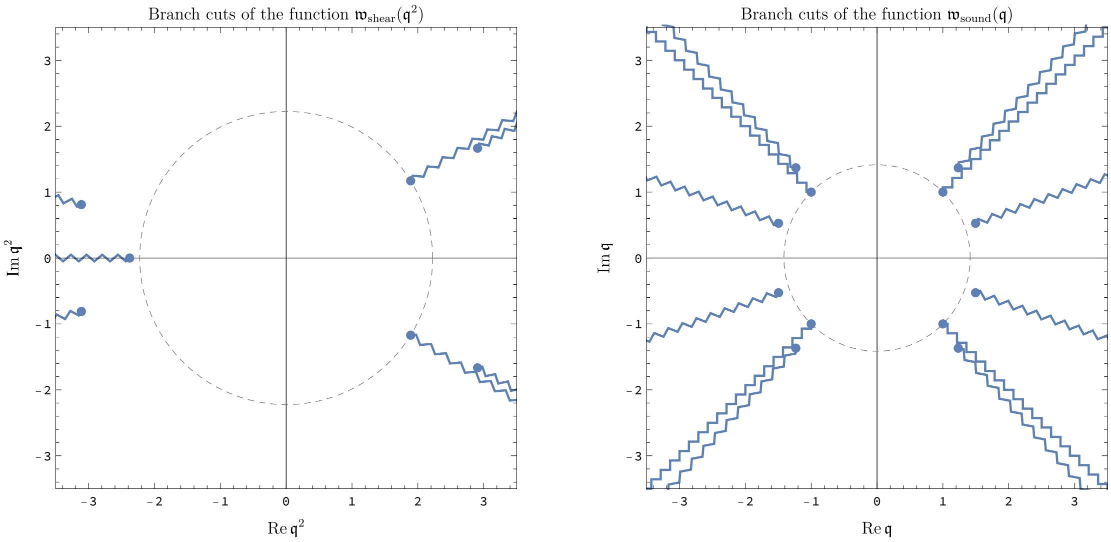

Figure 14. Branch point singularities, branch cuts and the domain of hydrodynamic series convergence for the shear mode in the complex $\mathfrak{q}^2$-plane (left panel) and the sound mode in the complex $\mathfrak{q}$-plane (right panel).

4.6 Analytic structure of the hydrodynamic dispersion relations

From the discussion above, it is clear that the shear mode dispersion relation $\mathfrak{w}_{\rm shear}(\mathfrak{q}^2)$ is an analytic function of complex $\mathfrak{q}^2$ in the circle $|\mathfrak{q}^2| < |\mathfrak{q}_c^2| \approx 2.224$. Since the appropriate second derivative of the spectral curve at the critical point is non-zero (i.e. $p = 2$ in the Puiseux language of section 2.3; see eq. (2.23)) which corresponds to a collision of *two* quasinormal modes, the critical point is the branch point singularity of the square root type, with the Puiseux series in powers of $\pm\sqrt{\mathfrak{q}^2 - \mathfrak{q}_c^2}$ providing the extension beyond the radius of convergence. We show the critical points, the radius of convergence and the appropriate branch cuts in the complex plane of $\mathfrak{q}^2$ in figure 14 (left panel).

For the sound mode dispersion relation, considered as a function of $\mathfrak{q}^2 \in \mathbb{C}$, the origin is a branch point, and the corresponding Puiseux series is given by eq. (2.21). It will be more convenient to consider the sound dispersion relation $\mathfrak{w}_{\rm sound}(\mathfrak{q})$ as a function of complexified magnitude $\mathfrak{q} \in \mathbb{C}$ of the wave-vector $\mathbf{q}$. The critical points, the radius of convergence and the appropriate branch cuts in the complex plane of $\mathfrak{q}$ are shown in figure 14 (right panel).

4.7 Pole-skipping in the full response functions

As already discussed in the Introduction, analytically continued hydrodynamic modes appear to be connected to the parameters of an OTOC related to the microscopic many-body quantum chaos. The apparent connection is provided by the phenomenon of pole-skipping [29–32], whereby a pole and a zero of a two-point correlation function collide for some $\mathfrak{w}, \mathfrak{q} \in \mathbb{C}$. In the sound channel of the energy-momentum tensor retarded two-point function, the pole-skipping has been studied in the context of holography [29, 31, 32], effective field theory [30] and two-dimensional conformal field theory in the limit of large central charge [71]. Here, we extend the discussion in refs. [29, 31, 32] to show that pole-skipping also occurs in other channels.

Consider a retarded two-point function $G^R(\mathfrak{w}, \mathfrak{q})$ of the energy-momentum tensor components at finite temperature. Schematically, and modulo tensor structure, the correlator can be written as

$$G^R(\mathfrak{w}, \mathfrak{q}) \sim \frac{b(\mathfrak{w}, \mathfrak{q})}{a(\mathfrak{w}, \mathfrak{q})}\,, \tag{4.39}$$

where $a(\mathfrak{w}, \mathfrak{q})$ necessarily contains a gapless hydrodynamic mode $\mathfrak{w} = \mathfrak{w}(\mathfrak{q})$ (either shear or sound) in the appropriate channels [41]. More generally, let $\mathcal{Z}_d = \{\mathfrak{w} = \mathfrak{w}(\mathfrak{q}) : a(\mathfrak{w}(\mathfrak{q}), \mathfrak{q}) = 0\}$ and $\mathcal{Z}_n = \{\mathfrak{w} = \mathfrak{w}(\mathfrak{q}) : b(\mathfrak{w}(\mathfrak{q}), \mathfrak{q}) = 0\}$, where we assume for simplicity that all zeros are simple. Then, pole-skipping occurs at generically complex $(\mathfrak{q}_*, \mathfrak{w}_*) \in \mathcal{P} = \mathcal{Z}_n \cap \mathcal{Z}_d$.

For theories with available gravity dual descriptions, the set $\mathcal{P}$ can be determined directly either by computing $\mathcal{Z}_n$ and $\mathcal{Z}_d$ (the set $\mathcal{Z}_d$ is nothing but the quasinormal spectrum) or from the dual gravity equations of motion (see below). In the case of energy-momentum tensor correlators of the $\mathcal{N} = 4$ SYM theory in the limit of infinite N_c and infinite 't Hooft coupling λ, pole-skipping in the three channels occurs at points $(\mathfrak{q}_*, \mathfrak{w}_*)$ given by

$$\text{Sound channel}: \qquad \mathfrak{q}_* = \sqrt{\frac{3}{2}}\, i\,, \qquad \mathfrak{w}_* = i\,, \tag{4.40}$$

$$\text{Shear channel}: \qquad \mathfrak{q}_* = \sqrt{\frac{3}{2}}\,, \qquad \mathfrak{w}_* = -i\,, \tag{4.41}$$

$$\text{Scalar channel}: \qquad \mathfrak{q}_* = \sqrt{\frac{3}{2}}\, i\,, \qquad \mathfrak{w}_* = -i\,. \tag{4.42}$$

We observe that $|\mathfrak{q}_*| = \sqrt{3/2}$, $|\mathfrak{w}_*| = 1$ in all three channels. The connection to the Lyapunov exponent λ_L and the butterfly velocity v_B is given by the formulae

$$\text{Sound channel}: \qquad \mathfrak{w}_*(\mathfrak{q}_*) = \frac{i\lambda_L}{2\pi T} = i\mathfrak{D}_*\,, \qquad \mathfrak{q}_* = i\ell_*\,, \tag{4.43}$$

$$\text{Shear channel}: \qquad \mathfrak{w}_*(\mathfrak{q}_*) = -\frac{i\lambda_L}{2\pi T} = -i\mathfrak{D}_*\,, \qquad \mathfrak{q}_* = \ell_*\,, \tag{4.44}$$

$$\text{Scalar channel}: \qquad \mathfrak{w}_*(\mathfrak{q}_*) = -\frac{i\lambda_L}{2\pi T} = -i\mathfrak{D}_*\,, \qquad \mathfrak{q}_* = i\ell_*\,, \tag{4.45}$$

where $\ell_* = \sqrt{3/2}$, $\mathfrak{D}_* = 1$, and $v_B = \mathfrak{D}_*/\ell_*$. It is clear from figures 2 and 1 that the sound and the shear dispersion relations pass through their respective "chaos" points (4.40) or (4.41). In the scalar channel, which has no hydrodynamic modes, pole-skipping is exhibited by (one of the pair of) the lowest-lying gapped modes in the spectrum. This can be seen from figure 3. Thus, in the $\mathcal{N} = 4$ SYM theory at infinite 't Hooft coupling, the values of λ_L and v_B defined by pole-skipping in the complexified dispersion relations of the lowest-lying modes (either hydrodynamic or gapped) coincide with those obtained from the appropriate limit of the OTOC:[14]

$$\lambda_L = 2\pi T\,, \qquad v_B = \sqrt{\frac{2}{3}}\,, \qquad \ell_* = \frac{\lambda_L}{2\pi T v_B} = \sqrt{\frac{3}{2}}\,. \tag{4.46}$$

[14] To subleading order in the inverse 't Hooft coupling expansion, the butterfly velocity is $v_B = \sqrt{2/3}\left(1 + \frac{23\zeta(3)}{16}\lambda^{-3/2}\right)$ while the relevant (long-distance) Lyapunov exponent remains uncorrected [32].

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In fact, irrespectively of the channel in question, we can define the (maximal holographic) Lyapunov exponent and the butterfly velocity through the pole-skipping location exhibited by the mode closest to the origin in the complex plane of $\mathfrak{w}$:

$$\lambda_L = 2\pi T \left|\mathfrak{w}_*\right| , \qquad v_B = \left|\frac{\mathfrak{w}_*}{\mathfrak{q}_*}\right| = \frac{|\mathfrak{w}_*|}{\ell_*} . \tag{4.47}$$

Pole-skipping points $(\mathfrak{q}_*, \mathfrak{w}_*)$ can be found directly from the dual gravity equations of motion [29, 31]. To show this explicitly for the $\mathcal{N} = 4$ SYM theory, we follow the arguments of ref. [31] and examine the horizon behaviour of Einstein's equations

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - 6 g_{\mu\nu} = 0 \tag{4.48}$$

in the infalling Eddington-Finkelstein (EF) coordinates (v, r, x^i) with

$$v = t + r_*(r) , \qquad \frac{dr_*}{dr} = \frac{1}{r^2 f(r)} . \tag{4.49}$$

We perturb the $5d$ AdS-Schwarzschild metric $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -r^2 f(r) dv^2 + 2dvdr + r^2 d\vec{x}^2$ to first order, $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}(r) e^{-i\omega v + ikz}$, and expand the (regular) metric fluctuations around the horizon $r = r_0$ as

$$\delta g_{\mu\nu}(r) = \sum_{n=0}^{\infty} \delta g_{\mu\nu}^{(n)} (r - r_0)^n . \tag{4.50}$$

Similarly to what was observed in ref. [31] for the sound channel, we find that in any channel, there exists a linear combination of the components $E_{\mu\nu}$, which vanishes identically at the horizon $r \to r_0$ at the pole-skipping values of the parameters $(\mathfrak{q}_*, \mathfrak{w}_*)$. Explicitly,

$$\text{Sound channel}: \quad \lim_{r\to r_0} E_{vv} = 0 \quad \text{at} \quad (\mathfrak{q}_*, \mathfrak{w}_*) = \left(\sqrt{3/2}\, i, i\right) , \tag{4.51}$$

$$\text{Shear channel}: \quad \lim_{r\to r_0} \left(E_{vx} + i\sqrt{\frac{2}{3}} E_{xz} \right) = 0 \quad \text{at} \quad (\mathfrak{q}_*, \mathfrak{w}_*) = \left(\sqrt{3/2}, -i\right) , \tag{4.52}$$

$$\text{Scalar channel}: \quad \lim_{r\to r_0} E_{xy} = 0 \quad \text{at} \quad (\mathfrak{q}_*, \mathfrak{w}_*) = \left(\sqrt{3/2}\, i, -i\right) . \tag{4.53}$$

In other words, pole-skipping occurs at values of the parameters $(\mathfrak{q}_*, \mathfrak{w}_*)$ for which the rank of the matrix $E_{\mu\nu}$ decreases at the horizon.

We note also that for the $\mathcal{N} = 4$ SYM theory, the chaos point $|\mathfrak{q}_*^2| = 3/2$ lies within the radius of convergence of the hydrodynamic series in both the shear ($|\mathfrak{q}_c^2| \approx 2.224$) and the sound ($|\mathfrak{q}_c^2| = 2$) channels.

5 Pole-skipping and level-crossing in $2d$ thermal CFT correlators

In a $2d$ CFT, the (equilibrium) retarded finite-temperature two-point correlation function of an operator of non-integer scaling dimension Δ and spin zero in momentum space is

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given by the expression[15,16] [34]

$$G^R(\mathfrak{w},\mathfrak{q}) = C_\Delta\,\Gamma\left(\frac{\Delta}{2}+\frac{i(\mathfrak{w}-\mathfrak{q})}{2}\right)\Gamma\left(\frac{\Delta}{2}+\frac{i(\mathfrak{w}+\mathfrak{q})}{2}\right)\Gamma\left(\frac{\Delta}{2}-\frac{i(\mathfrak{w}-\mathfrak{q})}{2}\right)$$
$$\times\,\Gamma\left(\frac{\Delta}{2}-\frac{i(\mathfrak{w}+\mathfrak{q})}{2}\right)\Big[\cosh(\pi\mathfrak{q})-\cos(\pi\Delta)\cosh(\pi\mathfrak{w})+i\sin(\pi\Delta)\sinh(\pi\mathfrak{w})\Big]\,, \tag{5.1}$$

where C_Δ is the normalisation constant, and we put $T_L = T_R = T$. Note also that here, in $1+1$ dimensions, the symbol $\mathfrak{q}$ denotes $q/2\pi T$, rather than $|q|/2\pi T$. Similar formulae can be written for integer Δ [34]. The correlator (5.1) has a sequence of poles at

$$\mathfrak{w}_n(\mathfrak{q}) = \pm\mathfrak{q} - i\,(2n+\Delta)\,, \tag{5.2}$$

where $n = 0, 1, 2, \ldots$. These are precisely the quasinormal frequencies of the dual BTZ black hole [34, 72]. In this section, we shall examine these correlators for their pole-skipping and level-crossing properties.[17,18]

5.1 Pole-skipping in the full response functions

The zeros of the correlator (5.1) come from the zeros of the expression in the square brackets,

$$\cosh(\pi\mathfrak{q}) - \cos(\pi\Delta)\cosh(\pi\mathfrak{w}) + i\sin(\pi\Delta)\sinh(\pi\mathfrak{w})$$
$$= 2\sin\left[\frac{\pi}{2}(\Delta+i\mathfrak{w}-i\mathfrak{q})\right]\sin\left[\frac{\pi}{2}(\Delta+i\mathfrak{w}+i\mathfrak{q})\right]\,, \tag{5.3}$$

and are given by

$$\Delta + i\mathfrak{w} - i\mathfrak{q} = 2n_1\,, \tag{5.4}$$
$$\Delta + i\mathfrak{w} + i\mathfrak{q} = 2n_2\,, \tag{5.5}$$

where $n_1, n_2 = 1, 2, 3, \ldots$. Note that the zeros of eq. (5.3) with $n_1, n_2 = 0, -1, -2, \ldots$ are not among the zeros of the correlator since they are identically (i.e. for arbitrary $\mathfrak{w}$, $\mathfrak{q}$) cancelled by the poles of the first two Gamma-functions in eq. (5.1).

[15] In ref. [34], the expression for $G^R(\omega, q)$ was derived holographically from dual gravity. For integer Δ, it was further checked in ref. [34] that thus obtained formula coincides (up to normalisation) with the retarded correlator obtained from $2d$ CFT.

[16] The expression for $G^R(\omega, q)$ in the form (4.16) of ref. [34] assumes $\mathfrak{w}, \mathfrak{q} \in \mathbb{R}$. To be valid for generic $\mathfrak{w}, \mathfrak{q} \in \mathbb{C}$, it has to be rewritten in the form (5.1). We would like to thank D. Vegh for pointing this out.

[17] Similar issues have been recently independently studied in ref. [73]. The results of ref. [73] agree with ours whenever they overlap.

[18] Here, we only consider correlation functions of $2d$ CFT operators with scaling dimension Δ and spin $s = 0$. The energy-momentum, having $\Delta = 2$ and $s = 2$, is not of this type. Its finite-temperature two-point function (see e.g. refs. [71, 74]) in momentum space has a pole corresponding to a mode propagating on the light cone. The corresponding dispersion relation line passes through the "chaos" point of that correaltor [71], just as it does in $4d$.

The pole-skipping phenomenon in G^R occures for $\mathfrak{w}$ and $\mathfrak{q}$ simultaneously satisfying the conditions (5.2) and (5.4), (5.5), i.e. for

$$\mathfrak{q}_* = \pm i(\Delta + n - n_*)\,, \tag{5.6}$$
$$\mathfrak{w}_* = -i(n + n_*)\,, \tag{5.7}$$

where $n = 0, 1, 2, \ldots$ and $n_* = 1, 2, \ldots$ (here n_* denotes either n_1 or n_2), and Δ is not an integer.[19]

We also note that the Euler reflection formula, $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$, can be used to rewrite the correlator (5.1) in the form

$$G^R(\mathfrak{w}, \mathfrak{q}) = C_\Delta \, \frac{2\pi^2 \Gamma\left(\frac{\Delta}{2} - \frac{i(\mathfrak{w}-\mathfrak{q})}{2}\right)\Gamma\left(\frac{\Delta}{2} - \frac{i(\mathfrak{w}+\mathfrak{q})}{2}\right)}{\Gamma\left(1-\frac{\Delta}{2} - \frac{i(\mathfrak{w}-\mathfrak{q})}{2}\right)\Gamma\left(1-\frac{\Delta}{2} - \frac{i(\mathfrak{w}+\mathfrak{q})}{2}\right)}\,. \tag{5.8}$$

For integer Δ, the poles of G^R are still given by eq. (5.2), but the functional form of the correlators is somewhat different from (5.1) (see ref. [34]). Here, we focus on the case of $\Delta = 2$. For $\Delta = 2$, one has [34]

$$G^R \sim \left(\mathfrak{w}^2 - \mathfrak{q}^2\right)\left[\psi\left(1 - \frac{i}{2}\left(\mathfrak{w} - \mathfrak{q}\right)\right) + \psi\left(1 - \frac{i}{2}\left(\mathfrak{w} + \mathfrak{q}\right)\right)\right]\,, \tag{5.9}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. The singularities of the correlator (5.9) are simple poles located at

$$\mathfrak{w}_n(\mathfrak{q}) = \pm\mathfrak{q} - 2i\,(n+1)\,, \qquad n = 0, 1, 2, \ldots\,. \tag{5.10}$$

In the case of pole-skipping, they are cancelled by the zeros coming from the numerator $\mathfrak{w}^2 - \mathfrak{q}^2$, which occur when $\mathfrak{w}_n = \mp\mathfrak{q}$. The discrete set of momenta $\mathfrak{q}$ that satisfies this condition is

$$\mathfrak{q} = \pm i(n+1)\,, \qquad n = 0, 1, 2, \ldots\,. \tag{5.11}$$

Therefore, the pole-skipping points for $\Delta = 2$ are given by

$$\mathfrak{q}_* = \pm i(n+1)\,, \tag{5.12}$$
$$\mathfrak{w}_* = -i(n+1)\,, \tag{5.13}$$

where $n = 0, 1, 2, \ldots$. In particular, for the pair of poles that lies closest to the origin in the complex $\mathfrak{w}$ plane (ones with $n = 0$ among those in eq. (5.10)),

$$\mathfrak{w}_0^\pm(\mathfrak{q}) = \pm\mathfrak{q} - 2i\,, \tag{5.14}$$

the branch $\mathfrak{w}_0^+$ passes through the (lowest-lying) pole-skipping point $\mathfrak{q}_* = i$, $\mathfrak{w}_* = -i$, whereas the branch $\mathfrak{w}_0^-$ does not. The branch $\mathfrak{w}_0^-$ passes through the pole-skipping point $\mathfrak{q}_* = -i$, $\mathfrak{w}_* = -i$, whereas the branch $\mathfrak{w}_0^+$ does not (see figure 15). Finally, we note that for $\Delta = 1$, the correlator is directly proportional to the sum of two $\psi-$functions [34], and there is no pole-skipping.

[19] Introducing $Q = n_*$ and $N = n + n_*$, the pole-skipping condition can be written as $\mathfrak{q}_* = \pm i(\Delta + N - 2Q)$, $\mathfrak{w}_* = -iN$, where $N = 1, 2, \ldots$ and $Q = 1, \ldots, N$. This coincides with the results in ref. [73]. We would like to thank R. Davison for pointing out an error in eqs. (5.6), (5.7) in the first version of this paper.

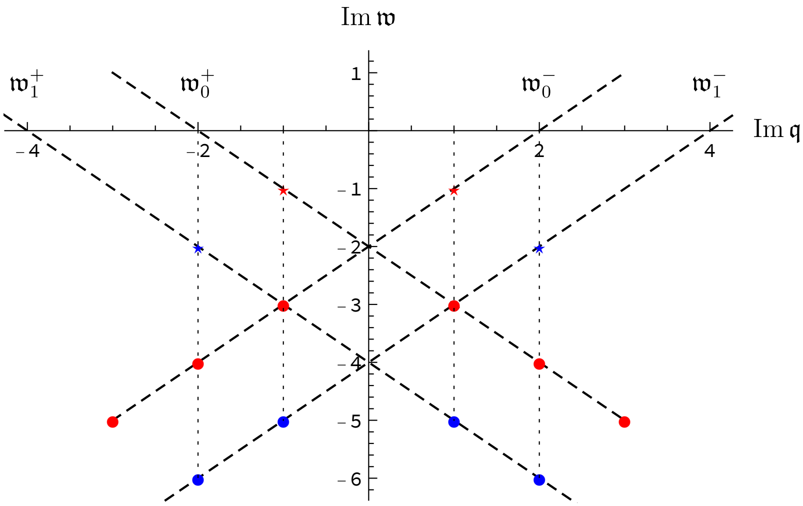

Figure 15. Pole-skipping and level-crossing points in a $2d$ CFT correlator of operators with conformal dimension $\Delta = 2$, for the smallest in magnitude poles $\mathfrak{w}_0^{\pm}$ and $\mathfrak{w}_1^{\pm}$. The red stars at $\operatorname{Im}\mathfrak{q} = \pm 1$ correspond to the pole-skipping points and the red dots label the critical points of level-crossings of $\mathfrak{w}_0^{\pm}$. The blue stars at $\operatorname{Im}\mathfrak{q} = \pm 2$ correspond to the pole-skipping points and the blue dots label the critical points of level-crossings of $\mathfrak{w}_1^{\pm}$.

5.2 BTZ spectrum level-crossing

Since the quasinormal spectrum $\mathfrak{w}_n^{\pm}(\mathfrak{q})$ is known explicitly (see eq. (5.2)), the level-crossing points can be found directly. Such level-crossing points were used above in theories with gapless excitations to determine the radius of convergence of their hydrodynamic series. Considering complex $\mathfrak{q} = |\mathfrak{q}|e^{i\theta}$, we have

$$\operatorname{Re}\mathfrak{w}_n^{\pm} = \pm|\mathfrak{q}|\cos\theta \equiv X\,, \tag{5.15}$$

$$\operatorname{Im}\mathfrak{w}_n^{\pm} = \pm|\mathfrak{q}|\sin\theta - 2n - \Delta \equiv Y\,, \tag{5.16}$$

and thus the orbits followed by the poles in the complex $\mathfrak{w}$ plane when the phase θ changes from 0 to 2π are circles

$$X^2 + (Y + 2n + \Delta)^2 = |\mathfrak{q}|^2\,, \qquad n = 0, 1, 2, \dots\,. \tag{5.17}$$

The poles move counter-clockwise and collide on the imaginary axis of $\mathfrak{w}$ (moreover, at integer values of $|\mathfrak{w}|$ if Δ is an integer). More precisely, two poles collide when $\mathfrak{w}_n^- = \mathfrak{w}_m^+$, $m \neq n$, i.e. when (the case $n = 0$ or $m = 0$ is treated separately below)

$$\mathfrak{q}_{\rm c} = i\,(m-n)\,, \tag{5.18}$$

$$\mathfrak{w}_{\rm c} = -i\,(m+n+\Delta)\,, \tag{5.19}$$

$$m, n = 1, 2, \dots, \text{ with } m \neq n\,, \tag{5.20}$$

with the first collision occurring for $m = n \pm 1$, i.e. for $\mathfrak{q}_{\rm c} = \pm i$, $\mathfrak{w}_{\rm c} = -i\,(2n + \Delta \pm 1)$, $n = 1, 2, \dots$. The mode with $n = 0$, i.e. $\mathfrak{w}_0^{\pm} = \pm\mathfrak{q} - i\Delta$, has only one neighbour, and the

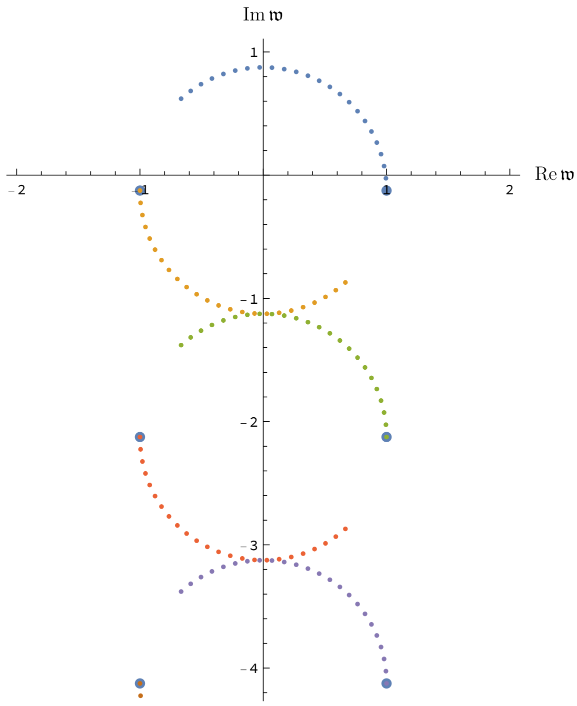

Figure 16. BTZ quasinormal spectrum for $\Delta = 1/8$ in the complex $\mathfrak{w}$-plane at complex momentum $\mathfrak{q} = |\mathfrak{q}|e^{i\theta}$, with $|\mathfrak{q}| = 1$ and θ changing from 0 to $3\pi/4$. The spectrum at $\theta = 0$ is shown by large dots.

critical points correspond to $\mathfrak{w}_0^- = \mathfrak{w}_n^+$ (i.e. $\mathfrak{q}_c = in$ and $\mathfrak{w}_c = -i(n+\Delta)$) or $\mathfrak{w}_n^- = \mathfrak{w}_0^+$ (i.e. $\mathfrak{q}_c = -in$ and $\mathfrak{w}_c = -i(n+\Delta)$, $n = 1, 2, \ldots$). Thus, the zero mode critical points are

$$\mathfrak{q}_c = \pm in \,, \tag{5.21}$$
$$\mathfrak{w}_c = -i\,(n+\Delta)\,, \tag{5.22}$$
$$n = 1, 2, \ldots\,. \tag{5.23}$$

The motion of poles in the complex frequency plane and their level-crossings are illustrated for $\Delta = 1/8$ and $|\mathfrak{q}| = 1$ in figure 16.

For larger $|\mathfrak{q}|$, the trajectories intersect but the poles miss each other, so there is no phenomenon of one trajectory crossing into and continuing as the other. In a sense, here, we have “level-touching” rather than “level-crossing”. The nearest critical points are thus

$$\mathfrak{q}_c = \pm i\,, \qquad \mathfrak{w}_c = -i\,(2n+\Delta\pm 1)\,, \qquad n = 1, 2, \ldots\,, \tag{5.24}$$
$$\mathfrak{q}_c = \pm i\,, \qquad \mathfrak{w}_c = -i\,(\Delta+1)\,, \qquad n = 0\,. \tag{5.25}$$

In particular, for $\Delta = 2$, we have[20]

$$\mathfrak{q}_c = i\,, \qquad \mathfrak{w}_c = -3i, -5i, -7i, -9i\ldots\,, \tag{5.26}$$
$$\mathfrak{q}_c = -i\,, \qquad \mathfrak{w}_c = -3i, -3i, -5i, -7i, \ldots\,. \tag{5.27}$$

[20]These critical points are single poles: the collisions may occur at the same point $\mathfrak{w}_c$, but the phases are different for different modes.

We observe that for non-integer Δ, the values of $\mathfrak{q}$ corresponding to the pole-skipping and the level-crossing do not coincide. The same is (trivially) true for $\Delta = 1$ as well, where there is no pole-skipping in the correlator at all. For $\Delta = 2$, however, a curious picture emerges. Consider again the "sound" mode $\mathfrak{w}_0^{\pm}$ (5.14). The imaginary parts of the two branches obey

$$\mathrm{Im}\,\mathfrak{w}_0^{\pm} = \pm\mathrm{Im}\,\mathfrak{q} - 2\,. \tag{5.28}$$

The pole-skipping points found in section 5.1 are $\mathfrak{q}_* = i$, $\mathfrak{w}_* = -i$ and $\mathfrak{q}_* = -i$, $\mathfrak{w}_* = -i$, and the critical points are $\mathfrak{q}_c = \pm in$, $\mathfrak{w}_c = -i(n+2)$, $n = 1, 2, \ldots$. For $\Delta = 2$, and the mode with $n = 1$,

$$\mathfrak{w}_1^{\pm}(\mathfrak{q}) = \pm\mathfrak{q} - 4i\,, \tag{5.29}$$

the pole-skipping occurs at $\mathfrak{q} = \pm 2i$ and $\mathfrak{w} = -2i$. The critical points are located at $\mathfrak{q}_c = \pm i(m-1)$, $\mathfrak{w}_c = -i(m+3)$, $m = 2, 3, \ldots$. This is illustrated in figure 15.

6 Discussion

In this paper, we introduced spectral curves as a useful tool for investigating analytic properties of gapless collective excitations in classical hydrodynamics.[21] We showed that the dispersion relations of hydrodynamic modes, such as shear and sound modes, are generically given by Puiseux series expansions in rational powers of the spatial momentum squared. These series are guaranteed to converge in some vicinity of the origin (the point with zero frequency $\omega = 0$ and zero spatial momentum $\mathbf{q}^2 = 0$ in the $(\omega, \mathfrak{q}^2) \in \mathbb{C}^2$ space), so long as the analyticity of the spectral curve at the origin can be proven (e.g. by holographic or other means). Thus, given the analyticity of the spectral curve, the asymptotic nature of the series for hydrodynamic modes in momentum space can be automatically ruled out. The radius of convergence of the series is given by the distance from the origin to the critical point of the spectral curve nearest to the origin. After developing the general theory, we then used holography as a theoretical laboratory where all these of features can be seen and analysed explicitly. Before focusing on the main example of the strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at finite temperature, to illustrate our method, we first considered the holographic model with broken translation symmetry, where an exact spectral curve is available. We have shown that the critical points of the spectral curves can be found by studying quasinormal spectra at complex values of spatial momentum: the critical points correspond to the collisions of quasinormal frequencies (poles of dual correlators) in the complex frequency plane at critical (generically, complex) values of spatial momentum. These values also set the radii of convergence for the dispersion relation. We call this phenomenon the quasinormal mode level-crossing, in analogy with the well-known phenomenon of level-crossing for eigenvalues of Hermitian operators.

Applying these methods to the strongly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, we found that the gradient expansions for the hydrodynamic shear and sound modes have finite radii of convergence given by $q^{c}_{\text{sound}} = \sqrt{2}\,\omega_0$ for the sound mode and

[21] See footnote 1.

by $q^{c}_{\text{shear}} \approx 1.49\,\omega_0$ for the shear mode, where $\omega_0 = 2\pi T$ is the fundamental Matsubara frequency. Thus, in both channels, the hydrodynamic series converge up to the order of $|\mathbf{q}|/T \sim O(10)$, which is a vast improvement over the naive expectation that $|\mathbf{q}|/T \ll 1$ provides a natural expansion parameter for hydrodynamic dispersion relations. The obstruction to convergence in the example of the $\mathcal{N} = 4$ SYM theory comes from the collision of poles of the two-point correlation function of the energy-momentum tensor at complex q^2.

As mentioned in the Introduction, the problem of extending the hydrodynamic modes in the complex momentum plane beyond the branch point singularity was recently investigated by Withers [27] in the context of a holographic model in $2 + 1$ dimensions with finite chemical potential. The shear-diffusion mode series could be Padé-resummed and extended beyond the branch point singularity, which was in that case located at an imaginary value of momentum. The main focus of ref. [27] was on the possibility of recovering the full spectrum from the hydrodynamic derivative expansion, similar to recovering the non-hydrodynamic modes from asymptotic series via Borel resummation and resurgence [17, 18, 25]. The quasinormal spectrum in the holographic models with finite temperature T and non-vanishing chemical potential μ such as the one considered in ref. [27] is rather complicated and changes qualitatively as the parameter T/μ is varied (in particular, it involves pole collisions even at real values of the momentum) [59, 66, 75–79]. In ref. [27], the shear-diffusion mode was found to have a radius of convergence inversely proportional to the chemical potential. Naively, this would imply infinite radius of convergence in the limit of vanishing μ, in apparent contradiction with our results. However, the result of ref. [27] was obtained at a specific fixed value of T/μ, and we expect it to change when the complex momentum behaviour of other gapped modes in the model is taken into account with T/μ increasing. This will require further study. It would be also interesting to extend the results of the present work to the sound channel (not considered in ref. [27]) as well as to other holographic models with finite chemical potential such as the STU model [80], and other models [81–83], including those in the large D limit [84].

Pole collisions in the correlation functions appear in holographic models in different contexts [33, 57, 59–68]. No less interesting are collisions among poles and zeros of the correlators known as pole-skipping [29–32]. What we have shown here is that this phenomenon, known to exist in the sound channel of strongly coupled $\mathcal{N} = 4$ SYM theory [29], exists also in the shear and scalar channels of the energy-momentum correlators. The conjectured connection to the OTOC thus allows one to determine the parameters quantifying microscopic many-body chaos (scrambling time and butterfly velocity) by considering the complexified behaviour of the lowest-lying modes (those with the smallest $|\omega|$ in the spectrum) in any channel, be it a channel with or without hydrodynamic modes. In general, the critical points and the pole-skipping points are different. We have analysed the $2d$ CFT finite-temperature correlators and the spectra of the dual BTZ black hole to demonstrate this explicitly. What this implies for the relation between chaos and hydrodynamics is that the "chaos" (or pole-skipping) points can lie within or outside of the radius of convergence of the hydrodynamic series. In particular, while this is not the case in the holographic model with broken translation symmetry considered in section 3, we found that in the

$\mathcal{N} = 4$ SYM theory, pole-skipping points for both of the two hydrodynamic modes lie within the radius of convergence of the corresponding dispersion relations. This provides an explanation for the observation of the fast convergence of the hydrodynamic series to the exact chaos point in ref. [29].

Can finiteness of the convergence radius of the hydrodynamic modes dispersion relations expansion be taken as a criterion for validity of hydrodynamics? By analogy, one may think of a free particle whose dispersion relation $\omega = \sqrt{p^2 + m^2} - m = p^2/2m + \ldots$ has branch points located at $p = \pm im$, and for which the failure of the convergence of the gradient expansion corresponds to the breakdown of the non-relativistic approximation. We hope our results may be of interest for studies of higher-order hydrodynamics necessary for improving the precision of hydrodynamic predictions and also for justifying the construction of the effective field theory of hydrodynamics formulated as a gradient expansion [85–95]. As already mentioned in ref. [8] in the context of the discussion of the "unreasonable effectiveness" of hydrodynamics as an effective theory, many previous studies have reported the divergence of the derivative expansion in relativistic hydrodynamics [15, 17, 18, 18, 25]. Possibly, the asymptotic nature of the expansion appearing in those publications should be viewed as a reflection of the singular nature of the state about which this expansion is performed, rather than a generic property of the hydrodynamic gradient expansion itself. On the other hand, even for a free particle, the momentum space and position space pictures look different in this respect: the small-momentum expansion of the corresponding dispersion relation has a finite radius of convergence, whereas e.g. for the position space propagator, the large-time expansion is only asymptotic.[22] This issue needs further investigation. The role of the non-hydrodynamic degrees of freedom is the common feature of the mentioned works and the present paper.

Of special interest is the dependence of the radii of convergence on coupling. In ref. [8], using eq. (2.27) as a crude approximation and the results of refs. [96, 97], we argued that in the $\mathcal{N} = 4$ SYM theory, the radius of convergence is smaller at weaker coupling. This, of course, requires the actual study of the spectrum at finite coupling. More generally, in the context of the problem of interpolating between weak and strong coupling regimes of the same theory at finite temperature [60, 98], one may note[23] that the problem of convergence of hydrodynamic series has been raised and partially investigated in the 1960s in kinetic theory [99]. This approach, together with recent studies of relevant issues at weak coupling [14, 100–102], may deserve more attention in the context of the problem of the validity of the hydrodynamic description at finite coupling.

Acknowledgments

S.G. and A.O.S. would like to thank the organisers of the programme "Bounding transport and chaos in condensed matter and holography" at Nordita, where part of this work was initiated. S.G. is supported by the U.S. Department of Energy under grant Contract Number DE-SC0011090. S.G. would also like to thank H. Liu, D. Sustretov and W. Taylor

[22]We thank Hong Liu for pointing out this example to us.

[23]We would like to thank J. Noronha for bringing ref. [99] to our attention.

for stimulating and illuminating discussions, and for suggestions on the relevant algebraic geometry literature. P.K. would like to thank the Rudolph Peierls Centre for Theoretical Physics at the University of Oxford for hospitality during the initial stage of this work, and the organisers of the KITP programme “Chaos and Order”, where part of this work was completed. P.K.'s work was supported in part by NSERC of Canada. A.O.S. would like to thank Moscow State University and especially A.V. Borisov, as well as the Kadanoff Center for Theoretical Physics at the University of Chicago for hospitality during the final stage of this work. He also thanks P. Glorioso, D.T. Son, M. Stephanov, P.B. Wiegmann and F. Essler, J. March-Russell, S. Parameswaran for discussions in Chicago and Oxford, respectively, and the participants of the seminars at the Institute for Nuclear Research, Steklov Mathematical Institute and Lebedev Physical Institute of the Russian Academy of Sciences for critical questions and useful suggestions. The work of P.T. is supported by an Ussher Fellowship from Trinity College Dublin. We would like to thank A. Buchel, M. Heller, A. Kurkela and J. Noronha for illuminating correspondence. We also would like to thank B. Withers for correcting our inadvertent partial misrepresentation of his results in the first version of our preprint [8].

A Analytic implicit function theorem and Puiseux series

Here, we collect the necessary information from complex analysis regarding the following problem. Given an implicit function $f(x,y)=0$, where $x,y\in\mathbb{C}$, we would like to find explicit solution(s) in the form $y=y(x)$, at least locally in the vicinity of some point (x_0,y_0), where $y(x)$ may be represented by a finite or infinite series in x. We would like to determine, furthermore, what sets the radius of convergence of such series.

A simple example is provided by the function $f(x,y)=x^2+y^2-1=0$. Since $f(x,y)$ is a polynomial, it determines a complex algebraic curve. Singular points of $f(x,y)$ are determined by the simultaneous solution of the equations $f(x,y)=0$, $f_{,x}(x,y)=0$, $f_{,y}(x,y)=0$, where the comma subscript denotes the partial derivative with respect to the argument after the comma. Clearly, this particular curve has no singular points. It does, however, have the so-called “points of multiplicity 1” or “one-fold points”, where $f_{,x}(x,y)=0$ or $f_{,y}(x,y)=0$ (but not both simultaneously). These are sometimes called critical points. We are interested in the local behaviour of $y=y(x)$ near a critical point defined by the conditions $f(x,y)=0$, $f_{,y}(x,y)=0$. In our example, there are two such points: $(x,y)=(\pm1,0)$. The series representation $y=y(x)$ in the vicinity of e.g. $(x,y)=(1,0)$ has two branches:

$$y=y_1(x)=i\sqrt{2}(x-1)^{\frac{1}{2}}+i2^{-\frac{3}{2}}(x-1)^{\frac{3}{2}}+\cdots\,, \tag{A.1}$$

$$y=y_2(x)=-i\sqrt{2}(x-1)^{\frac{1}{2}}-i2^{-\frac{3}{2}}(x-1)^{\frac{3}{2}}+\cdots\,. \tag{A.2}$$

This is an example of the Puiseux series, i.e. the power series with fractional exponents. These series converge in the circle with the centre at $(x,y)=(1,0)$ and radius $R=2$ which is the distance from $(1,0)$ to the nearest critical point, $(x,y)=(-1,0)$.

One may be interest in the behaviour $y=y(x)$ in the vicinity of a regular point, where $f_{,y}(x,y)\neq0$, for example, near $(x,y)=(0,1)$ in our example. Here, since $f_{,y}(x,y)\neq0$,

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the implicit function theorem guarantees that we can compute the derivatives $y'(x)$, $y''(x)$ and so on, and represent $y(x)$ by the Taylor series in the vicinity of $x = 0$,

$$y = y(x) = 1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots . \tag{A.3}$$

This series is convergent in the circle of radius $R = 1$, determined by the distance from the point $x = 0$ to the nearest critical point(s) at $x = \pm 1$.

In general, for an implicit function given by the equation $f(x, y) = 0$, where $f(x, y)$ is either a finite polynomial in x and y, or an analytic function at a point (x, y) (i.e. a polynomial of an infinite degree), the behaviour at a regular point is governed by the analytic implicit function theorem [103], and the behaviour in the vicinity of a critical point is determined by the Puiseux theorem. In the former case, $y = y(x)$ is represented by a Taylor series converging in some vicinity of a regular point. In the latter case, it is represented by a Puiseux series converging in some vicinity of a critical point. We now recall some facts from complex analysis [104] and explain the Puiseux construction [45, 47].

Definition. *A function, $f(x, y)$, from a neighbourhood of $(x_0, y_0) \in \mathbb{C}^2$ to $\mathbb{C}$ is called analytic at (x_0, y_0) if near (x_0, y_0) it is given by the uniformly convergent power series*

$$f(x, y) = \sum_{n,m=0}^{\infty} a_{nm}(x - x_0)^n (y - y_0)^m . \tag{A.4}$$

Theorem (Analytic implicit function). *If $f(x_0, y_0) = 0$ and $f_{,y}(x_0, y_0) \neq 0$, there exist $\epsilon > 0$ and $\delta > 0$ so that $\mathbb{D}_\epsilon(x_0) \times \mathbb{D}_\delta(y_0)$ is in the neighbourhood where f is defined, and g is a map of $\mathbb{D}_\epsilon(x_0)$ into $\mathbb{D}_\delta(y_0)$ so that $f(x, g(x)) = 0$ and for each $x \in \mathbb{D}_\epsilon(x_0)$, $g(x)$ is the unique solution of $f(x, g(x)) = 0$ with $g(x) \in \mathbb{D}_\delta(y_0)$. Moreover, $g(x)$ is analytic in $\mathbb{D}_\epsilon(x_0)$ and*

$$g'(x) = -\frac{\frac{\partial f}{\partial x}(x, g(x))}{\frac{\partial f}{\partial y}(x, g(x))} . \tag{A.5}$$

Similarly, one can compute higher derivatives of $g(x)$ and represent it by a Taylor series around $x = x_0$ convergent in $\mathbb{D}_\epsilon(x_0)$. Note that the statements of the theorem are *local*, e.g. the size of the domain $\mathbb{D}_\epsilon(x_0)$ is unspecified, it is only known that it exists for some $\epsilon > 0$. In other words, we know that the radius of convergence of the series of $g(x)$ around $x = x_0$ is non-zero but its value is left unspecified. In the example above, we saw that the value of the radius of convergence is determined by the distance from the centre of the expansion x_0 to the nearest critical point of $f(x, y)$. Note also that the statements of the theorem depend crucially on $f(x, y)$ being an analytic function at (x_0, y_0) (in particular, a finite order polynomial in x and y).

Now we return to the original problem: for $f(x, y) = 0$, where $x, y \in \mathbb{C}$, find explicit solution(s) in the form $y = y(x)$, at least locally in the vicinity of some point (x_0, y_0), where $y(x)$ may be represented by series (possibly infinite) at $x = x_0$. For simplicity, we set $x_0 = 0$. First, we check $f(0, y)$. If this is a polynomial in y of degree n, then the equation $f(0, y) = 0$ has n roots y_i, $i = 1, \ldots, n$. There are two possibilities.

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Local behaviour at regular points: all the roots y_i, $i = 1, \ldots, n$ of the equation $f(0, y) = 0$ are distinct. Then $f_{,y}(0, y_i) \neq 0$, and the analytic implicit function theorem guarantees the existence of a unique Taylor expansion $y = y(x)$ at $x = 0$.

Local behaviour at critical points: the equation $f(0, y) = 0$ has multiple roots. For simplicity, let $y_0 = 0$ be such a root. Then we have $f(0, 0) = 0$, $f_{,y}(0, 0) = 0$, $(\partial^2 f/\partial y^2)(0, 0) = 0, \ldots$ $(\partial^p f/\partial y^p)(0, 0) \neq 0$, if $y = 0$ is a zero of $f(0, y) = 0$ of order p. We expect p branches of the solutions $y = Y_j(x)$, $j = 1, \ldots, p$, at $x = 0$. They are given by Puiseux series

$$y = Y_j(x) = \sum_{k \geq k_0}^{\infty} a_k x^{\frac{k}{m_j}}\,, \qquad j = 1, \ldots, p\,, \tag{A.6}$$

where m_j are positive integers, and k_0 is a non-negative integer which in general depends on j. The exponents k_0/m_j, $(k_0 + 1)/m_j$, etc, and the coefficients a_{k_0}, a_{k_0+1}, etc, can be determined by the Newton polygon method (1671), as described e.g. in refs. [45, 47]. The Puiseux series are converging in some vicinity of the point $x = 0$ provided $f(x, y)$ is an analytic function at $(x, y) = (0, 0)$ (or a finite polynomial). If some $m_j > 1$, we necessarily have among those p branches a family of m_j solutions of the form

$$y = Y_l(x) = \sum_{k \geq k_0}^{\infty} a_k \left(e^{\frac{2\pi i l}{m_j}}\right)^k x^{\frac{k}{m_j}}\,, \qquad l = 0, 1, \ldots, m_j - 1\,. \tag{A.7}$$

As an example, consider the algebraic curve [48]

$$f(x, y) = y^5 - 4y^4 + 4y^3 + 2x^2y^2 - xy^2 + 2x^2y + 2xy + x^4 + x^3 = 0\,. \tag{A.8}$$

Since $f(0, y) = y^3(y-2)^2$, the points $(0, 0)$ and $(0, 2)$ are critical points with multiplicities 3 and 2, respectively. We expect $y = y(x)$ to be given by 3 branches of Puiseux series at $(0, 0)$ and by 2 branches at $(0, 2)$. Applying the Newton polygon method [45, 47, 48] at the point $(0, 0)$, we find $m_1 = m_2 = 2$ and $k_0 = 1$, $m_3 = 1$ and $k_0 = 2$, with appropriate coefficients:

$$y = Y_1(x) = i\frac{\sqrt{2}}{2}x^{\frac{1}{2}} - \frac{1}{8}\,x + i\frac{27\sqrt{2}}{128}x^{\frac{3}{2}} - \frac{7}{32}\,x^2 + \cdots\,, \tag{A.9}$$

$$y = Y_2(x) = -i\frac{\sqrt{2}}{2}x^{\frac{1}{2}} - \frac{1}{8}\,x - i\frac{27\sqrt{2}}{128}x^{\frac{3}{2}} - \frac{7}{32}\,x^2 + \cdots\,, \tag{A.10}$$

$$y = Y_3(x) = -\frac{1}{2}\,x^2 + \frac{1}{8}\,x^4 - \frac{1}{8}\,x^5 + \frac{1}{16}\,x^6 + \cdots\,. \tag{A.11}$$

At the point $(0, 2)$, we have 2 branches, as expected:

$$y = Y_4(x) = 2 + \frac{1 + i\sqrt{95}}{8}\,x + \cdots\,, \tag{A.12}$$

$$y = Y_5(x) = 2 + \frac{1 - i\sqrt{95}}{8}\,x + \cdots\,. \tag{A.13}$$

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B Perturbative solution of eq. (4.1)

Here, we list the explicit expressions for the components of the perturbative solution of eq. (4.1),

$$
\begin{aligned}
G_0(u) &= u\,, \\
G_1(u) &= \left(\frac{\mathfrak{q}^2}{2} - i\mathfrak{w}\right)(1-u) + i\mathfrak{w}\frac{u}{2}\ln\frac{1+u}{2} \\
&= \left(\frac{\mathfrak{q}^2}{2} - i\mathfrak{w}\right)(1-u) - i\mathfrak{w}\,\frac{u}{2}\,\mathrm{Li}_1\left(\frac{1-u}{2}\right)\,, \\
G_2(u) &= \mathfrak{w}^2\left[\frac{u}{2}\,\mathrm{Li}_2\left(\frac{1-u}{2}\right) + \frac{u}{8}\,\mathrm{Li}_1^2\left(\frac{1-u}{2}\right) + \frac{1+u}{2}\,\mathrm{Li}_1\left(\frac{1-u}{2}\right)\right] \\
&\quad + \mathfrak{q}^2\left(\mathfrak{q}^2 - \frac{3i\mathfrak{w}}{2}\right)\frac{u}{2}\,\mathrm{Li}_1\left(\frac{1-u}{2}\right) - \frac{i\mathfrak{w}\mathfrak{q}^2}{4}\,\mathrm{Li}_1\left(\frac{1-u}{2}\right) \\
&\quad + \mathfrak{q}^2\left(\frac{\mathfrak{q}^2}{2} - i\mathfrak{w}\right)u\ln u + \frac{\mathfrak{q}^4}{4}(1-u)\,,
\end{aligned}
$$

as well as the appropriate boundary values,

$$
\begin{aligned}
G_0(0) &= 0\,, \\
G_1(0) &= -i\mathfrak{w} + \frac{\mathfrak{q}^2}{2}\,, \\
G_2(0) &= \frac{\mathfrak{q}^4}{4} - \frac{i\mathfrak{w}\mathfrak{q}^2\ln 2}{4} + \frac{\mathfrak{w}^2\ln 2}{2}\,, \\
G_3(0) &= i\mathfrak{w}^3\left(\frac{\pi^2}{24} + \ln 2 - \frac{3}{8}\ln^2 2\right) + \mathfrak{q}^6\left(\frac{\ln 2}{4} - \frac{1}{8}\right) + i\mathfrak{w}\mathfrak{q}^4\left(\frac{1}{4} - \frac{\ln 2}{8}\right) \\
&\quad + \mathfrak{q}^2\mathfrak{w}^2\left(\frac{\pi^2}{48} - \frac{\ln 2}{2} - \frac{\ln^2 2}{16}\right)\,, \\
G_4(0) &= \mathfrak{q}^8\left(-\frac{1}{16} + \frac{\pi^2}{64} - \frac{\ln 2}{8}\right) - \mathfrak{q}^4\mathfrak{w}^2\left(\frac{\pi^2}{96} + (12 - 7\ln 2)\frac{\ln 8}{96}\right) \\
&\quad - i\mathfrak{q}^6\mathfrak{w}\left(\frac{\pi^2}{96} + (-5 + \ln 4)\frac{\ln 64}{96}\right) \\
&\quad + \mathfrak{w}^4\left((24 - 5\ln 2)\frac{\ln^2 2}{48} + \frac{\pi^2}{48}(-4 + \ln 8) - \frac{1}{2}\zeta(3)\right) \\
&\quad + i\mathfrak{q}^2\mathfrak{w}^3\left(-\frac{\pi^2\ln 2}{96} + (-24 + \ln 2)\frac{\ln^2 2}{96} + \frac{3}{16}\zeta(3)\right)\,.
\end{aligned}
$$

C Kepler's equation at complex eccentricity

The connection between algebraic curves, their critical points and non-analyticity of associated integrals has an interesting history [105]. Newton proved in “Principia” that every algebraically integrable oval must have singular points: all smooth ovals are algebraically non-integrable (hence the non-analyticity $T \propto a^{3/2}$ in the Kepler's third law). Moreover,

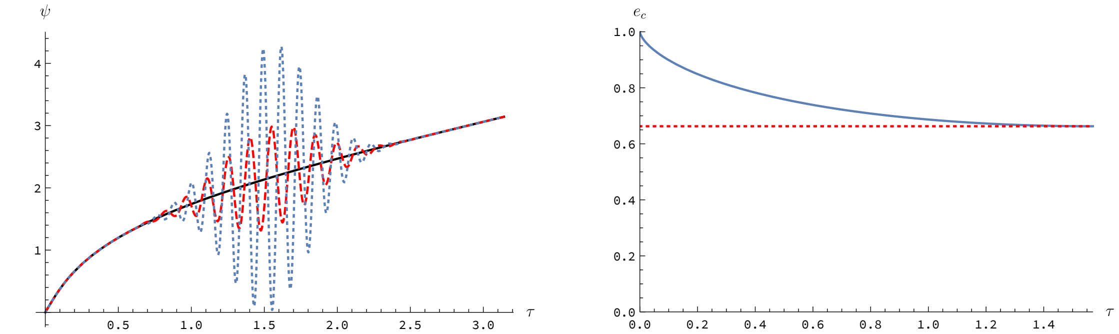

Figure 17. Left panel: solutions $\psi(\tau)$ of Kepler's equation (C.4) at $e = 0.75$ for $\tau \in [0, \pi]$ (half a period): "exact" numerical solution of (C.4) (solid black line), series solution (C.5) truncated at 50 terms (dashed red line) and 60 terms (dotted blue line). The rate of convergence is maximal at $\tau = \pi/2$. Right panel: the radius of convergence $e_c(\tau)$ of the series (C.5) as a function of τ for $\tau \in [0, \pi/2]$. The red dotted line coresponds to Laplace's value e_L. For $0 < e < e_L$, the series converges for all $\tau \in [0, 2\pi]$.

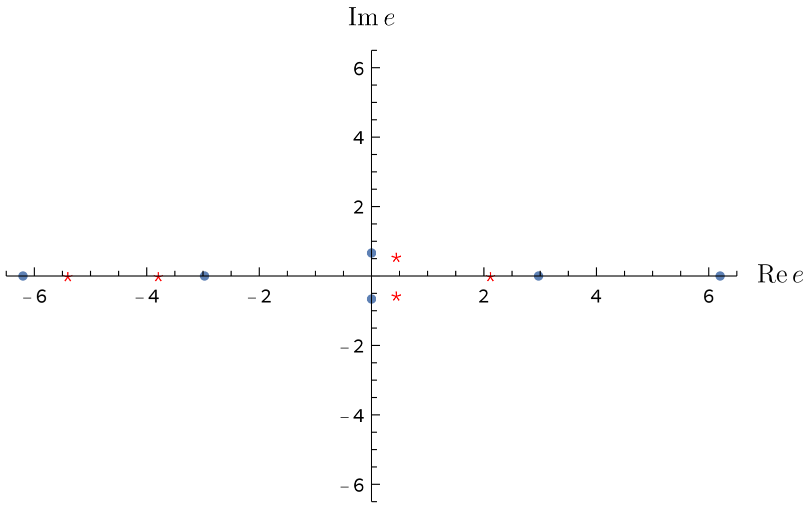

Figure 18. Critical points of the Kepler's curve (C.4) in the complex eccentricity plane. The blue dots are the critical points at $\tau = \pi/2$. The points closest to the origin are located at $e \approx \pm 0.662743i$. They determine the radius of convergence e_L of the series (C.5). For $\tau < \pi/2$, the critical points are located at a larger distance from the origin. For example, the critical points at $\tau = \pi/4$ are shown by red asterisks. They determine the radius of convergence $e_c(\tau)$, shown in the right panel of figure 17. With $\tau \to 0$, the three critical points merge at $e = 1$.

the radius of convergence of the series solving Kepler's equation is determined by the critical points in the complex eccentricity plane.

Kepler's law of motion of a planet in an elliptical orbit with eccentricity e, $0 < e < 1$, is usually expressed in a parametric form [106]

$$r = a\,(1 - e\cos\psi)\ , \tag{C.1}$$

$$t = \frac{T}{2\pi}\,(\psi - e\sin\psi)\ , \tag{C.2}$$

where r is the magnitude of the radius-vector from the centre of the force to the planet, a is the major semi-axis of the ellipse, T is the period of revolution and $\psi \in [0, 2\pi]$ is the parameter known in astronomy as the *eccentric anomaly*. Knowing $\psi(t)$, one can find the position of the planet in the polar coordinates $(r(t), \varphi(t))$ as a function of time using eq. (C.1) and the equation

$$\tan\frac{\varphi}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2}\,. \tag{C.3}$$

Introducing $\tau \equiv 2\pi t/T$, we rewrite eq. (C.2) as

$$K = \tau - \psi + e\sin\psi = 0\,. \tag{C.4}$$

Eq. (C.4) is known as Kepler's equation. The task of finding a solution $\psi = \psi(\tau)$ preoccupied Kepler, Newton, Lagrange, Laplace, Bessel, Cauchy and other great minds and led to progress in various mathematical disciplines. To quote V.I. Arnold [105]: *"This equation plays an important role in the history of mathematics. From the time of Newton, the solution has been sought in the form of a series in powers of the eccentricity e. The series converges when $|e| \leq 0.662743\ldots$. The investigation of the origin of this mysterious constant led Cauchy to the creation of complex analysis. Such fundamental mathematical concepts and results as Bessel functions, Fourier series, the topological index of a vector field, and the "principle of the argument" of the theory of functions of a complex variable also first appeared in the investigation of Kepler's equation"*.

A formal series solution of Kepler's equation was found by Lagrange [107] who apparently was not concerned with the series convergence (more details can be found in the book [108])

$$\psi(\tau, e) = \tau + \sum_{n=1}^{\infty} a_n(\tau)\frac{e^n}{n!}\,, \tag{C.5}$$

where

$$a_n = \frac{d^{n-1}(\sin^n\tau)}{d\tau^{n-1}}\,. \tag{C.6}$$

As pointed out by Laplace [109], the series (C.5) converges for all $\tau \in [0, 2\pi]$ as long as $|e| \leq e_L \approx 0.662743\ldots$. For $e > e_L$, the series diverges for some values of τ, in a rather peculiar manner (see figure 17, left panel).

What determines the radius of convergence $e_c(\tau)$ of the series (C.5)? This problem was investigated by Cauchy, Puiseux and Serret in a series of papers in 1849-1859 [108]. In modern language, the answer is the following. Treat Kepler's equation (C.4) as a complex analytic curve in the space of $\psi \in \mathbb{C}$, $e \in \mathbb{C}$, with τ remaining a real parameter. The critical points of the curve $K = 0$ obey eq. (C.4) as well as the equation

$$\frac{\partial K}{\partial\psi} = e\cos\psi - 1 = 0\,. \tag{C.7}$$

The critical points closest to the origin in the complex eccentricity plane are shown in figure 18. Their location is parametrised by τ. The radius of convergence $e_c(\tau)$ is given by the distance from the origin to the nearest singularity. This distance is a monotonic function of τ in the interval $[0, \pi]$ (half a period), with the minimum at $\tau = \pi/2$ given by $e_c(\frac{\pi}{2}) = e_L$. Thus, for $0 < e < e_L$, the series (C.5) converges for all $\tau \in [0, 2\pi]$. The dependence of the radius of convergence on τ is shown in figure 17 (right panel).

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Published for SISSA by Springer

Received: October 15, 2019
Accepted: November 26, 2019
Published: January 14, 2020

Leading multi-stress tensors and conformal bootstrap

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Abstract: Near lightcone correlators are dominated by operators with the lowest twist. We consider the contributions of such leading lowest twist multi-stress tensor operators to a heavy-heavy-light-light correlator in a CFT of any even dimensionality with a large central charge. An infinite number of such operators contribute, but their sum is described by a simple ansatz. We show that the coefficients in this ansatz can be determined recursively, thereby providing an operational procedure to compute them. This is achieved by bootstrapping the corresponding near lightcone correlator: conformal data for any minimal-twist determines that for the higher minimal-twist and so on. To illustrate this procedure in four spacetime dimensions we determine the contributions of double- and triple-stress tensors. We compute the OPE coefficients; whenever results are available in the literature, we observe complete agreement. We also compute the contributions of double-stress tensors in six spacetime dimensions and determine the corresponding OPE coefficients. In all cases the results are consistent with the exponentiation of the near lightcone correlator. This is similar to the situation in two spacetime dimensions for the Virasoro vacuum block.

Keywords: AdS-CFT Correspondence, Conformal Field Theory

ArXiv ePrint: 1909.05775

https://doi.org/10.1007/JHEP01(2020)076

JHEP01(2020)076

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1 Introduction and summary

1.1 Introduction

The two-point function of the stress tensor in Conformal Field Theories is proportional to a single parameter, the central charge C_T. It generally serves as a measure of the number of degrees of freedom in the theory. In two spacetime dimensions this statement can be made precise: one can define a c-function which monotonically decreases along Renormalization Group flows and reduces to the central charge at conformal fixed points [1]. In four spacetime dimensions the situation is a bit more subtle and it is the a-coefficient in the conformal anomaly which necessarily satisfies $a_{IR} \leq a_{UV}$ [2]. Nevertheless, in any unitary conformal field theory a and C_T can only differ by a number of $\mathcal{O}(1)$ (see [3] for the original argument and [4–10] for more recent field theoretic proofs.) Hence, to consider the limit of infinite number of degrees of freedom one needs to take C_T to infinity.

In two spacetime dimensions conformal symmetry is described by the infinite-dimensional Virasoro algebra. This symmetry strongly constrains correlators, especially when combined with the $C_T \to \infty$ limit. Of particular interest is the "heavy-heavy-light-light" correlator, which involves two "heavy" operators with conformal dimension $\Delta_H \sim C_T$ and two "light" operators with conformal dimension $\Delta_L \sim \mathcal{O}(1)$. In this case the contribution of the identity operator and all its Virasoro descendants is known as the Virasoro vacuum block and has been calculated in several ways [11–17]. The Virasoro vacuum block (and finite C_T corrections to it) is instrumental in a variety of settings, such as e.g. the problem of information loss [18–23] and properties of the Renyi and entanglement entropies [24–27] (see also [28, 29] for the original applications of large C_T correlators in this context).

The heavy-heavy-light-light Virasoro vacuum block exponentiates

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\rangle \sim e^{\Delta_L \mathcal{F}(\mu;z)}, \tag{1.1}$$

with $\mathcal{F}$ a known function which admits an expansion in powers of $\mu \sim \Delta_H/C_T$

$$\mathcal{F}(\mu;z) = \sum_k \mu^k \mathcal{F}^{(k)}(z). \tag{1.2}$$

One can consider contributions of various quasi-primaries made out of the stress tensor to $\mathcal{F}^{(k)}$. At $k=1$ the only such quasi-primary is the stress tensor itself, while for $k=2$ one needs to sum an infinite number of quasi-primaries quadratic in the stress tensor (double-stress operators) and labeled by spin. The situation is similar for all other values of k. It is possible to compute the OPE coefficients of the corresponding quasi-primaries, starting from the known result for the Virasoro vacuum block. Interestingly, at each order in μ, $\mathcal{F}^{(k)}$ can be written as a sum of particular terms [30][1]

$$\mathcal{F}^{(k)}(z) = \sum_{\{i_p\}} b_{i_1\ldots i_k} f_{i_1}(z)\ldots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = 2k, \tag{1.3}$$

where $f_a(z) = (1-z)^a {}_2F_1(a,a,2a,1-z)$.

It is an interesting question whether a similar structure appears when the number of spacetime dimensions d is greater than two. Unlike in two spacetime dimensions, in addition to spin, multi-stress tensor operators are also labeled by their twist. An interesting subset of multi-stress tensor operators is comprised out of those with minimal twist. These operators dominate in the lightcone limit over those of higher twist. In [32] an expression for the OPE coefficients of two scalars and minimal-twist double-stress tensor operators in $d=4$ was obtained, and the sum was performed to obtain a remarkably simple expression for the near lightcone $\mathcal{O}(\mu^2)$ term in the heavy-heavy-light-light correlator. It was shown to have a similar form to (1.3). One may now wonder if the minimal-twist multi-stress tensor part of the correlator in higher dimensions exponentiates

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\big|_{\text{multi-stress tensors}} \sim e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{1.4}$$

and whether $\mathcal{F}(\mu;z,\bar{z})$ can be expressed as

$$\mathcal{F}(\mu;z,\bar{z}) = \sum_k \mu^k \mathcal{F}^{(k)}(z,\bar{z}), \tag{1.5}$$

[1]Similar expressions in a slightly different context appeared in [31].

with

$$\mathcal{F}^{(k)}(z,\bar{z}) = (1-\bar{z})^{k\left(\frac{d-2}{2}\right)} \sum_{\{i_p\}} b_{i_1\dots i_k} f_{i_1}(z)\dots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = k\left(\frac{d+2}{2}\right), \tag{1.6}$$

and d an even number.

In this paper we investigate this further. We start by assuming that the multi-stress tensor sector of the heavy-heavy-light-light correlator in the near lightcone regime $\bar{z} \to 1$ admits an expansion in μ

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle|_{\text{multi-stress tensors}} \sim \sum_k \mu^k \mathcal{G}^{(k)}(z,\bar{z}), \tag{1.7}$$

where each coefficient function $\mathcal{G}^{(k)}(z,\bar{z})$ takes a particular form:

$$\mathcal{G}^{(k)}(z,\bar{z}) = \frac{(1-\bar{z})^{k\left(\frac{d-2}{2}\right)}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1\dots i_k} f_{i_1}(z)\dots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = k\left(\frac{d+2}{2}\right). \tag{1.8}$$

We subsequently use this ansatz to compute the contributions of the multi-stress tensor operators to the near lightcone correlator and extract the corresponding OPE coefficients.

For even d, the hypergeometric functions in (1.8) reduce to terms which contain at most one power of $\log(z)$ each. Their products contain multi-logs whose coefficients turn out to be rational functions of z. We use the conformal bootstrap approach initiated in [33] (for a review and references see e.g. [34–36]) to relate these functions to the anomalous dimensions and OPE coefficients of the heavy-light double-twist operators in the cross channel. The ansatz (1.8) has just a few coefficients at any finite k which can be determined completely from the cross-channel data derived using the $(k-1)$th term. This is related to the fact that all the $\log^m(z)$ terms with $2 \le m \le k$ are completely determined by the anomalous dimensions and OPE coefficients at $\mathcal{O}(\mu^{k-1})$. At each step, we obtain an overconstrained system of equations solved by the same set of $a_{i_1\dots i_k}$. This provides strong support to the ansatz (1.6). We then proceed to derive the OPE coefficients of the multi-stress tensor operators with two light scalars from our result. In practice, we complete this program to $\mathcal{O}(\mu^3)$ in $d=4$ and to $\mathcal{O}(\mu^2)$ in $d=6$. However the procedure outlined can be easily generalised to arbitrary order in μ and any even d.

In [37] the authors considered holographic CFTs dual to gravitational theories defined by the Einstein-Hilbert Lagrangian plus higher derivative terms and a scalar field minimally coupled to gravity in AdS_{d+1}. Interpreting the scalar propagator in an asymptotically AdS_{d+1} black hole background as a heavy-heavy-light-light four point function, enabled the authors of [37] to extract the OPE coefficients of a few multi-stress tensor operators from holography (see also [38–40] for related work). Ref. [37] also argued that the OPE coefficients of the leading, minimal-twist multi-stress operators are universal — they do not depend on the higher derivative terms in the Lagrangian. Their results agree with the general expressions obtained in this paper, upon substitution of the relevant quantum numbers. We do not use holography in our work; our major assumption is (1.8). It would be interesting to see what is the regime of applicability of our results.

1.2 Summary of the results

In this paper we argue that for a large class of CFTs (including holographic CFTs) in even d, the contribution of minimal-twist multi-stress tensors to the correlator in the lightcone limit can be written as a sum of products of certain hypergeometric functions. To be explicit, let us define functions $f_a(z)$ as

$$f_a(z) = (1-z)^a {}_2F_1(a,a,2a;1-z). \tag{1.9}$$

The stress tensor contribution to the correlator in the lightcone limit is given in any dimension d by

$$\mathcal{G}^{(1)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^{\frac{d-2}{2}}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{\Delta_L \Gamma\left(\frac{d}{2}+1\right)^2}{4\Gamma(d+2)} f_{\frac{d+2}{2}}(z). \tag{1.10}$$

At $\mathcal{O}(\mu^2)$ the contribution from twist-four double-stress tensor operators in $d=4$ is

$$\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L-2)}\right) \times \left((\Delta_L-4)(\Delta_L-3)f_3^2(z) + \frac{15}{7}(\Delta_L-8)f_2(z)f_4(z) + \frac{40}{7}(\Delta_L+1)f_1(z)f_5(z)\right). \tag{1.11}$$

This result agrees with the expression obtained by different methods in [32].

The contribution from twist-six triple-stress tensors in the lightcone limit in $d=4$ at order $\mathcal{O}(\mu^3)$ is

$$\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(a_{117} f_1(z)^2 f_7(z) + a_{126} f_1(z) f_2(z) f_6(z) + a_{135} f_1(z) f_3(z) f_5(z) + a_{225} f_2(z)^2 f_5(z) + a_{234} f_2(z) f_3(z) f_4(z) + a_{333} f_3(z)^3\Big), \tag{1.12}$$

where coefficients a_{ijk} are given by (3.18).

Furthermore, from (1.12) and (3.18), we find the OPE coefficients of twist-six triple-stress tensor operators as a finite sum (for details see section 3.4). Two such OPE coefficients for twist-6 triple-stress tensors were calculated holographically in [37] and agree with our results.

The contribution from twist-eight double-stress tensors to the correlator in the lightcone limit in $d=6$ at order $\mathcal{O}(\mu^2)$ is

$$\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^4}{[(1-z)(1-\bar{z})]^{\Delta_L}} \times \Big(a_{13} f_1(z) f_7(z) + a_{26} f_2(z) f_6(z) + a_{35} f_3(z) f_5(z) + a_{44} f_4(z)^2\Big), \tag{1.13}$$

where a_{mn} are given by (4.7). Using (1.13) and (4.7) we find the OPE coefficients for operators of type $:T_{\mu\nu}\partial_{\lambda_1}\ldots\partial_{\lambda_{2l}}T_{\alpha\beta}:$ in $d=6$ to be equal to:

$$P_{8,s}^{(HH,LL)} = \mu^2 \frac{c\Delta_L}{(\Delta_L-3)(\Delta_L-4)} (a_3\Delta_L^3 + a_2\Delta_L^2 + a_1\Delta_L + a_0), \tag{1.14}$$

where c and a_m, given by (4.15), are functions of the total spin $s = 4+2l$.

In general we propose that the contribution from minimal-twist multi-stress tensor operators to the correlator in even d at $\mathcal{O}(\mu^k)$ in the lightcone limit takes the form

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^{k\left(\frac{d}{2}-1\right)}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1\dots i_k} f_{i_1}(z)\dots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = k\left(\frac{d+2}{2}\right), \quad (1.15)$$

where the sum goes over all sets of $\{i_p\}$ with $i_p \le i_{p+1}$ and $a_{i_1\dots i_k}$ coefficients that need to be fixed.[2]

We also check that the stress tensor sector of the near lightcone correlator exponentiates

$$\langle \mathcal{O}_H(x_4)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle|_{\text{multi-stress tensors}} \underset{\bar{z}\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \quad (1.16)$$

where $\mathcal{F}(\mu; z, \bar{z})$ is a rational function of Δ_L that remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$. We explicitly verify this up to $\mathcal{O}(\mu^3)$ in $d = 4$ and $\mathcal{O}(\mu^2)$ in $d = 6$.

1.3 Outline

The rest of the paper is organized as follows. In section 2, we establish notation and write general expressions for the heavy-heavy-light-light correlator in both the direct channel (T-channel) and the cross channel (S-channel). We further write down the stress tensor contribution to the correlator in the lightcone limit in arbitrary spacetime dimensions d. In section 3, we find the contribution of minimal-twist double- and triple-stress tensor operators in $d = 4$ in the lightcone limit. We show that this contribution exponentiates and we write an expression for the OPE coefficients of minimal-twist triple-stress tensors of spin s with scalar operators, in the form of a finite sum. In section 4, we repeat this program up to $\mathcal{O}(\mu^2)$ in $d = 6$. Again we confirm exponentiation and we find a closed form expression for the OPE coefficients of minimal-twist double-stress tensors of arbitrary spin with scalar operators. We discuss our results in section 5.

2 Review of heavy-heavy-light-light correlator in the lightcone limit

Below we review the setup of a heavy-heavy-light-light correlator with focus on its behaviour in the lightcone limit. We mostly follow [30, 32, 41].

The object that we study is a four-point function of pairwise identical scalars $\langle \mathcal{O}_H(x_4)\mathcal{O}_L(x_3)\mathcal{O}_L(x_2)\mathcal{O}_H(x_1)\rangle$. Here $\mathcal{O}_H$ and $\mathcal{O}_L$ are scalar operators with scaling dimension $\Delta_H \propto \mathcal{O}(C_T)$ and $\Delta_L \propto \mathcal{O}(1)$, with $C_T \gg 1$ the central charge.

Using conformal transformations we define the stress tensor sector of the correlator by

$$\mathcal{G}(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H} \langle \mathcal{O}_H(x_4)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\Big|_{\text{multi-stress tensors}}, \quad (2.1)$$

[2] One only needs to sum the linearly independent products of functions f_a.

where z and $\bar{z}$ are the usual cross-ratios[3]

$$u = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2},$$
$$v = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}. \tag{2.2}$$

In (2.1) the "multi-stress tensor" subscript stands to indicate the contribution of the identity and all multi-stress tensor operators.

The correlator $\mathcal{G}(z,\bar{z})$ can be expanded in the "T-channel" $\mathcal{O}_L(1)\times\mathcal{O}_L(z,\bar{z}) \to \mathcal{O}_{\tau,s}$ as[4]

$$\mathcal{G}(z,\bar{z}) = [(1-z)(1-\bar{z})]^{-\Delta_L} \sum_{\mathcal{O}_{\tau,s}} P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z}), \tag{2.3}$$

where $\tau = \Delta - s$ and s denote the twist and spin of the exchanged operator, respectively, and $g^{(0,0)}_{\tau,s}(z,\bar{z})$ the conformal block of the primary operator $\mathcal{O}_{\tau,s}$. Moreover, $P^{(HH,LL)}_{\mathcal{O}_{\tau,s}}$ are defined as

$$P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} = \left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_{\tau,s}} \lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}_{\tau,s}}, \tag{2.4}$$

where $\lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}}$ and $\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}}$ denote the respective OPE coefficients.

We will mainly be interested in the lightcone limit defined by $u \ll 1$ or equivalently $\bar{z} \to 1$. In this limit the T-channel expansion (2.3) is dominated by minimal-twist operators as follows from the behaviour of the conformal blocks

$$\mathcal{G}(u,v) \underset{u\to 0}{\approx} u^{-\Delta_L} \sum_{\mathcal{O}_{\tau,s}} P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} u^{\frac{\tau}{2}} (1-v)^{-\frac{\tau}{2}} f_{\frac{\tau}{2}+s}(v), \tag{2.5}$$

where $\tau = \Delta - s$ is the twist and

$$f_{\frac{\tau}{2}+s}(v) = (1-v)^{\frac{\tau}{2}+s} {}_2F_1\left(\frac{\tau}{2}+s, \frac{\tau}{2}+s, \tau+2s, 1-v\right) \tag{2.6}$$

is a SL(2; R) conformal block.

For any CFT in $d > 2$ the leading contribution in the lightcone limit comes from the exchange of the identity operator with twist $\tau = 0$. Another operator present in any unitary CFT is the stress tensor with twist $\tau = d-2$. Its contribution to the correlator is completely fixed by a Ward identity and

$$P^{(HH,LL)}_{T_{\mu\nu}} = \mu \frac{\Delta_L}{4} \frac{\Gamma\left(\frac{d}{2}+1\right)^2}{\Gamma(d+2)}, \tag{2.7}$$

where

$$\mu := \frac{4\Gamma(d+2)}{(d-1)^2 \Gamma\left(\frac{d}{2}\right)^2} \frac{\Delta_H}{C_T}. \tag{2.8}$$

[3]Note that (u,v) are exchanged compared to the more common convention.

[4]For reasons of convenience, here and in the rest of the paper we refer to $\mathcal{G}(z,\bar{z})$ as the correlator; the reader should keep in mind that $\mathcal{G}(z,\bar{z})$ is not the full correlator but only its stress tensor sector, as defined in (2.1).

As explained in [30], the correlator admits a natural perturbative expansion in μ,

$$\mathcal{G}(z,\bar{z}) = \sum_k \mu^k \mathcal{G}^{(k)}(z,\bar{z})\,. \tag{2.9}$$

Using (2.5) and (2.7), we find the following contribution due to the stress tensor at $\mathcal{O}(\mu)$

$$\mathcal{G}^{(1)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^{\frac{d-2}{2}}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{\Delta_L \Gamma\left(\frac{d}{2}+1\right)^2}{4\Gamma(d+2)} (1-z)^{\frac{d+2}{2}} {}_2F_1\left(\frac{d+2}{2},\frac{d+2}{2};d+2;1-z\right). \tag{2.10}$$

Let us study the correlator perturbatively in μ in the lightcone limit. At k-th order in that expansion we expect contributions from minimal-twist multi-stress tensor operators of the schematic form $[T^k]_{\tau,s} =: T_{\mu_1\nu_1}\ldots\partial_{\lambda_1}\ldots\partial_{\lambda_l}T_{\mu_k\nu_k}$:, where the minimal-twist τ and spin s of these operators are given by

$$\begin{aligned} \tau &= k(d-2), \\ s &= 2k+l \end{aligned} \tag{2.11}$$

and l an even integer denoting the number of uncontracted derivatives. We moreover define the product of OPE coefficients for minimal-twist operators at order k as

$$P^{(HH,LL)}_{[T^k]_{\tau,s}} = \mu^k P^{(HH,LL);(k)}_{\tau,s}. \tag{2.12}$$

Compared to the $k=1$ case, there exists an infinite number of minimal-twist multi-stress tensor operators for each value of $k>1$. To obtain their contribution to the correlator in the lightcone limit, we thus have to sum over all these operators.

The correlator can likewise be expanded in the "S-channel" $\mathcal{O}_L(z,\bar{z}) \times \mathcal{O}_H(0) \to \mathcal{O}_{\tau',s'}$ as

$$\mathcal{G}(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{\mathcal{O}_{\tau',s'}} P^{(HL,HL)}_{\mathcal{O}_{\tau',s'}} g^{(\Delta_{HL},-\Delta_{HL})}_{\tau',s'}(z,\bar{z}). \tag{2.13}$$

where $P^{(HL,HL)}_{\mathcal{O}_{\tau',s'}}$ are the products of the corresponding OPE coefficients and $\Delta_{HL} = \Delta_H - \Delta_L$. Operators contributing in the S-channel are "heavy-light double-twist operators" [30, 41][5] that can be schematically written as $[\mathcal{O}_H\mathcal{O}_L]_{n,l} =: \mathcal{O}_H\partial^{2n}\partial^l\mathcal{O}_L$:, with scaling dimension $\Delta_{n,l} = \Delta_H+\Delta_L+2n+l+\gamma(n,l)$ and spin l. In the $\Delta_H\to\infty$ limit the $d=4$ blocks are given by

$$g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \approx \frac{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L+2n+\gamma)}}{\bar{z}-z}\left(\bar{z}^{l+1}-z^{l+1}\right), \tag{2.14}$$

and similarly in $d=6$

$$g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \approx \frac{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L+2n+\gamma(n,l))}}{(\bar{z}-z)^3}\left(\bar{z}^{l+3}-\frac{l+3}{l+1}\bar{z}^{l+2}z^1-(z\leftrightarrow\bar{z})\right). \tag{2.15}$$

[5]This the analogue of light-light double-twist operators that are present in the cross channel of $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\mathcal{O}_1\rangle$, with $\mathcal{O}_1$ and $\mathcal{O}_2$ both light, in any CFT [42, 43].

The anomalous dimensions $\gamma(n,l)$ admit an expansion in μ

$$\gamma(n,l)=\sum_{k=1}^{\infty}\mu^k\gamma_{n,l}^{(k)}. \tag{2.16}$$

Likewise, we expand the product of the OPE coefficients of the double-twist operators as

$$P_{n,l}^{(HL,HL)}=P_{n,l}^{(HL,HL);\mathrm{MFT}}\sum_{k=0}^{\infty}\mu^k P_{n,l}^{(HL,HL);(k)}, \tag{2.17}$$

with $P_{n,l}^{(HL,HL);(0)}=1$. The zeroth order OPE coefficients $P_{n,l}^{(HL,HL);\mathrm{MFT}}$ in the S-channel are those of Mean Field Theory found in [44]. In the limit $\Delta_H\to\infty$ they are given by

$$P_{n,l}^{(HL,HL);\mathrm{MFT}}\approx\frac{(\Delta_L-d/2+1)_n(\Delta_L)_{l+n}}{n!\,l!\,(l+d/2)_n}, \tag{2.18}$$

where $(a)_n$ denotes the Pochhammer symbol. For large l we further approximate (2.18) by

$$P_{n,l}^{(HL,HL);\mathrm{MFT}}\approx\frac{l^{\Delta_L-1}\left(\Delta_L-\frac{d}{2}+1\right)_n}{n!\,\Gamma(\Delta_L)}. \tag{2.19}$$

To reproduce the correct singularities manifest in the T-channel one has to sum over infinitely many heavy-light double-twist operators with $l\gg 1$. For such operators the dependence of the OPE data on the spin l for $l\gg 1$ is:[6]

$$\begin{aligned}P_{n,l}^{(HL,HL);(k)}&=\frac{P_n^{(k)}}{l^{\frac{k(d-2)}{2}}},\\ \gamma_{n,l}^{(k)}&=\frac{\gamma_n^{(k)}}{l^{\frac{k(d-2)}{2}}}.\end{aligned} \tag{2.20}$$

Note that generally the OPE data in the S-channel receives corrections needed to reproduce double-twist operators in the T-channel; however, since we are interested in the stress tensor sector we consider only contributions of the form given in (2.20).

3 Multi-stress tensors in four dimensions

In this section we describe how to use crossing symmetry to fix the contribution of minimal-twist multi-stress tensors to the heavy-heavy-light-light correlator in $d=4$ to $\mathcal{O}(\mu^3)$. The methods described generalize to other even spacetime dimensions, with the six-dimensional case to $\mathcal{O}(\mu^2)$ described in section 4. In principle the same technology can also be used to determine the correlator at higher orders. Moreover, the resulting expression can be decomposed into multi-stress tensor blocks of minimal-twist, allowing us at each order to read off the OPE coefficients of minimal-twist multi-stress tensors.

[6]This behavior in the large l limit is different from that of the OPE data of light-light double-twist operators [42, 43].

The idea is to study the S-channel expansion in (3.31) in the limit $1-\bar{z} \ll z \ll 1$. In this limit operators with $l \gg 1$ and low values of n dominate. Expanding the conformal blocks in (2.14) for small $\gamma(n,l)$ and $\bar{z} \to 1$, the blocks in $d=4$ reduce to

$$(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \bar{z}^l p(\log z, \gamma(n,l)) \frac{z^n}{1-z}, \tag{3.1}$$

where $p(\log z, \gamma(n,l))$ is given by

$$p(\log z, \gamma(n,l)) = z^{\frac{1}{2}\gamma(n,l)} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\gamma(n,l)\log z}{2} \right)^j. \tag{3.2}$$

Inserting (3.1) into (3.31) and converting the sum into an integral, we have the following expression for the correlator in the limit $\bar{z} \to 1$

$$\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \sum_{n=0}^{\infty} \frac{z^n}{1-z} \int_0^\infty dl P^{(HL,HL)}_{n,l} \bar{z}^l p(\log z, \gamma(n,l)). \tag{3.3}$$

In the following we consider an expansion of (3.3) around $z=0$. The key point is to note that by expanding the anomalous dimensions and OPE coefficients, as in (2.16) and (2.17) respectively, terms proportional to $z^p \log^i z$ with $i = 2,3,\ldots,k$ and any p at $\mathcal{O}(\mu^k)$, in (3.3) are completely determined in terms of OPE data at $\mathcal{O}(\mu^{k-1})$. Moreover, using (2.20) one sees that the integral over the spin l yields

$$\int_0^\infty dl l^{\Delta_L-1-k} \bar{z}^l = \frac{\Gamma(\Delta_L-k)}{(-\log \bar{z})^{\Delta_L-k}} \underset{\bar{z}\to 1}{\approx} \frac{\Gamma(\Delta_L-k)}{(1-\bar{z})^{\Delta_L-k}}, \tag{3.4}$$

at $\mathcal{O}(\mu^k)$ in the limit $\bar{z} \to 1$. This correctly reproduces the expected $\bar{z}$ behaviour of minimal-twist multi-stress tensors in the T-channel, thus verifying (2.20).

We now make the following ansatz for the correlator

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^k}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1\ldots i_k} f_{i_1}(z)\ldots f_{i_k}(z), \tag{3.5}$$

where the sum goes over all sets of $\{i_p\}$ with i_p integers and $i_p \le i_{p+1}$ such that $\sum_{p=1}^k i_p = 3k$ and $a_{i_1\ldots i_k}$ coefficients that need to be fixed. Generally $f_a(z)$ are given by

$$f_a(z) = q_{1,a}(z) + q_{2,a}(z)\log z, \tag{3.6}$$

where $q_{(1,2),a}(z)$ are rational functions and the ansatz (3.5) at $\mathcal{O}(\mu^k)$ is therefore a polynomial in $\log z$ of degree k. By crossing symmetry terms with $\log^a z$, with $2 \le a \le k$, are determined by OPE data at $\mathcal{O}(\mu^{k-1})$. This is what we will use to determine the coefficients $a_{i_1\ldots i_p}$.

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3.1 Stress tensor

We start by determining the OPE data at $\mathcal{O}(\mu)$. This is easily obtained by matching (3.3) at $\mathcal{O}(\mu)$ with the stress tensor contribution (2.10). Explicitly, multiplying both channels by $(1-z)$ we have at $\mathcal{O}(\mu)$

$$\frac{\Delta_L f_3(z)}{120[(1-z)(1-\bar{z})]^{\Delta_L-1}} = \frac{1}{(1-\bar{z})^{\Delta_L-1}} \sum_{n=0}^{\infty} \frac{\Gamma(\Delta_L+n-1)z^n}{\Gamma(\Delta_L)n!} \left(P_n^{(1)} + \frac{\gamma_n^{(1)}}{2} \log z \right). \quad (3.7)$$

Expanding the l.h.s. in (3.7) for $z \ll 1$ we find

$$\begin{aligned} \frac{\Delta_L/120}{[(1-z)(1-\bar{z})]^{\Delta_L-1}} f_3(z) = \frac{1}{(1-\bar{z})^{\Delta_L-1}} \Bigg(& -\frac{\Delta_L}{4}(3+\log z) \\ & - z\frac{\Delta_L}{4}(3(\Delta_L+1)+(\Delta_L+5)\log z) \\ & - z^2\frac{\Delta_L}{8}(3\Delta_L(\Delta_L+3)+(12+\Delta_L(\Delta_L+11))) \\ & + \mathcal{O}(z^3, z^3\log z) \Bigg), \end{aligned} \quad (3.8)$$

while the r.h.s. is given by

$$\begin{aligned} \frac{\sum_{n=0}^{\infty} \frac{\Gamma(\Delta_L+n-1)z^n}{\Gamma(\Delta_L)n!} \left(P_n^{(1)} + \frac{\gamma_n^{(1)}}{2}\log z \right)}{(1-\bar{z})^{\Delta_L-1}} = \frac{1}{(1-\bar{z})^{\Delta_L-1}} \Bigg(& \frac{P_0^{(1)} + \frac{\gamma_0^{(1)}}{2}\log z}{\Delta_L - 1} \\ & + z\left(P_1^{(1)} + \frac{\gamma_1^{(1)}}{2}\log z\right) \\ & + z^2\frac{\Delta_L}{2}\left(P_2^{(1)} + \frac{\gamma_2^{(1)}}{2}\log z\right) \\ & + \mathcal{O}(z^3, z^3\log z) \Bigg). \end{aligned} \quad (3.9)$$

Comparing (3.8) and (3.9) order-by-order in z one finds the following OPE data

$$\begin{aligned} \gamma_0^{(1)} &= -\frac{\Delta_L(\Delta_L-1)}{2}, \\ \gamma_1^{(1)} &= -\frac{\Delta_L(\Delta_L+5)}{2}, \\ \gamma_2^{(1)} &= -\frac{12+\Delta_L(\Delta_L+11)}{2}, \end{aligned} \quad (3.10)$$

which agrees with eq. (6.10) in [30], and the OPE coefficients

$$
\begin{aligned}
P_0^{(1)} &= -\frac{3\Delta_L(\Delta_L-1)}{4}, \\
P_1^{(1)} &= -\frac{3\Delta_L(\Delta_L+1)}{4}, \\
P_2^{(1)} &= -\frac{3\Delta_L(\Delta_L+3)}{4}.
\end{aligned}
\tag{3.11}
$$

It is straightforward to continue and compute the $\mathcal{O}(\mu)$ OPE data in the S-channel for any value of n.

3.2 Twist-four double-stress tensors

From (3.5) we infer the following expression for the contribution due to twist-four double-stress tensors to the heavy-heavy-light-light correlator in the limit $\bar{z} \to 1$:

$$
\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(a_{15} f_1(z) f_5(z) + a_{24} f_2(z) f_4(z) + a_{33} f_3^2(z) \Big). \tag{3.12}
$$

By expanding (3.12) further in the limit $z \ll 1$ and collecting terms that goes as $z^p \log^2 z$, we will match with known contributions obtained from (3.3).

Inserting (3.10) and (3.11) in the S-channel (3.3) fixes terms proportional to $z^p \log^2 z$ up to $\mathcal{O}(z^2 \log^2 z)$. Expanding the ansatz (3.12) and matching with the S-channel reproduces the result obtained in [32]:

$$
\begin{aligned}
\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L-2)} \right) \times \\
& \left\{ (\Delta_L-4)(\Delta_L-3) f_3^2(z) + \frac{15}{7}(\Delta_L-8) f_2(z) f_4(z) + \frac{40}{7}(\Delta_L+1) f_1(z) f_5(z) \right\}.
\end{aligned}
\tag{3.13}
$$

Using the $\mathcal{O}(\mu)$ OPE data in the S-channel for $n > 2$ in (3.8) and (3.9) one gets an overconstrained system which is still solved by (3.13). This is a strong argument in favor of the validity of our ansatz (3.5).

We can now use (3.13) to derive the $\mathcal{O}(\mu^2)$ OPE data in the S-channel by matching terms proportional to $z^p \log^i z$ as $z \to 0$, with $i = 0, 1$, by comparing with (3.3). This is done in the same way it was done for $\mathcal{O}(\mu)$ OPE data in the S-channel. For example, one finds the following data for $n = 0, 1, 2, 3$:

$$
\begin{aligned}
\gamma_0^{(2)} &= -\frac{(\Delta_L-1)\Delta_L(4\Delta_L+1)}{8}, \\
\gamma_1^{(2)} &= -\frac{\Delta_L(\Delta_L+1)(4\Delta_L+35)}{8}, \\
\gamma_2^{(2)} &= -\frac{(3+\Delta_L)(68+\Delta_L(69+4\Delta_L))}{8}, \\
\gamma_3^{(2)} &= -\frac{(5+\Delta_L)(204+\Delta_L(4\Delta_L+103))}{8},
\end{aligned}
\tag{3.14}
$$

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which agrees with eq. (6.39) in [30], and for the OPE coefficients

$$
\begin{aligned}
P_0^{(2)} &= \frac{(\Delta_L-1)\Delta_L(-28+\Delta_L(-145+27\Delta_L))}{96},\\
P_1^{(2)} &= \frac{\Delta_L(-596+\Delta_L(-399+\Delta_L(-64+27\Delta_L)))}{96},\\
P_2^{(2)} &= \frac{-1248+\Delta_L(-2252+\Delta_L(-699+\Delta_L(44+27\Delta_L)))}{96},\\
P_3^{(2)} &= \frac{-3744+\Delta_L(-4940+\Delta_L(-783+\Delta_L(152+27\Delta_L)))}{96}.
\end{aligned}
\tag{3.15}
$$

It is again straightforward to extract the OPE data for any value of n.

3.3 Twist-six triple-stress tensors

We now consider the multi-stress tensor sector of the correlator at $\mathcal{O}(\mu^3)$ and proceed similarly to the previous section. From (3.5) we infer the following expression for the contribution due to twist-six triple-stress tensors:

$$
\begin{aligned}
\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(& a_{117} f_1^2 f_7 + a_{126} f_1 f_2 f_6 + a_{135} f_1 f_3 f_5 \\
& + a_{225} f_2^2 f_5 + a_{234} f_2 f_3 f_4 + a_{333} f_3^3 \Big),
\end{aligned}
\tag{3.16}
$$

where $f_i = f_i(z)$ is given by (2.6).[7] Taking the limit $1-\bar{z} \ll z \ll 1$ of (3.16), we fix the coefficients by matching with terms proportional to $z^p \log^2 z$ and $z^p \log^3 z$, with $p = 0,1,2$ from (3.3). This requires using the OPE data of the heavy-light double-twist operators $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ for $n = 0,1,2$ and $l \gg 1$ to $\mathcal{O}(\mu^2)$, given in (3.10), (3.11), (3.14) and (3.15).

We find the following solution:

$$
\begin{aligned}
a_{117} &= \frac{5\Delta_L(\Delta_L+1)(\Delta_L+2)}{768768(\Delta_L-2)(\Delta_L-3)},\\
a_{126} &= \frac{5\Delta_L(5\Delta_L^2-57\Delta_L-50)}{6386688(\Delta_L-2)(\Delta_L-3)},\\
a_{135} &= \frac{\Delta_L(2\Delta_L^2-11\Delta_L-9)}{1209600(\Delta_L-3)},\\
a_{225} &= -\frac{\Delta_L(7\Delta_L^2-51\Delta_L-70)}{2903040(\Delta_L-2)(\Delta_L-3)},\\
a_{234} &= \frac{\Delta_L(\Delta_L-4)(3\Delta_L^2-17\Delta_L+4)}{4838400(\Delta_L-2)(\Delta_L-3)},\\
a_{333} &= \frac{\Delta_L(\Delta_L-4)(\Delta_L^3-16\Delta_L^2+51\Delta_L+24)}{10368000(\Delta_L-2)(\Delta_L-3)}.
\end{aligned}
\tag{3.18}
$$

[7] Note that we omitted a potential term of the form $f_1 f_4^2$. This can be written in terms of f_3^3, $f_1 f_3 f_5$, $f_2^2 f_5$ and $f_2 f_3 f_4$, as follows from:

$$
f_3^3(z) = \frac{20}{21} f_1(z) f_3(z) f_5(z) - \frac{27}{28} f_1(z) f_4^2(z) - \frac{20}{21} f_2^2(z) f_5(z) + \frac{55}{28} f_2(z) f_3(z) f_4(z). \tag{3.17}
$$

We can also consider higher values of p and obtain an overconstrained system of equations, whose solution is still (3.18). Inserting (3.18) into (3.16), we obtain the contribution from minimal-twist triple-stress tensor operators to the heavy-heavy-light-light correlator in the lightcone limit.

Note that for $\Delta_L \to \infty$, the correlator is determined by the exponentiation of the stress tensor discussed e.g. in [32], i.e.

$$\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{1}{3!} \left(\frac{\Delta_L}{120}(1-z)^3 {}_2F_1(3,3;6;1-z)\right)^3 + \cdots , \tag{3.19}$$

which one indeed obtains by taking $\Delta_L \to \infty$ of (3.16) with (3.18). Here ellipses denote terms subleading in Δ_L.

By analytically continuing $z \to e^{-2\pi i} z$ and sending $z \to 1$, one can access the large impact parameter regime of the Regge limit. To do this we use the following property of the hypergeometric function (see e.g. [4]):

$${}_2F_1(a,a,2a,1-ze^{-2\pi i}) = {}_2F_1(a,a,2a,1-z) + 2\pi i \frac{\Gamma(2a)}{\Gamma(a)^2} {}_2F_1(a,a,1,z). \tag{3.20}$$

Using (3.20) the leading term from (3.16) with the coefficients (3.18) in the limit $1-\bar{z} \ll 1-z \ll 1$ is given by

$$\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1, z\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} \times \\ \left(-\frac{9i\pi^3 \Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{2(\Delta_L-2)(\Delta_L-3)} \left(\frac{1-\bar{z}}{(1-z)^2}\right)^3\right). \tag{3.21}$$

This agrees with the holographic calculation in a shockwave background at $\mathcal{O}(\mu^3)$ given by eq. (45) in [39] based on techniques developed in [45–49].

3.4 Exponentiation of leading-twist multi-stress tensors

In $d = 2$ the heavy-heavy-light-light correlator is determined by the heavy-heavy-light-light Virasoro vacuum block. This block contains the exchange of any number of stress tensors and derivatives thereof in the T-channel [11, 12, 17], and therefore all multi-stress tensor contributions. This block, together with the disconnected part, exponentiates as

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\rangle = e^{\Delta_L \mathcal{F}(z)}, \tag{3.22}$$

for a known function $\mathcal{F}(z)$ independent of Δ_L. It is interesting to ask if something similar happens for the contribution of the minimal-twist multi-stress tensors in the lightcone limit of the correlator in higher dimensions. By this we mean whether the stress tensor sector of the correlator can be written as

$$\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{3.23}$$

for some function $\mathcal{F}(\mu;z,\bar{z})$ which is a rational function of Δ_L and remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$.

The $\bar{z}$ dependence implies the following form of $\mathcal{F}(\mu; z, \bar{z})$:

$$\mathcal{F}(\mu; z, \bar{z}) = \mu(1-\bar{z})\mathcal{F}^{(1)}(z) + \mu^2(1-\bar{z})^2\mathcal{F}^{(2)}(z) + \mu^3(1-\bar{z})^3\mathcal{F}^{(3)}(z) + \mathcal{O}(\mu^4). \tag{3.24}$$

At leading order we observe $\mathcal{F}^{(1)}(z) = \frac{1}{120} f_3(z)$, which is just the stress tensor contribution. At second order we find:

$$\mathcal{F}^{(2)}(z) = \frac{(12-5\Delta_L) f_3(z)^2 + \frac{15}{7}(\Delta_L - 8) f_2(z) f_4(z) + \frac{40}{7}(\Delta_L + 1) f_1(z) f_5(z)}{28800(\Delta_L - 2)}. \tag{3.25}$$

Note that $\mathcal{F}^{(2)}(z)$ is independent of Δ_L in the limit $\Delta_L \to \infty$.

To find $\mathcal{F}^{(3)}(z)$ we parametrise it as

$$\begin{aligned}\mathcal{F}^{(3)}(z) = \Big(& b_{117} f_1^2(z) f_7(z) + b_{126} f_1(z) f_2(z) f_6(z) + b_{135} f_1(z) f_3(z) f_5(z) \\ &+ b_{225} f_2^2(z) f_5(z) + b_{234} f_2(z) f_3(z) f_4(z) + b_{333} f_3^3(z)\Big).\end{aligned} \tag{3.26}$$

It is clear that for terms which do not contain a factor of $f_3(z)$, the coefficients b_{ijk} should satisfy $b_{ijk} = a_{ijk}/\Delta_L$. This is not true for terms which contain a factor of f_3. Inserting $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ and eq. (3.26) in (3.23), expanding in μ and matching with (3.16) yields

$$\begin{aligned} b_{117} &= \frac{a_{117}}{\Delta_L}, \\ b_{126} &= \frac{a_{126}}{\Delta_L}, \\ b_{225} &= \frac{a_{225}}{\Delta_L}, \\ b_{135} &= -\frac{11\Delta_L^2 - 19\Delta_L - 18}{1209600(\Delta_L - 2)(\Delta_L - 3)}, \\ b_{234} &= \frac{(\Delta_L - 2)(\Delta_L + 2)}{1209600(\Delta_L - 2)(\Delta_L - 3)}, \\ b_{333} &= \frac{7\Delta_L^2 - 18\Delta_L - 24}{2592000(\Delta_L - 2)(\Delta_L - 3)}. \end{aligned} \tag{3.27}$$

From (3.25) and (3.27), one finds that the correlator exponentiates to $\mathcal{O}(\mu^3)$ in the sense described above, i.e. $\mathcal{F}(\mu; z, \bar{z})$ is a rational function of Δ_L of $\mathcal{O}(1)$ as $\Delta_L \to \infty$.

To leading order in Δ_L, exponentiation for large Δ_L is a prediction of the AdS/CFT correspondence. The two-point function of the operator $\mathcal{O}_L$ in the state created by the heavy operator $\mathcal{O}_H$ is given in terms of the exponential of the (regularized) geodesic distance between the boundary points in the dual bulk geometry. For details on this, see e.g. [32].

3.5 OPE coefficients of triple-stress tensors

In this section we describe how to decompose the correlator (3.16) into an infinite sum of minimal-twist triple-stress tensor operators. In order to do this we use the following multiplication formula for hypergeometric functions [32]:

$${}_2F_1(a, a; 2a; w)\,{}_2F_1(b, b; 2b; w) = \sum_{m=0}^{\infty} p[a, b, m] w^{2m}\,{}_2F_1[a+b+2m, a+b+2m, 2a+2b+4m, w], \tag{3.28}$$

where

$$p[a,b,m] = \frac{2^{-4m}\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)\Gamma(a+m)\Gamma(b+m)\Gamma\left(a+b+m-\frac{1}{2}\right)\Gamma\left(a+b+2m\right)}{\sqrt{\pi}\Gamma(a)\Gamma(b)\Gamma(m+1)\Gamma\left(a+m+\frac{1}{2}\right)\Gamma\left(b+m+\frac{1}{2}\right)\Gamma(a+b+m)\Gamma\left(a+b+2m-\frac{1}{2}\right)}. \tag{3.29}$$

It is useful to note that by using (3.28) we can write a similar formula for the functions f_a defined in (2.6):

$$f_a(z)f_b(z) = \sum_{m=0}^{\infty} p[a,b,m]f_{a+b+2m}(z), \tag{3.30}$$

where $p[a,b,m]$ is defined in (3.29). It is now clear that the correlator (3.16) can be written as a double sum over functions $f_{9+2(n+m)}$. We can thus write the stress tensor sector of the correlator in the lightcone limit at $\mathcal{O}(\mu^3)$ as

$$\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{n,m=0}^{\infty} c[m,n]f_{9+2(n+m)}(z), \tag{3.31}$$

with

$$\begin{aligned} c[m,n] =& \big(a_{333}p[3,3,m]p[3,6+2m,n] + a_{117}p[1,7,m]p[1,8+2m,n] \\ &+ a_{126}p[2,6,m]p[1,8+2m,n] + a_{135}p[3,5,m]p[1,8+2m,n] \\ &+ a_{225}p[2,5,m]p[2,7+2m,n] + a_{234}p[3,4,m]p[2,7+2m,n]\big), \end{aligned} \tag{3.32}$$

where coefficients a_{ijk} are fixed in (3.18).

Comparing (3.31) with (2.5) we see that the contribution at $\mathcal{O}(\mu^3)$ comes from operators of the schematic form $:T_{\alpha\beta}T_{\gamma\delta}\partial_{\rho_1}\ldots\partial_{\rho_{2l}}T_{\mu\nu}:$. These operators have $\frac{\tau}{2}+s = 9+2l$, where s is total spin $s = 6+2l$. The corresponding OPE coefficients of such operators will be a sum of all contributions in (3.31) for which $n+m = l$.

Now, one can write OPE coefficients of operators of type $:T_{\alpha\beta}T_{\gamma\delta}\partial_{\rho_1}\ldots\partial_{\rho_{2l}}T_{\mu\nu}:$ as

$$P^{(HH,LL);(3)}_{6,6+2l} = \sum_{n=0}^{l} c[l-n,n]. \tag{3.33}$$

Let us write a few of the coefficients explicitly here:

$$\begin{aligned} \mu^3 P^{(HH,LL);(3)}_{6,6} &= \mu^3\frac{\Delta_L(3024+\Delta_L(7500+\Delta_L(7310+143\Delta_L(25+7\Delta_L))))}{10378368000(\Delta_L-2)(\Delta_L-3)}, \\ \mu^3 P^{(HH,LL);(3)}_{6,8} &= \mu^3\frac{\Delta_L(2688+\Delta_L(7148+\Delta_L(9029+13\Delta_L(464+231\Delta_L))))}{613476864000(\Delta_L-3)(\Delta_L-2)}, \\ \mu^3 P^{(HH,LL);(3)}_{6,10} &= \mu^3\frac{\Delta_L(888+\Delta_L(2216+\Delta_L(3742+17\Delta_L(181+143\Delta_L))))}{9468531072000(\Delta_L-3)(\Delta_L-2)}. \end{aligned} \tag{3.34}$$

We further find that $P^{(HH,LL);(3)}_{6,6}$ and $P^{(HH,LL);(3)}_{6,8}$ agree with the expression obtained holographically in [37].

4 Minimal-twist double-stress tensors in six dimensions

In this section we derive the contribution of minimal-twist double-stress tensors to the heavy-heavy-light-light correlator in the lightcone limit in $d = 6$. The method is analogous to the four-dimensional case described in section 3.

From (1.15) we make the following ansatz for the stress tensor sector in the lightcone limit:

$$\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^4}{[(1-z)(1-\bar{z})]^{\Delta_L}} \times \Big(a_{17}f_1(z)f_7(z) + a_{26}f_2(z)f_6(z) + a_{35}f_3(z)f_5(z) + a_{44}f_4^2(z)\Big). \tag{4.1}$$

The S-channel conformal blocks in six dimensions in the limit $\Delta_H \to \infty$ are given by (2.15). In the lightcone limit $\bar{z} \to 1$ operators with $l \gg 1$ dominate and the blocks can be approximated by

$$(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \simeq \frac{\bar{z}^l z^n p(\log z,\gamma)}{(1-z)^2}, \tag{4.2}$$

with p given by (3.2). Replacing the sum in (3.31) with an integral and inserting (4.2) we have

$$\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \sum_{n=0}^{\infty} \frac{z^n}{(1-z)^2} \int_0^{\infty} dl P^{(HL,HL)}_{n,l} \bar{z}^l p(\log z,\gamma). \tag{4.3}$$

As in $d = 4$ one finds that terms proportional to $\log^i z$ with $i = 2, 3, \ldots, k$ at $\mathcal{O}(\mu^k)$, are determined by the OPE data at $\mathcal{O}(\mu^{k-1})$.

At $\mathcal{O}(\mu)$ we can use the known contribution from the stress tensor exchange (2.10) to derive the anomalous dimensions $\gamma_n^{(1)}$ and the OPE coefficients $P_n^{(1)}$ just as it was done in four dimensions. This is done by matching (4.3) order by order in the small z expansion. Using (2.20) one can integrate over spin. E.g. for $n = 0, 1, 2, 3$:

$$\begin{aligned}
\gamma_0^{(1)} &= -\frac{(\Delta_L-2)(\Delta_L-1)\Delta_L}{2}, \\
\gamma_1^{(1)} &= -\frac{(\Delta_L-1)\Delta_L(\Delta_L+10)}{2}, \\
\gamma_2^{(1)} &= -\frac{\Delta_L(\Delta_L+2)(\Delta_L+19)}{2}, \\
\gamma_3^{(1)} &= -\frac{(\Delta_L+4)(\Delta_L(\Delta_L+29)+30)}{2},
\end{aligned} \tag{4.4}$$

these anomalous dimensions agree with eq. (6.10) in [30]. Similarly, we obtain the following OPE coefficients:

$$\begin{aligned}
P_0^{(1)} &= -\frac{11(\Delta_L-2)(\Delta_L-1)\Delta_L}{12}, \\
P_1^{(1)} &= -\frac{(\Delta_L-1)\Delta_L(11\Delta_L+38)}{12},
\end{aligned}$$

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$$
\begin{aligned}
P_2^{(1)} &= -\frac{\Delta_L(22+\Delta_L(87+11\Delta_L))}{12}, \\
P_3^{(1)} &= -\frac{\Delta_L(202+\Delta_L(147+11\Delta_L))}{12}.
\end{aligned}
\tag{4.5}
$$

It is straightforward to continue to higher values of n.

Plugging (4.4) into (4.3) in the limit $1-\bar{z} \ll z \ll 1$ one finds the following contribution to the terms proportional to $\frac{z^p \log^2 z}{(1-\bar{z})^{\Delta_L-4}}$ at $\mathcal{O}(\mu^2)$:

$$
\begin{aligned}
p=0 :& \frac{\Delta_L^2(\Delta_L-1)(\Delta_L-2)}{32(\Delta_L-3)(\Delta_L-4)}, \\
p=1 :& \frac{\Delta_L^2(\Delta_L-1)(\Delta_L+6)(\Delta_L+16)}{32(\Delta_L-3)(\Delta_L-4)} \\
p=2 :& \frac{\Delta_L^2(\Delta_L^4+46\Delta_L^3+599\Delta_L^2+1898\Delta_L+1056)}{64(\Delta_L-3)(\Delta_L-4)}, \\
p=3 :& \frac{\Delta_L^7+72\Delta_L^6+1651\Delta_L^5+13344\Delta_L^4+40180\Delta_L^3+41952{\Delta_L}^2+14400\Delta_L}{192(\Delta_L-3)(\Delta_L-4)}.
\end{aligned}
\tag{4.6}
$$

It is now straightforward to expand the ansatz (4.1) in the limit $1-\bar{z} \ll z \ll 1$, collect terms that behave as $z^p \log^2 z$ and compare them to the S-channel (4.6). This determines the coefficients:

$$
\begin{aligned}
a_{17} &= \frac{\Delta_L(\Delta_L+1)(\Delta_L+2)}{64064(\Delta_L-3)(\Delta_L-4)}, \\
a_{26} &= \frac{\Delta_L(-18+(-12+\Delta_L)\Delta_L)}{133056(\Delta_L-3)(\Delta_L-4)}, \\
a_{35} &= \frac{\Delta_L(\Delta_L-6)(\Delta_L-15)}{302400(\Delta_L-3)(\Delta_L-4)}, \\
a_{44} &= \frac{\Delta_L(\Delta_L-5)(\Delta_L-6)}{627200(\Delta_L-3)}.
\end{aligned}
\tag{4.7}
$$

One can consider higher values of p; eq. (4.7) is still the solution of the corresponding overconstrained system.

The double-stress tensor contribution to the correlator in the lightcone limit $\bar{z} \to 1$ is therefore given by

$$
\begin{aligned}
\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^4}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{\Delta_L}{(\Delta_L-3)(\Delta_L-4)} \left(\frac{1}{627200}\right) \\
& \times \Bigg\{ (\Delta_L-4)(\Delta_L-5)(\Delta_L-6) f_4^2(z) + \frac{56(\Delta_L-6)(\Delta_L-15)}{27} f_3(z) f_5(z) \\
& + \frac{1400(\Delta_L(\Delta_L-12)-18)}{297} f_2(z) f_6(z) + \frac{1400(\Delta_L+1)(\Delta_L+2)}{143} f_1(z) f_7(z) \Bigg\}.
\end{aligned}
\tag{4.8}
$$

Using (4.8) one can deduce the second order OPE data in the S-channel. The anomalous dimensions at this order can then be compared to the holographic calculations in [30] to reveal perfect agreement.

4.1 Exponentiation of minimal-twist multi-stress tensors in six dimensions

It is interesting to study whether the stress tensor sector of the correlator exponentiates in the lightcone limit

$$\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{4.9}$$

with $\mathcal{F}(\mu;z,\bar{z})$ a rational function of Δ_L that is of $\mathcal{O}(1)$ as $\Delta_L \to \infty$. In the lightcone limit $\mathcal{F}(\mu;z,\bar{z})$ admits an expansion

$$\mathcal{F}(\mu;z,\bar{z}) = \mu(1-\bar{z})^2 \mathcal{F}^{(1)}(z) + \mu^2 (1-\bar{z})^4 \mathcal{F}^{(2)}(z) + \mathcal{O}(\mu^3). \tag{4.10}$$

At $\mathcal{O}(\mu)$ one finds $\mathcal{F}^{(1)}(z) = \frac{\Gamma(\frac{6}{2}+1)^2}{4\Gamma(6+2)} f_4(z)$ from the stress tensor contribution. Using (4.8) we find

$$\mathcal{F}^{(2)}(z) = b_{17} f_1(z) f_7(z) + b_{26} f_2(z) f_6(z) + b_{35} f_3(z) f_5(z) + b_{44} f_4^2(z) \tag{4.11}$$

with

$$\begin{aligned} b_{17} &= \frac{a_{17}}{\Delta_L}, \\ b_{26} &= \frac{a_{26}}{\Delta_L}, \\ b_{35} &= \frac{a_{35}}{\Delta_L}, \\ b_{44} &= -\frac{4\Delta_L^2 - 31\Delta_L + 60}{313600(\Delta_L - 3)(\Delta_L - 4)}. \end{aligned} \tag{4.12}$$

From (4.12) we indeed see that the stress tensor sector of the correlator exponentiates at least to $\mathcal{O}(\mu^2)$ in $d = 6$.

4.2 OPE coefficients of minimal-twist double-stress tensors

In this section we decompose the stress tensor sector of the correlator (4.1) into a sum over minimal-twist double-stress tensors. The discussion follows that of section 3.5.

Applying (3.30) to (4.8), we find that $a + b + 2l = 8 + 2l$ which is $\frac{\tau}{2} + s + 2l$, with $\tau = 8$ and $s = 4$ being the twist and spin of the simplest minimal-twist double-stress tensor operator $:T_{\mu\nu}T_{\rho\lambda}:$. Non-zero value of l thus gives the contribution from operators of higher spin of the form $:T_{\mu\nu}\partial_{\rho_1}\ldots\partial_{\rho_{2l}}T_{\delta\lambda}:$, where no indices are contracted and only even spin operators contribute to the OPE between identical scalars.

It is now straightforward to write down the OPE coefficients for minimal-twist double-stress tensors in six dimensions. E.g. one finds for the lowest-spin operators the following

OPE coefficients:

$$\mu^2 P_{8,4}^{(HH,LL);(2)} = \mu^2 \frac{\Delta_L(600+\Delta_L(1394+\Delta_L(677+429\Delta_L)))}{269068800(\Delta_L-3)(\Delta_L-4)},$$

$$\mu^2 P_{8,6}^{(HH,LL);(2)} = \mu^2 \frac{\Delta_L(30+\Delta_L(187+\Delta_L(-120+143\Delta_L)))}{3430627200(\Delta_L-3)(\Delta_L-4)},$$

$$\mu^2 P_{8,8}^{(HH,LL);(2)} = \mu^2 \frac{\Delta_L(60+\Delta_L(1382+\Delta_L(-1857+1105\Delta_L)))}{657033721344(\Delta_L-3)(\Delta_L-4)}. \tag{4.13}$$

For general spin we have ($s = 4 + 2l$)

$$P_{8,s}^{(HH,LL)} = \mu^2 \frac{c\Delta_L}{(\Delta_L-3)(\Delta_L-4)}(a_3\Delta_L^3 + a_2\Delta_L^2 + a_1\Delta_L + a_0) \tag{4.14}$$

where

$$c = \frac{2^{-9-2s}\sqrt{\pi}s(s+2)\Gamma(s-1)}{(s-3)(s+4)(s+6)(s+8)(s+10)\Gamma(s+\frac{7}{2})},$$

$$a_3 = (s-2)s(s+2)(s+5)(s+7)(s+9),$$

$$a_2 = -3(2880 + s(s+7)(-276 + s(s+7)(-56 + s(s+7)))),$$

$$a_1 = 2(25920 + s(s+7)(3276 + s(s+7)(-80 + s(s+7)))),$$

$$a_0 = 675 \times 2^7. \tag{4.15}$$

5 Discussion

In this paper we consider the minimal-twist multi-stress tensor contributions to the heavy-heavy-light-light correlator of scalars in large C_T CFTs in even spacetime dimensions. We provide strong evidence for the conjecture that all such contributions are described by the ansatz (1.15) and determine the coefficients by performing a bootstrap procedure. In practice this is completed for twist-four double-stress tensors and twist-six triple-stress tensors in four dimensions as well as twist-eight double-stress tensors in six dimensions. In principle it is straightforward to use our technology to determine the coefficients $a_{i_1 \ldots i_k}$ to arbitrarily high order in μ; this must be related to the universality of the minimal-twist OPE coefficients.

In two dimensions the heavy-heavy-light-light Virasoro vacuum block exponentiates [see eq. (1.1)], with $\mathcal{F}(\mu; z)$ independent of Δ_L. In higher dimensions we observe a similar exponentiation with $\mathcal{F}(\mu; z, \bar{z})$ a rational function of Δ_L that remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$. It would be interesting to see whether it is possible to write down a closed-form recursion formula for $\mathcal{F}(\mu; z, \bar{z})$. Solving such a recursion formula would give a higher-dimensional analogue of the two-dimensional Virasoro vacuum block.

An immediate technical question concerns CFTs in odd spacetime dimensions. We could not immediately generalize our results in this context — the ansatz in eq. (1.15) fails in odd dimensions. However, the heavy-light conformal blocks are known [41], so a similar approach should be feasible.

It would be interesting to study the regime of applicability of our results. We have not used holography; our main assumption is the ansatz (1.8), known to be true for holographic CFTs to $\mathcal{O}(\mu^2)$ in $d = 4$ [32]. Yet, our general expressions for the OPE coefficients agree with the OPE coefficients computed in some holographic examples [37]. What happens once one goes beyond holographic CFTs — will our ansatz need to be modified by the inclusion of terms suppressed by the gap or the central charge? We leave these questions for subsequent investigations.

Another interesting direction concerns the study of the bulk scattering phase-shift in the presence of a black hole background. In the context of higher dimensional CFTs, this problem was first considered in [30] where the gravitational expression was given to all orders in μ and the CFT computation was performed to $\mathcal{O}(\mu)$. Subsequently, $\mathcal{O}(\mu^2)$ was discussed in [41]. In [39] the $\mathcal{O}(\mu)$ contribution was exponentiated to yield the scattering phase shift in the presence of a shock-wave geometry. A CFT computation of the phase shift to all orders in μ is still lacking. This would in principle involve understanding Regge theory beyond the leading order. It will be interesting to see whether the results of this article could be helpful in this regard.

Acknowledgments

We thank G.-S. Ng, E. Perlmutter, K. Sen and D. Simmons-Duffin for useful discussions. The work of R.K. and A.P. is supported in part by an Irish Research Council Laureate Award. The work of P.T. is supported in part by an Ussher Fellowship Award. M.K. and A.P. thank the Aspen Center for Physics, where part of this work was completed, for hospitality. Work at ACP is supported by the National Science Foundation under Grant No. NSF PHY-1607611.

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JHEP01(2020)076

Published for SISSA by Springer

Received: March 20, 2020
Revised: May 13, 2020
Accepted: May 27, 2020
Published: July 3, 2020

Stress tensor sector of conformal correlators operators in the Regge limit

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Abstract: An important part of a CFT four-point function, the stress tensor sector, comprises the exchanges of the stress tensor and its composites. The OPE coefficients of these multi-stress tensor operators and consequently, the complete stress tensor sector of four-point functions in CFTs with a large central charge, can be determined by computing a heavy-heavy-light-light correlator. We show how one can make substantial progress in this direction by bootstrapping a certain ansatz for the stress tensor sector of the correlator, iteratively computing the OPE coefficients of multi-stress tensor operators with increasing twist. Some parameters are not fixed by the bootstrap — they correspond to the OPE coefficients of multi-stress tensors with spin zero and two. We further show that in holographic CFTs one can use the phase shift computed in the dual gravitational theory to reduce the set of undetermined parameters to the OPE coefficients of multi-stress tensors with spin zero. Finally, we verify some of these results using the Lorentzian OPE inversion formula and comment on its regime of applicability.

Keywords: AdS-CFT Correspondence, Conformal Field Theory, Black Holes

ArXiv ePrint: 2002.12254

https://doi.org/10.1007/JHEP07(2020)019

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1 Introduction and summary

1.1 Introduction

Conformal field theories (CFTs) are the harmonic oscillators of our times; besides being significantly more amenable to analytic study compared to generic quantum field theories, they also provide a non-perturbative definition of gravity in negatively curved spacetimes via the AdS/CFT correspondence [1–3]. Their robust structure bears many important consequences which have come to light in recent years due to the development of conformal bootstrap techniques following [4–7]. This is especially pronounced in spacetime dimension $d > 2$ which this article is focused upon.

Conformal symmetry imposes highly non-trivial constraints on the theory. Two- and three-point correlation functions are fixed up to a handful of position-independent parameters [8]. Four- and higher-point functions [9–11] are determined as long as the CFT spectrum of local operators and the respective OPE coefficients are known (for recent techniques see the original works of [12, 13] and the modern approach developed in [14, 15]).

While computing four-point correlation functions is possible in principle, the amount of necessary data makes it difficult in practice. Consistency principles, such as crossing symmetry and unitarity, come to rescue. In fact, the idea of the conformal bootstrap programme is to use these consistency conditions to place restrictions on the CFT data (spectrum of operators and OPE coefficients) and, if possible, solve the theory completely.

One way to make use of crossing symmetry is to consider kinematic regimes which enhance the contribution of a limited number of operators in a given channel, and are typically reproduced by an infinite number of operators in another channel. A standard example is the lightcone limit where the initially spacelike separation between two operators is allowed to become null. Focusing on the lightcone limit of a four-point correlation function allows one to deduce the existence of double-twist operators at large spin in any CFT in dimensions $d > 2$ [16, 17].

A natural assumption when considering an arbitrary CFT is the existence of a stress tensor. The two-point function of the stress-tensor depends on a single parameter, the central charge C_T, which serves as a rough measure of the number of degrees of freedom in the theory. In this paper, we will consider local CFTs with a large number of degrees of freedom, a.k.a. large central charge $C_T \gg 1$.

Specifically, our goal herein is to study the contribution of the stress-tensor sector in scalar CFT correlation functions, $\langle \mathcal{O}_1\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\rangle$. What we mean here by the "stress-tensor sector" is the set of operators composed out of stress-tensors and derivatives,[1] schematically denoted by $:T_{\mu_1\nu_1}\cdots T_{\mu_{p-1}\nu_{p-1}}\partial^{2n}\partial_{\lambda_1}\cdots\partial_{\lambda_q}T_{\mu_p\nu_p}:$. Such operators are present in large C_T CFTs, but their contribution to a correlation function is of particular interest in CFTs with holographic duals since it is related to the contribution of multiple gravitons in the corresponding Witten diagrams.

We consider the four-point function $\langle \mathcal{O}_H\mathcal{O}_L\mathcal{O}_L\mathcal{O}_H\rangle$ of two pairwise identical scalar operators labeled as "light, L", and "heavy, H", depending on whether their conformal dimension scales with the number of degrees of freedom, $\Delta_H \propto \mathcal{O}(C_T)$, or not, $\Delta_L \propto \mathcal{O}(1)$.

[1]The identity operator is considered as the first trivial entry of the stress-tensor sector.

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The reason this correlator is well-suited to the exploration of the stress-tensor sector is the presence of an additional parameter, μ, proportional to the ratio of the conformal dimension of the heavy operators with the central charge, $\mu \propto \Delta_H/C_T$. This parameter naturally counts the number of stress-tensors in a composite multi-stress tensor operator. To distinguish the contribution of such operators from the full HHLL correlator in what follows we will denote it as $\mathcal{G}(z,\bar{z})$, i.e.,

$$\mathcal{G}(z,\bar{z}) = \langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\Big|_{\text{multi-stress tensors}} . \tag{1.1}$$

Note that from $\mathcal{G}(z,\bar{z})$ in (1.1) one can read off the OPE coefficients of multi-stress tensor operators to leading order in $1/C_T$ but exact in Δ_L.

The HHLL correlator is interesting in its own right. In the limit of a large number of degrees of freedom, it is related to the thermal two-point function $\langle \mathcal{O}_L\mathcal{O}_L\rangle_T$ — as long as the average energy of the canonical ensemble is roughly equal to the conformal dimension of the heavy operator. When the CFT is additionally characterised by an infinite gap, $\Delta_{\text{gap}} \to \infty$, in the spectrum of primary single-trace (non-composite) operators with spin greater than two, the situation is even more interesting. In this case, the theory has an equivalent description in terms of a classical, local gravitational theory in AdS [18]. Such a CFT is called holographic as a minimally defined realisation of the holographic paradigm. When a holographic CFT is considered at finite temperature, the appropriate gravitational description is that of an asymptotically AdS black hole [19]. In this case, the HHLL correlator, in a certain kinematical regime, is expected to describe the scattering of a light particle by the black hole in the dual gravitational theory [20].

To study the stress tensor sector of the HHLL correlator we will employ crossing symmetry and the conformal bootstrap. Specifically, we consider the lightcone limit where the separation between the two $\mathcal{O}_L$ operators is close to being null. In this limit, the dominant contribution in the direct channel (T-channel, where the pairwise identical operators approach each other) is coming from multi-stress tensor operators with low twist (where the twist τ is the difference between the conformal dimension Δ and the spin s of a given operator, $\tau = \Delta - s$). In the cross-channel (S-channel), an infinite number of double-twist operators of the schematic form $:\mathcal{O}_H\partial_{\mu_1}\ldots\partial_{\mu_l}\partial^{2n}\mathcal{O}_L:$ with $l \gg 1$ should be considered.

In [21], it was argued through a holographic calculation that the OPE coefficients of minimal-twist multi-stress tensors are "universal" in the sense that they are completely fixed in terms of just two CFT parameters: Δ_L and $\frac{1}{C_T}$ (see also [22]). In [23], a formula for the OPE coefficients of the minimal twist double-stress tensors was written. In [24], it was shown how one can, at least in principle, evaluate the contribution of the stress tensor sector to all orders in μ in arbitrary even number of spacetime dimensions d in the lightcone limit. The strategy there was based on proposing an ansatz for $\mathcal{G}$ with a few undetermined parameters and then fixing these parameters by means of the lightcone bootstrap. In the process, one can extract the OPE coefficients of all multi-stress tensors with minimal twist. A different approach based on the Lorentzian inversion formula [25, 26] for extracting the minimal-twist double- and triple-stress tensor OPE coefficients was used in [27][2] and also appears to confirm the universality of the minimal-twist stress tensor sector.

[2]One should exercise caution when using the Lorentzian inversion formula in the context of the HHLL correlator as the Regge behaviour of the correlator has not been rigorously established.

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In this paper, we investigate the stress tensor sector further by considering contributions from multi-stress tensors with non-minimal twist. Our goal is to determine the structure of the correlator to subleading orders in the lightcone limit and extract the relevant OPE coefficients. Once more, we motivate an ansatz similar to the one successfully describing the leading lightcone behavior of $\mathcal{G}(z, \bar{z})$ and show that most of the parameters in the ansatz can be fixed using lightcone bootstrap. A few parameters are, however, left undetermined and might depend on the details of the theory. They correspond to the OPE coefficients of multi-stress tensors with spin $s = 0, 2$. Our approach can be employed to study the stress-tensor sector to arbitrary orders in μ and $(1 - \bar{z})$. In this paper, we completed this program for the $\mathcal{O}(\mu^2)$ subleading, subsubleading and subsubsubleading terms as well as the $\mathcal{O}(\mu^3)$ subleading and subsubleading terms.

We also investigate a complementary approach to computing the OPE data of the stress tensor sector using the Lorentzian inversion formula. As noted earlier, the validity of the Lorentzian inversion formula for the HHLL correlator has not been rigorously established. It is however natural to expect that it is applicable in the large-C_T and small-μ expansion, as long as a Regge bound is observed. Here we assume that the Regge behavior of the correlator is given by σ^{-k} at $\mathcal{O}(\mu^k)$ in the large-C_T limit, which is consistent with the behaviour of the scattering phase shift from a black hole (or a massive star) computed classically in AdS. We then find that whenever the Lorentzian inversion formula is applicable, i.e., for operators of spin $s > k + 1$ at $\mathcal{O}(\mu^k)$, OPE data extracted with both methods are in perfect agreement. However, already at order $\mathcal{O}(\mu^3)$, our ansatz combined with the crossing symmetry or Lorentzian inversion formula is more powerful than the Lorentzian inversion formula alone. For instance, while the former procedure allows us to determine the OPE coefficient of a triple-stress tensor with spin $s = 4$ and twist $\tau = 8$, this is not possible using solely the Lorentzian inversion formula.

Finally, we explore the possibility of obtaining the unknown OPE data from the gravitational description of the CFT. We use the phase shift calculation in the dual gravitational theory. The scattering phase shift — acquired by a highly energetic particle travelling in the background of the AdS black hole — was first computed in the Regge limit in Einstein gravity in [20]. To explicitly see how the presence of higher derivative gravitational terms affects the OPE data, we work in Einstein-Hilbert + Gauss-Bonnet gravity with small Gauss-Bonnet coupling λ_{GB}. To combine the gravitational results with those of the CFT in the lightcone regime, we follow the approach first discussed in [23] and further developed in [24], which involves an analytic continuation of the lightcone results around $z = 0$ and an expansion around $z = 1$. Matching terms in the correlator obtained from the gravitational calculation to those obtained from the CFT enables us to completely fix the stress tensor sector of the HHLL correlator up to the OPE coefficients of the spin-0 multi-stress tensors which are left undetermined. Non-universality is manifest by the presence of the Gauss-Bonnet coupling in the expressions for the OPE coefficients.

1.2 Summary of results

In this paper, we show that the stress tensor sector of the HHLL correlator in $d = 4$ can be written in terms of products of $f_a(z)$ functions defined as

$$f_a(z) = (1 - z)^a {}_2F_1(a, a, 2a, 1 - z). \tag{1.2}$$

The stress tensor sector of the HHLL correlator can be expanded in powers of μ and then in powers of $(1-\bar{z})$ as

$$\mathcal{G}(z,\bar{z}) = \sum_{k=0}^{\infty} \mu^k \mathcal{G}^{(k)}(z,\bar{z}) = \frac{1}{((1-z)(1-\bar{z}))^{\Delta_L}} + \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} \mu^k (1-\bar{z})^{-\Delta_L+k+m} \mathcal{G}^{(k,m)}(z), \tag{1.3}$$

where we have explicitly separated the contribution of the identity operator.[3] We explain how one can write $\mathcal{G}^{(k,m)}(z)$ for arbitrary k and m.

We write an ansatz for each $\mathcal{G}^{(k,m)}(z)$ with a few unknown coefficients and fix all, but a handful of them, via lightcone bootstrap. The undetermined coefficients correspond to the OPE coefficients of spin-0 and spin-2 exchanged operators. We further show that in holographic CFTs one can use the phase shift computed in the dual gravitational theory to reduce the set of undetermined parameters to the OPE coefficients of multi-stress tensors with spin zero.

Operators of non-minimal twist give a subleading contribution in the lightcone limit, $1-\bar{z} \ll 1$, which can be expressed as a sum of products of the functions $f_a(z)$ (times an appropriate power of $(1-\bar{z})$). This form is similar to the contribution of minimal-twist multi-stress tensor operators considered in [24]. While our method can be used to address the contribution of operators of arbitrary twist, here we focus on determining the specific contributions of operators with twist $\tau = 6, 8, 10$, at $\mathcal{O}(\mu^2)$ and $\tau = 8, 10$, at $\mathcal{O}(\mu^3)$.

At $\mathcal{O}(\mu)$, the only operator that contributes to the stress tensor sector of the correlator is the stress tensor and its contribution is completely fixed by conformal symmetry. In $d = 4$ its exact (to all orders in $\bar{z}$) contribution is given by

$$\mathcal{G}^{(1)}(z,\bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L-1}} \frac{\Delta_L}{120(\bar{z}-z)} \Big(f_3(z) - f_3(\bar{z})\Big). \tag{1.4}$$

At $\mathcal{O}(\mu^2)$, the leading contribution in the lightcone limit, due to twist-four double-stress tensors, was evaluated in [23]

$$\mathcal{G}^{(2,0)}(z) = \frac{1}{(1-z)^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L-2)}\right) \times \\ \left[(\Delta_L-4)(\Delta_L-3) f_3^2(z) + \frac{15}{7}(\Delta_L-8) f_2(z) f_4(z) + \frac{40}{7}(\Delta_L+1) f_1(z) f_5(z)\right]. \tag{1.5}$$

We show that the subleading contribution in the lightcone limit, due to twist-four and twist-six double-stress tensors, is given by

$$\mathcal{G}^{(2,1)}(z) = \frac{1}{(1-z)^{\Delta_L}} \left[\left(\frac{3-z}{2(1-z)}\right) \left(a_{33} f_3(z)^2 + a_{24} f_2(z) f_4(z) + a_{15} f_1(z) f_5(z)\right) \right. \\ \left. + \left(b_{14} f_1(z) f_4(z) + c_{16} f_1(z) f_6(z) + c_{25} f_2(z) f_5(z) + c_{34} f_3(z) f_4(z)\right)\right], \tag{1.6}$$

with coefficients a_{mn} and c_{mn} given in (3.14). The coefficient b_{14} is non-universal and generically depends on the details of the theory. It corresponds to the OPE coefficient of

[3]The contribution of the identity operator is denoted with $k = 0$.

twist-six double-stress tensor with spin $s = 2$

$$b_{14} = P^{(2)}_{8,2}, \tag{1.7}$$

obtained holographically in [21] and here, via the gravitational phase-shift calculation in (5.48).

The subsubleading contribution in the lightcone limit, due to twist-four, six and eight double-stress tensor operators, is

$$\begin{aligned}\mathcal{G}^{(2,2)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(&\left(\frac{z(2z-7)+11}{6(z-1)^2}\right)(a_{33}f_3^2 + a_{24}f_2f_4 + a_{15}f_1f_5)\\ &+\left(\frac{2-z}{1-z}\right)(b_{14}f_1f_4 + c_{16}f_1f_6 + c_{25}f_2f_6 + c_{34}f_3f_4) + (d_{17}f_1f_7 + d_{26}f_2f_6\\ &+ d_{35}f_3f_5 + d_{44}f_4^2 + e_{15}f_1f_5 + g_{13}f_1f_3)\Bigg),\end{aligned} \tag{1.8}$$

with coefficients d_{mn} given in (3.19). By f_a we mean $f_a(z)$ which we will use for brevity. The coefficients g_{13} and e_{15} are theory dependent and are related to the OPE coefficients of twist-eight double-stress tensors with spin $s = 0, 2$ by

$$\begin{aligned} g_{13} &= P^{(2)}_{8,0},\\ e_{15} &= P^{(2)}_{10,2} - \frac{5}{252}P^{(2)}_{8,0}. \end{aligned} \tag{1.9}$$

These coefficients were also obtained by a gravitational computation in [21]. Here we have used the calculation of the phase shift in the dual gravitational theory to determine the OPE coefficient of the spin-2 operator, $P^{(2)}_{10,2}$, in (5.51).

The subsubsubleading contribution in the lightcone limit, due to double-stress tensors with twists $\tau = 4, 6, 8, 10$, is given by

$$\begin{aligned}\mathcal{G}^{(2,3)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(&\left(\frac{z((13-3z)z-23)+25}{12(1-z)^3}\right)(a_{33}f_3^2 + a_{24}f_2f_4 + a_{15}f_1f_5)\\ &+\left(\frac{1}{(1-z)^2}+\frac{1}{1-z}+\frac{9}{10}\right)(b_{14}f_1f_4 + c_{16}f_1f_6 + c_{25}f_2f_5 + c_{34}f_3f_4)\\ &+\left(\frac{1}{1-z}+\frac{3}{2}\right)(d_{17}f_1f_7 + d_{26}f_2f_6 + d_{35}f_3f_5 + d_{44}f_4^2 + e_{15}f_1f_5 + g_{13}f_1f_3+)\\ &+ g_{13}f_3 + (h_{18}f_1f_8 + h_{27}f_2f_7 + h_{36}f_3f_6 + h_{45}f_4f_5 + j_{16}f_1f_6 + i_{14}f_1f_4)\Bigg),\end{aligned} \tag{1.10}$$

with h_{mn} given in (3.25). The non-universal coefficients here are i_{14} and j_{16} which are related to the OPE coefficients of twist-ten double-stress tensor operators with spin $s = 0, 2$

$$\begin{aligned} i_{14} &= P^{(2)}_{10,0},\\ j_{16} &= P^{(2)}_{12,2} - \frac{2}{99}P^{(2)}_{10,0}. \end{aligned} \tag{1.11}$$

The OPE coefficient $P^{(2)}_{12,2}$ is determined in (5.52) using the phase shift calculation in the dual gravitational theory. Non-universality is manifest through dependence on the Gauss-Bonnet coupling.

Using the results above, we also extract the OPE coefficients $P^{(2)}_{\Delta,s}$ of double-stress tensors of given twist. For $\tau = 6$:

$$P^{(2)}_{10+2\ell,4+2\ell} = \frac{\sqrt{\pi}2^{-4\ell-17}\Gamma(2n+7)}{(\ell+4)(\ell+5)(\ell+6)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{13}{2}\right)} \times \frac{\Delta_L}{(\Delta_L-3)(\Delta_L-2)}(a_{1,\ell}\Delta_L^3 + b_{1,\ell}\Delta_L^2 + c_{1,\ell}\Delta_L + d_{1,\ell}), \tag{1.12}$$

where $a_{1,\ell}$, $b_{1,\ell}$, $c_{1,\ell}$, $d_{1,\ell}$ can be found in (3.17). For $\tau = 8$:

$$P^{(2)}_{12+2\ell,4+2\ell} = \frac{\sqrt{\pi}\Delta_L 2^{-4\ell-19}\Gamma(2\ell+7)}{3(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)(\ell+4)(\ell+5)} \times \frac{a_{2,\ell}\Delta_L^4 + b_{2,\ell}\Delta_L^3 + c_{2,\ell}\Delta_L^2 + d_{2,\ell}\Delta_L + e_{2,\ell}}{(\ell+6)(\ell+7)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{15}{2}\right)}, \tag{1.13}$$

with $a_{2,\ell}$, $b_{2,\ell}$, $c_{2,\ell}$, $d_{2,\ell}$ and $e_{2,\ell}$ given in (3.22). Similarly for $\tau = 10$:

$$P^{(2)}_{14+2\ell,4+2\ell} = \frac{\sqrt{\pi}2^{-4\ell-22}\Gamma(2\ell+9)}{5(2\ell+1)(2\ell+3)(2\ell+5)(2\ell+7)\Gamma\left(2\ell+\frac{17}{2}\right)} \times \frac{\Delta_L(\Delta_L+1)(a_{3,\ell}\Delta_L^4 + b_{3,\ell}\Delta_L^3 + c_{3,\ell}\Delta_L^2 + d_{3,\ell}\Delta_L + e_{3,\ell})}{(\ell+5)(\ell+6)(\ell+7)(\ell+8)(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \tag{1.14}$$

with $a_{3,\ell}$, $b_{3,\ell}$, $c_{3,\ell}$, $d_{3,\ell}$ and $e_{3,\ell}$ expressed in terms of Δ_L in (3.28). Note that in all of these formulas $\ell \geq 0$ and, therefore, the OPE coefficients of operators with spin $s = 0, 2$ are not included here. It appears that at $\mathcal{O}(\mu^2)$, the OPE coefficients of all operators with spin $s \geq 4$ are universal in the sense that they only depend on Δ_L and C_T. On the other hand, the OPE coefficients of double-stress tensors with $s = 0, 2$ are non-universal.

At $\mathcal{O}(\mu^3)$, the leading contribution of twist-six triple-stress tensors in the lightcone limit, was computed in [24]

$$\mathcal{G}^{(3,0)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Big(a_{117}f_1(z)^2 f_7(z) + a_{126}f_1(z)f_2(z)f_6(z) + a_{135}f_1(z)f_3(z)f_5(z) + a_{225}f_2(z)^2 f_5(z) + a_{234}f_2(z)f_3(z)f_4(z) + a_{333}f_3(z)^3\Big), \tag{1.15}$$

where the coefficients a_{ijk} can be found in (4.2).

The subleading contribution to the correlator is due to twist-eight and twist-six triple-stress tensors

$$\mathcal{G}^{(3,1)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(\left(\frac{2-z}{1-z}\right)(a_{117}f_1^2 f_7 + a_{126}f_1 f_2 f_6 + a_{135}f_1 f_3 f_5 + a_{225}f_2^2 f_5 + a_{234}f_2 f_3 f_4 + a_{333}f_3^3) + (b_{116}f_6 f_1^2 + c_{118}f_8 f_1^2 + c_{145}f_4 f_5 f_1 + c_{127}f_2 f_7 f_1 + c_{244}f_2 f_4^2 + c_{334}f_3^2 f_4 + c_{235}f_2 f_3 f_5 + c_{226}f_2^2 f_6)\Bigg), \tag{1.16}$$

with b_{ijk} and c_{ijk} given in (B.1). Terms proportional to a_{ijk} come from the subleading contribution due to the minimal-twist triple-stress tensors in (1.15). Note that all of these coefficients are non-universal, since they depend on b_{14} from the $\mathcal{O}(\mu^2)$ result. Accordingly, no OPE coefficients of non-minimal-twist triple-stress tensors are universal.

A similar story holds for the subsubleading contribution to the correlator at $\mathcal{O}(\mu^3)$. This is due to multi-stress tensors with twist six, eight and ten and takes the following form

$$\mathcal{G}^{(3,2)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(\left(\frac{144z^2-448z+464}{160(z-1)^2}\right)(a_{117}f_1^2f_7 + a_{126}f_1f_2f_6 + a_{135}f_1f_3f_5 + a_{225}f_2^2f_5 + a_{234}f_2f_3f_4 + a_{333}f_3^3) + \left(\frac{1}{1-z}+\frac{3}{2}\right)(b_{116}f_6f_1^2 + c_{118}f_8f_1^2 + c_{145}f_4f_5f_1 + c_{127}f_2f_7f_1 + c_{244}f_2f_4^2 + c_{334}f_3^2f_4 + c_{235}f_2f_3f_5 + c_{226}f_2^2f_6) + (d_{117}f_1^2f_7 + e_{115}f_1^2f_5 + g_{119}f_1^2f_9 + g_{128}f_1f_2f_8 + g_{155}f_1f_5^2 + g_{227}f_2^2f_7 + g_{236}f_2f_3f_6 + g_{245}f_2f_4f_5 + g_{335}f_3^2f_5 + g_{344}f_3f_4^2)\Bigg), \tag{1.17}$$

with d_{117} and g_{ijk} in (C.1)–(C.3) and e_{115} in (5.56).

We further explain how one can write an ansatz for the correlator at arbitrary order in μ and the lightcone expansion. All unknown coefficients in the ansatz, except those that correspond to OPE coefficients of spin-0 and spin-2 operators, can be fixed by means of the lightcone bootstrap. We further show that in holographic CFTs one can use the phase shift computed in the dual gravitational theory to reduce the set of undetermined parameters to the OPE coefficients of multi-stress tensors with spin zero. Our results for these OPE coefficients precisely match those in [21] whenever available in the latter.

The OPE coefficients of multi-stress tensors can also be calculated using the Lorentzian inversion formula as in [27]. In order to determine for which operators the formula can be applied, one should consider the behavior of the correlation function in the Regge limit. The Regge behavior of the correlator at $\mathcal{O}(\mu^k)$ is $1/\sigma^k$, implying that the Lorentzian inversion formula can be used to extract the OPE coefficients of the operators with spin $s > k+1$. Accordingly, already at $\mathcal{O}(\mu^3)$, fixing the relevant OPE coefficients by combining an ansatz with the lightcone bootstrap allows one to determine more OPE data compared to those obtained with the sole use of the Lorentzian inversion formula. We explicitly check that it is not possible to extract the OPE coefficient of a triple-stress tensor with spin $s = 4$ and twist $\tau = 8$ using the Lorentzian inversion formula. Note, however, that this coefficient is completely determined in this article (where an ansatz is additionally employed).

1.3 Outline

This paper is organized as follows. In section 2, we set up the notation and review the S- and T-channel expansions of the HHLL correlator. In section 3, we analyze the stress tensor sector of the correlator at $\mathcal{O}(\mu^2)$, where we compute the subleading, subsubleading

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and subsubsubleading contributions in the lightcone expansion. We also compute the OPE coefficients of double-stress tensors with twist $\tau = 6, 8, 10$ and spin $s > 2$. In section 4, we analyze the stress tensor sector of the correlator at $\mathcal{O}(\mu^3)$, where we explicitly calculate the subleading and subsubleading contributions in the lightcone expansion. In section 5, we investigate the Gauss-Bonnet dual gravitational theory and give additional evidence for the universality of the OPE coefficients of minimal-twist multi-stress tensors using the phase shift calculation. Furthermore, we calculate the OPE coefficients of double- and triple-stress tensors with spin $s = 2$ (up to undetermined spin zero data). In section 6, we show how one can use the Lorentzian inversion formula in order to extract the OPE coefficients of double-stress tensors with twist $\tau = 4, 6$. We discuss our results in section 7. Appendix A contains certain relations that products of f_a functions satisfy, while appendices B and C contain explicit expressions for the coefficients which determine the correlator in subleading and subsubleading lightcone order at $\mathcal{O}(\mu^3)$. Several OPE coefficients of twist-eight triple-stress tensors are listed in appendix D. Finally, in appendix E we clarify the relationship between the scattering phase shift as defined in [20] and the deflection angle.

2 Review of near lightcone heavy-heavy-light-light correlator

In this section, we review the procedure for extracting information about the stress tensor sector of a four-point correlation function between two pairwise identical scalars $\mathcal{O}_H$, $\mathcal{O}_L$, with scaling dimensions $\Delta_H \propto \mathcal{O}(C_T)$ and $\Delta_L \propto \mathcal{O}(1)$, respectively, via the lightcone bootstrap. We closely follow ref. [24]. Using conformal transformations to fix the positions of three of the operators at $0, 1, x_4 \to \infty$, we define the stress tensor sector of the correlator by

$$\mathcal{G}(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H} \langle \mathcal{O}_H(x_4)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\Big|_{\text{multi-stress tensors}}, \tag{2.1}$$

where $(z, \bar{z})$ are the invariant cross-ratios given by

$$\begin{aligned} z\bar{z} &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \\ (1-z)(1-\bar{z}) &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}. \end{aligned} \tag{2.2}$$

2.1 T-channel expansion

The notion of the stress-tensor sector comes from expanding the correlator in the T-channel defined as $\mathcal{O}_L(z,\bar{z}) \times \mathcal{O}_L(1) \to \mathcal{O}_{\tau,s}$:

$$\mathcal{G}(z,\bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\mathcal{O}_{\tau,s}} P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z, 1-\bar{z}), \tag{2.3}$$

where s and $\tau = \Delta - s$ denote the spin and the twist of the exchanged primary operator $\mathcal{O}_{\tau,s}$. $P^{(HH,LL)}_{\mathcal{O}_{\tau,s}}$ denotes the product of OPE coefficients

$$P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} = \left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_{\tau,s}} \lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}_{\tau,s}} \tag{2.4}$$

and $g^{(0,0)}_{\tau,s}(1-z, 1-\bar{z})$ the corresponding conformal block.

Consider the T-channel expansion (2.3) in $d = 4$. Conformal blocks in $d = 4$ are given by [28]

$$g_{\tau,s}^{(0,0)}(1-z,1-\bar{z}) = \frac{(1-z)(1-\bar{z})}{\bar{z}-z}\left(f_{\frac{\beta}{2}}(z)f_{\frac{\tau-2}{2}}(\bar{z}) - f_{\frac{\beta}{2}}(\bar{z})f_{\frac{\tau-2}{2}}(z)\right), \tag{2.5}$$

with conformal spin, $\beta = \Delta + s$, and

$$f_a(z) = (1-z)^a {}_2F_1(a,a,2a,1-z). \tag{2.6}$$

In the lightcone limit, defined by $\bar{z} \to 1$ and z fixed, the leading contribution to the conformal blocks (2.5) comes from the first term in parenthesis in (2.5)

$$g_{\tau,s}^{(0,0)}(1-z,1-\bar{z}) = (1-\bar{z})^{\frac{\tau}{2}}\left(f_{\frac{\beta}{2}}(z) + \mathcal{O}((1-\bar{z}))\right). \tag{2.7}$$

From (2.7) it is clear that the operators with the lowest twist in the T-channel dominate the correlator in the lightcone limit. In any unitary CFT in $d = 4$ the operator with the lowest twist is the identity operator with twist $\tau = 0$. Another operator with low twist present in any local CFT is the stress tensor operator with $\tau = 2$. In particular, the exchange of the stress tensor is completely fixed since the product of the relevant OPE coefficients is determined by Ward identities

$$P_{T_{\mu\nu}}^{(HH,LL)} = \mu\frac{\Delta_L}{120}, \tag{2.8}$$

where

$$\mu = \frac{160}{3}\frac{\Delta_H}{C_T}. \tag{2.9}$$

The central charge C_T is defined via the two-point function of the stress tensor

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{C_T}{\Omega_{d-1}^2 x^{2d}}\mathcal{I}_{\mu\nu,\rho\sigma}(x), \tag{2.10}$$

where

$$\mathcal{I}_{\mu\nu,\rho\sigma}(x) = \frac{1}{2}\left(I_{\mu\rho}(x)I_{\nu\sigma}(x) + I_{\mu\sigma}(x)I_{\nu\rho}(x)\right) - \frac{1}{d}\eta_{\mu\nu}\eta_{\rho\sigma},$$
$$I_{\mu\nu} = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}, \qquad \Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}. \tag{2.11}$$

Note that the only single-trace primaries with twist equal to or lower than that of the stress tensor are scalars $\mathcal{O}$ with dimension $1 \leq \Delta_{\mathcal{O}} \leq 2$, or conserved currents with twist $\tau = 2$. In a theory without supersymmetry there is no *a priori* reason for the contributions of these operators, even if they exist, to be enhanced by a factor of Δ_H, so generically we expect them to be subleading in $C_T \to \infty$ limit.[4]

[4]Interestingly, in [29] it is conjectured that OPE coefficients $\lambda_{\phi\psi\psi}$ of operators ϕ with conformal dimension $\Delta_\phi \ll \Delta_{\text{gap}}$ and ψ with conformal dimension Δ_ψ, such that $\Delta_\phi \ll \Delta_\psi \ll C_T^{\#>0}$, scale as $\lambda_{\phi\psi\psi} \propto \frac{\Delta_\psi}{\sqrt{C_T}}$. Note however that here we are working in different regime, as $\Delta_H \propto \mathcal{O}(C_T)$.

The stress tensor sector of the correlator (2.1) admits a perturbative expansion in μ given by

$$\mathcal{G}(z,\bar{z}) = \sum_{k=0}^{\infty} \mu^k \mathcal{G}^{(k)}(z,\bar{z}), \tag{2.12}$$

where the cases $k=0$ and $k=1$ correspond to the exchange of the identity and the stress tensor, respectively. For higher k we expect "multi-stress tensors" to contribute to $\mathcal{G}(z,\bar{z})$; the minimal-twist multi-stress tensor primaries are of the schematic form

$$[T^k]_{\tau_{k,\min},s} =: T_{\mu_1\nu_1} \dots T_{\mu_{k-1}\nu_{k-1}} \partial_{\lambda_1} \dots \partial_{\lambda_{2\ell}} T_{\mu_k\nu_k} : , \tag{2.13}$$

with twist $\tau_{k,\min}$ and spin s given by

$$\begin{aligned} \tau_{k,\min} &= 2k, \\ s &= 2k + 2\ell, \end{aligned} \tag{2.14}$$

with ℓ an integer. Since we are interested in the four-point function of pairwise identical scalar operators, only multi-stress tensor operators with even spin give a nonvanishing contribution. At $\mathcal{O}(\mu^2)$, the contribution of these operators was explicitly calculated in [23]. Following that, it was shown in [24] how one can write the contributions of these operators at arbitrary order in the μ-expansion, in the lightcone limit $(1-\bar{z}) \ll 1$, using an appropriate ansatz and lightcone bootstrap. We briefly review this procedure here since the contribution from non-minimal-twist operators is obtained in a similar manner.

At $\mathcal{O}(\mu^k)$, there are infinitely many minimal-twist multi-stress tensors with twist $2k$ according to (2.14) which are distinguished by their conformal spin $\beta = \Delta + s$ given by $\beta = 6k + 4\ell$ with $\ell = 0, 1, 2, \dots$. Inserting the leading behavior of the blocks (2.7) in (2.3) one finds

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^k}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\ell=0} P^{(k)}_{\Delta(\ell),s(\ell)} f_{\frac{\beta(\ell)}{2}}(z), \tag{2.15}$$

with

$$\mu^k P^{(k)}_{\Delta(\ell),s(\ell)} = P^{(HH,LL)}_{[T^k]_{\tau,s(\ell)}}, \tag{2.16}$$

where $\Delta(\ell) = \frac{\tau+\beta}{2}$, $\tau = 2k$, $s(\ell) = 2k + 2\ell$ and conformal spin $\beta = 6k + 4\ell$. Here $\underset{\bar{z}\to 1}{\approx}$ means that only the leading contribution as $\bar{z} \to 1$ is kept. It was shown in [24] that the infinite sum in (2.15) takes a particular form

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^k}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1 \dots i_k} f_{i_1}(z) \dots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = 3k, \tag{2.17}$$

with i_p being integers and $a_{i_1 \dots i_k}$ are coefficients that can be determined via lightcone bootstrap. Furthermore, using an identity for the product of two f_a functions (eq. (A.1) in [23]) one can express the $\mathcal{G}^{(k)}(z,\bar{z})$ in the form of (2.15) to read off the OPE coefficients for the exchange of minimal-twist multi-stress tensors of arbitrary conformal spin.

In this paper, we want to consider multi-stress tensors with non-minimal twist. These operators are obtained by contracting indices in (2.13) either between the derivatives or between the operators. At $\mathcal{O}\left(\mu^k\right)$ there exist operators $[T^k]_{\tau_{k,m},s}$ with twist

$$\tau_{k,m} = \tau_{k,\text{min}} + 2m, \tag{2.18}$$

for any non-negative integer m. For $m \neq 0$, these operators provide subleading contributions to the correlator in the lightcone limit. To consider these subleading contributions it is convenient to expand $\mathcal{G}^{(k)}(z,\bar{z})$ from (2.12) as

$$\mathcal{G}^{(k)}(z,\bar{z}) = \sum_{m=0}^{\infty} (1-\bar{z})^{-\Delta_L+k+m} \mathcal{G}^{(k,m)}(z), \tag{2.19}$$

where $\mathcal{G}^{(k,m)}(z)$ comes from operators of twists $\tau_{k,m}$ and less.

For illustration, let us consider the case $k=2$ with $m=1$. There exist two infinite families of operators with twist $\tau_{2,1}=6$ of the schematic form

$$\begin{aligned} \mathcal{O}_{6,2\ell_1+2} &\sim\, : T_{\mu\kappa}\partial_{\lambda_1}\ldots\partial_{\lambda_{2\ell_1}} T^{\kappa}{}_{\nu} :, \\ \mathcal{O}'_{6,2\ell_2+4} &\sim\, : T_{\mu\nu}\partial_{\lambda_1}\ldots\partial_{\lambda_{2\ell_2}}\partial^2 T_{\rho\sigma} : . \end{aligned} \tag{2.20}$$

These two families share the same twist and spin for $\ell_1 = \ell_2 + 1$. Hence, they are indistinguishable for $\ell_1 \geq 1$ at order $1/C_T$ in the large C_T expansion. A single operator stands out; it corresponds to $\ell_1 = 0$ and is of the schematic form $: T_{\mu\alpha}T^{\alpha}{}_{\nu} :$. Note that $: T_{\mu\alpha}T^{\alpha}{}_{\nu} :$ has minimal conformal spin $\beta = 10$, among the ones in (2.20), since $\beta_{\ell_1} = \beta_{\ell_2+1} = 10 + 4\ell_1$, for $\ell_1 \geq 1$.

Let us now move on to the case $k=2$ and $m=2$. Here, there are three infinite families $\mathcal{O}_{8,s}$, $\mathcal{O}'_{8,s}$ and $\mathcal{O}''_{8,s}$ with conformal spin $8+4\ell_1$, $12+4\ell_2$ and $16+4\ell_3$, respectively. Schematically, these families can be represented as

$$\begin{aligned} \mathcal{O}_{8,2\ell_1} &\sim\, : T_{\alpha\beta}\partial_{\lambda_1}\ldots\partial_{\lambda_{2\ell_1}} T^{\alpha\beta} :, \\ \mathcal{O}'_{8,2\ell_2+2} &\sim\, : T_{\mu\alpha}\partial_{\lambda_1}\ldots\partial_{\lambda_{2\ell_2}}\partial^2 T^{\alpha}{}_{\nu} :, \\ \mathcal{O}''_{8,2\ell_3+4} &\sim\, : T_{\mu\nu}\partial_{\lambda_1}\ldots\partial_{\lambda_{2\ell_3}}(\partial^2)^2 T_{\rho\sigma} : . \end{aligned} \tag{2.21}$$

Notice once more that the infinite families are indistinguishable for conformal spin $\beta \geq 16$. Here, operators with $\beta = 8, 12$ stand out. The operator with $\beta = 8$ is of the schematic form $: T_{\alpha\beta}T^{\alpha\beta} :$. For $\beta = 12$, there are two indistinguishable operators of the schematic form $: T_{\mu\alpha}\partial^2 T^{\alpha}{}_{\nu} :$ and $: T_{\alpha\beta}\partial_\mu\partial_\nu T^{\alpha\beta} :$.

The same holds for $m \geq 3$ (and $\tau \geq 10$) since there is no other independent way to contract stress tensor indices. The discussion above generalizes straightforwardly to $\mathcal{O}(\mu^k)$ with $k+1$ number of infinite families at high enough twist.

2.2 S-channel expansion

The correlator (2.1) can also be expanded in the S-channel defined as $\mathcal{O}_L(z,\bar{z}) \times \mathcal{O}_H(0) \to \mathcal{O}_{\tau',s'}$,

$$\mathcal{G}(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{\mathcal{O}_{\tau',s'}} P^{(\text{HL},\text{HL})}_{\mathcal{O}_{\tau',s'}} g^{(\Delta_{\text{HL}},-\Delta_{\text{HL}})}_{\tau',s'}(z,\bar{z}), \tag{2.22}$$

where $P^{(\mathrm{HL,HL})}_{\mathcal{O}_{\tau',s'}}$ denotes the product of OPE coefficients in the S-channel, $\Delta_{\mathrm{HL}} = \Delta_H - \Delta_L$, and $g^{(\Delta_{\mathrm{HL}},-\Delta_{\mathrm{HL}})}_{\tau',s'}(z,\bar{z})$ are the relevant conformal blocks. Operators contributing in the S-channel expansion are "heavy-light double-twist" operators [20, 30][5] of the schematic form $[\mathcal{O}_H\mathcal{O}_L]_{n,l} =: \mathcal{O}_H(\partial^2)^n\partial_{\mu_1}\ldots\partial_{\mu_l}\mathcal{O}_L :$, with conformal dimensions $\Delta = \Delta_H+\Delta_L+2n+l+\gamma$. The conformal blocks for these heavy-light double-twist operators in $d=4$ are given by

$$g^{(\Delta_{\mathrm{HL}},-\Delta_{\mathrm{HL}})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) = \frac{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L+2n+\gamma_{n,l})}}{\bar{z}-z}\left(\bar{z}^{l+1}-z^{l+1}\right)+\mathcal{O}\left(\frac{1}{\Delta_H}\right). \tag{2.23}$$

The anomalous dimensions and the product of OPE coefficients for heavy-light double-twist operators admit an expansion in powers of μ:

$$\gamma_{n,l} = \sum_{k=1}^{\infty}\mu^k\gamma^{(k)}_{n,l}\,,$$
$$P^{(\mathrm{HL,HL})}_{n,l} = P^{(\mathrm{HL,HL});\mathrm{MFT}}_{n,l}\sum_{k=0}^{\infty}\mu^k P^{(\mathrm{HL,HL});(k)}_{n,l}\,, \tag{2.24}$$

where $P^{(\mathrm{HL,HL});\mathrm{MFT}}_{n,l}$ are the Mean Field Theory coefficients [31], which can be found by matching with the exchange of the identity in the T-channel, and $P^{(\mathrm{HL,HL});(0)}_{n,l} = 1$. Explicitly, in $d=4$ and for $\Delta_H \gg 1$,

$$P^{(\mathrm{HL,HL});\mathrm{MFT}}_{n,l} = \frac{(\Delta_L-1)_n(\Delta_L)_{l+n}}{n!\,l!\,(l+2)_n}+\mathcal{O}\left(\frac{1}{\Delta_H}\right), \tag{2.25}$$

where $(a)_n$ is the Pochhammer symbol defined by $(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)}$.

We begin by briefly reviewing the calculation in the lightcone expansion, i.e. due to the multi-stress tensors in the T-channel. Inserting the blocks (2.23) in the S-channel expansion (2.22) one finds that

$$\mathcal{G}(z,\bar{z}) = \sum_{n=0}^{\infty}\frac{(z\bar{z})^n}{\bar{z}-z}\int_0^{\infty}dl P^{(\mathrm{HL,HL})}_{n,l}(z\bar{z})^{\frac{1}{2}\gamma_{n,l}}(\bar{z}^{l+1}-z^{l+1}), \tag{2.26}$$

where the sum was approximated by an integral over l. Expanding the OPE data in (2.26) according to (2.24) and noting that

$$(z\bar{z})^{\frac{1}{2}\gamma_{n,l}} = \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\gamma_{n,l}\log(z\bar{z})}{2}\right)^j, \tag{2.27}$$

it follows that terms proportional to $\log^i z$ at $\mathcal{O}(\mu^k)$, with $i=2,3,\ldots k$, in (2.26) are determined by OPE data at $\mathcal{O}(\mu^{k-1})$. These terms can therefore be matched with the T-channel in order to fix the coefficients in the ansatz.

[5] In the lightcone limit of $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\mathcal{O}_1\rangle$, with $\mathcal{O}_1,\mathcal{O}_2$ both light, it was found in [16, 17] that the there exists "light-light double-twist" operators $[\mathcal{O}_1\mathcal{O}_2]_{n,l} =: \mathcal{O}_1(\partial^2)^n\partial_{\mu_1}\ldots\partial_{\mu_l}\mathcal{O}_2 :$ for $l \gg 1$. These are found by matching with the identity exchange in the S-channel. The same is true for the heavy-heavy-light-light case.

In [24], the leading contribution of the OPE data of heavy-light double-twist operators as $l \to \infty$, together with the leading contribution of the conformal blocks as $\bar{z} \to 1$, was used to determine the minimal-twist contributions in the stress tensor sector of the T-channel. This paper extends that analysis by considering subleading corrections in the lightcone expansion and therefore probing non-minimal-twist contributions in the T-channel. In particular, the S-channel OPE data have the following dependence on the spin l as $l \to \infty$:

$$\gamma^{(k)}_{n,l} = \frac{1}{l^k}\sum_{p=0}^{\infty}\frac{\gamma^{(k,p)}_n}{l^p},$$
$$P^{(\text{HL,HL});(k)}_{n,l} = \frac{1}{l^k}\sum_{p=0}^{\infty}\frac{P^{(\text{HL,HL});(k,p)}_n}{l^p}, \tag{2.28}$$

which is necessary in order to reproduce the correct power of $(1-\bar{z})$ as $\bar{z} \to 1$. This can be seen by substituting the expansion of (2.25) in the large-l limit

$$P^{(\text{HL,HL});\text{MFT}}_{n,l} = l^{\Delta_L}\left(\frac{(\Delta_L-1)_n}{n!\Gamma(\Delta_L)l} + \frac{(2n(\Delta_L-2)+\Delta_L(\Delta_L-1))(\Delta_L-1)_n}{2(n!)\Gamma(\Delta_L)l^2} + \mathcal{O}\left(\frac{1}{l^3}\right)\right), \tag{2.29}$$

and (2.28) in (2.26) which result in integrals of the form

$$\int_0^{\infty} dl \bar{z}^l l^{\Delta_L-m-1} = \frac{\Gamma(\Delta_L-m)}{(-\log\bar{z})^{\Delta_L-m}}, \tag{2.30}$$

where m is a positive integer. Expanding (2.30) for $\bar{z} \to 1$, the correct $\bar{z}$-behavior of the stress tensor sector in the T-channel is reproduced from the S-channel.

3 Double-stress tensors in four dimensions

In this section, we analyze the stress tensor sector of the HHLL correlator at $\mathcal{O}(\mu^2)$ in $d=4$. The operators that contribute at this order in the T-channel are the double-stress tensors. Here, we investigate the subleading contributions that are coming from families of operators with nonminimal twist, specifically, $\tau_{2,1}=6$, $\tau_{2,2}=8$ and $\tau_{2,3}=10$, according to (2.18).

The dominant contribution in the lightcone limit at $\mathcal{O}(\mu^2)$ was calculated in [23]. It comes from the operators with minimal twist $\tau_{2,\text{min}} = 4$ and they are of the schematic form $:T_{\mu\nu}\partial_{\alpha_1}\ldots\partial_{\alpha_{2\ell}}T_{\rho\sigma}:$. These operators have conformal dimension $\Delta = 8+2\ell$ and spin $s = 4+2\ell$. The result is [23]

$$\mathcal{G}^{(2,0)}(z) = \frac{1}{(1-z)^{\Delta_L}}\left(\frac{\Delta_L}{28800(\Delta_L-2)}\right)\times \left[(\Delta_L-4)(\Delta_L-3)f_3^2(z) + \frac{15}{7}(\Delta_L-8)f_2(z)f_4(z) + \frac{40}{7}(\Delta_L+1)f_1(z)f_5(z)\right], \tag{3.1}$$

where $f_a(z) = (1-z)^a {}_2F_1(a,a,2a,1-z)$.

3.1 Twist-six double-stress tensors

Twist-six double-stress tensors contribute at $\mathcal{O}(\mu^2)$ and at subleading order in the lightcone expansion $\sim (1-\bar{z})^{-\Delta_L+3}$ as $\bar{z}\to 1$. As shown in this section, this contribution again takes a particular form with a few undetermined coefficients which, except for a single one, can be fixed using lightcone bootstrap. The undetermined data is shown to correspond to a single OPE coefficient due to the exchange of the twist-six and spin-two double-stress tensor $:T_\mu{}^\rho T_{\rho\nu}:$.

We will now motivative an ansatz for the subleading contribution to the stress tensor sector at $\mathcal{O}(\mu^2)$. Let us focus first on corrections due to the leading lightcone contribution of twist-four double-stress tensors. These corrections originate from subleading terms in the lightcone expansion of the conformal blocks in (2.7). Note however that they are purely kinematical and do not contain any new data. Explicitly, the subleading corrections to the blocks of twist-four double-stress tensors are given by

$$\begin{aligned} g^{(0,0)}_{4,s}(1-z,1-\bar{z}) \underset{\bar{z}\to 1}{\approx} &(1-\bar{z})^2\left(1+(1-\bar{z})\left(\frac{3-z}{2(1-z)}\right)+\mathcal{O}\left((1-\bar{z})^2\right)\right) f_{\frac{\beta}{2}}(z) \\ &-(1-\bar{z})^{s+3}\left(1+(1-\bar{z})\left(\frac{s+2}{2}+\frac{1}{1-z}\right)+\mathcal{O}((1-\bar{z})^2)\right) f_1(z). \end{aligned} \tag{3.2}$$

Since we are interested in the subleading contribution, i.e. terms that behave as $(1-\bar{z})^3$ as $\bar{z}\to 1$ in (3.2), only the first line in (3.2) needs to be considered. (Note that $s\geq 4$ for minimal-twist double-stress tensors.)

Next, consider the contribution of twist-six double-stress tensors. Recall that the form of the minimal-twist double-stress tensors' contribution to (3.1) can be motivated by decomposing products of the type $f_a(z)f_b(z)$ in terms of the lightcone conformal blocks. This decomposition is explicitly given by [23]:

$$f_a(z)f_b(z)=\sum_{\ell=0}^{\infty} p(a,b,\ell) f_{a+b+2\ell}(z), \tag{3.3}$$

where

$$p(a,b,\ell)=\frac{2^{-4\ell}\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)\Gamma\left(\ell+\frac{1}{2}\right)\Gamma(a+\ell)\Gamma(b+\ell)\Gamma\left(a+b+\ell-\frac{1}{2}\right)\Gamma(a+b+2\ell)}{\sqrt{\pi}\Gamma(a)\Gamma(b)\Gamma(\ell+1)\Gamma\left(a+\ell+\frac{1}{2}\right)\Gamma\left(b+\ell+\frac{1}{2}\right)\Gamma(a+b+\ell)\Gamma\left(a+b+2\ell-\frac{1}{2}\right)}. \tag{3.4}$$

Using the leading behavior of the conformal blocks (3.2) in the lightcone limit, it was found that $a+b+2\ell$ should be identified with $\frac{\beta}{2}=\frac{\Delta+s}{2}$. In order to reproduce twist-six double-stress tensors of the form $:T_{\mu\nu}\partial^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T_{\rho\sigma}:$ we should therefore consider products f_af_b with $a+b=7$. Likewise, to take into account operators of the form $:T_{\mu\beta}\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^\beta{}_\nu:$ we include products f_af_b with $a+b=5$.

From the arguments above, we make the following ansatz for the subleading correction in the lightcone expansion due to double-stress tensors:

$$\begin{aligned} \mathcal{G}^{(2,1)}(z)=\frac{1}{(1-z)^{\Delta_L}}\Bigg[&\left(\frac{3-z}{2(1-z)}\right)\left(a_{33}f_3(z)^2+a_{24}f_2(z)f_4(z)+a_{15}f_1(z)f_5(z)\right) \\ &+\left(b_{14}f_1(z)f_4(z)+b_{23}f_2(z)f_3(z)+c_{16}f_1(z)f_6(z)+c_{25}f_2(z)f_5(z)+c_{34}f_3(z)f_4(z)\right)\Bigg], \end{aligned} \tag{3.5}$$

where b_{ij}, c_{ij} are coefficients that will be determined using lightcone bootstrap and encode the contribution from twist-six double-stress tensors. Once b_{ij} and c_{ij} are determined, one can use the decomposition in (3.3) to read off the OPE coefficients of twist-six double-stress tensors with any given spin. Moreover, a_{ij} in (3.5) are coefficients that can be read off from the minimal-twist contribution in (3.1) and do therefore not contain any new information.

We proceed with the S-channel calculation to fix the unknown coefficients in (3.5). Let us first mention that the products of $f_a(z)$ functions in the second line of (3.5) are not linearly independent as one can see from (A.1), so we set $b_{23} = 0$. Moreover, the coefficients a_{ij} must be the same as in (3.1). We will momentarily keep them undetermined to have an extra consistency check of our calculation.

In the S-channel we have double-twist operators of the form $:\mathcal{O}_H \partial^{2n}\partial^l \mathcal{O}_L:$ with conformal dimension $\Delta = \Delta_H + \Delta_L + 2n + l + \gamma_{n,l}$. The relevant anomalous dimensions $\gamma_{n,l}$ and OPE coefficients are given in (2.24) and (2.28) ($k = 2$ in this case). In the lightcone limit, the dominant contribution comes from operators with large spin l, $l \gg n$. The mean field theory OPE coefficients are given by (2.29). The conformal blocks of these operators in the limit $1 - \bar{z} \ll z \ll 1$ are

$$g_{n,l}^{(\Delta_{\rm HL},-\Delta_{\rm HL})}(z,\bar{z}) \approx \frac{(z\bar{z})^{\frac{\Delta_H+\Delta_L+\gamma(n,l)}{2}}}{\bar{z}-z} z^n \bar{z}^{l+n+1}. \tag{3.6}$$

We first need to fix the OPE data at $\mathcal{O}(\mu)$. Coefficients $\gamma_n^{(1,p)}$ and $P_n^{(1,p)}$ can be determined for every p and n by matching the S-channel correlator with the correlator in the T-channel at $\mathcal{O}(\mu)$. This is just the stress tensor block times its OPE coefficient and it is known for arbitrary z and $\bar{z}$. As we saw earlier

$$(\bar{z}-z)\mathcal{G}^{(1)}(z,\bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L-1}} \frac{\Delta_L}{120} \left(f_3(z) - f_3(\bar{z})\right). \tag{3.7}$$

Expanding (3.7) near $\bar{z} \to 1$ leads to

$$(\bar{z}-z)\mathcal{G}^{(1)}(z,\bar{z}) = \frac{(1-\bar{z})}{((1-z)(1-\bar{z}))^{\Delta_L}} \Bigg(-\Delta_L \left(\frac{3}{4}(1+z) + \frac{1+z(z+4)}{4(1-z)}\log(z)\right) - \sum_{p=1}^{\infty} \frac{\Delta_L(p-2)(p-1)(1-z)}{4p(p+1)(p+2)}(1-\bar{z})^p \Bigg). \tag{3.8}$$

On the other hand, we expand the integrand of (2.26) up to the $\mathcal{O}(\mu)$, integrate this expansion over l, and then expand in the lightcone limit $\bar{z} \to 1$ to obtain a result of the form

$$(\bar{z}-z)\mathcal{G}^{(1)}(z,\bar{z}) = \frac{1}{(1-\bar{z})^{\Delta_L-1}} \sum_{p=0}^{\infty} \left(\sum_{n=0}^{\infty} r_{n,p}(z) z^n (1-\bar{z})^p \right). \tag{3.9}$$

The functions $r_{n,p}(z)$ can be explicitly calculated. Here $r_{n,0}(z)$, $r_{n,1}(z)$ and $r_{n,2}(z)$ are given by

$$\begin{aligned}
r_{n,0}(z) &= \frac{\Gamma(\Delta_L+n-1)}{2\Gamma(\Delta_L)\Gamma(n+1)}\left(2P_n^{(1,0)}+\log(z)\gamma_n^{(1,0)}\right),\\
r_{n,1}(z) &= \frac{\Gamma(\Delta_L+n-1)}{2\Gamma(\Delta_L)\Gamma(n+1)(\Delta_L-2)}\Big(2(P_n^{(1,0)}+P_n^{(1,1)})-(\Delta_L-2)\gamma_n^{(1,0)}\\
&\qquad +\log(z)(\gamma_n^{(1,0)}+\gamma_n^{(1,1)})\Big),\\
r_{n,2}(z) &= \frac{\Gamma(\Delta_L+n-1)}{2(\Delta_L-2)(\Delta_L-3)\Gamma(\Delta_L)\Gamma(n+1)}\Big(2(\Delta_L+n-1)P_n^{(1,0)}+2(\Delta_L+n)P_n^{(1,1)}\\
&\quad +2P_n^{(1,2)}-\frac{1}{2}(\Delta_L-3)(\Delta_L\gamma_n^{(1,0)}+2\gamma_n^{(1,1)})+\log(z)((\Delta_L+n-1)\gamma_n^{(1,0)}\\
&\quad +(\Delta_L+n)\gamma_n^{(1,1)}+\gamma_n^{(1,2)})\Big).
\end{aligned} \tag{3.10}$$

Similarly, one can calculate any $r_{n,p}(z)$ for arbitrary p. In each $r_{n,p}(z)$ the z-dependence enters only through a single logarithmic term as in (3.10). In order to extract the OPE data we match (3.8) and (3.9) and obtain the following relations

$$\begin{aligned}
\sum_{n=0}^{\infty} z^n r_{n,0}(z) &= -\frac{\Delta_L}{(1-z)^{\Delta_L}}\left(\frac{3}{4}(1+z)+\frac{1+z(z+4)}{4(1-z)}\log(z)\right),\\
\sum_{n=0}^{\infty} z^n r_{n,p}(z) &= -\frac{\Delta_L}{(1-z)^{\Delta_L}}\frac{(p-2)(p-1)(1-z)}{4p(p+1)(p+2)},
\end{aligned} \tag{3.11}$$

for $p\geq 1$. To solve these equations, we start from the first line, expand the right-hand side in $z\to 0$ limit and match term by term on both sides. From terms with $\log(z)$ we extract the $\gamma_n^{(1,0)}$ and from terms without $\log(z)$, we extract the $P_n^{(1,0)}$. We move on to $p=1$ case, where we again expand the right-hand side of the second line in (3.11) in $z\to 0$ limit. Using $\gamma_n^{(1,0)}$ and $P_n^{(1,0)}$, we extract $\gamma_n^{(1,1)}$ and $P_n^{(1,1)}$. Straightforwardly, one can continue this process and extract OPE data for any value of p.

By proceeding with this calculation to high enough values and p one can notice that there is a simple expression for $\gamma_n^{(1,p)}$ given by

$$\gamma_n^{(1,p)} = (-1)^{p+1}\left(\frac{1}{2}(\Delta_L-1)\Delta_L+3n^2-3(1-\Delta_L)n\right), \tag{3.12}$$

for all $p\geq 0$ and $n\geq 0$. Note that for $p=0$ this expression agrees with the one in [27]. There is no similar expression for $P_n^{(1,p)}$ so we list results for first p-s:

$$\begin{aligned}
P_n^{(1,0)} &= -\frac{3}{4}(\Delta_L-1)\Delta_L-\frac{3\Delta_L n}{2},\\
P_n^{(1,1)} &= 3(n-1)n-\frac{1}{4}\Delta_L\left(\Delta_L\left(\Delta_L+6n-6\right)+6(n-4)n+5\right),\\
P_n^{(1,2)} &= \frac{1}{8}(\Delta_L(\Delta_L(\Delta_L^2+8n\Delta_L+6n(3n-1)-13)+2(n(3n(2n-5)-25)+6))\\
&\quad -12n(2n^2+n-3)),
\end{aligned}$$

$$
\begin{aligned}
P_n^{(1,3)} &= \frac{1}{120}(180n(n(3-(n-3)n)+5)-234)\Delta_L + 3n(n^3+n^2-2) \\
&+ \frac{1}{120}\Delta_L^2(-\Delta_L(\Delta_L(11\Delta_L+90n-20)+90n(3n-1)+55) \\
&+ 90(3-4n)n^2+280).
\end{aligned} \tag{3.13}
$$

After the calculation of the OPE data at $\mathcal{O}(\mu)$, one can fix the coefficients in the ansatz (3.5) by expanding the integrand of (2.26) up to $\mathcal{O}(\mu^2)$ and then integrating the obtained expression over l. The result of the integration is expanded near $\bar{z} \to 1$ and we collect the term that behaves as $(1-\bar{z})^{-\Delta_L+3}$. It depends on z, n and OPE data $P_n^{(k,p)}$ and $\gamma_n^{(k,p)}$ for $k = 1, 2$ and $p = 0, 1$, but we are interested only in the part of this term that contains $\log^2(z)$. This part only depends on OPE data at $\mathcal{O}(\mu)$, so it will be completely determined. We collect terms that behave as $(1-\bar{z})^{-\Delta_L+3}\log^2(z)z^m$. By expanding the ansatz (3.5) near $z \to 0$ we can collect terms that behave as $\log^2(z)z^m$ and by matching these to the ones calculated through S-channel, we obtain a system of linear equations for the coefficients in the ansatz. This system will be over-determined by taking m to be large enough. Solving it for $m \leq 20$, we obtain

$$
\begin{aligned}
a_{33} &= \frac{(\Delta_L-4)(\Delta_L-3)\Delta_L}{28800(\Delta_L-2)}, \\
a_{24} &= \frac{(\Delta_L-8)\Delta_L}{13440(\Delta_L-2)}, \\
a_{15} &= \frac{\Delta_L(\Delta_L+1)}{5040(\Delta_L-2)}, \\
c_{16} &= \frac{25}{396}b_{14} + \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(83-7\Delta_L\right)+158\right)+108\right)}{3193344\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
c_{25} &= -\frac{1}{12}b_{14} + \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L+19\right)-146\right)-108\right)}{1451520\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
c_{34} &= \frac{\left(\Delta_L-4\right)\Delta_L\left(11\left(\Delta_L-4\right)\Delta_L-27\right)}{2419200\left(\Delta_L-3\right)\left(\Delta_L-2\right)}.
\end{aligned} \tag{3.14}
$$

As expected, the coefficients a_{mn} are identical to those in (3.1). We are left with one undetermined coefficient. This is perhaps not surprising since we know from [21] that the OPE coefficients of the subleading twist multi-stress tensor operators are not universal. This non-universality is introduced in our correlator through coefficient b_{14}. One can check that after inserting (3.14) to (3.5) the term that multiplies the unknown coefficient b_{14} corresponds to the lightcone limit of the conformal block of the operator with dimension $\Delta = 8$ and spin $s = 2$. We thus conclude that b_{14} is the OPE coefficient of $:T_{\mu\alpha}T^{\alpha}{}_{\nu}:$,

$$
b_{14} = P_{8,2}^{(2)}. \tag{3.15}
$$

Now, using (3.3) we can write the T-channel OPE coefficients for the remaining double-stress tensor operators with twist $\tau_{2,1} = 6$ and conformal spin $\Delta + s \geq 14$. Explicitly, these are found to be given by

$$P^{(2)}_{10+2\ell,4+2\ell} = \frac{\sqrt{\pi}2^{-4\ell-17}\Gamma(2\ell+7)}{(\ell+4)(\ell+5)(\ell+6)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{13}{2}\right)} \times \frac{\Delta_L}{(\Delta_L-3)(\Delta_L-2)}(a_{1,\ell}\Delta_L^3+b_{1,\ell}\Delta_L^2+c_{1,\ell}\Delta_L+d_{1,\ell}), \tag{3.16}$$

where

$$\begin{aligned} a_{1,\ell} &= (\ell+2)(2\ell+9)(\ell(2\ell+13)+9), \\ b_{1,\ell} &= 144-2\ell(2\ell+13)(\ell(2\ell+13)+12), \\ c_{1,\ell} &= \ell(2\ell+13)(\ell(2\ell+13)+33)+558, \\ d_{1,\ell} &= 216. \end{aligned} \tag{3.17}$$

Here $\ell \geq 0$ and $P^{(2)}_{\Delta,s}$ is the sum of OPE coefficients of all operators with conformal dimension Δ and spin s. There is no way to distinguish operators with the same quantum numbers Δ and s at this level in the large C_T expansion. This type of degeneracy occurs for each conformal spin greater than 10 for twist $\tau_{2,1} = 6$. Also, perfect agreement between (3.16) and all the OPE coefficients of double-stress tensor operators of twist $\tau_{2,1} = 6$ and spin $s > 2$ calculated in [21] is observed. Note that $P^{(2)}_{8,2}$ can not be found from (3.16) by setting $\ell = -1$, this would not agree with the result in [21]. In section 6 we rederive (3.16) using the Lorentzian inversion formula.

3.2 Twist-eight double-stress tensors

We follow the same logic as in the previous section in order to write the subsubleading part of the stress tensor sector of the HHLL correlator in the lightcone limit at $\mathcal{O}(\mu^2)$. This part scales as $(1-\bar{z})^{-\Delta_L+4}$. Here, we include contributions coming from operators with twist $\tau_{2,2} = 8$. These operators can be grouped in three families and they are schematically written as : $T_{\mu\nu}(\partial^2)^2\partial_{\alpha_1}\ldots\partial_{\alpha_{2\ell}}T_{\rho\sigma}$: with $\Delta = 12+2\ell$ and $s = 4+2\ell$, : $T_{\mu\beta}\partial^2\partial_{\alpha_1}\ldots\partial_{\alpha_{2\ell}}T^{\beta}{}_{\nu}$: with $\Delta = 10+2\ell$ and $s = 2+2\ell$ and finally : $T_{\beta\gamma}\partial_{\alpha_1}\ldots\partial_{\alpha_{2\ell}}T^{\beta\gamma}$: with $\Delta = 8+2\ell$ and $s = 2\ell$. Subtleties with regard to the contributions of the different families are discussed in section 2.1.

Once more, we need to include the contributions of lower twist operators, i.e. by expanding their conformal blocks as $\bar{z} \to 1$ up to order $(1-\bar{z})^4$ and collect the additional z dependence. Accordingly, we write the following ansatz

$$\begin{aligned} \mathcal{G}^{(2,2)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(&\left(\frac{z(2z-7)+11}{6(z-1)^2}\right)(a_{33}f_3^2+a_{24}f_2f_4+a_{15}f_1f_5) \\ &+\left(\frac{2-z}{1-z}\right)(b_{14}f_1f_4+c_{16}f_1f_6+c_{25}f_2f_6+c_{34}f_3f_4) \\ &+(d_{17}f_1f_7+d_{26}f_2f_6+d_{35}f_3f_5+d_{44}f_4^2+e_{15}f_1f_5+e_{24}f_2f_4+e_{33}f_3^2 \\ &+g_{13}f_1f_3+g_{22}f_2^2)\Bigg), \end{aligned} \tag{3.18}$$

where f_a means $f_a(z)$. Coefficients a_{mn} and c_{mn} are already calculated, while b_{14} is undetermined from the bootstrap. The linear dependence between certain products of $f_a(z)$ functions (for more details see appendix A, in particular (A.2)) allows us to set three coefficients to zero, e.g., $g_{22}=0$, $e_{33}=0$ and $e_{24}=0$.

To fix the unknown coefficients in (3.18) we match terms that behave as $(1-\bar{z})^{-\Delta_L+4}z^m \cdot \log^2 z$ from the S-channel calculation of the correlator to terms with the same behavior in (3.18) for small z. For the S-channel calculation, we need the OPE data at $\mathcal{O}(\mu)$ up to $p=2$, given by (3.12) and (3.13). We obtain an over-constrained system of linear equations, whose solution is

$$\begin{aligned}
d_{17} &= \frac{9e_{15}}{143}+\frac{5g_{13}}{4004}+\frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(232-17\Delta_L\right)+1009\right)+1908\right)+1008\right)}{115315200\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
d_{26} &= -\frac{e_{15}}{12}+\frac{5g_{13}}{1386}-\frac{\Delta_L\left(\Delta_L\left(\left(\Delta_L-7\right)\Delta_L\left(11\Delta_L-179\right)+3636\right)+2736\right)}{119750400\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
d_{35} &= -\frac{g_{13}}{180}+\frac{\Delta_L\left(\Delta_L\left(\left(\Delta_L-7\right)\Delta_L\left(37\Delta_L-13\right)+1332\right)+3312\right)}{108864000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
d_{44} &= \frac{\left(\Delta_L-6\right)\Delta_L\left(\Delta_L+2\right)}{9408000\left(\Delta_L-2\right)}.
\end{aligned} \tag{3.19}$$

The undetermined coefficients g_{13} and e_{15} are related to the T-channel OPE coefficients $P^{(2)}_{8,0}$ and $P^{(2)}_{10,2}$ by the following relations

$$\begin{aligned}
g_{13} &= P^{(2)}_{8,0},\\
e_{15} &= P^{(2)}_{10,2}-\frac{5}{252}P^{(2)}_{8,0}.
\end{aligned} \tag{3.20}$$

Here $P^{(2)}_{8,0}$ is the T-channel OPE coefficient of the operator of the schematic form $:T_{\alpha\beta}T^{\alpha\beta}:$, while $P^{(2)}_{10,2}$ is related to the OPE coefficients of the operators $:T_{\alpha\beta}\partial_{\mu_1}\partial_{\mu_2}T^{\alpha\beta}:$ and $:T_{\mu\alpha}\partial^2 T^{\alpha}{}_{\nu}:$ which have the same quantum numbers Δ and s and are thus indistinguishable at this order in large C_T expansion. After inserting (3.20) and (3.19) into (3.18) one can check that both $P^{(2)}_{8,0}$ and $P^{(2)}_{10,2}$ will be multiplied by the relevant lightcone conformal blocks.

Exactly as in the previous section, we can now extract the OPE coefficients $P^{(2)}_{\Delta,s}$ for operators with twist $\tau_{2,2}=8$ and $\Delta=12+2\ell$, $s=4+2\ell$, for $\ell\geq 0$,[6]

$$\begin{aligned}
P^{(2)}_{12+2\ell,4+2\ell} = &\frac{\sqrt{\pi}\Delta_L 2^{-4\ell-19}\Gamma(2\ell+7)}{3(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)(\ell+4)(\ell+5)}\\
&\times\frac{a_{2,\ell}\Delta_L^4+b_{2,\ell}\Delta_L^3+c_{2,\ell}\Delta_L^2+d_{2,\ell}\Delta_L+e_{2,\ell}}{(\ell+6)(\ell+7)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{15}{2}\right)},
\end{aligned} \tag{3.21}$$

[6] For each $\Delta=12+2\ell$ and $s=4+2\ell$ with $\ell\geq 0$ there is a triple degeneracy, because all three families of operators with twist $\tau_{2,2}=8$ will be mixed.

where

$$
\begin{aligned}
a_{2,\ell} &= \ell(2\ell+15)(\ell(2\ell+15)(\ell(2\ell+15)+59)+1084)+6012,\\
b_{2,\ell} &= 14004-2\ell(2\ell+15)(\ell(2\ell+15)(\ell(2\ell+15)+32)-131),\\
c_{2,\ell} &= \ell(2\ell+15)(\ell(2\ell+15)(\ell(2\ell+15)+113)+4594)+60984,\\
d_{2,\ell} &= 216(11\ell(2\ell+15)+302),\\
e_{2,\ell} &= 864(\ell(2\ell+15)+34).
\end{aligned}
\tag{3.22}
$$

It is quite remarkable that these OPE coefficients are fixed purely by the bootstrap.

3.3 Twist-ten double-stress tensors

Now we want to go one step further and analyze the subsubsubleading contribution to the stress tensor sector of the HHLL correlator. This contribution scales as $(1-\bar{z})^{-\Delta_L+5}$ in the lightcone limit. We have to take in to account the double-stress tensor operators of twist $\tau_{2,3}=10$ in order to calculate this contribution. These operators can again be grouped in three families of the schematic form $:T_{\mu\nu}(\partial^2)^3\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T_{\rho\sigma}:$ with $\Delta=14+2\ell$ and $s=4+2\ell$, $:T_{\mu\beta}(\partial^2)^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\beta}{}_{\nu}:$ with $\Delta=12+2\ell$ and $s=2+2\ell$ and finally $:T_{\beta\gamma}\partial^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\beta\gamma}:$ with $\Delta=10+2\ell$ and $s=2\ell$.

In order to include contributions from lower twist operators we have to expand their conformal blocks up to $(1-\bar{z})^5$ for $\bar{z}\to 1$. The ansatz takes the following form

$$
\begin{aligned}
\mathcal{G}^{(2,3)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(&\left(\frac{z((13-3z)z-23)+25}{12(1-z)^3}\right)(a_{33}f_3^2+a_{24}f_2f_4+a_{15}f_1f_5)\\
&+\left(\frac{1}{(1-z)^2}+\frac{1}{1-z}+\frac{9}{10}\right)(b_{14}f_1f_4+c_{16}f_1f_6+c_{25}f_2f_5+c_{34}f_3f_4)\\
&+\left(\frac{1}{1-z}+\frac{3}{2}\right)(d_{17}f_1f_7+d_{26}f_2f_6+d_{35}f_3f_5+d_{44}f_4^2+e_{15}f_1f_5+g_{13}f_1f_3)\\
&-g_{13}f_3+(h_{18}f_1f_8+h_{27}f_2f_7+h_{36}f_3f_6+h_{45}f_4f_5+j_{16}f_1f_6+j_{25}f_2f_5\\
&+j_{34}f_3f_4+i_{14}f_1f_4+i_{23}f_2f_3)\Bigg),
\end{aligned}
\tag{3.23}
$$

with h_{mn}, j_{mn} and i_{mn}, coefficients that we need to determine, and with b_{14}, e_{15} and g_{13} undetermined from the bootstrap. The term $g_{13}f_3(z)$ in the next-to-last line of the previous equation has its origin in the correction to the conformal block of operator $:T_{\alpha\beta}T^{\alpha\beta}:$. This operator has $\beta=\tau_{2,2}=8$ which implies that both lines in the following expansion of the conformal block

$$
\begin{aligned}
g^{(0,0)}_{8,0}(1-z,1-\bar{z}) = (1-\bar{z})^4&\left(1+(1-\bar{z})\left(\frac{3}{2}+\frac{1}{1-z}\right)+\mathcal{O}\left((1-\bar{z})^2\right)\right)f_4(z)\\
&-(1-\bar{z})^5\left(1+(1-\bar{z})\left(2+\frac{1}{1-z}\right)+\mathcal{O}((1-\bar{z})^2)\right)f_3(z)
\end{aligned}
\tag{3.24}
$$

contribute. The contribution from the first line of (3.24) is included in the third line of (3.23), while we had to explicitly add the contribution from the second line. Using (A.1) and (A.3) we set $i_{23} = 0$, $j_{34} = 0$ and $j_{25} = 0$.

From the S-channel calculation, we collect the terms in the correlator which behave as $(1-\bar{z})^{-\Delta_L+5}\log^2(z)z^m$ and are fixed in terms of OPE data at $\mathcal{O}(\mu)$ for $p \leq 3$. By expanding (3.23) near $z \to 0$ we obtain terms with the same behavior as linear functions of unknown coefficients and by matching them with the terms from the S-channel, we determine the unknown coefficients. These are

$$\begin{aligned}
h_{18} &= \frac{49 i_{14}}{38610} + \frac{49 j_{16}}{780} - \frac{\Delta_L(\Delta_L+1)(\Delta_L(\Delta_L(\Delta_L(47\Delta_L-721)-5182)-15204)-13680)}{4942080000(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \\
h_{27} &= \frac{5 i_{14}}{1404} - \frac{j_{16}}{12} - \frac{\Delta_L(\Delta_L+1)(\Delta_L(\Delta_L(\Delta_L(8\Delta_L-229)+1097)+7224)+10080)}{1383782400(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \\
h_{36} &= -\frac{i_{14}}{180} + \frac{\Delta_L(\Delta_L+1)(\Delta_L(\Delta_L(\Delta_L(34\Delta_L-137)-1829)+5712)+23040)}{2661120000(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \\
h_{45} &= \frac{(\Delta_L-6)\Delta_L(\Delta_L+1)(\Delta_L+2)}{62720000(\Delta_L-3)(\Delta_L-2)}.
\end{aligned} \tag{3.25}$$

Our approach does not allow us to determine the coefficients j_{16} and i_{14}. These are related to the T-channel OPE coefficients of operators with twist $\tau_{2,3} = 10$ and minimal conformal spin by

$$\begin{aligned}
i_{14} &= P^{(2)}_{10,0}, \\
j_{16} &= P^{(2)}_{12,2} - \frac{2}{99} P^{(2)}_{10,0}.
\end{aligned} \tag{3.26}$$

Notice that, despite the fact that the h_{mn} depend on the undetermined OPE data, we are able to extract all the OPE coefficients of double-stress tensors with twist $\tau_{2,3} = 10$ and conformal spin $\Delta + s \geq 18$. Explicitly, they are given by:

$$\begin{aligned}
P^{(2)}_{14+2\ell,4+2\ell} = &\frac{\sqrt{\pi}2^{-4\ell-22}\Gamma(2\ell+9)}{5(2\ell+1)(2\ell+3)(2\ell+5)(2\ell+7)\Gamma\left(2\ell+\frac{17}{2}\right)} \\
&\times \frac{\Delta_L(\Delta_L+1)(a_{3,\ell}\Delta_L^4 + b_{3,\ell}\Delta_L^3 + c_{3,\ell}\Delta_L^2 + d_{3,\ell}\Delta_L + e_{3,\ell})}{(\ell+5)(\ell+6)(\ell+7)(\ell+8)(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)},
\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}
a_{3,\ell} &= \ell(2\ell+17)(\ell(2\ell+17)(\ell(2\ell+17)+70)+1513)+9756, \\
b_{3,\ell} &= 38232 - 2(\ell-1)\ell(2\ell+17)(2\ell+19)(\ell(2\ell+17)+44), \\
c_{3,\ell} &= 196164 + \ell(17+2\ell(11647+\ell(17+2\ell)(196+\ell(17+2\ell)))), \\
d_{3,\ell} &= 504(647+19\ell(17+2\ell)), \\
e_{3,\ell} &= 4320(53+\ell(17+2\ell)).
\end{aligned} \tag{3.28}$$

We expect that a similar picture is true for all subleading twist double-stress tensor operators. At $\mathcal{O}(\mu^2)$, the ansatz for $\mathcal{G}^{(2,m)}(z)$ will naturally include products of the type

$f_a(z)f_b(z)$, such that $a+b=6+m$, together with $f_1(z)f_{3+m}(z)$ and $f_1(z)f_{1+m}(z)$. The coefficients of the latter two will be left undetermined from the lightcone bootstrap at every order in the lightcone expansion. Such coefficients will be related to the non-universal OPE coefficients of double-stress tensors with spin $s=0,2$ for a given twist. On the other hand, the coefficients of the products $f_a(z)f_b(z)$, with $a+b=6+m$, once determined, will allow us to extract the OPE coefficients of all double-stress tensors with conformal spin $\beta \geq 12+2m$. We expect them to be universal, despite the fact that the coefficients of the products $f_a(z)f_b(z)$, with $a+b=6+m$, will be plagued by the ambiguities present in the determination of the OPE coefficients of operators spin $s=0,2$ — just as herein.

4 Triple-stress tensors in four dimensions

In this section, we consider the stress tensor sector of the HHLL correlator at $\mathcal{O}(\mu^3)$ in $d=4$. The operators which contribute in the T-channel are triple-stress tensors. Since we are interested in the lightcone limit $1-\bar{z} \ll 1$, we consider contributions of operators with low twist. Triple-stress tensors with minimal twist can be written in the schematic form $:T_{\mu\nu}T_{\rho\sigma}\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T_{\eta\xi}:$. These operators have twist $\tau_{3,\text{min}}=6$ and their contribution to the HHLL correlator in the lightcone limit was found in [24]:

$$\mathcal{G}^{(3,0)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Big(a_{117}f_1(z)^2f_7(z)+a_{126}f_1(z)f_2(z)f_6(z) \\ +a_{135}f_1(z)f_3(z)f_5(z)+a_{225}f_2(z)^2f_5(z)+a_{234}f_2(z)f_3(z)f_4(z)+a_{333}f_3(z)^3\Big), \tag{4.1}$$

where the coefficients a_{ikl} are

$$\begin{aligned}
a_{117} &= \frac{5\Delta_L(\Delta_L+1)(\Delta_L+2)}{768768(\Delta_L-2)(\Delta_L-3)},\\
a_{126} &= \frac{5\Delta_L(5\Delta_L^2-57\Delta_L-50)}{6386688(\Delta_L-2)(\Delta_L-3)},\\
a_{135} &= \frac{\Delta_L(2\Delta_L^2-11\Delta_L-9)}{1209600(\Delta_L-3)},\\
a_{225} &= -\frac{\Delta_L(7\Delta_L^2-51\Delta_L-70)}{2903040(\Delta_L-2)(\Delta_L-3)},\\
a_{234} &= \frac{\Delta_L(\Delta_L-4)(3\Delta_L^2-17\Delta_L+4)}{4838400(\Delta_L-2)(\Delta_L-3)},\\
a_{333} &= \frac{\Delta_L(\Delta_L-4)(\Delta_L^3-16\Delta_L^2+51\Delta_L+24)}{10368000(\Delta_L-2)(\Delta_L-3)}.
\end{aligned} \tag{4.2}$$

4.1 Twist-eight triple-stress tensors

We now consider the subleading contributions at $\mathcal{O}(\mu^3)$ coming from triple-stress tensor operators with twist $\tau_{3,1}=8$. There are two families of such operators, these can be schematically written as $:T_{\mu\nu}T_{\rho\alpha}\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\alpha}{}_{\xi}:$ with $\Delta=12+2\ell$ and spin $s=4+2\ell$

and $:T_{\mu\nu}T_{\rho\sigma}\partial^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T_{\eta\xi}:$ with $\Delta = 14+2\ell$ and spin $s=6+2\ell$. The conformal spins of these families are $\beta = 16+4\ell$ and $\beta = 20+4\ell$, respectively, so we expect products of three $f_a(z)$ functions such that their indices add up to 8 and 10. The contribution to the correlator of these operators scales as $(1-\bar{z})^{-\Delta_L+4}$ for $\bar{z}\to 1$. This implies that one needs to include the contribution from the minimal twist triple-stress tensor operators (due to corrections to their conformal blocks).

Our ansatz takes the form

$$\begin{aligned}\mathcal{G}^{(3,1)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(&\left(\frac{2-z}{1-z}\right)(a_{117}f_1^2f_7 + a_{126}f_1f_2f_6 + a_{135}f_1f_3f_5 + a_{225}f_2^2f_5\\ &+ a_{234}f_2f_3f_4 + a_{333}f_3^3) + (b_{116}f_6f_1^2 + b_{134}f_3f_4f_1 + b_{125}f_2f_5f_1 + b_{233}f_2f_3^2\\ &+ b_{224}f_2^2f_4 + c_{118}f_8f_1^2 + c_{145}f_4f_5f_1 + c_{136}f_3f_6f_1 + c_{127}f_2f_7f_1 + c_{244}f_2f_4^2\\ &+ c_{334}f_3^2f_4 + c_{235}f_2f_3f_5 + c_{226}f_2^2f_6)\Bigg),\end{aligned} \tag{4.3}$$

where a_{jkl} are given in (4.2). The linear dependence between products of three f_a functions, with explicit relations given in appendix A, allows us to set the following coefficients to zero

$$b_{125} = b_{134} = b_{224} = b_{233} = c_{136} = 0. \tag{4.4}$$

To fix the coefficients b_{116} and c_{jkl} we perform an S-channel calculation up to $\mathcal{O}(\mu^3)$. The relevant terms now scale as $(1-\bar{z})^{-\Delta_L+4}\log^3(z)z^m$ and $(1-\bar{z})^{-\Delta_L+4}\log^2(z)z^m$ when $\bar{z}\to 1$ and $z\to 0$.

We fix the S-channel OPE data at $\mathcal{O}(\mu^2)$ using the results of the previous section, specifically eqs. (3.5), (3.18) and (3.23). Since the OPE coefficients of double-stress operators of spin 0 and 2 are left undetermined, the S-channel OPE data is fixed in terms of these. Concretely, $\gamma_n^{(2,0)}$ and $P_n^{(2,0)}$ are completely determined since the leading-twist OPE coefficients are known and universal, while $\gamma_n^{(2,1)}$ and $P_n^{(2,1)}$ depend on b_{14}, $\gamma_n^{(2,2)}$ and $P_n^{(2,2)}$ depend on b_{14}, g_{13} and e_{15} and so on.[7]

We were able to fix all the unknown coefficients in the ansatz (4.3) using bootstrap. Crucially, there are no spin $s=0,2$ operators that contribute at this level. Here, we list two of the coefficients while all others can be found in appendix B.

$$\begin{aligned}b_{116} &= -\frac{\Delta_L(\Delta_L+3)(\Delta_L(\Delta_L(\Delta_L(1001\Delta_L+387)-4326)+13828)+5040)}{10378368000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}\\ &\quad+\frac{b_{14}(\Delta_L(143\Delta_L+427)+540)}{17160(\Delta_L-4)},\\ c_{118} &= \frac{7(\Delta_L+3)\left(604800b_{14}\left(\Delta_L^2-5\Delta_L+6\right)+\Delta_L\left(-21\Delta_L^3+229\Delta_L^2+414\Delta_L+284\right)\right)}{856627200\left(\Delta_L^3-9\Delta_L^2+26\Delta_L-24\right)}.\end{aligned} \tag{4.5}$$

[7]Explicit expressions for the S-channel OPE data are too cumbersome to quote here.

Notice that they depend on b_{14}. This is because the anomalous dimensions at $\mathcal{O}(\mu^2)$, $\gamma_n^{(2,2)}$ depend on it. Moreover, no OPE coefficient of triple-stress tensors with twist $\tau_{3,1}=10$ is universal since all of them depend on b_{14}. These OPE coefficients can be written in the form of a finite sum, similarly to what happens for the OPE coefficients of leading twist triple-stress tensor, given in [24]. We define $i_1(r,q)$ and $i_2(r,q)$ as

$$i_1(r,q)=b_{116}p(1,1,r)p(2r+2,6,q), \tag{4.6}$$

and

$$\begin{aligned} i_2(r,q)=\;& c_{118}p(1,1,r)p(2r+2,8,q)+c_{127}p(1,2,r)p(2r+3,7,q) \\ &+c_{145}p(1,4,r)p(2r+5,5,q)+c_{226}p(2,2,r)p(2r+4,6,q) \\ &+c_{235}p(2,3,r)p(2r+5,5,q)+c_{244}p(2,4,r)p(2r+6,4,q) \\ &+c_{334}p(3,3,r)p(2r+6,4,q), \end{aligned} \tag{4.7}$$

where $p(a,b,\ell)$ are given by (3.4). The OPE coefficients can be written as

$$P^{(3)}_{14+2\ell,6+2\ell}=\sum_{r=0}^{\ell+1} i_1(r,\ell+1-r)+\sum_{r=0}^{\ell} i_2(r,\ell-r), \tag{4.8}$$

for $k\geq 0$, while $P^{(3)}_{12,4}=i_1(0,0)=b_{116}$. We give the explicit expressions for some OPE coefficients in appendix D.

4.2 Twist-ten triple-stress tensors

Here, we consider the contribution of triple-stress tensor operators of twist $\tau_{3,2}=10$. These operators can be divided in three families of the schematic form $:T_{\mu\nu}T_{\alpha\beta}\partial_{\mu_1}\dots\partial_{\mu_{2\ell}}(\partial^2)^2T_{\rho\sigma}:$ with conformal dimension $\Delta=16+2\ell$ and spin $s=6+2\ell$, $:T_{\mu\nu}T_{\alpha\beta}\partial_{\mu_1}\dots\partial_{\mu_{2\ell}}\partial^2 T^{\beta}{}_{\rho}:$ with $\Delta=14+2\ell$ and $s=4+2\ell$ and finally $:T_{\mu\alpha}T_{\nu\beta}\partial_{\mu_1}\dots\partial_{\mu_{2\ell}}T^{\alpha\beta}:$ with $\Delta=12+2\ell$ and $s=2+2\ell$. One can see that in the last family an operator of spin $s=2$ is included.

An appropriate ansatz in this case is

$$\begin{aligned} \mathcal{G}^{(3,2)}(z,\bar z)=\;&\frac{1}{(1-z)^{\Delta_L}}\Bigg(\left(\frac{144z^2-448z+464}{160(z-1)^2}\right)(a_{117}f_1^2f_7+a_{126}f_1f_2f_6+a_{135}f_1f_3f_5 \\ &+a_{225}f_2^2f_5+a_{234}f_2f_3f_4+a_{333}f_3^3)+\left(\frac{1}{1-z}+\frac{3}{2}\right)(b_{116}f_6f_1^2+c_{118}f_8f_1^2+c_{145}f_4f_5f_1 \\ &+c_{127}f_2f_7f_1+c_{244}f_2f_4^2+c_{334}f_3^2f_4+c_{235}f_2f_3f_5+c_{226}f_2^2f_6)+(d_{117}f_1^2f_7+e_{115}f_1^2f_5 \\ &+g_{119}f_1^2f_9+g_{128}f_1f_2f_8+g_{155}f_1f_5^2+g_{227}f_2^2f_7+g_{236}f_2f_3f_6+g_{245}f_2f_4f_5+g_{335}f_3^2f_5 \\ &+g_{344}f_3f_4^2)\Bigg), \end{aligned} \tag{4.9}$$

where $f_a=f_a(z)$ and we have included only the linearly independent products of these functions.

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The lightcone bootstrap fixes all coefficients except e_{115}. One can check that this is exactly the OPE coefficient $P^{(3)}_{12,2}$ of the spin-2 operator $:T_{\mu\alpha}T_{\nu\beta}T^{\alpha\beta}:$ with $\Delta = 12$ and spin $s = 2$

$$e_{115} = P^{(3)}_{12,2}. \tag{4.10}$$

All other coefficients can be found in appendix B. Notice that all coefficients depend on b_{14}, g_{13} and e_{15} because the S-channel OPE data at $\mathcal{O}(\mu^2)$ depend on them.

Again, we write the OPE coefficients for all triple-stress tensor operators with twist $\tau_{3,2} = 10$ and $\beta \geq 18$ in the form of a finite sum. We define $j_1(r,q)$, $j_2(r,q)$ and $j_3(r,q)$ as

$$j_1(r,q) = e_{115}p(1,1,r)p(2r+2,5,q), \tag{4.11}$$

$$j_2(r,q) = d_{117}p(1,1,r)p(2r+2,7,q) \tag{4.12}$$

and

$$\begin{aligned} j_3(r,q) = {} & g_{119}p(1,1,r)p(2r+2,9,q) + g_{128}p(1,2,r)p(2r+3,8,q) \\ & + g_{155}p(1,5,r)p(2r+6,5,q) + g_{227}p(2,2,r)p(2r+4,7,q) \\ & + g_{236}p(2,3,r)p(2r+5,6,q) + g_{245}p(2,4,r)p(2r+6,5,q) \\ & + g_{335}p(3,3,r)p(2r+6,5,q) + g_{344}p(3,4,r)p(2r+7,4,q), \end{aligned} \tag{4.13}$$

where $p(a,b,\ell)$ is given by (3.4). The OPE coefficients can now be written as

$$P^{(3)}_{16+2\ell,6+2\ell} = \sum_{r=0}^{\ell+2} j_1(r,\ell+2-r) + \sum_{r=0}^{\ell+1} j_2(r,\ell+1-r) + \sum_{r=0}^{\ell} j(r,\ell-r), \tag{4.14}$$

for $\ell \geq 0$, while

$$P^{(3)}_{14,4} = j_1(0,1) + j_1(1,0) + j_2(0,0). \tag{4.15}$$

Finally, we conclude that the stress tensor sector of the HHLL correlator to all orders in μ and in the lightcone expansion will take a similar form in terms of products of f_a functions. One should be able to completely fix the coefficients, except for terms that correspond to the OPE coefficients of multi-stress tensor operators with spin $s = 0, 2$, using the lightcone bootstrap.

5 Holographic phase shift and multi-stress tensors

In this section, we demonstrate how to calculate the T-channel OPE coefficients of spin-2 operators (up to undetermined spin-0 data) which are left undetermined after the lightcone bootstrap, using a gravitational calculation of the scattering phase shift. We are interested in the scattering phase shift — or eikonal phase — resulting from the eikonal resummation of graviton exchanges when a fast particle is scattered by a black hole.[8] Seeking to explore the universality properties of the undetermined OPE coefficients of the previous section, we perform the calculation in Gauss-Bonnet gravity extending the results of [20] to this case. We argue that the phase shift in the large impact parameter limit is independent of

[8]For CFT approach to the Regge scattering of scalar particles in pure AdS see [32–38].

higher-derivative corrections to the dual gravitational lagrangian. This is consistent with the universality of the minimal-twist multi-stress tensor sector in the dual CFT. On the other hand, we observe that the subleading OPE data of spin-2 multi-stress tensors depend explicitly on the Gauss-Bonnet coupling $\lambda_{\rm GB}$.

The computation involves performing an inverse Fourier transform of the exponential of the phase shift in the large impact parameter expansion, to obtain the HHLL correlator in position space.[9] This is done following the approach of [39]. Comparison with the expressions for the HHLL correlator in the lightcone limit requires analytically continuing the results of sections 3 and 4 and taking the limit $z \to 1$. Identifying terms in the HHLL four-point function with the same large impact parameter and $z \to 1$ behavior allows us to extract the spin-2 OPE coefficients of the double- and triple-stress tensor operators (up to undetermined spin zero data).

5.1 Universality of the phase shift in the large impact parameter limit

In this subsection, we consider Gauss-Bonnet gravity in $(d+1)$-dimensions and argue that the phase shift obtained by a highly energetic particle traveling in a spherical AdS-Schwarzschild background is independent of the Gauss-Bonnet coupling $\lambda_{\rm GB}$ in the large impact parameter limit.

The action of Gauss-Bonnet gravity in $(d+1)$-dimensional spacetime is

$$S = \frac{1}{16\pi G}\int d^{d+1}\sqrt{-g}\left(R + \frac{d(d-1)}{\ell^2} + \frac{\tilde\lambda_{\rm GB}}{(d-2)(d-3)}(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\right), \tag{5.1}$$

where the coupling parameter $\tilde\lambda_{\rm GB}$ is measured in units of the cosmological constant ℓ: $\tilde\lambda_{\rm GB} = \lambda_{\rm GB}\ell^2$, with $\lambda_{\rm GB}$ being a dimensionless coefficient. The AdS-Schwarzschild black hole metric which is a solution of the Gauss-Bonnet theory is given by [40, 41]:

$$ds^2 = -r_{\rm AdS}^2 f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \tag{5.2}$$

where

$$f(r) = 1 + \frac{r^2}{2\lambda_{\rm GB}}\left(1 - \sqrt{1 - 4\lambda_{\rm GB}\left(1 - \frac{\tilde\mu}{r^d}\right)}\right), \tag{5.3}$$

with

$$\tilde\mu = \frac{16\pi GM}{(d-1)\Omega_{d-1}\ell^{d-2}}, \qquad \mu = \frac{\tilde\mu}{r_{\rm AdS}^{d-2}\sqrt{1-4\lambda_{\rm GB}}}, \tag{5.4}$$

and

$$r_{\rm AdS} = \left(\frac{1}{2}(1 + \sqrt{1 - 4\lambda_{\rm GB}})\right)^{1/2} \tag{5.5}$$

where Ω_{d-1} is the surface area of a $(d-1)$-dimensional unit sphere embedded in d-dimensional Euclidean space. The metric is normalized such that the speed of light is

[9] Recall that the exponential of the phase shift corresponds to the Regge limit of HHLL four-point function in momentum space [20].

equal to 1 at the boundary (i.e. $g_{tt}/g_{\phi\phi} \to 1$ as $r \to \infty$) and all dimensionful parameters are measured in units of ℓ. The product ($\ell r_{\rm AdS}$) is the radius of the asymptotic Anti-de Sitter space.

The two conserved charges along the geodesics, p^t and p^ϕ, are

$$p^t = r_{\rm AdS}^2 f(r)\frac{dt}{d\lambda},$$
$$p^\phi = r^2\frac{d\phi}{d\lambda}, \tag{5.6}$$

where λ denotes an affine parameter. Null geodesics are described by the following equation,

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{(p^\phi)^2}{2r^2}f(r) = \frac{1}{2}\frac{(p^t)^2}{r_{\rm AdS}^2}, \tag{5.7}$$

similarly to Einstein gravity.

A light particle, starting from the boundary, traversing the bulk and reemerging on the boundary experiences a time delay and a path deflection given by:

$$\Delta t = 2\int_{r_0}^{\infty}\frac{dr}{r_{\rm AdS}f(r)\sqrt{1-\alpha^2\frac{r_{\rm AdS}^2}{r^2}f(r)}},$$
$$\Delta\phi = 2\alpha\, r_{\rm AdS}\int_{r_0}^{\infty}\frac{dr}{r^2\sqrt{1-\alpha^2\frac{r_{\rm AdS}^2}{r^2}f(r)}}, \tag{5.8}$$

where $\alpha = p^\phi/p^t$ and r_0 the impact parameter determined by $\frac{dr}{d\lambda}\big|_{r(\lambda)=r_0} = 0$, i.e.,

$$1-\alpha^2\frac{r_{\rm AdS}^2}{r_0^2}f(r_0) = 0. \tag{5.9}$$

Defining the phase shift as $\delta = -p\cdot\Delta x = p^t\Delta t - p^\phi\Delta\phi$, we find that

$$\delta = 2\frac{p^t}{r_{\rm AdS}}\int_{r_0}^{\infty}\frac{dr}{f(r)}\sqrt{1-\alpha^2\frac{r_{\rm AdS}^2}{r^2}f(r)}. \tag{5.10}$$

Just as in [20], we are interested in expanding the phase shift order by order in μ. It is easy to see that in terms of CFT data μ can be expressed as

$$\mu = \frac{4}{(d-1)^2}\frac{\Gamma(d+2)}{\Gamma(d/2)^2}\frac{\Delta_H}{C_T}, \tag{5.11}$$

which is consistent with (2.9). Here C_T is the central charge of the dual conformal theory [42]:

$$C_T = \frac{\pi^{\frac{d}{2}-1}}{2(d-1)}\frac{\Gamma(d+2)}{\Gamma(d/2)^3 G}(r_{\rm AdS}\ell)^{d-1}\sqrt{1-4\lambda_{\rm GB}}, \tag{5.12}$$

and $\Delta_H = M\ell r_{\rm AdS}$.

In order to calculate the phase shift, we introduce a new variable y, given by $y = \frac{r_0}{r}$. Using this variable (5.10) can be written as:

$$\delta = 2\frac{p^t r_0}{r_{\rm AdS}} \int_0^1 \frac{dy}{y^2 f\left(\frac{r_0}{y}\right)} \left(1 - \alpha^2 \frac{r_{\rm AdS}^2 y^2}{{r_0}^2} f\left(\frac{r_0}{y}\right)\right)^{1/2} . \tag{5.13}$$

Expanding the phase shift

$$\delta = \sum_{k=0}^{\infty} \mu^k \delta^{(k)}, \tag{5.14}$$

and solving (5.9) perturbatively in μ reads

$$r_0 = b - \frac{b^{3-d}}{2r_{\rm AdS}^{2-d}}\mu + \frac{b^{3-2d}}{8r_{\rm AdS}^{4-2d}}\left(b^2(3-2d) + \frac{4\lambda_{\rm GB}}{\sqrt{1-4\lambda_{\rm GB}}}\right)\mu^2 + \mathcal{O}(\mu^3). \tag{5.15}$$

Generically, we get an expansion of the form

$$r_0 = b + \sum_{k=1}^{\infty} a_k \mu^k, \tag{5.16}$$

where the a_k, which depend on b, in the large impact parameter limit ($b \to \infty$) behave as

$$a_k \propto b\left(\frac{r_{\rm AdS}}{b}\right)^{k(d-2)} . \tag{5.17}$$

Notice that there is no explicit $\lambda_{\rm GB}$ dependence in the leading term,[10] since the metric (5.2) approaches the one in pure GR.

To study the leading behavior of the phase shift for large impact parameters it is convenient to define a function $g(x)$ as

$$g(x) = r_{\rm AdS}^2 \frac{f(x)}{x^2}, \tag{5.18}$$

with f given by (5.3), and denote the integrand of (5.13) by $h\left(g\left(\frac{r_0}{y}\right)\right)$, with

$$h(x) = \frac{1}{x}\sqrt{1-\alpha^2 x}, \tag{5.19}$$

to express (5.13) as

$$\delta = 2p^t\left(\frac{r_{\rm AdS}}{r_0}\right)\int_0^1 h\left(g\left(\frac{r_0}{y}\right)\right) dy. \tag{5.20}$$

In practice, to calculate the phase shift in the large impact parameter limit, we first expand the integrand of (5.20) in powers of μ, perform the integration with respect to y, and then expand the result in powers of b. The b-dependence of $\delta^{(k)}$ is therefore fixed before the integration and the integral just determines the overall numerical factor (assuming that it is convergent).

[10] Except the overall dependence on $r_{\rm AdS}$.

We can immediately see that $g\left(\frac{r_0}{y}\right)$ depends on μ explicitly and implicitly through $r_0(\mu)$ in (5.15). In order to make this clear we write $g\left(\frac{r_0}{y},\mu\right)$ instead of just $g\left(\frac{r_0}{y}\right)$. Defining $g^{(n,m)}\left(\frac{b}{y},0\right)$ as

$$g^{(n,m)}\left(\frac{b}{y},0\right)=\frac{\partial^n\partial^m}{\partial r_0^n\partial\mu^m}g\left(\frac{r_0}{y},\mu\right)\Big|_{r_0=b,\mu=0}, \tag{5.21}$$

allows us to write the following expansion for $h\left(g\left(\frac{r_0}{y},\mu\right)\right)$:

$$\begin{aligned}h\left(g\left(r_0/y,\mu\right)\right)=&\,h(g(b/y,0))+\mu h'(g(b/y,0))\left(g^{(0,1)}(b/y,0)+a_1g^{(1,0)}(b/y,0)\right)\\&+\frac{\mu^2}{2}h''(g(b/y,0))\left(g^{(0,1)}(b/y,0)+a_1g^{(1,0)}(b/y,0)\right)^2\\&+\frac{\mu^2}{2}h'(g(b/y,0))\Big(g^{(0,2)}(b/y,0)+2a_2g^{(1,0)}(b/y,0)\\&+2a_1g^{(1,1)}(b/y,0)+a_1^2g^{(2,0)}(b/y,0)\Big)+\mathcal{O}(\mu^3),\end{aligned} \tag{5.22}$$

where a_k are the coefficients appearing in (5.16). It is clear that at each order in the μ-expansion we will have a sum of products composed from derivatives of $h(x)$ and sums of the form

$$\sum_{\left\{k_i:\sum\limits_{i=1}^{p}k_i\leqslant n\right\}}a_{k_1}a_{k_2}\dots a_{k_p}g^{(p,n-\sum_{i=1}^{p}k_i)}(b/y,0)\,. \tag{5.23}$$

Notice first that $g(b/y,0)$, $g^{(m,0)}(b/y,0)$ and $g^{(m,1)}(b/y,0)$ do not depend on $\lambda_{\rm GB}$ as can be seen from (5.18). The same is true for $h^{(n)}(g(b/y,0))$ for any n as follows from (5.19). On the contrary, $g^{(m,n)}(b/y,0)$ with $n\geq 2$ depend explicitly on $\lambda_{\rm GB}$. It is then evident that any dependence on $\lambda_{\rm GB}$ will come from terms like the ones in parenthesis in (5.22) which are of the type (5.23). We will now show that all the terms in such sums which contain $\lambda_{\rm GB}$, are subleading in the large impact parameter limit.

Recall that $a_k\propto b^{1-k(d-2)}$ for $k\geq 1$. Using (5.18) one can check that $g^{(m,n)}(b/y,0)\propto b^{-m-nd}$ for $n>0$ and $g^{(m,0)}(b/y,0)\propto b^{-m-2}$. We thus need to spearately consider two cases: products of the form $a_{k_1}a_{k_2}\dots a_{k_p}g^{(p,n-q)}(b/y,0)$, with $q=\sum_{i=1}^p k_i$ and $q<n$ and products of the form $a_{k_1}a_{k_2}\dots a_{k_p}g^{(p,0)}(b/y,0)$ for which $q=n$.

The former behave as

$$a_{k_1}a_{k_2}\dots a_{k_p}g^{(p,n-q)}(b/y,0)\propto\frac{1}{b^{nd-2q}}. \tag{5.24}$$

Clearly, the leading behavior in the large impact parameter regime corresponds in this case to $q=n-1$, recall, however, that $g^{(p,1)}$ does not depend on $\lambda_{\rm GB}$. The behavior of the latter terms is

$$a_{k_1}a_{k_2}\dots a_{k_p}g^{(p,0)}(b/y,0)\propto\frac{1}{b^{nd-2(n-1)}}, \tag{5.25}$$

which is again independent of $\lambda_{\rm GB}$. The conclusion is that the leading behavior in the large impact parameter regime comes from terms containing $g^{(p,0)}(b/y,0)$ and $g^{(p,1)}(b/y,0)$ that do not contain $\lambda_{\rm GB}$.

One can extend these considerations straightforwardly to any gravitational theory that contains a spherical black hole with a metric given by

$$ds^2 = -(1 + r^2 \tilde{f}(r))dt^2 + \frac{dr^2}{1 + r^2 \tilde{h}(r)} + r^2 d\Omega_{d-1}^2 \tag{5.26}$$

where the functions $\tilde{f}(r)$ and $\tilde{h}(r)$ admit an expansion of the following form in the large r limit:

$$\begin{aligned} \tilde{f}(r) &= 1 - \sum_{n=0}^{\infty} \frac{\tilde{f}_{nd}}{r^{nd}} = 1 - \frac{\tilde{f}_0}{r^d} - \frac{\tilde{f}_d}{r^{2d}} - \dots \\ \tilde{h}(r) &= 1 - \sum_{n=0}^{\infty} \frac{\tilde{h}_{nd}}{r^{nd}} = 1 - \frac{\tilde{h}_0}{r^d} - \frac{\tilde{h}_d}{r^{2d}} - \dots , \end{aligned} \tag{5.27}$$

for some constants $\tilde{f}_{nd}$ and $\tilde{h}_{nd}$ (these are the spherical black hole metrics considered in eqs. (5.1) and (5.10) in [21]).

5.2 Spin-2 multi-stress tensor OPE data from the gravitational phase shift

The gravitational phase shift in a black hole background is related to the lightcone HHLL four-point function discussed extensively in this article. In the following, we will exploit the precise relationship between the two to extract the OPE data of multi-stress tensor operators of spin-2 in the dual conformal field theory (modulo spin zero data). While the explicit procedure can be worked out for arbitrary multi-stress tensors, we will herein focus on double and triple-stress tensor operators, which control the $\mathcal{O}(\mu^2)$ and $\mathcal{O}(\mu^3)$ lightcone behavior of the HHLL correlation function.

5.2.1 The phase shift in Gauss-Bonnet gravity to $\mathcal{O}(\mu^3)$

In this section, we focus on the gravity side and determine the phase shift order by order in μ up to $\mathcal{O}(\mu^3)$ relevant for this article. Starting from $\mathcal{O}(\mu^0)$ we consider the following expression

$$\delta^{(0)} = 2b\, p^t\, r_{\text{AdS}} \sqrt{1-\alpha^2} \int_0^1 \frac{\sqrt{1-y^2}}{b^2 + r_{\text{AdS}}^2 y^2} dy. \tag{5.28}$$

Evaluating this integral and using the following notation $p^{\pm} = p^t \pm p^{\phi}$, $-p^2 = p^+ p^-$, leads to

$$\delta^{(0)} = \pi p^- . \tag{5.29}$$

This is of course none other but the "phase shift" in pure AdS space.

At $\mathcal{O}(\mu)$ the result is the same as in [20], where Einstein gravity was considered,

$$\delta^{(1)} = \sqrt{-p^2} \left(\frac{b}{r_{\text{AdS}}}\right)^{1-d} \left(\frac{d-1}{2}\right) B\left[\frac{d-1}{2}, \frac{3}{2}\right] {}_2F_1\left(1, \frac{d-1}{2}, \frac{d}{2}+1, -\frac{r_{\text{AdS}}^2}{b^2}\right). \tag{5.30}$$

At this order, the phase shift depends only on the single graviton exchange, which is unaffected by the higher derivative terms in the gravitational action. According to the holographic dictionary, the exchange of a single graviton is related to the exchange of a single stress tensor in the T-channel. The corresponding OPE coefficient is fixed by the Ward identity, so it does not depend on the details of the theory.

We now consider the phase shift at higher orders in μ. For convenience herein all results are presented in $d=4$. At $\mathcal{O}(\mu^2)$, using the technique presented in the previous subsection, we find that:

$$\delta^{(2)} = \frac{7\pi}{8}\sqrt{-p^2}\left[5\frac{b}{r_{\text{AdS}}}\left(\sqrt{1+\frac{r_{\text{AdS}}^2}{b^2}}-1\right)-\frac{5}{2}\frac{r_{\text{AdS}}}{b}+\frac{5}{4}\frac{r_{\text{AdS}}^3}{b^3}\right.$$
$$\left.+\frac{\lambda_{\text{GB}}}{r_{\text{AdS}}^2\sqrt{1-4\lambda_{\text{GB}}}}\left(4\frac{b}{r_{\text{AdS}}}\left(\sqrt{1+\frac{r_{\text{AdS}}^2}{b^2}}-1\right)-2\frac{r_{\text{AdS}}}{b}+\frac{1}{2}\frac{r_{\text{AdS}}^3}{b^3}-\frac{1}{4}\frac{r_{\text{AdS}}^5}{b^5}\right)\right]. \tag{5.31}$$

In the lightcone limit ($b\to\infty$) this reduces to

$$\delta^{(2)} \underset{b\to\infty}{\approx} \frac{35\pi\sqrt{-p^2}r_{\text{AdS}}^5}{128b^5}-\frac{35\pi\sqrt{-p^2}r_{\text{AdS}}^7}{1024b^7}\left(5+\frac{4\lambda_{\text{GB}}}{r_{\text{AdS}}^2\sqrt{1-4\lambda_{\text{GB}}}}\right)+\dots. \tag{5.32}$$

We explicitly see that the leading contribution does not depend on λ_{GB}, while the subleading does.

Let us denote $\delta^{(2)}_{\text{GR}}$ to be equal to (5.31) when $\lambda_{\text{GB}}=0$,

$$\delta^{(2)}_{\text{GR}} = \frac{35\pi r_{\text{AdS}}^5\sqrt{-p^2}}{128b^5}\,{}_2F_1\left(1,\frac{5}{2},4,-\frac{r_{\text{AdS}}^2}{b^2}\right), \tag{5.33}$$

which is the pure Einstein gravity result for the phase shift at $\mathcal{O}(\mu^2)$. Then $\delta^{(2)}$ can be written as

$$\delta^{(2)} = \delta^{(2)}_{\text{GR}}\left(1+\frac{4\lambda_{\text{GB}}}{5r_{\text{AdS}}^2\sqrt{1-4\lambda_{\text{GB}}}}\right)-\frac{7\pi\sqrt{-p^2}\lambda_{\text{GB}}}{32r_{\text{AdS}}^2\sqrt{1-4\lambda_{\text{GB}}}}\left(\frac{r_{\text{AdS}}}{b}\right)^5. \tag{5.34}$$

The phase shift at $\mathcal{O}(\mu^3)$ is given by

$$\delta^{(3)} = \delta^{(3)}_{\text{GR}}\left(1+\frac{12\lambda_{\text{GB}}}{7r_{\text{AdS}}^2\sqrt{1-4\lambda_{\text{GB}}}}+\frac{16\lambda_{\text{GB}}^2}{21r_{\text{AdS}}^4(1-4\lambda_{\text{GB}})}\right)$$
$$-\sqrt{-p^2}\left(\frac{r_{\text{AdS}}}{b}\right)^7\left(\frac{495\pi\lambda_{\text{GB}}}{512r_{\text{AdS}}^2\sqrt{1-4\lambda_{\text{GB}}}}+\frac{55\pi\lambda_{\text{GB}}^2}{128r_{\text{AdS}}^4(1-4\lambda_{\text{GB}})}\right)$$
$$+\sqrt{-p^2}\left(\frac{r_{\text{AdS}}}{b}\right)^9\frac{77\pi\lambda_{\text{GB}}^2}{256r_{\text{AdS}}^4(1-4\lambda_{\text{GB}})}, \tag{5.35}$$

where

$$\delta^{(3)}_{\text{GR}} = \frac{231r_{\text{AdS}}^7}{16b^7}\sqrt{-p^2}B\left(\frac{7}{2},\frac{3}{2}\right){}_2F_1\left(1,\frac{7}{2},5,-\frac{r_{\text{AdS}}^2}{b^2}\right). \tag{5.36}$$

By expanding (5.35) in the large impact parameter limit, one again explicitly sees that the leading term does not depend on λ_{GB}.

5.2.2 Inverse Fourier transform of the phase shift at $\mathcal{O}(\mu^2)$

To make contact with the position space HHLL correlation function, one needs to perform a Fourier transform of the phase shift. According to [20], the HHLL four-point function in the Regge limit $\sqrt{-p^2} \gg 1$ is given by

$$\tilde{\mathcal{G}}(x) = \int \frac{d^d p}{(2\pi)^d} e^{ipx} \mathcal{B}(p), \tag{5.37}$$

where $\tilde{\mathcal{G}}(x) = \langle \mathcal{O}_H(x_1)\mathcal{O}_L(x_2)\mathcal{O}_L(x_3)\mathcal{O}_H(x_4)\rangle_{\text{Regge limit}}$ and $\mathcal{B}(p) = \mathcal{B}_0(p)e^{i\delta}$. The factor $\mathcal{B}_0(p)$ reproduces the disconnected correlator and it is given by

$$\mathcal{B}_0(p) = C(\Delta_L)\theta(p^0)\theta(-p^2)e^{i\pi\Delta_L}(-p^2)^{\Delta_L - \frac{d}{2}}, \tag{5.38}$$

with normalization

$$C(\Delta_L) = \frac{2^{d+1-2\Delta_L}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta_L)\Gamma\left(\Delta_L - \frac{d}{2} + 1\right)}. \tag{5.39}$$

We expand the integrand of (5.37) in powers of μ using (5.14), explicitly

$$\begin{aligned}\mathcal{B}(p) = \mathcal{B}_0(p)\Bigg(&1 + \mu i\delta^{(1)} + \mu^2\left(i\delta^{(2)} - \frac{1}{2}{\delta^{(1)}}^2\right)\\ &+ \mu^3\left(i\delta^{(3)} - \delta^{(1)}\delta^{(2)} - \frac{i}{6}{\delta^{(1)}}^3\right) + \mathcal{O}(\mu^4)\Bigg).\end{aligned} \tag{5.40}$$

This generates an expansion for $\tilde{\mathcal{G}}(x)$ from (5.37) as

$$\tilde{\mathcal{G}}(x) = \sum_{k=0}^{\infty} \mu^k \tilde{\mathcal{G}}^{(k)}(x). \tag{5.41}$$

Let us start by studying the correlator at $\mathcal{O}(\mu^2)$. The imaginary part of the correlator in the Regge limit at this order comes from $i\delta^{(2)}$ in (5.40) while the real part comes from $-\frac{1}{2}{\delta^{(1)}}^2$.

Consider first the imaginary part. To perform the inverse Fourier transform it is convenient to first expand $\delta^{(2)}$ as follows:

$$\begin{aligned}\delta^{(2)} = 7\pi^2\sqrt{-p^2}\Bigg(&\frac{5}{2}\Pi_{5,3}(L) + \left(\frac{15}{4} - \frac{5\lambda_{\text{GB}}}{r_{\text{AdS}}^2\sqrt{1-4\lambda_{\text{GB}}}}\right)\Pi_{7,3}(L)\\ &+ \left(5 - \frac{16\lambda_{\text{GB}}}{r_{\text{AdS}}^2\sqrt{1-4\lambda_{\text{GB}}}}\right)\Pi_{9,3}(L) + \dots\Bigg).\end{aligned} \tag{5.42}$$

In (5.42) $b/r_{\text{AdS}} = \sinh(L)$ and

$$\Pi_{\Delta-1;d-1}(x) = \frac{\pi^{1-\frac{d}{2}}\Gamma(\Delta-1)}{2\Gamma\left(\Delta - \frac{d-2}{2}\right)} e^{-(\Delta-1)x}\ {}_2F_1\left(\frac{d}{2}-1, \Delta-1, \Delta - \frac{d-2}{2}, e^{-2x}\right), \tag{5.43}$$

the three-dimensional hyperbolic space propagator of a massive particle with mass square equal to $(\Delta-1)^2$. The dots in (5.42) stand for terms with hyperbolic space propagators with $\Delta > 10$. We can now perform the inverse Fourier transform of (5.42) with the help of eqs. (3.23) in [20] and (3.4) in [39].

The term which contains $\Pi_{5,3}(L)$ includes (after the inverse Fourier transform) the contribution of double-stress tensors with minimal twist $\tau = 4$. As we have already shown it does not depend on $\lambda_{\rm GB}$, which we can also explicitly see in (5.42). The next term, that contains $\Pi_{7,3}(L)$, includes the contribution from the double-stress tensor operators of twist $\tau_{2,1} = 6$. We can use this term to fix the coefficient b_{14} which was left undetermined in (3.5). Similar reasoning applies to all the higher-order terms in the large impact parameter expansion of (5.42). Namely, the term proportional to $\Pi_{2m+1,3}(L)$ is related to double-stress tensor operators of twist $\tau = 2m$.

Performing the inverse Fourier transform following [39] leads to

$$\begin{aligned} i\text{Im}\left(\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)\right) &= \int \frac{d^4p}{(2\pi)^4} e^{ipx} \mathcal{B}_0(p) i\delta^{(2)} = \frac{2i}{\Gamma(\Delta_L)\Gamma(\Delta_L-1)\sigma^{2\Delta_L+1}} \\ &\times \Big(a_1 \Pi_{5,3}(\rho)\Gamma(\Delta_L-2)\Gamma(\Delta_L+2) + b_1 \Pi_{7,3}(\rho)\Gamma(\Delta_L-3)\Gamma(\Delta_L+3) \\ &+ c_1 \Pi_{9,3}(\rho)\Gamma(\Delta_L-4)\Gamma(\Delta_L+4) + \dots\Big) + \dots, \end{aligned} \tag{5.44}$$

where $a_1 = \frac{35}{2}\pi^2$, $b_1 = 7\pi^2\left(\frac{15}{4} - \frac{5\lambda_{\rm GB}}{r^2_{\rm AdS}\sqrt{1-4\lambda_{\rm GB}}}\right)$ and $c_1 = 7\pi^2\left(5 - \frac{16\lambda_{\rm GB}}{r^2_{\rm AdS}\sqrt{1-4\lambda_{\rm GB}}}\right)$. The ellipses outside the parenthesis in (5.44) denote contributions due to double-trace operators in the T-channel that are not important for studying the stress tensor sector. The position space coordinates σ and ρ are defined as

$$z = 1 - \sigma e^{\rho}, \qquad \bar{z} = 1 - \sigma e^{-\rho}. \tag{5.45}$$

after the analytic continuation $z \to z e^{-2i\pi}$. Once more, notice that the dominant contribution in the large impact parameter regime, $\rho \to \infty$, comes from the factor $\Pi_{5,3}(\rho)$ in (5.44) which exactly matches the imaginary part of the correlator (3.1) in [20].

5.2.3 Comparison with the HHLL correlation function in the lightcone limit at $\mathcal{O}(\mu^2)$

A few simple steps are required before we can finally relate (5.44) with the results of section 3 and determine the OPE coefficients of the spin-2 double-stress tensor operators. As explained in [20], one has to analytically continue $\mathcal{G}^{(2,1)}$, $\mathcal{G}^{(2,2)}$ and $\mathcal{G}^{(2,3)}$ (defined in section 2) around the origin by taking $z \to z e^{-2i\pi}$ and expand the result in the vicinity of $\sigma \to 0$. The relevant term, which corresponds to the imaginary part of the correlator (3.5) as $\sigma \to 0$, reads:

$$\begin{aligned} i\text{Im}\left((\sigma e^{-\rho})^{3-\Delta_L}\mathcal{G}^{(2,1)}(1-\sigma e^{\rho})\right) &= 7i\pi \frac{e^{-7\rho}}{\sigma^{2\Delta_L+1}}\Bigg(12600 b_{14} \\ &+ \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(123-7\Delta_L\right)+78\right)-12\right)}{16\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\Bigg). \end{aligned} \tag{5.46}$$

Comparing this with the subleading term of (5.44) as $\rho \to \infty$, i.e.,

$$i\mathrm{Im}\left(\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)\right)|_{e^{-7\rho}} = -\frac{35 i\pi e^{-7\rho}\Delta_L\left(\Delta_L+1\right)\left(8\lambda_{\mathrm{GB}}+\Delta_L\left(4\lambda_{\mathrm{GB}}-5\sqrt{1-4\lambda_{\mathrm{GB}}}r_{\mathrm{AdS}}^2\right)\right)}{4\sigma^{2\Delta_L+1}\sqrt{1-4\lambda_{\mathrm{GB}}}r_{\mathrm{AdS}}^2\left(\Delta_L^2-5\Delta_L+6\right)} + \dots, \tag{5.47}$$

with the ellipses again denoting double-trace operators, allows one to obtain the following expression for the unknown parameter b_{14}:

$$b_{14} = P_{8,2}^{(2)} = \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(7\Delta_L-23\right)+22\right)+12\right)}{201600\left(\Delta_L-3\right)\left(\Delta_L-2\right)} - \frac{\lambda_{\mathrm{GB}}\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)}{2520\sqrt{1-4\lambda_{\mathrm{GB}}}r_{\mathrm{AdS}}^2\left(\Delta_L-3\right)\left(\Delta_L-2\right)}. \tag{5.48}$$

Note that this precisely matches the OPE coefficient of the double trace operator of conformal dimension $\Delta = 8$ and $s = 2$ calculated in [21] from gravity by other means. As expected, the OPE coefficient in (5.48) explicitly depends on λ_{GB}.

Let us now go one step further and fix $P_{10,2}^{(2)}$ contributing to $\mathcal{G}^{(2,2)}(z)$ through (3.20). Analytically continuing (3.18) and taking the limit $\sigma \to 0$, yields

$$i\mathrm{Im}\left(\left(\sigma e^{-\rho}\right)^{4-\Delta_L}\mathcal{G}^{(2,2)}(1-\sigma e^{\rho})\right) = i\frac{49}{400}\frac{\pi e^{-9\rho}}{\sigma^{2\Delta_L+1}}\left(720000 b_{14}+11404800\frac{P_{10,2}^{(2)}}{\mu^2} + \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(6327-362\Delta_L\right)+749\right)+12888\right)+12288\right)}{7\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\right). \tag{5.49}$$

For reasons that will be explained later, we only consider here the imaginary part of the subsubleading term in the correlator. To extract the OPE data we need to compare (5.49) with the subsubleading contribution in the large impact parameter limit of (5.44), which is

$$i\mathrm{Im}\left(\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)\right)|_{e^{-9\rho}} = i\frac{7}{4}\frac{\pi e^{-9\rho}}{\sigma^{2\Delta_L+1}}\left(\frac{10\Delta_L\left(\Delta_L+1\right)}{\Delta_L-2} - \frac{7\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)\left(16\lambda_{\mathrm{GB}}+\Delta_L\left(12\lambda_{\mathrm{GB}}-5\sqrt{1-4\lambda_{\mathrm{GB}}}r_{\mathrm{AdS}}^2\right)\right)}{\sqrt{1-4\lambda_{\mathrm{GB}}}r_{\mathrm{AdS}}^2\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\right). \tag{5.50}$$

Substituting (5.48) in (5.49) and matching to (5.50) enables us to determine the OPE coefficient $P_{10,2}^{(2)}$,

$$P_{10,2}^{(2)} = \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(187\Delta_L-552\right)+901\right)+1012\right)+912\right)}{79833600\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} - \frac{\lambda_{\mathrm{GB}}\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)\left(\Delta_L+3\right)}{12474\sqrt{1-4\lambda_{\mathrm{GB}}}r_{\mathrm{AdS}}^2\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}. \tag{5.51}$$

This precisely matches the one calculated in [21].

Similarly, one can match the CFT expression for $\text{Im}\left((\sigma e^{-\rho})^{5-\Delta_L}\mathcal{G}^{(2,3)}(1-\sigma e^{\rho})\right)$ in (3.23), to its gravitational counterpart $\text{Im}\left(\mathcal{G}^{(2)}(x)\right)|_{e^{-11\rho}}$, by expanding (5.42) and (5.44) up to $\mathcal{O}(e^{-11\rho})$. This allows one to additionally determine $P^{(2)}_{12,2}$ in (3.26)

$$P^{(2)}_{12,2} = \frac{\Delta_L(\Delta_L+1)(\Delta_L(\Delta_L(\Delta_L(6721\Delta_L-15603)+46474)+100828)+143760)}{44396352000(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)} - \frac{5\lambda_{\text{GB}}\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{453024\sqrt{1-4\lambda_{\text{GB}}}r^2_{\text{AdS}}(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}. \tag{5.52}$$

Notice that we did not use the real part of $\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)$, which comes from the term $-\frac{1}{2}{\delta^{(1)}}^2$ in (5.40) and behaves as $\sigma^{-2\Delta_L-2}$ for $\sigma\to 0$. This term matches the corresponding term with the same σ behavior in the correlator. It does not give us any new information, because it is independent of the OPE coefficients of operators with spin $s=0,2$.

5.2.4 Extracting OPE data from the gravitational phase shift at $\mathcal{O}(\mu^3)$

Let us now consider the $\mathcal{O}(\mu^3)$ terms in the correlator. Focusing on the gravity side, we start by performing an inverse Fourier transform. (5.40) instructs us to consider three terms $i\delta^{(3)}$, $\delta^{(1)}\delta^{(2)}$ and $i(\delta^{(1)})^3$, which give rise to terms that behave as $\sigma^{-2\Delta_L-1}$, $\sigma^{-2\Delta_L-2}$ and $\sigma^{-2\Delta_L-3}$, respectively. Performing the relevant computations, we observe that $\delta^{(1)}\delta^{(2)}$ and $i(\delta^{(1)})^3$ do not provide additional information because the corresponding terms in the correlators are already fixed by bootstrap (these terms simply give us an extra consistency check). Focusing on the inverse Fourier transform of $i\delta^{(3)}$, we expand (5.35) in terms of the hyperbolic space propagators, $\Pi_{m,3}(L)$,

$$\delta^{(3)} = \sqrt{-p^2}\Big(a_2\Pi_{7,3}(L)+b_2\Pi_{9,3}(L)+c_2\Pi_{11,3}(L)+\dots\Big), \tag{5.53}$$

where

$$\begin{aligned} a_2 &= \frac{1155}{8}\pi^2, \\ b_2 &= 231\pi^2\left(-\frac{3\lambda_{\text{GB}}}{r^2_{\text{AdS}}\sqrt{1-4\lambda_{\text{GB}}}}+2\right), \\ c_2 &= \frac{231\pi^2}{8}\left(\frac{32\lambda^2_{\text{GB}}}{r^4_{\text{AdS}}(1-4\lambda_{\text{GB}})}-\frac{120\lambda_{\text{GB}}}{r^2_{\text{AdS}}\sqrt{1-4\lambda_{\text{GB}}}}+35\right), \end{aligned} \tag{5.54}$$

which leads to

$$\begin{aligned} i\text{Im}\left(\tilde{\mathcal{G}}^{(3)}(\sigma,\rho)\right)\Big|_{\frac{1}{\sigma^{2\Delta_L+1}}} &= \int\frac{d^4p}{(2\pi)^4}e^{ipx}\mathcal{B}_0(p)i\delta^{(3)} = \frac{2i}{\Gamma(\Delta_L)\Gamma(\Delta_L-1)\sigma^{2\Delta_L+1}} \\ &\times\Big(a_2\Pi_{7,3}(\rho)\Gamma(\Delta_L-3)\Gamma(\Delta_L+3)+b_2\Pi_{9,3}(\rho)\Gamma(\Delta_L-4)\Gamma(\Delta_L+4) \\ &+c_2\Pi_{11,3}(\rho)\Gamma(\Delta_L-5)\Gamma(\Delta_L+5)+\dots\Big)+\text{double traces}, \end{aligned} \tag{5.55}$$

The leading and subleading contributions in the large impact parameter limit $\rho\to\infty$ come from $\Pi_{7,3}(\rho)$ and $\Pi_{9,3}(\rho)$ and behave as $\frac{i\pi e^{-7\rho}}{\sigma^{2\Delta_L+1}}$ and $\frac{i\pi e^{-9\rho}}{\sigma^{2\Delta_L+1}}$, respectively. They are precisely matched by the relevant terms in (4.1) in the vicinity of $\sigma \to 0$ after analytic continuation [39]. This is another sanity check of the procedure described herein, since these terms do not incorporate contributions from spin-2 operators.

To extract further OPE data, we proceed to match the subsubleading correction of (5.55) in the large impact parameter limit to the term in (4.9) which behaves as $\sim \frac{i\pi e^{-11\rho}}{\sigma^{2\Delta_L+1}}$. This allows us to determine the coefficient $e_{115} = P^{(3)}_{12,2}$ in (4.9) which corresponds to the OPE coefficient of the triple-stress tensors of spin $s = 2$ with conformal dimension $\Delta = 12$:

$$\begin{aligned} e_{115} = &-\frac{117\Delta_L^6 - 439\Delta_L^5 + 407\Delta_L^4 + 859\Delta_L^3 + 202\Delta_L^2 + 696\Delta_L}{172972800(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\ &- \frac{\lambda_{\rm GB}(143\Delta_L^6 - 231\Delta_L^5 - 3597\Delta_L^4 - 9489\Delta_L^3 - 11186\Delta_L^2 - 4920\Delta_L)}{43243200 r_{\rm AdS}^2\sqrt{1-4\lambda_{\rm GB}}(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\ &+ \frac{\lambda_{\rm GB}^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{24024 r_{\rm AdS}^4(1-4\lambda_{\rm GB})(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\ &+ P^{(2)}_{8,0}\frac{76 + \frac{400}{\Delta_L-5} + 11\Delta_L}{1320}. \end{aligned} \tag{5.56}$$

Notice that e_{115} is not completely determined by the above procedure since the spin-0 OPE data, $P^{(2)}_{8,0}$, is not fixed. Summarising, we conclude that we are able to fix all coefficients in the ansatz except those that correspond to the OPE coefficients of operators of spin-0. However, using the expression for $P^{(2)}_{8,0}$ found in [21] one finds

$$\begin{aligned} P^{(3)}_{12,2} = &\frac{1001\Delta_L^7 - 6864\Delta_L^6 + 12615\Delta_L^5 - 3980\Delta_L^4 - 6156\Delta_L^3 - 11736\Delta_L^2 - 1440\Delta_L}{3459456000(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\ &- \frac{\lambda_{\rm GB}(143\Delta_L^6 - 206\Delta_L^5 - 1631\Delta_L^4 - 3622\Delta_L^3 - 3540\Delta_L^2 - 1200\Delta_L)}{28828800 r_{\rm AdS}^2\sqrt{1-4\lambda_{\rm GB}}(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\ &+ \frac{\lambda_{\rm GB}^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{24024 r_{\rm AdS}^4(1-4\lambda_{\rm GB})(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)}. \end{aligned} \tag{5.57}$$

6 Lorentzian inversion formula

It was recently shown in [27] that one can obtain the OPE coefficients of minimal twist double and triple-stress tensors using the Lorentzian inversion formula. Here, we review this method and show how it can be generalized to extract the OPE coefficients of twist-six double-stress tensors. In principle, it can also be generalized to multi-stress tensors of arbitrarily high twist.

6.1 Twist-four double-stress tensors

Consider the correlation function

$$(w\bar{w})^{-\Delta_L} G(w,\bar{w}) = \langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(w,\bar{w})\mathcal{O}_L(0)\rangle. \tag{6.1}$$

The Lorentzian inversion formula is given by [25, 26]

$$c(\tau,\beta) = \frac{1+(-1)^{\frac{\beta-\tau}{2}}}{2}\kappa_\beta \int_0^1 dw d\bar{w}\mu^{(0,0)}(w,\bar{w}) \times g^{(0,0)}_{-\tau+2(d-1),\frac{\beta+\tau}{2}-d+1}(w,\bar{w})\mathrm{dDisc}[G(w,\bar{w})], \tag{6.2}$$

where

$$\mu^{(0,0)}(w,\bar{w}) = \frac{|w-\bar{w}|^{d-2}}{(w\bar{w})^d}, \tag{6.3}$$

$$\kappa_\beta = \frac{\Gamma(\frac{\beta}{2})^4}{2\pi^2\Gamma(\beta)\Gamma(\beta-1)}, \tag{6.4}$$

where $\tau = \Delta - s$ and $\beta = \Delta + s$. Here $g^{(0,0)}_{\tau,s}$ is a conformal block given with $\Delta \to s+d-1$ and $s \to \Delta - d + 1$ and in $d=4$ is given by (2.5). Moreover, dDisc denotes the double-discontinuity of $G(w,\bar{w})$ in (6.1), which is equal to the correlator of a double commutator, and it is given by

$$\mathrm{dDisc}[G(w,\bar{w})] = G(w,\bar{w}) - \frac{1}{2}G^{\circlearrowleft}(w,\bar{w}) - \frac{1}{2}G^{\circlearrowright}(w,\bar{w})\,. \tag{6.5}$$

Here $G^{\circlearrowleft}$ and $G^{\circlearrowright}$ correspond to the same correlator analytically continued in two different ways around $w=1$, namely $(1-w) \to (1-w)e^{\pm 2\pi i}$. The OPE data, $P_{\frac{\tau'+\beta}{2},\frac{\beta-\tau'}{2}}$, can be extracted from $c(\tau,\beta)$ via[11]

$$P_{\frac{\tau'+\beta}{2},\frac{\beta-\tau'}{2}} = -\mathrm{Res}_{\tau=\tau'}c(\tau,\beta), \tag{6.6}$$

where τ' and β denote the twist and conformal spin of operators in the physical spectrum of the theory exchanged in the channel $\mathcal{O}_L \times \mathcal{O}_L \to \mathcal{O}_{\tau',J'} \to \mathcal{O}_H \times \mathcal{O}_H$.

We would like to apply the Lorentzian inversion formula to the HHLL correlator to extract the OPE data of the double-stress tensors. To this end, we will use information of the correlator from the channel where $\mathcal{O}_H\mathcal{O}_L$ merge. The function $G(z,\bar{z})$ can be obtained from $\mathcal{G}(z,\bar{z})$ via

$$G(w,\bar{w}) = (w\bar{w})^{\Delta_L}\mathcal{G}(1-w,1-\bar{w}). \tag{6.7}$$

To apply the Lorentzian inversion formula we first need to calculate $\mathcal{G}(z,\bar{z})$ using the S-channel operator product expansion (2.22). First, let us start with the leading contribution of $\mathcal{G}(z,\bar{z})$ in the lightcone limit $\bar{z} \to 1$ at $\mathcal{O}(\mu^2)$. These give the leading contributions when $\bar{w} \to 0$ in $G(w,\bar{w})$. After the integration with respect to $\bar{w}$ in (6.2), these contributions fix the position of the pole and residue of $c(\tau,\beta)$ that corresponds to lowest-twist double-stress tensors. Subleading contributions in $\bar{z} \to 1$ (or $\bar{w} \to 0$) only create new poles, without changing the residue of existing ones, therefore, they do not affect the OPE coefficients of lowest-twist operators. The leading contribution in the $(1-\bar{z})$-expansion comes from the leading contribution of the $1/l$-expansion of the S-channel OPE

[11] In principle there is an extra term in this relation when $\tau - d = 0, 1, 2, \ldots$ [25], however, it vanishes in the cases considered.

data. Only the term proportional to $\log^2(z)$ contributes to the double-discontinuity and we denote it by $\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)}$. The number in the superscript denotes the power of μ in which we are working. Substituting in to (2.26) equations (2.24), (2.29), (2.27) and (2.28), we find that

$$\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)} = \log^2(z\bar{z}) \int_0^\infty dl \sum_{n=0}^{\infty} \frac{(z\bar{z})^n l^{\Delta_L-3}\left(z^{l+1}-\bar{z}^{l+1}\right)\Gamma\left(n+\Delta_L-1\right)}{8(z-\bar{z})\Gamma(n+1)\Gamma\left(\Delta_L-1\right)\Gamma\left(\Delta_L\right)} \times \left(\left(\gamma_n^{(1,0)}\right)^2 + \mathcal{O}\left(\frac{1}{l}\right)\right). \tag{6.8}$$

In the lightcone limit, the dominant contribution to this expression comes from operators with large spin $l \gg 1$, we can, therefore, approximate the sum over l by an integral. Note that only $\mathcal{O}(\mu)$ OPE data, i.e., $\gamma_n^{(1,0)}$, appears in (6.8). Using (3.12) we evaluate (6.8) and collect the leading term as $\bar{z} \to 1$,

$$\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)} = \log^2(z) \frac{(1-\bar{z})^{2-\Delta_L}(1-z)^{-\Delta_L-4}}{32\left(\Delta_L-2\right)} \times \Delta_L\left(\Delta_L\left((z(z+4)+1)^2\Delta_L + z(z(54-(z-28)z)+28)-1\right)+72z^2\right) + \mathcal{O}\left((1-\bar{z})^{3-\Delta_L}\right). \tag{6.9}$$

With the help of (6.7) one obtains

$$G^{(2)}(w,\bar{w})\big|_{\log^2(1-w)} = \frac{\Delta_L \bar{w}^2 \log^2(1-w)}{32 w^4(\Delta_L-2)} \times \left(\Delta_L\left(((w-6)w+6)^2\Delta_L - w(w(w(w+24)-132)+216)+108\right)+72(w-1)^2\right) + \mathcal{O}(\bar{w}^3), \tag{6.10}$$

which agrees with (4.12) in [27]. Now, it is easy to see that

$$\mathrm{dDisc}[G^{(2)}(w,\bar{w})] = \frac{\pi \bar{w}^2 \Delta_L}{8w^4(\Delta_L-2)} \times \left(\Delta_L\left(((w-6)w+6)^2\Delta_L - w(w(w(w+24)-132)+216)+108\right)+72(w-1)^2\right) + \mathcal{O}(\bar{w}^3). \tag{6.11}$$

To compute the integral (6.2) we substitute

$$\mu^{(0,0)}(w,\bar{w}) = \frac{1}{w^2\bar{w}^4} + \mathcal{O}\left(\frac{1}{\bar{w}^3}\right), \tag{6.12}$$

$$g^{(0,0)}_{-\tau+2(d-1),\frac{\tau+\beta}{2}-d+1}(w,\bar{w}) = \bar{w}^{3-\frac{\tau}{2}}\left(f_{\frac{\beta}{2}}(1-w)+\mathcal{O}(\bar{w})\right), \tag{6.13}$$

valid in the lightcone limit $\bar{w} \to 0$ (or $\bar{z} \to 1$), and set $(-1)^{\frac{\beta-\tau}{2}} = 1$ since only even-spin operators contribute. Combining the above we arrive at the following expression for $c(\tau,\beta)$

$$c_0(\tau,\beta) = -\frac{\sqrt{\pi}2^{-\beta+1}\Delta_L\Gamma\left(\frac{\beta}{2}\right)}{(\tau-4)(\beta-10)(\beta-6)(\beta-2)\beta(\beta+4)}\times \left(\frac{384\left(\Delta_L-7\right)\Delta_L+4608}{(\beta+8)\left(\Delta_L-2\right)\Gamma\left(\frac{1}{2}(\beta-1)\right)}+\frac{(\beta-2)\beta\Delta_L\left((\beta-2)\beta\left(\Delta_L-1\right)-56\Delta_L+200\right)}{(\beta+8)\left(\Delta_L-2\right)\Gamma\left(\frac{1}{2}(\beta-1)\right)}\right), \tag{6.14}$$

where the subscript denotes that this result is obtained in the leading order of the lightcone expansion. The OPE coefficients of the minimal-twist double-stress tensors are given by

$$P^{(2)}_{\frac{\beta}{2}+2,\frac{\beta}{2}-2} = -\mathrm{Res}_{\tau=4}c_0(\tau,\beta), \tag{6.15}$$

where $\beta = 12+4\ell$, $\ell \geq 0$, and are in precise agreement with (1.6) in [23] and (4.15) in [27].

6.2 Twist-six double-stress tensors

Here we use the same method to obtain the OPE coefficients of double-stress tensors with twist $\tau_{2,1} = 6$. We first need to compute the subleading contribution in the lightcone limit to eqs. (6.11), (6.12) and (6.13). Specifically, the integration measure

$$\mu^{(0,0)}(w,\bar{w}) = \frac{1}{w^2\bar{w}^4} - \frac{2}{w^3\bar{w}^3} + \mathcal{O}\left(\bar{w}^{-2}\right), \tag{6.16}$$

and the conformal block,

$$g^{(0,0)}_{-\tau+2(d-1),\frac{\tau+\beta}{2}-d+1}(w,\bar{w}) = \bar{w}^{3-\frac{\tau}{2}}f_{\frac{\beta}{2}}(1-w)\left(1+\bar{w}\left(1-\frac{\tau}{4}+\frac{1}{w}\right)+\mathcal{O}(\bar{w}^2)\right), \tag{6.17}$$

were obtained from the explicit expressions given in (6.3) and (2.5).

To evaluate the subleading term in $\mathrm{dDisc}[G^{(2)}(w,\bar{w})]$ we reconsider the S-channel computation. Similarly to the case of leading twist, only the part of the correlator with $\log^2(z)$ contributes to the discontinuity. However, we now have to include the subleading corrections in the $1/l$-expansion of the S-channel OPE data. With the help of (2.26), (2.24), (2.27), (2.28) and (2.29) one finds that

$$\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)} = \frac{\log^2(z\bar{z})}{16(z-\bar{z})\Gamma(\Delta_L)\Gamma(\Delta_L-1)}\sum_{n=0}^{\infty}(z\bar{z})^n\frac{\Gamma(\Delta_L-1+n)}{\Gamma(n+1)}\int_0^{\infty}dl \; l^{\Delta_L-6}\left(z^{l+1}-\bar{z}^{l+1}\right)\left(2(l-2n)+\Delta_L\left(\Delta_L+2n-1\right)\right)\left(l\gamma_n^{(1,0)}+\gamma_n^{(1,1)}\right)^2+\mathcal{O}\left(l^{\Delta_L-7}\right). \tag{6.18}$$

To proceed, one evaluates (6.18) using (3.12) and collects the leading and subleading contributions as $\bar{z}\to 1$, which behave as $(1-\bar{z})^{2-\Delta_L}$ and $(1-\bar{z})^{3-\Delta_L}$ respectively. Using (6.7) it is then simple to obtain $G^{(2)}(w,\bar{w})\big|_{\log^2(1-w)}$ up to $\mathcal{O}(\bar{w}^4)$ and evaluate its double-discontinuity:

$$\begin{aligned}\mathrm{dDisc}[G^{(2)}(w,\bar{w})] &= -\frac{\pi^2\bar{w}^2\Delta_L}{8w^5\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\Big(-3w^5\Delta_L-72w^4\Delta_L+324w^3\Delta_L\\
&-504w^2\Delta_L+252w\Delta_L+216w^3-432w^2+216w+4w^5\Delta_L^2-12w^4\Delta_L^2+12w^3\Delta_L^2\\
&-36w\Delta_L^3-w^5\Delta_L^3+12w^4\Delta_L^3-48w^3\Delta_L^3+72w^2\Delta_L^3+\bar{w}(-144\Delta_L+612w\Delta_L+216w^3\\
&-432w^2+216w-w^5\Delta_L-52w^4\Delta_L+324w^3\Delta_L-744w^2\Delta_L+540w\Delta_L^2-216\Delta_L^2\\
&-72\Delta_L^3+w^5\Delta_L^2-18w^4\Delta_L^2+156w^3\Delta_L^2-456w^2\Delta_L^2+144w\Delta_L^3-2w^4\Delta_L^3+24w^3\Delta_L^3\\
&-96w^2\Delta_L^3)\Big)+\mathcal{O}(\bar{w}^4)\,. \end{aligned} \tag{6.19}$$

Substituting (6.16), (6.17) and (6.19) in (6.2) and integrating leads to an analytic expression for $c(\tau,\beta)$. The relevant part of this expression — the one with non-zero residue at $\tau=6$ — turns out to be:

$$\begin{aligned}c_1(\tau,\beta) = &-\frac{2^{4-\beta}\sqrt{\pi}\Gamma\left(\frac{\beta}{2}\right)\Delta_L}{(\beta-12)(\beta-8)(\beta-4)(\tau-10)(\tau-8)(\tau-6)(\tau-4)}\\
&\times\left(\frac{\beta^4\Delta_L-4\beta^3\Delta_L-68\beta^2\Delta_L-960\beta\Delta_L^2+144\beta\Delta_L-14976\Delta_L^2}{(\beta+2)(\beta+6)(\beta+10)\Gamma\left(\frac{\beta-1}{2}\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\right.\\
&+\frac{\beta^4\Delta_L^3-2\beta^4\Delta_L^2-4\beta^3\Delta_L^3+8\beta^3\Delta_L^2-116\beta^2\Delta_L^3+472\beta^2\Delta_L^2}{(\beta+2)(\beta+6)(\beta+10)\Gamma\left(\frac{\beta-1}{2}\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\\
&\left.+\frac{240\beta\Delta_L^3+2304\Delta_L^3+19584\Delta_L+13824}{(\beta+2)(\beta+6)(\beta+10)\Gamma\left(\frac{\beta-1}{2}\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\right)+\ldots,\end{aligned} \tag{6.20}$$

where the ellipsis stands for the terms with zero residue at $\tau=6$ and 1 in the subscript denotes that this expression is obtained in the subleading order of the lightcone expansion.

It is now straightforward to read off the OPE coefficients of double-stress tensors with twist $\tau_{2,1}=6$ from

$$P^{(2)}_{\frac{\beta}{2}+3,\frac{\beta}{2}-3} = -\mathrm{Res}_{\tau=6}c_1(\tau,\beta). \tag{6.21}$$

For $\beta=14+4\ell$ (3.16) is reproduced. It is already stated in section 3 that this formula does not reproduce the right OPE coefficient $P^{(2)}_{8,2}$ for $\ell=-1$. Thus, we explicitly see that the Lorentzian inversion formula does not allow us to obtain the OPE data of spin-2 double-stress tensors with twist $\tau=6$.

In general, to determine for which operators at $\mathcal{O}(\mu^k)$ the Lorentzian inversion formula can be applied, one has to consider the behavior of the correlator in the Regge limit. At $\mathcal{O}(\mu^k)$ the correlator in the Regge limit behaves like $1/\sigma^{2\Delta_L+k}$. Therefore, the Lorentzian inversion formula correctly produces the OPE coefficients of multi-stress tensor operators with spin $s>k+1$. Accordingly, already at order $\mathcal{O}(\mu^3)$, fixing the OPE coefficients by combining an ansatz for the correlator with the crossing symmetry (or Lorentzian inversion formula) appears more powerful than the Lorentzian inversion formula alone. Namely, we

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were able to fix the OPE coefficients of spin-4 operators and the one with twist $\tau = 8$ is given by (D.1), while using the Lorentzian inversion formula one can only fix the OPE coefficients of operators with spin $s > 4$.

7 Discussion

In this paper, we consider the stress tensor sector of a four-point function of pairwise identical scalars in a class of CFTs with a large central charge. It is completely determined by the OPE coefficients of multi-stress tensor operators, which can be read off the result for a heavy-heavy-light-light correlator. The stress tensor sector of the HHLL correlator is naturally expanded perturbatively in $\mu \sim \frac{\Delta_H}{C_T}$, where Δ_H is the scaling dimension of the heavy operator. The power of μ counts the number of stress tensors within the exchanged multi-stress tensor operators. By further expanding the HHLL stress tensor sector in the lightcone limit, the multi-stress tensor operators can be organized into sectors of different twists. Similarly to the minimal-twist sector, combining an appropriate ansatz with the lightcone bootstrap, we show that the contribution from the non-minimal twist multi-stress tensors is almost completely determined. Unlike the minimal twist case, a few coefficients are not fixed by the bootstrap — these correspond to the OPE coefficients of multi-stress tensors with spin $s = 0, 2$.

An extra check is provided by applying the Lorentzian OPE inversion formula (see [27] for an earlier application of the inversion formula in this context). It gives the same results but has less predictive power than the ansatz.

The OPE coefficients for double-stress tensors are particularly simple and we provide closed-form expressions for those with twist $\tau = 4, 6, 8, 10$ and any spin greater than 2. All of these OPE coefficients are completely fixed by the bootstrap. This is related to their independence of the higher-derivative terms in the dual bulk gravitational Lagrangian. The OPE coefficients for double-stress tensors with spin $s = 0, 2$ are not fixed by the bootstrap and do depend on such higher derivative terms. It is interesting that at the level of double-stress tensors, only the OPE coefficients with spin $s = 0, 2$ are not fixed by the bootstrap (non-universal). On the other hand, all non-minimal twist triple-stress tensor OPE coefficients are non-universal.[12]

Assuming a holographic dual, we show that the OPE coefficients for spin-2 multi-stress tensors can be determined by studying the large impact parameter regime of the Regge limit, following [20, 30, 39] (modulo the spin zero OPE data). This is done explicitly in Einstein Hilbert+Gauss-Bonnet gravity. Some of these OPE coefficients are known [21] and agree with our results.

It would be interesting if one could compute the spin zero and spin two multi stress tensor OPE coefficients with CFT techniques. Perhaps the conglomeration approach first discussed in [31] or the more recent work [45, 46] will be useful in this direction.

The regime of applicability of the ansatz (and the exact meaning of universality) used in this paper remains unsettled (the ansatz seems to work in holographic CFTs, but does it

[12] Here we use universality and "fixed by the bootstrap" terms interchangeably. However, it remains to be determined what is the universality class and whether it the same as the set of unitary holographic theories.

also apply for other CFTs with a large central charge?). This question appears already in the leading twist case studied in [24]. To address this issue, it would be interesting to investigate the OPE coefficients of multi-stress tensors in CFTs with a large central charge, but not necessarily holographic. A related question is the existence of an infinite-dimensional algebra responsible for the form of the near-lightcone correlator. In two dimensions the relevant algebra is simply the Virasoro algebra. The Virasoro vacuum block has been computed in several ways [47–53]. Recently an algebraic way of reproducing the near lightcone contribution of the stress tensor was discussed in [54] — it would be interesting to investigate this further.

Returning to holographic theories, one interesting question would be to understand the critical behavior of geodesics in the vicinity of the circular light orbit, recently studied in [55], from the CFT point of view. This corresponds to the situation where the deflection angle is very large. The deflection angle φ in asymptotically flat Schwarzschild geometries is supposed to be related to the eikonal phase δ via

$$2\sin\frac{\varphi}{2} = -\frac{1}{E}\frac{\partial\delta}{\partial b} \tag{7.1}$$

where E is the incoming particle energy and b is the impact parameter (see e.g. [56] for a recent discussion). This agrees with eq. (E.1) for small deflection angles, but deviations might occur for large deflection angles. It would be interesting to investigate this further.

Acknowledgments

We thank G.S. Ng, K. Sen and A. Zhiboedov for useful discussions. The work of R.K. and A.P. is supported in part by an Irish Research Council Laureate Award. The work of P.T. is supported in part by an Ussher Fellowship Award.

A Linear relations between products of $f_a(z)$ functions

Here we list some linear relations between products of the $f_a(z)$ functions used in the main text.

$$f_1(z)f_4(z) + \frac{1}{15}f_3(z)f_4(z) - \frac{4}{63}f_2(z)f_5(z) - f_2(z)f_3(z) = 0, \tag{A.1}$$

$$\begin{aligned}\frac{308}{25}f_2^2(z) - \frac{308}{25}f_1(z)f_3(z) + \frac{5929}{375}f_3^2(z) - \frac{2673}{2500}f_4^2(z) - \frac{396}{25}f_1(z)f_5(z) + f_2(z)f_6(z) = 0,\\ 245f_2^2(z) - 245f_1(z)f_3(z) - \frac{7}{12}f_3^2(z) - \frac{81}{80}f_4^2(z) + f_3(z)f_5(z) = 0,\\ \frac{140}{9}f_2^2(z) - \frac{140}{9}f_1(z)f_3(z) - \frac{28}{27}f_3^2(z) + f_2(z)f_4(z) = 0,\end{aligned} \tag{A.2}$$

$$\frac{3991680}{16000}f_2(z)f_3(z) - \frac{99}{125}f_4(z)f_3(z) + f_6(z)f_3(z) - \frac{6237}{25}f_1(z)f_4(z) - \frac{891}{875}f_4(z)f_5(z) = 0,$$

$$f_2(z)f_7(z)+\frac{7007}{500}f_2(z)f_3(z)+\frac{39611}{2500}f_4(z)f_3(z)-\frac{7007}{500}f_1(z)f_4(z)-\frac{4719}{4375}f_4(z)f_5(z)-\frac{143}{9}f_1(z)f_6(z)=0, \tag{A.3}$$

$$-\frac{1}{15}f_6(z)f_2(z)^2+\frac{297}{4375}f_4(z)^2f_2(z)+f_1(z)f_5(z)f_2(z)+\frac{44}{625}f_3(z)f_5(z)f_2(z)+\frac{9}{143}f_1(z)f_7(z)f_2(z)-\frac{44}{625}f_3(z)^2f_4(z)-\frac{297}{4375}f_1(z)f_4(z)f_5(z)-f_1(z)f_1(z)f_6(z)=0, \tag{A.4}$$

$$-f_6(z)f_1(z)^2+f_3(z)f_4(z)f_1(z)-\frac{297}{4375}f_4(z)f_5(z)f_1(z)+\frac{9}{143}f_2(z)f_7(z)f_1(z)+\frac{9}{2500}f_2(z)f_4(z)^2-\frac{7}{1875}f_3(z)^2f_4(z)+\frac{7}{1875}f_2(z)f_3(z)f_5(z)-\frac{7}{1980}f_2(z)^2f_6(z)=0, \tag{A.5}$$

$$-f_6(z)f_1(z)^2+\frac{9}{143}f_2(z)f_7(z)f_1(z)-\frac{297}{4375}f_4(z)f_5(z)f_1(z)+\frac{297}{4375}f_2(z)f_4(z)^2+f_2(z)^2f_4(z)-\frac{44}{625}f_3(z)^2f_4(z)+\frac{7}{1875}f_2(z)f_3(z)f_5(z)-\frac{7}{1980}f_2(z)^2f_6(z)=0, \tag{A.6}$$

$$-f_6(z)f_1(z)^2+\frac{9}{143}f_2(z)f_7(z)f_1(z)-\frac{297}{4375}f_4(z)f_5(z)f_1(z)+f_2(z)f_3(z)^2+\frac{9}{2500}f_2(z)f_4(z)^2-\frac{44}{625}f_3(z)^2f_4(z)+\frac{2647}{39375}f_2(z)f_3(z)f_5(z)-\frac{7}{1980}f_2(z)^2f_6(z)=0, \tag{A.7}$$

$$-f_6(z)f_2(z)^2+\frac{891}{875}f_4(z)^2f_2(z)+\frac{132}{125}f_3(z)f_5(z)f_2(z)-\frac{132}{125}f_3(z)^2f_4(z)-\frac{891}{875}f_1(z)f_4(z)f_5(z)+f_1(z)f_3(z)f_6(z)=0. \tag{A.8}$$

B Coefficients in $\mathcal{G}^{(3,1)}(z)$

Here we list the coefficients in $\mathcal{G}^{(3,1)}(z)$:

$$b_{116}=-\frac{\Delta_L(\Delta_L+3)(\Delta_L(\Delta_L(\Delta_L(1001\Delta_L+387)-4326)+13828)+5040)}{10378368000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}+\frac{b_{14}(\Delta_L(143\Delta_L+427)+540)}{17160(\Delta_L-4)},$$

$$c_{118}=\frac{7(\Delta_L+3)\left(604800b_{14}\left(\Delta_L^2-5\Delta_L+6\right)+\Delta_L\left(-21\Delta_L^3+229\Delta_L^2+414\Delta_L+284\right)\right)}{856627200\left(\Delta_L^3-9\Delta_L^2+26\Delta_L-24\right)},$$

$$c_{127}=\frac{\Delta_L(\Delta_L(\Delta_L(\Delta_L(\Delta_L(14\Delta_L-15)+6040)-36125)-75814)-49620)}{2306304000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}-\frac{3b_{14}(\Delta_L(2\Delta_L+3)+135)}{11440(\Delta_L-4)},$$

$$c_{145}=\frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left((32680-1183\Delta_L)\,\Delta_L-183605\right)+34900\right)+570808\right)+436440\right)}{47040000000\,(\Delta_L-4)\,(\Delta_L-3)\,(\Delta_L-2)}+\frac{3b_{14}\left(\Delta_L\left(257\Delta_L-2227\right)+510\right)}{700000\,(\Delta_L-4)},$$

$$c_{226}=\frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left((40020-1337\Delta_L)\,\Delta_L-274845\right)+96350\right)+2323212\right)+1910160\right)}{71850240000\,(\Delta_L-4)\,(\Delta_L-3)\,(\Delta_L-2)}+\frac{b_{14}\left(\Delta_L\left(22\Delta_L-267\right)+960\right)}{39600\,(\Delta_L-4)},$$

$$c_{235}=\frac{b_{14}\left((10283-1153\Delta_L)\,\Delta_L-5790\right)}{900000\,(\Delta_L-4)}+\frac{\Delta_L\left(51463\Delta_L^5-846480\Delta_L^4+1320405\Delta_L^3\right)}{1632960000000\left(\Delta_L^3-9\Delta_L^2+26\Delta_L-24\right)}+\frac{\Delta_L\left(22381100\Delta_L^2-46886088\Delta_L-46446840\right)}{1632960000000\left(\Delta_L^3-9\Delta_L^2+26\Delta_L-24\right)},$$

$$c_{244}=\frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(1337\Delta_L-32145\right)+160095\right)+19525\right)-266712\right)-182160\right)}{70560000000\,(\Delta_L-4)\,(\Delta_L-3)\,(\Delta_L-2)}+\frac{9b_{14}\left(\Delta_L\left(71-11\Delta_L\right)+270\right)}{175000\,(\Delta_L-4)},$$

$$c_{334}=\frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(509\Delta_L-1515\right)+83415\right)-808325\right)+823116\right)+902880\right)}{90720000000\,(\Delta_L-4)\,(\Delta_L-3)\,(\Delta_L-2)}+\frac{b_{14}\left(\Delta_L\left(11\Delta_L-71\right)-270\right)}{18750\,(\Delta_L-4)}. \tag{B.1}$$

C Coefficients in $\mathcal{G}^{(3,2)}(z)$

Here we list the coefficients in $\mathcal{G}^{(3,2)}(z)$:

$$g_{119}=\frac{g_{13}\left(7\Delta_L\left(128-77\Delta_L\right)+6720\right)}{16409250\,(\Delta_L-5)}+\frac{49b_{14}\left(\Delta_L\left(\Delta_L\left(170-11\Delta_L\right)+981\right)+1620\right)}{16409250\,(\Delta_L-5)\,(\Delta_L-4)}+\frac{196e_{115}}{49725}+\frac{539\Delta_L^7-15386\Delta_L^6+54215\Delta_L^5+951510\Delta_L^4+2911426\Delta_L^3}{472586400000\,(\Delta_L-5)\,(\Delta_L-4)\,(\Delta_L-3)\,(\Delta_L-2)}+\frac{98e_{15}\left(\Delta_L+4\right)}{16575\,(\Delta_L-5)}+\frac{3737076\Delta_L^2+1779120\Delta_L}{472586400000\,(\Delta_L-5)\,(\Delta_L-4)\,(\Delta_L-3)\,(\Delta_L-2)},$$

$$g_{128}=-\frac{7g_{13}\left(\Delta_L\left(4\Delta_L-469\right)+930\right)}{12355200\,(\Delta_L-5)}-\frac{7b_{14}\left(\Delta_L\left(22\Delta_L^2-64\Delta_L+4197\right)+11745\right)}{6177600\,(\Delta_L-5)\,(\Delta_L-4)}+\frac{462\Delta_L^7-24203\Delta_L^6+1044630\Delta_L^5-3466005\Delta_L^4-24181012\Delta_L^3-39855972\Delta_L^2}{1779148800000\,(\Delta_L-5)\,(\Delta_L-4)\,(\Delta_L-3)\,(\Delta_L-2)}-\frac{49e_{15}\left(\Delta_L\left(\Delta_L+2\right)+102\right)}{93600\,(\Delta_L-5)}-\frac{61201\Delta_L}{4942080000\,(\Delta_L-5)\,(\Delta_L-4)\,(\Delta_L-3)\,(\Delta_L-2)},$$

$$g_{155}=\frac{11e_{15}\left(\Delta_L\left(278\Delta_L-2789\right)+126\right)}{2756250\,(\Delta_L-5)}+\frac{11g_{13}\left(\Delta_L\left(2279\Delta_L-7400\right)-8370\right)}{231525000\,(\Delta_L-5)}-\frac{3146e_{115}}{275625}+\frac{b_{14}\left(12063\Delta_L^3-88048\Delta_L^2-131165\Delta_L+196110\right)}{77175000\,(\Delta_L-5)\,(\Delta_L-4)}$$

$$
\begin{aligned}
&+\frac{-244401285\Delta_L^4+853023786\Delta_L^3+2178372216\Delta_L^2+1399907880\Delta_L}{233377200000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\\
&+\frac{-1406986\Delta_L^7+28367309\Delta_L^6-123035140\Delta_L^5}{233377200000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
g_{227}=&\frac{e_{15}\left(\Delta_L\left(52\Delta_L-751\right)+3234\right)}{93600\left(\Delta_L-5\right)}-\frac{e_{115}}{240}+\frac{g_{13}\left(\Delta_L\left(1051\Delta_L-12370\right)-52530\right)}{86486400\left(\Delta_L-5\right)}\\
&+\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(3131\Delta_L-33896\right)-62985\right)+1236870\right)}{86486400\left(\Delta_L-5\right)\left(\Delta_L-4\right)}\\
&+\frac{-213549\Delta_L^7+6031106\Delta_L^6-23990385\Delta_L^5-205647690\Delta_L^4}{87178291200000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\\
&+\frac{853227874\Delta_L^3+2135805744\Delta_L^2+1445776920\Delta_L}{87178291200000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
g_{236}=&\frac{e_{15}\left(\left(15074-1223\Delta_L\right)\Delta_L-39816\right)}{6804000\left(\Delta_L-5\right)}+\frac{g_{13}\left(\Delta_L\left(186926\Delta_L-1951295\right)+5891220\right)}{6286896000\left(\Delta_L-5\right)}\\
&+\frac{143e_{115}}{340200}+\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(23001\Delta_L-469741\right)+3383740\right)-7782480\right)}{1047816000\left(\Delta_L-5\right)\left(\Delta_L-4\right)}\\
&-\frac{9324749\Delta_L^7-433851406\Delta_L^6+5233472135\Delta_L^5-21967190310\Delta_L^4}{6337191168000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\\
&-\frac{10644674676\Delta_L^3+72859312056\Delta_L^2+65903302080\Delta_L}{6337191168000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \qquad \text{(C.1)}\\
g_{245}=&-\frac{99e_{15}\left(\Delta_L\left(83\Delta_L-754\right)-1064\right)}{4900000\left(\Delta_L-5\right)}+\frac{g_{13}\left(73\Delta_L\left(275-274\Delta_L\right)+170060\right)}{137200000\left(\Delta_L-5\right)}\\
&+\frac{5577e_{115}}{245000}+\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(79801-14981\Delta_L\right)+410980\right)-55320\right)}{68600000\left(\Delta_L-5\right)\left(\Delta_L-4\right)}\\
&+\frac{1300313\Delta_L^7-22489422\Delta_L^6+63989995\Delta_L^5+399569530\Delta_L^4}{138297600000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\\
&+\frac{-690996588\Delta_L^3-2276065528\Delta_L^2-1491467040\Delta_L}{138297600000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
g_{335}=&\frac{1144e_{115}}{5315625}+\frac{g_{13}\left(\Delta_L\left(6426275-894839\Delta_L\right)+685170\right)}{17860500000\left(\Delta_L-5\right)}\\
&-\frac{11e_{15}\left(\Delta_L\left(11143\Delta_L-143659\right)+451206\right)}{212625000\left(\Delta_L-5\right)}\\
&-\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(446853\Delta_L-4788638\right)+4992635\right)+44234910\right)}{5953500000\left(\Delta_L-5\right)\left(\Delta_L-4\right)}\\
&+\frac{43544683\Delta_L^7-877022702\Delta_L^6+4877336920\Delta_L^5-1356232020\Delta_L^4}{9001692000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\\
&+\frac{-28767381333\Delta_L^3-34411007748\Delta_L^2-12217009140\Delta_L}{9001692000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},
\end{aligned}
$$

$$g_{344}=\frac{11e_{15}\left(\Delta_L\left(278\Delta_L-2789\right)+126\right)}{2625000\left(\Delta_L-5\right)}+\frac{g_{13}\left(\Delta_L\left(17194\Delta_L-10525\right)-249570\right)}{220500000\left(\Delta_L-5\right)}$$
$$-\frac{1573e_{115}}{131250}+\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(9438\Delta_L-48673\right)-325415\right)+511110\right)}{73500000\left(\Delta_L-5\right)\left(\Delta_L-4\right)}$$
$$+\frac{-1593347\Delta_L^7+27045868\Delta_L^6-6670280\Delta_L^5-1193221320\Delta_L^4}{444528000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}$$
$$+\frac{1878076947\Delta_L^3+5698801932\Delta_L^2+3877115760\Delta_L}{444528000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \tag{C.2}$$

$$d_{117}=-\frac{9}{220}e_{115}+\frac{84+\Delta_L(53+13\Delta_L)}{1560(\Delta_L-5)}e_{15}+\frac{13\Delta_L\left(209\Delta_L+409\right)+8340}{7207200\left(\Delta_L-5\right)}g_{13}$$
$$-\frac{4641\Delta_L^7+22727\Delta_L^6+44901\Delta_L^5+67569\Delta_L^4+519742\Delta_L^3}{290594304000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}$$
$$-\frac{828876\Delta_L^2+333648\Delta_L}{290594304000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}$$
$$+\frac{\Delta_L\left(\Delta_L\left(5317\Delta_L+18140\right)+68763\right)+69660}{7207200\left(\Delta_L-5\right)\left(\Delta_L-4\right)}b_{14}. \tag{C.3}$$

D OPE coefficients of twist-eight triple-stress tensors

Here we list a few OPE coefficients of twist-eight triple-stress tensors which are found using (4.8):

$$P^{(3)}_{12,4}=\frac{P^{(2)}_{8,2}\left(\Delta_L\left(143\Delta_L+427\right)+540\right)}{17160\left(\Delta_L-4\right)}$$
$$-\frac{1001\Delta_L^6+3390\Delta_L^5-3165\Delta_L^4+850\Delta_L^3+46524\Delta_L^2+15120\Delta_L}{10378368000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \tag{D.1}$$

$$P^{(3)}_{14,6}=\frac{9P^{(2)}_{8,2}\left(\Delta_L\left(13\Delta_L+11\right)+12\right)}{544544\left(\Delta_L-4\right)}$$
$$+\frac{7917\Delta_L^6+38174\Delta_L^5+140795\Delta_L^4+266390\Delta_L^3+253908\Delta_L^2+97776\Delta_L}{548900352000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \tag{D.2}$$

$$P^{(3)}_{16,8}=\frac{5P^{(2)}_{8,2}\left(\Delta_L\left(17\Delta_L+2\right)+6\right)}{9876048\left(\Delta_L-4\right)}$$
$$+\frac{362593\Delta_L^6+881129\Delta_L^5+2782307\Delta_L^4+4155839\Delta_L^3+3518084\Delta_L^2+1198176\Delta_L}{438022480896000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \tag{D.3}$$

$$P^{(3)}_{18,10}=\frac{P^{(2)}_{8,2}\left(\Delta_L\left(323\Delta_L-77\right)+54\right)}{823727520\left(\Delta_L-4\right)}+\frac{17413253\Delta_L^6+23717684\Delta_L^5+79039447\Delta_L^4}{377794389772800000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}$$
$$+\frac{92754344\Delta_L^3+73231064\Delta_L^2+22535496\Delta_L}{377794389772800000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}. \tag{D.4}$$

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Assuming Einstein-Hilbert + Gauss-Bonnet gravity in the bulk, the OPE coefficient $P^{(2)}_{8,2}$ was derived in (5.48) and can be inserted in (D.1)–(D.4).

E Derivation of the deflection angle from the phase shift

Here we simply show that the bulk phase shift, defined as $\delta = p^t(\Delta t) - p^\phi(\Delta\phi)$ in [20] is consistent with the standard equation relating the eikonal phase and the scattering angle

$$\frac{\partial\delta}{\partial b} = -p^t\,\Delta\phi \tag{E.1}$$

obtained with the use of the stationary phase approximation for small scattering angles. Our discussion is focused on asymptotically flat space. In this case, the formulas in classical gravity which provide the deflection angle and the time delay are:

$$\begin{aligned} \Delta t &= 2\int_{r_0}^{\infty} \frac{dr}{f\sqrt{1-\frac{b^2 f}{r^2}}} \\ \Delta\phi &= 2b\int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{1-\frac{b^2 f}{r^2}}}\,. \end{aligned} \tag{E.2}$$

They can be obtained from eq. (2.9) in [20] with the substitution $\frac{p^\phi}{p^t} = b$ (and the appropriate definition of the blackening factor $f(r)$). Note that the equation for the turning point of the geodesic, r_0, reduces in Schwarzchild geometry to:

$$1 - \frac{b^2}{r^2 f(r_0)} = 0 \tag{E.3}$$

Defining the bulk phase shift via $\delta = p^t(\Delta t) - p^\phi(\Delta\phi)$, leads to

$$\delta = p^t(\Delta t) - p^\phi(\Delta\phi) = p^t\left(\Delta t - b\Delta\phi\right) = 2p^t\int_{r_0}^{\infty}\frac{dr}{f}\sqrt{1-\frac{b^2 f}{r^2}}\,. \tag{E.4}$$

Differentiating the bulk phase shift with respect to the impact parameter yields:

$$\frac{\partial\delta}{\partial b} = -2p^t\,b\int_{r_0}^{\infty}\frac{dr}{r^2\sqrt{1-\frac{b^2 f}{r^2}}} - 2p^t\frac{1}{f(r_0)}\sqrt{1-\frac{b^2 f(r_0)}{r_0^2}} = -p^t(\Delta\phi)\,, \tag{E.5}$$

where to arrive at the last equality we used the equation satisfied by the turning point r_0. Hence,

$$\Delta\phi = -\frac{1}{p^t}\frac{\partial\delta}{\partial b}\,. \tag{E.6}$$

Finally note that assuming the classical relation $J \equiv p_\phi = b\,p^t$, the deflection angle can also be computed through

$$\Delta\phi = -\frac{\partial\delta}{\partial J}\,. \tag{E.7}$$

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Published for SISSA by Springer
Received: June 4, 2021
Accepted: September 7, 2021
Published: September 29, 2021

JHEP09(2021)205

Thermalization in large-N CFTs

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Abstract: In d-dimensional CFTs with a large number of degrees of freedom an important set of operators consists of the stress tensor and its products, multi stress tensors. Thermalization of such operators, the equality between their expectation values in heavy states and at finite temperature, is equivalent to a universal behavior of their OPE coefficients with a pair of identical heavy operators. We verify this behavior in a number of examples which include holographic and free CFTs and provide a bootstrap argument for the general case. In a free CFT we check the thermalization of multi stress tensor operators directly and also confirm the equality between the contributions of multi stress tensors to heavy-heavy-light-light correlators and to the corresponding thermal light-light two-point functions by disentangling the contributions of other light operators. Unlike multi stress tensors, these light operators violate the Eigenstate Thermalization Hypothesis and do not thermalize.

Keywords: $1/N$ Expansion, AdS-CFT Correspondence, Conformal Field Theory

ArXiv ePrint: 2102.04953

https://doi.org/10.1007/JHEP09(2021)205

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1 Introduction and summary

Holography [1–3] provides us with a useful tool to study d-dimensional CFTs at large central charge C_T, especially when combined with modern CFT techniques (see e.g. [4–6] for reviews). One of the basic objects in this setup is a Witten diagram with a single graviton exchange which contributes to four-point functions. It can be decomposed into the conformal blocks of the stress-tensor and of the double-trace operators made out of external fields [7].

When a pair of the external operators denoted by $\mathcal{O}_H$ is taken to be heavy, with the conformal dimension $\Delta_H \sim C_T$, and the other pair denoted by $\mathcal{O}_L$ stays light, the resulting heavy-heavy-light-light (HHLL) correlator describes a light probe interacting with a heavy state. In this case, operators which are comprised out of many stress tensors (multi stress tensor operators) contribute, together with the multi-trace operators involving $\mathcal{O}_L$. As we review below, the OPE coefficients of the scalar operators with a (unit-normalized) multi stress tensor operator $T^k_{\tau,s}$, which contains k stress tensors and has twist τ and spin s, scale like $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^k_{\tau,s}} \sim \Delta^k / C_T^{k/2}$ for large Δ.

The contribution of a given multi stress tensor operator to the HHLL four-point function $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$ can be compared to the contribution of the same operator to the corresponding two-point function at finite temperature[1] β^{-1}, $\langle \mathcal{O}_L \mathcal{O}_L \rangle_\beta$. In this paper we argue that they are the same in generic large-C_T CFTs. As we explain later, this means that OPE coefficients of $T^k_{\tau,s}$ with the two heavy operators $\mathcal{O}_H$, $\langle \mathcal{O}_H T^k_{\tau,s} \mathcal{O}_H \rangle$, are equal to their finite temperature expectation values, $\langle T^k_{\tau,s} \rangle_\beta$. The relation between the inverse temperature β and the conformal dimension Δ_H is set by considering the stress tensor ($k=1, \tau = d-2, s=2$), but the equality between the thermal expectation values and the OPE coefficients for all other multi stress tensor operators is a nontrivial statement. We call it "the thermalization of the stress tensor sector".[2] It is directly related to the Eigenstate Thermalization Hypothesis (ETH) [26–30], as we review below. Hence, we argue that all multi stress tensor operators in the large-C_T CFTs satisfy the ETH. In $d=2$ the ETH and thermalization have been studied in e.g. [31–61].

[1] See [8–25] for some previous work on finite temperature conformal field theories in $d > 2$.

[2] We show this explicitly for certain primary heavy operators $\mathcal{O}_H$ in free CFTs. We also observe that other light operators do not satisfy the thermalization property that the stress tensor sector enjoys.

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Here we want to address the $d > 2$ case. In holographic theories CFT and bootstrap techniques provide a lot of data which indicates that the thermalization of the stress tensor sector happens [62–73]. Some of the OPE coefficients in holographic CFTs were computed using two-point functions in a black hole background [63] — these are thermal correlators according to the standard holographic dictionary. It is also worth noting that the leading Δ behavior of the OPE coefficients in holographic models does not depend on the coefficients of the higher derivative terms in the bulk lagrangian [71] (this should not be confused with the universality of the OPE coefficients of the minimal-twist multi stress tensors [63]). Such a universality follows from the thermalization of the stress tensor sector as we discuss below.

A natural question is whether the thermalization of the stress tensor sector is just a property of holographic CFTs or if it holds more generally. In this paper we argue for the latter scenario. We compute the OPE coefficients (and the thermal expectation values) for a number of multi stress tensor operators in a free CFT and observe thermalization as well as universality of OPE coefficients. We also provide a bootstrap argument for all CFTs with a large central charge.

The rest of the paper is organized as follows. In section 2, we begin by considering the thermalization of multi stress tensor operators $T^k_{\tau,s}$. The heavy state we consider is created by a scalar operator $\mathcal{O}_H$ with dimension $\Delta_H \sim C_T$ and by thermalization of a multi stress tensor operator we mean[3]

$$\langle \mathcal{O}_H | T^k_{\tau,s} | \mathcal{O}_H \rangle \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{\tau,s}} \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \langle T^k_{\tau,s} \rangle_\beta, \tag{1.1}$$

where the heavy state $|\mathcal{O}_H\rangle$ on the sphere of unit radius is created by the operator $\mathcal{O}_H$, $\lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{\tau,s}}$ are the OPE coefficients of $T^k_{\tau,s}$ in the $\mathcal{O}_H \times \mathcal{O}_H$ OPE and $|_{\Delta_H^k/C_T^{k/2}}$ means we keep only leading terms that scale like $\Delta_H^k/C_T^{k/2} \sim C_T^{k/2}$. In (1.1) $\langle T^k_{\tau,s} \rangle_\beta$ is the one-point function on the sphere at finite temperature β^{-1}. Note that the OPE coefficients involving the stress tensor are fixed by the Ward identity, and hence eq. (1.1) for the stress tensor establishes a relation between the temperature β^{-1} and Δ_H. By the large-C_T factorization,[4] the thermal one-point functions of multi stress tensors can be related to the thermal one-point function of the stress tensor itself. Explicitly,

$$\langle T^k_{\tau,s} \rangle_\beta = c^k_{\tau,s} (\langle T^1_{d-2,2} \rangle_\beta)^k = c^k_{\tau,s} \left(\lambda_{\mathcal{O}_H \mathcal{O}_H T^1_{d-2,2}} \right)^k, \tag{1.2}$$

where $c^k_{\tau,s}$ are theory-independent coefficients that appear because of the index structure in $\langle T^k_{\tau,s} \rangle_\beta$. In the second equality in (1.2) we used (1.1) for the stress tensor. Note that (1.1) and (1.2) imply that the leading Δ_H behavior of the multi stress tensor OPE coefficients is universal, i.e. it does not depend on the theory.[5] We provide a bootstrap argument for

[3] Here we are suppressing the tensor structure. Note that all terms scale like $C_T^{k/2}$ which is consistent with $T^k_{\tau,s}$ being unit-normalized.

[4] See [74] for a general discussion of large-N factorization and [75, 76] and [8] for the discussion in the context of gauge theories and CFTs respectively. The factorization holds in adjoint models in the 't Hooft limit at finite temperature, but there are counterexamples, like e.g. a direct product of low-C_T CFTs. However the factorization of multi stress tensors would still apply in these models.

[5] This amounts to the large-C_T factorization of correlators $\langle \mathcal{O}_H | T_{\mu\nu} \dots T_{\alpha\beta} | \mathcal{O}_H \rangle$ in heavy states.

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this universality in all large-C_T theories. Also note that (1.2) is written for multi-trace operators $T^k_{\tau,s}$ which do not contain derivatives, but the presence of derivatives does not affect the statement of universality.

In section 3, we check the universality by computing a number of the multi stress tensor OPE coefficients in a free SU(N) adjoint scalar theory in $d = 4$ dimensions. We compare the leading Δ_H behavior in the free theory with results from holography/bootstrap and find perfect agreement in all cases listed below. After fixing the coefficients for the stress tensor case in section 3.1, we look at the first nontrivial case, $T^2_{4,4}$ in section 3.2. Section 3.3 is devoted to the double stress tensor with two derivatives, $T^2_{4,6}$. This is an operator whose finite temperature expectation value vanishes in the large volume limit (on the plane), but is finite on the sphere. In section 3.4 we consider minimal twist multi stress tensors of the type $T^k_{2k,2k}$. Section 3.5 is devoted to multi stress tensors with non-minimal twist, $T^2_{6,2}$ and $T^2_{8,0}$.

In section 4, we verify that (1.1) holds in the free adjoint scalar theory for a variety of operators. In this section we again consider $d = 4$, but in addition, take the infinite volume limit. This is for technical reasons — it is easier to compute a finite temperature expectation value on the plane than on the sphere. We spell out the index structure in (1.1) in detail and go over all the examples discussed in the previous section. In addition, we discuss some triple stress tensor operators.

We continue in section 5 by studying thermal two-point functions in the free adjoint scalar model in $d = 4$. By decomposing the correlator into thermal blocks we read off the product of thermal one-point functions and the OPE coefficients for several operators of low dimension and observe agreement with the results of sections 3 and 4. Due to the presence of multiple operators with the same dimension and spin, we have to solve a mixing problem to find which operators contribute to the thermal two-point function.

In section 6 we explain the relation between our results and the Eigenstate Thermalization Hypothesis. We observe that unlike multi stress tensors, other light operators explicitly violate the Eigenstate Thermalization Hypothesis and do not thermalize. We end with a discussion in section 7.

Appendices A, B, and C contain explicit calculations of OPE coefficients while in appendices D and E thermal one-point functions are calculated. In appendix F we review the statement that the thermal one-point functions of multi-trace operators with derivatives vanish on $S^1 \times \mathbf{R}^{d-1}$. In appendix G we study a free scalar in two dimensions and calculate thermal two-point functions of certain quasi-primary operators. In appendix H we consider a free scalar vector model in four dimensions. Appendix I discusses the factorization of multi-trace operators in the large volume limit.

2 Thermalization and universality

In the following we consider large-C_T CFTs on a $(d-1)$-dimensional sphere of radius R, which we set to unity for most of this section. As reviewed in [71], the stress tensor sector of conformal four-point functions consists of the contributions of the stress tensor and all its composites (multi stress tensors). The HHLL correlators we consider involve two heavy operators inserted at $x^0_E = \pm\infty$ and two light operators inserted on the Euclidean cylinder, with angular separation φ and time separation x_E^0. The correlator in a heavy state (the HHLL correlator on the cylinder) is related to the correlator on the plane by a conformal transformation

$$\langle\mathcal{O}_H|\mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)|\mathcal{O}_H\rangle = \lim_{x_4\to\infty} x_4^{2\Delta_H}(z\bar{z})^{-\Delta/2}\langle\mathcal{O}_H(x_4)\mathcal{O}(1)\mathcal{O}(z,\bar{z})\mathcal{O}_H(0)\rangle, \tag{2.1}$$

where the cross-ratios $(z,\bar{z})$ on the plane are related to the coordinates (x_E^0,φ) via

$$z = e^{-x_E^0 - i\varphi}, \qquad \bar{z} = e^{-x_E^0 + i\varphi}. \tag{2.2}$$

The stress tensor sector of the HHLL correlator is given by

$$\mathcal{G}(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H}\langle\mathcal{O}_H(x_4)\mathcal{O}(1)\mathcal{O}(z,\bar{z})\mathcal{O}_H(0)\rangle\Big|_{\text{multi stress tensors}} \tag{2.3}$$

and can be expanded in conformal blocks

$$\mathcal{G}(z,\bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta}}\sum_{T^k_{\tau,s}} P^{(HH,LL)}_{T^k_{\tau,s}}\, g^{(0,0)}_{\tau,s}(1-z,1-\bar{z}), \tag{2.4}$$

where τ, s, k label the twist, spin, and multiplicity of multi stress tensors. We are interested in the double scaling limit where the central charge and the dimension of $\mathcal{O}_H$ are large, $C_T, \Delta_H \to \infty$ with their ratio $\mu \propto \Delta_H/C_T$ fixed. In this limit the products of the OPE coefficients which appear in (2.4) are given by

$$P^{(HH,LL)}_{T^k_{\tau,s}} = \left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}}\lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^k}, \tag{2.5}$$

where we only keep the leading, $\left(\frac{\Delta_H}{\sqrt{C_T}}\right)^k$ term in the OPE coefficients $\lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}$, but retain all terms in the OPE coefficients of the light operators $\lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}}$. The contribution of the conformal family of a multi stress operator $T^k_{\tau,s}$ to the HHLL correlator is therefore

$$\langle\mathcal{O}_H|\mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)|\mathcal{O}_H\rangle|_{T^k_{\tau,s}} = \frac{P^{(HH,LL)}_{T^k_{\tau,s}}\, g^{(0,0)}_{\tau,s}(1-z,1-\bar{z})}{[\sqrt{z\bar{z}}(1-z)(1-\bar{z})]^{\Delta}}. \tag{2.6}$$

We now consider these CFTs at finite temperature β^{-1}. To isolate the contribution of the conformal family associated with $T^k_{\tau,s}$, we can write the thermal correlator as

$$\begin{aligned}\langle\mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)\rangle_\beta &= \frac{1}{Z(\beta)}\sum_i e^{-\beta\Delta_i}\langle\mathcal{O}_i|\mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)|\mathcal{O}_i\rangle \\ &= \frac{1}{[\sqrt{z\bar{z}}(1-z)(1-\bar{z})]^{\Delta}}\sum_{T^k_{\tau,s}}\left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}}\, g^{(0,0)}_{\tau,s}(1-z,1-\bar{z})\ \langle T^k_{\tau,s}\rangle_\beta \\ &\quad + \dots,\end{aligned} \tag{2.7}$$

where

$$\langle T^k_{\tau,s}\rangle_\beta = \frac{1}{Z(\beta)}\sum_i e^{-\beta\Delta_i}\lambda_{\mathcal{O}_i\mathcal{O}_i T^k_{\tau,s}} \tag{2.8}$$

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is the finite temperature one-point function on the sphere of the $T^k_{\tau,s}$ operator and the dots denote contributions from other operators. In (2.8) $Z(\beta)$ is the partition function and the sum runs over all operators, including descendants.[6] Note that

$$\langle T^k_{\tau,s}\rangle_\beta = \beta^{-(\tau+s)} f^k_{\tau,s}(\beta). \tag{2.9}$$

Here and below the indices are suppressed (see e.g. [17] for the explicit form) and $f^k_{\tau,s}(\beta) \sim C_T^{k/2}$ is a theory-dependent nontrivial function of β which approaches a constant $f^k_{\tau,s}(0)$ in the large volume ($\beta \to 0$) limit.

Consider the thermalization of the stress tensor sector:

$$\langle \mathcal{O}_H | T^k_{\tau,s} | \mathcal{O}_H \rangle \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{\tau,s}} \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \langle T^k_{\tau,s}\rangle_\beta. \tag{2.10}$$

Note that $T^k_{\tau,s}$ is unit-normalized, so all terms in (2.10) scale like $C_T^{k/2}$. Eq. (2.10) implies the equality between (2.6) and the corresponding term in (2.7). Note that the left-hand side of (2.10) is a function of the energy density while the right-hand side is a function of temperature. The relationship is fixed by considering the stress tensor case: the corresponding function $f^1_{d-2,2}(\beta)$ is determined by the free energy on the sphere (see section 6).

In the following, we will first discuss the case where the multi stress operators $T^k_{\tau,s}$ do not have any derivatives inserted, and then show that the derivatives do not change the conclusions. Assuming large-C_T factorization, the leading C_T behavior of $\langle T^k_{\tau,s}\rangle_\beta$ on the sphere is determined by that of the stress tensor. Schematically,

$$\langle T^k_{\tau,s}\rangle_\beta = c^k_{\tau,s} \left(\langle T^1_{d-2,2}\rangle_\beta\right)^k + \dots, \tag{2.11}$$

where $c^k_{\tau,s}$ are numerical coefficients, which depend on k, τ, s, but are independent of the details of the theory and the dots stand for terms subleading in C_T^{-1}. By combining (2.11) and (2.10), one can formulate a universality condition

$$\lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{\tau,s}} \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = c^k_{\tau,s} (\lambda_{\mathcal{O}_H \mathcal{O}_H T^1_{d-2,2}})^k = c^k_{\tau,s} \left(\frac{d}{1-d}\right)^k \frac{\Delta_H^k}{C_T^{\frac{k}{2}}}, \tag{2.12}$$

where the last equality follows from the stress tensor Ward identity for the three-point function which fixes $\lambda_{\mathcal{O}_H \mathcal{O}_H T^1_{d-2,2}}$ ($T^1_{d-2,2}$ here is unit-normalized). In other words, thermalization and large-C_T factorization imply that the leading $\Delta^k/C_T^{k/2}$ behavior of the multi stress tensor OPE coefficients is completely fixed and given by (2.12) in all large-C_T CFTs.

In the paragraph above we considered multi stress tensor operators that did not contain any derivatives in them. However, the story largely remains the same when the derivatives are included, as long as their number does not scale with C_T. Indeed, the three-point function involving the stress-tensor with added derivatives, $\partial_\alpha \dots \partial_\beta T_{\mu\nu}$ still behaves like $\lambda_{\mathcal{O}_H \mathcal{O}_H \partial_\alpha \dots \partial_\beta T_{\mu\nu}} \simeq \Delta_H/\sqrt{C_T}$ up to a theory-independent coefficient. Hence, (2.12) still holds, provided thermalization and large-C_T factorization hold on the sphere.

[6] The corresponding conformal blocks can be obtained in the usual way by applying the quadratic conformal Casimir and solving the resulting differential equation [77].

Note that due to conformal invariance, correlators on the sphere depend on R only through the ratio β/R. Moreover, in the large volume limit, factors of R need to drop out of (2.6) and (2.7) to have a well defined limit. To see this we use that $(1-z)\to 0$ and $(1-\bar{z})\to 0$ when $R\to\infty$ and the conformal blocks behave as (see e.g. [6])

$$\begin{aligned} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z}) &\sim \mathcal{N}_{d,s}[(1-z)(1-\bar{z})]^{\frac{\tau+s}{2}} C_s^{(d/2-1)}\left(\frac{(1-z)+(1-\bar{z})}{2\sqrt{(1-z)(1-\bar{z})}}\right) \\ &\sim \mathcal{N}_{d,s}\frac{|x|^{\tau+s}}{R^{\tau+s}} C_s^{(d/2-1)}\left(\frac{x_E^0}{|x|}\right), \end{aligned} \tag{2.13}$$

where $|x| = \sqrt{(x_E^0)^2+\mathbf{x}^2}$, $C_s^{(d/2-1)}(\frac{x_E^0}{|x|})$ is a Gegenbauer polynomial and $\mathcal{N}_{d,s} = \frac{s!}{(d/2-1)_s}$. Including the factor $[(1-z)(1-\bar{z})]^{-\Delta}$ from (2.6) in (2.13) this agrees with the thermal block on $S^1\times\mathbf{R}^{d-1}$ in [13]. Now from the thermalization of the stress tensor we will find in the large volume limit that

$$\frac{\Delta_H}{C_T} \propto \left(\frac{R}{\beta}\right)^d, \tag{2.14}$$

and from (2.12) and (2.13) it follows that

$$g^{(0,0)}_{\tau,s}(1-z,1-\bar{z})\lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}}\lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}\Big|_{\frac{\Delta_H^k}{C_T^k}} \propto R^{dk-(\tau+s)}\beta^{-dk}. \tag{2.15}$$

The dimension of multi stress tensors $T^k_{\tau,s}$ is given by $\tau+s = dk+n$ where $n=0,2,\ldots$. Therefore, the only multi stress tensors that contribute in the large volume limit have dimensions dk. Restoring R in (2.6)–(2.7) and inserting (2.15) one finds that R drops out in the large volume limit. The correct dependence $\beta^{-(\tau+s)}$ from (2.9) in the $R\to\infty$ limit is also recovered in (2.6) using (2.15). The multi stress tensor operators that contribute in the large volume limit are therefore of the schematic form $T_{\mu_1\nu_1}T_{\mu_2\nu_2}\cdots T_{\mu_k\nu_k}$ with arbitrarily many contractions and no derivatives.

In holographic theories thermalization and the Wilson line prescription for the correlator allows one to compute the universal part of the OPE coefficients (see [66, 78] for explicit computations in the $d=4$ case). It is also easy to check explicitly that the universality (2.12) holds for holographic theories with a Gauss-Bonnet gravitational coupling added. While the statement was shown to be true for the leading twist OPE coefficients in [63], it was not immediately obvious for multi stress tensors of non-minimal twist. Some such OPE coefficients were computed in [63, 71]. (See e.g. eqs. (5.48), (5.51), (5.52), (5.57) and (D.1)-(D.5) in [71]). Indeed, the leading $\Delta^k/C_T^{k/2}$ behavior of these OPE coefficients is independent of the Gauss-Bonnet coupling.

What about a general large-C_T theory? We first consider the OPE coefficients of double-stress tensors. To this end, consider the four point function[7] $\langle\mathcal{O}T_{\mu\nu}T_{\rho\sigma}\mathcal{O}\rangle$ where $\mathcal{O}$ is a scalar operator with scaling dimension Δ. In the direct channel $\mathcal{O}\times\mathcal{O}\to\mathcal{O}'\to T_{\mu\nu}\times T_{\rho\sigma}$ for finite Δ and large C_T, the leading contribution in the large-C_T limit comes from the

[7]This correlator for finite Δ was recently considered in holographic CFTs with $\Delta_{\rm gap}\gg 1$ and $\Delta\ll\Delta_{\rm gap}$ in [73].

identity operator $\mathcal{O}\times\mathcal{O}\to\mathbf{1}\to T_{\mu\nu}\times T_{\rho\sigma}$. The subleading contributions in the direct-channel are due to single trace operators as well as double trace operators made out of the external operators of the schematic form $T^2_{\tau,s}$ and $[\mathcal{O}\mathcal{O}]_{n,l} =: \mathcal{O}\partial^{2n}\partial_1\ldots\partial_l\mathcal{O}:$. The exchange of the identity operator is reproduced in the cross-channel $\mathcal{O}\times T_{\mu\nu}\to[\mathcal{O}T_{\alpha\beta}]_{n,l}\to\mathcal{O}\times T_{\rho\sigma}$ by mixed double-trace operators $[\mathcal{O}T_{\alpha\beta}]_{n,l}$ with OPE coefficients fixed by the MFT [79–81]. The subleading contributions in $1/C_T$ are then due to corrections to the anomalous dimension and OPE coefficients of $[\mathcal{O}T_{\alpha\beta}]_{n,l}$ and single trace operators in the $\mathcal{O}\times T_{\mu\nu}$ OPE. An important example of the latter is the exchange of the single trace operator $\mathcal{O}$, whose contribution is universally fixed by the stress tensor Ward identity to be $(\lambda_{\mathcal{O}T^1_{d-2,2}\mathcal{O}})^2\propto\Delta^2/C_T$ times the conformal block. This gives a universal contribution to $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$ as was also noted in [73].

We now want to consider the case where $\Delta\sim C_T$ and study the OPE coefficients of the double-stress tensor operators in the $\mathcal{O}\times\mathcal{O}$ OPE. Firstly, note that the contribution from $T^2_{\tau,s}$ to the four-point function expanded in the direct channel is proportional to $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}\lambda_{TTT^2_{\tau,s}}$. The OPE coefficients $\lambda_{TTT^2_{\tau,s}}$ are fixed by the MFT and are independent of Δ and therefore the dependence on the scaling dimension comes solely from the OPE coefficients $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$. In the cross-channel, we analyze two kinds of contributions: from the exchanged operator $\mathcal{O}$ and from all other operators $\mathcal{O}'\neq\mathcal{O}$. From the operator $\mathcal{O}$ we get a universal contribution to the OPE coefficients in the direct channel $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$, that we denote by $\lambda^{(1)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$. This contribution is universal since it only depends on $(\lambda_{\mathcal{O}T^1_{d-2,2}\mathcal{O}})^2\propto\Delta^2/C_T$ in the cross-channel, which is fixed by the Ward identity. The contributions from other operators $\mathcal{O}'$ to the same OPE coefficient will be denoted by $\lambda^{(2)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$, such that $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}} = \lambda^{(1)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}+\lambda^{(2)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$. Note that it also follows from the stress tensor Ward identity that the only scalar primary that appears in the cross-channel is $\mathcal{O}$. The operator $\mathcal{O}'$ therefore necessarily has spin $s\neq 0$.

To prove universality we need to show that $\lambda^{(2)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}\ll\Delta^2/C_T$ in limit $1\ll\Delta\propto C_T$ by studying the Δ dependence of the OPE coefficients $\lambda_{\mathcal{O}T^1_{d-2,2}\mathcal{O}'}$ in the cross-channel. For operators $\mathcal{O}'$, such that $\Delta_{\mathcal{O}'}\ll\Delta$, we expect that these OPE coefficients are heavily suppressed. It would be interesting to understand if one could put a general bound on the contribution of these operators in the cross-channel in any large-C_T theory. On the other hand, assuming thermalization, the OPE coefficients due to operators $\mathcal{O}'$ such that $\Delta_{\mathcal{O}'}\sim\Delta$ have been calculated in [23]. The obtained results are in agreement with our expectation, namely, these OPE coefficients are suppressed in $1\ll\Delta\propto C_T$ limit. Additionally, in the cross-channel we have double-trace operators $[\mathcal{O}T_{\alpha\beta}]_{n,l}$, whose OPE is fixed by the MFT and it does not get Δ-enhanced. Note that in holographic theories with a large gap, $1\ll\Delta_{\text{gap}}\ll C_T$, in the regime $\Delta\ll\Delta_{\text{gap}}$ there is a coupling $\lambda_{\mathcal{O}T^1_{d-2,2}T^1_{d-2,2}}$ which scales like $\frac{1}{\Delta^2_{\text{gap}}\sqrt{C_T}}$ and its contribution to multi-stress tensor OPE coefficients was studied in [73]. This is different from the regime considered in this paper where $\Delta\gg\Delta_{\text{gap}}$.

One can iteratively extend the argument given here to multi stress tensors operators (with $k>2$) by considering multi stress tensors as external operators. For example, to argue the universality of $\lambda_{\mathcal{O}\mathcal{O}T^3_{\tau,s}}$ one may consider $\langle\mathcal{O}T^1_{d-2,2}T^2_{\tau,s}\mathcal{O}\rangle$. The bootstrap argument above can be applied again by using the fact that OPE coefficients $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$ are universal, and the OPE coefficients $\lambda_{\mathcal{O}T^2_{\tau,s}\mathcal{O}'}$ are again expected to be subleading.

3 OPE coefficients in the free adjoint scalar model

In this section we consider a four-dimensional theory of a free scalar in the adjoint representation of SU(N), see [82–87] for related work. The relation between N and the central charge C_T in this theory is [88]

$$C_T = \frac{4}{3}(N^2-1), \tag{3.1}$$

and we consider the large-N (large-C_T) limit. The propagator for the scalar field ${\phi^i}_j$ is given by

$$\langle {\phi^i}_j(x){\phi^k}_l(y)\rangle = \left({\delta^i}_l{\delta^k}_j - \frac{1}{N}{\delta^i}_j{\delta^k}_l\right)\frac{1}{|x-y|^2}. \tag{3.2}$$

A single trace scalar operator with dimension Δ is given by

$$\mathcal{O}_\Delta(x) = \frac{1}{\sqrt{\Delta}N^{\frac{\Delta}{2}}} : Tr(\phi^\Delta) : (x), \tag{3.3}$$

where $: \ldots :$ denotes the oscillator normal ordering and the normalization is fixed by

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(y)\rangle = \frac{1}{|x-y|^{2\Delta}}. \tag{3.4}$$

The CFT data that we compute in this section are the OPE coefficients of multi stress tensors in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. Assuming we can take $\Delta \to \Delta_H \sim C_T$, the large-$\Delta$ limit of these OPE coefficients is shown to be universal. One may worry that for $\Delta_H \sim C_T$ we can no longer trust the planar expansion, but, as we show in appendix C, the large-Δ limit of the planar result yields the correct expression even for $\Delta_H \sim C_T$.

3.1 Stress tensor

The stress tensor operator is given by

$$T_{\mu\nu}(x) = \frac{1}{3\sqrt{C_T}} : Tr\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\phi\partial_\mu\partial_\nu\phi - (\text{trace})\right) : (x), \tag{3.5}$$

where the normalization

$$\langle T^{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{1}{|x|^8}\left({I^{(\mu}}_\rho(x){I^{\nu)}}_\sigma(x) - (\text{traces})\right), \tag{3.6}$$

with ${I^\mu}_\nu(x) := {\delta^\mu}_\nu - \frac{2x^\mu x_\nu}{|x|^2}$. The OPE coefficient is fixed by the stress tensor Ward identity to be

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^1_{2,2}} = -\frac{4\Delta}{3\sqrt{C_T}}. \tag{3.7}$$

It is also useful to find (3.7) using Wick contractions since an analogous calculation will be necessary for multi stress tensors. We do this explicitly in appendix A.

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3.2 Double-stress tensor with minimal twist

In this section we study the minimal-twist composite operator made out of two stress tensors

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu} T_{\rho\sigma)} : (x) - (\text{traces}), \tag{3.8}$$

with the normalization

$$\langle (T^2)^{\mu\nu\rho\sigma}(x)(T^2)_{\kappa\lambda\delta\omega}(0)\rangle = \frac{1}{|x|^{16}} \left(I^{(\mu}{}_{\kappa} I^{\nu}{}_{\lambda} I^{\rho}{}_{\delta} I^{\sigma)}{}_{\omega} - (\text{traces})\right). \tag{3.9}$$

Consider the following three-point function

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}}{|x_{12}|^{2\Delta-4}|x_{13}|^4|x_{23}|^4} \left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})\right), \tag{3.10}$$

where $Z^\mu = \frac{x_{13}^\mu}{|x_{13}|^2} - \frac{x_{12}^\mu}{|x_{12}|^2}$. It is shown in appendix A that the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}$ is given at leading order in the large-C_T limit by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} = \frac{8\sqrt{2}\Delta(\Delta-1)}{9C_T}. \tag{3.11}$$

Evaluating $P^{(HH,LL)}_{T^2_{4,4}}$ defined by (2.5) in the large-Δ limit,[8] we obtain

$$\begin{aligned} P^{(HH,LL)}_{T^2_{4,4}} &= \left(-\frac{1}{2}\right)^4 \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{8}{81}\frac{\Delta_H^2}{C_T^2}\left(\Delta^2 + \mathcal{O}(\Delta)\right) = \mu^2\left(\frac{\Delta^2}{28800} + \mathcal{O}(\Delta)\right), \end{aligned} \tag{3.12}$$

where we use the following relation

$$\mu = \frac{160}{3}\frac{\Delta_H}{C_T}. \tag{3.13}$$

The result (3.12) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [63] and bootstrap in [66, 68].

3.3 Double-stress tensor with minimal twist and spin $s = 6$

We consider double-stess tensor operator with two (uncontracted) derivatives inserted

$$(T^2)_{\mu\nu\rho\sigma\eta\kappa}(x) = \frac{1}{2\sqrt{182}} : \left(T_{(\mu\nu}\partial_\rho\partial_\sigma T_{\eta\kappa)}(x) - \frac{7}{6}\left(\partial_{(\rho} T_{\mu\nu}\right)\left(\partial_\sigma T_{\eta\kappa)}\right)(x) - (\text{traces})\right) : . \tag{3.14}$$

Using the conformal algebra eq. (C.2), it is straightforward to check that this operator is primary. It is unit-normalized such that

$$\langle (T^2)^{\mu\nu\rho\sigma\eta\kappa}(x)(T^2)_{\alpha\beta\gamma\delta\xi\epsilon}(0)\rangle = \frac{1}{|x|^{20}} \left(I^{(\mu}{}_{\alpha} I^{\nu}{}_{\beta} I^{\rho}{}_{\gamma} I^{\sigma}{}_{\delta} I^{\eta}{}_{\xi} I^{\kappa)}{}_{\epsilon} - (\text{traces})\right). \tag{3.15}$$

[8] By the large-Δ limit, we strictly speaking mean $1 \ll \Delta \ll C_T$. However in this paper we often extrapolate this to the $\Delta \sim C_T$ regime.

By a calculation similar to those summarized in appendix A, we observe that the OPE coefficient of $(T^2)_{\mu\nu\rho\sigma\eta\kappa}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE is given at leading order in the large-C_T limit by

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,6}} = \frac{8}{3}\sqrt{\frac{2}{91}}\frac{\Delta(\Delta-1)}{C_T}. \tag{3.16}$$

Evaluating $P^{(HH,LL)}_{T^2_{4,6}}$, defined by (2.5), in the large-Δ limit, we obtain

$$\begin{aligned} P^{(HH,LL)}_{T^2_{4,6}} &= \left(-\frac{1}{2}\right)^6 \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,6}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,6}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{2}{819}\frac{\Delta_H^2}{C_T^2}\left(\Delta^2 + \mathcal{O}\left(\Delta\right)\right) = \mu^2\left(\frac{\Delta^2}{1164800} + \mathcal{O}(\Delta)\right). \end{aligned} \tag{3.17}$$

The result (3.17) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [63] and bootstrap in [66, 68].

3.4 Minimal-twist multi stress tensors

We now consider multi stress tensors $T^k_{2k,2k}$. Just like the double stress tensor ($k=2$), we show that these have universal OPE coefficients in the large-Δ limit for any k.

Consider the unit-normalized minimal-twist multi stress tensor operator given by

$$(T^k)_{\mu_1\mu_2\ldots\mu_{2k}}(x) = \frac{1}{\sqrt{k!}} : T_{(\mu_1\mu_2}T_{\mu_3\mu_4}\cdots T_{\mu_{2k-1}\mu_{2k})} : (x) - (\text{traces}). \tag{3.18}$$

The OPE coefficient of $(T^k)_{\mu_1\mu_2\ldots\mu_{2k}}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE, in the large-C_T limit is given by[9]

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^k_{2k,2k}} = \left(-\frac{4}{3}\right)^k \frac{1}{\sqrt{k!}C_T^{k/2}}\frac{\Gamma(\Delta+1)}{\Gamma(\Delta-k+1)}. \tag{3.19}$$

First, we write $P^{(HH,LL)}_{T^3_{6,6}}$, defined by (2.5), in the large-Δ limit. We obtain this OPE coefficient from (3.19) for $k=3$,

$$\begin{aligned} P^{(HH,LL)}_{T^3_{6,6}} &= \left(-\frac{1}{2}\right)^6 \lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{6,6}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{6,6}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^3} \\ &= \frac{32}{2187}\frac{\Delta_H^3}{C_T^3}\left(\Delta^3 + \mathcal{O}\left(\Delta^2\right)\right) = \mu^3\left(\frac{\Delta^3}{10368000} + \mathcal{O}(\Delta^2)\right). \end{aligned} \tag{3.20}$$

The result (3.20) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [63] and bootstrap in [68].

[9] See appendix A for detailed computations of similar OPE coefficients.

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Additionally, we consider the OPE coefficient $P^{(HH,LL)}_{T^k_{2k,2k}}$ in the large-Δ limit for general k,

$$\begin{aligned} P^{(HH,LL)}_{T^k_{2k,2k}} &= \left(-\frac{1}{2}\right)^{2k} \lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{2k,2k}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^k_{2k,2k}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^k} \\ &= \frac{1}{k!}\left(\frac{2}{3}\right)^{2k} \frac{\Delta_H^k}{C_T^k}\left(\Delta^k + \mathcal{O}\left(\Delta^{k-1}\right)\right) = \mu^k \left(\frac{\Delta^k}{120^k k!} + \mathcal{O}(\Delta^{k-1})\right). \end{aligned} \tag{3.21}$$

If we consider the limit $1-\bar{z} \ll 1-z \ll 1$, such that $\mu(1-\bar{z})(1-z)^3$ is held fixed, only operators $T^k_{2k,2k}$ contribute to the heavy-heavy-light-light four-point function given by eq. (2.3). The conformal blocks of $T^k_{2k,2k}$ in this limit are given by

$$g^{(0,0)}_{2k,2k}(1-z, 1-\bar{z}) \approx (1-\bar{z})^k (1-z)^{3k}, \tag{3.22}$$

and we can sum all contributions in eq. (2.4) explicitly to obtain

$$\mathcal{G}(z,\bar{z}) \approx \frac{1}{((1-z)(1-\bar{z}))^\Delta} e^{\frac{\mu\Delta}{120}(1-\bar{z})(1-z)^3}. \tag{3.23}$$

Notice that the term in the exponential is precisely the stress-tensor conformal block in the limit $1-\bar{z} \ll 1-z \ll 1$ times its OPE coefficient. Therefore, the OPE coefficients (3.19) imply the exponentiation of stress-tensor conformal block. We conclude that these OPE coefficients are the same as the ones computed using holography and bootstrap in the limit of large Δ.

3.5 Double-stress tensors with non-minimal twist

So far we have shown that the minimal-twist multi stress tensor OPE coefficients are universal in the limit of large Δ. In this subsection, we extend this to show that the simplest non-minimal twist double-stress tensors also have universal OPE coefficients at large Δ.

The subleading twist double-stress tensor with twist $\tau = 6$ is of the schematic form $:T^\mu{}_\alpha T^{\alpha\nu}:$ and has dimension $\Delta = 8$ and spin $s = 2$. It is given by

$$(T^2)_{\mu\nu}(x) = \frac{1}{\sqrt{2}} :T_{\mu\alpha}T^\alpha{}_\nu:(x) - (\text{trace}). \tag{3.24}$$

The normalization in (3.24) is again chosen such that $(T^2)_{\mu\nu}$ is unit-normalized, see appendix B for details.

The OPE coefficient of $(T^2)_{\mu\nu}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE is found from the three-point function in the large-C_T limit, for details see appendix B,

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle = \frac{4\sqrt{2}\Delta(\Delta-1)}{9C_T} \frac{Z^\mu Z^\nu - (\text{trace})}{|x_{12}|^{2\Delta-6}|x_{13}|^6|x_{23}|^6}, \tag{3.25}$$

from which we read off the OPE coefficient

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{6,2}} = \frac{4\sqrt{2}\Delta(\Delta-1)}{9C_T}. \tag{3.26}$$

Evaluating $P^{(HH,LL)}_{T^2_{6,2}}$, defined by (2.5), in the large-Δ limit, we obtain

$$\begin{aligned} P^{(HH,LL)}_{T^2_{6,2}} &= \left(-\frac{1}{2}\right)^2 \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{6,2}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{6,2}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{8}{81}\frac{\Delta_H{}^2}{C_T^2}\left(\Delta^2+\mathcal{O}\left(\Delta\right)\right) = \mu^2\left(\frac{\Delta^2}{28800}+\mathcal{O}(\Delta)\right). \end{aligned} \tag{3.27}$$

The result (3.27) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [63] and bootstrap in [71].

We further consider the scalar double-stress tensor with $\Delta = 8$ and spin $s = 0$ which is given by

$$(T^2)(x) = \frac{1}{3\sqrt{2}} : T_{\mu\nu}T^{\mu\nu} : (x). \tag{3.28}$$

The three point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle$ is found in appendix B to be

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle = \frac{2\sqrt{2}\Delta(\Delta-1)}{9C_T}\frac{1}{|x_{12}|^{2\Delta-8}|x_{13}|^8|x_{23}|^8}, \tag{3.29}$$

from which we read off the OPE coefficient

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{8,0}} = \frac{2\sqrt{2}\Delta(\Delta-1)}{9C_T}. \tag{3.30}$$

We write $P^{(HH,LL)}_{T^2_{8,0}}$ in the large-Δ limit

$$\begin{aligned} P^{(HH,LL)}_{T^2_{8,0}} &= \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{8,0}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{8,0}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{8}{81}\frac{\Delta_H{}^2}{C_T^2}\left(\Delta^2+\mathcal{O}(\Delta)\right) = \mu^2\left(\frac{\Delta^2}{28800}+\mathcal{O}(\Delta)\right). \end{aligned} \tag{3.31}$$

The result (3.31) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [63] and bootstrap in [71].

4 Thermal one-point functions in the free adjoint scalar model

In this section we explicitly show that multi stress tensor operators thermalize in the free theory by calculating the thermal one-point function of some of these operators on $S^1 \times \mathbf{R}^3$. One-point functions of primary symmetric traceless operators at finite temperature are fixed by symmetry up to a dimensionless coefficient $b_\mathcal{O}$ (see e.g. [8, 13])

$$\langle \mathcal{O}_{\mu_1\cdots\mu_{s_\mathcal{O}}}\rangle_\beta = \frac{b_\mathcal{O}}{\beta^{\Delta_\mathcal{O}}}\left(e_{\mu_1}\cdots e_{\mu_{s_\mathcal{O}}} - (\text{traces})\right). \tag{4.1}$$

Here e_μ is a unit vector along the thermal circle.

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To compare the thermal one-point functions and OPE coefficients from the previous section, we need to derive a relation between $\frac{\Delta_H}{C_T}$ and the temperature[10] β^{-1}. Here $\Delta_H \sim N^2$ refers to the scaling dimension of a heavy operator $\mathcal{O}_H$ with OPE coefficients given by the large-Δ limit of those obtained in section 3. One can relate the inverse temperature β to the parameter $\mu = \frac{160}{3}\frac{\Delta_H}{C_T}$ using the Stefan-Boltzmann's law $E/\mathrm{vol}(S^3) = N^2\pi^2/30\beta^4$. The energy of the state E is related to its conformal dimension Δ via $E = \Delta/R$. One can then use $\mathrm{vol}(S^3) = 2\pi^2 R^3$ and the relation between N and C_T given by (3.1), to find

$$\mu = \frac{160}{3}\frac{\Delta_H}{C_T} = \frac{160}{3}E\,\frac{R}{C_T} = \frac{8}{3}\left(\frac{\pi R}{\beta}\right)^4 . \tag{4.2}$$

4.1 Stress tensor

The thermal one-point function for the stress tensor $T^1_{2,2} = T_{\mu\nu}$ is calculated in appendix D where we find that $b_{T^1_{2,2}}$ is given by

$$b_{T^1_{2,2}} = -\frac{2\pi^4 N}{15\sqrt{3}}. \tag{4.3}$$

Using (4.2) and (D.6) one arrives at

$$b_{T^1_{2,2}}\beta^{-4} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^1_{2,2}}. \tag{4.4}$$

4.2 Double-stress tensor with minimal twist

In this section we calculate the thermal one-point function of the double-stress tensor operator with $\tau = 4$ and spin $s = 4$. The operator is written explicitly in (3.8). The leading contribution to the thermal one-point function of $(T^2)_{\mu\nu\rho\sigma}$ follows from the large-N factorization and is given by

$$\begin{aligned}\langle (T^2)_{\mu\nu\rho\sigma}\rangle_\beta &= \frac{1}{\sqrt{2}}\langle T_{(\mu\nu}\rangle_\beta \langle T_{\rho\sigma)}\rangle_\beta - (\text{traces}) \\ &= \frac{2\sqrt{2}\pi^8 N^2}{675\beta^8}\left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right).\end{aligned} \tag{4.5}$$

Using the relation (4.2) and the OPE coefficient (3.11), we observe the thermalization of this operator,

$$b_{T^2_{4,4}}\beta^{-8} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}}\Big|_{\frac{\Delta_H^2}{C_T}}. \tag{4.6}$$

4.3 Minimal-twist multi stress tensors

Consider now multi stress tensors $T^k_{2k,2k}$ with twist $\tau = 2k$ and spin $s = 2k$. We show that these operators thermalize for any k by calculating their thermal one-point functions:

$$\langle (T^k)_{\mu_1\mu_2\ldots\mu_{2k}}\rangle_\beta = \frac{b_{T^k_{2k,2k}}}{\beta^{4k}}\left(e_{\mu_1}e_{\mu_2}\cdots e_{\mu_{2k}} - (\text{traces})\right), \tag{4.7}$$

[10] See also section 6 and appendix D for alternative derivations.

where the leading behavior of $b_{T^k_{2k,2k}}$ follows from the large-N factorization:

$$b_{T^k_{2k,2k}} = \frac{1}{\sqrt{k!}}(b_{T^1_{2,2}})^k = \frac{(-\frac{2}{5})^k N^k \pi^{4k}}{3^{\frac{3k}{2}}\sqrt{k!}}. \tag{4.8}$$

Eqs. (4.2) and (4.8) may be combined to yield

$$b_{T^k_{2k,2k}}\beta^{-4k} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{2k,2k}}\Big|_{\frac{\Delta_H^k}{C_T^{k/2}}}. \tag{4.9}$$

4.4 Double-stress tensors with non-minimal twist

The subleading twist double-stress tensor is of the schematic form $:T^\mu{}_\alpha T^{\alpha\nu}:$ and has twist $\tau = 6$ and spin $s = 2$. The explicit form can be found in (3.24). The leading term in the thermal one-point function is given by

$$\begin{aligned}
\langle (T^2)^{\mu\nu}\rangle_\beta &= \frac{1}{\sqrt{2}}\langle T^{\mu\alpha}\rangle_\beta \langle T^\nu{}_\alpha\rangle_\beta - (\text{trace}) \\
&= \frac{b^2_{T^1_{2,2}}}{2\sqrt{2}\beta^8}\left(e^\mu e^\nu - \frac{1}{4}\delta^{\mu\nu}\right) \\
&= \frac{\sqrt{2}N^2\pi^8}{675\beta^8}\left(e^\mu e^\nu - \frac{1}{4}\delta^{\mu\nu}\right),
\end{aligned} \tag{4.10}$$

therefore,

$$b_{T^2_{6,2}} = \frac{\sqrt{2}N^2\pi^8}{675}. \tag{4.11}$$

Taking the large-Δ limit of the OPE coefficient in (3.26) and substituting (4.2), we observe thermalization,

$$b_{T^2_{6,2}}\beta^{-8} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{6,2}}\Big|_{\frac{\Delta_H^2}{C_T}}. \tag{4.12}$$

We further consider the scalar double-stress tensor with $\tau = 8$ and $s = 0$ which is given by (3.28). The thermal one-point function for this operator is

$$\begin{aligned}
\langle (T^2)\rangle_\beta &= \frac{1}{3\sqrt{2}}\langle T_{\mu\nu}\rangle_\beta \langle T^{\mu\nu}\rangle_\beta \\
&= \frac{1}{3\sqrt{2}}\frac{3}{4}b^2_{T^1_{2,2}}\beta^{-8} = \frac{\pi^8 N^2}{675\sqrt{2}\beta^8},
\end{aligned} \tag{4.13}$$

where the factor of $\frac{3}{4}$ in the first line comes from the index contractions. Hence,

$$b_{T^2_{8,0}} = \frac{\pi^8 N^2}{675\sqrt{2}}. \tag{4.14}$$

Using (4.14), (3.30) and (4.2), we again observe thermalization,

$$b_{T^2_{8,0}}\beta^{-8} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{8,0}}\Big|_{\frac{\Delta_H^2}{C_T}}. \tag{4.15}$$

4.5 Triple-stress tensors with non-minimal twist

We consider the triple stress tensors with $\tau = 8, s = 4$ and $\tau = 10, s = 2$. The unit-normalized triple stress tensor with $\tau = 8$ can be written as

$$(T^3)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{3}}\Big(: T_{(\mu\nu}T_{\rho|\alpha|}T^{\alpha}{}_{\sigma)} : (x) - (\text{traces})\Big)\,, \tag{4.16}$$

where $|\alpha|$ denotes that index α is excluded from the symmetrization. The thermal one-point function follows from large-N factorization

$$\begin{aligned}\langle (T^3)_{\mu\nu\rho\sigma}\rangle_\beta &= \frac{1}{\sqrt{3}}\Big(\langle T_{(\mu\nu}\rangle_\beta \langle T_{\rho|\alpha|}\rangle_\beta \langle T^{\alpha}{}_{\sigma)}\rangle_\beta - (\text{traces})\Big)\\ &= \frac{1}{2\sqrt{3}}\frac{b^3_{T^1_{2,2}}}{\beta^{12}}\left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right)\\ &= -\frac{4\pi^{12}N^3}{30375\beta^{12}}\left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right),\end{aligned} \tag{4.17}$$

therefore,

$$b_{T^3_{8,4}} = -\frac{4\pi^{12}N^3}{30375}. \tag{4.18}$$

The OPE coefficient of the operator with same quantum numbers ($\Delta = 12$, $s = 4$) is calculated holographically and is given by (D.1) in [71]. In the large-Δ limit it can be written as

$$P^{(HH,LL)}_{T^3_{8,4}} = \left(-\frac{1}{2}\right)^4 \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{8,4}}\lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{8,4}}\bigg|_{\left(\frac{\Delta_H}{C_T}\right)^3} = \frac{64}{2187}\frac{\Delta_H^3\Delta^3}{C_T^3} + \mathcal{O}(\Delta^2). \tag{4.19}$$

Now, one can easily read-off $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{8,4}}$ in the large-Δ limit

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{8,4}} = -\frac{32\Delta^3}{27\sqrt{3}{C_T}^{3/2}} + \mathcal{O}(\Delta^2) = -\frac{4\Delta^3}{9N^3} + \mathcal{O}(\Delta^2), \tag{4.20}$$

where we use the relation between central charge C_T and N given by (3.1). Using (4.2) one can obtain

$$b_{T^3_{8,4}}\beta^{-12} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{8,4}}\Big|_{\frac{\Delta_H^3}{C_T^{3/2}}}. \tag{4.21}$$

We also consider the triple stress tensors with quantum numbers $\Delta = 12$ and $s = 2$. There are two linearly independent such operators that schematically can be written as $: T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu} :$ and $: T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu} :$. We write the following linear combinations of these operators

$$(T^3)_{\mu\nu}(x) = \frac{1}{10\sqrt{2}}\Big(: T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu} : (x) + 4 : T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu} : (x) - (\text{trace})\Big)\,, \tag{4.22}$$

$$(\tilde{T}^3)_{\mu\nu}(x) = \frac{7}{20}\Big(: T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu} : (x) - \frac{12}{7} : T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu} : (x) - (\text{trace})\Big)\,. \tag{4.23}$$

Both $(T^3)_{\mu\nu}$ and $(\tilde{T}^3)_{\mu\nu}$ are unit-normalized and their overlap vanishes in the large-N limit

$$\langle (T^3)_{\mu\nu}(x)(\tilde{T}^3)^{\rho\sigma}(y)\rangle = \mathcal{O}(1/N^2). \tag{4.24}$$

The thermal one-point functions of these operators, obtained by large-N factorization, in the large-N limit are given by

$$\begin{aligned} \langle (T^3)_{\mu\nu}\rangle_\beta &= -\sqrt{\frac{2}{3}}\frac{N^3\pi^{12}}{10125\beta^{12}}\left(e_\mu e_\nu - (\text{trace})\right), \\ \langle (\tilde{T}^3)_{\mu\nu}\rangle_\beta &= \mathcal{O}(N), \end{aligned} \tag{4.25}$$

therefore,

$$\begin{aligned} b_{T^3_{10,2}} &= -\sqrt{\frac{2}{3}}\frac{N^3\pi^{12}}{10125}, \\ b_{\tilde{T}^3_{10,2}} &= 0. \end{aligned} \tag{4.26}$$

The holographic OPE coefficient of the operator with the same quantum numbers ($\Delta = 12$, $s = 2$), with external scalar operators is given by (5.57) in [71]. In the large-Δ limit it can be written as

$$P^{(HH,LL)}_{T^3_{10,2}} = \left(-\frac{1}{2}\right)^2 \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{10,2}}\lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{10,2}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^3} = \frac{32}{729}\frac{\Delta_H^3\Delta^3}{C_T^3} + \mathcal{O}(\Delta^2). \tag{4.27}$$

We can read-off $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{10,2}}$:

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{10,2}} = -\frac{8\sqrt{2}}{27}\frac{\Delta^3}{C_T^{3/2}} + \mathcal{O}(\Delta^2) = -\frac{\sqrt{2}}{3\sqrt{3}}\frac{\Delta^3}{N^3} + \mathcal{O}(\Delta^2). \tag{4.28}$$

Again, using (4.2), one can confirm that this operator thermalizes

$$b_{T^3_{10,2}}\beta^{-12} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{10,2}}\Big|_{\frac{\Delta_H^3}{C_T^{3/2}}}. \tag{4.29}$$

5 Thermal two-point function and block decomposition

In this section we study the thermal two-point function $\langle\mathcal{O}_\Delta\mathcal{O}_\Delta\rangle_\beta$ and decompose it in thermal blocks. We determine the contributions of a few low-lying operators, including the stress tensor $T^1_{2,2}$ and the double stress tensor $T^2_{4,4}$. They exactly match the corresponding OPE coefficients and thermal expectation values computed in previous sections. Due to the presence of multiple operators with equal scaling dimension and spin, there is a mixing problem which we solve explicitly in a few cases. Related appendices include appendix F, where we review the statement that the thermal one-point functions of multi-trace operators with derivatives vanish on $S^1 \times \mathbf{R}^{d-1}$ and appendix G, where we consider two-dimensional thermal two-point functions. In appendix H we do a similar analysis for the vector model in four dimensions.

5.1 Thermal two-point function of a single trace scalar operator

The correlator at finite temperature β^{-1} in the free theory can be calculated by Wick contractions using the propagators on $S^1 \times \mathbf{R}^3$. Explicitly, the two-point function at finite temperature is given by[11]

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \tilde{g}(x_E^0, |\mathbf{x}|)^\Delta + \frac{\pi^4\Delta(\Delta-2)}{9\beta^4}\tilde{g}(x_E^0, |\mathbf{x}|)^{\Delta-2} + \dots , \tag{5.1}$$

where

$$\begin{aligned} \tilde{g}(x_E^0, |\mathbf{x}|) &= \sum_{m=-\infty}^{\infty} \frac{1}{(x_E^0 + m\beta)^2 + \mathbf{x}^2} \\ &= \frac{\pi}{2\beta|\mathbf{x}|}\left[\mathrm{Coth}\left(\frac{\pi}{\beta}(|\mathbf{x}| - ix_E^0)\right) + \mathrm{Coth}\left(\frac{\pi}{\beta}(|\mathbf{x}| + ix_E^0)\right)\right]. \end{aligned} \tag{5.2}$$

The dots in (5.1) contain contributions due to further self-contractions which will not be important below.[12] Taking the $\beta \to \infty$ limit of (5.1) we can read off the decomposition of the two-point function in terms of thermal conformal blocks on $S^1 \times \mathbf{R}^3$ with coordinates $x = (x_E^0, \mathbf{x})$.

Following [13], if $|x| = \sqrt{(x_E^0)^2 + \mathbf{x}^2} \leq \beta$ the two-point function can be evaluated using the OPE:

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \sum_{\mathcal{O}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}} |x|^{\tau-2\Delta} x_{\mu_1}\cdots x_{\mu_{s_\mathcal{O}}} \langle \mathcal{O}^{\mu_1\cdots\mu_{s_\mathcal{O}}}\rangle_\beta , \tag{5.3}$$

where $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}}$ is the OPE coefficient, τ and $s_\mathcal{O}$ is the twist and spin of $\mathcal{O}$, respectively. Using (4.1) together with (5.3), the two-point function on $S^1 \times \mathbf{R}^3$ can be organized in the following way [13]:

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \sum_{\mathcal{O}_{\tau,s}\in\mathcal{O}_\Delta\times\mathcal{O}_\Delta} \frac{a_{\mathcal{O}_{\tau,s}}}{\beta^{\Delta_\mathcal{O}}} \frac{1}{|x|^{2\Delta-\tau+s}} C_s^{(1)}\left(\frac{x_E^0}{|x|}\right), \tag{5.4}$$

where we sum over primary operators $\mathcal{O}_{\tau,s}$, with twist τ and spin s, appearing in the OPE $\mathcal{O}_\Delta \times \mathcal{O}_\Delta \sim \mathcal{O}_{\tau,s} + \dots$. In (5.4) $C_s^{(1)}(x_E^0/|x|)$ is a Gegenbauer polynomial which, together with a factor of $|x|^{-2\Delta+\tau-s}$, forms a thermal conformal block in $d = 4$ dimensions and the coefficients $a_{\mathcal{O}_{\tau,s}}$ are given by

$$a_{\mathcal{O}_{\tau,s}} = \left(\frac{1}{2}\right)^s \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{\tau,s}} b_{\mathcal{O}_{\tau,s}} . \tag{5.5}$$

Expanding (5.1) for $\beta \to \infty$ one finds:

$$\begin{aligned} \langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \frac{1}{|x|^{2\Delta}}\bigg[1 &+ \frac{\pi^2\Delta}{3\beta^2}|x|^2 \\ &+ \frac{\pi^4\Delta}{90\beta^4}|x|^2(3\mathbf{x}^2(5\Delta-9) + (15\Delta-19)(x_E^0)^2) + \mathcal{O}(\beta^{-6})\bigg]. \end{aligned} \tag{5.6}$$

[11] Here and below we assume that $\Delta > 4$. We further drop the disconnected term $\langle \mathcal{O}_\Delta\rangle_\beta^2 \sim N^2$.

[12] These terms will be proportional to $\beta^{-2a}\tilde{g}(x_E^0, |\mathbf{x}|)^{\Delta-a}$, with $a \geq 4$. When decomposed into thermal blocks, these will not affect the operators with dimension $\Delta < 8$ or $\Delta = 8$ with non-zero spin $s \neq 0$.

From the expansion (5.6), we can read off the coefficients $a_{\tau',s'} := \sum_{\mathcal{O}_{\tau',s'}} a_{\mathcal{O}_{\tau',s'}}$ where we sum over all operators with twist τ' and spin s':

$$\begin{aligned}
a_{2,0} &= \frac{\pi^2\Delta}{3}, \\
a_{4,0} &= \frac{\pi^4\Delta(3\Delta-5)}{18}, \\
a_{2,2} &= \frac{\pi^4\Delta}{45}.
\end{aligned} \tag{5.7}$$

For future reference, expanding (5.1) to $\mathcal{O}(\frac{1}{\beta^8})$ one finds

$$\begin{aligned}
a_{2,4} &= \frac{2\pi^6\Delta}{945}, \\
a_{4,4} &= \frac{\pi^8\Delta(\Delta-1)}{1050}.
\end{aligned} \tag{5.8}$$

Note that due to the mixing of operators with the same twist and spin, $a_{\tau,s}$ generically contains the contribution from multiple operators. In the following section we calculate the OPE coefficients and thermal one-point functions of operators which are not multi stress tensors but contribute to (5.7) and (5.8).

5.2 CFT data of scalar operators with dimensions two and four

We explicitly calculate the thermal one-point functions $\langle\mathcal{O}\rangle_\beta = b_{\mathcal{O}}\beta^{-\Delta_{\mathcal{O}}}$ and OPE coefficients $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}}$ for scalar operators $\mathcal{O}$ with twist $\tau'=2$ and $\tau'=4$ using Wick contractions. This is done to find which operators contribute to the thermal two-point function and to resolve a mixing problem.

For $\tau'=2$ there is only one such operator, the single trace operator $\mathcal{O}_2(x) = \frac{1}{\sqrt{2}N} : Tr(\phi^2) : (x)$ given in (3.3). The OPE coefficient is found by considering the three-point correlator

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_2(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}. \tag{5.9}$$

The three-point function is calculated in appendix A, in the large-N limit, and it is given by

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_2(x_3)\rangle = \frac{\sqrt{2}\Delta}{N}\frac{1}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \tag{5.10}$$

and therefore $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2} = \frac{\sqrt{2}\Delta}{N}$ to leading order in $1/N$. To calculate the thermal one-point function $\propto \langle Tr(\phi^2)\rangle_\beta$, we include self-contractions, i.e. contractions of fundamental fields within the same composite operator separated by a distance $m\beta$ along the thermal circle for $m\neq 0$ and integer. Explicitly, the one-point function of $\mathcal{O}_2$ is given by

$$\langle\mathcal{O}_2(x)\rangle_\beta = \frac{1}{\sqrt{2}N}\sum_{m\neq 0}\frac{N^2}{(m\beta)^2} = \frac{\pi^2 N}{3\sqrt{2}\beta^2}, \tag{5.11}$$

therefore,

$$b_{\mathcal{O}_2} = \frac{\pi^2 N}{3\sqrt{2}}. \tag{5.12}$$

The contribution to the thermal two-point function $a_{\mathcal{O}_2}$ is found using (5.10) and (5.12)

$$a_{2,0} = b_{\mathcal{O}_2} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_2} = \frac{\pi^2 \Delta}{3}. \tag{5.13}$$

This agrees with (5.7) which was obtained from the thermal two-point function.

We now continue with scalar operators of twist four. There are two such linearly independent operators appearing in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. In order to construct an orthonormal basis, consider the following single and double trace operators:

$$\begin{aligned} \mathcal{O}_4(x) &= \frac{1}{2N^2} : Tr(\phi^4) : (x), \\ \mathcal{O}_{4,\mathrm{DT}}(x) &= \frac{1}{2\sqrt{2}N^2} : Tr(\phi^2) Tr(\phi^2) : (x). \end{aligned} \tag{5.14}$$

We further construct the operator $\tilde{\mathcal{O}}_4$ that has vanishing overlap with $\mathcal{O}_{4,\mathrm{DT}}(x)$ as follows:

$$\tilde{\mathcal{O}}_4 = \mathcal{N} \Big[\mathcal{O}_4 - c_{\mathcal{O}_4 \mathcal{O}_{4,\mathrm{DT}}} \mathcal{O}_{4,\mathrm{DT}} \Big], \tag{5.15}$$

with $\mathcal{N}$ a normalization constant and $c_{\mathcal{O}_4 \mathcal{O}_{4,\mathrm{DT}}}$ is the overlap defined by

$$\langle \mathcal{O}_4(x) \mathcal{O}_{4,\mathrm{DT}}(y) \rangle = \frac{c_{\mathcal{O}_4 \mathcal{O}_{4,\mathrm{DT}}}}{|x-y|^8}. \tag{5.16}$$

Explicit calculation gives $c_{\mathcal{O}_4 \mathcal{O}_{4,\mathrm{DT}}} = \frac{2\sqrt{2}}{N}$ and $\mathcal{N} = \frac{1}{\sqrt{2}}$ in the large-N limit, and the scalar dimension four operator orthogonal to the double trace operator $\mathcal{O}_{4,\mathrm{DT}}$ is therefore

$$\tilde{\mathcal{O}}_4 = \frac{1}{\sqrt{2}} \left[\mathcal{O}_4 - \frac{2\sqrt{2}}{N} \mathcal{O}_{4,\mathrm{DT}} \right]. \tag{5.17}$$

Note that even though the second term in (5.15) is suppressed by $1/N$, it can still contribute to the thermal two-point function due to the scaling of OPE coefficients and one-point function of a k-trace operator $\mathcal{O}^{(k)}$:

$$\begin{aligned} b_{\mathcal{O}^{(k)}} &\sim N^k, \\ \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}^{(k)}} &\sim \frac{1}{N^k}, \end{aligned} \tag{5.18}$$

in the limit $N \to \infty$.

The one-point function and the OPE coefficient for $\mathcal{O}_4$ is found analogously to that of $\mathcal{O}_2$ in the large-N limit

$$\begin{aligned} b_{\mathcal{O}_4} &= \frac{\pi^4 N}{9}, \\ \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_4} &= \frac{2\Delta}{N}. \end{aligned} \tag{5.19}$$

Consider now the double trace operator given in (5.14). The one-point function factorizes in the large-N limit:

$$\begin{aligned} \langle \mathcal{O}_{4,\mathrm{DT}}(x) \rangle_\beta &= \frac{1}{\sqrt{2}} (\langle \mathcal{O}_2(x) \rangle_\beta)^2 \\ &= \frac{\pi^4 N^2}{18\sqrt{2}\beta^4}. \end{aligned} \tag{5.20}$$

Likewise, the OPE coefficient can be computed in the large-N limit (see appendix A)

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{4,\mathrm{DT}}} = \frac{\sqrt{2}\Delta(3\Delta-5)}{N^2}. \tag{5.21}$$

Consider now the thermal one-point function of $\tilde{\mathcal{O}}_4$ in (5.17)

$$\begin{aligned} \langle \tilde{\mathcal{O}}_4 \rangle_\beta &= \frac{1}{\sqrt{2}\beta^4} \left[b_{\mathcal{O}_4} - \frac{2\sqrt{2}}{N} b_{\mathcal{O}_{4,\mathrm{DT}}} \right] \\ &= \mathcal{O}(N^{-1}), \end{aligned} \tag{5.22}$$

where we have used (5.19) and (5.20). Since the corresponding OPE coefficient is suppressed by N^{-1}, it follows that the only scalar operator with dimension four contributing to the thermal two-point function is the double trace operator $\mathcal{O}_{4,\mathrm{DT}}$. From the OPE coefficient and thermal one-point function of this double trace operator, using (5.20) and (5.21), we find the following contribution to the thermal two-point function

$$a_{4,0} = \frac{\pi^4 \Delta(3\Delta-5)}{18}, \tag{5.23}$$

which agrees with (5.7).

5.3 CFT data of single-trace operator with twist two and spin four

The primary single trace operator $\Xi = \mathcal{O}_{2,4}$ with twist $\tau = 2$ and spin $s = 4$ is given by

$$\begin{aligned} \Xi_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{35}N} :Tr\big(&\phi(\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16(\partial_{(\mu}\phi)(\partial_\nu\partial_\rho\partial_{\sigma)}\phi) \\ &+ 18(\partial_{(\mu}\partial_\nu\phi)(\partial_\rho\partial_{\sigma)}\phi) - (\text{traces})\big) : (x). \end{aligned} \tag{5.24}$$

The relative coefficients follow from requiring that the operator is a primary, see appendix E for details.

The thermal one-point function of this operator is found from Wick contractions in the large-N limit to be

$$\langle \Xi_{\mu\nu\rho\sigma} \rangle_\beta = \frac{8\pi^6 N}{27\sqrt{35}\beta^6} \left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces}) \right). \tag{5.25}$$

Moreover, the OPE coefficient in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE can again be calculated using Wick contractions similarly to how it was done for $T^2_{4,4}$ in appendix A. By explicit calculation one finds

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\Xi_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{4\Delta}{\sqrt{35}N} \frac{Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \tag{5.26}$$

and therefore the OPE coefficient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}}$ is given by

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}} = \frac{4\Delta}{\sqrt{35}N}. \tag{5.27}$$

Now, it is easy to check that

$$\frac{1}{2^4} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}} b_{\mathcal{O}_{2,4}} = \frac{2\pi^6\Delta}{945}, \tag{5.28}$$

which agrees with $a_{2,4}$ in (5.8).

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5.4 CFT data of double-trace operators with twist and spin equal to four

To find the full contribution to the thermal two-point function from the operators with $\tau = 4$ and $s = 4$ we need to take into account the contribution of all operators with these quantum numbers and solve a mixing problem. In addition to the double-stress tensor operator with these quantum numbers, the other double trace primary operator which contributes is given by

$$\begin{aligned}\mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70}N^2} :Tr(\phi^2)\Big(Tr(\phi\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16Tr(\partial_{(\mu}\phi\partial_\nu\partial_\rho\partial_{\sigma)}\phi)\\ +18Tr(\partial_{(\mu}\partial_\nu\phi\partial_\rho\partial_{\sigma)}\phi)(x) - (\text{traces})\Big):(x),\end{aligned} \tag{5.29}$$

where the operator is unit-normalized. Notice that this is the double trace operator obtained by taking the normal ordered product of two single trace operators, the scalar operator with dimension 2 and the single trace spin-4 operator with dimension 6. There are more double trace operators with these quantum numbers which are, however, not simply products of single trace operators. These do not contribute to the thermal two-point function to leading order in $\frac{1}{N^2}$ (see appendix F).

Note that it follows from large-N factorization that the overlap of this operator with $(T^2)_{\mu\nu\rho\sigma}$ is suppressed by powers of $\frac{1}{N}$; since both of these are double trace operators and obey the scaling (5.18), to study the thermal two-point function to leading order in N^2, one can therefore neglect this overlap.

The thermal one-point function of $\mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}$ follows from the large-N factorization and we find that

$$b_{\mathcal{O}^{\text{DT}}_{4,4}} = \sqrt{\frac{2}{35}}\frac{4\pi^8 N^2}{81}, \tag{5.30}$$

where we used the thermal one-point functions for each single trace operator given by (5.11) and (5.25). The OPE coefficient is calculated in appendix A,

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}^{\text{DT}}_{4,4}} = \sqrt{\frac{2}{35}}\frac{4\Delta(\Delta-1)}{N^2}. \tag{5.31}$$

Using the thermal one point function and the OPE coefficient in (5.30) and (5.31) respectively, it is found that it the operator $\mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}$ gives the following contribution to the thermal two point function:

$$a_{\mathcal{O}^{\text{DT}}_{4,4}} = \left(\frac{1}{2}\right)^4 b_{\mathcal{O}^{\text{DT}}_{4,4}}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}^{\text{DT}}_{4,4}} = \frac{2\pi^8\Delta(\Delta-1)}{2835}. \tag{5.32}$$

The total contribution from $T^2_{4,4}$ together with that of $\mathcal{O}^{\text{DT}}_{4,4}$, using (3.11), (4.5) and (5.32), is

$$a_{4,4} = (a_{T^2_{4,4}} + a_{\mathcal{O}^{\text{DT}}_{4,4}}) = \frac{\pi^8\Delta(\Delta-1)}{1050}. \tag{5.33}$$

This agrees with $a_{4,4}$ in (5.8).

6 Comparison with the eigenstate thermalization hypothesis

In this section we discuss the relation of our results to the eigenstate thermalization hypothesis (ETH). We argue that the stress tensor sector of the free SU(N) adjoint scalar theory in $d = 4$ satisfies the ETH to leading order in $C_T \sim N^2 \gg 1$. We explain the equivalence of the micro-canonical and canonical ensemble when $\Delta_H \sim C_T$ in large-C_T theories. In this regime, the diagonal part of the ETH is (up to exponentially suppressed terms which we do not consider), equivalent to thermalization. Note that in two dimensions the Virasoro descendants of the identity satisfy the ETH (see e.g. [42] for a recent discussion).

We begin by showing the equivalence between the micro-canonical and the canonical ensemble on $S^1_\beta \times S^{d-1}$ when $\Delta_H \sim C_T \gg 1$. See [9, 11, 12, 14, 89] for a similar discussion at infinite volume as well as [47] in the two-dimensional case. The expectation value in the micro-canonical ensemble of an operator $\mathcal{O}$, which we take to be a scalar for simplicity, at energy $E = \Delta_H/R$ is given by

$$\langle \mathcal{O} \rangle_E^{(\mathrm{micro})} = \frac{1}{N(E)} \sum_{\tilde{\mathcal{O}}} \langle \tilde{\mathcal{O}} | \mathcal{O} | \tilde{\mathcal{O}} \rangle, \tag{6.1}$$

where we sum over states $|\tilde{\mathcal{O}}\rangle$ with energy $(E, E + \delta E)$ and $N(E)$ is the number of states in this interval. On the other hand, consider the partition function at inverse temperature β given by

$$Z(\beta) = \sum_{\tilde{\mathcal{O}}} e^{-\frac{\beta \tilde{\Delta}}{R}} = \int d\tilde{\Delta} \rho(\tilde{\Delta}) e^{-\frac{\beta \tilde{\Delta}}{R}}, \tag{6.2}$$

where we sum over all states in the theory. In the second line in (6.2) we have approximated the sum of delta-functions by a continuous function $\rho(\tilde{\Delta})$. Expectation values in the canonical ensemble is then computed by[13]

$$\langle \mathcal{O} \rangle_\beta = Z(\beta)^{-1} \int d\tilde{\Delta} \rho(\tilde{\Delta}) \langle \mathcal{O} \rangle_E^{(\mathrm{micro})} e^{-\frac{\beta \tilde{\Delta}}{R}}. \tag{6.3}$$

Consider the partition function in (6.2) with a free energy $F = -\beta^{-1} \log Z(\beta)$. By an inverse Laplace transform of (6.2) we find the density of states

$$\rho(\Delta_H) = \frac{1}{2\pi i R} \int d\beta' e^{\beta' \left(\frac{\Delta_H}{R} - F(\beta') \right)}. \tag{6.4}$$

For $\Delta_H \sim C_T$ and a large free energy[14] $F \sim C_T$, we can evaluate (6.4) using a saddlepoint approximation with the saddle at β given by

$$\frac{\Delta_H}{R} = \partial_{\beta'} (\beta' F)|_\beta. \tag{6.5}$$

Consider now the thermal expectation value in (6.3), multiplying both sides by $Z(\beta)$ and doing an inverse Laplace transform evaluated at $\Delta_H \sim C_T$ we find

$$\rho(\Delta_H) \langle \mathcal{O} \rangle_{\Delta_H/R}^{(\mathrm{micro})} = \frac{1}{2\pi i R} \int d\beta' \langle \mathcal{O} \rangle_{\beta'} e^{\beta' \left(\frac{\Delta_H}{R} - F(\beta') \right)}. \tag{6.6}$$

[13] It was argued in [14] that the existence of the thermodynamic limit implies that we only need to sum over operators with low spin.

[14] We consider a CFT in a high temperature phase.

For $F \sim C_T \gg 1$ we again use a saddlepoint approximation to evaluate (6.6) with the saddle at β determined by (6.5), assuming $\langle \mathcal{O} \rangle_{\beta'}$ does not grow exponentially with C_T. The r.h.s. of (6.6) is therefore the thermal expectation value $\langle \mathcal{O} \rangle_\beta$ multiplied by the saddlepoint approximation of the density of states in (6.4). It then follows that

$$\langle \mathcal{O} \rangle^{(\text{micro})}_{\Delta_H/R} \approx \langle \mathcal{O} \rangle_\beta, \tag{6.7}$$

with β determined by (6.5). In particular, in the infinite volume limit $R \to \infty$, the free energy is given by[15]

$$F = \frac{b_{T^{(\text{can})}_{\mu\nu}} S_d R^{d-1}}{d\beta^d}, \tag{6.8}$$

where $S_d = Vol(S^{d-1}) = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$. Inserting (6.8) in (6.5) we find [9]

$$\frac{\beta}{R} = \left(\frac{-(d-1) b_{T^{(\text{can})}_{\mu\nu}} S_d}{d\Delta_H} \right)^{\frac{1}{d}}. \tag{6.9}$$

We can use (6.7) to see the thermalization of the stress tensor. The free energy is related to the expectation value of the stress tensor $T^{(\text{can})}_{\mu\nu}$ [5]

$$\langle T^{(\text{can})}_{00} \rangle_\beta = \frac{1}{S_d R^{d-1}} \partial_\beta (-\beta F(\beta)). \tag{6.10}$$

On the other hand, the expectation value of $T^{(\text{can})}_{00}$ in a heavy state $|\mathcal{O}_H\rangle$ is fixed by the Ward identity to be

$$\langle \mathcal{O}_H | T^{(\text{can})}_{00} | \mathcal{O}_H \rangle = -\frac{\Delta_H}{S_d R^d}. \tag{6.11}$$

Multiplying (6.5) with $(S_d R^{d-1})^{-1}$ and comparing with (6.10)–(6.11) we find that

$$\langle \mathcal{O}_H | T^{(\text{can})}_{00} | \mathcal{O}_H \rangle = \langle T^{(\text{can})}_{00} \rangle_\beta. \tag{6.12}$$

This shows the thermalization of the stress tensor in heavy states where $F \sim \Delta_H \sim C_T$ in large-C_T theories. Note that this follows from (6.7) since we can replace the micro-canonical expectation value at $E = \Delta_H/R$, on the l.h.s., with the expectation value in any single heavy state with dimension Δ_H due to the Ward identity, independent of the heavy state. Put differently, the stress tensor satisfies the ETH as we will review below.

We now consider the eigenstate thermalization hypothesis for CFTs at finite temperature on the sphere S^{d-1} of radius R. The diagonal part of the ETH is given by

$$\langle \mathcal{O}_H | \mathcal{O}_{\tau,s} | \mathcal{O}_H \rangle = \langle \mathcal{O}_{\tau,s} \rangle^{(\text{micro})}_E + \mathcal{O}\left(e^{-S(E)}\right), \tag{6.13}$$

where $\mathcal{O}_H$ and $\mathcal{O}_{\tau,s}$ are local primary operators and $\langle \mathcal{O}_{\tau,s} \rangle^{(\text{micro})}_E$ is the expectation value of $\mathcal{O}_{\tau,s}$ in the micro-canonical ensemble at energy $E = \frac{\Delta_H}{R}$. Here we assume that the

[15] Here we denote the canonically normalized stress tensor by $T^{(\text{can})}_{\mu\nu}$, whose two-point function is given by $\langle T^{(\text{can})\mu\nu}(x) T^{(\text{can})}_{\rho\sigma}(y) \rangle = \frac{C_T}{S_d^2} (I^\mu{}_{(\rho} I^\nu{}_{\sigma)} - (\text{trace}))$.

operator $\mathcal{O}_H$ is a heavy scalar operator with large conformal dimension $\Delta_H \propto C_T \gg 1$. The operator $\mathcal{O}_{\tau,s}$ on the other hand can have non-zero spin.[16] In (6.13), $e^{S(E)}$ is the density of states at energy $E = \Delta_H/R$. As shown in (6.7), in the limit $\Delta_H \sim C_T \gg 1$, the micro-canonical ensemble is equivalent to the canonical ensemble at inverse temperature β determined by (6.5). It then follows from (6.13) that the diagonal part of the ETH can written in terms of OPE coefficients and thermal one-point functions:

$$\frac{\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_{\tau,s}}}{R^{\tau+s}} = \frac{b_{\mathcal{O}_{\tau,s}} f_{\mathcal{O}_{\tau,s}}(\beta/R)}{\beta^{\tau+s}} + \mathcal{O}\left(e^{-S(E)}\right), \tag{6.14}$$

where $f_{\mathcal{O}_{\tau,s}}$ also appears in (2.9). This is equivalent to the statement of thermalization discussed in the rest of the paper.

In this paper we observed that the multi stress tensor operators satisfy (6.14). One can also ask if (6.14) holds for any operator in the specific heavy state we considered. By comparing eqs. (5.10) and (5.11) using (4.2), one can check that operator $\mathcal{O}_2 = \frac{1}{\sqrt{2}N} : Tr(\phi^2) :$ does not satisfy (6.14). Since this is a free theory, it is not a surprise that the ETH is not satisfied by all operators in the spectrum which is seen explicitly in this case.

7 Discussion

In this paper we argued that multi stress tensor operators $T^k_{\tau,s}$ in CFTs with a large central charge C_T thermalize: their expectation values in heavy states are the same as their thermal expectation values. This is equivalent to the statement that multi stress tensor operators in higher-dimensional CFTs satisfy the diagonal part of the ETH in the thermodynamic limit. The analogous statement in the $d = 2$ case is that the Virasoro descendants of the identity satisfy the ETH condition in the large-C_T limit.

We observed that the operator $\mathcal{O}_2 = \frac{1}{\sqrt{2}N} : Tr(\phi^2) :$ does not satisfy the ETH. This is seen by comparing eqs. (5.10) and (5.11) using (4.2). While this operator does not thermalize in the heavy states we considered, the OPE coefficient averaged over all operators with $\Delta_H \sim C_T$ is expected to be proportional to the thermal one-point function. The averaged OPE coefficients should therefore scale like $\sim \sqrt{\Delta_H}$ compared to $\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_2} \sim \Delta_H/\sqrt{C_T}$ for the heavy states we considered. It would be interesting to exhibit heavy operators that produce the former scaling.

We provided a bootstrap argument in favor of the thermalization of multi stress tensor operators. One should be able to refine it to give an explicit form for leading behavior of the multi stress tensor OPE coefficients — we leave it for future work. The holographic/bootstrap OPE coefficients for the leading twist double stress tensor operators can be found in e.g. [66] — they are nontrivial functions of the spin. As explained in [66, 68], the leading Δ behavior of the minimal-twist double- and triple-stress tensor OPE coefficients is consistent with the exponentiation of the near lightcone stress tensor conformal block. One can go beyond the leading twist multi stress tensors. In holographic HHLL correlators each term of the type $(\Delta\mu)^k \sim (\Delta\Delta_H/C_T)^k$ comes from the exponentiation of

[16] The tensor structure in (6.13) is suppressed.

the stress-tensor block — this follows from the Wilson line calculation of the correlator in the AdS-Schwarzschild background [66, 78, 90].

In this paper we argue that this behavior is universal, and is not just confined to holographic theories. Hence, one can formulate another statement equivalent to the thermalization of multi stress tensor operators. Namely, scalar correlators of pairwise identical operators of dimensions $\Delta_{1,2}$ in large-C_T theories in the limit $\Delta_{1,2} \gg 1$, $\Delta_1\Delta_2/C_T$ fixed are given by the exponentials of the stress tensor conformal block.[17] This is similar to what happens in two-dimensional CFTs.

Note that the universality of the OPE coefficients is naively in tension with the results of [73], where finite gap (Δ_{gap}) corrections to the multi stress tensor OPE coefficients were considered. In particular, for double stress tensors, such corrections behave like $\Delta^3/\Delta_{\text{gap}}$ which is clearly at odds with the universality statement. Of course, the results of [73] are obtained in the limit $\Delta \ll \Delta_{\text{gap}}$, while in this paper we consider the opposite regime $\Delta \gg \Delta_{\text{gap}}$.

One may also wonder what happens with the universality of the OPE coefficients beyond leading order in Δ. In particular, in [78], it was shown that the bootstrap result for the HHLL correlator exactly matches the holographic Wilson line calculation (in the double scaling limit where only the minimal twist multi stress tensor operators contribute). This corresponds to including terms beyond the exponential of the stress tensor block — one needs to compute the HHLL correlator, take a logarithm of the result, divide by Δ, and then take the large-Δ limit. The result is sensitive to terms subleading in the large-Δ limit of the multi stress tensor OPE coefficients. In four spacetime dimensions the result in [78] is given by an elliptic integral — is it applicable beyond holography?

In [66] terms subleading in Δ were shown to be important for the computation of the phase shift. The simplest nontrivial case in two spacetime dimensions is the operator Λ_4 which is a level four Virasoro descendant of the identity (see e.g. [91]). One could also get it by using the CFT normal ordering and imposing the quasi-primary condition [92]. Consider now the case of minimal twist (twist four) operators in four dimensions. How do we determine the analog of Λ_4? There is no Virasoro algebra now.

Presumably, one can reconstruct the analog of Λ_4 in four spacetime dimensions by considering a CFT normal ordered product of stress tensors, and adding a single trace term to ensure that the resulting operator is a primary and is orthogonal to the stress tensor itself. Note that the CFT normal ordering differs from the oscillator normal ordering in a free theory by the addition of a single trace operator, as reviewed in appendix G. This procedure can then be generalized to other multi-trace operators. We leave it for future work.

It is also helpful to imagine what happens in a theory like $\mathcal{N} = 4$ Super Yang-Mills, where there is a marginal line connecting the weak and the strong coupling (the latter admits a holographic description). Presumably, as the coupling is turned on, only one operator remains light (with dimension eight and spin four), while others get anomalous dimensions. It would be interesting to see this explicitly even to the leading nontrivial order in the 't Hooft coupling. It would also be interesting to see how the corresponding OPE coefficient interpolates between its free and strong coupling values.

[17] See [33] for previous work on the eikonalization of the multi stress tensor OPE coefficients at large spin.

Using crossing symmetry, we argued that the universality of multi stress tensor OPE coefficients is related to the OPE coefficients $\lambda_{\mathcal{O}_H T_{\mu\nu} \mathcal{O}'}$, with $\mathcal{O}' \neq \mathcal{O}_H$ being either heavy or light, present in the cross-channel expansion. Such OPE coefficients with at least one operator being heavy were recently studied in [23, 93]. It would be interesting to further study the connection of our results to this work.

Another interesting question concerns the fate of the double trace operators of the schematic form $[\mathcal{O}_\Delta \mathcal{O}_\Delta]_{n,l}$. Consider the $d = 4$ case in the large volume limit and $n, l = 0$, for simplicity. We expect that the corresponding OPE coefficients in the free theory behave like $\lambda_{\mathcal{O}_H \mathcal{O}_H [\mathcal{O}_\Delta \mathcal{O}_\Delta]_{0,0}} \propto \Delta_H^2 / C_T \propto C_T \mu^2$,[18] while their thermal one-point functions behave like $\langle [\mathcal{O}_\Delta \mathcal{O}_\Delta]_{0,0} \rangle_\beta \propto C_T \beta^{-2\Delta}$. Comparing the two results with the help of (4.2) one observes that such operators do not thermalize in the free theory for generic Δ. The situation is more nontrivial in holography where we do not know the large μ behavior of the OPE coefficients.[19] As pointed out in [63], the contribution of double-trace operators to thermal two-point functions is different from that of multi stress tensors. The latter is only sensitive to the behavior of the metric near the boundary, but the former knows about the full black hole metric. This seems to indicate that the thermalization of the double trace operators in holographic theories is also unlikely.[20]

It is a natural question how generic are the heavy states for which the stress tensor sector thermalizes. The results of our paper seem to suggest that such thermalization is more generic than the thermalization of other light operators.[21] Other interesting questions include generalizations to the case of finite but large central charge and to non-conformal quantum field theories.

Acknowledgments

We thank Aleksandar Bukva, Ilija Burić, Sašo Grozdanov, Manuela Kulaxizi, Eric Perlmutter and Larry Yaffe for discussions, correspondence and comments on the draft. The work of R.K. and A.P. is supported in part by an Irish Research Council Laureate Award. The work of P.T. is supported in part by an Ussher Fellowship Award.

A OPE coefficients from Wick contractions

In this appendix we go through the calculations needed for finding the OPE coefficients of various operators using Wick contractions. This mainly amounts to counting the number of contractions leading to a planar diagram. For simplicity, the figures are shown for external

[18] This scaling is obtained by computing the OPE coefficient $\lambda_{\mathcal{O}_H \mathcal{O}_H [\mathcal{O}_\Delta \mathcal{O}_\Delta]_{0,0}}$ for $1 \ll \Delta_H \ll C_T$ and extrapolating it to the $\Delta_H \sim C_T$ regime.

[19] Note that the large-N scaling in holography is different. Both the OPE coefficients and the thermal expectation values behave like C_T^0 as opposed to $C_T \sim N^2$.

[20] A simple way to decouple such operators is to take the large-Δ limit.

[21] A closely related question of finding "typical" states where the stress tensor sector thermalizes in the large volume limit in $d = 2$ was recently discussed in [58]. There it was observed that such states are Virasoro descendants when the central charge is finite.

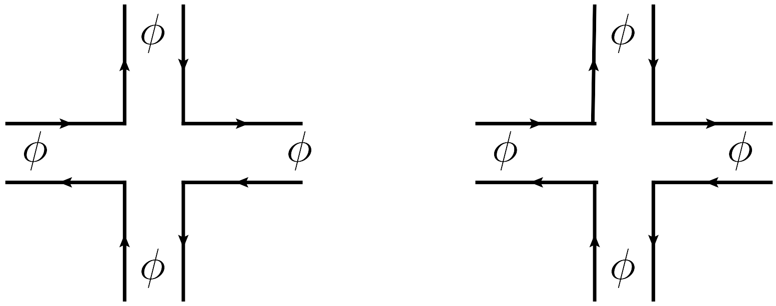

Figure 1. The two-point function $\langle: Tr(\phi^4) :: Tr(\phi^4) :\rangle$ before any contractions.

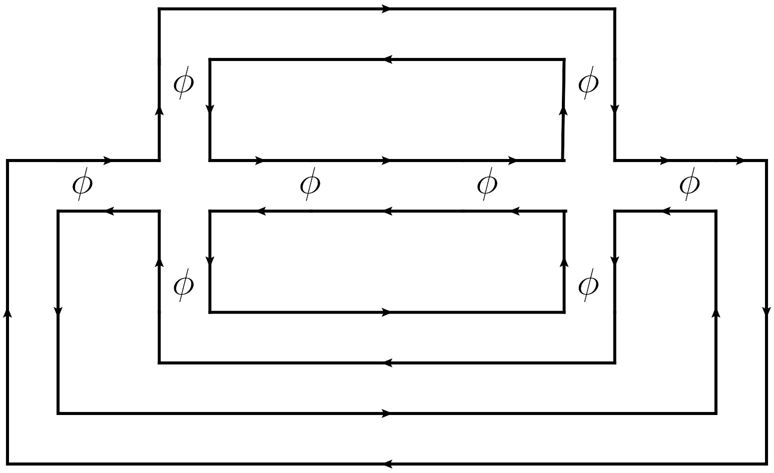

Figure 2. The two-point function $\langle: Tr(\phi^4) :: Tr(\phi^4) :\rangle$ completely contracted.

operators with $\Delta = 4$ while we write down the result for general Δ as this is needed for the main body of the paper.

To begin with, since we consider a large-N matrix theory, it is convenient to use the double-line notation for fundamental field propagators. In figure 1 the two-point function $\langle: Tr(\phi^4) :: Tr(\phi^4) :\rangle$ is visualised. In figure 2, the planar diagram is shown for $\Delta = 4$ and there are Δ number of such contractions giving a planar diagram

$$P_{\langle:Tr(\phi^\Delta)::Tr(\phi^\Delta):\rangle} = \Delta, \tag{A.1}$$

where the $P_{\langle\ldots\rangle}$ denotes the number of planar diagrams for $\langle\ldots\rangle$.

We further need the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}$. This is shown in figure 3 for $\Delta = 4$ and there are 2Δ possibilities for step (1), Δ number of possibilites for step (2) after which everything is fixed assuming that the diagram is planar. This gives

$$P_{\langle:Tr(\phi^\Delta)::Tr(\phi^\Delta)::Tr(\phi^2):\rangle} = 2\Delta^2. \tag{A.2}$$

In figure 4 the three-point function $\langle: Tr(\phi^\Delta) :: Tr(\phi^\Delta) :: Tr(\phi^4) :\rangle$ for $\Delta = 4$ is shown. For the first contraction (1) there are 2Δ possibilites, for the second contraction there are Δ and for step (3) there are two possibilites. This gives overall

$$P_{\langle:Tr(\phi^\Delta)::Tr(\phi^\Delta)::Tr(\phi^4):\rangle} = 4\Delta^2. \tag{A.3}$$

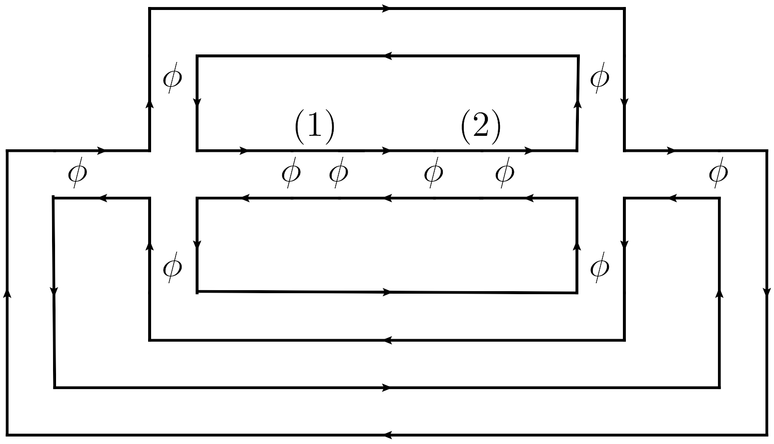

Figure 3. The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2) : \rangle$ completely contracted.

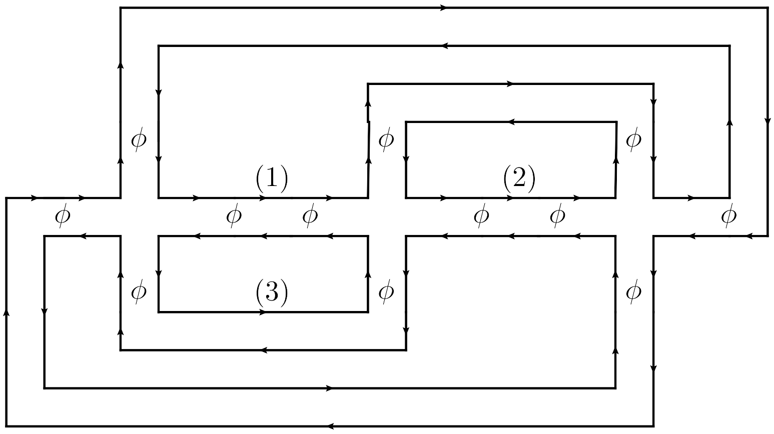

Figure 4. The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^4) : \rangle$ completely contracted.

In figure 5 and figure 6, the three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : Tr(\phi^2)Tr(\phi^2) : \rangle$ is shown. The reason for there being two different types of diagrams is because each trace term in the double trace operator $: Tr(\phi^2)Tr(\phi^2) :$ can either be contracted with the same $: Tr(\phi^4) :$ (figure 5, type B), or to both (figure 6, type A).

Consider first the type of diagrams in figure 5. For the first contraction there are 2Δ such terms and the second contraction gives another factor of 2. Contraction (3) and (4) contributes factors of Δ and 2 respectively. What remains is equivalent to the two-point function $\langle : Tr(\phi^{\Delta-2}) :: Tr(\phi^{\Delta-2}) : \rangle$ which further give a factor of $(\Delta - 2)$ and therefore there are $8\Delta^2(\Delta - 2)$ contractions of type B in figure 5.

Continuing with figure 6, the first contraction gives a factor of 2Δ, the second contraction Δ and the third one a factor of $2(\Delta - 1)$. What remains is then fixed by imposing that the diagram is planar. The type A diagrams in figure 6 therefore further contributes $4\Delta^2(\Delta - 1)$ planar diagrams to $\langle : Tr(\phi^\Delta) :: Tr(\phi^\Delta) : Tr(\phi^2)Tr(\phi^2) : \rangle$. It is therefore found that

$$P_{\langle : Tr(\phi^\Delta) :: Tr(\phi^\Delta) : Tr(\phi^2)Tr(\phi^2) : \rangle} = 4\Delta^2(3\Delta - 5). \tag{A.4}$$

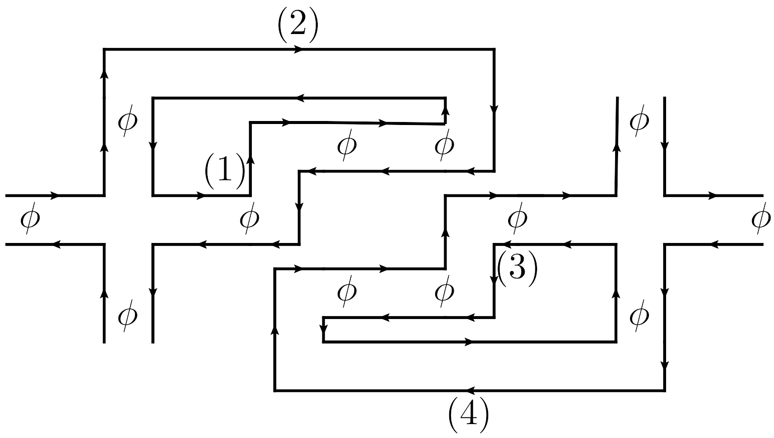

Figure 5. The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2)Tr(\phi^2) : \rangle$. There are two such types of contractions that give planar diagrams, here it shown when each $: Tr(\phi^2) :$ connect to a separate $: Tr(\phi^4) :$.

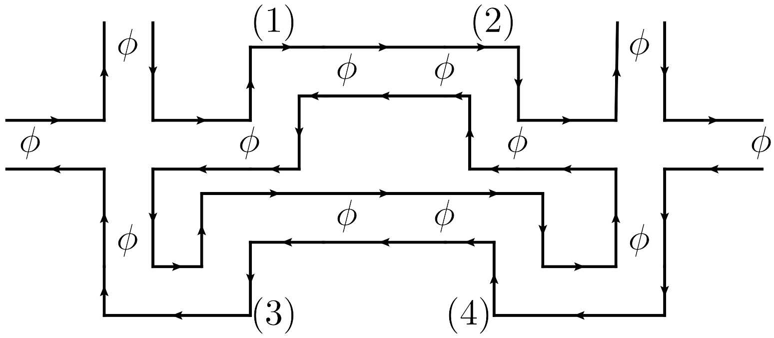

Figure 6. The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2)Tr(\phi^2) : \rangle$. There are two such types of contractions that give planar diagrams, here it shown when each $: Tr(\phi^2) :$ connect to both $: Tr(\phi^4) :$ operators.

Consider now the stress tensor OPE coffiecient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}}$ where

$$T_{\mu\nu}(x) = \frac{1}{2\sqrt{3}N} : Tr\left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}\phi \partial_\mu \partial_\nu \phi - (\text{trace})\right) : (x) \tag{A.5}$$

and the three-point function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu} \rangle$:

$$\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) T_{\mu\nu}(x_3) \rangle = \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}} \frac{Z_\mu Z_\nu - \text{traces}}{|x_{12}|^{2\Delta-2}|x_{23}|^2|x_{13}|^2}, \tag{A.6}$$

where $Z_\mu = \frac{x_{13\mu}}{|x_{13}|^2} - \frac{x_{12\mu}}{|x_{12}|^2}$. From the definition of $T_{\mu\nu}$ in (A.5) it is clear that the only term that contributes to term $x_{13\mu}x_{13\nu}$ comes from the second term in (A.5) that is of the form $\propto Tr(\phi \partial_\mu \partial_\nu \phi)$. Up to the derivatives, the diagram will look like those visualised in figure 3. The number of diagrams is half of that given in (A.2) since we restrict to terms proportional to $x_{13\mu}x_{13\nu}$:

$$P_{\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu} \rangle |_{x_{13\mu}x_{13\nu}}} = \Delta^2, \tag{A.7}$$

from which we reproduce (3.7).

Now we want to find the OPE coefficient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4}}$ for the double-stress tensor $T^2_{4,4}$. This is done similarly to the way the stress tensor OPE coefficient was found. First, the operator $(T^2)_{\mu\nu\rho\sigma}$ was given in (3.8) to be

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu} T_{\rho\sigma)} : (x) - (\text{traces}) \tag{A.8}$$

and the three-point function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta (T^2)_{\mu\nu\rho\sigma} \rangle$ is fixed by conformal symmetry to be

$$\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) (T^2)_{\mu\nu\rho\sigma}(x_3) \rangle = \frac{\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4}}}{|x_{12}|^{2\Delta-4} |x_{13}|^4 |x_{23}|^4} \left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces}) \right). \tag{A.9}$$

Consider the term in (A.9) proportional to $x_{13\mu} x_{13\nu} x_{13\rho} x_{13\sigma}$. This will be due to the term in $(T^2)_{\mu\nu\rho\sigma}$ of the form $Tr(\phi \partial_{(\mu} \partial_\nu \phi) Tr(\phi \partial_\rho \partial_{\sigma)} \phi)$. Using this we find that

$$\begin{aligned} \langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) (T^2)_{\mu\nu\rho\sigma}(x_3) \rangle|_{x_{13\mu} x_{13\nu} x_{13\rho} x_{13\sigma}} &= \frac{1}{\Delta N^\Delta} \frac{1}{\sqrt{2}} \left(\frac{-1}{4\sqrt{3} N} \right)^2 8^2 N^\Delta \\ &\times \frac{P_{\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4} \rangle|_{x_{13\mu} x_{13\nu} x_{13\rho} x_{13\sigma}}}}{|x_{12}|^{2(\Delta-2)} |x_{23}|^4 |x_{13}|^{12}}. \end{aligned} \tag{A.10}$$

The number of contractions giving a planar diagram, $P_{\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4} \rangle|_{x_{13\mu} x_{13\nu} x_{13\rho} x_{13\sigma}}}$, come from diagrams of the form given in figure 6. Since we are considering the term proportional $x_{13\mu} x_{13\nu} x_{13\rho} x_{13\sigma}$, the number of such diagrams are reduced compared to scalar double trace operator. Instead the first contraction, (1) in figure 6, give a factor of Δ, the second contraction, (2), a factor of $(\Delta - 1)$, the third contraction (3) gives a further factor Δ after which everything is fixed by imposing that the diagram is planar. We therefore find that

$$P_{\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4} \rangle|_{x_{13\mu} x_{13\nu} x_{13\rho} x_{13\sigma}}} = \Delta^2 (\Delta - 1), \tag{A.11}$$

and inserting this in (A.10) gives

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4}} = \frac{2\sqrt{2} \Delta (\Delta - 1)}{3N^2}, \tag{A.12}$$

and therefore reproduces (3.11).

Similar to the double-stress tensor, consider the dimension-eight spin-four double trace operator

$$\begin{aligned} \mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70} N^2} : Tr(\phi^2) \Big(& Tr(\phi \partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi) - 16 Tr(\partial_{(\mu} \phi \partial_\nu \partial_\rho \partial_{\sigma)} \phi) \\ & + 18 Tr(\partial_{(\mu} \partial_\nu \phi \partial_\rho \partial_{\sigma)} \phi)(x) - (\text{traces}) \Big) : (x). \end{aligned} \tag{A.13}$$

The three-point function $\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) \mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}(x_3) \rangle$ is given by

$$\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) \mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}(x_3) \rangle = \frac{\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}}}{|x_{12}|^{2\Delta-4} |x_{13}|^4 |x_{23}|^4} \left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces}) \right). \tag{A.14}$$

By again considering terms in (A.14) proportional to $x_{13\mu} x_{13\nu} x_{13\rho} x_{13\sigma}$ we find that each term in (A.13) will contribute planar diagram of the type in figure 5, while only the term $\sim Tr(\phi\partial^4\phi)$ also give a contribution of the type in figure 6. Considering first the terms coming from the diagram in figure 5, one finds that this contribution vanishes. The remaining contribution to the term (A.14) proportional to $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$ comes from the first term in (A.13) and the planar diagram pictured in figure 6; there are $2\Delta^2(\Delta-1)$ contractions giving such a planar diagram leading to

$$\begin{aligned}\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}(x_3)\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}} &= \frac{1}{\Delta N^\Delta}\frac{384}{96\sqrt{70}N^2}N^\Delta \\ &\times\frac{2\Delta^2(\Delta-1)}{|x_{12}|^{2(\Delta-2)}|x_{23}|^4|x_{13}|^{12}},\end{aligned} \tag{A.15}$$

where the 384 in the numerator come from the derivatives. This gives the OPE coefficient:

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}} = \sqrt{\frac{2}{35}}\frac{4\Delta(\Delta-1)}{N^2}+\mathcal{O}(N^{-4}). \tag{A.16}$$

B Subleading twist double-stress tensors

In this appendix we study the subleading twist double-stress tensors, both with dimension 8 and spin $s=0,2$ denoted (T^2) and $(T^2)^{\mu\nu}$ respectively. The calculations needed to find the OPE coefficient in the $\mathcal{O}_\Delta\times\mathcal{O}_\Delta$ OPE are reviewed as well as the normalization of $(T^2)^{\mu\nu}$.

The $(T^2)^{\mu\nu}$ was defined in (3.24) which we repeat here:

$$(T^2)^{\mu\nu}(x) = \frac{1}{\sqrt{2}}:T^\mu{}_\alpha T^{\alpha\nu}:(x) - \frac{\delta^{\mu\nu}}{4\sqrt{2}}:T^\beta{}_\alpha T^\alpha{}_\beta:(x). \tag{B.1}$$

The operator $(T^2)^{\mu\nu}$ can be seen to be unit-normalized to leading order in N:

$$\begin{aligned}\langle(T^2)^{\mu\nu}(x_1)(T^2)_{\rho\sigma}(x_2)\rangle &= \frac{1}{\sqrt{2}}\langle T^{\mu\alpha}(x_1)T_{\rho\beta}(x_2)\rangle\langle T^\nu{}_\alpha(x_1)T^\beta{}_\sigma\rangle \\ &+(\rho\longleftrightarrow\sigma)-(\text{traces})+\mathcal{O}(N^{-2}).\end{aligned} \tag{B.2}$$

Using the two-point function of the stress tensor in (3.6) and $I^\mu{}_\alpha I^\alpha{}_\rho = \delta^\mu{}_\rho$ one finds

$$\langle(T^2)^{\mu\nu}(x_1)(T^2)_{\rho\sigma}(x_2)\rangle = \frac{1}{|x|^{16}}\left(I^{(\mu}{}_\rho I^{\nu)}{}_\sigma - (\text{traces})\right), \tag{B.3}$$

from which it is seen that $(T^2)^{\mu\nu}$ is unit-normalised.

We now want to find the OPE coefficient of $(T^2)^{\mu\nu}$ in the $\mathcal{O}_\Delta\times\mathcal{O}_\Delta$ OPE. It can be found from the basic objects $I^{(1)}_{\mu\nu\rho\sigma}$, $I^{(2)}_{\mu\nu\rho\sigma}$ and $I^{(3)}_{\mu\nu\rho\sigma}$ which we calculate below.

We first consider a similar quantity $J^{(1)\mu\nu\rho\sigma}$:

$$\begin{aligned}J^{(1)\mu\nu\rho\sigma} &= \langle:Tr(\phi^\Delta):(x_1):Tr(\phi^\Delta):(x_2)::Tr(\partial_\mu\phi\partial_\nu\phi)Tr(\partial_\rho\phi\partial_\sigma\phi):(x_3)\rangle \\ &= \frac{2^4N^\Delta}{|x_{13}|^8|x_{23}|^8|x_{12}|^{2\Delta-4}}\times\Big[(2\Delta)^2(\Delta-2)(x^\mu_{13}x^\nu_{13}x^\rho_{23}x^\sigma_{23}+x^\mu_{23}x^\nu_{23}x^\rho_{13}x^\sigma_{13})+ \\ &\quad\Delta^2(\Delta-1)(x^\mu_{13}x^\nu_{23}(x^\rho_{13}x^\sigma_{23}+x^\rho_{23}x^\sigma_{13})+x^\mu_{23}x^\nu_{13}(x^\rho_{13}x^\sigma_{23}+x^\rho_{23}x^\sigma_{13}))\Big].\end{aligned} \tag{B.4}$$

Definining $X_{13}^{\mu\nu} = \frac{1}{|x_{13}|^4}(-\delta^{\mu\nu} + 4\frac{x_{13}^\mu x_{13}^\nu}{|x_{13}|^2})$ we then study $J^{(2)\mu\nu\rho\sigma}$:

$$\begin{aligned}J^{(2)\mu\nu\rho\sigma} &= \langle : Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\phi\partial_\mu\partial_\nu\phi)Tr(\phi\partial_\rho\partial_\sigma\phi) : (x_3)\rangle \\ &= \frac{N^\Delta}{|x_{12}|^{2\Delta-4}}\Big[\Delta^2(\Delta-1)2^2\left(X_{13}^{\mu\nu}\frac{1}{|x_{23}|^2}X_{13}^{\rho\sigma}\frac{1}{|x_{23}|^2} + X_{13}^{\mu\nu}\frac{1}{|x_{23}|^2}X_{23}^{\rho\sigma}\frac{1}{|x_{13}|^2}\right) \\ &\quad + ((2\Delta)^2(\Delta-2))2^2 X_{13}^{\mu\nu}\frac{1}{|x_{13}|^2}X_{23}^{\rho\sigma}\frac{1}{|x_{23}|^2} \\ &\quad + (13) \longleftrightarrow (23)\Big]. \end{aligned} \tag{B.5}$$

And lastly $J^{(3)\mu\nu\rho\sigma}$:

$$\begin{aligned}J^{(3)\mu\nu\rho\sigma} &= \langle : Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\phi\partial_\mu\partial_\nu\phi)Tr(\partial_\rho\phi\partial_\sigma\phi) : (x_3)\rangle \\ &= \frac{N^\Delta}{|x_{12}|^{2\Delta-4}}\Big[((2\Delta)^2(\Delta-2))2^3 X_{13}^{\mu\nu}\frac{1}{|x_{13}|^2}\frac{x_{23}^\rho x_{23}^\sigma}{|x_{23}|^8} + \\ &\quad + \Delta^2(\Delta-1)2^3 X_{13}^{\mu\nu}\frac{1}{|x_{23}|^2}\frac{x_{13}^\rho x_{23}^\sigma + x_{23}^\rho x_{13}^\sigma}{|x_{13}|^4|x_{23}|^4} \\ &\quad + (13) \longleftrightarrow (23)\Big]. \end{aligned} \tag{B.6}$$

We further need to make (B.4)–(B.6) traceless in the pairs (μ,ν) and (ρ,σ) and therefore define $I^{(i)\mu\nu\rho\sigma}$ as

$$I^{(i)\mu\nu\rho\sigma} = J^{(i)\mu\nu\rho\sigma} - \frac{\delta^{\mu\nu}}{4}J^{(i)\alpha}{}_\alpha{}^{\rho\sigma} - \frac{\delta^{\rho\sigma}}{4}J^{(i)\mu\nu\alpha}{}_\alpha + \frac{\delta^{\mu\nu}\delta^{\rho\sigma}}{16}J^{(i)\alpha}{}_\alpha{}^\gamma{}_\gamma. \tag{B.7}$$

From (B.4)–(B.6), the three-point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle$ is given by

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}\rangle = \frac{1}{12\sqrt{2}\Delta N^{\Delta+2}}\left(I^{(1)(\mu|\alpha}{}_\alpha{}^{|\nu)} - I^{(3)(\mu|\alpha}{}_\alpha{}^{|\nu)} + \frac{1}{4}I^{(2)(\mu|\alpha}{}_\alpha{}^{|\nu)} - (\text{trace})\right). \tag{B.8}$$

Explicitly we find that

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle = \frac{\sqrt{2}\Delta(\Delta-1)}{3N^2}\frac{Z^\mu Z^\nu - (\text{trace})}{|x_{12}|^{2\Delta-6}|x_{13}|^6|x_{23}|^6} + \mathcal{O}(N^{-4}). \tag{B.9}$$

Consider now the scalar operator (T^2) defined by

$$(T^2)(x) = \frac{1}{36\sqrt{2}N^2} : T_{\mu\nu}T^{\mu\nu} : (x). \tag{B.10}$$

The three-point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle$ can be found using $I^{(i)}$ defined in (B.7) as follows

$$\begin{aligned}\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle &= \frac{1}{36\sqrt{2}\Delta N^{2+\Delta}}\left(I^{(1)\mu\nu}{}_{\mu\nu} - I^{(3)\mu\nu}{}_{\mu\nu} + \frac{1}{4}I^{(2)\mu\nu}{}_{\mu\nu}\right) + \mathcal{O}(N^{-4}) \\ &= \frac{\Delta(\Delta-1)}{3\sqrt{2}N^2}\frac{1}{|x_{12}|^{2\Delta-8}|x_{13}|^8|x_{23}|^8} + \mathcal{O}(N^{-4}). \end{aligned} \tag{B.11}$$

C Single trace operator with dimension $\Delta \sim C_T$

In this appendix we study the single trace scalar operator $\mathcal{O}_{\Delta_H}$ given by

$$\mathcal{O}_H(x) = \frac{1}{\sqrt{\mathcal{N}_{\Delta_H}}} :Tr(\phi^{\Delta_H}):(x), \tag{C.1}$$

with $\Delta_H \sim C_T$ and $\mathcal{N}_{\Delta_H}$ a normalization constant.[22] When calculating the normalization constant $\mathcal{N}_{\Delta_H}$ as well as the three-point functions $\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\mathcal{O}(x_3)\rangle$, non-planar diagrams generically gets enhanced by powers of Δ_H and therefore invalidates the naive planar expansion. The goal of this appendix is to show that

$$\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\hat{\mathcal{O}}(x_3)\rangle = \langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\hat{\mathcal{O}}(x_3)\rangle|_{\Delta=\Delta_H}, \tag{C.2}$$

where $\hat{\mathcal{O}}$ is either $:Tr(\phi^2):$ or, more importantly, minimal-twist multi stress tensors with any spin. Moreover, note that the l.h.s. in (C.2) is in principle exact in $C_T \sim N^2$ while the r.h.s. is obtained by keeping only planar diagrams with $\Delta \ll C_T$ and then setting $\Delta = \Delta_H$ in the end.

The propagator for the field ϕ was given in (3.2) by

$$\langle {\phi^i}_j(x){\phi^k}_l(y)\rangle = \left({\delta^i}_l{\delta^k}_j - \frac{1}{N}{\delta^i}_j{\delta^k}_l\right)\frac{1}{|x-y|^2}. \tag{C.3}$$

Consider now the three-point function $\langle :Tr(\phi^{\Delta_H}):(x_1):Tr(\phi^{\Delta_H}):(x_2):Tr(\phi^2):(x_3)\rangle$. Due to the normal ordering, one ϕ field in $:Tr(\phi^2):(x_3)$ need to be contracted with $:Tr(\phi^{\Delta_H}):(x_1):$ and the other one with $:Tr(\phi^{\Delta_H}):(x_2):$. Note that for this contraction the second term in (C.3) give a contribution proportional to $Tr(\phi(x_3)) = 0$. It is therefore seen that

$$\langle :Tr(\phi_1^{\Delta_H})::Tr(\phi_2^{\Delta_H})::Tr(\phi_3^2):\rangle = 2\Delta_H\langle :Tr(\phi_3\phi_1^{\Delta_H-1})::Tr(\phi_2^{\Delta_H}):\rangle, \tag{C.4}$$

where we introduced the notation $\phi_i = \phi(x_i)$ and dropped the $|x_{ij}|^{-2}$ coming from (C.3). The position dependence is easily restored in the end. Now it is seen that the r.h.s. of (C.4) is proportional to the two-point function[23] of $\mathcal{O}_H$ and we therefore find that

$$\langle :Tr(\phi_1^{\Delta_H})::Tr(\phi_2^{\Delta_H})::Tr(\phi_3^2):\rangle = 2\Delta_H\mathcal{N}_{\Delta_H}, \tag{C.5}$$

which is exact to all orders in C_T. Including the normalization factor of $\mathcal{O}_H$ in (C.1) and $\mathcal{O}_2$ from (3.3) we find that

$$\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\mathcal{O}_2(x_3)\rangle = \frac{\sqrt{2}\Delta_H}{N}\frac{1}{|x_{12}|^{2\Delta_H-2}|x_{13}|^2|x_{23}|^2} + \mathcal{O}(N^{-3}). \tag{C.6}$$

By comparing (C.6) with (5.10) we find that

$$\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_2} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}|_{\Delta=\Delta_H}. \tag{C.7}$$

[22] Mixing with other operators with $\Delta \sim C_T$ is not important for this discussion.

[23] Up to the position dependence.

Note that in (C.6) the normalization of $\mathcal{O}_H$ cancels the contribution from non-planar diagrams in limit $\Delta_H \sim C_T$. For $\Delta = 2$ in (3.3), it is trivial to compute the normalization exact in N to get the correction to $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}$ in (C.6).

Consider now the stress tensor operator defined in (3.5) and the three-point function $\langle\mathcal{O}_H(x_1)\mathcal{O}_H(x_2)T_{\mu\nu}(x_3)\rangle$. This is fixed by the Ward identity but is an instructive example before considering more general multi stress tensors. In the same way as the OPE coefficient was found in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE, due to the tensor structure being fixed by conformal symmetry, we consider the term proportional to $x_{13}^\mu x_{13}^\nu$ in the three-point function. This comes from the $-\frac{1}{6\sqrt{C_T}}Tr(\phi\partial_\mu\partial_\nu\phi)$ term in the stress tensor when $\partial_\mu\partial_\nu\phi$ is contracted with one of the Δ_H number of $\phi(x_1)$ fields. Doing this contraction we therefore see that

$$\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) :: Tr(\phi_3\partial_\mu\partial_\nu\phi_3) :\rangle|_{x_{13}^\mu x_{13}^\nu} = 8\Delta_H\langle : Tr(\phi_3\phi_1^{\Delta_H-1}) :: Tr(\phi_2^{\Delta_H}) :\rangle, \tag{C.8}$$

where the factor 8 comes from the derivatives and we again suppress the spacetime dependence. The r.h.s. of (C.8) is also proportional to the normalization constant of $\mathcal{O}_H$. Including the normalization factor of the stress tensor in (3.5) and that of $\mathcal{O}_H$ in (C.1), the three-point function $\langle\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}\rangle$ can be obtained from (C.8) from which we read off the OPE coefficient

$$\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}} = -\frac{4\Delta_H}{3\sqrt{C_T}}. \tag{C.9}$$

This agrees with (3.7).

We now want to show that is true for minimal-twist multi stress tensors with any spin. For simplicity, consider the double-stress tensor with spin 4 defined in (3.8)

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : (x) - (\text{traces}). \tag{C.10}$$

Similarly to the calculation of the three-point function with the stress tensor, we can obtain the three-point function $\langle\mathcal{O}_H(x_1)\mathcal{O}_H(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle$ by considering the term proportional to $x_{13}^\mu x_{13}^\nu x_{13}^\rho x_{13}^\sigma$. This will be due to the term $\frac{1}{\sqrt{2}6^2 C_T}Tr(\phi\partial_\mu\partial_\nu\phi)Tr(\phi\partial_\rho\partial_\sigma\phi)$ when contracting $\partial_\mu\partial_\nu\phi$ with some $\phi(x_1)$ and likewise contracting $\partial_\rho\partial_\sigma\phi$ with some other $\phi(x_1)$. The number of such contractions is given by $\Delta_H(\Delta_H-1)$ and we find that

$$\begin{aligned}\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) : : Tr(\phi_3\partial_\mu\partial_\nu\phi_3)Tr(\phi_3\partial_\rho\partial_\sigma\phi_3) :\rangle|_{x_{13}^\mu x_{13}^\nu x_{13}^\rho x_{13}^\sigma} \\ = 8^2\Delta_H(\Delta_H-1)\langle : Tr(\phi_3^2\phi_1^{\Delta_H-2}) :: Tr(\phi_2^{\Delta_H}) :\rangle,\end{aligned} \tag{C.11}$$

where the factor of 8^2 again is due to acting with the derivatives and note that the position of the ϕ_3 fields in the last line is not important. It is again seen that the r.h.s. of (C.11) is proportional to the normalization constant of $\mathcal{O}_H$. Including the normalization in (3.8) and (C.1) we find the three-point function $\langle\mathcal{O}_H\mathcal{O}_H(T^2)_{\mu\nu\rho\sigma}\rangle$ and read off the OPE coefficient:

$$\lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}} = \frac{8\sqrt{2}\Delta_H(\Delta_H-1)}{9C_T} + \mathcal{O}(C_T^{-3/2}), \tag{C.12}$$

which is seen to agree with (3.11) when setting $\Delta_H = \Delta$. Note that the corrections in (C.12) are solely due to corrections in the normalization of $T^2_{4,4}$ and therefore $\lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}} =$

$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,4}}$ to all orders in C_T. These arguments generalize straightforwardly to minimal-twist multi stress tensor with any spin such that the results are the same as those obtained in the planar limit for $\Delta \ll C_T^2$ in section 3 by setting $\Delta_H = \Delta$. The only correction in C_T is then due to the normalization of the multi stress tensor.

The same argument applies to any scalar primary multi-trace operator $\mathcal{O}_\Delta$, without any derivatives, with OPE coefficients given by (C.6), (C.9) and (C.12).

D Stress tensor thermal one-point function

In order to calculate thermal one-point functions in the free adjoint scalar model we use the fact that the thermal correlation function is related to the zero-temperature case by summing over images. Consider now the thermal one-point function of the stress tensor. Generally, the one-point function of a spin-s symmetric traceless operator with dimension $\Delta_{\mathcal{O}}$ on $S^1 \times \mathbf{R}^{d-1}$ is given by [13]

$$\langle \mathcal{O}^{\mu_1 \cdots \mu_s}(x) \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} (e^{\mu_1} \ldots e^{\mu_s} - (\text{traces})), \tag{D.1}$$

where e^{μ_1} is a unit-vector along the thermal circle. Consider first the canonically normalized stress tensor given by $T^{(\text{can})}_{\mu\nu} = \frac{1}{3S_d}(Tr(\partial_\mu \phi \partial_\mu \phi) - \frac{1}{2} Tr(\phi \partial_\mu \partial_\nu \phi) - (\text{traces}))$. In order to find the one-point function, use the following:

$$\langle Tr(\partial^{(x)}_\mu \phi(x) \partial^{(y)}_\nu \phi(y)) \rangle = \frac{2(N^2-1)}{|x-y|^4} \left(\delta_{\mu\nu} - 4(y-x)_\mu (y-x)_\nu \frac{1}{|x-y|^2} \right) \tag{D.2}$$

and

$$\langle Tr(\partial^{(x)}_\mu \partial^{(x)}_\nu \phi(x) \phi(y)) \rangle = \frac{2(N^2-1)}{|x-y|^4} \left(-\delta_{\mu\nu} + 4(y-x)_\mu (y-x)_\nu \frac{1}{|x-y|^2} \right). \tag{D.3}$$

To get the thermal correlator, we use (D.2) and (D.3) with x, y along the thermal circle separated by a distance $m\beta$, with m integer, and sum over $m \neq 0$. The relevant terms for calculating the one-point functions in terms of fundamental fields are therefore

$$\begin{aligned} \langle Tr(\partial_\mu \phi \partial_\nu \phi) \rangle_{\beta,m} &= -\frac{8(N^2-1)}{(m\beta)^4} e^\mu e^\nu + \frac{2(N^2-1)}{(m\beta)^4} \delta_{\mu\nu}, \\ \langle Tr(\partial_\mu \partial_\nu \phi \phi) \rangle_{\beta,m} &= \frac{8(N^2-1)}{(m\beta)^4} e^\mu e^\nu - \frac{2(N^2-1)}{(m\beta)^4} \delta_{\mu\nu}, \end{aligned} \tag{D.4}$$

where we note that only the first term in each equation in (D.4) contribute to the non-trace term in (D.1).

We therefore find for the stress tensor one-point function:

$$\begin{aligned} \langle T^{(\text{can})}_{\mu\nu} \rangle_\beta &= \frac{1}{3S_d} (\langle Tr(\partial_\mu \phi \partial_\nu \phi) \rangle_\beta - \frac{1}{2} \langle Tr(\partial_\mu \partial_\nu \phi \phi) \rangle_\beta - \text{trace}) \\ &= \frac{-12(N^2-1)}{3S_d} \frac{2\zeta(4)}{\beta^4} (e_\mu e_\nu - (\text{trace})), \end{aligned} \tag{D.5}$$

where the $2\zeta(4)$ comes from summing over images and we therefore have

$$b_{T^{(\text{can})}_{\mu\nu}} = -\frac{4(N^2-1)}{S_d}2\zeta(4) = -\frac{4\pi^4}{45S_d}(N^2-1). \tag{D.6}$$

This agrees with $f = \frac{b_{T^{(\text{can})}_{\mu\nu}}}{d}$ in eq. (2.17) in [13] for (N^2-1) free scalar fields. This also agrees with $a_{2,2} = \frac{\pi^4\Delta}{45}$ found from the two-point thermal correlator using:

$$a_{2,2} = \frac{\pi^4\Delta}{45} = \left(\frac{1}{2}\right)^2 \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^{(\text{can})}} b_{T^{(\text{can})}_{\mu\nu}}}{\frac{C_T}{S_d^2}}, \tag{D.7}$$

using $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^{(\text{can})}} = -\frac{4\Delta}{3S_d}$ in this normalization and $C_T = \frac{4}{3}(N^2-1)$. This is simply related to the one-point function for the unit-normalized stress tensor by (to leading order in N)

$$\begin{aligned} b_{T_{\mu\nu}} &= \frac{b_{T^{(\text{can})}_{\mu\nu}}}{\frac{\sqrt{C_T}}{S_d}} \\ &\approx -\frac{2\pi^4 N}{15\sqrt{3}}. \end{aligned} \tag{D.8}$$

Let us now consider the thermalization of the stress tensor, keeping all the index structures. To compare the thermal two-point function with the heavy-heavy-light-light correlator, we want to relate the dimension of the heavy operator, Δ_H, to the inverse temperature β. Consider the expectation value of the stress tensor in a heavy state created by $\mathcal{O}_H$ on the cylinder $\mathbf{R}\times S^3$

$$\langle\mathcal{O}_H|T^{\mu\nu}(x^0_{E,2},\hat{n})|\mathcal{O}_H\rangle_{\text{cyl}} = \lim_{x_3\to\infty}|x_3|^{2\Delta_H}|x_2|^4\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}\frac{Z^\mu Z^\nu - \frac{1}{4}\delta^{\mu\nu}Z^\rho Z_\rho}{|x_{13}|^{2\Delta_H-2}|x_{23}|^2|x_{12}|^2}, \tag{D.9}$$

where the r.h.s. is found by a conformal transformation to the plane with $Z^\mu = \left(\frac{x^\mu_{12}}{|x_{12}|^2} + \frac{x^\mu_{23}}{|x_{23}|^2}\right)$. When $x_1=0$ and $x_3\to\infty$, it is seen that $Z^\mu = -\frac{x^\mu_2}{|x_2|^2}$ and (D.9) only depends on $\hat{x}^\mu = \frac{x^\mu_{21}}{|x_{21}|} = \hat{r}$, where $\hat{r}$ is a radial unit vector. In radial quantization it follows that

$$\langle\mathcal{O}_H|T^{\mu\nu}(x^0_{E,2},\hat{n})|\mathcal{O}_H\rangle_{\text{cyl}} = \frac{\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}}{R^4}\left(\hat{e}_\mu\hat{e}_\nu - \frac{1}{4}\delta_{\mu\nu}\right) \tag{D.10}$$

where we reintroduced the radius of the sphere R, $\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}$ is the OPE coefficient of $T_{\mu\nu}$ in the $\mathcal{O}_H\times\mathcal{O}_H$ OPE and $\hat{e}_\mu = (1,0,0,0)$.

The thermal one-point function of an operator $\mathcal{O}_{\tau,s}$, with twist τ and spin s, on $S^1\times S^3$ is fixed by conformal symmetry [13]

$$\langle\mathcal{O}_{\tau,s}(x)\rangle_\beta = \frac{b_{\mathcal{O}_{\tau,s}} f_{\mathcal{O}_{\tau,s}}(\frac{\beta}{R})}{\beta^{\tau+s}}(e^{\mu_1}\cdots e^{\mu_s} - (\text{traces})), \tag{D.11}$$

where $f_{\mathcal{O}_{\tau,s}}(0) = 1$ and $e^\mu = (1,0,0,0)$.

We assume thermalization of the stress tensor in the heavy state:

$$\langle \mathcal{O}_H | T_{\mu\nu}(x) | \mathcal{O}_H \rangle = \langle T_{\mu\nu}(x) \rangle_\beta \tag{D.12}$$

where $\langle T^{\mu\nu}(x)\rangle_\beta$ is the thermal one-point function at inverse temperature β evaluated on $S^1 \times S^3$, with R being the radius of S^3. Using (D.10)–(D.12) we find

$$\frac{\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}}}{R^4} = \frac{b_{T_{\mu\nu}} f_{T_{\mu\nu}}(\frac{\beta}{R})}{\beta^4}. \tag{D.13}$$

Using (D.13) for $R \to \infty$ in the free adjoint scalar theory, together with the one-point function $b_{T_{\mu\nu}} = -\frac{2\pi^4 N}{15\sqrt{3}}$ and the OPE coefficient $\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}} = -\frac{4\Delta_H}{3\sqrt{C_T}}$, one finds the following relation between $\mu = \frac{160\Delta_H}{3C_T}$ and the inverse temperature β:

$$\mu = \frac{8}{3}\left(\frac{\pi R}{\beta}\right)^4. \tag{D.14}$$

This agrees with (4.2).

E Dimension-six spin-four single trace operator

We want to calculate the contribution of the single trace operator with $\tau = 2$ and $s = 4$. The unit-normalised $\mathcal{O}_{2,4}$ operator is given by[24]

$$\begin{aligned}\Xi_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{35N}} : Tr(&\phi(\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16(\partial_{(\mu}\phi)(\partial_\nu\partial_\rho\partial_{\sigma)}\phi) \\ &+ 18(\partial_{(\mu}\partial_\nu\phi)(\partial_\rho\partial_{\sigma)}\phi) - (\text{traces})) : (x).\end{aligned} \tag{E.1}$$

The relative coefficients are fixed by demanding that it is a primary operator $[K_\alpha, \Xi_{\mu\nu\rho\sigma}] = 0$. Explicitly, this is done using the conformal algebra

$$\begin{aligned}[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu} D - M_{\mu\nu}), \\ [M_{\mu\nu}, P_\rho] &= -i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu),\end{aligned} \tag{E.2}$$

and the action on the fundamental field ϕ

$$\begin{aligned}P_\mu \phi(0) &= -i\partial_\mu \phi(0), \\ D\phi(0) &= i\phi(0).\end{aligned} \tag{E.3}$$

The relevant commutators in order to fix $\Xi_{\mu\nu\rho\sigma}$ are

$$\begin{aligned}[K_\alpha, P_\mu\phi] =& -2\eta_{\alpha\mu}\phi, \\ [K_\alpha, P_\mu P_\nu\phi] =& -4\eta_{\alpha\mu}P_\nu\phi - 4\eta_{\alpha\nu}P_\mu\phi + 2\eta_{\mu\nu}P_\alpha\phi, \\ [K_\alpha, P_\mu P_\nu P_\rho\phi] =& -6\eta_{\alpha\mu}P_\nu P_\rho\phi - 6\eta_{\alpha\nu}P_\mu P_\rho\phi - 6\eta_{\alpha\rho}P_\nu P_\mu\phi \\ &+ 2\eta_{\mu\nu}P_\rho P_\alpha\phi + 2\eta_{\rho\nu}P_\mu P_\alpha\phi + 2\eta_{\mu\rho}P_\nu P_\alpha\phi, \\ [K_\alpha, P_\mu P_\nu P_\rho P_\sigma\phi] =& -8\eta_{\alpha\mu}P_\nu P_\rho P_\sigma\phi - 8\eta_{\alpha\nu}P_\mu P_\rho P_\sigma\phi - 8\eta_{\alpha\rho}P_\nu P_\mu P_\sigma\phi - 8\eta_{\alpha\sigma}P_\nu P_\rho P_\mu\phi \\ &+ 2\eta_{\mu\nu}P_\rho P_\sigma P_\alpha\phi + 2\eta_{\mu\rho}P_\nu P_\sigma P_\alpha\phi + 2\eta_{\mu\sigma}P_\rho P_\nu P_\alpha\phi + 2\eta_{\nu\rho}P_\mu P_\sigma P_\alpha\phi \\ &+ 2\eta_{\nu\sigma}P_\mu P_\rho P_\alpha\phi + 2\eta_{\rho\sigma}P_\mu P_\nu P_\alpha\phi,\end{aligned} \tag{E.4}$$

which can also be found in e.g. appendix F in [94].

[24] We denote this operator either as $\mathcal{O}_{2,4}$ or $\Xi_{\mu\nu\rho\sigma}$ depending whether we want to explicitly list the indices or not.

The thermal one-point function of this operator is found from Wick contractions to be

$$\langle \Xi_{\mu\nu\rho\sigma} \rangle_\beta = \frac{8(\pi T)^6 N}{27\sqrt{35}} \left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right). \tag{E.5}$$

Moreover, the three-point function with operators $\mathcal{O}_\Delta(x) = \frac{1}{\sqrt{\Delta N^\Delta}} : Tr\left(\phi^\Delta\right) : (x)$ can again be calculated using Wick contractions similarly to how it was done for $T^2_{\mu\nu\rho\sigma}$ in appendix A. By explicit calculation one finds

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\Xi_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{4\Delta}{\sqrt{35}N} \frac{Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \tag{E.6}$$

and therefore the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}}$ is given by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}} = \frac{4\Delta}{\sqrt{35}N}. \tag{E.7}$$

Now, it is easy to check that

$$\frac{1}{2^4}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}} b_{\mathcal{O}_{2,4}} = \frac{2\pi^6\Delta}{945}, \tag{E.8}$$

which agrees with $a_{2,4}$ in (5.8).

F Thermal one-point functions of multi-trace operators in the large-N limit

In (5.33), it was shown that $a_{4,4}$ was due to double trace operators which were normal ordered products of single trace operators without any derivatives. There are, however, other double trace operators that have the same quantum numbers and are schematically represented as $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$. Concretely, the double trace operators with twist and spin four besides $(T^2)_{\mu\nu\rho\sigma}$ and $(\mathcal{O}^{\text{DT}})_{\mu\nu\rho\sigma}$ are $[\mathcal{O}_2\mathcal{O}_2]_{0,4}$ and $[\mathcal{O}_2 T_{\mu\nu}]_{0,2}$. We argue that the thermal one-point functions of these operators are subleading in the large-N limit when evaluated on the plane.

Consider the thermal one-point function of a double trace operator $[\mathcal{O}_a\mathcal{O}_b]_{n,l} = \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b + \ldots$, where $\mathcal{O}_a$ and $\mathcal{O}_b$ are single trace primary operators and dots represent terms where derivatives acts on $\mathcal{O}_a$ as well, in order to make $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$ a primary operator. The term in the thermal one-point function that behaves as N^k (N^2 for double trace operators) comes from contracting the fundamental field within each trace separately. Therefore we have

$$\langle \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta \approx \langle \mathcal{O}_a\rangle_\beta\langle\partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta + \mathcal{O}(1), \tag{F.1}$$

which is simply due to large-N factorization. As $\partial^{2n}\partial^l\mathcal{O}_b$ is a descendant of $\mathcal{O}_b$, it is easy to explicitly show that $\langle\partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta = 0$ for $n \neq 0$ or $l \neq 0$, from which it follows that

$$\langle \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta = \mathcal{O}(1). \tag{F.2}$$

Similar reasoning holds for all terms in $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$, so we conclude for $n \neq 0$ or $l \neq 0$ that

$$\langle[\mathcal{O}_a\mathcal{O}_b]_{n,l}\rangle_\beta = \mathcal{O}(1). \tag{F.3}$$

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It is easy to generalise (n and/or l non-zero)

$$\langle [\mathcal{O}_{a_1} \dots \mathcal{O}_{a_k}]_{n,l} \rangle_\beta = \mathcal{O}(N^{k-2}). \tag{F.4}$$

Using the canonical scaling for the OPE coefficients (5.18) it is found that these multi-trace operators give a suppressed contribution to the thermal two point function in the large-N limit:

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta [\mathcal{O}_{a_1} \dots \mathcal{O}_{a_k}]_{n,l}} \langle [\mathcal{O}_{a_1} \dots \mathcal{O}_{a_k}]_{n,l} \rangle_\beta = \mathcal{O}\Big(\frac{1}{N^2}\Big). \tag{F.5}$$

The conclusion is that these operators with $n \neq 0$ or $l \neq 0$ do not contribute to the thermal two-point functions to leading order in N. Note that for $n = l = 0$, the operator is just $:\mathcal{O}_{a_1} \mathcal{O}_{a_2} \dots \mathcal{O}_{a_k}:$ and it does contribute to the thermal 2pt function since

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta [\mathcal{O}_{a_1} \dots \mathcal{O}_{a_k}]_{n=0,l=0}} \langle [\mathcal{O}_{a_1} \dots \mathcal{O}_{a_k}]_{n=0,l=0} \rangle_\beta = \mathcal{O}(1). \tag{F.6}$$

From (F.5) it is seen that multi stress tensor operators of the schematic form $[T^k]_{n,l}$ with either n or l, or both, being non-zero will not contribute to the thermal correlator to leading order in N on the plane.

G Free boson in two dimensions

In this appendix we discuss free scalars in two dimensions. We first consider a single scalar and then the case of the SU(N) adjoint scalar. We compute two-point functions of a particular class of quasi-primary operators at finite temperature $1/\beta$. These two-point functions are not determined by the conformal symmetry, because the quasi-primary operators do not transform covariantly from the plane to the cylinder. They transform covariantly only with respect to the global conformal transformations. The only operators that have the non-zero thermal one-point functions are the Virasoro descendants of the vacuum and therefore, only these operators contribute to the thermal two-point function of the quasi-primary operators.[25] Virasoro descendants of the vacuum have different OPE coefficients with external quasi-primary operators compared with the case when primary external operators are considered.[26]

G.1 Review free boson in two dimensions

We consider single free boson $\phi(z)$ in two dimensions. The stress tensor can be written in terms of Virasoro modes as

$$T(z) = \sqrt{2} \sum_n z^{-n-2} L_n. \tag{G.1}$$

This stress tensor is unit-normalized

$$\langle T(z) T(w) \rangle = \frac{1}{(z-w)^4}. \tag{G.2}$$

[25] We check this explicitly up to the $\mathcal{O}(1/\beta^4)$.

[26] Deviation from the Virasoro vacuum block in the Regge limit of four-point HHLL correlator is observed in [95] as well.

The fundamental field can be expressed as Laurent series

$$\partial\phi(z) = \sum_{n=-\infty}^{+\infty} z^{-n-1}\alpha_n, \tag{G.3}$$

where oscillators α_n obey the following algebra

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}. \tag{G.4}$$

They act on the vacuum as

$$\alpha_n|0\rangle = 0, \qquad n \geq 0. \tag{G.5}$$

The two-point function of the fundamental fields is given by

$$\langle \partial\phi(z)\partial\phi(w)\rangle = \frac{1}{(z-w)^2}. \tag{G.6}$$

The unit-normalized stress tensor can be expressed in terms of the fundamental field as

$$T(z) = \frac{1}{\sqrt{2}} : \partial\phi\partial\phi : (z) = \frac{1}{\sqrt{2}} \sum_{m,n} z^{-m-n-2} : \alpha_m\alpha_n :, \tag{G.7}$$

where $: ab :$ denotes product of operators a and b with the corresponding free theory oscillators being normally ordered such that the operators annihilating the vacuum are put at the rightmost position. Then, it follows

$$L_n = \frac{1}{2}\sum_m : \alpha_{n-m}\alpha_m := \frac{1}{2}\left(\sum_{m\geq 0} \alpha_{n-m}\alpha_m + \sum_{m<0} \alpha_m\alpha_{n-m}\right). \tag{G.8}$$

G.2 Thermal two-point function of quasi-primary operator

We are interested in computing the thermal two-point function of quasi-primary operators at temperature $1/\beta$. Quasi-primary operators $\mathcal{O}(z)$ are defined as $[L_1, \mathcal{O}(z)] = 0$, or equivalently, in therms of their asymptotic in-states $\mathcal{O}(0)|0\rangle = |\mathcal{O}\rangle$, as $L_1|\mathcal{O}\rangle = 0$. We denote the quantum numbers of quasi-primary operators that correspond to eigenvalues of L_0 and $\bar{L}_0$ by $(h, \bar{h})$. We consider the following unit-normalized quasi-primary operator with quantum numbers $(h, 0)$

$$\mathcal{O}_h(z) = \frac{1}{\sqrt{h!}} : (\partial\phi)^h : (z) = \frac{1}{\sqrt{h!}} \sum_{m_1,m_2,\ldots,m_h} z^{-\sum_{i=1}^h m_i - h} : \alpha_{m_1}\ldots\alpha_{m_h} :, \tag{G.9}$$

which is properly defined when h is a positive integer. Its asymptotic in-state is given by

$$|\mathcal{O}_h\rangle = \mathcal{O}_h(0)|0\rangle = \frac{1}{\sqrt{h!}}(\alpha_{-1})^h|0\rangle. \tag{G.10}$$

One can check that this operator is a quasi-primary but not a Virasoro primary.

The thermal two-point function of this operator for even h is given by

$$\begin{aligned} \langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = &\sum_{n=0}^{\frac{1}{2}(h-2)} \frac{h!}{4^n(h-2n)!}\left(\frac{2\zeta(2)}{\beta^2}\right)^{2n}\left(\sum_{m=-\infty}^{\infty}\frac{1}{(z+m\beta)^2}\right)^{h-2n} \\ &+ \frac{2^h\pi}{\Gamma\left(\frac{1}{2}-\frac{h}{2}\right)^2\Gamma(h+1)}\left(\frac{2\zeta(2)}{\beta^2}\right)^h. \end{aligned} \tag{G.11}$$

This expression is obtained by writing all possible Wick contractions between fundamental fields $\partial\phi$, including those that belong to same operator $\mathcal{O}_h$, that we call self-contractions. Fundamental fields are separated along the thermal circle in all Wick contractions. Factors $\left(\frac{2\zeta(2)}{\beta^2}\right)$ are due to the self-contractions,

$$\sum_{m=-\infty,m\neq 0}^{\infty} \frac{1}{\beta^2 m^2} = \left(\frac{2\zeta(2)}{\beta^2}\right). \tag{G.12}$$

The sum over n comes from doing n self-contractions within each of the external operators. Term $\frac{h!}{4^n(h-2n)!}$ counts the number of Wick contractions with n self-contractions for each external operator, including $1/\sqrt{h!}$ normalization factors. The term in the second line of (G.11) is due to the case when we take $n = h/2$ self-contractions in both external operators, i.e. it represents the disconnected contribution.

Since the state $\mathcal{O}_h$ is quasi-primary, it transforms properly only with respect to the global conformal transformation. These are just the Möbius transformations in two-dimensional spacetime $z \to \frac{az+b}{cz+d}$, with $ad - bc = 1$. On the other hand, the usual way to calculate the thermal two-point function of primary operators in two dimensions is to do a conformal transformation from the plane to the cylinder with radius β, $z \to \frac{\beta}{2\pi}\log(z)$. This transformation is clearly not one of the Möbius transformations and that is why we can not use this method to compute the thermal two-point functions of quasi-primary operators.

Expanding (G.11) for $T = \frac{1}{\beta} \to 0$ one finds

$$z^{2h}\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = 1 + \frac{h}{3}\frac{(\pi z)^2}{\beta^2} + \frac{h(h-\frac{1}{5})}{12}\frac{(\pi z)^4}{\beta^4} + \mathcal{O}\left(\frac{1}{\beta^6}\right). \tag{G.13}$$

G.3 Quasi-primaries, OPE coefficients, and thermal one-point functions

In expansion (G.13), terms $\mathcal{O}(z^{h_1})$ are due to the quasi-primary operator with quantum numbers $(h_1, 0)$ in the operator product expansion $\mathcal{O}_h \times \mathcal{O}_h$. Identity in the expansion is due to the identity operator. We show that the second term on the r.h.s. is due to the stress tensor. The quantum numbers of stress tensor $T(z)$ are $(2, 0)$. First, we evaluate the thermal one-point function of the stress tensor

$$\langle T\rangle_\beta = \frac{1}{\sqrt{2}} \sum_{m=-\infty,m\neq 0}^{\infty} \frac{1}{\beta^2 m^2} = \frac{\pi^2}{3\sqrt{2}\beta^2}. \tag{G.14}$$

This is obtained by the Wick contractions of fundamental fields in the stress tensor, that are separated along the thermal circle. The same result can be obtained by the transform of the stress tensor from the plane to the cylinder using the Schwarzian derivative.

We define the OPE coefficient of unit-normalized operator $\mathcal{O}$, with quantum numbers $(h_\mathcal{O}, 0)$, with two $\mathcal{O}_h$ operators as

$$\langle \mathcal{O}_h(z_1)\mathcal{O}_h(z_2)\mathcal{O}(z_3)\rangle = \frac{\lambda_{\mathcal{O}_h\mathcal{O}_h\mathcal{O}}}{(z_1-z_3)^{h_\mathcal{O}}(z_2-z_3)^{h_\mathcal{O}}(z_1-z_2)^{2h-h_\mathcal{O}}}. \tag{G.15}$$

Next, we evaluate its OPE coefficient of the stress tensor with $\mathcal{O}_h$ by doing the Wick contractions between fundamental fields

$$\langle \mathcal{O}_h(z_1)\mathcal{O}_h(z_2)T(z_3)\rangle = \sqrt{2}h\frac{1}{(z_1-z_3)^2(z_2-z_3)^2(z_1-z_2)^{2(h-1)}}, \tag{G.16}$$

therefore $\lambda_{\mathcal{O}_h\mathcal{O}_h T} = \sqrt{2}h$. This OPE coefficient is fixed by the Ward identity. Now, it follows

$$z^2\lambda_{\mathcal{O}_h\mathcal{O}_h T}\langle T\rangle_\beta = \frac{h}{3}\frac{(\pi z)^2}{\beta^2}, \tag{G.17}$$

which reproduces the second term on the r.h.s. of (G.13).

We are now interested in the contributions of quasi-primary operators with quantum numbers $(4,0)$. There are only two linearly independent operators with these quantum numbers given by[27]

$$:TT:(z) = \frac{1}{\sqrt{24}}:(\partial\phi)^4:(z) = \frac{1}{\sqrt{24}}\sum_{a,b,c,d} z^{-a-b-c-d-4}:\alpha_a\alpha_b\alpha_c\alpha_d:, \tag{G.18}$$

$$\Lambda_4(z) = \sqrt{\frac{10}{27}}\left(\sum_{m,n=-\infty}^{\infty} z^{-m-n-4} * L_m L_n * -\frac{3}{10}\sum_{m=-\infty}^{\infty} z^{-m-4}(m+2)(m+3)L_m\right), \tag{G.19}$$

where $*ab*$ denotes the product where the relevant Virasoro generators are normally ordered. It should be noted that the operator $\Lambda_4(z)$ is Virasoro descendant of unity, while $:TT:(z)$ is not. The relevant asymptotic in-states are given by

$$\begin{aligned} |:TT:\rangle &=: TT:(0)|0\rangle = \frac{1}{\sqrt{24}}(\alpha_{-1})^4|0\rangle, \\ |\Lambda_4\rangle &= \Lambda_4(0)|0\rangle = \sqrt{\frac{10}{27}}\left(L_{-2}^2 - \frac{3}{5}L_{-4}\right)|0\rangle. \end{aligned} \tag{G.20}$$

In terms of oscillators, $|\Lambda_4\rangle$ state can be represented as

$$|\Lambda_4\rangle = \sqrt{\frac{10}{27}}\left(\frac{1}{4}(\alpha_{-1})^4 + \frac{2}{5}\alpha_{-1}\alpha_{-3} - \frac{3}{10}(\alpha_{-2})^2\right)|0\rangle. \tag{G.21}$$

From eqs. (G.20) and (G.21) one can see that $|:TT:\rangle$ and $|\Lambda_4\rangle$ are the only quasi-primary states with quantum numbers $(4,0)$. Namely, all such states have to be linear combinations of the following states

$$\alpha_{-4}|0\rangle, \qquad \alpha_{-3}\alpha_{-1}|0\rangle, \qquad \alpha_{-2}^2|0\rangle, \qquad \alpha_{-2}\alpha_{-1}^2|0\rangle, \qquad \alpha_{-1}^4|0\rangle, \tag{G.22}$$

because

$$L_0\left(\prod_{i=1}^N \alpha_{-k_i}\right)|0\rangle = \left(\sum_{i=1}^N k_i\right)\left(\prod_{i=1}^N \alpha_{-k_i}\right)|0\rangle, \tag{G.23}$$

where $k_i > 0$. It is straightforward to check

$$\begin{aligned} L_1\alpha_{-4}|0\rangle &= 4\alpha_{-3}|0\rangle, \\ L_1\alpha_{-3}\alpha_{-1}|0\rangle &= 3\alpha_{-2}\alpha_{-1}|0\rangle, \\ L_1\alpha_{-2}^2|0\rangle &= 4\alpha_{-2}\alpha_{-1}|0\rangle, \\ L_1\alpha_{-2}\alpha_{-1}^2|0\rangle &= 2\alpha_{-1}^3|0\rangle, \\ L_1\alpha_{-1}^4|0\rangle &= 0. \end{aligned} \tag{G.24}$$

[27] Both of them are unit-normalized.

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It follows that $\alpha_{-1}^4|0\rangle$ is already quasi-primary and one can make only one more as $\alpha_{-3}\alpha_{-1}|0\rangle - \frac{3}{4}\alpha_{-2}\alpha_{-2}|0\rangle$.[28] $|:TT:\rangle$ and $|\Lambda_4\rangle$ are just the linear combination of these two states with overall normalization.

Now, one can calculate the overlap of $|:TT:\rangle$ and $|\Lambda_4\rangle$ states as

$$\langle 0|\Lambda_4(0):TT:(0)|0\rangle = \frac{\sqrt{5}}{3}. \tag{G.25}$$

The state orthogonal to $|\Lambda_4\rangle$ can be written as

$$|\tilde{\Lambda}_4\rangle = \frac{3}{2}\left(:TT:(0) - \frac{\sqrt{5}}{3}\Lambda_4(0)\right)|0\rangle. \tag{G.26}$$

Using (G.20) and (G.21), it can be written in terms of free theory oscillators.

We compute the OPE coefficients of $:TT:$ and Λ_4 with two $\mathcal{O}_h$ operators. We express all states in terms of free theory oscillators and use algebra (G.4) to find

$$\lambda_{\mathcal{O}_h\mathcal{O}_h:TT:} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|:TT:\rangle = \frac{\sqrt{6}}{2}h(h-1), \tag{G.27}$$

$$\lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|\Lambda_4\rangle = \sqrt{\frac{5}{6}}h\left(h-\frac{1}{5}\right), \tag{G.28}$$

$$\lambda_{\mathcal{O}_h\mathcal{O}_h\tilde{\Lambda}_4} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|\tilde{\Lambda}_4\rangle = \frac{2}{\sqrt{6}}h\,(h-2)\,. \tag{G.29}$$

Now, we evaluate the thermal one-point functions of Λ_4 and $\tilde{\Lambda}_4$. From (3.4) in [58] we have

$$\langle *T^2*\rangle_\beta = \frac{3\pi^4}{20\beta^4}, \tag{G.30}$$

which is the thermal one-point function of the first term on the r.h.s. of (G.19). The second term can be written as $-\frac{3}{10}\sum_{m=-\infty}^{\infty} z^{-m-4}(m+2)(m+3)L_m = -\frac{3}{10\sqrt{2}}\partial^2 T(z)$. It is clear that it will not affect the thermal one-point function of $\Lambda_4(z)$, as $\langle\partial^2 T\rangle_\beta = 0$.

Therefore, from (G.19), we have

$$\langle\Lambda_4\rangle_\beta = \sqrt{\frac{10}{27}}\langle *T^2*\rangle_\beta = \frac{\pi^4}{2\sqrt{30}\beta^4}. \tag{G.31}$$

Now, it follows

$$z^4\langle\Lambda_4\rangle_\beta\lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \frac{\pi^4 z^4}{12\beta^4}h\left(h-\frac{1}{5}\right), \tag{G.32}$$

which is the third therm at the r.h.s. of (G.13). On the other hand, we can evaluate the thermal one-point function of $:TT:(z)$ operator by Wick contractions of fundamental fields separated along the thermal circle

$$\langle :TT:\rangle_\beta = \frac{\pi^4}{6\sqrt{6}\beta^4}. \tag{G.33}$$

[28] These states are not unit-normalized.

Using (G.26), it is straightforward to confirm that $\langle \tilde{\Lambda}_4 \rangle_\beta = 0$. Therefore, as we expected, operator $\tilde{\Lambda}_4$ does not contribute to the thermal two-point function of $\mathcal{O}_h$ operators, even thought it is present in the operator product expansion $\mathcal{O}_h \times \mathcal{O}_h$.

This is a general property of two-dimensional CFTs, that only the operators in the Virasoro vacuum module have non-zero expectation value on the cylinder.

G.4 Free adjoint scalar model in two dimensions

In this subsection we study a large-c theory. Consider the free adjoint SU(N) scalar in 2d with

$$\partial\phi(z)^a{}_b = \sum_m z^{-m-1}(\alpha_m)^a{}_b \tag{G.34}$$

with

$$[(\alpha_m)^a{}_b, (\alpha_n)^c{}_d] = m\delta_{m+n}\left(\delta^a{}_d\delta^c{}_b - \frac{1}{N}\delta^a{}_b\delta^c{}_d\right). \tag{G.35}$$

The thermal two point of the quasi-primary operator $\mathcal{O}_h = \frac{1}{\sqrt{hN^h}} : Tr((\partial\phi)^h) :$ follows immediately from the result in four dimensions upon replacing the propagator of fundamental fields. We find that

$$\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = g_{2d}(z)^h + \frac{\pi^4 h(h-2)}{9\beta^4}g_{2d}(z)^{h-2} + \dots, \tag{G.36}$$

where

$$\begin{aligned} g_{2d}(z) &= \sum_{m=-\infty}^{\infty}\frac{1}{(z+m\beta)^2} \\ &= \left(\frac{\pi}{\beta\sin(\pi z/\beta)}\right)^2. \end{aligned} \tag{G.37}$$

Expanding (G.36) for $\beta \to \infty$ we find

$$\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = z^{-2h}\left[1 + \frac{\pi^2 h}{3\beta^2}z^2 + \frac{\pi^4 h(15h-19)}{90\beta^4}z^4 + \mathcal{O}(\beta^{-6})\right]. \tag{G.38}$$

Consider first the normalized stress tensor which is given by

$$T = \frac{1}{\sqrt{2}N} : Tr(\partial\phi\partial\phi) :, \tag{G.39}$$

with $c = N^2$ so that $\langle T(z)T(0)\rangle = \frac{1}{z^4}$. By calculating the OPE coefficient with $\mathcal{O}_h$ and the thermal one-point function of T, one finds that these are the same as those for the scalar $Tr(\phi^2)$ operator in four dimensions so that $\langle T\rangle_\beta = \frac{\pi^2 N}{3\sqrt{2}\beta^2}$ and $\lambda_{\mathcal{O}_h\mathcal{O}_h T} = \frac{\sqrt{2}h}{N}$, and the product reproduces the weight two term in (G.38):

$$\langle T\rangle_\beta \lambda_{\mathcal{O}_h\mathcal{O}_h T} = \frac{\pi^2 h}{3\beta^2}. \tag{G.40}$$

Consider now $*TT*$ defined by

$$*TT*(0) = \lim_{z\to 0} T(z)T(0) - (\text{sing. terms}). \tag{G.41}$$

The OPE of the stress tensor in (G.39) can be found in the free theory by first performing Wick contractions

$$\begin{aligned} T(z)T(0) &= \frac{1}{2N^2} : Tr(\partial\phi(z)\partial\phi(z)) :: Tr(\partial\phi(0)\partial\phi(0)) : \\ &=: TT : (0) + \ldots + \frac{2}{N^2 z^2} : Tr(\partial\phi(z)\partial\phi(0)) : + \frac{1}{z^4}, \end{aligned} \tag{G.42}$$

and expanding the second term in (G.42) for $z \to 0$ we find

$$\begin{aligned} T(z)T(0) =&: TT : (0) + \ldots + \frac{2}{N^2 z^2} : Tr(\partial\phi(0)\partial\phi(0)) : \\ &+ \frac{2}{N^2 z} : Tr(\partial^2\phi(0)\partial\phi(0)) : + \frac{1}{N^2} : Tr(\partial^3\phi(0)\partial\phi(0)) : + \ldots \\ &+ \frac{1}{z^4}, \end{aligned} \tag{G.43}$$

where the dots refer to higher order terms in z. Inserting the OPE (G.43) in (G.41) we find that

$$* TT * (0) =: TT : (0) + \frac{1}{N^2} : Tr(\partial^3\phi(0)\partial\phi(0)) : . \tag{G.44}$$

Consider the state $*TT*(0)|0\rangle$, which is given in terms of oscillator modes by

$$* TT * (0)|0\rangle = \frac{1}{2N^2} Tr(\alpha_{-1}^2) Tr(\alpha_{-1}^2)|0\rangle + 2\frac{1}{N^2} Tr(\alpha_{-3}\alpha_{-1})|0\rangle. \tag{G.45}$$

Now $Tr(\alpha_{-1}^m)|0\rangle$ is a quasi-primary while $Tr(\alpha_{-3}\alpha_{-1})|0\rangle$ is not. One way to make it a quasi-primary is to simply remove the second term in (G.45) and then we get a quasi-primary state which is just $: TT : |0\rangle$. Another option is to remove a descendant of the stress tensor to construct $|\Lambda_4\rangle$. To do the latter we need to remove the descendant of the stress tensor with weight 4 given by $\partial^2 T$

$$\partial^2 T = \frac{\sqrt{2}}{N} : Tr(\partial^3\phi\partial\phi) : + \frac{\sqrt{2}}{N} : Tr(\partial^2\phi\partial^2\phi) : . \tag{G.46}$$

Acting on the vacuum we find

$$\partial^2 T(0)|0\rangle = \frac{2\sqrt{2}}{N} Tr(\alpha_{-3}\alpha_{-1})|0\rangle + \frac{\sqrt{2}}{N} Tr(\alpha_{-2}^2)|0\rangle. \tag{G.47}$$

Consider now $L_1 = \frac{\sqrt{2}}{N}(Tr(\alpha_{-1}\alpha_2) + Tr(\alpha_{-2}\alpha_3 + \ldots))$ which acts as $L_1 Tr(\alpha_{-2}^2)|0\rangle = \frac{4\sqrt{2}}{N} Tr(\alpha_{-1}\alpha_{-2})|0\rangle$ and as $L_1 Tr(\alpha_{-3}\alpha_{-1})|0\rangle = \frac{3\sqrt{2}}{N} Tr(\alpha_{-1}\alpha_{-2})|0\rangle$. We can therefore construct a quasi-primary state annihilated by L_1: $Tr(\alpha_{-3}\alpha_{-1})|0\rangle - \frac{3}{4} Tr(\alpha_{-2}^2)|0\rangle$. The quasi-primary $|\Lambda_4\rangle$ is then given by:

$$\begin{aligned} |\Lambda_4\rangle &= \frac{1}{\sqrt{2}} \left[*TT*(0)|0\rangle - \frac{3}{5\sqrt{2}N} \partial^2 T(0)|0\rangle \right] \\ &= \frac{1}{2\sqrt{2}N^2} \left[Tr(\alpha_{-1}^2) Tr(\alpha_{-1}^2)|0\rangle - \frac{6}{5} Tr(\alpha_{-2}^2)|0\rangle + \frac{8}{5} Tr(\alpha_{-1}\alpha_{-3})|0\rangle \right] \end{aligned} \tag{G.48}$$

There are two more weight 4 single trace quasi-primary operators given by

$$\begin{aligned}\mathcal{O}^{(1)} &= \frac{1}{2N^2}Tr((\partial\phi)^4)\\ \mathcal{O}^{(2)} &= \frac{n_{\mathcal{O}^{(2)}}}{N}\left(Tr(\partial^3\phi\partial\phi)-\frac{3}{2}Tr(\partial^2\phi\partial^2\phi)\right),\\ &= \frac{n_{\mathcal{O}^{(2)}}}{N}\left(\frac{1}{2}\partial^2Tr(\partial\phi\partial\phi)-\frac{5}{2}Tr(\partial^2\phi\partial^2\phi)\right),\end{aligned} \tag{G.49}$$

where $n_{\mathcal{O}^{(2)}}$ is some N-independent normalization constant. The state $|\Lambda_4\rangle$ can be written in terms of $:TT:(0)|0\rangle + a\mathcal{O}_2(0)|0\rangle$ in the following way

$$|\Lambda_4\rangle = \frac{1}{\sqrt{2}}\left[:TT:(0)|0\rangle + \frac{2}{5Nn_{\mathcal{O}^{(2)}}}\mathcal{O}^{(2)}|0\rangle\right]. \tag{G.50}$$

The OPE coefficient for $:TT:$ is up to a normalization the same as the scalar dimension 4 double trace operator in 4d and is given by

$$\begin{aligned}\langle\mathcal{O}_h\mathcal{O}_h:TT:\rangle &= \frac{1}{hN^h}\frac{1}{2N^2}4h^2(3h-5)N^h\frac{1}{z_{13}^4z_{23}^4z_{12}^{2h-4}}\\ &= \frac{1}{N^2}2h(3h-5)\frac{1}{z_{13}^4z_{23}^4z_{12}^{2h-4}},\end{aligned} \tag{G.51}$$

where $4h^2(3h-5)$ come from the number of contractions giving planar diagrams. Consider now the OPE coefficient for $\mathcal{O}^{(2)}$. One finds

$$\begin{aligned}\langle\mathcal{O}_h\mathcal{O}_h\mathcal{O}^{(2)}\rangle &= \frac{n_{\mathcal{O}^{(2)}}N^h}{hN^{h+1}z_{13}^4z_{23}^4z_{12}^{2h-2}}\left[(-2)(-3)h^2(z_{13}^2+z_{23}^2)-\frac{3}{2}2h^2(-2)^2z_{13}z_{23}\right]\\ &= \frac{6hn_{\mathcal{O}^{(2)}}}{Nz_{13}^4z_{23}^4z_{12}^{2h-4}}.\end{aligned} \tag{G.52}$$

Using (G.51), (G.52) and (G.50) we find the OPE coefficient for $|\Lambda_4\rangle$

$$\langle\mathcal{O}_h\mathcal{O}_h\Lambda_4\rangle = \frac{\sqrt{2}h(15h-19)}{5N^2}. \tag{G.53}$$

Note that the h dependence matches that of the weight 4 term in the two-point function (G.38). Additionally, the OPE coefficient given by (G.53) can not be extrapolated to the limit when $h\sim C_T$, as in this limit the planar expansion used for calculating (G.53) breaks down. For this reason, we can not test the thermalization of Λ_4 in heavy state $\mathcal{O}_{h_H}$. Let us consider the thermal one-point function which is given by

$$\langle\Lambda_4\rangle_\beta = \left[\frac{1}{\sqrt{2}}b_T^2+\mathcal{O}(1)\right] = \frac{\pi^4N^2}{18\sqrt{2}\beta^4}, \tag{G.54}$$

where the term $\propto \frac{1}{N}\langle\mathcal{O}^{(2)}\rangle_\beta$ is subleading since it is single trace. We find that

$$\langle\Lambda_4\rangle_\beta\lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \frac{\pi^4h(15h-19)}{90\beta^4}, \tag{G.55}$$

which agrees with the weight 4 term in (G.38).

Note that it is explicitly seen that one can write Λ_4 either as $*TT*$ +(desc. of T) or as $:TT: +\frac{1}{N}\mathcal{O}_{ST}$ with $\mathcal{O}_{ST}$ a quasi-primary single trace operator. In this case the single trace operator which one needs to add to $:TT:$ to get Λ_4 can be written as a sum of descendants $\mathcal{O}^{(2)} \propto \partial^2 T - \frac{5}{\sqrt{2}} Tr(\partial^2\phi\partial^2\phi)$. Explicitly, we have

$$\begin{aligned} |\Lambda_4\rangle &= \frac{1}{\sqrt{2}}\left[*TT*(0) - \frac{3}{5\sqrt{2}N}\partial^2 T(0)\right]|0\rangle \\ &= \frac{1}{\sqrt{2}}\left[:TT:(0) + \frac{2}{5Nn_{\mathcal{O}^{(2)}}}\mathcal{O}^{(2)}\right]|0\rangle. \end{aligned} \tag{G.56}$$

As we saw above, using the second line in (G.56) it is straightforward to calculate correlation functions using Wick contractions to see that Λ_4 gives the full weight four contributions to the thermal two-point function for large-N theories.

Now, we consider the following quasi-primary operator

$$\mathcal{O}_\Delta(z,\bar{z}) = \frac{\sqrt{2}}{\sqrt{\Delta}N^{\Delta/2}} : Tr\left((\partial\phi\bar{\partial}\bar{\phi})^{\frac{\Delta}{2}}\right):(z,\bar{z}), \tag{G.57}$$

where we denote the anti-holomorphic part of the free field by $\bar{\phi} = \bar{\phi}(\bar{z})$. The thermal two-point function of this operator, up to the terms subleading in large-N expansion, is given by

$$\begin{aligned} \langle\mathcal{O}_\Delta(z,\bar{z})\mathcal{O}_\Delta(0,0)\rangle_\beta &= \frac{\pi^{2\Delta}}{\beta^{2\Delta}\sin^\Delta\left(\frac{\pi z}{\beta}\right)\sin^\Delta\left(\frac{\pi\bar{z}}{\beta}\right)} \\ = \frac{1}{(z\bar{z})^\Delta}&\left(1 + \frac{\pi^2\Delta(z^2+\bar{z}^2)}{6\beta^2} + \frac{\pi^4\Delta(5\Delta+2)}{360\beta^4}(z^4+\bar{z}^4) + \frac{\pi^4\Delta^2}{36\beta^4}z^2\bar{z}^2 + \mathcal{O}\left(\frac{1}{\beta^6}\right)\right). \end{aligned} \tag{G.58}$$

One can easily check that the OPE coefficients of stress tensor T and its anti-holomorphic partner $\bar{T}$ with $\mathcal{O}_\Delta$ are given by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{T}} = \frac{\Delta}{\sqrt{2}N}, \tag{G.59}$$

while their thermal one-point function are given by

$$\langle T\rangle_\beta = \langle\bar{T}\rangle_\beta = \frac{\pi^2 N}{3\sqrt{2}\beta^2}. \tag{G.60}$$

It is easy to check that terms proportional to β^{-2} in (G.58) are contributions of T and $\bar{T}$ operators

$$\langle T\rangle_\beta\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T}z^2 + \langle\bar{T}\rangle_\beta\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{T}}\bar{z}^2 = \frac{\pi^2\Delta(z^2+\bar{z}^2)}{6\beta^2}. \tag{G.61}$$

We compute the OPE coefficient of operators Λ_4, defined by (G.48), and its anti-holomorphic partner $\bar{\Lambda}_4$ with $\mathcal{O}_\Delta$ and obtain

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\Lambda_4} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{\Lambda}_4} = \frac{\Delta(5\Delta+2)}{10\sqrt{2}N^2}, \tag{G.62}$$

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which agrees with (C.26) in [66]. Its thermal one-point function (which is the same as $\langle\bar{\Lambda}_4\rangle_\beta$) is given by (G.54). Another operator that contributes to thermal two-point function (G.58) is $:T\bar{T}:$. Its OPE coefficient with $\mathcal{O}_\Delta$ and thermal one-point function are given by

$$\begin{aligned}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta:T\bar{T}:} &= \frac{\Delta^2}{2N^2}\\ \langle :T\bar{T}:\rangle_\beta &= \frac{\pi^4 N^2}{18\beta^4}.\end{aligned} \tag{G.63}$$

Again, it is easy to check

$$\begin{aligned}\langle\Lambda_4\rangle_\beta\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\Lambda_4}z^4+\langle\bar{\Lambda}_4\rangle_\beta\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{\Lambda}_4}\bar{z}^4+\langle :T\bar{T}:\rangle_\beta\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta:T\bar{T}:}z^2\bar{z}^2 =\\ =\frac{\pi^4\Delta(5\Delta+2)}{360\beta^4}(z^4+\bar{z}^4)+\frac{\pi^4\Delta^2}{36\beta^4}z^2\bar{z}^2,\end{aligned} \tag{G.64}$$

which matches with the corresponding terms in (G.58).

The OPE coefficients $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\Lambda_4}$, $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{\Lambda}_4}$, and $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta:T\bar{T}:}$ can be extrapolated to the limit $\Delta\sim N^2$, by the same logic as in appendix C. Then, we can explicitly check the thermalization property of Λ_4, $\bar{\Lambda}_4$, and $:T\bar{T}:$. To establish a relation between the inverse temperature β and the conformal dimension Δ_H of heavy state $\mathcal{O}_H=\mathcal{O}_{\Delta\sim N^2}$, we assume the thermalization of stress tensor

$$\langle T\rangle_\beta=\lambda_{\mathcal{O}_H\mathcal{O}_H T}, \tag{G.65}$$

which implies

$$\frac{\Delta_H}{N^2}=\frac{\pi^2}{3\beta^2}. \tag{G.66}$$

Using this relation, it is easy to show

$$\begin{aligned}\langle\Lambda_4\rangle_\beta &= \lambda_{\mathcal{O}_H\mathcal{O}_H\Lambda_4}\Big|_{\frac{\Delta_H^2}{N^2}},\\ \langle\bar{\Lambda}_4\rangle_\beta &= \lambda_{\mathcal{O}_H\mathcal{O}_H\bar{\Lambda}_4}\Big|_{\frac{\Delta_H^2}{N^2}},\\ \langle :T\bar{T}:\rangle_\beta &= \lambda_{\mathcal{O}_H\mathcal{O}_H:T\bar{T}:}\Big|_{\frac{\Delta_H^2}{N^2}}.\end{aligned} \tag{G.67}$$

This means that operators Λ_4, $\bar{\Lambda}_4$, and $:T\bar{T}:$ thermalize in the quasi-primary state $\mathcal{O}_H$ similarly to the thermalization in a Virasoro primary states in large-c theory, that was analyzed in [42].

H Vector model

In this section we study the free scalar vector model at large-N. Consider the scalar operator

$$\mathcal{O}_\Delta=\frac{1}{\sqrt{\mathcal{N}(\Delta)}}:(\varphi^i\varphi^i)^{\frac{\Delta}{2}}:(x), \tag{H.1}$$

where $\mathcal{N}(\Delta)$ is a normalization constant which to leading order in N is given by

$$\mathcal{N}(\Delta) \approx (\Delta)!!N^{\frac{\Delta}{2}}. \tag{H.2}$$

The thermal two-point function is given by

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \tilde{g}(x_E^0, |\mathbf{x}|)^\Delta + \left(\frac{\Delta}{2}\right)^2 \frac{1}{\Delta}\tilde{g}(x_E^0, |\mathbf{x}|)^{\Delta-2} + \dots, \tag{H.3}$$

where

$$\begin{aligned}\tilde{g}(x_E^0, |\mathbf{x}|) &= \sum_{m=-\infty}^{\infty} \frac{1}{(x_E^0 + m\beta)^2 + \mathbf{x}^2} \\ &= \frac{\pi}{2\beta|\mathbf{x}|}\left[\mathrm{Coth}\left(\frac{\pi}{\beta}(|\mathbf{x}| - ix_E^0)\right) + \mathrm{Coth}\left(\frac{\pi}{\beta}(|\mathbf{x}| + ix_E^0)\right)\right].\end{aligned} \tag{H.4}$$

The thermal $a_{\tau,J}$ coefficients $a_{2,2}$ and $a_{4,4}$ are the same as in the adjoint model (this is so since the second term in (H.3) does not affect these):

$$\begin{aligned} a_{2,2} &= \frac{\pi^4\Delta}{45}, \\ a_{4,4} &= \frac{\pi^8\Delta(\Delta-1)}{1050}. \end{aligned} \tag{H.5}$$

The unit-normalized stress tensor is given by

$$T_{\mu\nu}(x) = \frac{1}{3\sqrt{C_T}} : \left(\partial_\mu\varphi^i\partial_\nu\varphi^i - \frac{1}{2}\varphi^i\partial_\mu\partial_\nu\varphi^i - (\mathrm{trace})\right) : (x), \tag{H.6}$$

where $C_T = \frac{4}{3}N$. The OPE coefficient of the stress tensor is again found by Wick contractions to be

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T_{\mu\nu}} = -\frac{4\Delta}{3\sqrt{C_T}}, \tag{H.7}$$

in agreement with the stress tensor Ward identity. The double-stress tensor is given by

$$T^2_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : -(\mathrm{traces}), \tag{H.8}$$

and the OPE coefficient is calculated precisely as for the adjoint model and we find

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} = \frac{8\sqrt{2}}{9C_T}\Delta(\Delta-1). \tag{H.9}$$

There is another double-trace operator with twist 4 and spin 4 and takes the same form $: \mathcal{O}_2\mathcal{O}_{2,4} :$ as for the adjoint model

$$\begin{aligned}\mathcal{O}^{\mathrm{DT}}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70}N} : \varphi^i\varphi^i\Big(&\varphi^j\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\varphi^j - 16\partial_{(\mu}\varphi^j\partial_\nu\partial_\rho\partial_{\sigma)}\varphi^j \\ &+ 18\partial_{(\mu}\partial_\nu\varphi^j\partial_\rho\partial_{\sigma)}\varphi^j - (\mathrm{traces})\Big) : (x).\end{aligned} \tag{H.10}$$

The OPE coefficient and the thermal one-point function yields the same result as for the corresponding operator in the adjoint model.[29] It then follows that the $a_{4,4}$ extracted from (H.3) is reproduced by the sum of the double stress tensor and (H.10).

[29]Note that this is not true for all operators but is in line with the fact that $a_{4,4}$ is unaffected by the second term in (H.3).

I Factorization of thermal correlators

In this appendix we argue for the factorization of thermal expectation values of multi-trace operators in large-C_T theories on $S^1 \times \mathbf{R}^{d-1}$. Consider the thermal two-point function of a scalar operator $\mathcal{O}$ with dimension Δ:

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_\beta = \langle \mathcal{O}\rangle_\beta \langle \mathcal{O}\rangle_\beta + \langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{\beta,c}, \tag{I.1}$$

where the second term consist of the connected part of the correlator. Note that the disconnected term in (I.1) is independent of the position x. On the other hand we can evaluate (I.1) using the OPE on the plane which takes the form

$$\mathcal{O}(x)\mathcal{O}(0) = \frac{1}{|x|^{2\Delta}} + \sum_{n,l} \lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{n,l}} x^{2n+l} [\mathcal{O}\mathcal{O}]_{n,l} + \dots, \tag{I.2}$$

when written in terms of primaries and the dots refer to terms surpressed in the large-C_T limit. Note that $\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{n,l}}$ are the MFT OPE coefficient which are of order 1. The term in (I.2) that is independent of x is due to the $n = l = 0$ term in (I.2) and inserting the OPE on the l.h.s. of (I.2), we find that

$$\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{0,0}} \langle [\mathcal{O}\mathcal{O}]_{0,0}\rangle_\beta = \langle \mathcal{O}\rangle_\beta^2. \tag{I.3}$$

When $[\mathcal{O}\mathcal{O}]_{0,0}$ is unit-normalized the OPE coefficient is given by $\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{0,0}} = \sqrt{2}$ and it follows that

$$\langle [\mathcal{O}\mathcal{O}]_{0,0}\rangle_\beta = \frac{1}{\sqrt{2}} \langle \mathcal{O}\rangle_\beta^2. \tag{I.4}$$

We therefore see the that the thermal one-point function of the double-trace operator factorizes on the plane. We expect a similar argument to hold for multi stress tensors.

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JHEP09(2021)205

Published for SISSA by Springer

Received: April 30, 2021
Accepted: June 13, 2021
Published: June 30, 2021

Hydrodynamic dispersion relations at finite coupling

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Abstract: By using holographic methods, the radii of convergence of the hydrodynamic shear and sound dispersion relations were previously computed in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at infinite 't Hooft coupling and infinite number of colours. Here, we extend this analysis to the domain of large but finite 't Hooft coupling. To leading order in the perturbative expansion, we find that the radii grow with increasing inverse coupling, contrary to naive expectations. However, when the equations of motion are solved using a qualitative non-perturbative resummation, the dependence on the coupling becomes piecewise continuous and the initial growth is followed by a decrease. The piecewise nature of the dependence is related to the dynamics of branch point singularities of the energy-momentum tensor finite-temperature two-point functions in the complex plane of spatial momentum squared. We repeat the study using the Einstein-Gauss-Bonnet gravity as a model where the equations can be solved fully non-perturbatively, and find the expected decrease of the radii of convergence with the effective inverse coupling which is also piecewise continuous. Finally, we provide arguments in favour of the non-perturbative approach and show that the presence of non-perturbative modes in the quasinormal spectrum can be indirectly inferred from the analysis of perturbative critical points.

Keywords: Black Holes in String Theory, Effective Field Theories, Gauge-gravity correspondence, Holography and quark-gluon plasmas

ArXiv ePrint: 2104.11035

https://doi.org/10.1007/JHEP06(2021)180

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1 Introduction

In the hydrodynamic regime, quantum field theory is expected to contain collective excitations such as sound waves [1, 2]. These hydrodynamic modes are characterised in momentum space by their gapless dispersion relations $\omega = \omega(\mathbf{q})$, where ω is the frequency of the mode and $\mathbf{q}$ is its wave-vector. In the simplest case of a relativistic neutral isotropic fluid, two hydrodynamic modes known as shear and sound modes have dispersion relations

$$\omega_{\text{shear}}(\mathbf{q}^2) = -iD\mathbf{q}^2 + \cdots \,, \tag{1.1}$$

$$\omega_{\text{sound}}(\mathbf{q}^2) = \pm v_s(\mathbf{q}^2)^{\frac{1}{2}} - i\frac{\Gamma}{2}\mathbf{q}^2 + \cdots \,. \tag{1.2}$$

These modes arise as linearised fluctuations of an equilibrium state and describe transverse momentum (shear) and longitudinal energy-momentum (sound) transfer. The coefficients of the series such as the speed of sound v_s, the transverse momentum diffusion constant $D = \eta/sT$ and the sound attenuation constant $\Gamma = (\zeta + 4\eta/3)/sT$, where η and ζ are, respectively, shear and bulk viscosities and s is the equilibrium entropy density at temperature T, are determined by the underlying microscopic quantum field theory [2]. In the following, it will be convenient to use the frequency $\mathfrak{w} = \omega/2\pi T$ and the spatial momentum $\mathfrak{q} = \mathbf{q}/2\pi T$ normalised by the Matsubara frequency.

Recently, in the context of exploring the domain of applicability of hydrodynamics, the radii of convergence of the series (1.1), (1.2) have been investigated in some strongly interacting quantum field theories by using their dual gravitational descriptions in refs. [3–8] and by using field theory methods in refs. [9, 10]. In particular, by promoting $\mathfrak{q}^2$ to a complex variable and analysing critical points of the associated spectral curves, in refs. [4, 5], it was found that for the $\mathcal{N} = 4$ supersymmetric SU(N_c) Yang-Mills theory (SYM) in the limit of infinite number of colours $N_c \to \infty$ and infinite 't Hooft coupling $\lambda = g^2_{YM}N_c \to \infty$, the radii of convergence $R = |\mathfrak{q}^2|$ of the hydrodynamic series $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)$ in the complex $\mathfrak{q}^2$-plane are given by

$$R^{(\infty)}_{\text{shear}} \approx 2.22\,, \tag{1.3}$$

$$R^{(\infty)}_{\text{sound}} = 2\,. \tag{1.4}$$

The physical reason behind the breakdown of the convergence of hydrodynamic series is the presence of the gapped non-hydrodynamic degrees of freedom whose spectra "cross levels" with the hydrodynamic degrees of freedom at some (generically complex) value of $\mathfrak{q}^2$.

Our main goal in this paper is to find the 't Hooft coupling constant corrections to the infinite coupling results (1.3), (1.4), similar to the coupling constant corrections to the entropy [11, 12], shear viscosity [13, 14] and other transport coefficients (see ref. [15] and references therein) computed for the $\mathcal{N} = 4$ SYM theory earlier. Our methods are discussed in detail in section 2, and in appendices A and B.

Naively, one may expect that the radius of convergence $R(\lambda)$ decreases with the coupling decreasing from its infinite value. Indeed, schematically [16], while at infinite coupling the characteristic spectral distance $\nu^{(\infty)}$ (set by the location of quasinormal modes in the

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dual gravity theory [17–19]) is coupling-independent, its counterpart $\nu^{(0)}$ at small coupling (set by the eigenvalues of a suitable linearised collision operator [20, 21], [22]) is coupling-dependent and parametrically small, hence,

$$R^{(\infty)} \sim \nu^{(\infty)}/T \sim 1\,, \tag{1.5}$$

$$R^{(0)} \sim \nu^{(0)}/T \sim \lambda^2 \ln \lambda^{-1} \ll 1\,, \tag{1.6}$$

where eq. (1.5) is clearly consistent with the results (1.3), (1.4).

However, these expectations are shattered by a concrete calculation. Using perturbative methods only, in section 3 we find instead that at large coupling the radius of convergence increases with the coupling decreasing from its infinite value, namely,

$$R_{\text{shear}}(\lambda) = R^{(\infty)}_{\text{shear}} \left(1 + 674.15\,\lambda^{-3/2} + \cdots\right)\,, \tag{1.7}$$

$$R_{\text{sound}}(\lambda) = R^{(\infty)}_{\text{sound}} \left(1 + 481.68\,\lambda^{-3/2} + \cdots\right)\,, \tag{1.8}$$

where $R^{(\infty)}_{\text{shear}}$ and $R^{(\infty)}_{\text{sound}}$ are given by eqs. (1.3) and (1.4). This result is unexpected. Admittedly, the large numerical coefficients in eqs. (1.7) and (1.8) may reflect the necessity of a non-perturbative "resummation" along the lines of ref. [23]. Indeed, applying such a resummation (discussed in detail below), we find that $R_{\text{shear}}(\lambda)$ and $R_{\text{sound}}(\lambda)$ become *decreasing* functions of the decreasing λ after the initial growth that is well-approximated by eqs. (1.7) and (1.8) (see figure 1). In consequence, for the $\mathcal{N} = 4$ SYM theory, our analysis implies that $R(\lambda)$ should be a non-monotonic and piecewise continuous function. The dependence of R_{shear} and R_{sound} on $\gamma \sim \lambda^{-3/2}$ shown in figure 1 constitutes the main result of this paper. The non-perturbative part of the curve $R_{\text{shear}}(\gamma)$ (the red segment of the curve in the right panel of figure 1) coincides with the boundary of validity of hydrodynamics previously discussed in ref. [16]. We note that the piecewise character of this dependence is similar to the one recently observed for infinitely strongly coupled theories with finite chemical potential [7, 24] and the Sachdev-Ye-Kitaev chain at finite coupling [9].

To understand the non-perturbative aspects of the analysis better, in section 4, we compute the radii of convergence of hydrodynamic series using the Einstein-Gauss-Bonnet gravity in five dimensions as a theoretical laboratory. There, the second-order bulk equations of motion can be solved fully non-perturbatively in the Gauss-Bonnet coupling [16, 25, 26], and thus the outcome of relevant perturbative resummations can be compared with exact results. We find (see figures 13 and 16) that the radius of convergence decreases (and the dependence is piecewise continuous) with what can be phenomenologically identified as the direction of decreasing CFT coupling [16, 26–28], which is qualitatively similar to the results obtained for the $\mathcal{N} = 4$ SYM theory in section 3.

The question of the hydrodynamic series convergence at finite coupling was recently addressed in ref. [10] for experimentally realisable fluids, and in ref. [9] for the Sachdev-Ye-Kitaev chain. The calculations of ref. [10] are based on estimating the size of the k-gap [29] and show an increasing R with increasing Coulomb coupling strength, while ref. [9] finds a non-monotonic dependence, with R growing towards weak coupling. Radii of convergence

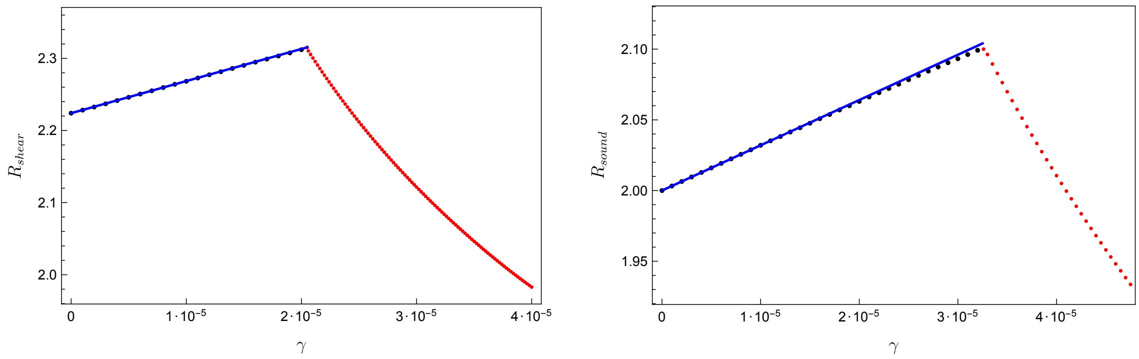

Figure 1. Radii of convergence R_{shear} and R_{sound} of the hydrodynamic shear (left panel) and sound (right panel) modes in the $\mathcal{N} = 4$ SYM theory as a function of the coupling $\gamma \propto \lambda^{-3/2}$. Solid blue curves correspond to the perturbative results of eqs. (1.7), (1.8), black dots are the non-perturbative results. The red curves are determined by the level-crossings of the hydrodynamic modes with the modes not present in the perturbative spectrum.

of hydrodynamic series in relativistic kinetic theory (in the relaxation time approximation) were recently studied in ref. [30].

This paper is structured as follows. In section 2, we briefly review the method of critical points of spectral curves introduced in refs. [4, 5] to compute the radii of convergence, as well as the non-perturbative resummation approach for theories with higher-derivative equations of motion. In section 3, we use the dual higher-derivative gravity to compute the radii of convergence for the shear and sound modes in the $\mathcal{N} = 4$ SYM theory at large but finite 't Hooft coupling. This analysis is done perturbatively and non-perturbatively by using the "resummed" version of the first-order theory. In section 4, we perform similar calculations in Einstein-Gauss-Bonnet gravity to check the validity of our approach. We also demonstrate level-crossings at higher momenta. Then, in section 5, we discuss the validity of non-perturbative "resummations" used in holography. We first consider a toy algebraic example and then study in detail the shear channel of the Einstein-Gauss-Bonnet theory. These examples are used to draw plausible conclusions about the $\mathcal{N} = 4$ SYM theory and the emergence of purely relaxing gapped modes in that theory. We conclude with a discussion of open problems in section 6. Appendix A is a short review of the methods involving critical points and quasinormal level-crossing. Appendix B introduces a useful method for determining the Puiseux exponent at a critical point by analysing the coefficients of hydrodynamic series of a dispersion relation $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)$. Finally, appendices C and D contain the coefficients of the differential equations used in the paper.

2 Critical points, quasinormal level-crossings and the "resummation"

In this section, we briefly review the methods used to obtain the main results of the paper. These methods were formulated in refs. [4, 5], where the interested reader can find more details and examples.

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2.1 Critical points, Puiseux series and quasinormal level-crossing

In momentum space, the hydrodynamic dispersion relations arise from the hydrodynamic spectral curve $P_H(\mathfrak{q}^2, \mathfrak{w}) = 0$ given by the zeros of the determinant of the matrix of linearised fluctuations around an equilibrium state [4, 5]. Using the symmetry of the system and applying the Newton polygon method, one can write generic expressions for the dispersion relations describing transverse momentum and longitudinal energy-momentum fluctuations in terms of the converging Puiseux series centred at the origin:

$$\mathfrak{w}_{\text{shear}}(\mathfrak{q}^2) = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n , \tag{2.1}$$

$$\mathfrak{w}_{\text{sound}}(\mathfrak{q}^2) = -i \sum_{n=1}^{\infty} a_n e^{\pm \frac{i\pi n}{2}} \left(\mathfrak{q}^2\right)^{n/2} . \tag{2.2}$$

The modes (2.1) and (2.2) are gapless. The coefficients c_n and a_n of the series are real functions proportional to the transport coefficients. In particular, $c_1 = 2\pi T D$, $a_1 = \pm v_s$, $a_2 = -\Gamma \pi T$. In the underlying quantum field theory, the full spectral curve $P(\mathfrak{q}^2, \mathfrak{w})$, which reduces to $P_H(\mathfrak{q}^2, \mathfrak{w})$ in the hydrodynamic limit, is proportional to the denominator of the two-point retarded correlation function of the corresponding conserved current (here, the energy-momentum tensor). The spectral curve equation $P(\mathfrak{q}^2, \mathfrak{w}) = 0$ contains the full spectrum of modes $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)$, gapless and gapped. Identifying gapless modes with (2.1) and (2.2), one reads off the transport coefficients in terms of the quantum field theory parameters.

Each Puiseux series is an expansion around a critical point $(\mathfrak{q}_c^2, \mathfrak{w}_c)$ of order p which is a solution of the following set of equations:

$$P(\mathfrak{q}_c^2, \mathfrak{w}_c) = 0\,, \;\; \partial_{\mathfrak{w}} P(\mathfrak{q}_c^2, \mathfrak{w}_c) = 0\,, \;\; \cdots\,, \;\; \partial_{\mathfrak{w}}^p P(\mathfrak{q}_c^2, \mathfrak{w}_c) \neq 0\,. \tag{2.3}$$

The order p determines the number of branches of the curve at the critical point. The analytic properties of the branches can be found by using e.g. the Newton polygon method, as explained in ref. [5] and references therein. Accordingly, a critical point may constitute a branch point singularity (or worse) or be a regular point depending on the coefficients of the original complex curve. For example, a point with $p = 1$ is always a regular point, as guaranteeed by the implicit function theorem, in which case a Puiseux series is the ordinary Taylor series. In terms of eq. (2.3), the shear mode is a Puiseux series in $\mathfrak{q}^2$ of order $p = 1$ around the origin $(\mathfrak{q}_c^2, \mathfrak{w}_c) = (0, 0)$ (i.e., a Taylor series around a regular point), whereas for the sound mode, the origin is a branch point singularity generating a Puiseux series of order $p = 2$ [4, 5].

The radii of convergence of the series (2.1) and (2.2) are set by the locations of the closest to the origin singularities of the functions $\mathfrak{w}_{\text{shear}}(\mathfrak{q}^2)$ and $\mathfrak{w}_{\text{sound}}(\mathfrak{q}^2)$, respectively, in the complex $\mathfrak{q}^2$-plane. Critical points of the spectral curve $P(\mathfrak{q}^2, \mathfrak{w}) = 0$ having branch points at $\mathfrak{q}^2 = \mathfrak{q}_c^2$ are the common source of such singularities. At the critical points with $p > 1$, the equation $P(\mathfrak{q}_c^2, \mathfrak{w}_c) = 0$ has multiple roots, and hence the hydrodynamic mode "collides" with one or more gapped modes in the complex $\mathfrak{w}$-plane. If the corresponding

$\mathfrak{q}^2 = \mathfrak{q}_c^2$ is a branch point, this is the level-crossing phenomenon,[1] albeit happening here at complex values of frequency and momentum squared. We illustrate this with simple examples in appendix A. In appendix B, we also introduce a method based on the Darboux theorem which allows one to compute the Puiseux exponent at a critical point closest to (but different from) the origin by analysing the coefficients of a series centred at the origin.

Computing a spectral curve in quantum field theory, even perturbatively, is a difficult problem. However, for strongly interacting theories with gravity dual descriptions, this task is rather straightforward. Indeed, the recipe for computing the two-point retarded correlators from dual gravity [31] implies that the spectral curve $P(\mathfrak{q}^2, \mathfrak{w}) = 0$ is determined by the boundary value $Z(u = 0, \mathfrak{q}^2, \mathfrak{w})$ of the solution $Z(u, \mathfrak{q}^2, \mathfrak{w})$ to the bulk equations of motion for the fluctuations coupled to the relevant conserved current: $P(\mathfrak{q}^2, \mathfrak{w}) = Z(u = 0, \mathfrak{q}^2, \mathfrak{w}) = 0$ [4, 5]. Analytic properties of the spectral curve such as the location of branch point singularities are thus inherited from the properties of the bulk ODEs. In practice, the ODEs are sufficiently complicated and have to be solved numerically. Having such a solution, one first solves eqs. (2.3) to find the critical points in the complex $\mathfrak{q}^2$-plane, and then determines the degree of singularity at the critical points by considering the quasinormal mode behaviour in the complex $\mathfrak{w}$-plane under the monodromy $\mathfrak{q}^2 = |\mathfrak{q}^2| e^{i\varphi}$, where $\varphi \in [0, 2\pi]$ (see appendix A). The closest to the origin (in the complex $\mathfrak{q}^2$-plane) critical point exhibiting a branch point singularity sets the radius of convergence of the series (2.1), (2.2). In the $\mathcal{N} = 4$ SYM theory at infinite N_c and infinite 't Hooft coupling, the critical points closest to the origin in the shear and sound channels are located at

$$\text{Shear}: \quad \mathfrak{q}_{\rm c}^2 \approx 1.8906469 \pm 1.1711505i, \quad \mathfrak{w}_{\rm c} \approx \pm 1.4436414 - 1.0692250i\,, \tag{2.4}$$

$$\text{Sound}: \quad \mathfrak{q}_{\rm c}^2 = \pm 2i, \quad \mathfrak{w}_{\rm c} = \pm 1 - i\,, \tag{2.5}$$

leading to the radii of convergence (1.3) and (1.4). In general, a multitude of critical points is expected to exist in the complex $\mathbf{q}^2$-plane, representing level-crossings among two or more branches of the spectrum. Moreover, at finite N_c or at weak coupling described by kinetic theory, one may also expect other types of singularities to appear [30].

Here, we continue working in the limit $N_c \to \infty$, and extend the approach of refs. [4, 5] to bulk gravity theories with higher derivative terms, i.e., to the domain of large but finite 't Hooft coupling.

2.2 Non-perturbative "resummation"

Inverse 't Hooft coupling corrections in the $\mathcal{N} = 4$ SYM theory arise from higher-derivative terms in the dual type IIB string theory low energy effective action (see e.g. refs. [11–15]). In a holographic calculation of a quasinormal spectrum, the bulk equations of motion typically produce a differential equation for the background fluctuation $Z = Z(u, \mathfrak{w}, \mathfrak{q}^2)$ of the form [13–15]

$$\partial_u^2 Z + \mathcal{A}(u, \mathfrak{w}, \mathfrak{q}^2)\partial_u Z + \mathcal{B}(u, \mathfrak{w}, \mathfrak{q}^2) Z = \gamma H\left[Z, \partial_u Z, \partial_u^2 Z, \partial_u^3 Z, \ldots\right]\,, \tag{2.6}$$

[1] At the critical points with regular branches we have "level-touching" rather than "level-crossing", as happens e.g. for the BTZ background [4]. See appendix A for details.

where u is the radial coordinate in the bulk with $u = 0$ the location of the boundary, γ is a small parameter proportional to the inverse coupling (e.g. $\gamma \sim \lambda^{-3/2}$ in the $\mathcal{N} = 4$ SYM with the 't Hooft coupling λ), and the right hand side comes from the leading higher-derivative correction to Einsten-Hilbert action (e.g. from the R^4 term in type IIB supergravity). To avoid issues such as Ostrogradsky instability (see e.g. ref. [26] and references therein), the higher-derivative terms in eq. (2.6) are usually treated as perturbations of the second-order ODE, and its left-hand-side is used to eliminate all derivatives higher than the first one from the right-hand-side, ignoring contributions of order γ^2 and higher. The resulting equation,

$$\partial_u^2 Z + \bar{\mathcal{A}}(u, \mathfrak{w}, \mathfrak{q}^2, \gamma)\partial_u Z + \bar{\mathcal{B}}(u, \mathfrak{w}, \mathfrak{q}^2, \gamma) Z = 0\,, \tag{2.7}$$

is a homogeneous linear second-order ODE whose coefficients $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ now depend on γ. As discussed above, the spectral curve is then determined by the boundary value of the solution $Z(u, \mathfrak{w}, \mathfrak{q}^2, \gamma)$ to that ODE, i.e. $P(\mathfrak{q}^2, \mathfrak{w}) \equiv Z(u \to 0, \mathfrak{w}, \mathfrak{q}^2, \gamma) = 0$.

In the standard approach, one looks for a perturbative solution to eq. (2.7) in the form $Z = Z_0 + \gamma Z_1$. Similarly, the perturbative ansatz for the spectrum is $\mathfrak{w} = \mathfrak{w}_0 + \gamma\mathfrak{w}_1$, where $\mathfrak{w}_0$ is the quasinormal frequency at $\gamma = 0$. Alternatively, one can solve eq. (2.7) without assuming a perturbative ansatz. Such a non-perturbative solution, if it can be expanded in series in powers of $\gamma \ll 1$, will not be fully quantitatively correct beyond linear order in γ, since both in the original equation (2.6) and in the steps leading to eq. (2.7) terms of order γ^2 and higher were ignored. Quantitatively, the solution only captures the non-perturbative effects in γ related to eq. (2.7) and in this sense only partially "resums" the contributions nonlinear in γ in the approximation to the full solution. However, such a solution may provide a more faithful qualitative approximation to the exact solution at finite γ. Moreover, if the exact solution is non-perturbative in $\gamma \ll 1$, any perturbative ansatz would necessarily miss it completely, whereas the non-perturbative approach is capable of describing the situation qualitatively correctly. The choice of a correct ansatz is the crucial step in singular perturbation theory [32]. We discuss these issues in more detail in section 5 and illustrate them with simple examples. In the context of holography, partial "resummations" have been used in refs. [23], [16, 33] and criticised in ref. [34]. A crucial feature of such a "resummation" in holography, first pointed out in [16], is that the quasinormal spectrum now contains new, non-pertubative gapped modes which seem to play an important role in describing physics at finite coupling qualitatively correctly [16, 33, 35, 36]. We shall see in section 3 that the situation with the radii of convergence is similar: the non-perturbative "resummation" reverses the tendency seen in eqs. (1.7), (1.8), making the radii to decrease (after an initial rise) with the coupling decreasing. In section 4, we compare this behaviour with that in the Einstein-Gauss-Bonnet theory, where both perturbative and non-perturbative results are available, using it as a theoretical laboratory to test our methods, and find a qualitative agreement with the $\mathcal{N} = 4$ SYM case. Curiously, in section 5 we find that it is in fact possible in some cases to infer the existence of non-perturbative critical points by using perturbative data. While we are able to explicitly demonstrate this in the Einstein-Gauss-Bonnet theory, for the $\mathcal{N} = 4$ SYM theory, this may serve as an indicative argument that the same behaviour is plausible.

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3 Convergence of hydrodynamic series in the $\mathcal{N} = 4$ SYM theory

We begin by studying the coupling dependence of the radii of convergence of the hydrodynamic shear and sound modes of the $\mathcal{N} = 4$ SU(N_c) SYM theory in the $N_c \to \infty$ limit and at large but finite 't Hooft coupling λ. Our analysis uses its gravitational dual, namely, the type IIB supergravity with higher-derivative terms in the action. For the $\mathcal{N} = 4$ SYM theory, the source of finite 't Hooft coupling corrections is the ten-dimensional low-energy effective action of type IIB string theory

$$S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4\cdot 5!}F_5^2 + \gamma e^{-\frac{3}{2}\phi}\mathcal{W} + \dots\right), \tag{3.1}$$

where $\gamma = \alpha'^3\zeta(3)/8$, with α' set by the length of the fundamental string, and the term $\mathcal{W}$ proportional to the contractions of the four copies of the Weyl tensor,

$$\mathcal{W} = C^{\alpha\beta\gamma\delta}C_{\mu\beta\gamma\nu}C_{\alpha}{}^{\rho\sigma\mu}C^{\nu}{}_{\rho\sigma\delta} + \frac{1}{2}C^{\alpha\delta\beta\gamma}C_{\mu\nu\beta\gamma}C_{\alpha}{}^{\rho\sigma\mu}C^{\nu}{}_{\rho\sigma\delta}\,. \tag{3.2}$$

Considering corrections to the AdS-Schwarzschild black brane background and its fluctuations, potential α' corrections to supergravity fields other than the metric and the five-form field have been argued to be irrelevant [37]. Moreover, as discussed in [38], for the purposes of computing the corrected quasinormal spectrum, one can use the Kaluza-Klein reduced five-dimensional action

$$S = \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left(R + \frac{12}{L^2} + \gamma\mathcal{W}\right), \tag{3.3}$$

where $\mathcal{W}$ is now given by eq. (3.2) in $5d$. The effective five-dimensional gravitational constant is related to the rank of the gauge group SU(N_c) by the expression $\kappa_5 = 2\pi/N_c$. The parameter γ is related to the value of the 't Hooft coupling constant λ in the $\mathcal{N} = 4$ SYM theory via $\gamma = \lambda^{-3/2}\zeta(3)L^6/8$. This parameter is dimensionless in units of L. Higher derivative terms in the equations of motion are treated as perturbations in γ. In the following, we shall use λ and

$$\gamma = \frac{\zeta(3)}{8\lambda^{3/2}} \ll 1 \tag{3.4}$$

interchangeably.

The black brane solution to the equations of motion following from the action (3.3), which is dual to an equilibrium thermal state of the CFT at temperature T, is given by [11, 12]

$$ds^2 = \frac{(\pi T L)^2}{u}\left(-e^{A(u)}f(u)dt^2 + dx^2 + dy^2 + dz^2\right) + e^{B(u)}\frac{L^2 du^2}{4u^2 f}, \tag{3.5}$$

where $f(u) = 1 - u^2$. The radial coordinate is denoted by u, with the boundary located at $u = 0$ and the horizon at $u = 1$. To leading order in γ, the functions $A(u)$ and $B(u)$ were found to be

$$A(u) = -15\gamma\left(5u^2 + 5u^4 - 3u^6\right), \qquad B(u) = 15\gamma\left(5u^2 + 5u^4 - 19u^6\right). \tag{3.6}$$

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The correction to the $\mathcal{N}=4$ SYM entropy density is then given by [11]

$$\frac{s}{s_0} = \frac{3}{4}\left(1+15\gamma+\cdots\right), \tag{3.7}$$

where $s_0 = 2\pi^2 N_c^2 T^3/3$ is the Stefan-Boltzmann entropy density of the ideal gas of particles in the $\mathcal{N}=4$ SYM theory (i.e., in the theory at $\lambda = 0$). The metric (3.5) was also used to compute 't Hooft coupling constant corrections to the ratio of shear viscosity η to entropy density [13, 14],

$$\frac{\eta}{s} = \frac{1}{4\pi}\left(1+120\gamma+\cdots\right), \tag{3.8}$$

and to all of the second-order transport coefficients of the $\mathcal{N}=4$ SYM plasma.[2] Computing transport coefficients and, in general, correlation functions of the energy-momentum tensor in the $\mathcal{N}=4$ SYM plasma requires considering small fluctuations of the metric $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}(u,t,x,y,z)$, where $g^{(0)}_{\mu\nu}$ is the background (3.5). Due to translational invariance and spatial isotropy of the background, we can Fourier transform the fluctuations and choose the direction of spatial momentum along z, so that

$$h_{\mu\nu}(u,t,z) = \int \frac{d\omega dq}{(2\pi)^2}\, e^{-i\omega t+iqz}\, h_{\mu\nu}(u,\omega,q)\,. \tag{3.9}$$

Following the recipes of ref. [19] and choosing the radial gauge $h_{u\nu}=0$, one can write down the linearised equations of motion for the three gauge-invariant linear combinations Z_i, $i=1,2,3$, of the modes $h_{\mu\nu}(u,\omega,q)$ in the scalar, shear and sound channels, respectively [16, 19, 39].

The linearised equations of motion obtained following the procedure outlined in section 2 are given in the three channels by [16]

$$\partial_u^2 Z_i + \mathcal{A}_{(i)}(u,\mathfrak{w},\mathfrak{q}^2,\gamma)\partial_u Z_i + \mathcal{B}_{(i)}(u,\mathfrak{w},\mathfrak{q}^2,\gamma) Z_i = 0\,, \tag{3.10}$$

where $\mathcal{A}_{(i)}(u,\mathfrak{w},\mathfrak{q}^2,\gamma) = \mathcal{A}^{(0)}_{(i)}(u,\mathfrak{w},\mathfrak{q}^2) + \gamma\mathcal{A}^{(1)}_{(i)}(u,\mathfrak{w},\mathfrak{q}^2)$ and $\mathcal{B}_{(i)}(u,\mathfrak{w},\mathfrak{q}^2,\gamma) = \mathcal{B}^{(0)}_{(i)}(u,\mathfrak{w},\mathfrak{q}^2) + \gamma\mathcal{B}^{(1)}_{(i)}(u,\mathfrak{w},\mathfrak{q}^2)$. The coefficients are given explicitly in appendix C. As discussed in section 2, using the ODEs (3.10), the quasinormal spectrum can now be computed either perturbatively by expanding $Z = Z_0 + \gamma Z_1$ along with $\mathfrak{w} = \mathfrak{w}_0 + \gamma\mathfrak{w}_1$ and $\mathfrak{q}^2 = \mathfrak{q}_0^2 + \gamma\mathfrak{q}_1^2$ [40], or non-perturbatively by treating eq. (3.10) as being exact in the parameter γ [23], [16, 33].

As an example, the shear channel quasinormal spectrum for $\gamma = 1\cdot 10^{-5}$ is shown in figure 2. Its novel feature, discussed in detail in ref. [16], is the existence of the non-perturbative (in γ) gapped modes on the imaginary axis. The highest (closest to the real axis) of those modes is shown in figure 2 by the red square: with real $\mathfrak{q}^2$ increasing, this mode moves up the axis and at $\mathfrak{q}^2 = \mathfrak{q}_*^2$ it collides with the hydrodynamic shear mode (shown in figure 2 by the red circle). For $\mathfrak{q}^2 > \mathfrak{q}_*^2$, the two modes move off the imaginary axis, effectively destroying the diffusive pole of the correlator. In ref. [16], this was interpreted as the end of the hydrodynamic regime at sufficiently large spatial momentum (small wavelength), where the microscopic effects prevail over the collective ones. The dependence

[2] The complete list of the coefficients can be found e.g. in refs. [15, 26].

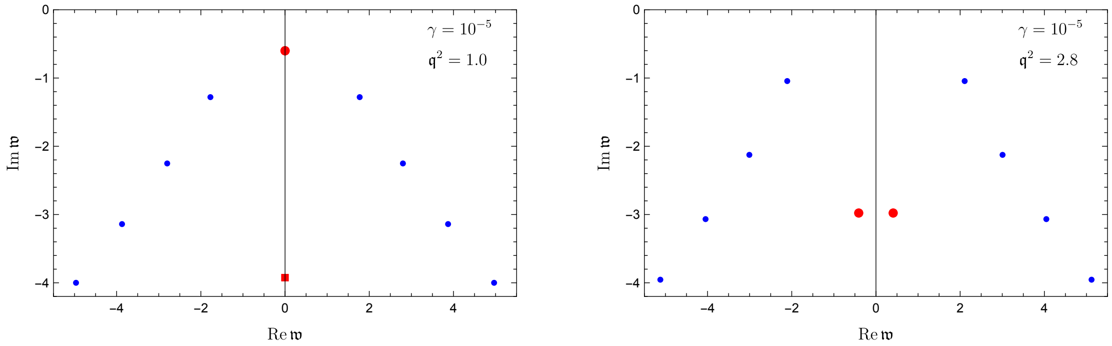

Figure 2. Quasinormal spectrum in the shear channel of the $\mathcal{N}=4$ SYM for $\gamma = 1 \cdot 10^{-5}$ and $\mathfrak{q}^2 = 1$ (left panel). The hydrodynamic shear mode at $\mathfrak{w} \approx -0.60064i$ is shown by the red circle. The new feature, not seen in a perturbative calculation, is the appearance of an extra mode on the imaginary axis (shown by the red square), ascending from complex infinity with γ increasing [16]. With real $\mathfrak{q}^2$ increasing, the hydrodynamic mode moves down the imaginary axis, while the new non-perturbative mode moves up. They collide at $\mathfrak{q}_*^2 \approx 2.72$ and move off the axis for $\mathfrak{q}^2 > \mathfrak{q}_*^2$ (right panel).

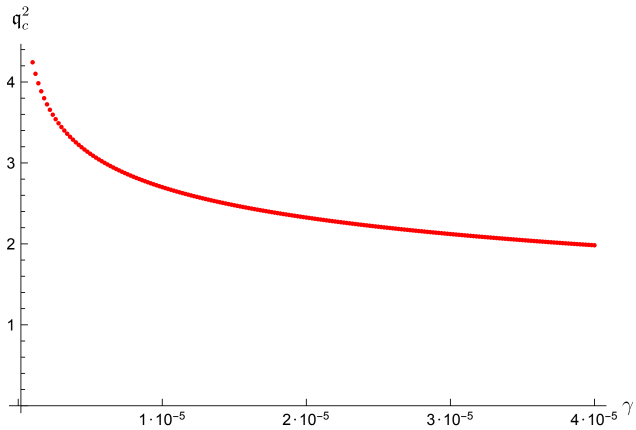

Figure 3. The value of the (real) spatial momentum squared, limiting the hydrodynamic regime, as a function of the (inverse) coupling γ in the shear channel of the $\mathcal{N}=4$ SYM [16]. Hydrodynamics has a wider range of applicability in $\mathfrak{q}^2$ at smaller γ (larger 't Hooft coupling).

$\mathfrak{q}_*^2 = \mathfrak{q}_*^2(\gamma)$, shown in figure 3, suggests that the domain of applicability of the hydrodynamic description is smaller at larger γ (i.e., at smaller 't Hooft coupling), but it seems to extend to arbitrarily large momentum in the limit of infinite coupling (at $\gamma \to 0$) [16].

To see how this qualitative picture is amended at very large but finite 't Hooft coupling, we now consider the radius of convergence of the hydrodynamic shear and sound dispersion series in this theory, which requires us to solve eq. (2.3) and look for critical points with $p = 2$. With $P(\mathfrak{q}^2, \mathfrak{w})$ given by $P(\mathfrak{q}^2, \mathfrak{w}) = Z(\mathfrak{w}, \mathfrak{q}^2) \equiv Z(u = 0, \mathfrak{w}, \mathfrak{q}^2)$ for any channel (we omit the index "i" labeling the channel), we are therefore looking for solutions $\mathfrak{w} = \mathfrak{w}_c$ and $\mathfrak{q}^2 = \mathfrak{q}_c^2$ to the system

$$Z(\mathfrak{w}, \mathfrak{q}^2) = 0\,, \qquad \partial_{\mathfrak{w}} Z(\mathfrak{w}, \mathfrak{q}^2) = 0\,, \tag{3.11}$$

with $\partial^2_{\mathfrak{w}} Z(\mathfrak{w}, \mathfrak{q}^2) \neq 0$. In the non-perturbative approach (in γ), critical points follow directly

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from eq. (3.11), where $Z(\mathfrak{w}, \mathfrak{q}^2)$ can be found by constructing a Frobenius series solution $Z(u, \mathfrak{w}, \mathfrak{q}^2)$ to eq. (3.10) around the horizon in the standard way [17–19], and setting $u = 0$. For $\gamma = 0$, this is the same procedure as the one used in refs. [4, 5].

To find the critical points perturbatively, we expand equations (3.11) as

$$\begin{aligned} Z_0(\mathfrak{w}, \mathfrak{q}^2) + \gamma Z_1(\mathfrak{w}, \mathfrak{q}^2) &= 0\,, \\ \partial_{\mathfrak{w}} Z_0(\mathfrak{w}, \mathfrak{q}^2) + \gamma \partial_{\mathfrak{w}} Z_1(\mathfrak{w}, \mathfrak{q}^2) &= 0\,, \end{aligned} \tag{3.12}$$

and also expand

$$\mathfrak{w}_{\rm c} = \mathfrak{w}_{\rm c,0} + \gamma \mathfrak{w}_{\rm c,1}\,, \qquad \mathfrak{q}^2_{\rm c} = \mathfrak{q}^2_{\rm c,0} + \gamma \mathfrak{q}^2_{\rm c,1}\,. \tag{3.13}$$

Together, these expansions yield a system of equations at $\mathcal{O}(\gamma^0)$:

$$\begin{aligned} Z_0(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0}) &= 0\,, \\ \partial_{\mathfrak{w}} Z_0(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0}) &= 0\,, \end{aligned} \tag{3.14}$$

and a system at $\mathcal{O}(\gamma)$:

$$\begin{aligned} Z_1(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0}) + \partial_{\mathfrak{q}^2} Z_0(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0})\, \mathfrak{q}^2_{\rm c,1} &= 0\,, \\ \partial_{\mathfrak{w}} Z_1(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0}) + \partial^2_{\mathfrak{w}} Z_0(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0})\, \mathfrak{w}_{\rm c,1} + \partial_{\mathfrak{w}} \partial_{\mathfrak{q}^2} Z_0(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0})\, \mathfrak{q}^2_{\rm c,1} &= 0\,. \end{aligned} \tag{3.15}$$

Eqs. (3.14) and (3.15) are sufficient to find $\mathfrak{w}_{\rm c,0}$, $\mathfrak{w}_{\rm c,1}$, $\mathfrak{q}^2_{\rm c,0}$ and $\mathfrak{q}^2_{\rm c,1}$. More explicitly, eq. (3.15) allows us to express

$$\begin{aligned} \mathfrak{q}^2_{\rm c,1} &= -\frac{Z_1(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0})}{\partial_{\mathfrak{q}^2} Z_0(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0})}\,, \\ \mathfrak{w}_{\rm c,1} &= \frac{1}{\partial^2_{\mathfrak{w}} Z_0(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0})} \left(\frac{Z_1(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0}) \partial_{\mathfrak{w}} \partial_{\mathfrak{q}^2} Z_0(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0})}{\partial_{\mathfrak{q}^2} Z_0(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0})} - \partial_{\mathfrak{w}} Z_1(\mathfrak{w}_{\rm c,0}, \mathfrak{q}^2_{\rm c,0}) \right)\,. \end{aligned} \tag{3.16}$$

As before, the function $Z_0(\mathfrak{w}, \mathfrak{q}^2)$ can be obtained as the boundary value of the Frobenius solution to eq. (3.10) with $\gamma = 0$ and $Z_1(\mathfrak{w}, \mathfrak{q}^2)$ as the boundary value of the Frobenius solution to the corresponding inhomogeneous equation.

3.1 Shear channel

At infinite 't Hooft coupling, the shear mode dispersion relation $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)$ has numerous branch point singularities [4, 5]. At finite coupling, we expect those singularities, now parametrised by $\gamma \propto \lambda^{-3/2}$, to move in the complex $\mathfrak{q}^2$-plane with γ varying. As discussed in section 2.2, one can compute relevant corrections by using either the "conservative" perturbative or the non-perturbative approach.

3.1.1 Perturbative calculation

For a perturbative calculation of the coupling constant correction to the radius of convergence in the shear channel of the $\mathcal{N}=4$ SYM theory, we use eq. (3.10) with $i=2$. To first order in $\gamma \propto \lambda^{-3/2}$, from eqs. (3.16) we find the following first set of critical points closest to the origin in the complex $\mathfrak{q}^2$-plane:

$$\mathfrak{q}_c^2 \approx 1.89065 \pm 1.17115i + \gamma(4081.99 \pm 1862.06i)\,, \tag{3.17}$$
$$\mathfrak{w}_c \approx \pm 1.44364 - 1.06923i + \gamma(\pm 3671.27 + 1360.52i)\,. \tag{3.18}$$

The value of $\mathfrak{q}_c^2$ in eq. (3.17) gives the convergence radius $|\mathfrak{q}_c^2|$ quoted in eq. (1.7). The radius increases with γ increasing (i.e. with the coupling λ decreasing from its infinite value). The numerical coefficients multiplying the parameter γ in eqs. (3.17), (3.18) are large: the perturbative terms give small corrections to the $\gamma=0$ result only for $\gamma \lesssim 10^{-4}-10^{-5}$.

The next closest to the origin critical point (i.e. the critical point with larger value of $|\mathfrak{q}^2|$ than (3.17)) is located on the negative real axis of $\mathfrak{q}^2$:

$$\mathfrak{q}_{c,1}^2 \approx -2.37737 + \gamma\, 2608.88\,, \tag{3.19}$$
$$\mathfrak{w}_{c,1} \approx -1.64659i - \gamma\, 6599.64i\,. \tag{3.20}$$

At $\gamma=0$, this critical point plays no role in determining the radius of convergence. At finite γ, the point (3.19) moves closer to the origin with γ increasing, whereas the pair of points (3.17) moves away from it. At $\gamma=\gamma_* \approx 2.172\cdot 10^{-5}$, the critical point (3.19) formally becomes dominant (closest to the origin), changing the situation qualitatively. We view this as an indication of the breakdown of linear (in γ) approximation.

The next two sets of critical points with yet larger values of $|\mathfrak{q}^2|$ are

$$\mathfrak{q}_{c,2}^2 \approx -3.11051 \mp 0.81050i + \gamma(-52560.3 \pm 77406.3i)\,, \tag{3.21}$$
$$\mathfrak{w}_{c,2} \approx \pm 1.41043 - 2.87086i + \gamma(\pm 31019.2 - 11091.7i)\,, \tag{3.22}$$
$$\mathfrak{q}_{c,3}^2 \approx 2.90684 \pm 1.66612i + \gamma(40520.1 \pm 17681.1i)\,, \tag{3.23}$$
$$\mathfrak{w}_{c,3} \approx \pm 2.38819 - 2.13154i + \gamma(\pm 26733.9 + 13539.1i)\,. \tag{3.24}$$

Notice again the large numerical coefficients multiplying the perturbative parameter γ in eqs. (3.21)–(3.24). For illustration, several perturbative closest to the origin critical points in the complex $\mathfrak{q}^2$-plane for $\gamma = 1\cdot 10^{-5}$ are shown in figure 4 in blue colour.

3.1.2 Non-perturbative calculation

For a non-perturbative calculation, we solve eqs. (3.10) and (3.11) numerically without assuming γ to be small. We observe three qualitatively different scenarios of quasinormal modes' behaviour, and illustrate them by showing the modes at $\gamma = 1\cdot 10^{-5}$, $\gamma = 2\cdot 10^{-5}$ and $\gamma = 3\cdot 10^{-5}$, respectively (see figures 5, 7, 8):

(a) At $\gamma = 1\cdot 10^{-5}$, the top modes in the spectrum are shown in figure 5 in the complex plane of $\mathfrak{w}$, for complex values of the spatial momentum squared $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\varphi}$, where the phase φ is varied from 0 to 2π. The figure shows how the critical points (we show

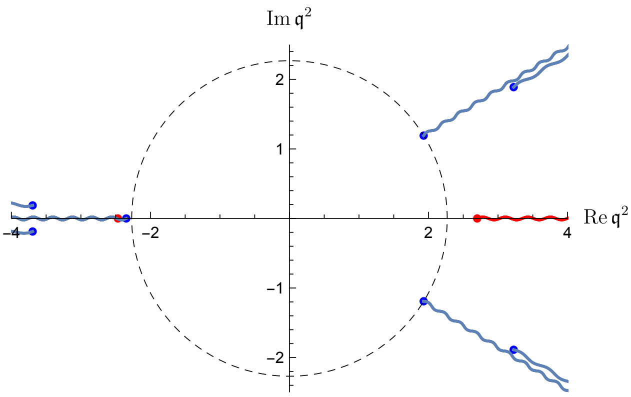

Figure 4. The closest to the origin branch points (see table 1) of the $\mathcal{N} = 4$ SYM shear mode dispersion relation $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)$ in the complex $\mathfrak{q}^2$-plane at $\gamma = 1 \cdot 10^{-5}$. The critical points seen in perturbation theory and their branch cuts are shown by blue colour, the two non-perturbative critical points on the real axis are shown in red. The radius of convergence at $\gamma = 1 \cdot 10^{-5}$, $R_{\text{shear}} \approx 2.27$, is determined by the pair of branch points at $\mathfrak{q}^2 \approx 1.93 \pm 1.19i$. We note that it is the sole inclusion of the non-perturbative red critical point on the positive real $\mathfrak{q}^2$-axis that leads to the non-perturbative result of figure 3.

#	$\mathfrak{q}_c^2$ (non-pert.)	$\lvert\mathfrak{q}_c^2\rvert$ (non-pert.)	$\mathfrak{q}_c^2$ (pert.)	$\lvert\mathfrak{q}_c^2\rvert$ (pert.)
1	$1.93027 \pm 1.19123i$	2.268	$1.93147 \pm 1.18977i$	2.269
2	-2.34715	2.347	-2.35128	2.351
3	-2.46848	2.469	n/a	n/a
4	2.70094	2.701	n/a	n/a
5	$-3.69434 \pm 0.18770i$	3.699	$-3.63611 \pm 0.03644i$	3.528
6	$3.22474 \pm 1.88845i$	3.737	$3.31204 \pm 1.84293i$	3.790

Table 1. The six closest to the origin (in the complex $\mathfrak{q}^2$-plane) critical points for $\gamma = 1 \cdot 10^{-5}$.

the first four points closest to the origin) arise from the collision of quasinormal modes trajectories as the phase φ varies. The closest to the origin (in the complex $\mathfrak{q}^2$-plane; see figure 4) pair of critical points sets the radius of convergence $R_{\text{shear}} = |\mathfrak{q}_c^2| \approx 2.27$ of the hydrodynamic series (in the complex $\mathfrak{w}$-plane, this point is shown in the top left panel in figure 5). The location of the six closest to the origin critical points at $\gamma = 1 \cdot 10^{-5}$ is given in tables 1 and 2, where a comparison between perturbative and non-perturbative results is also made. For $\gamma = 1 \cdot 10^{-5}$, the location of the critical points is in a reasonably good agreement with the perturbative results (3.17), (3.18) and (3.19), (3.20), except when the collision of modes involves the mode on the imaginary axis which has no perturbative analogue.

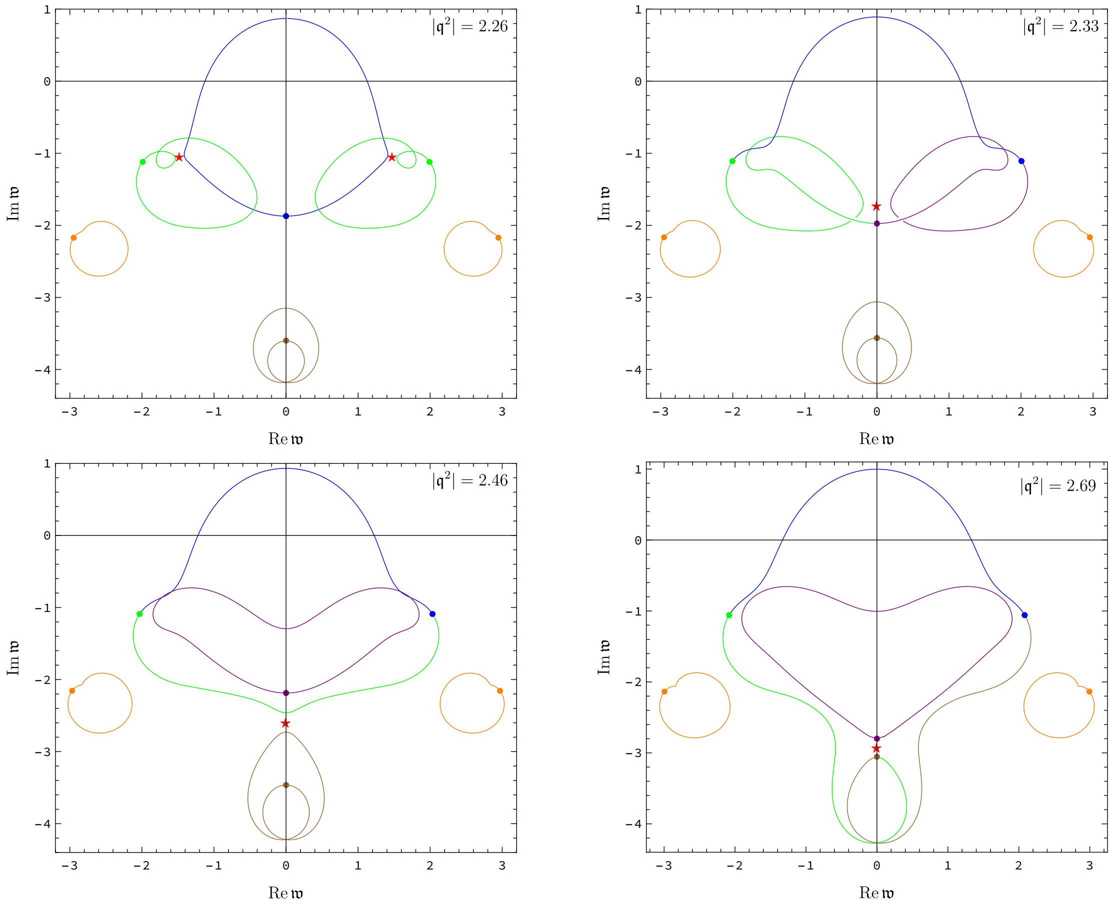

Figure 5. Quasinormal spectrum in the shear channel at $\gamma = 1\cdot 10^{-5}$, computed non-perturbatively in γ. The trajectories are plotted for complex values of the spatial momentum squared, $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\varphi}$, where phase φ is varied from 0 to 2π. The positions of quasinormal modes at $\varphi = 0$ are shown by dots. The positions of the two critical points (closest to the origin in the complex $\mathfrak{q}^2$-plane) are shown by red stars. The first pair of critical points corresponds to the collision of trajectories of the two top gapped modes (shown in green) with the hydrodynamic shear mode trajectory (shown in blue) at $\mathfrak{q}_c^2 \approx 1.93027 \pm 1.19123i$ and $\mathfrak{w}_c \approx \pm 1.47755 - 1.05400i$. The corresponding value $|\mathfrak{q}^2| \approx 2.27$ sets the radius of convergence of the hydrodynamic mode (top left panel). The second critical point arises from the collision on the imaginary axis between the parts of the common curve involving the three top modes including the shear mode (top right panel). The two plots in the bottom row show the third and the fourth critical points. Both points arise on the imaginary axis of complex frequency from the collision involving the new, non-perturbative mode in the quasinormal spectrum.

(b) At $\gamma = 2\cdot 10^{-5}$, the top modes in the spectrum are shown in figure 7. Here, the first level-crossing (i.e., the level-crossing with the minimal value of $|\mathfrak{q}^2|$) occurs between the two top gapped modes and the non-perturbative mode on the imaginary axis, as shown in the top left panel of figure 7. However, the shear mode is not affected by this crossing: its first non-analyticity still arises as a result of the collision with the top two gapped modes as shown in the top right panel of figure 7. This collision sets the radius of convergence of the shear mode at $R_{\text{shear}}(\gamma) = |\mathfrak{q}_c^2| \approx 2.31$. In the

#	$\mathfrak{w}_c$ (non-perturbative)	$\mathfrak{w}_c$ (perturbative)
1	$\pm 1.47755 - 1.05400i$	$\pm 1.48035 - 1.05562i$
2	$-1.73447i$	$-1.71258i$
3	$-2.60274i$	n/a
4	$-2.93397i$	n/a
5	$\pm 1.53654 - 2.71700i$	$\pm 1.72062 - 2.98177i$
6	$\pm 2.54972 - 1.99267i$	$\pm 2.65553 - 1.99615i$

Table 2. The six closest to the origin (in the complex $\mathfrak{w}$-plane) critical points for $\gamma = 1 \cdot 10^{-5}$.

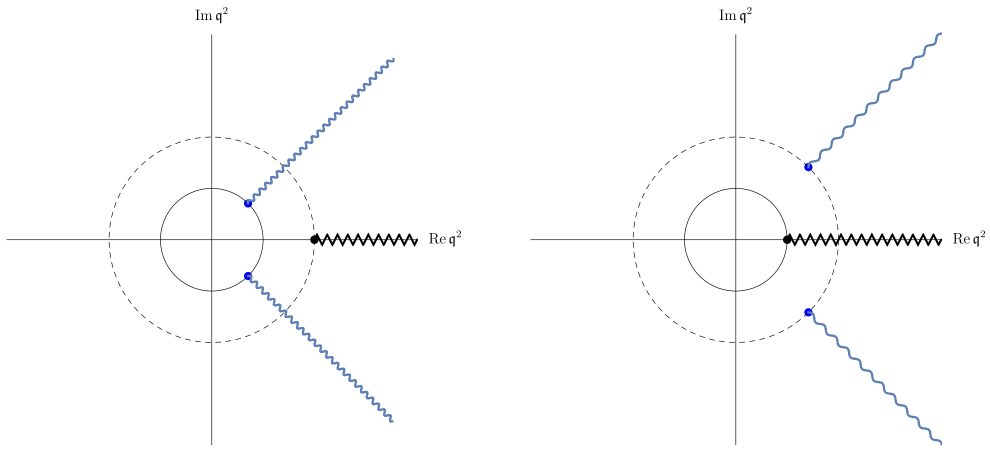

Figure 6. The closest to the origin branch points of the shear mode dispersion relation $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)$ in the complex $\mathfrak{q}^2$-plane (shown schematically). The radius of convergence is determined by the pair of branch points (left panel) or the branch point on the real axis (right panel).

complex $\mathfrak{q}^2$-plane, the position of the branch point singularities is thus qualitatively the same as at $\gamma = 1 \cdot 10^{-5}$ (see figure 6, left panel), but the radius of convergence $R_{\text{shear}}(\gamma)$ increases with γ increasing (i.e. with the coupling λ decreasing).

c) Finally, at $\gamma = 3 \cdot 10^{-5}$, the top modes in the spectrum are shown in figure 8. In this case, the situation is qualitatively different. Now the first level-crossing occurs at a *real* value of $\mathfrak{q}_c^2 \approx 2.12157$ and at $\mathfrak{w}_c \approx -2.096i$, which is a result of the collision between the shear mode and the non-perturbative mode on the imaginary axis (top left panel in figure 8). This is the collision of the type shown in figure 2 and interpreted in ref. [16], where it was discovered, as the end point $\mathfrak{q}_*^2(\gamma)$ of the hydrodynamic regime (in the sense that for real $\mathfrak{q}^2 > \mathfrak{q}_*^2$ the hydrodynamic purely imaginary shear mode does not exist). The radius of convergence in the $\mathfrak{q}^2$-plane is set by the corresponding value of $|\mathfrak{q}_*^2| = |\mathfrak{q}_c^2| \approx 2.12157$ (see figure 6, right panel, where this situation is shown schematically). Thus, at $\gamma = 3 \cdot 10^{-5}$, the radius of convergence is determined by the non-perturbative mode. In this regime, the convergence radius decreases with γ increasing.

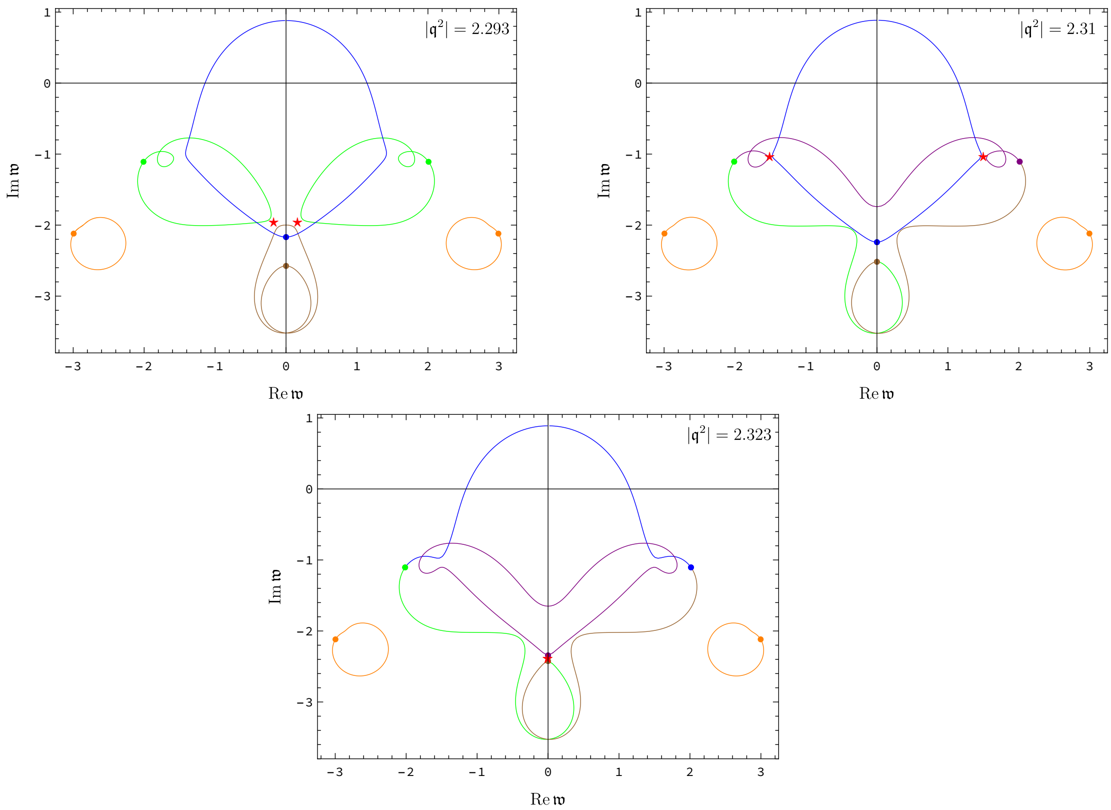

Figure 7. Quasinormal spectrum in the shear channel at $\gamma = 2\cdot 10^{-5}$, computed non-perturbatively in γ. The level-crossing occurring at the smallest value of $|\mathfrak{q}^2|$ (top left panel) does not affect the hydrodynamic mode (shown in blue colour). The first (smallest in $|\mathfrak{q}^2|$) critical point of the shear mode arises from the level-crossing with the top gapped modes (top right panel). This point sets the radius of convergence of the hydrodynamic series. The critical point with an even higher value of $|\mathfrak{q}^2|$ is shown in the bottom panel.

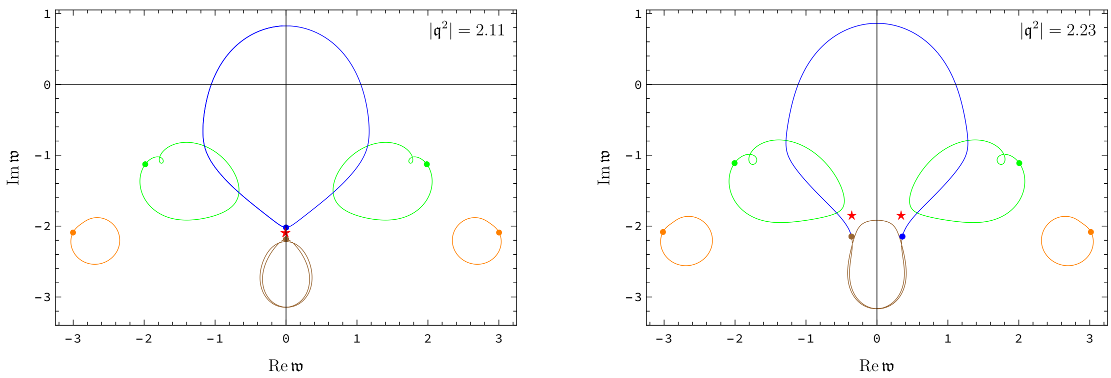

Figure 8. Quasinormal spectrum in the shear channel at $\gamma = 3\cdot 10^{-5}$, computed non-perturbatively in γ. The radius of convergence is set by the level-crossing between the shear mode and the non-perturbative mode on the imaginary axis (left panel). This critical point coincides with the endpoint of the hydrodynamic regime discussed in ref. [16]. An example of critical points with a larger $|\mathfrak{q}^2|$ is shown in the right panel.

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The three examples considered above illustrate the general situation fully. For infinitesimally small γ, the radius of convergence of the shear mode dispersion relation is an increasing function of γ. In the complex $\mathfrak{q}^2$-plane, the obstacle to convergence is the pair of critical points, as shown in figure 6 (left panel). These points move away from the origin with γ increasing. At $\gamma = \gamma_* \approx 2.05 \cdot 10^{-5}$, the picture changes qualitatively, as the transition between the regimes a) and c) occurs. Now the new critical point arising from the level-crossing with the non-perturbative mode is located closer to the origin in the complex $\mathfrak{q}^2$-plane than the previous pair of critical points (figure 6, right panel). This new critical point is located on the positive real axis of $\mathfrak{q}^2$ and corresponds to the end point of hydrodynamic regime as discussed in ref. [16]. This point moves closer to the origin with γ increasing. The dependence of the radius of convergence on γ is thus a piecewise continuous function,[3] shown in figure 1 (left panel), which is the main result of this section. The results of the perturbative and non-perturbative calculations for the closest to the origin critical points (the ones seen in perturbation theory) are in good agreement, as can be observed from table 1.

We emphasise that the non-perturbative effects discussed in this section are at best qualitative, since terms of order γ^2 and higher will inevitably modify them. We believe these effects are qualitatively correct as they fit well with various physical expectations [16, 23, 33, 36]. In particular, the existence of the non-perturbative critical point makes the radius of convergence decrease with the ‘t Hooft coupling decreasing from its infinite value. Admittedly, an alternative conservative point of view would simply limit any considerations by the range of γ up to $\gamma \lesssim 2 \cdot 10^{-5}$, beyond which the perturbative treatment becomes unreliable.

3.2 Sound channel

For the sound channel, the analysis follows the strategy used in the previous section and in refs. [4, 5] very closely. The relevant equation of motion is eq. (3.10) with $i = 3$.

3.2.1 Perturbative calculation

Solving the equations perturbatively to linear order γ, we find the following correction to the location of the closest to the origin critical point

$$\mathfrak{q}_c^2 \approx \pm 2i + \gamma(166.844 \pm 3201.39i)\,, \tag{3.25}$$

$$\mathfrak{w}_c \approx \pm 1 - i + \gamma(\pm 2948.55 + 1459.36i)\,. \tag{3.26}$$

Eq. (3.25) gives the radius of convergence $R_{\text{sound}} = |\mathfrak{q}_c^2|$ stated in eq. (1.8). The next critical point is given by

$$\mathfrak{q}_{c,1}^2 \approx -0.01681 \pm 3.12967i + \gamma(9108.90 \pm 36862.6i)\,, \tag{3.27}$$

$$\mathfrak{w}_{c,1} \approx \pm 1.90135 - 2.04492i + \gamma(\pm 24615.9 + 12589.5i)\,. \tag{3.28}$$

[3] Curiously, the same type of a piecewise smooth dependence is observed when considering the radius of convergence as a function of chemical potential [7, 24] and as a function of the coupling in the Sachdev-Ye-Kitaev chain [9].

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As in the case of the shear mode, the coefficients of the perturbative expansion are quite large: the correction becomes comparable to the $\gamma = 0$ result already for $\gamma \sim 10^{-4}$–10^{-5}.

3.2.2 Non-perturbative calculation

Non-perturbative treatment implies solving the equation of motion (3.10) without assuming γ to be small. As in the shear mode case, there are essentially two qualitatively different scenarios of the distribution of critical points. We illustrate them by showing trajectories of quasinormal modes in the complex $\mathfrak{w}$-plane at complex $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\varphi}$, $\varphi \in [0, 2\pi]$, for $\gamma = 2 \cdot 10^{-5}$ and $\gamma = 4.5 \cdot 10^{-5}$ in figures 9 and 10, respectively.

In figure 9, left panel, the critical point limiting the radius of convergence of the sound mode's dispersion relation arises from the level-crossing between that mode and the top gapped modes (this situation is qualitatively the same as at $\gamma = 0$ [4, 5]). The pair of critical points closest to the origin in the complex $\mathfrak{q}^2$-plane is well approximated by the perturbative expression (3.27) for $\gamma \lesssim 3 \cdot 10^{-5}$ (figure 1, right panel). The radius of convergence, $R_{\text{sound}} = |\mathfrak{q}_{\text{c}}^2|$, increases with γ (see the blue curve in figure 1, right panel).

For $\gamma > \gamma_* \approx 3.225 \cdot 10^{-5}$, the situation changes qualitatively, as illustrated in figure 10. Now the sound mode first collides with the non-perturbative mode (in figure 10, i.e. at $\gamma = 4.5 \cdot 10^{-5}$, this happens at $\mathfrak{q}_c^2 = 0.9568 \pm 1.7083i$, $\mathfrak{w}_c \approx \mp 0.16513 - 1.8681i$, corresponding to $R_{\text{sound}} = |\mathfrak{q}_{\text{c}}^2| \approx 1.958$). In this regime, the radius of convergence, $R_{\text{sound}} = |\mathfrak{q}_{\text{c}}^2|$, decreases with γ (see the red curve in figure 1, right panel). This dependence is very similar, although not identical, to the one observed in figure 1 (left panel) for the shear channel.

As before, in a conservative approach we would limit ourselves to the blue part of the curve in the right panel of figure 1, which ends in the region where perturbation theory becomes unreliable. We believe, however, that the red part of the curve, although not quantitatively precise, reflects the dependence of the radius of convergence on coupling qualitatively correctly. The piecewise smooth dependence on the coupling, shown in figure 1 (right panel), is likely to persist even with γ^2 and higher terms in the action taken into account quantitatively correctly, since it corresponds to the discrete change of "status" of the closest to the origin critical point, even though the critical points move continuously in the complex plane with varying coupling.

4 Convergence of hydrodynamic series in the Einstein-Gauss-Bonnet theory

We now consider the radii of convergence of gapless quasinormal modes in five-dimensional Einstein-Gauss-Bonnet gravity. Although this theory may not have a healthy QFT dual [41], it is nevertheless a very useful theoretical laboratory for relevant bulk calculations in higher-derivative gravity: by design, its equations of motion are second-order in derivatives and can thus be solved fully non-perturbatively in terms of the higher-derivative coupling.

The Einstein-Gauss-Bonnet action in $5d$ is

$$S_{GB} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[R + \frac{12}{L^2} + \frac{l_{GB}^2}{2} \left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) \right], \tag{4.1}$$

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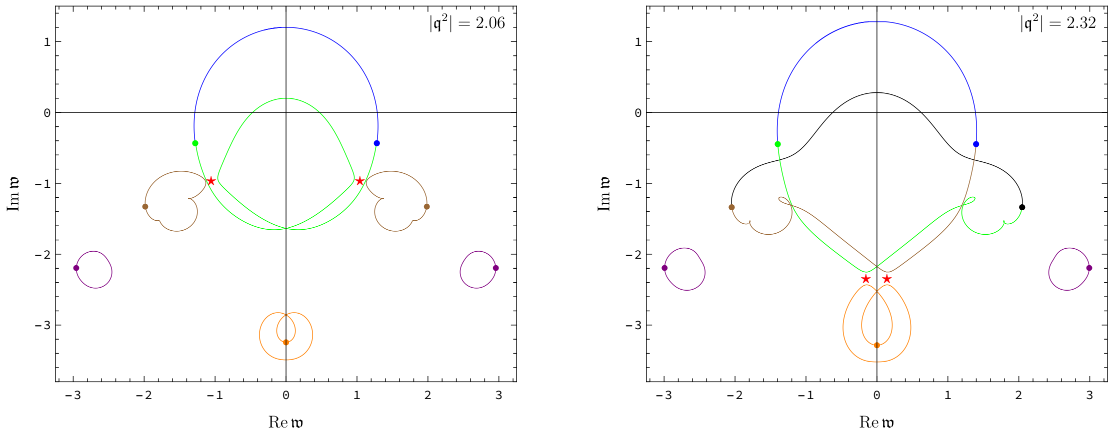

Figure 9. Quasinormal spectrum in the sound channel at $\gamma = 2{\cdot}10^{-5}$, computed non-perturbatively in γ. The positions of critical points are shown by red stars.

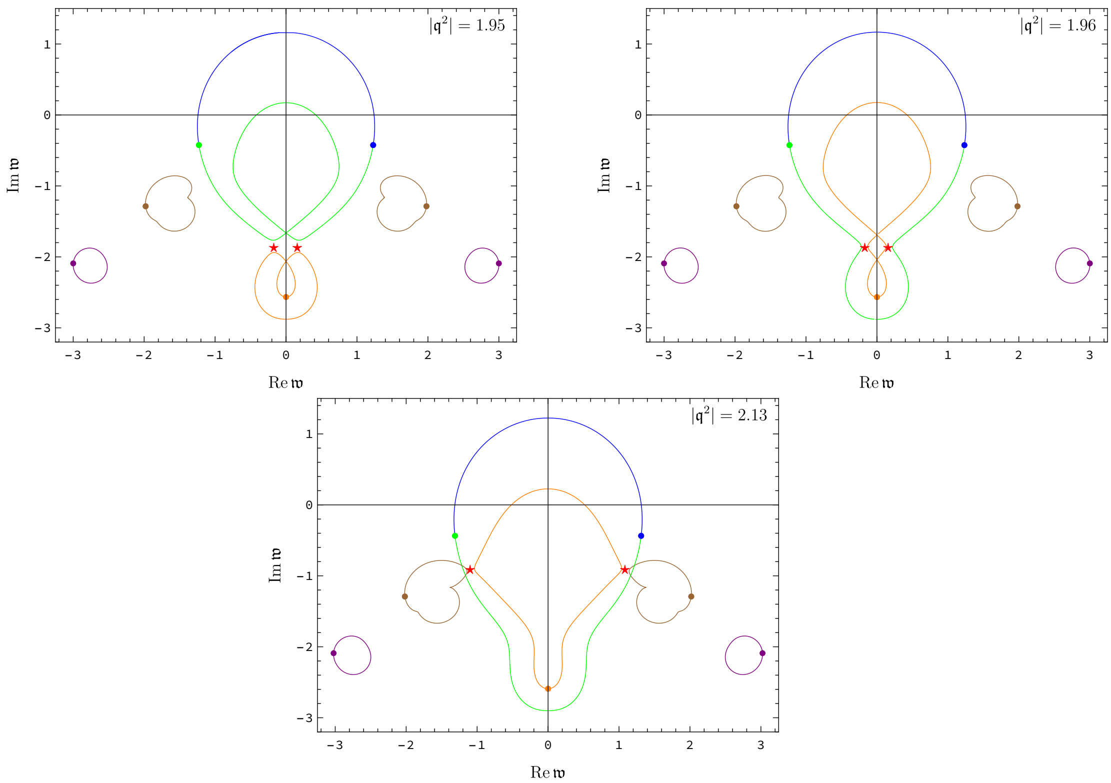

Figure 10. Quasinormal spectrum in the sound channel at $\gamma = 4.5 \cdot 10^{-5}$, computed non-perturbatively in γ.

where the scale l_{GB}^2 of the higher-derivative term can be chosen to be set by a cosmological constant, $l_{GB}^2 = \lambda_{GB} L^2$, where λ_{GB} is a dimensionless parameter.

The black brane metric solution of Einstein-Gauss-Bonnet equations of motion can be found analytically and is given by[4]

$$ds^2 = -f(r)N_{GB}^2 dt^2 + \frac{1}{f(r)}dr^2 + \frac{r^2}{L^2}\left(dx^2 + dy^2 + dz^2\right) , \tag{4.2}$$

where

$$f(r) = \frac{r^2}{L^2}\frac{1}{2\lambda_{GB}}\left[1 - \sqrt{1 - 4\lambda_{GB}\left(1 - \frac{r_0^4}{r^4}\right)}\right] \tag{4.3}$$

and the constant N_{GB} can be chosen to normalise the speed of light at the boundary to $c = 1$:

$$N_{GB}^2 = \frac{1}{2}\left(1 + \sqrt{1 - 4\lambda_{GB}}\right) . \tag{4.4}$$

The position of the horizon is at $r = r_0$. The Hawking temperature corresponding to the solution (4.2) is given by

$$T = \frac{N_{GB} r_0}{\pi L^2} = \frac{r_0\sqrt{1 + \gamma_{GB}}}{\sqrt{2}\pi L^2} , \tag{4.5}$$

where we introduced the notation $\gamma_{GB} \equiv \sqrt{1 - 4\lambda_{GB}}$. We shall use λ_{GB} and γ_{GB} interchangeably in the following. The range $\lambda_{GB} < 0$ corresponds to $\gamma_{GB} \in (1, \infty)$ and the interval $\lambda_{GB} \in (0, 1/4]$ maps onto $\gamma_{GB} \in [0, 1)$, with $\lambda_{GB} = 0$ corresponding to $\gamma_{GB} = 1$.

To compute the quasinormal mode spectrum, we again consider linearised metric perturbations. In analogy with our discussion in section 3, we arrive at three gauge-invariant equations of motion for the scalar, shear and sound channels ($i = 1, 2, 3$, respectively):

$$\partial_u^2 Z_i + \mathcal{A}_{(i)}(u, \mathfrak{w}, \mathfrak{q}^2, \lambda_{GB})\partial_u Z_i + \mathcal{B}_{(i)}(u, \mathfrak{w}, \mathfrak{q}^2, \lambda_{GB}) Z_i = 0 , \tag{4.6}$$

where the coefficients $\mathcal{A}_{(i)}$ and $\mathcal{B}_{(i)}$ are given in appendix D. All relevant details regarding the theory and the derivation of these equations can be found in ref. [26].

4.1 Shear channel

The shear channel spectrum is determined by eq. (4.6) with $i = 2$. As in the case of the $\mathcal{N} = 4$ theory, we compare calculations done perturbatively and non-perturbatively in the Gauss-Bonnet coupling λ_{GB}.

[4] Exact solutions and thermodynamics of black branes and black holes in Einstein-Gauss-Bonnet gravity were considered in [42] (see also [43–46]).

4.1.1 Perturbative calculation

Solving eqs. (3.16) (with the eq. (4.6), $i = 2$, as the underlying equation of motion) perturbatively in λ_{GB}, we find the critical point closest to the origin in the complex $\mathfrak{q}^2$-plane:

$$\mathfrak{q}_c^2 \approx 1.89065 \pm 1.17115i + \lambda_{GB}(-2.01742 \pm 22.5317i) + \mathcal{O}(\lambda_{GB}^2)\,, \tag{4.7}$$

$$\mathfrak{w}_c \approx \pm 1.44364 - 1.06923i + \lambda_{GB}(\mp 1.69340 + 8.39996i) + \mathcal{O}(\lambda_{GB}^2)\,. \tag{4.8}$$

The radius of convergence for the shear hydrodynamic mode is then given by $R_{\text{shear}} = |\mathfrak{q}_c^2|$:

$$R_{\text{shear}}(\lambda_{GB}) \approx 2.22 + 22.6\,\lambda_{GB} + \mathcal{O}(\lambda_{GB}^2)\,. \tag{4.9}$$

The coefficient in front of λ_{GB} is a positive number, which is similar to the $\mathcal{N} = 4$ case. However, the "physical" regime of the hypothetical field theory dual to Gauss-Bonnet gravity corresponds to $\lambda_{GB} < 0$ [16, 26–28, 47]. In this case, the radius of convergence decreases with $|\lambda_{GB}|$ increasing, the trend opposite to the one found for the $\mathcal{N} = 4$ SYM theory in section 3.1.

We also compute perturbative λ_{GB} corrections to the next three higher critical points, finding

$$\mathfrak{q}_{c,1}^2 \approx -2.37737 - 2.59245\,\lambda_{GB} + \mathcal{O}(\lambda_{GB}^2)\,, \tag{4.10}$$

$$\mathfrak{w}_{c,1} \approx -1.64659 - 3.44719\,\lambda_{GB} + \mathcal{O}(\lambda_{GB}^2)\,, \tag{4.11}$$

for the first point,

$$\mathfrak{q}_{c,2}^2 \approx -3.11051 \mp 0.81050i + \lambda_{GB}(-1.70074 \pm 5.98722i) + \mathcal{O}(\lambda_{GB}^2)\,, \tag{4.12}$$

$$\mathfrak{w}_{c,2} \approx \pm 1.41043 - 2.87086i + \lambda_{GB}(\mp 0.31643 - 4.26447i) + \mathcal{O}(\lambda_{GB}^2)\,, \tag{4.13}$$

for the second, and

$$\mathfrak{q}_{c,3}^2 \approx 2.90684 \pm 1.66612i + \lambda_{GB}(-2.16892 \pm 68.0434i) + \mathcal{O}(\lambda_{GB}^2)\,, \tag{4.14}$$

$$\mathfrak{w}_{c,3} \approx \pm 2.38819 - 2.13154i + \lambda_{GB}(\mp 3.28617 + 19.0509i) + \mathcal{O}(\lambda_{GB}^2)\,, \tag{4.15}$$

for the third.

4.1.2 Non-perturbative calculation

Eqs. (3.16) and (4.6) can be solved fully non-perturbatively in λ_{GB}. Several examples of the shear channel Gauss-Bonnet quasinormal spectrum are shown in figure 11 and figure 12. Their characteristic feature, explored in refs. [16, 26], is the presence of non-perturbative modes located (for real $\mathfrak{q}^2$) on the imaginary axis in the complex $\mathfrak{w}$-plane. At sufficiently small values of $|\lambda_{GB}|$, these modes lead to new critical points only for large values of $|\mathfrak{q}^2|$: the closest to the origin critical points setting the radius of convergence of the shear mode are not affected by them (see figure 11, where the spectrum is shown for $\lambda_{GB} = -0.01$ and various values of complex $\mathfrak{q}^2$). At larger values of $|\lambda_{GB}|$, however, the situation changes qualitatively. Now the closest to the origin critical point and thus the radius of convergence are set by the non-perturbative mode (this is illustrated by figure 12, where the spectrum is shown for $\lambda_{GB} = -0.03$). The transition between the two regimes occurs at $\lambda_{GB} \approx -0.0198$. The dependence of the radius of convergence of the shear mode's dispersion relation on Gauss-Bonnet coupling is shown in figure 13.

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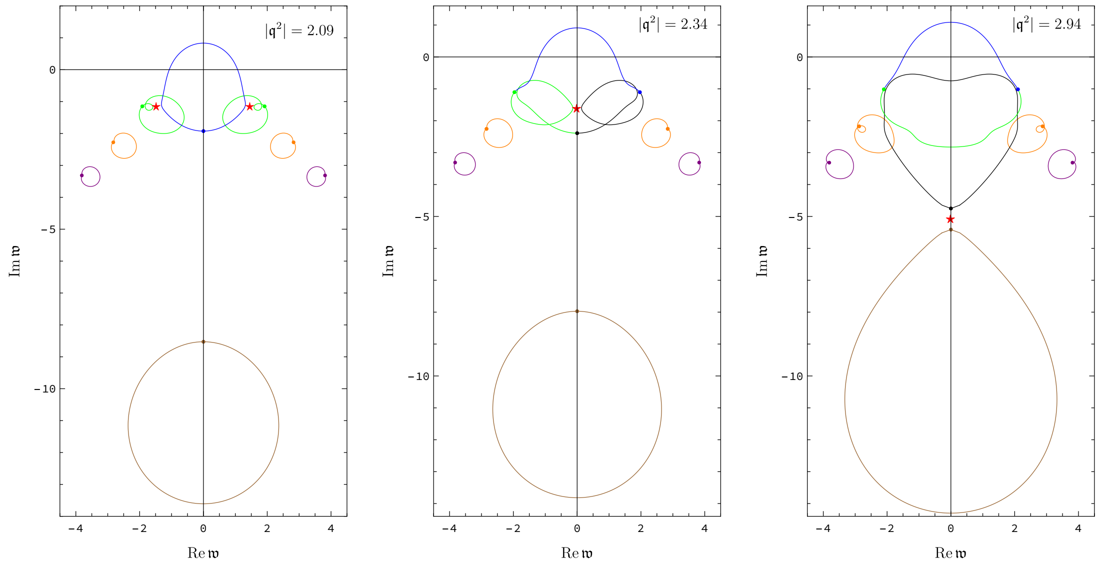

Figure 11. Quasinormal spectrum in the shear channel of Gauss-Bonnet theory, computed non-perturbatively in λ_{GB}, at $\lambda_{GB} = -0.01$. The trajectories are plotted for complex values of the spatial momentum squared, $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\varphi}$, where phase φ is varied from 0 to 2π. The positions of quasinormal modes at $\varphi = 0$ are shown by dots. The positions of the critical points are shown by red stars.

4.2 Sound channel

Finally, we repeat the same analysis for the sound channel of the Einstein-Gauss-Bonnet theory using eqs. (3.16) and the equation of motion (4.6) with $i = 3$.

4.2.1 Perturbative calculation

To linear order in the perturbative expansion in λ_{GB}, we find the closest to the origin pair of critical points at

$$\mathfrak{q}_c^2 \approx \pm 2i + \lambda_{GB}(-10.8809 \pm 10.4314i) + \mathcal{O}(\lambda_{GB}^2)\,, \tag{4.16}$$

$$\mathfrak{w}_c \approx \pm 1 - i + \lambda_{GB}(\mp 2.05394 + 3.49495i) + \mathcal{O}(\lambda_{GB}^2)\,. \tag{4.17}$$

Hence, the radius of convergence in the sound channel is given (perturbatively) by

$$R_{\text{sound}}(\lambda_{GB}) \approx 2 + 15.0735\,\lambda_{GB} + \mathcal{O}(\lambda_{GB}^2)\,. \tag{4.18}$$

For "physical" values of λ_{GB} ($\lambda_{GB} < 0$), this is a decreasing function of $|\lambda_{GB}|$ (which is different from the $\mathcal{N} = 4$ SYM theory).

4.2.2 Non-perturbative calculation

Computing critical points in the sound channel of the Einstein-Gauss-Bonnet theory non-perturbatively in λ_{GB}, we find that for sufficiently small $|\lambda_{GB}|$, the situation remains qualitatively the same as in the $\lambda_{GB} = 0$ case: the radius of convergence is determined by the level-crossing between sound modes and the top gapped modes in the complex $\mathfrak{w}$-plane, as

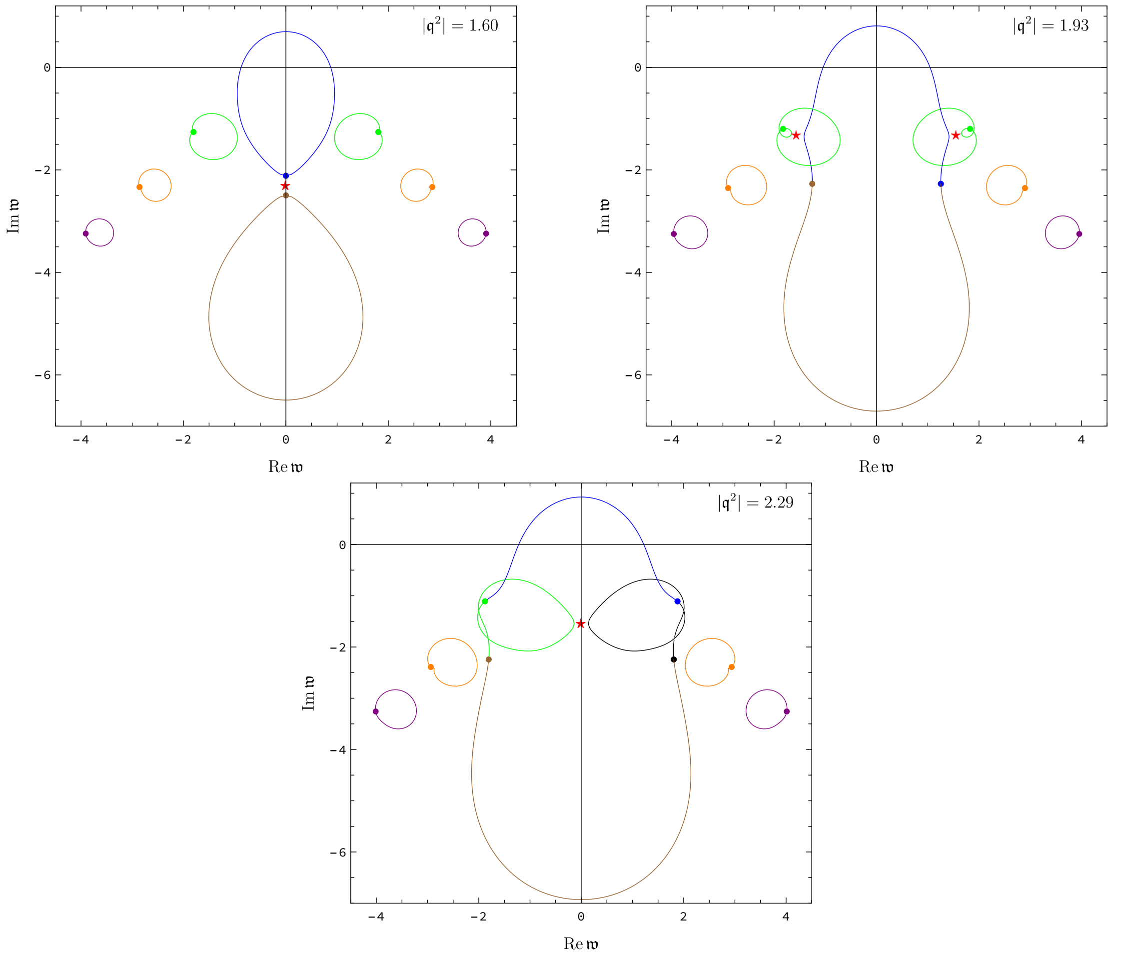

Figure 12. Quasinormal spectrum in the shear channel of Gauss-Bonnet theory, computed non-perturbatively in λ_{GB}, at $\lambda_{GB} = -0.03$.

shown in figure 14 (left panel). The non-perturbative mode also leads to critical points, but this happens at larger value of $|\mathfrak{q}^2|$ (see the right panel of figure 14).

At larger value of the coupling $|\lambda_{GB}|$, the picture changes qualitatively and the radius of convergence of the sound dispersion relation is now determined by the level-crossing with the non-perturbative quasinormal mode, as shown in the left panel of figure 15.

The transition between the two regimes happens at $\lambda_{GB} \approx -0.0338$. The dependence of R_{sound} (including both perturbative and non-perturbative results) is shown in figure 16.

5 Non-perturbative quasinormal modes and singular perturbation theory

In our analysis of the radii of convergence in the $\mathcal{N} = 4$ SYM theory, as well as in previous works on higher-derivative holography [23], [16, 33, 35, 36], the non-perturbative "resummation" and the appearance of the non-perturbative quasinormal modes played a rather prominent role. In particular, the existence of these non-perturbative features seems to be fully consistent with the requirement of a physically reasonable interpolation

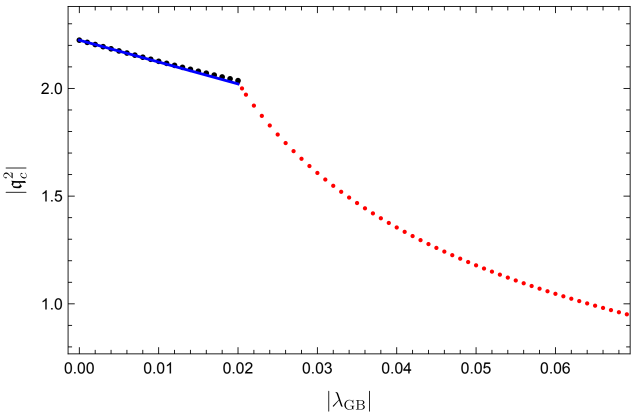

Figure 13. Radius of convergence of the hydrodynamic shear mode in the Einstein-Gauss-Bonnet theory as a function of the higher-derivative coupling λ_{GB} ($\lambda_{GB} < 0$). The blue line is the perturbative result (4.9). Black dots on top of the blue line denote the radius of convergence computed non-perturbatively in λ_{GB} (see figure 11, top left panel). The red dots correspond to the critical point involving the shear mode and the non-perturbative mode (see figure 12, top left panel). The transition between the two regimes occurs at $\lambda_{GB} \approx -0.0198$.

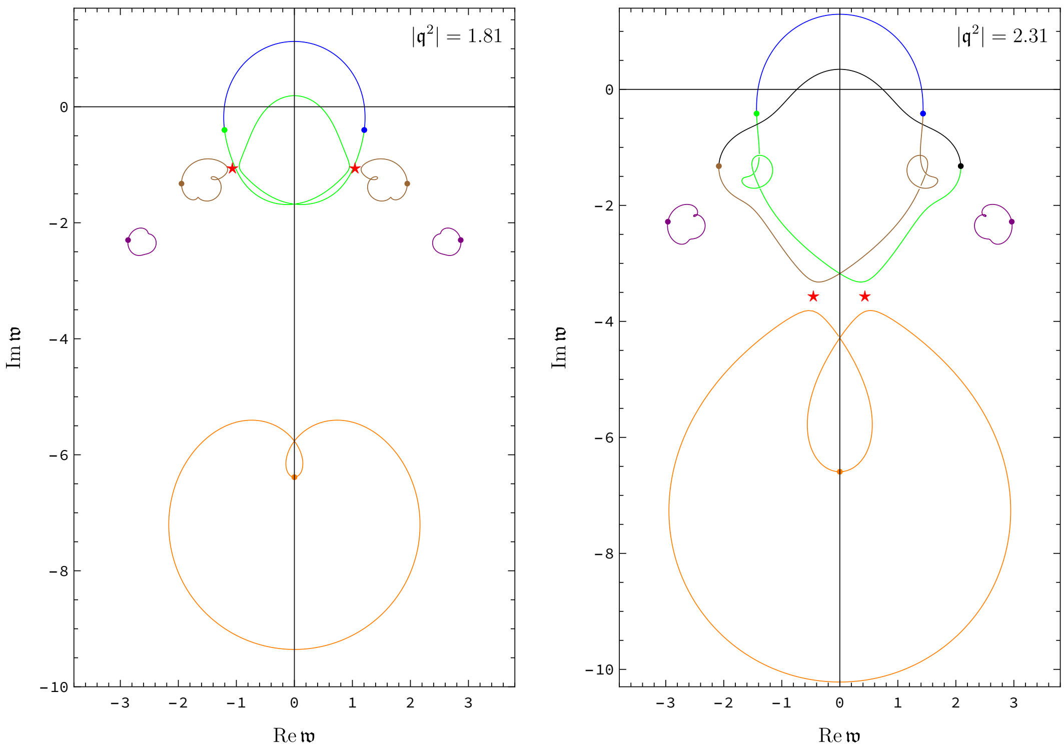

Figure 14. Quasinormal spectrum in the sound channel of Gauss-Bonnet theory, computed non-perturbatively in λ_{GB}, at $\lambda_{GB} = -0.02$. The trajectories are plotted for complex values of the spatial momentum squared, $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\varphi}$, where phase φ is varied from 0 to 2π. The positions of quasinormal modes at $\varphi = 0$ are shown by dots. The positions of the critical points are shown by red stars.

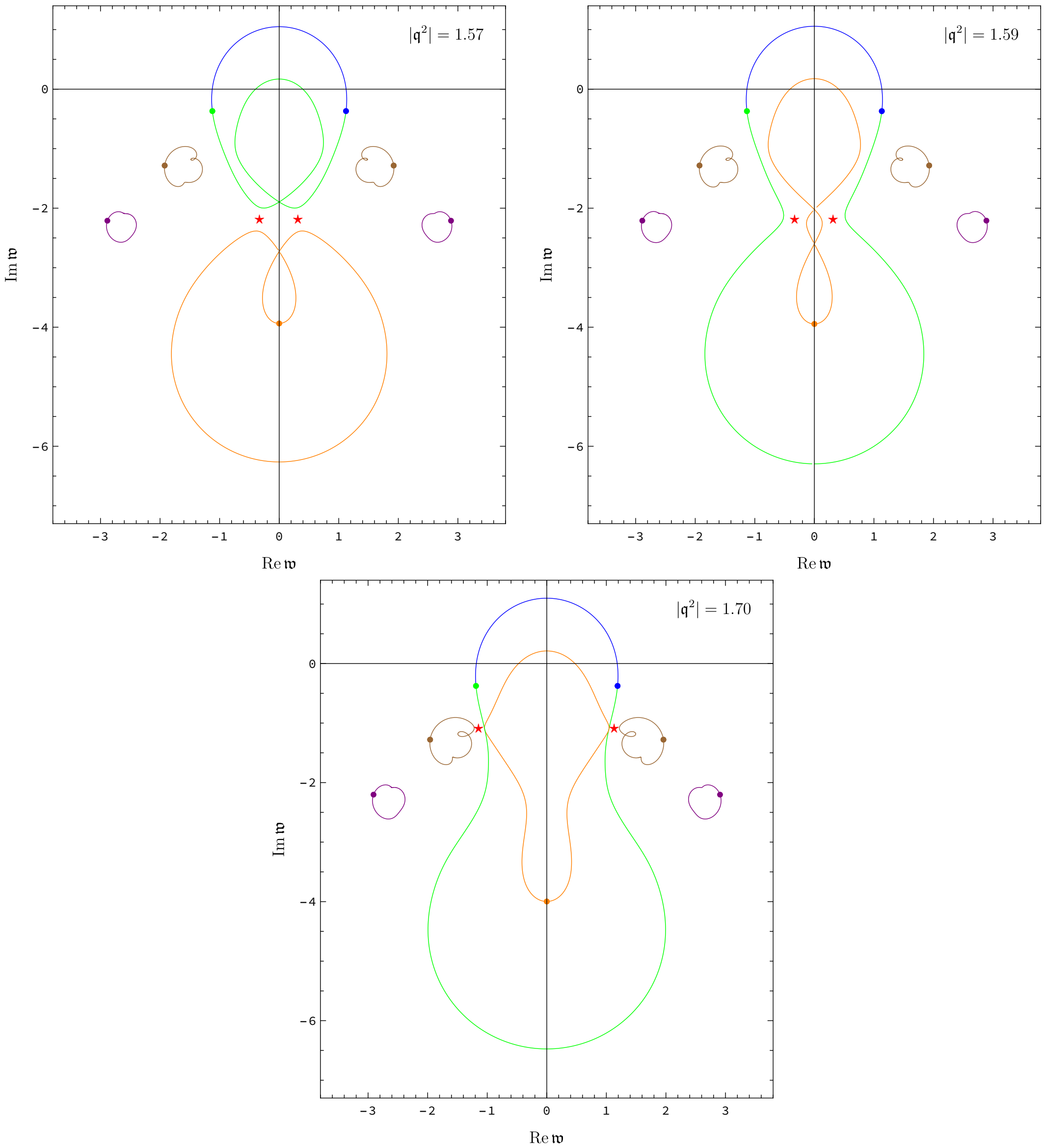

Figure 15. Quasinormal spectrum in the sound channel of Gauss-Bonnet theory, computed non-perturbatively in λ_{GB}, at $\lambda_{GB} = -0.04$. The trajectories are plotted for complex values of the spatial momentum squared, $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\varphi}$, where phase φ is varied from 0 to 2π. The positions of the quasinormal modes at $\varphi = 0$ are shown by dots. The positions of the critical points are shown by red stars.

between strongly coupled (holographic) regime and weakly coupled (e.g. kinetic) regime in the same theory. This interpolation, even for simplest theories such as CFTs considered at finite temperature, is not fully understood [16, 33, 36, 48–52]. Admittedly, relying — even only qualitatively — on the non-perturbative treatment in models arising as truncations of a perturbative expansion may seem to be unwarranted [34]. However, before dismissing such an approach as ineffable nonsense, one may wish to consider examples where it is known to be successful.

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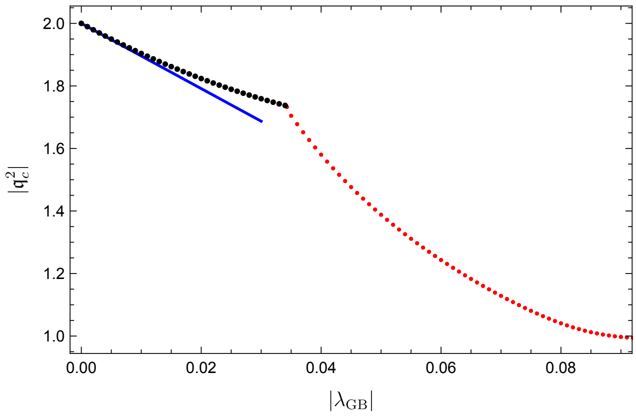

Figure 16. Radius of convergence of the hydrodynamic sound mode in the Einstein-Gauss-Bonnet theory as a function of the magnitude of the higher-derivative coupling λ_{GB} ($\lambda_{GB} < 0$). The blue line is the linear approximation (4.18). Black dots are the non-perturbative result for the radius of convergence arising as shown in figure 14, left panel. Red dots are the radius of convergence arising as a result of the level-crossing between the sound mode and the non-perturbative mode (see figure 15, top left panel). The transition between the two regimes occurs at $\lambda_{GB} \approx -0.0338$.

In this section, we first outline the problem as we see it, and then discuss in detail a simple example of an algebraic equation containing a small parameter, where similar issues arise.

Consider the full set of solutions to an equation (algebraic or differential), which we schematically write as

$$\mathcal{L}[x,\epsilon] = 0\,. \tag{5.1}$$

We denote this (possibly infinite) set of solutions by $X = \{x_1(\epsilon), x_2(\epsilon), \ldots\}$. Here ϵ is a small parameter such that $\mathcal{L}[x,\epsilon]$ can be formally expanded in a series,

$$\mathcal{L}[x,\epsilon] = \mathcal{L}^{(0)}[x] + \epsilon\mathcal{L}^{(1)}[x] + \epsilon^2\mathcal{L}^{(2)}[x] + \ldots = 0\,. \tag{5.2}$$

Truncating the series (5.1) at order ϵ^0, ϵ^1, ϵ^2, ..., and solving the corresponding equations, we obtain sets of solutions $X^{(0)} = \{x_1^{(0)}, x_2^{(0)}, \ldots\}$, $X^{(1)} = \{x_1^{(1)}(\epsilon), x_2^{(1)}(\epsilon), \ldots\}$, $X^{(2)} = \{x_1^{(2)}(\epsilon), x_2^{(2)}(\epsilon), \ldots\}$ and so on. A natural question to ask is in what sense the solutions $X^{(0)}$, $X^{(1)}$, $X^{(2)}, \ldots$ approximate the true solution X. Note that even the number of roots in each "truncated" set $X^{(n)}$ depends on n. For example, the equation

$$\mathcal{L}[x] = \mathcal{L}^{(0)}[x] = 1 - x = 0 \tag{5.3}$$

has a single solution $X^{(0)} = \{x_1^{(0)} = 1\}$, whereas extending $\mathcal{L}[x]$ by adding a term ϵx^2,

$$\mathcal{L}[x,\epsilon] = \mathcal{L}^{(0)}[x] + \epsilon\mathcal{L}^{(1)}[x] = 1 - x + \epsilon x^2 = 0\,, \tag{5.4}$$

leads to two solutions, $X^{(1)} = \{x_1^{(1)}(\epsilon), x_2^{(1)}(\epsilon)\}$, one of which is perturbative and another one is non-perturbative in ϵ:

$$x_1^{(1)}(\epsilon) = \frac{1}{2\epsilon}\left(1 - \sqrt{1-4\epsilon}\right) = 1 + \epsilon + O(\epsilon^2)\,, \tag{5.5}$$

$$x_2^{(1)}(\epsilon) = \frac{1}{2\epsilon}\left(1 + \sqrt{1-4\epsilon}\right) = \frac{1}{\epsilon} - 1 - \epsilon + O(\epsilon^2)\,. \tag{5.6}$$

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The solution $x_1^{(1)}(\epsilon)$ can be constructed order by order in ϵ via standard perturbation theory (i.e., assuming a *perturbative ansatz* for the solution), whereas the $x_2^{(1)}(\epsilon)$ is "invisible" in the standard perturbative approach yet it can be built consistently using singular perturbation theory [32]. Now imagine that eq. (5.4) is a truncation of the equation

$$\mathcal{L}^{(N)}[x,\epsilon] = x - 1 + \sum_{n=1}^{N} \epsilon^n x^{n+1} = 1 - x + \epsilon x^2 + O(\epsilon^2) = 0\,. \tag{5.7}$$

At any finite N, among the $N+1$ roots of the equation (5.7), one is perturbative in ϵ (it is a regular perturbation of the solution $x = 1$ of the equation (5.3)) and the remaining N roots are non-perturbative: they disappear to infinity in the limit $\epsilon \to 0$. These extra N roots are located (approximately) along the circle $|x| = 1/\epsilon$ in the complex x-plane. Finally, we note that eq. (5.7) can be regarded as a truncation at order ϵ^N of an exact function

$$\mathcal{L}^{(\infty)}[x,\epsilon] = x - 1 + \frac{\epsilon x^2}{1-\epsilon x} = 0\,. \tag{5.8}$$

Note that eq. (5.8) has only *one* solution,

$$x = \frac{1}{1+\epsilon}\,, \tag{5.9}$$

whose small ϵ expansion coincides (for $|\epsilon| < 1$) with the perturbative solution of the equation (5.7) at the appropriate order. Thus, in this example, truncating the small ϵ expansion of eq. (5.8) at order N produces one correct and N spurious roots located approximately at the boundary of analyticity of the function $\mathcal{L}^{(\infty)}[x,\epsilon]$, i.e. at $|x| = 1/\epsilon$.

This simple example seems to reinforce the "conservative" approach to the quasinormal spectrum in higher-derivative gravity suggesting that only perturbative solutions can be trusted. Non-perturbative solutions exist (and can be constructed via singular perturbation theory) at each given order of the expansion in a small parameter, but these solutions appear to be artefacts of the expansion and disappear when the full function is considered. However, such a verdict might be too quick, as the example in the next subsection shows.

5.1 An algebraic equation example

Consider the algebraic equation

$$\mathcal{L}[x,\epsilon] = ix - 1 - x\sinh\epsilon x = 0\,, \tag{5.10}$$

where ϵ is a parameter. At $\epsilon = 0$, eq. (5.10) has a single root, $x = x_0 = -i$. For $\epsilon > 0$, however, there are infinitely many solutions, parametrised by ϵ (see figure 17, left panel, where the roots of eq. (5.10) closest to the origin in the complex x-plane are shown for $\epsilon = 0.5$). Note that all these solutions but one are non-perturbative in ϵ, since they must disappear from the set of solutions in the limit $\epsilon \to 0$ leaving the single root $x = x_0$ at $\epsilon = 0$.

The solutions to eq. (5.10) can be constructed as series in $\epsilon \ll 1$. One such solution is the finite ϵ correction to the solution $x_0 = -i$ of the equation at $\epsilon = 0$. Using the standard perturbation theory, we find

$$x_0(\epsilon) = -i\left[1 - \epsilon + 2\epsilon^2 - \frac{29}{6}\epsilon^3 + 13\epsilon^4 - \frac{4481}{120}\epsilon^5 + \frac{5048}{45}\epsilon^6 + O(\epsilon^7)\right]. \tag{5.11}$$

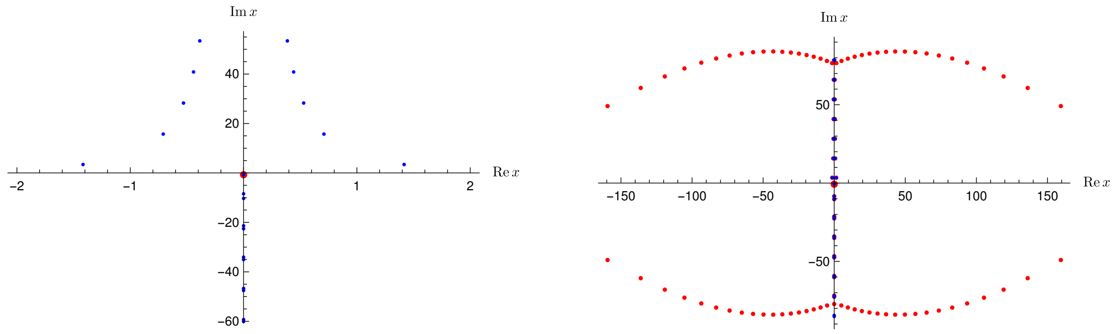

Figure 17. The roots of eq. (5.10) (at $\epsilon = 0.5$) closest to the origin in the complex x-plane (left panel). The large red dot corresponds to the perturbative root at $x \approx -0.735544i$. (Note that the perturbative series (5.11) fails to converge for this value of ϵ.) The other roots on the imaginary axis are at $x \approx -8.4414i$, $x \approx -10.3128i$, $x \approx -21.3770i$, $x \approx -22.5885i$, $x \approx -34.0718i$, $x \approx -35.0365i$, $x \approx -46.7093i$, $x \approx -47.5349i$. In the right panel, the same set of roots as in the left panel is approximated by the first 100 roots of the polynomial (5.14). The blue dots accurately reproduce the actual values from the left panel. The purpose of the zoomed-out plot is to show the spurious, unphysical red dots which move to infinity as the order of approximation is increased.

The non-perturbative roots can be found analytically by using the methods of singular perturbation theory [32]. Introducing a new variable $x = \bar{x}/\epsilon$ and taking the limit of $\epsilon \to 0$ in eq. (5.10) while keeping $\bar{x}$ fixed, we find the equation

$$i - \sinh \bar{x} = 0\,. \tag{5.12}$$

The infinite set of solutions to this equation, $\bar{x}_n = i\pi(1+4n)/2$, where $n \in \mathbb{Z}$, then gives all the non-perturbative roots of the original eq. (5.10) as

$$x_n^{\pm}(\epsilon) = \frac{1}{\epsilon}\left[\frac{i\pi\,(1+4n)}{2} \pm \frac{2}{\sqrt{\pi\,(1+4n)}}\epsilon^{1/2} + \frac{4i}{[\pi\,(1+4n)]^2}\epsilon \pm \mathcal{O}(\epsilon^{3/2})\right]. \tag{5.13}$$

The series in eqs. (5.11) and (5.13) converge[5] for $\epsilon < |\epsilon_c| \approx 0.2625$.

We now replace $\mathcal{L}[x,\epsilon]$ by a finite polynomial (a truncated Taylor expansion of eq. (5.10)),

$$\mathcal{L}^{(2N+2)}[x,\epsilon] = ix - 1 - \epsilon x^2 \sum_{n=0}^{N} \frac{\epsilon^{2n} x^{2n}}{(2n+1)!} = 0\,, \tag{5.14}$$

and ask whether the $2N+2$ roots of eq. (5.14) approximate the exact solutions of eq. (5.10) as we increase N. Our previous discussion implies that at least one such approximation, a regular perturbative correction to the zeroth-order root $x_0 = -i$, should exist. Indeed, solving the corresponding equations $\mathcal{L}^{(2)}[x,\epsilon] = 0$, $\mathcal{L}^{(4)}[x,\epsilon] = 0$, etc., perturbatively in ϵ, we reproduce the series (5.11) term by term. All others roots are non-perturbative in ϵ, disappearing to complex infinity in the limit $\epsilon \to 0$.

To mimic our approach to the quasinormal spectra in higher-derivative gravity, we now solve eq. (5.14) numerically at each order of N, without assuming ϵ to be small. A

[5]The radius of convergence is determined by the closest to the origin critical point of the function (5.10).

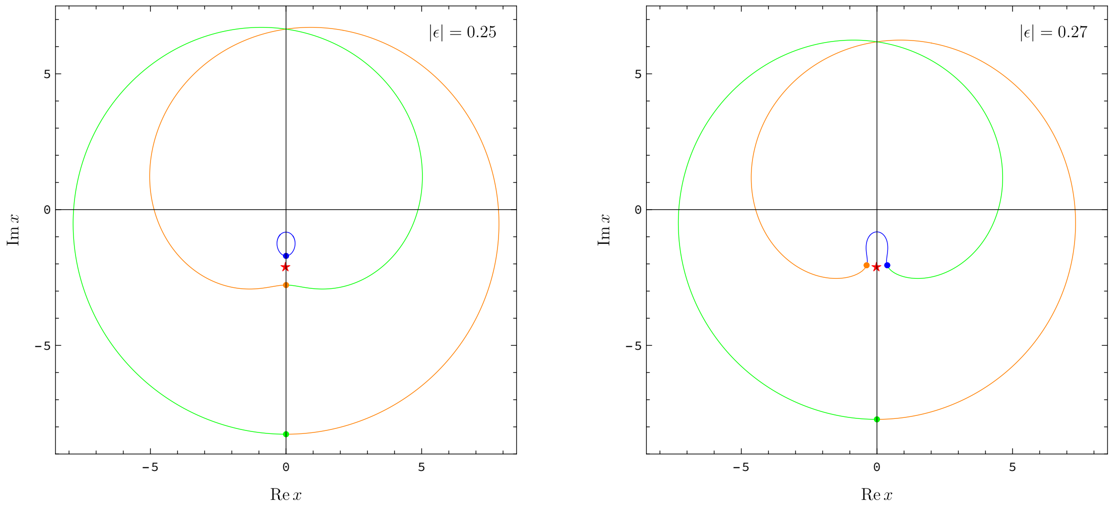

Figure 18. The two closest to the origin solutions x_0 and x_1 to eq. (5.10) in the complex x-plane as functions of the complexified parameter $\epsilon = -|\epsilon|e^{i\varphi}$, where $\varphi \in [0, 2\pi]$ for $|\epsilon| = 0.25$ (left panel) and $|\epsilon| = 0.27$ (right panel). Coloured dots correspond to $\varphi = 0$. The critical point setting the radius of convergence of the series (5.11) is located at $x \approx -2.1176i$ (it is shown by the red star).

generic result is illustrated in the right panel of figure 17, where $\epsilon = 0.5$, and both the exact solutions to eq. (5.10) and the roots of the polynomial (5.14) with $N = 50$ are shown. It is clear that in a (large) region of the complex plane containing the origin, the solutions of eq. (5.10) (both perturbative and non-perturbative) are well approximated by some of the roots of the polynomial (5.14). There are also "spurious roots", forming the top and the bottom red "arcs" in figure 17, right panel: they do not approximate any solution. The exact solutions of eq. (5.10) located outside of the domain bounded by the red arcs are not approximated by any of the roots of the polynomial (5.14) and remain "invisible". The domain bounded by the red arcs of spurious roots increases with N increasing. Our main conclusion is that it is possible in principle to approximate at least some of the exact non-perturbative solutions by the non-perturbative roots of the perturbative truncation of the exact equation. Of course, this example is not a justification of our "resummation" of quasinormal spectra but we hope it shows that such an approach has the right to exist.

We now ask a different question. As noted in footnote 5, the radius of convergence of the series (5.11) representing the perturbative solution $x_0 = x_0(\epsilon)$ of eq. (5.10) is set by the closest to the origin (in the complex ϵ-plane) critical point of the complex curve (5.10) determined by the conditions

$$\mathcal{L}[x, \epsilon] = 0\,, \qquad \partial_x \mathcal{L}[x, \epsilon] = 0\,. \tag{5.15}$$

The solution of eqs. (5.15) with the smallest $|\epsilon|$ is $x \approx -2.1176i$, $\epsilon \approx -0.2625$. In the complex x-plane, the critical point corresponds to the level-crossing between the perturbative mode x_0 and the nearest *non-perturbative* solution of eq. (5.10), as shown in figure 18. Now suppose that we only have access to the successive polynomial approximations (5.14) to the full equation (5.10). Can we detect the existence of the correct

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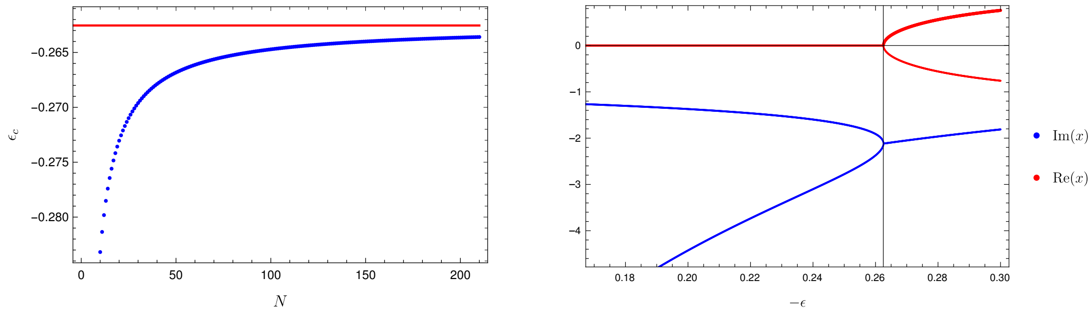

Figure 19. The critical points $\epsilon_c(N)$ computed perturbatively using the polynomial approximations (5.14) of order $2N+2$ (blue line) converges to the "exact" critical point $\epsilon_c \approx -0.2625$ (flat red line) of the full equation (5.10) (left panel). The dependence of real and imaginary parts of the root x on ϵ (right panel). The level-crossing occurs at $\epsilon_c \approx -0.2625$. The same transition in the complex x-plane is shown in figure 18.

critical point and hence the non-perturbative solution by using only these polynomial approximations and the regular perturbation theory? The answer is affirmative: solving perturbatively eqs. (5.15) with $\mathcal{L}[x,\epsilon]$ replaced by the polynomial approximation (5.14) of order $2N+2$, we find that the (smallest in magnitude) critical value $\epsilon_c(N)$ converges to the "exact" result $\epsilon_c \approx -0.2625$ with N increasing (see figure 19, left panel). Since the critical point corresponds to the level-crossing (see figure 19, right panel), i.e., a collision between perturbative and non-perturbative roots in the complex x-plane (see figure 18), the above observation suggests that the perturbative calculation "knows" about this collision, even though the non-perturbative root responsible for it is inaccessible in perturbation theory and remains "invisible". This raises a question of whether the non-perturbative quasinormal modes arising in higher-derivative gravity can be indirectly detected by a perturbative analysis of the relevant critical points. We investigate this question in the next section.

5.2 A signature of non-perturbative modes from the perturbative analysis of the Einstein-Gauss-Bonnet critical points

The quasinormal spectrum in the shear channel of Einstein-Gauss-Bonnet gravity contains modes located on the imaginary axis which are non-perturbative in the Gauss-Bonnet coupling λ_{GB} (see figure 11 and refs. [16, 26]). As discussed in section 4, for some range of parameters, the top (closest to the origin) non-perturbative mode collides with the hydrodynamic shear mode at real $\mathfrak{q}_c^2$ and purely imaginary $\mathfrak{w}_c$, as illustrated in figure 12. This collision is followed by another collision between the emergent pair of propagating modes with the pair of gapped modes from the standard "Christmas tree" sequence. For a fixed value of λ_{GB}, the second collision again occurs at real $\mathfrak{q}_c^2$. We illustrate this in figure 20, where the shear channel Einstein-Gauss-Bonnet spectrum is shown at $\lambda_{GB} \approx -0.04956$ for several values of real $\mathfrak{q}^2$: as $\mathfrak{q}^2$ is increased from $\mathfrak{q}^2 = 0.81$ to $\mathfrak{q}^2 = 1.17$, the hydrodynamic mode and the non-perturbative mode on the imaginary axis approach each other, colliding at $\mathfrak{q}_c^2 \approx 1.18544$ and forming a pair of propagating modes, which move off the axis and at $\mathfrak{q}_c^2 \approx 1.87785$ collide again, now with the pair of the "Christmas tree" gapped modes. The two (sets of) critical points resulting from these collisions are given by

$$\lambda_{GB} \approx -0.04956 \ : \tag{5.16}$$

$$\mathfrak{q}_c^2 \approx 1.18544\,, \qquad \mathfrak{w}_c \approx -1.63793i\,, \tag{5.17}$$

$$\mathfrak{q}_c^2 \approx 1.87785\,, \qquad \mathfrak{w}_c \approx \pm 1.67253 - 1.41811i\,. \tag{5.18}$$

We emphasise that the existence of the critical points (5.17) and (5.18) is the direct consequence of the existence of the non-perturbative mode on the imaginary axis, absent at $\lambda_{GB} = 0$. Indeed, for $\lambda_{GB} = 0$ (which corresponds to the $\mathcal{N} = 4$ SYM theory at infinite 't Hooft coupling), the hydrodynamic shear mode travels unobstructedly along the imaginary $\mathfrak{w}$ axis towards negative infinity as the real $\mathfrak{q}^2$ is increased: no critical point exists in the theory for purely imaginary $\mathfrak{w}_c$ at purely real and positive $\mathfrak{q}_c^2$. In contrast, at finite λ_{GB}, the shear mode collides with the non-perturbative mode on the imaginary axis at real and positive $\mathfrak{q}_c^2$ given by eq. (5.17), and then the resulting two propagating modes give rise to the *pair of critical points* at another real value of $\mathfrak{q}_c^2$ (eq. (5.18)). The pair of the propagating modes leading to the pair of critical points in eq. (5.18) does not exist in the perturbative spectrum: it is created by the collision of the shear mode and the non-perturbative mode on the imaginary axis.

Can the existence of non-perturbative critical points such as the ones in eq. (5.18) be inferred from the perturbative data (i.e., from eqs. (2.3), solved perturbatively in λ_{GB}), in analogy with what has been observed in section 5.1? The goal is to find a perturbative approximation to the pair of critical points (5.18) with real $\mathfrak{q}^2$.

Perturbatively, to leading order in λ_{GB}, the closest to the origin critical point is given by eq. (4.7), reproduced here for convenience:

$$\mathfrak{q}_c^2 \approx 1.89065 \pm 1.17115i + \lambda_{GB}(-2.01742 \pm 22.5317i) + \mathcal{O}(\lambda_{GB}^2)\,, \tag{5.19}$$

$$\mathfrak{w}_c \approx \pm 1.44364 - 1.06923i + \lambda_{GB}(\mp 1.69340 + 8.39996i) + \mathcal{O}(\lambda_{GB}^2)\,. \tag{5.20}$$

Eq. (5.19) implies that the critical value of $\mathfrak{q}^2$ is purely real for $\lambda_{GB} \approx -0.0519780$, and is given by

$$\lambda_{GB} \approx -0.0519780 \ : \tag{5.21}$$

$$\mathfrak{q}_c^2 \approx 1.9955 + \mathcal{O}(\lambda_{GB}^2)\,, \qquad \mathfrak{w}_c \approx \pm 1.53166 - 1.50584i + \mathcal{O}(\lambda_{GB}^2)\,. \tag{5.22}$$

We conjecture that the critical point (5.22) at $\lambda_{GB} \approx -0.0519780$ is the perturbative approximation to the "exact" critical point (5.18) at $\lambda_{GB} \approx -0.04956$. To see that this is indeed the case, we extend the perturbative expansion in eqs. (5.19), (5.20) to order $\mathcal{O}(\lambda_{GB}^N)$. At each N, we then numerically compute the value of λ_{GB} (denoted by $\lambda_{GB}(N)$) from the algebraic condition that sets $\operatorname{Im} \mathfrak{q}_c^2(\lambda_{GB}) = 0$ (among multiple roots $\lambda_{GB}(N)$, we choose the solution which is real and closest to the non-perturbative value $\lambda_{GB} \approx -0.04956$). Then, we use thus obtained $\lambda_{GB}(N)$ to evaluate $\mathfrak{w}_c(N)$ and $\mathfrak{q}_c^2(N)$ from the perturbative series in λ_{GB}. The numerical sequences $\mathfrak{w}_c(N)$, $\mathfrak{q}_c^2(N)$ do not converge and have to be Padé-resummed. The Padé-resummed values then converge to the corresponding values for the non-perturbative critical point (5.18) with N increasing (see figure 21).

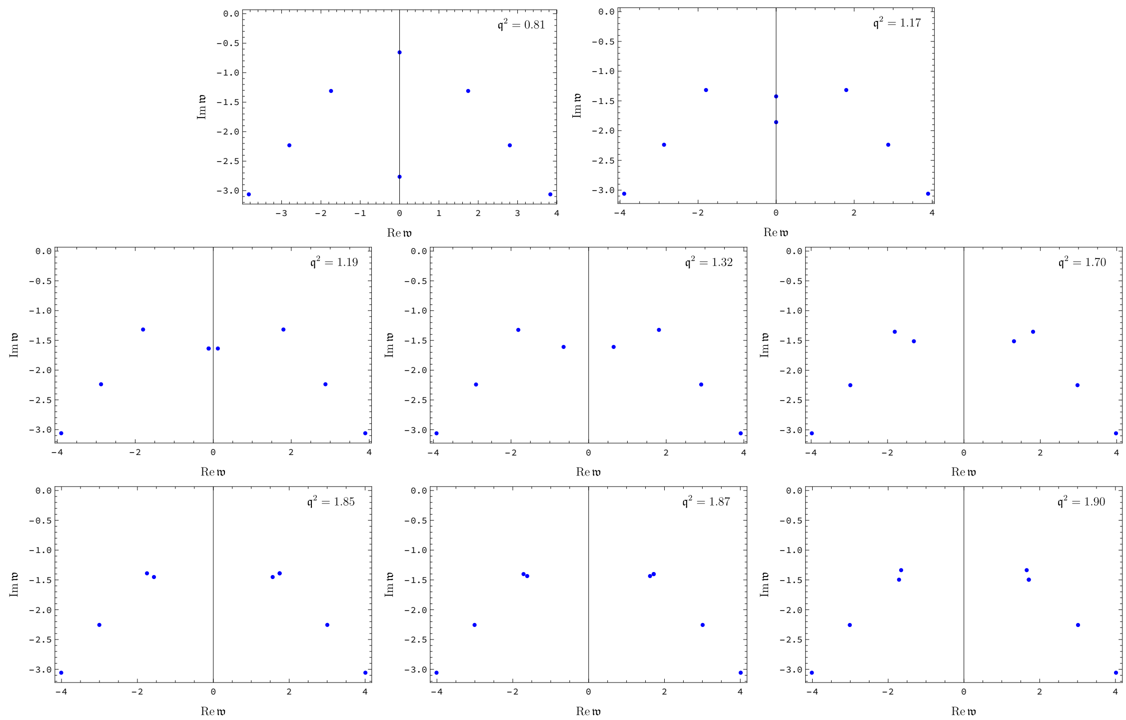

Figure 20. The non-perturbative quasinormal spectrum in the shear channel of the Einstein-Gauss-Bonnet theory at $\lambda_{GB} \approx -0.04956$, plotted for several (increasing) values of $\mathfrak{q}^2 \in \mathbb{R}$. At $\mathfrak{q}^2 = 0.81$, the shear mode and the non-perturbative mode are present on the imaginary axis; with $\mathfrak{q}^2$ increasing, the two modes approach each other on the imaginary axis, collide at the critical point (5.17), move off the axis as the pair of propagating modes (shown in plots with $\mathfrak{q}^2 = 1.19; 1.32; 1.70; 1.85; 1.87$) and then collide with the pair of the "Christmas tree" gapped modes at the second critical point (5.18).

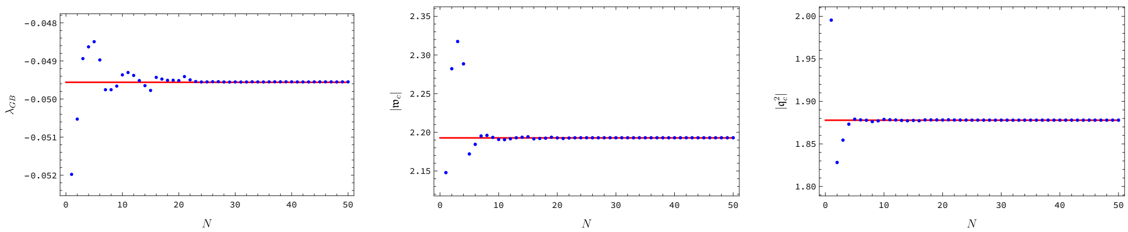

Figure 21. Parameters of the critical point $\lambda_{GB}(N)$, $|\mathfrak{w}_c(N)|$, $\mathfrak{q}_c^2(N)$ obtained from the perturbative analysis of eqs. (2.3) supplemented by a Padé resummation (blue dots). The orders of the Padé approximant $[a/b]$ were chosen as follows: $[N/N]$ for $1 \leq N < 5$; $[N-3/2]$ for $5 \leq N < 15$; $[N-11/10]$ for $15 \leq N \leq 50$. The red lines are the non-perturbative results given by eqs. (5.16), (5.17) and (5.18).

Hence, a perturbative analysis of eqs. (2.3) (admittedly, aided by a Padé resummation) seems to be capable of reproducing at least one of the non-perturbative critical points of the full quasinormal spectrum. Since such a point can only occur due to the presence of a non-perturbative mode in the spectrum, we conclude that the calculation indirectly confirms the existence of the non-perturbative mode itself.

Another example of an inherently non-perturbative critical point that can be reproduced this way is the critical point located at

$$\begin{aligned} &\lambda_{GB} \approx -0.02327\ : \\ &\quad \mathfrak{q}_c^2 \approx 2.8462\,, \quad \mathfrak{w}_c \approx \pm 2.6329 - 2.4987i\,. \end{aligned} \tag{5.23}$$

The appropriate perturbative critical point is given by eq. (4.14). Requiring $\operatorname{Im}\mathfrak{q}_c^2(\lambda_{GB}) = 0$, to first order in λ_{GB} we have

$$\begin{aligned} &\lambda_{GB} \approx -0.024486\ : \\ &\quad \mathfrak{q}_c^2 = 2.95995 + \mathcal{O}(\lambda_{GB}^2)\,, \quad \mathfrak{w}_c = \pm 2.46865 - 2.59802i + \mathcal{O}(\lambda_{GB}^2)\,. \end{aligned} \tag{5.24}$$

Continuing the expansion (5.24) to higher orders of λ_{GB} as described above, we find that the sequence of order-N critical points $\lambda_{GB}(N)$, $\mathfrak{w}_c(N)$, $\mathfrak{q}_c^2(N)$ converges to the non-perturbative result (5.23) with N increasing.

5.3 A signature of non-perturbative modes from the perturbative analysis of critical points in the $\mathcal{N} = 4$ SYM theory

We now turn to the perturbative analysis of critical points in the $\mathcal{N} = 4$ SYM theory. Can we detect the existence of non-perturbative modes in the quasinormal spectrum by using the approach of the previous section? The recipe is to look for a pair of critical points with real value of $\mathfrak{q}^2$. Considering the lowest perturbative critical point (3.17) and finding γ from the condition $\operatorname{Im}\mathfrak{q}_c^2(\gamma) = 0$, we find to leading order in γ:

$$\begin{aligned} &\gamma \approx -0.0006290\ : \\ &\quad \mathfrak{q}_c^2 \approx -0.67674\,, \quad \mathfrak{w}_c \approx \pm 0.86542 - 1.92493i\,. \end{aligned} \tag{5.25}$$

Such a pair of critical points at a single real $\mathfrak{q}_c^2$ does not exist in an infinitely strongly coupled theory (at $\gamma = 0$). At finite γ, it can arise as a result of the collision on the imaginary axis between the shear mode and the non-perturbative mode, followed by the second collision between the pair of the resulting propagating modes and the two gapped "Christmas tree" modes, similarly to what happens in the Einstein-Gauss-Bonnet case as described in section 5.2. We conjecture that eqs. (5.25) are approximations to the non-perturbative critical point appearing in the right panel of figure 8. Of course, unlike in the Einstein-Gauss-Bonnet theory, here we have no access to γ^2 and higher terms in the action and thus cannot verify this conjecture. Nevertheless, we can interpret the existence of the point (5.25) as an indirect evidence for the existence of the non-perturbative modes in the full quasinormal spectrum of the gravitational background dual to the $\mathcal{N} = 4$ SYM theory.

6 Discussion

In this paper, we have determined the dependence on the coupling of the radii of convergence of the shear and sound hydrodynamic dispersion relations in the strongly coupled $\mathcal{N} = 4$ SYM theory. This dependence is shown in figure 1. Limiting ourselves to the results

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obtained using the standard perturbation theory only, the coupling constant dependence of the radii in the shear and sound channels is given by eqs. (1.7), (1.8) (the blue lines in figure 1), respectively. These perturbative results suggest that the radii of convergence increase with the 't Hooft coupling decreasing from its infinite value.

However, we argued that the presence of non-perturbative modes in the quasinormal spectrum of the dual black brane background at large but finite coupling will modify the perturbative result by making the radii's dependence on the coupling piecewise continuous, as shown in figure 1. This type of dependence is familiar from the finite density examples of holographic theories at infinite coupling [7, 24] and from the Sachdev-Ye-Kitaev (SYK) chain at finite coupling [9]. The origin of this similarity is not entirely clear to us.[6] In the SYK case, it is the coupling dependence that gives rise to the piecewise behaviour, in parallel to what we observe in this paper. The strong-weak dependence in that case is, however, reversed. In the charged case, the chemical potential normalised by temperature plays the role of the coupling in the present paper being responsible both for the extra modes on the imaginary axis and the piecewise continuous dependence of the radius of convergence. One plausible guess is the emergence of approximate symmetries in all these cases in line with what was discussed in ref. [35].

Curiously, in the regime where the non-perturbative mode becomes relevant, the radius of convergence of the shear mode dispersion relation coincides with the endpoint $\mathfrak{q}_c^2(\lambda)$ of the hydrodynamic regime introduced earlier in ref. [16]. We then repeated our analysis for the case of the Einstein-Gauss-Bonnet theory using it as a theoretical laboratory to test our methods. Again, due to the presence of non-perturbative quasinormal modes in the spectrum, we found the piecewise dependence of the radii of convergence on the coupling (see figures 13 and figure 16 for the shear and sound channel results, respectively). The role of the non-perturbative modes in a quasinormal spectrum thus appears to be significant. Their properties are similar to the properties of non-perturbative roots of algebraic equations with a small parameter, where singular perturbation theory is often useful. We have discussed in detail a simple example of an algebraic equation whose perturbative and non-perturbative roots can be consistently found using singular perturbation theory. Although at present we cannot offer a generalisation of such a discussion to differential operators, we believe a qualitatively similar picture should hold there, too, and therefore the non-perturbative quasinormal modes can represent the qualitatively correct feature of the full theory. Finally, we show that the presence of the non-perturbative modes in the spectrum can be indirectly inferred by the analysis of critical points using the perturbative data only.

From the work in refs. [4, 5] it is clear that thermal two-point functions of the energy-momentum tensor in the $\mathcal{N} = 4$ SYM theory at infinite 't Hooft coupling contain multiple branch point singularities in the complex plane of $\mathfrak{q}^2$. At large but finite coupling, the location of these singularities acquires a dependence on the coupling. Our analysis in the present paper suggests that the property of being a singularity closest to the origin (and thus setting the radius of convergence) may change discretely as a function of the coupling, leading to the piecewise nature of the dependence of the radii of convergence on coupling,

[6] We would like to thank the anonymous referee for raising this issue.

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and that such a change is induced by the presence of non-perturbative quasinormal modes in the spectrum of a dual gravitational theory. It would be interesting to investigate whether a similar phenomenon is observed at small but finite coupling. Of special interest are the studies of the hydrodynamic series convergence in strongly coupled theories with non-zero chemical potential [6, 7], [53], where the coupling dependence has not yet been explored.

Acknowledgments

The work of S.G. was supported by the research programme P1-0402 of Slovenian Research Agency (ARRS). The work of P.T. is supported by an Ussher Fellowship from Trinity College Dublin. P.T. would like to thank Robin Karlsson for discussions. We also thank Jorge Noronha for comments on the draft of the paper.

A Critical points and the radius of convergence in holography

Here, we briefly review the main points of the method introduced in refs. [4, 5] to determine the radii of convergence of hydrodynamic dispersion relations in holography. In the limit where the dual gravity description of a QFT is valid (e.g. in the $N_c \to \infty$ limit of the $\mathcal{N} = 4$ SU(N_c) SYM theory), information about the hydrodynamic and other dispersion relations is contained in the quasinormal spectrum of the bulk equations of motion. In terms of a gauge-invariant variable Z, a typical bulk equation of motion is a second-order ODE

$$\partial_u^2 Z + A(u, \mathfrak{w}, \mathfrak{q}^2)\partial_u Z + B(u, \mathfrak{w}, \mathfrak{q}^2) Z = 0\,, \tag{A.1}$$

where u is the bulk radial coordinate with $u = 0$ corresponding to the boundary of the dual gravity background. The dependence of the coefficients of the equation on $\mathfrak{q}^2$ reflects the rotation invariance of the theory, whereas the dependence on $\mathfrak{w}$ is a consequence of the choice of the boundary condition at the horizon. The quasinormal spectrum $\mathfrak{w}_i = \mathfrak{w}_i(\mathfrak{q}^2)$ is determined by the equation

$$P(\mathfrak{q}^2, \mathfrak{w}) \equiv Z(u = 0, \mathfrak{q}^2, \mathfrak{w}) = 0\,, \tag{A.2}$$

which also defines the spectral curve. The critical point condition (2.3) means that $p \geq 2$ branches $\mathfrak{w}_i = \mathfrak{w}_i(\mathfrak{q}^2)$ collide at $(\mathfrak{w}_c, \mathfrak{q}_c^2) \in \mathbb{C}^2$. The branches are locally represented by Puiseux series whose analytic structure is determined by the coefficients of the spectral curve via, e.g., the Newton polygon method. The situations when $\mathfrak{w}_i \sim (\mathfrak{q}^2 - \mathfrak{q}_c^2)^{-\nu}$, where ν is a fractional power, e.g. $\nu = -1/2$, are common. The closest to the origin (in the complex $\mathfrak{q}^2$-plane) critical point with such a branch point singularity limits the convergence of the series $\mathfrak{w}_i = \mathfrak{w}_i(\mathfrak{q}^2)$ centered at $\mathfrak{q}^2 = 0$ and thus sets its radius of convergence, $R = |\mathfrak{q}_c^2|$. This is the phenomenon of quasinormal level-crossing, analogous to the quantum-mechanical level-crossing [4, 5]. When ν is a negative integer or zero, the branches are locally analytic, and we have "level-touching" rather than "level-crossing" [5].

In practice, the bulk ODEs are sufficiently complicated and have to be solved numerically. With such a solution in hand, one first solves eqs. (2.3) to find the critical points

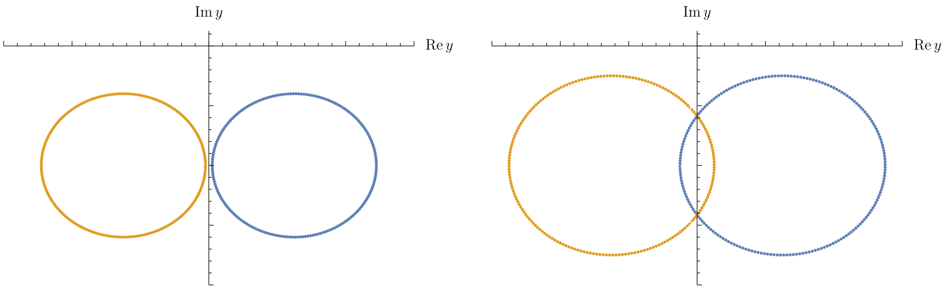

Figure 22. Level-touching: the two branches $y_1^{(1,2)}(x^2)$ of the spectral curve P_1 at complex $x^2 = |x^2|e^{i\varphi}$, with φ varying from 0 to 2π, at fixed $|x_1^2| < |x_c^2|$ (left panel) and $|x_2^2| > |x_c^2|$ (right panel).

in the complex $\mathfrak{q}^2$-plane, and then determines the degree of the singularity at the critical points by considering the quasinormal mode behaviour in the complex $\mathfrak{w}$-plane under the monodromy $\mathfrak{q}^2 = |\mathfrak{q}^2|e^{i\varphi}$, where $\varphi \in [0, 2\pi]$. We illustrate the difference between "level-crossing" and "level-touching" by the following simple example.

Consider the complex curves

$$P_1(x^2, y) = a^2 - b^2 + 2bcx^2 - c^2x^4 - 2ay + y^2 = 0\,, \tag{A.3}$$

$$P_2(x^2, y) = a^2 - b + cx^2 - 2ay + y^2 = 0\,, \tag{A.4}$$

where $x^2 \in \mathbb{C}$, $y \in \mathbb{C}$, and the coefficients a, b, c are some fixed complex numbers. Applying the criterium (2.3) to P_1 and P_2, we find that both curves have the $p = 2$ type crtitical point at $(x_c^2, y_c) = (b/c, a)$. The two branches of the curve P_1 are given by

$$y_1^{(1)}(x^2) = a + b - cx^2 = y_c - c(x^2 - x_c^2)\,, \tag{A.5}$$

$$y_1^{(2)}(x^2) = a - (b - cx^2) = y_c + c(x^2 - x_c^2)\,, \tag{A.6}$$

whereas the two branches of P_2 are

$$y_2^{(1)}(x^2) = a - \sqrt{b - cx^2} = y_c - i\sqrt{c}(x^2 - x_c^2)^{1/2}\,, \tag{A.7}$$

$$y_2^{(2)}(x^2) = a + \sqrt{b - cx^2} = y_c + i\sqrt{c}(x^2 - x_c^2)^{1/2}\,. \tag{A.8}$$

For the curve P_1, the two branches at the critical point are analytic functions of x^2. For the curve P_2, the critical point is a branch point singularity. Numerically, one can distinguish between the regular and the singular behaviour by finding local solutions $y_i = y_i(x^2)$ to eqs. (A.3), (A.4) at complex $x^2 = |x^2|e^{i\varphi}$, with fixed $|x^2|$ and $\varphi \in [0, 2\pi]$. The "trajectories" traced by the branches as the phase φ varies from 0 to 2π are shown in figure 22 for the curve P_1 and in figure 23 for the curve P_2 at fixed $|x_1^2| < |x_c^2| < |x_2^2|$. The regular branches of P_1 approach and touch each other at $x^2 = x_c^2$, with individual trajectories remaining closed under monodromy at $|x^2| > |x_c^2|$, as shown in figure 22. This is "level-touching", observed e.g. in the case of the BTZ quasinormal spectrum [5]. In case of a branch point singularity at the critical point, the two branches merge into a single trajectory for $|x^2| > |x_c^2|$, as they are mapped into each other under the monodromy (figure 23). This is the level-crossing phenomenon [4, 5].

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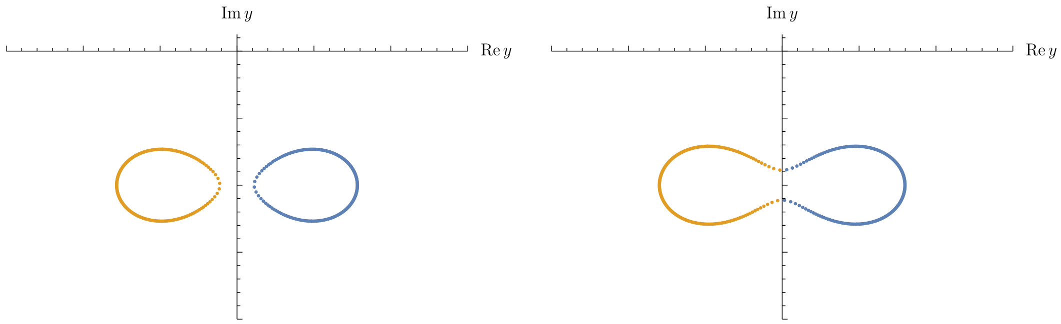

Figure 23. Level-crossing: the two branches $y_2^{(1,2)}(x^2)$ of the spectral curve P_2 at complex $x^2 = |x^2|e^{i\varphi}$, with φ varying from 0 to 2π, at fixed $|x_1^2| < |x_c^2|$ (left panel) and $|x_2^2| > |x_c^2|$ (right panel).

B How to reconstruct the Puiseux exponent from the power series

Given the hydrodynamic expansion (e.g. (2.1) or (2.2)) of the dispersion relation $\mathfrak{w} = \mathfrak{w}(\mathfrak{q}^2)$ around $\mathfrak{q}^2 = 0$, one can determine, at least in principle, the exponent of the Puiseux expansion $\mathfrak{w} \sim (\mathfrak{q}^2 - \mathfrak{q}_c^2)^{-\nu}$ at the nearest to the origin critical point $\mathfrak{q}_c^2$, given by eq. (2.3). This can be done by using the Darboux theorem (see e.g. [54], Theorem 11.10b) which states that if a function $p(t)$ has a singularity at $t = t_0$ of the form

$$p(t) \sim r(t)\left(1 - \frac{t}{t_0}\right)^{-\nu}, \qquad t \to t_0\,, \tag{B.1}$$

where $r(t)$ is an analytic function and ν is not a negative integer or zero, then the coefficients a_n of the Taylor expansion of $p(t)$ at the origin, $p(t) = \sum\limits_{n=0}^{\infty} a_n t^n$, have the following asymptotic form as $n \to \infty$

$$a_n \sim \frac{\Gamma(n+\nu)}{\Gamma(\nu) t_0^n n!}\left[r(t_0) - \frac{(\nu-1)t_0 r'(t_0)}{(n+\nu-1)} + \frac{(\nu-1)(\nu-2)t_0^2 r''(t_0)}{2!(n+\nu-1)(n+\nu-2)} + \cdots\right]. \tag{B.2}$$

Keeping only the leading term in eq. (B.2), we find

$$\nu = \lim_{n\to\infty}\left(t_0(n+1)\frac{a_{n+1}}{a_n} - n\right). \tag{B.3}$$

Then, using the subdominant terms in eq. (B.2), one can in principle reconstruct the function $r(t)$ by recovering its derivatives. In practice, a finite (and often relatively small) number of the coefficients a_n (e.g. computed numerically) is sufficient to determine ν with a good precision. Thus, if t_0 is the critical point closest to the origin, one can determine (or, strictly speaking, conjecture) whether this point is a singularity of $p(t)$ (i.e., whether ν is a fractional or positive integer number) and therefore sets the radius of convergence and reconstruct the Puiseux expansion around t_0 by analysing the coefficients a_n.

A complication arises when there are two or more critical points located at the same distance from the origin. This is the case for the hydrodynamic dispersion relations in the $\mathcal{N} = 4$ SYM theory at infinite 't Hooft coupling, where for the shear and sound modes we have a pair of complex conjugate critical points in the complex $\mathfrak{q}^2$-plane [4]. Such cases were studied for example in ref. [55]. Instead, we find it more convenient to reduce the case with two critical points to the previous one with the help of a conformal transformation. Indeed, let $t = t_1$ and $t = t_2$, where $|t_1| = |t_2|$, be the critical points located at the same distance from the origin $t = 0$. By performing a Möbius transformation

$$t \to z = \frac{at+b}{ct+d} \tag{B.4}$$

and requiring that under the map, $0 \to 0$, $t_1 \to 1$ and $t_2 \to z_2$, where $|z_2| > 1$, we reduce the situation to the one of the Darboux theorem: e.g., the exponent ν_1 of the branch point $t = t_1$ is determined by applying the Darboux procedure to the point $z = 1$. The inverse transformation is

$$t = \frac{dz-b}{a-cz}\,. \tag{B.5}$$

We must also require that the singularity at $z \equiv z_s = a/c$ stays outside of the unit circle in the complex z-plane, so that any analytic part of $p(t)$ remains analytic after the transformation. This implies the requirement $|a/c| > 1$. Moreover, removing z_s sufficiently far from the unit circle is advisable from the following technical point of view: analytic functions such as e^t acquire an essential singularity $\sim \exp[1/(z - z_s)]$ at $z = z_s$, and this would complicate numerics if z_s were too close to the unit circle. Explicitly, we have

$$t \to z = \frac{t z_2 (t_1 - t_2)}{t(t_1 z_2 - t_2) + t_1 t_2 (1 - z_2)}\,, \tag{B.6}$$

whereas the inverse transformation is given by

$$t = \frac{z t_1 t_2 (z_2 - 1)}{(t_1 z_2 - t_2) z + (t_2 - t_1) z_2}\,. \tag{B.7}$$

The singularity z_s is at

$$z_s = \frac{(t_1 - t_2) z_2}{t_1 z_2 - t_2} = \frac{(\alpha - 1) z_2}{\alpha z_2 - 1}\,,$$

where $\alpha \equiv t_1/t_2$, with $|\alpha| = 1$. We need to choose $|z_2| > 1$. Let $z_2 = x + iy$. Then,

$$|z_s|^2 = \frac{|\alpha - 1|^2 (x^2 + y^2)}{x^2 + y^2 + 1 - 2\,x\,\mathrm{Re}\,\alpha + 2\,y\,\mathrm{Im}\,\alpha}\,. \tag{B.8}$$

The zero of the denominator of (B.8) is at $x = \mathrm{Re}\,\alpha$, $y = -\mathrm{Im}\,\alpha$, with $|z_2|^2 = x^2 + y^2 = 1$, since $|\alpha| = 1$. By choosing $x = \mathrm{Re}\,\alpha + \epsilon$ and $y = -\mathrm{Im}\,\alpha - \epsilon$, where $\epsilon > 0$, we can satisfy the requirements $|z_2| > 1$ and $|z_s| \gg 1$. Indeed,

$$|z_s|^2 = \frac{|\alpha - 1|^2}{2\epsilon^2} \left(1 + 2\,\epsilon\,\mathrm{Re}\,\alpha + 2\,\epsilon\,\mathrm{Im}\,\alpha + 2\epsilon^2\right)\,, \tag{B.9}$$

and so $|z_s| \sim |\alpha - 1|/\sqrt{2}\epsilon$ for small ϵ (in numerical calculations, ϵ should not be too small, otherwise we need a large number of the coefficients a_n to achieve a satisfactory convergence in eq. (B.3)).

Applying this procedure to the hydrodynamic series of the $\mathcal{N} = 4$ SYM theory using the data of ref. [4], we find $\nu = -1/2$ to a good precision, confirming that the dispersion relations have branch point singularities of the square root type at the closest to the origin critical points. This is fully consistent with the characteristic quasinormal mode behaviour described in appendix A.

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C Coefficients $\mathcal{A}_{(i)}$ and $\mathcal{B}_{(i)}$ of eq. (3.10) in the $\mathcal{N}=4$ SYM theory

The coefficients can be written in the form

$$\mathcal{A}_{(i)}(u,\mathfrak{w},\mathfrak{q},\gamma)=\mathcal{A}^{(0)}_{(i)}(u,\mathfrak{w},\mathfrak{q})+\gamma\mathcal{A}^{(1)}_{(i)}(u,\mathfrak{w},\mathfrak{q})\,, \tag{C.1}$$

$$\mathcal{B}_{(i)}(u,\mathfrak{w},\mathfrak{q},\gamma)=\mathcal{B}^{(0)}_{(i)}(u,\mathfrak{w},\mathfrak{q})+\gamma\mathcal{B}^{(1)}_{(i)}(u,\mathfrak{w},\mathfrak{q})\,, \tag{C.2}$$

where $i=1,2,3$ for the scalar, shear and sound channels, respectively. We have for the scalar channel:

$$\mathcal{A}^{(0)}_{(1)}(u,\mathfrak{w},\mathfrak{q})=-\frac{1+u^2}{u\,(1-u^2)}\,, \tag{C.3}$$

$$\mathcal{B}^{(0)}_{(1)}(u,\mathfrak{w},\mathfrak{q})=\frac{\mathfrak{w}^2-\mathfrak{q}^2\,(1-u^2)}{u\,(1-u^2)^2}\,, \tag{C.4}$$

$$\mathcal{A}^{(1)}_{(1)}(u,\mathfrak{w},\mathfrak{q})=6u\left(160\mathfrak{q}^2u^3+129u^4+94u^2-25\right)\,, \tag{C.5}$$

$$\mathcal{B}^{(1)}_{(1)}(u,\mathfrak{w},\mathfrak{q})=\frac{192\mathfrak{q}^4u^5-\mathfrak{q}^2\,(851u^6-789u^4+75u^2+30)+6\,(-89u^4+30u^2+5)\,\mathfrak{w}^2}{u\,(1-u^2)}\,. \tag{C.6}$$

For the shear channel:

$$\mathcal{A}^{(0)}_{(2)}(u,\mathfrak{w},\mathfrak{q})=-\frac{(1+u^2)\,\mathfrak{w}^2-\mathfrak{q}^2\,(1-u^2)^2}{u\,(1-u^2)\,(\mathfrak{w}^2-\mathfrak{q}^2\,(1-u^2))}\,, \tag{C.7}$$

$$\mathcal{B}^{(0)}_{(2)}(u,\mathfrak{w},\mathfrak{q})=\frac{\mathfrak{w}^2-\mathfrak{q}^2\,(1-u^2)}{u\,(1-u^2)^2}\,, \tag{C.8}$$

$$\begin{aligned}\mathcal{A}^{(1)}_{(2)}(u,\mathfrak{w},\mathfrak{q})=\;&\frac{2u}{(\mathfrak{w}^2-\mathfrak{q}^2\,(1-u^2))^2}\Bigg[640\mathfrak{q}^6u^3\left(u^2-1\right)^2\\&-4\mathfrak{q}^4u^2\left(135u^6-450u^4-248u^3\mathfrak{w}^2+495u^2+200u\mathfrak{w}^2-180\right)\\&+\mathfrak{q}^2\mathfrak{w}^2\left(-462u^6+1374u^4+160u^3\mathfrak{w}^2-1002u^2+75\right)\\&+3\left(129u^4+94u^2-25\right)\mathfrak{w}^4\Bigg]\,,\end{aligned} \tag{C.9}$$

$$\begin{aligned}\mathcal{B}^{(1)}_{(2)}(u,\mathfrak{w},\mathfrak{q})=\;&-\frac{3}{u\,(1-u^2)\,(\mathfrak{w}^2-\mathfrak{q}^2\,(1-u^2))}\Bigg[-64\mathfrak{q}^6u^5\left(u^2-1\right)\\&+\mathfrak{q}^4\left(425u^8-880u^6-64u^5\mathfrak{w}^2+480u^4-15u^2-10\right)\\&+\mathfrak{q}^2\left(699u^6-693u^4+75u^2+20\right)\mathfrak{w}^2+2\left(89u^4-30u^2-5\right)\mathfrak{w}^4\Bigg]\,.\end{aligned} \tag{C.10}$$

For the sound channel:

$$\mathcal{A}^{(0)}_{(3)}(u,\mathfrak{w},\mathfrak{q}) = -\frac{3\left(1+u^2\right)\mathfrak{w}^2-\mathfrak{q}^2\left(3-2u^2+3u^4\right)}{u\left(1-u^2\right)\left(3\mathfrak{w}^2-\mathfrak{q}^2\left(3-u^2\right)\right)}\,, \tag{C.11}$$

$$\mathcal{B}^{(0)}_{(3)}(u,\mathfrak{w},\mathfrak{q}) = \frac{3\mathfrak{w}^4-2\left(3-2u^2\right)\mathfrak{w}^2\mathfrak{q}^2-\mathfrak{q}^2\left(1-u^2\right)\left(4u^3+\mathfrak{q}^2\left(u^2-3\right)\right)}{u\left(1-u^2\right)^2\left(3\mathfrak{w}^2-\mathfrak{q}^2\left(3-u^2\right)\right)}\,, \tag{C.12}$$

$$\begin{aligned}\mathcal{A}^{(1)}_{(3)}(u,\mathfrak{w},\mathfrak{q}) = {}& \frac{2u}{\left(3\mathfrak{w}^2-\mathfrak{q}^2\left(3-u^2\right)\right)^3}\Big[32\mathfrak{q}^8u^3\left(35u^6-291u^4+753u^2-585\right)\\ &-3\mathfrak{q}^6\left(3741u^{10}-27911u^8-2720u^7\mathfrak{w}^2+60804u^6+12992u^5\mathfrak{w}^2-50112u^4\right.\\ &\left.-12960u^3\mathfrak{w}^2+16887u^2-225\right)+3\mathfrak{q}^4\mathfrak{w}^2\left(-19401u^8+59832u^6+4960u^5\mathfrak{w}^2\right.\\ &\left.-53892u^4-7200u^3\mathfrak{w}^2+26094u^2-1125\right)+9\mathfrak{q}^2\mathfrak{w}^4\left(-1263u^6+99u^4\right.\\ &\left.+160u^3\mathfrak{w}^2-3915u^2+525\right)+81\left(129u^4+94u^2-25\right)\mathfrak{w}^6\Big]\,,\end{aligned} \tag{C.13}$$

$$\begin{aligned}\mathcal{B}^{(1)}_{(3)}(u,\mathfrak{w},\mathfrak{q}) = {}& \frac{1}{u\left(1-u^2\right)\left(3\mathfrak{w}^2-\mathfrak{q}^2\left(3-u^2\right)\right)^3}\Big[192\mathfrak{q}^{10}u^5\left(u^2-3\right)^3\\ &-\mathfrak{q}^8\left(u^2-3\right)\left(5811u^{10}-41287u^8-1728u^7\mathfrak{w}^2+74004u^6+5184u^5\mathfrak{w}^2\right.\\ &\left.-35169u^4+495u^2+270\right)-3\mathfrak{q}^6\left(11184u^{13}-90072u^{11}+17099u^{10}\mathfrak{w}^2\right.\\ &+223952u^9-106323u^8\mathfrak{w}^2-16u^7\left(108\mathfrak{w}^4+12971\right)+185876u^6\mathfrak{w}^2\\ &\left.+1728u^5\left(3\mathfrak{w}^4+34\right)-91107u^4\mathfrak{w}^2+1800u^3+2835u^2\mathfrak{w}^2+1080\mathfrak{w}^2\right)\\ &+3\mathfrak{q}^4\mathfrak{w}^2\left(-68316u^{11}+279504u^9-40333u^8\mathfrak{w}^2-319056u^7+121158u^6\mathfrak{w}^2\right.\\ &\left.+36u^5\left(48\mathfrak{w}^4+2713\right)-81018u^4\mathfrak{w}^2+3600u^3+6075u^2\mathfrak{w}^2+1620\mathfrak{w}^2\right)\\ &-9\mathfrak{q}^2\mathfrak{w}^4\left(21708u^9-37140u^7+7003u^6\mathfrak{w}^2+12972u^5-10017u^4\mathfrak{w}^2\right.\\ &\left.+600u^3+1755u^2\mathfrak{w}^2+360\mathfrak{w}^2\right)-162\left(89u^4-30u^2-5\right)\mathfrak{w}^8\Big]\,.\end{aligned} \tag{C.14}$$

See ref. [16] for details.

D Coefficients $\mathcal{A}_{(i)}$ and $\mathcal{B}_{(i)}$ of eq. (4.6) in the Einstein-Gauss-Bonnet theory

In the scalar, shear and sound channels ($i = 1,2,3$) we have, correspondingly,

$$\mathcal{A}_{(1)} = -\frac{1}{u}-u\left[\frac{1}{\left(\gamma_{GB}^2-1\right)\left(1-u^2\right)^2+1-u^2}+\frac{1}{\left(1-u^2\right)\sqrt{\gamma_{GB}^2-\left(\gamma_{GB}^2-1\right)u^2}}\right]\,, \tag{D.1}$$

$$\begin{aligned}\mathcal{B}_{(1)} = {}& \frac{(\gamma_{GB}-1)(\gamma_{GB}+1)^2\left(3\left(\gamma_{GB}^2-1\right)u^2-\gamma_{GB}^2\right)\left(-\gamma_{GB}^2+\left(\gamma_{GB}^2-1\right)u^2+U\right)}{4u\left(\gamma_{GB}^2-\left(\gamma_{GB}^2-1\right)u^2\right)^{3/2}\left(-\gamma_{GB}^2+\left(\gamma_{GB}^2-1\right)u^2+2U-1\right)}\mathfrak{q}^2\\ &+\frac{\left(\gamma_{GB}^2-1\right)^2\left(-\gamma_{GB}^2+\left(\gamma_{GB}^2-1\right)u^2+U\right)}{4u(U-1)\sqrt{\gamma_{GB}^2-\left(\gamma_{GB}^2-1\right)u^2}\left(-\gamma_{GB}^2+\left(\gamma_{GB}^2-1\right)u^2+2U-1\right)}\mathfrak{w}^2\,,\end{aligned} \tag{D.2}$$

$$\mathcal{A}_{(2)} = -\frac{2\gamma_{GB}^4(\gamma_{GB}+1)\left[\frac{1}{2}\left(1-\gamma_{GB}^2\right)\left(u^2-1\right)(U-2)+U-1\right]}{u(U-1)U^3\left[\gamma_{GB}^2(\gamma_{GB}+1)(U-1)\mathfrak{q}^2-\left(\gamma_{GB}^2-1\right)U^2\mathfrak{w}^2\right]}\mathfrak{q}^2$$
$$-\frac{\left(1-\gamma_{GB}^2\right)\left(\gamma_{GB}^4+\left(1-\gamma_{GB}^2\right)^2u^4-2\left(1-\gamma_{GB}^2\right)u^2\left(U-\gamma_{GB}^2\right)-\gamma_{GB}^2U\right)}{u(U-1)U\left[\gamma_{GB}^2(\gamma_{GB}+1)(U-1)\mathfrak{q}^2-\left(\gamma_{GB}^2-1\right)U^2\mathfrak{w}^2\right]}\mathfrak{w}^2\,, \tag{D.3}$$

$$\mathcal{B}_{(2)} = \frac{\gamma_{GB}^2(\gamma_{GB}+1)(U+1)}{4u\left(u^2-1\right)U^2}\mathfrak{q}^2+\frac{\left(U^2+2U+1\right)}{4u\left(u^2-1\right)^2}\mathfrak{w}^2\,, \tag{D.4}$$

$$\mathcal{A}_{(3)} = \frac{3}{2u}+\frac{3(\gamma_{GB}-1)\left[\left(\gamma_{GB}^2-1\right)u^2-\gamma_{GB}^2\right]\left[\left(\gamma_{GB}^2-1\right)u^2(5U-7)-5\gamma_{GB}^2(U-1)\right]}{2u(U-1)U^2D_1}\mathfrak{w}^2$$
$$+\frac{\left(\gamma_{GB}^2-1\right)^2u^4\left(-3\gamma_{GB}^2+5U-7\right)+\gamma_{GB}^2\left(\gamma_{GB}^2-1\right)u^2\left(18\gamma_{GB}^2-13U+10\right)}{2u(U-1)U^2D_1}\mathfrak{q}^2$$
$$-\frac{15\gamma_{GB}^4\left(\gamma_{GB}^2-2U+1\right)}{2u(U-1)U^2D_1}\mathfrak{q}^2\,, \tag{D.5}$$

$$\mathcal{B}_{(3)} = \frac{\left(\gamma_{GB}^2-1\right)^2}{D_0}\Bigg\{12(\gamma_{GB}-1)^2\gamma_{GB}^2(\gamma_{GB}+1)\mathfrak{q}^2u^5-4(\gamma_{GB}-1)\gamma_{GB}^2\mathfrak{q}^2u^3\left(3\gamma_{GB}^2-7U+4\right)$$
$$+\left(\gamma_{GB}^2-1\right)^3\mathfrak{q}^2u^6\left(3(\gamma_{GB}-1)\mathfrak{w}^2+\mathfrak{q}^2\right)$$
$$-u^2\gamma_{GB}^2\left(\gamma_{GB}^2-1\right)\left[\mathfrak{q}^4\left(\gamma_{GB}^2+2U\right)+(\gamma_{GB}-1)\mathfrak{q}^2\mathfrak{w}^2\left(9\gamma_{GB}^2-4U\right)-6(\gamma_{GB}-1)^2U\mathfrak{w}^4\right]$$
$$+\left(\gamma_{GB}^2-1\right)^2u^4\left[\mathfrak{q}^4\left(3\gamma_{GB}^2(U-2)+U\right)+2(\gamma_{GB}-1)\mathfrak{q}^2U\mathfrak{w}^2-3(\gamma_{GB}-1)^2U\mathfrak{w}^4\right]$$
$$-3\gamma_{GB}^4\left[\mathfrak{q}^4\left(\gamma_{GB}^2(U-2)+U\right)+2(\gamma_{GB}-1)\mathfrak{q}^2\mathfrak{w}^2\left(U-\gamma_{GB}^2\right)+(\gamma_{GB}-1)^2U\mathfrak{w}^4\right]\Bigg\}, \tag{D.6}$$

where we have defined

$$D_1 \equiv \left(\gamma_{GB}^2-1\right)u^2\left(3(\gamma_{GB}-1)\mathfrak{w}^2+\mathfrak{q}^2\right)+3\gamma_{GB}^2\left(\mathfrak{q}^2(U-1)-(\gamma_{GB}-1)\mathfrak{w}^2\right)\,, \tag{D.7}$$
$$D_0 \equiv 4(\gamma_{GB}-1)u(U-1)^2U^3D_1\,. \tag{D.8}$$

In the above expressions, we also used $U^2 = u^2+\gamma_{GB}^2-u^2\gamma_{GB}^2$, as well as the dimensionless frequency and momentum $\mathfrak{w} = \omega/2\pi T$, $\mathfrak{q} = q/2\pi T$, where T is the Hawking temperature of the black brane background. See ref. [26] for details.

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JHEP06(2021)180

Published for SISSA by Springer

Received: *April 2, 2022*
Accepted: *May 23, 2022*
Published: *June 29, 2022*

CFT correlators, $\mathcal{W}$-algebras and generalized Catalan numbers

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Abstract: In two spacetime dimensions the Virasoro heavy-heavy-light-light (HHLL) vacuum block in a certain limit is governed by the Catalan numbers. The equation for their generating function can be generalized to a differential equation which the logarithm of the block satisfies. We show that a similar story holds for the HHLL $\mathcal{W}_N$ vacuum blocks, where a suitable generalization of the Catalan numbers plays the main role. Moreover, the $\mathcal{W}_N$ blocks have the same form as the stress tensor sector of HHLL near lightcone conformal correlators in $2(N-1)$ spacetime dimensions. In the latter case the Catalan numbers are generalized to the numbers of linear extensions of certain partially ordered sets.

Keywords: AdS-CFT Correspondence, Conformal and W Symmetry, $1/N$ Expansion

ArXiv ePrint: 2111.07924

https://doi.org/10.1007/JHEP06(2022)162

JHEP06(2022)162

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1 Introduction and summary of results

The local conformal algebra in two dimensions is the infinite-dimensional Virasoro algebra and is generated by the modes of the stress tensor operator $T(z)$. It induces a natural decomposition of correlation functions into Virasoro conformal blocks which capture the contribution from a given Virasoro primary and all its Virasoro descendants. With respect to the global conformal algebra, each Virasoro representation contains an infinite number of quasi-primaries — the Virasoro symmetry therefore imposes strong constraints on the theory as seen from the perspective of someone that only knew about its global part.

Further, the presence of symmetries in CFTs is deeply connected to universal features. An example is Cardy's formula for the density of high energy of states in two-dimensional CFTs [1]. It follows from the large conformal transformation of the torus and the dominance of the lowest dimension operator in the partition function in the low-temperature limit. Another example of universality is the presence of large-spin double-twist composite operators in any unitary $d > 2$-dimensional CFT [2, 3]. This follows from studying the lightcone limit of a four-point function of scalar operators and utilizing crossing symmetry. In one channel the identity operator dominates because it is the operator with the smallest twist $\tau = 0$. Interpreting this in a different channel leads to the existence of double-twist operators with large spin ℓ and universal OPE data to leading order in the ℓ^{-1} expansion. Furthermore, in the absence of light scalars ($\Delta_{\text{min}} > d-2$), the correction in the lightcone limit is due to conserved currents, in particular, the stress tensor operator. Its OPE coefficient in the OPE $T \subset \mathcal{O}_\Delta \times \mathcal{O}_\Delta$ of identical scalars is fixed by Ward identities in terms of the scaling dimension Δ and the central charge C_T. This leads to further universal corrections for the OPE data of the double-twist operators in the other channel.

In two dimensions, the Virasoro vacuum block of a four-point function $\mathcal{O}_1 \times \mathcal{O}_1 \to [T^k] \to \mathcal{O}_2 \times \mathcal{O}_2$ contains contributions from the stress tensor and an infinite family of composite operators of the schematic form $[T^k]$ for each k which are completely fixed by the symmetries. In higher-dimensional CFTs with a large central charge C_T, there are similar composite operators $[T^k]_{\tau,s}$ — with τ and s denoting the twist and spin, respectively. A priori, the multi-stress tensor OPE coefficients in the OPE of identical scalar operators, $[T^k]_{\tau,s} \subset \mathcal{O}_\Delta \times \mathcal{O}_\Delta$, are not fixed by symmetries in contrast to the two-dimensional case. These operators are, however, ubiquitous in theories with gravity duals since they are related to the exchange of multi-graviton states in the bulk. Therefore in order to understand the emergence of gravity in the bulk from the CFT data on the boundary, these operators play a vital role. It is further interesting to ask if there is a notion of universality in the exchanges of multi-stress tensors in holographic CFTs with large C_T and a large gap in the spectrum of higher-spin single trace operators.

An important case where the exchange of these multi-stress tensors is expected to dominate compared to that of generic operators is when considering heavy states. This is so because the OPE coefficients of multi-stress tensors $[T^k]$ in a scalar OPE $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ scale like Δ^k for large Δ. An extreme example of this is when the heavy states have dimension Δ of order C_T. Such heavy states are expected to thermalize in holographic CFTs and according to the AdS/CFT dictionary, thermal states on the boundary are dual to black holes in the bulk. Correlation functions of light operators in heavy states therefore provide a possible window into one of the most interesting questions in the AdS/CFT correspondence, the physics of black holes.

In two dimensions, the heavy-heavy-light-light (HHLL) Virasoro vacuum block was found in [4, 5] and contains a wealth of information that can be used to shed light on black hole information loss and the thermalization of heavy states, entanglement entropy and much more [6–22]. In four dimensions,[1] recent progress has been made in studying the contribution of multi-stress tensor operators to HHLL correlators, both using conformal bootstrap techniques and the gravitational dual description [23–36]. In [29] following [27], it was argued that the contribution of all minimal-twist operators $[T^k]_{\tau_{\text{min}},s}$, with $\tau_{\text{min}} = 2k$ and spin $s = 2k + l$ for $l = 0, 2, 4, \ldots$, in holographic CFTs, takes a specific form which is reminiscent to that obtained from the Virasoro vacuum block. It repackages an infinite

[1]Similar results also holds for $d > 2$ with d even.

number of minimal-twist multi-stress tensor OPE coefficients in the HHLL correlator and it is natural to ask if this is governed by an underlying emergent symmetry in the lightcone limit similar to the Virasoro symmetry.

In this work we present further progress in this direction by studying the HHLL vacuum blocks of two-dimensional CFTs with $\mathcal{W}_N$ higher-spin symmetry,[2] see [37–41] for related work. The semi-classical vacuum blocks were found for $N = 3$ in [39, 42] and for general N in [40]. In this case, the charges of the "light" operator are large but much smaller than those of the heavy operator which scale with the central charge $c \gg 1$. Expanding the $\mathcal{W}_N$ vacuum blocks in $\frac{q_H^{(i)}}{c}$, where $q_H^{(i)}$ is the spin$-i$ charge of the heavy operator, we find that the result is again similar to the expansion of the Virasoro vacuum block, with a decomposition in terms of composite operators with the correct weight under the global conformal algebra. In particular, when $q_H^{(3)} \sim c \gg q_H^{(i\neq 3)}$, the dominant contributions[3] are due to composite quasi-primary operators with the schematic form $[W^k]_{2l}$ made out the spin-3 current $W(z)$. The resulting functions, which are linear combinations of products of hypergeometric functions, are also present in the result for the minimal-twist stress tensor sector of the $d = 4$ HHLL correlator. This is one of the main motivations for our work.

We further explicitly compute the first few terms of the $\mathcal{W}_N$ HHLL vacuum blocks for $N = 3, 4$ in the limit $q_H^{(3)} \sim c \gg q_H^{(i\neq 3)}$ using an explicit mode calculation. This limit has the advantage that the charges of the light operators are kept fixed as $c \to \infty$ and sheds further light on how the resulting structure that appears in the four-dimensional stress tensor sector of the HHLL correlator could appear from an underlying symmetry algebra. The results agree with those obtained from the expansion of the semi-classical vacuum blocks which assumed that the charges of the light operators were large. This gives further evidence that those results remain true also for finite charge. The mode calculation presented in this work can in principle also be used to compute $\frac{1}{c}$ corrections to the HHLL vacuum blocks.

Focusing on the logarithm of the $\mathcal{W}_3$ HHLL vacuum block we further show that it satisfies a non-linear differential equation which, in a certain limit, reduces to a cubic equation for the generating function for the sequence of integers given by A085614 in [43]. The $\mathcal{W}_3$ HHLL vacuum block can also be obtained from a set of diagrammatic rules similar to the Virasoro vacuum block [12]. The story can be generalized in the case of the $\mathcal{W}_4$ HHLL block both in the limit where the spin-4 charge scales with the central charge and is parametrically larger than all other charges and in the limit where the spin-3 charge scales with the central charge and is parametrically larger than the rest of the charges. We expect a similar story to hold for all $\mathcal{W}_N$ blocks. From a mathematician's point of view, the $\mathcal{W}_N$ vacuum blocks provide generating functions for several new sequences which can be understood as different generalizations of the Catalan numbers' sequence.

[2]We will mainly consider $N = 3, 4$ but the methods used and the structure remains similar for any N.

[3]Note that it is only the spin-3 charge of the "heavy" operators that scales with c and, in particular, their scaling dimension is small compared to c. We will still refer to these as heavy. It is possible to extend our results to the case when all the charges of the heavy operators are large but we will not attempt to do so since it is the spin-3 sector that resembles the stress tensor sector in four dimensions.

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Further, we examine the stress tensor sector of the four-dimensional HHLL correlator when the conformal dimension of the light operator vanishes, $\Delta_L \to 0$.[4] A similar picture emerges with the relevant sequence of numbers given by the number of linear extensions of the one-level grid partially ordered set (poset) $G[(1^{k-1}), (0^{k-2}), (0^{k-2})]$.[5] We observe the same structure appearing in $d = 6, 8$ as well. In this case, the sequences of numbers are related to the linear extensions of the $G[(\frac{d-2}{2})^{k-1}, (0)^{k-2}, (0^{k-2})]$ posets. In the spirit of the two-dimensional cases examined here, one would hope that knowing the algebraic equation satisfied by the generating function of this sequence, would allow the determination of a differential equation satisfied by the all-orders stress-tensor sector of the HHLL correlator in the lightcone limit for $\Delta_L \to 0$. However, to our knowledge, the generating functions of the number of linear extensions of $G[(\frac{d-2}{2})^{k-1}, (0)^{k-2}, (0^{k-2})]$ are not known.

1.1 Summary of results

Consider a heavy-heavy-light-light (HHLL) four-point function in a two-dimensional CFT with a large central charge c and a higher-spin $\mathcal{W}_N$ symmetry $\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle$. The operators $\mathcal{O}_H$ and $\mathcal{O}_L$ are $\mathcal{W}_N$ primaries and carry higher-spin charges $q_H^{(i)}$ and $q^{(i)}$, with $i = 2, 3, \ldots, N$, respectively. Such a four-point function can be decomposed into blocks which contain contributions from a $\mathcal{W}_N$ primary $\mathcal{O}$ and all its $\mathcal{W}_N$-descendants. We define $\mathcal{G}_N(z)$ as the holomorphic part of the HHLL correlator restricted to the identity block contribution in the direct channel $\mathcal{O}_L \times \mathcal{O}_L \to 1_{\mathcal{W}_N} \to \mathcal{O}_H \times \mathcal{O}_H$. We specify our discussion to the cases $N = 3, 4$ although it can be generalized to any N.

1.1.1 HHLL blocks by mode decomposition

We start by considering the case $N = 3$ where the CFT protagonists are the stress tensor $T(z)$ and a spin-3 field $W(z)$. $\mathcal{G}_3(z)$ contains the exchange of all states schematically denoted by

$$|\{a_i, b_j\}\rangle := W_{a_1} W_{a_2} \ldots W_{a_n} L_{b_1} L_{b_2} \ldots L_{b_k} |0\rangle - (\ldots)|0\rangle, \tag{1.1}$$

where L_b and W_a are the modes of $T(z)$ and $W(z)$, respectively, and the ellipses ensure that these states are mutually orthogonal. In particular, the subsector consisting of only states with modes L_b acting on the vacuum is that of the Virasoro vacuum block and was studied in detail in [12]. We are interested in heavy states with a large spin-3 charge $w_H \equiv q_H^{(3)}$ with[6]

$$\begin{aligned} &h_H \ll w_H \sim c \to \infty, \\ &h, w \ll c, \end{aligned} \tag{1.2}$$

[4] Note that this is below the unitarity bound. There are, however, certain observables such as the phase shift [23] that do not depend on Δ_L that one might be able to extract from the $\Delta_L \to 0$ limit.

[5] The Catalan numbers are also the numbers of linear extensions of the one-level grid poset $G[(0^{k-1}), (0^{k-2}), (0^{k-2})]$.

[6] It is straightforward to extend our results to the case when all the heavy charges are $\mathcal{O}(c)$ but we will not attempt to do so. See however appendix A and B.

where h_H and h are the conformal weights of the heavy and light operator, respectively, and w is the spin-3 charge of the light operator. The effect of using (1.2) is that the dominant contribution to $\mathcal{G}_3(z)$ is due to states of the form

$$|\{a_i\}\rangle = W_{a_1} W_{a_2} \dots W_{a_n}|0\rangle - (\dots)|0\rangle \tag{1.3}$$

because each W-mode will to leading order contribute a factor of w_H when acting on the heavy operators. Inserting the projection on the single mode states $W_{-m}|0\rangle$ in the correlator one finds the $\mathcal{O}(\frac{w_H}{c})$ term of the vacuum block

$$\mathcal{G}_3(z)\Big|_{\frac{w_H}{c}} = \frac{3ww_H}{c}\frac{f_3(z)}{z^{2h}}, \tag{1.4}$$

where z^{-2h} is the disconnected correlator and $f_a(z)$ is an SL$(2;R)$ conformal block given by

$$f_a(z) = z^a {}_2F_1(a, a; 2a, z). \tag{1.5}$$

The result in (1.4) is the conformal block due to the exchange of the quasi-primary $W(z)$ and all its descendants under the global conformal group.

It is useful to recall the behavior of a d-dimensional conformal block, $g^{(0,0)}_{\tau,s}(z,\bar{z})$, in the lightcone limit $\bar{z} \to 0$

$$g^{(0,0)}_{\tau,s}(z,\bar{z}) \sim \bar{z}^{\frac{\tau}{2}} f_{\frac{\tau}{2}+s}(z). \tag{1.6}$$

In four dimensions, the stress tensor block with $\tau = s = 2$ has the same z-dependence as (1.4) (as can be seen from (1.6)).

Going back to $d = 2$, we consider the $\mathcal{O}(\frac{w_H^2}{c^2})$ contribution to $\mathcal{G}_3(z)$. This is due to the (unnormalized) states

$$|Y_{m,n}\rangle = \left[W_{-n}W_{-m} - \frac{(3n+2m)\, m\,(m^2-1)\,(m^2-4)}{30(m+n)((m+n)^2-1)} L_{-m-n}\right]|0\rangle, \tag{1.7}$$

where the second term ensures that they are orthogonal to the states $L_{-n-m}|0\rangle$. Projecting onto these states one finds that

$$\mathcal{G}_3\left(z\right)\Big|_{\frac{w_H^2}{c^2}} = \left[\frac{1}{2}\left(\frac{3ww_H}{c} f_3\left(z\right)\right)^2 - \frac{9w_H^2 h}{70c^2} w_3\left(z\right)\right] z^{-2h}, \tag{1.8}$$

where $w_3 = -14f_3^2 + 15f_2f_4$. The resulting simple-looking expression can be decomposed into global conformal blocks of $[W^2]_{2l}$, with weights $h = 6, 8, \dots$, with the use of a product formula for hypergeometric functions found in [27].

Eq. (1.8) shows that the vacuum block contribution to the correlation function at quadratic order in the heavy charge expansion can be written as a sum of products f_af_b such that $a + b = 6$, where $h = 6$ is the weight of the lightest operator $[W^2]_0$. In higher, even spacetime dimension a similar picture emerges. In particular it was shown in [27, 29] that the minimal-twist double-stress tensor contributions to HHLL correlators in four dimensions can be written as $\mathcal{G}_{d=4}|_{\Delta_H^2/C_T^2} \propto a_{15}f_1f_5 + a_{24}f_2f_4 + a_{33}f_3^2$, for some Δ_L dependent coefficients a_{ij}.

Let us now include a spin-4 current U(z). With the four-dimensional results quoted above in mind, we consider the $\mathcal{W}_4$ HHLL vacuum block in the limit where the spin-3 charge is parametrically larger than the rest (this is done in appendix B). The states (1.7) have a non-vanishing overlap with the single mode states $U_{-m-n}|0\rangle$ and by removing this overlap, one finds that the correction to the $\mathcal{O}(\frac{w_H^2}{c^2})$ term in (1.8) is proportional to the spin-4 charge u of the light operator. The result takes the form

$$\mathcal{G}_4(z)\Big|_{\frac{w_H^2}{c^2}} \propto a_{4,15} f_1 f_5 + a_{4,24} f_2 f_4 + a_{4,33} f_3^2, \tag{1.9}$$

with coefficients $a_{4,ij}$ linear in the charges (h, u) of the light operator and quadratic in w due to the first term in (1.8).[7]

The results herein, obtained using explicit mode calculations, are in agreement with those for the $\mathcal{W}_N$ semi-classical vacuum blocks obtained in [40]. While the mode calculation becomes tedious at higher orders in $\frac{w_H}{c}$, the expansion of the semi-classical vacuum block is straightforward. Generally, we find that the expansion of the logarithm of the HHLL vacuum block in powers of $\frac{w_H}{c}$ can be written as a linear combination of products of hypergeometric:

$$\log\left(z^{2h}\mathcal{G}_N(z)\right) = \sum_{k=1}^{\infty}\left(\frac{w_H}{c}\right)^k \sum_{\{i_p\}} b_{N,i_1\ldots i_k} f_{i_1}(z)\ldots f_{i_k}(z), \tag{1.10}$$

where we have normalized the expression by the (holomorphic) part of the disconnected correlator z^{-2h}. i_p are integers such that $i_1+\ldots+i_k = 3k$ and the coefficients $b_{N,i_1,\ldots i_p}$ are linear in the charges $q^{(i)}$ of the light operator.[8]

1.1.2 Generalized Catalan numbers and differential equations

It is instructive to examine the behavior of the vacuum blocks when $z \to 1$. Similarly to the case of the Virasoro vacuum block, we observe that the logarithm of the $\mathcal{W}_N$ vacuum block, with one of the heavy charges $q_H \sim c \to \infty$ and all other charges fixed and parametrically smaller, has the following behavior in the limit $z \to 1$:

$$\log(\mathcal{G}_N(z)) \sim B_N\left(q^{(i)}, \frac{q_H}{c}\right)\log(1-z), \tag{1.11}$$

where the function B_N is linear in the light charges $q^{(i)}$ and can be perturbatively expanded in $\frac{q_H}{c}$. This behavior is non-trivial since generally a product of k functions f_a is a k-th order polynomial in $\log(1-z)$ with coefficients that are rational functions of z.

For the Virasoro case, the corresponding function B_2 is the generating function of the Catalan numbers. For $\mathcal{W}_3$ in the limit $w_H \sim c \to \infty$, with the other charges parametrically smaller and for certain values of the ratio of the charges of the light operator, we find that B_3 satisfies a cubic equation. Inspired by it, one can construct similarly to the Virasoro

[7] Whilst the form of the $\mathcal{G}_4(z)$ at quadratic order matches that of the four-dimensional result (notice the presence of the $f_1 f_5$-term), there is no choice of the charges of the light operators which would yield an exact match.

[8] Although the form of the $\mathcal{W}_N$ vacuum block expansion resembles that of the four-dimensional one, there is no value of N that would yield an exact match.

case, a cubic differential equation satisfied by $\mathcal{F}_3 \equiv \log \mathcal{G}_3$ with (1.2). We present it below in the case $h = 3w$:

$$\frac{1}{6w}\frac{d^3}{dz^3}\mathcal{F}_3(z) = -\frac{1}{54w^3}\left(\frac{d}{dz}\mathcal{F}_3(z)\right)^3 + \frac{1}{6w^2}\left(\frac{d^2}{dz^2}\mathcal{F}_3(z)\right)\left(\frac{d}{dz}\mathcal{F}_3(z)\right) + \frac{2x}{(1-z)^3}, \tag{1.12}$$

where $x = 6\frac{w_H}{c}$. We also derive diagrammatic rules for the $\mathcal{W}_3$ HHLL vacuum block satisfies.

We also consider the $\mathcal{W}_4$ HHLL vacuum block in appendix B. We study its behavior in the region $z \sim 1$ in two different cases; when the spin-4 charge, $u_H \sim c \gg 1$ while $h_H, w_H \ll c$ and when the spin-3 charge scales with c, $w_H \sim c \gg 1$ but $u_H, h_H \ll c$. In both cases the logarithm of the HHLL vacuum block behaves as $\mathcal{F}_4 \sim \log(1-z)$ in the limit $z \to 1$. In the former case, the generating function B_4 defined according to (1.11), satisfies a quartic equation for four different choices of the ratio h/u. In particular, when $h = 5u$ one can show that $\log \mathcal{G}(z)$ solves a differential equation whose form is inspired by the algebraic equation satisfied by B_4. The situation is similar but slightly more involved when the spin-3 charge, $w_H \sim c$.

Finally, we study the stress tensor sector of the HHLL correlator in d-spacetime dimensions in the limit $z \to 1$. In this case, we further have to take the $\Delta_L \to 0$ limit in order to remove higher log terms and find that the corresponding sequence of numbers are those of the number of linear extensions of posets $G[(\frac{d-2}{2})^{k-1}, (0)^{k-2}, (0^{k-2})]$. These are generalizations of the Catalan numbers which can be obtained as the number of linear extensions of the simpler poset $G[(0^{k-1}), (0^{k-2}), (0^{k-2})]$.

1.2 Outline

Section 2 is devoted to explicit mode calculations of the HHLL vacuum blocks. Specifically, in section 2.1 we review the Virasoro result and in section 2.2 we generalize this calculation to the case of the $\mathcal{W}_3$ HHLL vacuum block. In section 3, we study the behavior of the HHLL vacuum blocks in the region $z \sim 1$. After a short review of the Virasoro case, in section 3.2 we focus on the $\mathcal{W}_3$ vacuum block. We observe the appearance of a generalized Catalan sequence, determine its generating function and the algebraic equation the latter satisfies. Inspired by this algebraic equation, we determine a cubic differential equation satisfied by the logarithm of the $\mathcal{W}_3$ vacuum block for certain ratios of the charges of the light operators. We conclude the discussion of the spin-3 case with new diagrammatic rules for the $\mathcal{W}_3$ vacuum block expansion. In section 3.3 we investigate in a similar manner the stress tensor sector of the four-dimensional HHLL correlator in holographic CFTs. We conclude with a discussion in section 4. In appendix A one finds further details on the explicit mode calculations for the $\mathcal{W}_3$ HHLL vacuum block. In appendix B we consider the $\mathcal{W}_4$ HHLL vacuum block. When w_H is the only large charge, we show using the $\mathcal{W}_4$-algebra that one gets an extension of the $\mathcal{W}_3$ result which takes a form similar to that of the stress tensor sector of the HHLL correlator in $d = 4$. When u_H is the only large charge, we show that the HHLL vacuum block and a specific choice of the light charges is again governed by a generalization of the Catalan numbers, and that a corresponding non-linear differential equation can be written down analogous to the $\mathcal{W}_3$ case. A similar albeit more involved story emerges in the $z \to 1$ limit when the only large charge is w_H.

2 HHLL blocks by mode decomposition

In this section we perform a mode calculation of $\mathcal{W}_N$ higher-spin vacuum blocks in two-dimensional CFTs with large central charge. We review the calculation of the Virasoro vacuum block in section 2.1 following [4, 5] and extend this to include higher-spin currents in section 2.2 and appendix B. The semi-classical vacuum block, for large charges, in $\mathcal{W}_N$ theories has been calculated in [39, 42] for $N=3$ and in [40] for general N in the dual bulk theory using a Wilson line prescription. Expanding these known results we find agreement with those obtained from the mode calculation. The calculation of the $\mathcal{W}_N$ vacuum block using an explicit mode expansion can in principle be extended to include finite central charge as well as finite charges of the external operators.

2.1 Review of the Virasoro vacuum block

In this section we use the Virasoro modes to explicitly calculate the first terms due to Virasoro descendants of the vacuum following [4, 5].

We consider a four point function of pair-wise identical operators $\mathcal{O}_H$ and $\mathcal{O}_L$ with conformal weight H and h, respectively, given by $\langle\mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle$, where we suppress the anti-holomorphic part, have used conformal symmetry to fix the operators at $0, z, 1, \infty$ and set $\mathcal{O}_H(\infty) = \lim_{z\to\infty} z^{2H}\mathcal{O}_H(z)$. The limit that will be considered is $c\to\infty$ with h and $\frac{H}{c}$ fixed.

We are interested in the contribution due to Virasoro descendants of the vacuum, i.e. states of the schematic form

$$\mathcal{G}_2(z) = \langle\mathcal{O}_H(\infty)\mathcal{O}_H(1)\sum_{\{m_i\},\{n_j\}}\frac{L_{-m_1}L_{-m_2}\dots L_{-m_i}|0\rangle\langle 0|L_{n_j}\dots L_{n_2}L_{n_1}}{\mathcal{N}_{\{m_i\},\{n_j\}}}\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle, \tag{2.1}$$

where $\mathcal{N}_{\{m_i\},\{n_j\}}$ is a normalization factor and $\mathcal{G}_2(z)$ is defined as the HHLL correlator restricted to the contribution of the identity block in the direct channel (the subscript (2) here stands for the Virasoro algebra as opposed to (N) for the $\mathcal{W}_N$). In [12] an orthogonal basis was constructed in the limit $c\to\infty$ and it was shown how to perform this sum using a recursion relation. The correlator organizes into powers of $\frac{H}{c}$ and we will study the first two terms in this expansion. These are due to single and double mode states respectively.

To begin with, consider the contribution from states of the form $L_{-n}|0\rangle$ in (2.1). To calculate this, we need the Virasoro algebra

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n,0}, \tag{2.2}$$

as well as the action on primary operators

$$[L_n, \mathcal{O}\,(z)] = z^n\,[h(n+1)+z\partial]\,\mathcal{O}(z). \tag{2.3}$$

It is straightforward to evaluate $\langle 0|L_n\mathcal{O}(z)\mathcal{O}(0)\rangle$ and $\langle\mathcal{O}_H(\infty)\mathcal{O}_H(1)L_{-n}|0\rangle$ for $n\geq 2$ with the help of (2.3). We find that

$$\begin{aligned}\langle 0|L_n\mathcal{O}(z)\mathcal{O}(0)\rangle &= z^n[h(n+1)+z\partial]z^{-2h} = h(n-1)z^{n-2h}\\ \langle\mathcal{O}_H(\infty)\mathcal{O}_H(1)L_{-n}|0\rangle &= H(n-1).\end{aligned} \tag{2.4}$$

The norm of these states is given by the central term

$$\mathcal{N}_{n,n} = \langle L_n L_{-n} \rangle = \frac{c}{12} n \left(n^2 - 1 \right). \tag{2.5}$$

Combining the above allows one to obtain the single mode state contribution to the vacuum block

$$\mathcal{G}_2(z)|_{\frac{H}{c}} = z^{-2h} \sum_{n=2}^{\infty} \frac{12Hh}{c} \frac{(n-1)}{(n+1)} \frac{z^n}{n} = \frac{2Hh}{c} f_2(z) z^{-2h}, \tag{2.6}$$

where the SL(2, R) blocks f_a are given by

$$f_a(z) = z^a {}_2F_1(a, a; 2a; z). \tag{2.7}$$

Consider now states of the schematic form $L_{-m}L_{-n}|0\rangle$. These are not orthogonal to the single mode states $L_{-m-n}|0\rangle$ since

$$\langle L_{m+n} L_{-n} L_{-m} \rangle = (2n - m) \frac{c}{12} m \left(m^2 - 1 \right) \neq 0. \tag{2.8}$$

Removing this overlap one can construct states $|X_{m,n}\rangle$,[9] that are orthogonal to $L_{-m-n}|0\rangle$:

$$|X_{m,n}\rangle = \left[L_{-n}L_{-m} - \frac{\langle L_{m+n}L_{-n}L_{-m}\rangle}{\langle L_{m+n}L_{-m-n}\rangle} L_{-m-n} \right] |0\rangle, \tag{2.9}$$

which contribute at $\mathcal{O}(\frac{H^2}{c^2})$ to $\mathcal{G}(z)$. The contribution of these states can be found from (for details see appendix A, as well as [12])

$$\begin{aligned}
\langle 0|L_m L_n \mathcal{O}_L(z)\mathcal{O}_L(0)\rangle &= \left[h^2(n-1)(m-1) + hm(m-1) \right] z^{s-2h}, \\
\langle 0|L_{m+n} \mathcal{O}_L(z)\mathcal{O}_L(0)\rangle &= h(s-1) z^{s-2h},
\end{aligned} \tag{2.10}$$

where $s = m + n$. With the help of (2.10) one finds that

$$\begin{aligned}
\langle X_{m,n}| \mathcal{O}_L(z)\, \mathcal{O}_L(0)\, |0\rangle &= \Bigg[h^2 (n-1)(m-1) + hm(m-1) \\
&\quad - \frac{(2n-m) \frac{c}{12} m (m^2 - 1)}{\frac{c}{12} s (s^2 - 1)} h (s-1) \Bigg] z^{s-2h} \\
&= \left[h^2 (m-1)(n-1) + h \frac{n(n-1)\, m(m-1)}{s(s+1)} \right] z^{s-2h}
\end{aligned} \tag{2.11}$$

as in [12]. Furthermore, keeping only the leading term for large H gives

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)|X_{m,n}\rangle = H^2(n-1)(m-1). \tag{2.12}$$

The norm of the states $|X_{m,n}\rangle$ in the large-c limit is given by the square of the central terms, i.e. ,

$$\mathcal{N}_{X_{m,n}} = \langle L_m L_n L_{-n} L_{-m} \rangle = \left(\frac{c}{12} \right)^2 m \left(m^2 - 1 \right) n \left(n^2 - 1 \right) + \ldots, \tag{2.13}$$

[9] Note that the states $|X_{m,n}\rangle$ thus defined are not unit normalised.

where the ellipses refer to terms subleading in c. Combining the above one finds the contribution of the states $|X_{m,n}\rangle$ to the vacuum block in (2.1) to be

$$\mathcal{G}_2(z)|_{\frac{H^2}{c^2}} = \frac{z^{-2h}}{2}\left(\frac{12Hh}{c}\right)^2 \sum_{m,n=2}^{\infty} \frac{(m-1)(n-1)}{(m+1)(n+1)}\frac{z^{m+n}}{mn} + z^{-2h}\frac{72H^2h}{c^2}\sum_{m,n=2}^{\infty}\frac{(m-1)(n-1)}{(m+1)(n+1)}\frac{z^{m+n}}{(m+n)(m+n+1)}, \tag{2.14}$$

where we have included a symmetry factor of $\frac{1}{2}$ due to the exchange symmetry $(m \leftrightarrow n)$. The first line in (2.14) comes from the exponentiation of the first term, i.e., it is the square of (2.6) divided by 2

$$\mathcal{G}_2(z)|_{\frac{H^2h^2}{c^2}} = \frac{1}{2}\left(\frac{2Hh}{c}f_2\right)^2 z^{-2h}. \tag{2.15}$$

The second line in (2.14) can be written as a sum of products of functions $f_a f_b$ such that $a+b=4$ in the following way

$$\mathcal{G}_2(z)|_{\frac{H^2h}{c^2}} = z^{-2h}\frac{2H^2h}{c^2}\left[-f_2^2 + \frac{6}{5}f_1f_3\right] \tag{2.16}$$

as was noted in [23].

The relative coefficient between the terms in the bracket of (2.16) is precisely such that in the limit $z \to 1$ the coefficient in front of $\log^2(1-z)$ vanishes and (2.16) behaves as

$$\mathcal{G}_2(z)|_{\frac{H^2h}{c^2}} \underset{z\to 1}{\approx} \log(1-z). \tag{2.17}$$

In [12] it was noticed that this behavior persists to all orders, *i.e*, the coefficients of all the $\log^p(1-z)$ with $p>1$ vanish in the limit $z \to 1$ and hence $\mathcal{G}_2(z)$ has a simple logarithmic behavior in this limit. Moreover, the authors of [12] observed that the coefficients in front of the $\log(1-z)$ terms at each order in $\frac{H}{c}$ form the Catalan numbers' sequence. In the following sections we will see a similar statement being true for $\mathcal{W}_{N=3,4}$ vacuum blocks.[10]

2.2 $\mathcal{W}_3$ vacuum block

In an effort to elucidate the connection between the structure of the vacuum block in the $\frac{H}{c}$ expansion and the underlying symmetry algebra, we consider now a 2d CFT with a spin-3 current $W(z)$. The spin-3 modes are defined by

$$W(z) = \sum_n W_n z^{-n-3}, \tag{2.18}$$

and satisfy the $\mathcal{W}_3$ algebra

$$\begin{aligned}
[L_m, W_n] &= (2m-n)W_{m+n},\\
[W_m, W_n] &= \frac{c}{360}m\left(m^2-1\right)\left(m^2-2^2\right)\delta_{m+n}+\\
&\quad + (m-n)\left[\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right]L_{m+n}\\
&\quad + \frac{16}{22+5c}(m-n)\,\Lambda_{m+n},
\end{aligned} \tag{2.19}$$

[10]We expect this to be true for arbitrary N.

where $\Lambda_m = \sum_p : L_{m-p}L_p : -\frac{3}{10}(m+2)(m+3)L_m$. The spin-3 current $W(z)$ is a primary operator normalised so that $\langle W(z)W(0)\rangle = \frac{c}{3z^6}$. Note that the non-linear terms in (2.19) are suppressed in the large-c limit.

We will study the $\mathcal{W}_3$ vacuum block $\mathcal{G}_3$ contribution to the four point function of pairwise identical scalars $\mathcal{O}_H$ and $\mathcal{O}_L$. These are $\mathcal{W}_3$ primaries and have conformal weights H and h, as before, as well spin-3 charges $\pm w_H$ and $\pm w$, respectively, with the following scaling as $c \to \infty$:[11]

$$w_H \gg H, h, w, \qquad \frac{w_H}{c} = \text{fixed}. \tag{2.20}$$

As we will see, the contribution from the pure Virasoro modes considered in the previous section is suppressed compared to that containing the spin-3 charge modes of the "heavy" operator and is due to states of the schematic form $W_{-m_1}\dots W_{-m_i}L_{-n_1}\dots L_{-n_j}|0\rangle$. To evaluate the contribution of such states explicitly, we need to construct an orthogonal basis using the algebra (2.19) and find the commutator $[W_m, \mathcal{O}]$.

Consider first the commutator $[W_m, \mathcal{O}]$. This is determined by the singular terms in the OPE

$$\begin{aligned} W(z)\,\mathcal{O}(0)\,|0\rangle &= z^{-3}W_0|h,w\rangle + z^{-2}W_{-1}|h,w\rangle + z^{-1}W_{-2}|h,w\rangle + \mathcal{O}\left(z^0\right) \\ &= z^{-3}w\mathcal{O}|0\rangle + z^{-2}\left(\mathcal{O}_{h+1} + \frac{3w}{2h}\partial\mathcal{O}\right)|0\rangle \\ &\quad + z^{-1}\left(\mathcal{O}_{h+2} + \frac{2}{h+1}\partial\mathcal{O}_{h+1} + \frac{3w}{h\,(2h+1)}\partial^2\mathcal{O}\right)|0\rangle + \dots, \end{aligned} \tag{2.21}$$

where $\mathcal{O}_{h+1}$ and $\mathcal{O}_{h+2}$ are quasi-primary operators with conformal weight $h+1$ and $h+2$, respectively, and are given by

$$\begin{aligned} \mathcal{O}_{h+1}(0)\,|0\rangle &:= \left[W_{-1}\mathcal{O} - \frac{3w}{2h}L_{-1}\mathcal{O}\right]|0\rangle, \\ \mathcal{O}_{h+2}(0)\,|0\rangle &:= \left[W_{-2}\mathcal{O} - \frac{2}{h+1}L_{-1}\mathcal{O}_{h+1} - \frac{3w}{h\,(2h+1)}L_{-1}^2\mathcal{O}\right]|0\rangle. \end{aligned} \tag{2.22}$$

Being quasi-primaries, they satisfy $[L_1, \mathcal{O}_{h+1}(0)] = [L_1, \mathcal{O}_{h+2}(0)] = 0$ which can be verified using the algebra (2.19). The commutator $[W_n, \mathcal{O}]$ can be found using translation invariance, multiplying with $\int_{\mathcal{C}(z)} \frac{dw}{2\pi i} w^{n+2}$ and using the OPE (2.21)

$$\begin{aligned} [W_m, \mathcal{O}(z)] =& \frac{w\,(m+1)\,(m+2)}{2} z^m\mathcal{O}(z) + (m+2)\,z^{m+1}\left(\mathcal{O}_{h+1}(z) + \frac{3w}{2h}\partial\mathcal{O}(z)\right) \\ &+ z^{m+2}\left(\mathcal{O}_{h+2}(z) + \frac{2}{h+1}\partial\mathcal{O}_{h+1}(z) + \frac{3w}{h\,(2h+1)}\partial^2\mathcal{O}(z)\right). \end{aligned} \tag{2.23}$$

Consider now the contribution to $\mathcal{G}(z)$ from states $W_{-n}|0\rangle$. In order to calculate $\langle W_n\mathcal{O}(z)\mathcal{O}(0)\rangle$,[12] we note that $\langle \mathcal{O}_{h+1}(z)\mathcal{O}(0)\rangle = \langle \mathcal{O}_{h+2}(z)\mathcal{O}(0)\rangle = 0$ since these and $\mathcal{O}$

[11]In [44] it was shown that unitary representations have weight $\tilde{h} \sim c$ and therefore neither the heavy nor the light operators we consider are unitary.

[12]We denote $\mathcal{O}_L \equiv \mathcal{O}$ to simplify the notation.

are quasi-primaries with different conformal weights. It follows that only $\mathcal{O}$ and its global descendants in (2.23) contribute to $\langle W_m \mathcal{O}(z)\mathcal{O}(0)\rangle$, leading to

$$\begin{aligned}\langle W_n \mathcal{O}(z)\,\mathcal{O}(0)\rangle &= z^n\left[\frac{w}{2}(n+1)(n+2)+\frac{3w}{2h}(n+2)\,z\partial_z+\frac{3w}{h(2h+1)}z^2\partial_z^2\right]z^{-2h}\\ &=\frac{w}{2}(n-1)(n-2)\,z^{n-2h},\end{aligned}\tag{2.24}$$

where the operator at z has spin-3 charge w and the operator at 0 has charge $(-w)$. On the other hand, for the heavy part, one finds that

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)W_n\rangle = \frac{w_H}{2}(n-1)(n-2),\tag{2.25}$$

where the operator at $z=1$ carries spin-3 charge $(-w_H)$ and the one at $z\to\infty$, charge w_H. Multiplying (2.24) with (2.25), dividing with the norm given by the central term in (2.19) and summing over $n=3,4,\ldots$, one finds the expected result for the $\mathcal{W}_3$ vacuum block due to the exchange of a spin-3 quasi-primary

$$\mathcal{G}_3(z)\,|_{\frac{w_H w}{c}} = z^{-2h}\frac{90 w_H w}{c}\sum_{n=3}^{\infty}\frac{(n-1)(n-2)}{(n+1)(n+2)}\frac{z^n}{n}=\frac{3w_H w}{c}f_3(z)\,z^{-2h}.\tag{2.26}$$

Consider now states of the form $W_{-n}W_{-m}|0\rangle$. These are orthogonal to $W_{-n}|0\rangle$ since $W(0)$ does not appear in the OPE $W(z)W(0)$. On the other hand, the stress tensor appears in this OPE and the overlap $\langle L_{m+n}W_{-n}W_{-m}\rangle$ is non-zero. The overlap can be calculated using the fact that $W(z)$ is a primary field. With the help of the first line in (2.19) one finds

$$\langle L_{m+n}W_{-n}W_{-m}\rangle = \frac{c}{360}(3n+2m)\,m\left(m^2-1\right)\left(m^2-4\right).\tag{2.27}$$

Removing this overlap leads to states orthogonal to the single-mode ones

$$|Y_{m,n}\rangle = \left[W_{-n}W_{-m}-\frac{(3n+2m)\,m\,(m^2-1)\,(m^2-4)}{30\,(m+n)\left((m+n)^2-1\right)}L_{-m-n}\right]|0\rangle,\tag{2.28}$$

with norm $\mathcal{N}_{Y_{m,n}}=\langle Y_{m,n}|Y_{m,n}\rangle = (\frac{c}{360})^2 m(m^2-1)(m^2-4)n(n^2-1)(n^2-4)$. The overlap with the double-mode states $L_{-m}L_{-n}|0\rangle$ is suppressed in the large-c limit.

The next step is to compute $\langle W_m W_n \mathcal{O}(z)\mathcal{O}(0)\rangle$ using the commutator $[W_n,\mathcal{O}(z)]$ in (2.23). We find that

$$\begin{aligned}&\langle W_m W_n \mathcal{O}(z)\,\mathcal{O}(0)\rangle\\ &= z^n\left[\frac{w}{2}(n+1)(n+2)+\frac{3w}{2h}(n+2)\,z\partial_z+\frac{3w}{h(2h+1)}z^2\partial_z^2\right]\langle W_m\mathcal{O}(z)\,\mathcal{O}(0)\rangle\\ &\quad + z^{n+1}\left[(n+2)+\frac{2}{h+1}z\partial\right]\langle W_m\mathcal{O}_{h+1}(z)\,\mathcal{O}(0)\rangle\\ &\quad + z^{n+2}\langle W_m\mathcal{O}_{h+2}(z)\,\mathcal{O}(0)\rangle.\end{aligned}\tag{2.29}$$

To evaluate (2.29) one may use the commutators $[W_m,\mathcal{O}_{h+1}(z)]$ and $[W_m,\mathcal{O}_{h+2}(z)]$ which are found in appendix A. Alternatively, recall that the three-point functions $\langle W(z)\mathcal{O}_{h+1}(z)\mathcal{O}(z)\rangle$,

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and $\langle W(z)\mathcal{O}_{h+2}(z)\mathcal{O}(z)\rangle$, are fixed by conformal symmetry up to the respective OPE coefficients. This gives

$$\begin{aligned} & z^{n+1}\left[(n+2)+\frac{2}{h+1}z\partial\right]\int\frac{dz_3}{2\pi i}z_3^{m+2}\langle W\left(z_3\right)\mathcal{O}_{h+1}\left(z\right)\mathcal{O}\left(0\right)\rangle \\ & =\lambda_{W\mathcal{O}_{h+1}\mathcal{O}}\frac{m\left(m-1\right)\left(m-2\right)\left(h\left(n-2\right)+2m+n\right)}{6\left(h+1\right)}z^{m+n-2h}, \end{aligned} \tag{2.30}$$

where $\lambda_{W\mathcal{O}_{h+1}\mathcal{O}}$ is the OPE coefficient of $\mathcal{O}$ in the OPE $W\times\mathcal{O}_{h+1}$. Likewise, $\langle W_m\mathcal{O}_{h+2}(z)\mathcal{O}(0)\rangle$ is given by

$$z^{n+2}\langle W_m\mathcal{O}_{h+2}(z)\mathcal{O}(0)\rangle=\frac{\lambda_{W\mathcal{O}_{h+1}\mathcal{O}}}{24}(m-2)(m-1)m(m+1)z^{m+n-2h}. \tag{2.31}$$

The OPE coefficients are found with the help of the algebra, (2.19), by taking the limit $z\to 0$

$$\begin{aligned}\langle\mathcal{O}\left(z_3\right)W\left(z\right)\mathcal{O}_{h+1}\left(0\right)\rangle &\approx z^{-4}\langle\mathcal{O}\left(z_3\right)W_1\left(W_{-1}-\frac{3w}{2h}L_{-1}\right)\mathcal{O}\left(0\right)\rangle \\ &=z^{-4}z_3^{-2h}\left[\frac{h\left(2-c+32h\right)}{22+5c}-\frac{9w^2}{2h}\right],\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}&\langle\mathcal{O}\left(z_3\right)W\left(z\right)\mathcal{O}_{h+2}\left(0\right)\rangle \\ &\approx z^{-5}\langle\mathcal{O}\left(z_3\right)W_2\left(W_{-2}-\frac{2}{h+1}L_{-1}W_{-1}+\frac{3w}{\left(h+1\right)\left(2h+1\right)}L_{-1}^2\right)\mathcal{O}\left(0\right)\rangle \\ &=z^{-5}z_3^{-2h}\left[\frac{8h\left(6+c+8h\right)}{22+5c}-\frac{2}{h+1}\frac{4h\left(2-c+32h\right)}{22+5c}+\frac{36w^2}{\left(h+1\right)\left(2h+1\right)}\right].\end{aligned} \tag{2.33}$$

From (2.32) and (2.33) we deduce that for large-c

$$\begin{aligned}\lambda_{W\mathcal{O}_{h+1}\mathcal{O}}&=-\frac{h}{5}-\frac{9w^2}{2h}, \\ \lambda_{W\mathcal{O}_{h+2}\mathcal{O}}&=\frac{8h}{5}+\frac{8h}{5(h+1)}+\frac{36w^2}{(h+1)(2h+1)}.\end{aligned} \tag{2.34}$$

Using (2.30) and (2.31) and the OPE coefficients given in (2.34) to evaluate (2.29), we find that $\langle Y_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle$ is given by

$$\begin{aligned}\langle Y_{m,n}|\mathcal{O}\left(z\right)\mathcal{O}\left(0\right)\rangle 6=&\left[\frac{w^2}{4}\left(m-1\right)\left(m-2\right)\left(n-1\right)\left(n-2\right)\right.\\ &\left.-\frac{h}{30}\frac{m\left(m-1\right)\left(m-2\right)n\left(n-1\right)\left(n-2\right)}{\left(m+n\right)\left(m+n+1\right)}\right]z^{m+n-2h},\end{aligned} \tag{2.35}$$

with $|Y_{m,n}\rangle$ defined in (2.28). The heavy part $\langle\mathcal{O}_H(\infty)\mathcal{O}_H(1)|Y_{m,n}\rangle$ can be calculated in a similar manner,

$$\langle\mathcal{O}_H(\infty)\mathcal{O}_H(1)|Y_{m,n}\rangle=\frac{w_H^2}{4}(m-1)(m-2)(n-1)(n-2), \tag{2.36}$$

in the limit $w_H \gg H$. Multiplying (2.35) and (2.36), dividing by the norm $(\frac{c}{360})^2 m(m^2-1)(m^2-4)n(n^2-1)(n^2-4)$ and summing over $m,n=3,4,\ldots$ we determine the contribution of the states $|Y_{m,n}\rangle$ to the $\mathcal{W}_3$ vacuum block to be:

$$\mathcal{G}_3(z)\,|_{\frac{w_H^2}{c^2}} = \frac{z^{-2h}}{2}\sum_{m,n=3}^{\infty}\left[\left(\frac{90w_H w}{c}\right)^2 \frac{(m-1)(m-2)(n-1)(n-2)}{(m+1)(m+2)(n+1)(n+2)}\frac{1}{mn} - \frac{540w_H^2 h}{c^2}\frac{(m-1)(m-2)(n-1)(n-2)}{(m+1)(m+2)(n+1)(n+2)}\frac{1}{s(s+1)}\right]z^s, \tag{2.37}$$

where $s=m+n$. The first line in (2.37) is the exponentiated term analogous to the Virasoro case:

$$\mathcal{G}_3(z)\,|_{\frac{w_H^2 w^2}{c^2}} = \frac{1}{2}\left(\frac{3w_H w}{c}f_3\right)^2 z^{-2h}, \tag{2.38}$$

while the second line can be summed to

$$\mathcal{G}_3(z)|_{\frac{w_H^2 h}{c^2}} = -\frac{9w_H^2 h}{70c^2}w_3(z)z^{-2h}, \tag{2.39}$$

where $w_3(z)$ is a sum of products $f_a f_b$ with $a+b=6$:

$$\begin{aligned} w_3(z) \equiv& -14f_3^2(z)+15f_2(z)f_4(z) \\ =& 4200\sum_{m,n=3}^{\infty}\frac{(m-1)(m-2)(n-1)(n-2)}{(m+1)(m+2)(n+1)(n+2)}\frac{z^s}{s(s+1)}. \end{aligned} \tag{2.40}$$

Similar to the Virasoro case, it is easy to verify that the non-exponentiated term $w_3(z)$ behaves as $\log(1-z)$ when $z\to 1$.

We can also calculate the contribution to the $\mathcal{W}_3$ vacuum block from states of the form $\left[L_{-m}W_{-n} - \frac{\langle W_{m+n}L_{-m}W_{-n}\rangle}{\langle W_{m+n}W_{-m-n}\rangle}W_{-m-n}\right]|0\rangle$. This results in a term that contributes to exponentation and takes the form $\propto \frac{w_H H w h}{c^2}f_2 f_3$, as well as a term $\propto \frac{w_H H w}{c^2}(f_1 f_4 - \frac{7}{9}f_2 f_3)$. Such terms are subleading in the limit $w_H \gg H$ (see appendix A.1 for further details).

3 Generalized Catalan numbers and differential equations

In this section we study the logarithm of the correlator defined by $\mathcal{F}_N \equiv \log\mathcal{G}_N$. We start by reviewing the behavior of the logarithm of the Virasoro vacuum block, $\mathcal{F}_2 = \log\mathcal{G}_2$, in the limit $z\to 1$, the appearance of the Catalan numbers's sequence, and the differential equation satisfied by $\mathcal{F}_2$, following [12]. Next, we focus on the case $N=3$ where a very similar story emerges. Besides a certain generalization of the Catalan sequence, we also find a set of diagrammatic rules governing the expansion of the $\mathcal{W}_3$ vacuum block along with a differential equation satisfied by $\mathcal{F}_3$ for certain ratios of the values of the charges of the light operators. We also consider the logarithm of the stress-tensor sector of the four-dimensional correlator in the lightcone limit, which we denote by $\mathcal{G}_{d=4}$ and $\mathcal{F}_{d=4}$ respectively. We investigate the behavior in the limit $z\to 1$ and observe similarities with the two-dimensional cases when $\Delta_L \to 0$.

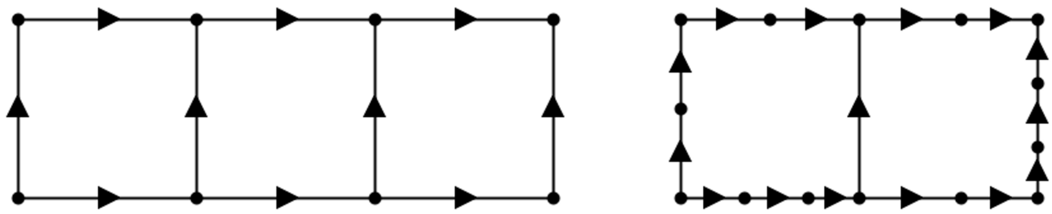

Figure 1. Posets denoted by $G([0,0,0,0],[0,0,0],[0,0,0])$ and $G([1,0,2],[1,1],[2,1])$, respectively.

3.1 The Virasoro vacuum block

In [12] it was shown how one can derive a differential equation satisfied by the logarithm of the Virasoro vacuum block, by studying its behavior in the $z \to 1$ limit. Expanding $\mathcal{F}_2$ in powers of h_H/c the authors of [12] observed that $\mathcal{F}_2$ behaves logarithmically when $z \to 1$. Furthermore they noticed that the sequence of the numerical coefficients multiplying the logarithm at the each order forms the sequence of Catalan numbers given by $c_{2,k}$:

$$c_{2,k} = \frac{\Gamma(2k-1)}{\Gamma(k)\Gamma(k+1)}, \qquad k \geq 1. \tag{3.1}$$

These numbers are generated by the following generating function

$$B_2(x) = \sum_{k=1}^{\infty} c_{2,k} x^k = \frac{1-\sqrt{1-4x}}{2}, \tag{3.2}$$

which satisfies

$$B_2(x) = B_2(x)^2 + x. \tag{3.3}$$

The Catalan numbers $c_{2,k}$ are known to appear in various problems in combinatorics. Here we would like to point out that they can also be understood as the numbers of linear extensions of one-level grid posets[13] $G([0^{k-1}],[0^{k-2}],[0^{k-2}])$, for $k \geq 1$. Generally, one-level grid-like posets $G[\mathbf{v},\mathbf{t},\mathbf{b}]$, where $\mathbf{v} = (v_1,\ldots,v_n)$, $\mathbf{t} = (t_1,\ldots,t_{n-1})$ and $\mathbf{b} = (b_1,\ldots,b_{n-1})$, can be represented with Hasse diagrams of the following type:

The numbers v_i denote the number of nodes in the i-th vertical edge, t_i denote the number of nodes in the i-th top edge and b_i denote the number of nodes in the i-th bottom edge, with the endpoints excluded. The Catalan numbers are the numbers of linear extensions of posets of the type depicted in the left Hasse diagram of figure 1.

The logarithm of the correlator $\mathcal{F}_2(z) = \log \mathcal{G}_2(z)$ when $z \to 1$ therefore behaves as

$$\mathcal{F}_2(z) \underset{z\to 1}{\approx} -2hB_2(x)\log(1-z), \tag{3.4}$$

with $x = 6\frac{h_H}{c}$. Inspired by (3.3) and (3.4) the authors of [12] find a differential equation satisfied by $\mathcal{F}_2(z)$ for all z:

$$\frac{1}{2h}\frac{d^2}{dz^2}\mathcal{F}_2(z) = \frac{1}{4h^2}\left(\frac{d}{dz}\mathcal{F}_2(z)\right)^2 + \frac{x}{(1-z)^2}. \tag{3.5}$$

[13] Partially ordered sets (posets) have a notion of ordering between some of the elements but not necessarily all of them. A linear extension of a partial ordering is a linear extension to a totally ordered set where all the elements are ordered in such a way that the original partial ordering is preserved.

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3.2 The $\mathcal{W}_3$ vacuum block

Here we uncover a similar story for the $\mathcal{W}_3$ vacuum block $\mathcal{G}_3$. Expanding in powers of $\frac{w_H}{c}$,

$$\log \mathcal{G}_3 \equiv \mathcal{F}_3(z) = \sum_{k=0}^{\infty} \left(\frac{w_H}{c}\right)^k \mathcal{F}_3^{(k)}(z), \tag{3.6}$$

with

$$\mathcal{F}_3^{(0)}(z) = -2h\log(z), \tag{3.7}$$

and using the exact expression known for the $\mathcal{W}_3$ vacuum block (see for example eq. (4.24) in [41]) one finds that

$$\begin{aligned}&\left\{ \lim_{z\to 1} \left(-\frac{\mathcal{F}_3^{(k)}(z)}{6^{k+1}\log(1-z)}\right) \middle| k = 1, 2, \dots \right\}\\ &= w \times \{1, n, 16, 35n, 768, 2002n, 49152, 138567n, \dots\},\end{aligned} \tag{3.8}$$

where we set $n \equiv h/w$. $\mathcal{F}_3$ in the limit $z \to 1$ is given by

$$\mathcal{F}_3(z) \underset{z\to 1}{\approx} -6w\log(1-z)B_3(x,n), \tag{3.9}$$

where $B_3(x,n)$ is the generating function of the sequence (3.8)

$$B_3(x,n) = \sum_{k=1}^{\infty} c_{3,k}x^k = \frac{1}{6}\left(\sqrt{3}\sin\left(\frac{1}{3}\arcsin\left(6\sqrt{3}x\right)\right) - n\cos\left(\frac{1}{3}\arcsin\left(6\sqrt{3}x\right)\right) + n\right). \tag{3.10}$$

Remarkably, there exist exactly three values of n for which $B_3(x,n)$ satisfies a cubic equation; these are $n = \pm 3$ and $n = 0$. For these values of the ratios of the light charges, the $\mathcal{W}_3$ vacuum block simplifies dramatically; it can be expressed in terms of a single function of z raised to a given power.[14]

For $n = \pm 3$ the sequence of (3.8) reduces to

$$\begin{aligned}&\left\{ \lim_{z\to 1} \left[-\left(\pm\frac{1}{6}\right)^{k+1} \frac{\mathcal{F}_3^{(k)}(z)}{\log(1-z)}\right] \middle| k = 1, 2, \dots \right\}\\ &= w \times \{1, \pm 3, 16, \pm 105, 768, \pm 6006, 49152, \pm 415701, \dots\}.\end{aligned} \tag{3.11}$$

Each term in this sequence can be derived from the following formula

$$c_{3,k} = \frac{(\pm 2)^{k-1}(3k-3)!!}{k!\,(k-1)!!}, \qquad k \geq 1. \tag{3.12}$$

Moreover, one can check that function (3.10) with $n = \pm 3$ satisfies the following relation

$$B_3(x,\pm 3) = -2B_3(x,\pm 3)^3 \pm 3B_3(x,\pm 3)^2 + x. \tag{3.13}$$

[14]For other values of n the generating function satisfies a sixth order algebraic equation. As a result writing a differential equation becomes cumbersome.

with $x = 6\frac{w_H}{c}$. Inspired by (3.13) we search for a cubic differential equation satisfied by $\mathcal{F}_3(z)$. It is easy to see, using the exact expression for the $\mathcal{W}_3$ block given for example in eq. (4.24) of [41], that $\mathcal{F}_3(z, n = 3) \equiv \hat{\mathcal{F}}_3(z)$ satisfies the following differential equation

$$\frac{1}{6w}\frac{d^3}{dz^3}\hat{\mathcal{F}}_3(z) = -\frac{1}{54w^3}\left(\frac{d}{dz}\hat{\mathcal{F}}_3(z)\right)^3 + \frac{1}{6w^2}\left(\frac{d^2}{dz^2}\hat{\mathcal{F}}_3(z)\right)\left(\frac{d}{dz}\hat{\mathcal{F}}_3(z)\right) + \frac{2x}{(1-z)^3}. \tag{3.14}$$

When $\frac{h}{w} = -3$ a similar equation can be found by taking $w \to -w$ and $1 - z \to \frac{1}{1-z}$. The case $n = 0$ is special and is discussed in appendix B.

3.2.1 Diagrammatic rules for the $\mathcal{W}_3$ block

Here we formulate diagrammatic rules for computing the logarithm of $\mathcal{W}_3$ vacuum block $\mathcal{F}_3(z) = \log \mathcal{G}_3(z)$, in the limit where $w_H \sim c \gg 1$ and all other charges are parametrically suppressed. The ratio of the charges of the light operator, n, is left arbitrary. The rules are similar to those in [12] for computing the logarithm of the Virasoro vacuum block.

We now have cubic and quartic vertices and the exchanged states are modes of the stress tensor and spin-3 current, which we refer to collectively as currents. The only relevant diagrams in the limit we consider, are those where a single propagator connects to the light operator $\mathcal{O}_L$. The rules can be stated as follows:

1. Label the k initial currents connected to operator $\mathcal{O}_H$ with integers $a_1, a_2, \ldots, a_k$.

2. Draw all diagrams where the k initial currents combine via 3-pt and 4-pt vertices to become a single current, which connects with the light operators.

3. For each propagator define its momentum p as the sum of the a_i flowing through it. Momentum is conserved at vertices. Each propagator comes with a factor
$$\frac{1}{(p+1)(p+2)}.$$

4. For each vertex coupling a current of momentum a_i to the external operator $\mathcal{O}_H$, include a factor of
$$\frac{w_H}{\sqrt{c}}(a_i - 1)(a_i - 2).$$

5. For each vertex coupling a current of momentum p to the external operator $\mathcal{O}_L$, include a factor of
$$\frac{1}{6\sqrt{c}}\Big((-1)^k(h - 3w) + h + 3w\Big)(p - 1)(p - 2).$$

6. For each 4-current vertex, include a factor of $-2/3c$. For each 3-current vertex, where two currents carry momentum m and n, while the third current carries momentum $m + n$ (see figure 3), include a factor of
$$\frac{1}{\sqrt{c}}(m + n + 2).$$

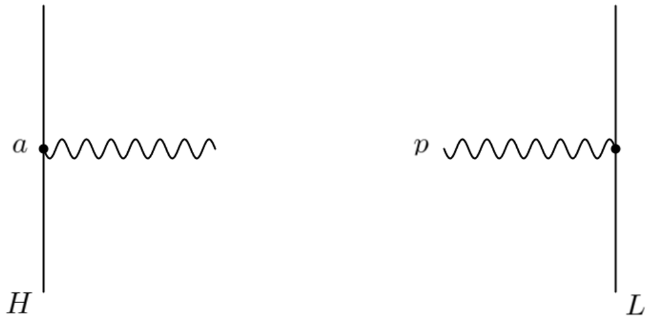

Figure 2. Vertices denoting the coupling of an exchanged current with the external states $\mathcal{O}_H$ and $\mathcal{O}_L$, respectively.

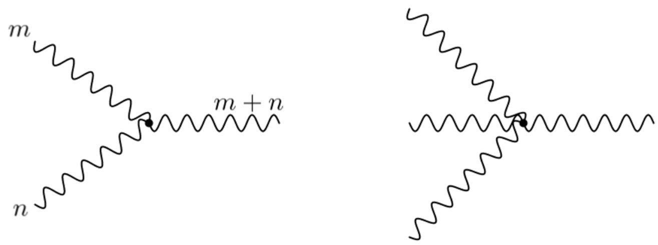

Figure 3. Vertices denoting 3-pt and 4-pt coupling of currents, respectively.

7. Take the product of the propagators and vertices and then multiply the result by

$$\frac{36^k}{k!}\frac{z^s}{s(s-1)(s-2)},$$

where $s = \sum_{i=1}^{k} a_i$.

8. Sum the resulting tree diagrams over all a_i from 3 to ∞ to obtain the $\frac{w_H^k}{c^k}$ term in $\mathcal{F}(z)|_{\mathcal{W}_3}$.

At orders w_H/c and w_H^2/c^2 there is just one diagram to take into account, while at order w_H^3/c^3 there are two different types of diagrams, see figure 4 and 5. This way, one obtains the expansion of the logarithm of $\mathcal{W}_3$ vacuum block, which is given by eq. (4.24) in [41].[15]

[15]We explicitly checked this up to $\mathcal{O}(w_H^4/c^4)$.

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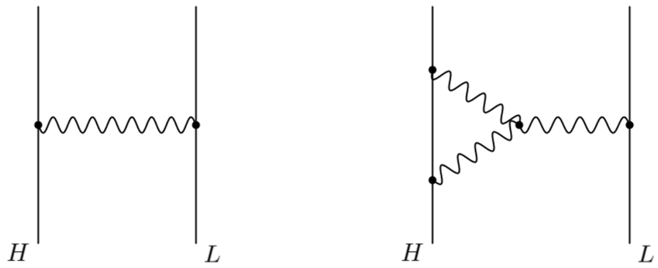

Figure 4. Diagrams at orders w_H/c and w_H^2/c^2, respectively.

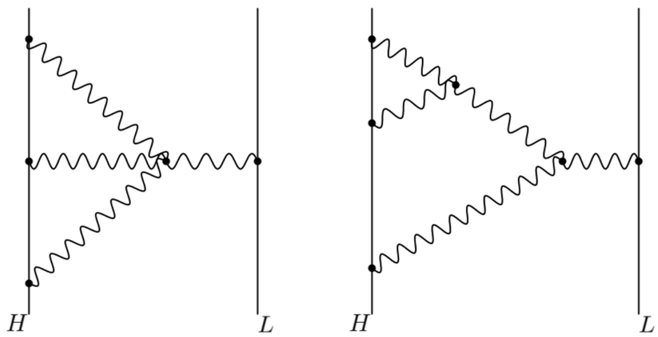

Figure 5. Diagrams at order w_H^3/c^3.

3.3 Stress tensor sector in $d = 4$

The stress tensor sector of the HHLL correlator in four-dimensional spacetime and in the lightcone limit ($\bar{z} \to 0$) is given according to [29] by

$$\mathcal{G}_{d=4}(z,\bar{z}) = \frac{1}{(z\bar{z})^{\Delta_L}} \left(1 + \sum_{k=1}^{\infty} \mu^k \bar{z}^k \mathcal{G}_{d=4}^{(k)}(z)\right), \tag{3.15}$$

$$\mathcal{G}_{d=4}^{(k)}(z) = \sum_{\{i_p\}} a_{i_1\dots i_k} f_{i_1}(z)\dots f_{i_k}(z), \tag{3.16}$$

where the sum goes over all sets of $\{i_p\}$ with $i_p \leq i_{p+1}$ and $a_{i_1\dots i_k}$ coefficients that depend on Δ_L, and the expansion parameter μ is given by

$$\mu \equiv \frac{160}{3}\frac{\Delta_H}{C_T}. \tag{3.17}$$

Explicit expressions for $\mathcal{G}_{d=4}^{(k)}$ with $k = 1, 2, 3$ are given in [29]. There it was also shown that $\mathcal{G}_{d=4}(z,\bar{z})$ can be written as

$$\mathcal{G}_{d=4}(z,\bar{z}) = e^{\Delta_L \mathcal{F}_{d=4}(z,\bar{z})}, \tag{3.18}$$

$\mathcal{F}_{d=4}(z,\bar{z})$ being of $\mathcal{O}(1)$ in the limit $\Delta_L \to \infty$ and which can be expanded as follows

$$\mathcal{F}_{d=4}(z,\bar{z}) = \mathcal{F}_{d=4}^{(0)}(z,\bar{z}) + \sum_{k=1}^{\infty} \mu^k \bar{z}^k \mathcal{F}_{d=4}^{(k)}(z). \tag{3.19}$$

with $\mathcal{F}^{(k)}_{d=4}$ being schematically of the same form as the $\mathcal{G}^{(k)}_{d=4}$ in (3.16). For $k = 0, 1, 2, 3$ for instance, we have

$$\begin{aligned}
\mathcal{F}^{(0)}_{d=4}(z, \bar{z}) &= -\log(z\bar{z}), \\
\mathcal{F}^{(1)}_{d=4}(z) &= \frac{1}{120} f_3(z), \\
\mathcal{F}^{(2)}_{d=4}(z) &= \frac{(12 - 5\Delta_L) f_3(z)^2 + \frac{15}{7}(\Delta_L - 8) f_2(z) f_4(z) + \frac{40}{7}(\Delta_L + 1) f_1(z) f_5(z)}{28800(\Delta_L - 2)}, \\
\mathcal{F}^{(3)}_{d=4}(z) &= b_{117} f_1^2(z) f_7(z) + b_{126} f_1(z) f_2(z) f_6(z) + b_{135} f_1(z) f_3(z) f_5(z) \\
&\quad + b_{225} f_2^2(z) f_5(z) + b_{234} f_2(z) f_3(z) f_4(z) + b_{333} f_3^3(z),
\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
b_{117} &= \frac{5(\Delta_L + 1)(\Delta_L + 2)}{768768(\Delta_L - 2)(\Delta_L - 3)}, \\
b_{126} &= \frac{5(5\Delta_L^2 - 57\Delta_L - 50)}{6386688(\Delta_L - 2)(\Delta_L - 3)}, \\
b_{225} &= -\frac{7\Delta_L^2 - 51\Delta_L - 70}{2903040(\Delta_L - 2)(\Delta_L - 3)}, \\
b_{135} &= -\frac{11\Delta_L^2 - 19\Delta_L - 18}{1209600(\Delta_L - 2)(\Delta_L - 3)}, \\
b_{234} &= \frac{(\Delta_L - 2)(\Delta_L + 2)}{1209600(\Delta_L - 2)(\Delta_L - 3)}, \\
b_{333} &= \frac{7\Delta_L^2 - 18\Delta_L - 24}{2592000(\Delta_L - 2)(\Delta_L - 3)}.
\end{aligned} \tag{3.21}$$

Inspired by the two-dimensional case, we consider the $\mathcal{F}^{(k)}_{d=4}(z)$ in the limit $z \to 1$. We observe that all terms proportional to $\log^i(1-z)$ with $i \geq 2$ vanish in this limit as long as $\Delta_L \to 0$. In this special case, one can show that

$$\left\{ \lim_{z\to 1, \Delta_L \to 0} \frac{(-4)^k (k!) \mathcal{F}^{(k)}_{d=4}(z)}{\log(1-z)} \middle| k = 1, 2, 3, 4, 5, \dots \right\} = \{1, 1, 6, 71, 1266, \dots\}. \tag{3.22}$$

The sequence of numbers in the (3.22) is known as the number of linear extensions of the one-level grid poset $G[(1^{k-1}), (0^{k-2}), (0^{k-2})]$, for $k \geq 1$, given by A274644 in [43]. As an example, the $k = 5$ case is represented by the Hasse diagram in figure 6. We do not explicitly discuss it here but the relevant posets in even number of dimension d are $G[(\frac{d-2}{2})^{k-1}, (0)^{k-2}, (0^{k-2})]$. The generating functions and the general formulas for the numbers of linear extensions of posets $G[(\frac{d-2}{2})^{k-1}, (0)^{k-2}, (0^{k-2})]$ are not (currently) known.

4 Discussion

We consider the $\mathcal{W}_N$ vacuum block contributions to heavy-heavy-light-light correlators in two-dimensional CFTs with higher-spin symmetries. We perform explicit mode calculations

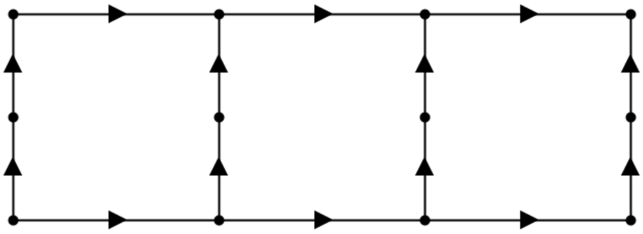

Figure 6. The poset denoted by $G([1,1,1,1],[0,0,0],[0,0,0])$.

for $\mathcal{W}_3$ and $\mathcal{W}_4$ blocks and show that they reproduce the semi-classical vacuum blocks whose explicit form can be found in e.g. [40]. We observe that terms in the expansion of these blocks in powers of $(q_H^{(i)}/c)$ satisfy the suitably modified ansatz which was used to compute the stress tensor sector of the $d=4$ HHLL correlator in [29].

The HHLL Virasoro vacuum block is governed by the Catalan numbers whose generating function satisfies a quadratic equation allowing the construction of a non-linear differential equation for the logarithm of the vacuum block [12]. We show that the $\mathcal{W}_3$ and $\mathcal{W}_4$ HHLL vacuum blocks are governed by generalizations of the Catalan numbers; for certain values of the light operator charges, their generating functions satisfy cubic and quartic algebraic equations respectively. We further show that these equations uplift to non-linear differential equations satisfied by the logarithm of the blocks. What's more, the leading twist stress tensor sector of HHLL correlators in even number of spacetime dimensions d has the same structure in the limit $\Delta_L \to 0$. The relevant generalization of the Catalan numbers is now the number of linear extensions of partially ordered sets $G[(\frac{d-2}{2})^{k-1},(0)^{k-2},(0)^{k-2}]$. For $d>2$ the generating functions for these sequences are not known.

The appearance of the generating function $B_N(x)$ comes from the limit $z\to 1$ of the logarithm $\mathcal{F}_N$ of the block, where $\mathcal{F}_N \sim B_N(x)\log(1-z)$. For example, eq. (3.13) defines generalizations of Catalan numbers; this and similar equations were studied in [45]. For the $\mathcal{W}_3$ case, we observe that for generic light charges h and w, the generating function satisfies a polynomial equation of degree 6, rather than 3, which however does not take the form studied in [45]. The numbers relevant for the $d=4$ result also do not seem to come from equations of this form; it would be interesting to understand this better.

Note that in the $d=4$ case, the logarithm of the minimal-twist stress tensor sector of HHLL correlators, $\mathcal{F}_{d=4}$, is a rational function of Δ_L which is $\mathcal{O}(1)$ for large Δ_L. An important difference with the $d=2$ $\mathcal{W}_N$ result is that in the limit $z\to 1$, at k-th order in the $\mu \simeq \frac{\Delta_H}{C_T}$ expansion, $\mathcal{F}^{(k)}_{d=4} \sim g(\Delta_L)\log^k(1-z)$ for some function $g(\Delta_L)$. However, in the limit $\Delta_L \to 0$, we do find that $\mathcal{F}_{d=4} \sim B_{d=4}(\mu)\log(1-z)$ with $B_{d=4}$ being the generating function of the number of linear extensions of the $G[(1^{k-1}),(0^{k-2}),(0^{k-2})]$ posets (this is also the number of Young tableaux with restrictions; similar numbers were recently studied in [46]).[16] If we knew an algebraic equation satisfied by $B_{d=4}$, we could perhaps construct a differential equation whose solution would give the full minimal-twist stress tensor sector in $d=4$ large-N CFTs in the limit $\Delta_L \to 0$.

[16] A similar story holds in d dimensions with the relevant poset now being $G[(\frac{d-2}{2})^{k-1},(0)^{k-2},(0^{k-2})]$.

Heavy-heavy-light-light $\mathcal{W}_N$ vacuum blocks where the spin-3 charge $q_H^{(3)} \sim c$ and $q_H^{i\neq 3} \ll c$ take a form similar to the minimal-twist stress tensor sector in four spacetime dimensions. In both cases, at order $(\frac{q_H^{(3)}}{c})^k$ in $d=2$ and order $\mu^k \simeq (\frac{\Delta_H}{C_T})^k$ in $d=4$, the result is a sum of products $f_{a_1} f_{a_2} \ldots f_{a_k}$ with $a_1 + a_2 + \ldots + a_k = 3k$. In two dimensions, we have shown how at $k=1,2$ and $N=3,4$, this follows from an explicit mode calculation and the knowledge of the higher-spin algebra. It would be interesting to understand if the $d=4$ minimal-twist stress tensor sector can also be related to an emergent symmetry algebra in the lightcone limit. Recently there have been several works devoted to the lightray operators made out of the stress tensor and to the study of the algebra of such operators [47–51]. It would be interesting to understand if there is a connection to our work.

In [21] the Chern-Simons description of pure gravity on AdS_3 and on Euclidean BTZ was related to the quantization of a certain co-adjoint orbit of the Virasoro group [52]. In this framework the HHLL Virasoro vacuum block, and corrections to it, can be computed. It would be interesting to explore a similar framework in the setup of our paper, where higher-spin currents are present.

Acknowledgments

We thank C. Banderier, K-W Huang, P. Kraus and S. Mozgovoy for discussions and correspondence. The work of R.K. and A.P. is supported in part by an Irish Research Council Laureate Award. The work of P.T. is supported in part by an Ussher Fellowship Award and by the U.S. DOE grant DE-SC00-17660.

A Some details on the calculation of the $\mathcal{W}_3$ block

We now make explicit the contribution of the operator $\mathcal{O}$ to the commutator $[W_m, \mathcal{O}_{h+j}(z)]$. To this end, consider the OPE between two quasiprimaries $\phi_i(z_1) \times \phi_j(z_2)|_{\phi^k}$:

$$\phi_i(z_1) \times \phi_j(z_2)|_{\phi^k} = \lambda_{ijk} \sum_{p=0}^{\infty} \frac{a_p(h_i, h_j, h_k)}{p!} \frac{\partial_{z_2}^p \phi^k(z_2)}{(z_1 - z_2)^{h_i + h_j - h_k - p}}, \tag{A.1}$$

where $a_p(h_i, h_j, h_k) = (h_i - h_j + h_k)_p (2h_k)_p^{-1}$. Setting $\phi_i(z_1) = W(z_1)$, $\phi_j = \mathcal{O}_{h+j}(z_2)$, $\phi^k = \mathcal{O}$ and integrating against $\int_{\mathcal{C}(z_2)} \frac{dz_1}{2\pi i} z_1^{m+2} W(z_1) \mathcal{O}_{h+j}(z_2)$ we find that

$$[W_m, \mathcal{O}_{h+j}(z_2)]|_{\mathcal{O}} = \lambda_{W\mathcal{O}_{h+j}\mathcal{O}} \int_{\mathcal{C}(z_2)} \frac{dz_1}{2\pi i} z_1^{m+2} \sum_{n=0}^{j+2} \frac{a_p(3, h+j, h) \partial_{z_2}^p \mathcal{O}(z_2)}{(z_1 - z_2)^{3+j-p} p!}, \tag{A.2}$$

and performing the integral we find that

$$[W_m, \mathcal{O}_{h+j}(z_2)]|_{\mathcal{O}} = \lambda_{W\mathcal{O}_{h+j}\mathcal{O}} \sum_{p=0}^{j+2} \frac{a_p(3, h+j, h)(m+2)!}{(m+p-j)!(j+2-p)!p!} z_2^{m+n+p-j} \partial_{z_2}^p \mathcal{O}(z_2). \tag{A.3}$$

A.1 Mixed states $W_{-n}L_{-m}|0\rangle$

We now consider the following states

$$|A_{m,n}\rangle = L_{-m}W_{-n}|0\rangle - \frac{\langle W_{m+n}L_{-m}W_{-n}\rangle}{\langle W_{n+m}W_{-n-m}\rangle}W_{-m-n}|0\rangle, \tag{A.4}$$

where (for $c \to \infty$)

$$\begin{aligned}
\langle W_{m+n}L_{-m}W_{-n}\rangle &= (3m+n)\frac{c}{360}n\left(n^2-1\right)\left(n^2-4\right), \\
\langle W_{n+m}W_{-n-m}\rangle &= \frac{c}{360}(m+n)\left((m+n)^2-1\right)\left((m+n)^2-4\right), \\
\langle W_n L_m L_{-m} W_{-n}\rangle &= \frac{c^2}{12\times 360}n\left(n^2-1\right)\left(n^2-4\right)m(m^2-1).
\end{aligned} \tag{A.5}$$

Now, one finds that $\langle A_{m,n}|\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle$

$$\begin{aligned}
\langle A_{m,n}|\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle &= \mathcal{D}_{L,m}\mathcal{D}_{W,n}\langle \mathcal{O}_L(z)\mathcal{O}_L(0)\rangle \\
&\quad - \frac{\langle W_{m+n}L_{-m}W_{-n}\rangle}{\langle W_{n+m}W_{-n-m}\rangle}\mathcal{D}_{W,m+n}\langle \mathcal{O}_L(z)\mathcal{O}_L(0)\rangle \\
&= \frac{1}{2}(m-1)(n-1)(n-2)whz^{m+n-2h}+ \\
&\quad + \frac{(m-1)m(n-2)(n-1)n(4+m+3n)}{2(m+n)(m+n+1)(m+n+2)}wz^{m+n-2h},
\end{aligned} \tag{A.6}$$

where

$$\begin{aligned}
[Q_m^{(N)}, \mathcal{O}_{h,q^{(N)}}(z)]|_{\mathcal{O}_{h,q^{(N)}}} &= q^{(N)}\int_{\mathcal{C}(z)}\frac{dz_1}{2\pi i}z_1^{m+N-1}\sum_{p=0}^{N-1}\frac{a_p(N,h,h)}{(z_1-z)^{N-p}p!}\partial_z^p\mathcal{O}_{h,q^{(N)}}(z) \\
&= q^{(N)}\sum_{p=0}^{N-1}\frac{a_p(N,h,h)}{p!}\frac{(m+N-1)!}{(N-p-1)!(m+p)!}z^{m+p}\partial_z^p\mathcal{O}_{h,q^{(N)}}(z) \\
&:= \mathcal{D}_{N,m}\mathcal{O}_{h,q^{(N)}}(z).
\end{aligned} \tag{A.7}$$

For the heavy part, we keep only the quadratic part in the charges such that

$$\lim_{z_4\to\infty} z_4^{2H}\langle \mathcal{O}_H(z_4)\mathcal{O}_H(1)|A_{m,n}\rangle = \frac{1}{2}(m-1)(n-2)(n-1)w_H H. \tag{A.8}$$

Multiplying (A.6) with (A.8) and dividing by the norm $\langle W_n L_m L_{-m} W_{-n}\rangle$ in (A.5), we find that

$$\begin{aligned}
&\sum_{m,n=2}^{\infty}\lim_{z_4\to\infty} z_4^{2h_H}\frac{\langle \mathcal{O}_H(z_4)\mathcal{O}_H(1)|A_{m,n}\rangle\langle A_{m,n}|\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle}{\langle W_n L_m L_{-m} W_{-n}\rangle}\Big|_{\frac{w_H H w h}{c^2}} \\
&= \frac{1080 w_H H w h}{c^2}z^{-2h}\sum_{m,n=2}^{\infty}\frac{(m-1)(n-1)(n-2)}{(m+1)(n+1)(n+2)}\frac{z^{m+n}}{mn} \\
&= \frac{6w_H H w h}{c^2}f_2 f_3,
\end{aligned} \tag{A.9}$$

which as expected is the "exponentiated term". On the other hand, consider

$$\sum_{m,n=2}^{\infty} \lim_{z_4\to\infty} z_4^{2H} \frac{\langle \mathcal{O}_H(z_4)\mathcal{O}_H(1)|A_{m,n}\rangle\langle A_{m,n}|\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle}{\langle W_n L_m L_{-m} W_{-n}\rangle}\Big|_{\frac{w_H H w}{c^2}}$$
$$= \frac{1080 w_H H w}{c^2} z^{-2h} \sum_{m,n=2}^{\infty} \frac{(m-1)(n-1)(n-2)}{(m+1)(n+1)(n+2)} \frac{(4+m+3n)z^{m+n}}{(m+n)(m+n+1)(m+n+2)}$$
$$\propto \frac{w_H H w}{c^2}\left(f_1 f_4 - \frac{7}{9} f_2 f_3\right). \tag{A.10}$$

Note that in both sums we have trivially extended the summation from $m \geqslant 3$ to $m \geqslant 2$.

On the other hand, by expanding the vacuum block we find precisely the same structure

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle\big|_{1_{\mathcal{W}_3},\frac{w_H H w h}{c^2}} \propto f_2 f_3,$$
$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle\big|_{1_{\mathcal{W}_3},\frac{w_H H w}{c^2}} \propto \left(f_1 f_4 - \frac{7}{9} f_2 f_3\right). \tag{A.11}$$

B $\mathcal{W}_4$ vacuum block

In this appendix we further include a spin-4 current and consider the $\mathcal{W}_4$ algebra. We will show that including a spin-4 current modifies the term proportional to $\frac{w_H^2}{c^2}$ discussed in section 2. The result can again be written as a sums of the following combination $f_a(z)f_b(z)$, with $a+b=6$. Compared to the case of $\mathcal{W}_3$, the term proportional to $\frac{w_H^2}{c^2}$ in the vacuum block will now depend also on the spin-4 charge u of the light operator.

We denote the spin-4 current by $\mathrm{U}(z)$ and the external operators carry eigenvalues $\pm u_H$ and $\pm u$. The heavy operator again has a spin-3 charge of $\mathcal{O}(c)$ while the conformal weight H and the spin-4 charge are small compared to w_H, i.e. $H, u_H \ll w_H$. In this limit, there are no new contributions due to the states $U_{-m}|0\rangle$ since they will be proportional to $\frac{u_H u}{c} f_4 z^{-2h}$, which is suppressed as $c\to\infty$. The first contribution will appear at $\mathcal{O}(\frac{w_H^2}{c^2})$ and is due to the fact that the modes $|Y_{m,n}\rangle$ are not orthogonal to $U_{-m-n}|0\rangle$. In this section we will therefore study the contribution due to the following states:

$$|\tilde{Y}_{m,n}\rangle = \left[W_{-n}W_{-m} - \frac{\langle L_{m+n}W_{-n}W_{-m}\rangle}{\langle L_{m+n}L_{-m-n}\rangle} L_{-m-n} - \frac{\langle U_{m+n}W_{-n}W_{-m}\rangle}{\langle U_{m+n}U_{-m-n}\rangle} U_{-m-n}\right]|0\rangle. \tag{B.1}$$

There are two new contributions to $\langle \tilde{Y}_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle$ compared to $\langle Y_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle$, one is simply that we need to include the last term in (B.1). The second is a correction to the OPE coefficients $\lambda_{W\mathcal{O}_{h+1}\mathcal{O}}$ and $\lambda_{W\mathcal{O}_{h+2}\mathcal{O}}$, these pick up a contribution that depends on the spin-4 charge u due to the fact that $[W_m, W_{-m}]$ contain the spin-4 zero mode U_0. Note that the heavy part remains unchanged since $w_H \gg u_H$ and is therefore given by (2.36):

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)|\tilde{Y}_{m,n}\rangle = \frac{w_H^2}{4}(m-1)(m-2)(n-1)(n-2), \tag{B.2}$$

and the norm of $|\tilde{Y}_{m,n}\rangle$ is also the same as that of $|Y_{m,n}\rangle$ (to leading order in c):

$$\mathcal{N}_{\tilde{Y}_{m,n}} = \langle \tilde{Y}_{m,n}|\tilde{Y}_{m,n}\rangle = \left(\frac{c}{360}\right)^2 m\left(m^2-1\right)\left(m^2-4\right)n\left(n^2-1\right)\left(n^2-4\right). \tag{B.3}$$

We therefore only need to calculate $\langle \tilde{Y}_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle$.

The modes U_m of $\mathrm{U}(z)$ are defined by

$$\mathrm{U}(z) = \sum_m U_m z^{-m-4}, \tag{B.4}$$

and since U is primary we know that

$$[L_m, U_n] = (3m-n)U_{m+n}. \tag{B.5}$$

Consider now various OPEs of the spin-3 and spin-4 field,[17] in terms of quasi-primaries

$$\begin{aligned} W(z)W(0) &= \frac{c}{3z^6} + \frac{2T(0)}{z^4} + \frac{\lambda_{WWU}\mathrm{U}(0)}{z^2} + \dots, \\ W(z)\mathrm{U}(0) &= \frac{\lambda_{WUW}W(0)}{z^4} + \dots, \\ \mathrm{U}(z)\mathrm{U}(0) &= \frac{c}{4z^8} + \frac{2T(0)}{z^6} + \lambda_{UUU}\frac{\mathrm{U}(0)}{z^4} + \dots, \end{aligned} \tag{B.6}$$

where $\lambda_{WUW} = \frac{3}{4}\lambda_{WWU} = \frac{3}{4}\frac{4}{\sqrt{3}}\sqrt{\frac{(2+c)(114+7c)}{(7+c)(22+5c)}} \approx \sqrt{\frac{21}{5}}$ and the ellipses denote non-linear terms that will be suppressed when $c \to \infty$. From (B.6), we can derive the commutator of the various modes. Especially, we want to consider $[W_n, U_m]$, $[W_n, W_m]$ and $[U_n, U_m]$. The last one is given by

$$\begin{aligned} [U_m, U_n] = &\frac{c}{20160}m\left(m^2-1\right)\left(m^2-4\right)\left(m^2-9\right)\delta_{m+n} \\ &+ \frac{(m-n)}{1680}\Big[3\left(m^4+n^4\right) + 4m^2n^2 - (2mn+39)\left(m^2+n^2\right) \\ &+ 20mn + 108\Big]L_{m+n} + \dots, \end{aligned} \tag{B.7}$$

while $[W_n, W_m]|_U$ is given by

$$[W_m, W_n]|_U = \lambda_{WWU}\frac{m-n}{2}U_{n+m}, \tag{B.8}$$

as well as

$$[W_m, U_n]|_W = \frac{\lambda_{WUW}}{84}\left[5m^3 + 9n - 5m^2n - n^3 - 17m + 3mn^2\right]W_{m+n} \tag{B.9}$$

Using (B.8) and (B.9), we find that

$$\begin{aligned} \langle U_{m+n}W_{-n}W_{-m}\rangle = &\frac{\lambda_{WUW}cm\left(m^2-1\right)\left(m^2-4\right)}{30240} \\ &\times\left[-9m + m^3 - 26n + 6m^2n + 14mn^2 + 14n^3\right] \end{aligned} \tag{B.10}$$

[17] See e.g. appendix A.2 in [53] for the $\mathcal{W}_4$ algebra.

and

$$\langle U_{m+n}U_{-m-n}\rangle = \frac{c}{20160}s\left(s^2-1\right)\left(s^2-4\right)\left(s^2-9\right). \tag{B.11}$$

From the three-point function $\langle \mathrm{U}(z_3)\mathcal{O}(z)\mathcal{O}(0)\rangle$ and $\lambda_{U\mathcal{O}\mathcal{O}} = u$ one finds that

$$\langle U_{m+n}\mathcal{O}(z)\mathcal{O}(0)\rangle = \frac{u}{6}(m+n-1)(m+n-2)(m+n-3)z^{m+n-2h}. \tag{B.12}$$

Lastly, we need to compute the corrections to the OPE coefficients $\lambda_{W\mathcal{O}_{h+1}\mathcal{O}}$ and $\lambda_{W\mathcal{O}_{h+2}\mathcal{O}}$. This is similar to the calculation in the $\mathcal{W}_3$ case and one finds that ($c\to\infty$, $z\to 0$)

$$\begin{aligned}\langle\mathcal{O}(z_3)\,W(z)\,\mathcal{O}_{h+1}(0)\rangle &\approx z^{-4}\langle\mathcal{O}(z_3)\,W_1\left(W_{-1}-\frac{3w}{2h}L_{-1}\right)\mathcal{O}(0)\rangle\\ &= z^{-4}z_3^{-2h}\left[-\frac{h}{5}+\lambda_{WWU}u-\frac{9w^2}{2h}\right],\end{aligned} \tag{B.13}$$

where we used $[W_1,W_{-1}] = \ldots + \lambda_{WWU}U_0$ and that $U_0\mathcal{O}(0)|0\rangle = u\mathcal{O}|0\rangle$. Likewise, one finds that

$$\begin{aligned}&\langle\mathcal{O}(z_3)W(z)\mathcal{O}_{h+2}(0)\rangle\\ &\approx z^{-5}\langle\mathcal{O}_h(z_3)\,W_2\left(W_{-2}-\frac{2}{h+1}L_{-1}W_{-1}+\left[\frac{3w}{h(h+1)}-\frac{3w}{h(2h+1)}\right]L_{-1}^2\right)\mathcal{O}(0)\rangle\\ &= z^{-5}z_3^{-2h}\left[\frac{8h}{5}+\frac{8h}{5(h+1)}+\frac{36w^2}{(2h+1)(h+1)}+2\lambda_{WWU}u-\frac{8u}{h+1}\lambda_{WWU}\right],\end{aligned} \tag{B.14}$$

to leading order when $c\to\infty$ and using $[W_2,W_{-2}]|_U = \ldots + 2U_0$. Putting this altogether gives

$$\begin{aligned}\langle\tilde{Y}_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle &= \langle Y_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle\\ &\quad+\frac{u\lambda_{WWU}(m-2)(m-1)m(n-2)(n-1)nz^{m+n-2h}}{12s(s+1)(s+2)(s+3)}\\ &\quad\times(17+2m^2+15n+2n^2+15m+9mn)+\ldots.\end{aligned} \tag{B.15}$$

Given (B.15), (B.2) and (B.3), we find the contribution to the vacuum block from the states $|\tilde{Y}_{m,n}\rangle$ proportional to u is given by

$$\mathcal{G}(z)|_{\frac{w_H^2 u}{c^2}} = \frac{37800w_H^2u\lambda_{WWU}z^{-2h}}{c^2}\left[25\tilde{w}_4(z)+3w_3(z)\right], \tag{B.16}$$

where w_3 is given by (2.40) and $\tilde{w}_4$ is a sum of products of functions f_af_b with $a+b=6$ given by

$$\begin{aligned}\tilde{w}_4 &= 3\left(-f_2f_4+\frac{4}{3}f_1f_5\right)\\ &= \sum_{m=3}^{\infty}\sum_{n=3}^{\infty}1260\frac{(m-2)(n-2)(n-1)n\left(m^2+6(n+2)(n+3)+m(9+4n)\right)}{m(n+2)(n+3)(n+4)s(s+1)(s+2)(s+3)}z^{m+n}.\end{aligned} \tag{B.17}$$

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B.1 Differential equation for the $\mathcal{W}_4$ vacuum block

Here we study the $\mathcal{W}_4$ vacuum block, or rather its logarithm, as $z \to 1$. The $\mathcal{W}_4$ HHLL vacuum block is known exactly. One can find it for instance in eq. (C.1) of [41]. In this case, we can choose to scale the spin-3 charge w_H with the central charge c — as in appendix A — with the hope of uncovering relations similar to those valid for the stress-tensor sector of the four-dimensional correlator in the light cone limit. However, we may also choose to consider the limit $u_H \sim c \gg 1$, with all other charges parametrically smaller.

Remarkably, $\mathcal{F}_4(z)$ behaves logarithmically in the limit $z \to 1$ in both cases. A sequence of numbers, the numerical coefficients of $\log(1-z)$ in the expansion of the relevant heavy charge can be determined, and a quartic differential equation satisfied by the logarithm of the block for certain ratios of the light charges can be found.

Let us first consider the scaling $u_H \sim c \gg 1$ and expand $\mathcal{F}_4(z) = \log \mathcal{G}_4(z)$ in powers of u_H/c as $\mathcal{F}_4(z) = \sum_{k=0}^{\infty} \left(\frac{u_H}{c}\right)^k \mathcal{F}_4^{(k)}(z)$ to obtain in the limit $z \to 1$:

$$\left\{ \lim_{z\to 1} \left(-\frac{\mathcal{F}^{(k)}(z)}{20 \times 6^k \log(1-z)} \right) \middle| k = 1, 2, \ldots \right\}$$
$$= u \times \{1, n-7, 458-14n, 1001n-13307, 732374-34034n, 1939938n-31667622, \ldots\}, \tag{B.18}$$

where we set

$$n = \frac{18}{5}\frac{h}{u}. \tag{B.19}$$

If $B_4(x,n)$ with $x \equiv 6\frac{u_H}{c}$ is the generating function of (B.18), then $\mathcal{F}(z)$ behaves in the limit $z \to 1$ as

$$\mathcal{F}_4(z) \underset{z\to 1}{\approx} -20u\log(1-z)B_4(x,n) \tag{B.20}$$

There exist four different values of n for which the generating function $B_4(x,n)$ satisfies a quartic equation. These are: $n = \{18, 3, -2, -12\}$.

When $n = 18$, we find the following quartic order equation for the generating function:

$$B_4(x,18) = 36B_4(x,18)^4 - 36B_4(x,18)^3 + 11B_4(x,18)^2 + x. \tag{B.21}$$

Inspired by this relation one finds that $\mathcal{F}_4(z, n = 18) \equiv \widetilde{\mathcal{F}}_4(z)$ satisfies the following differential equation

$$\widetilde{\mathcal{F}}''''(z) = 120u\left(\frac{x}{(1-z)^4} + \frac{9\widetilde{\mathcal{F}}'(z)^4}{40000u^4} - \frac{9\widetilde{\mathcal{F}}'(z)^2\widetilde{\mathcal{F}}''(z)}{2000u^3} + \frac{3\widetilde{\mathcal{F}}''(z)^2 + 4\widetilde{\mathcal{F}}'''(z)\widetilde{\mathcal{F}}'(z)}{400u^2}\right), \tag{B.22}$$

which reduces to the equation (B.21) in the limit $z \to 1$ using (B.20).

When $n = -12$ the generating function $B_4(x,-12)$ satisfies

$$B_4(x,-12) = -144B_4(x,-12)^4 - 96B_4(x,-12)^3 - 19B_4(x,-12)^2 + x, \tag{B.23}$$

whilst $\mathcal{F}_4(z, n=-12) \equiv \hat{\mathcal{F}}(z)$ is a solution of the following differential equation

$$\hat{\mathcal{F}}''''(z) = 120u\left(\frac{x}{(1-z)^4} - \frac{9\hat{\mathcal{F}}'(z)^4}{10000u^4} - \frac{3\hat{\mathcal{F}}'(z)^2\hat{\mathcal{F}}''(z)}{250u^3} - \frac{7\hat{\mathcal{F}}''(z)^2 + 6\hat{\mathcal{F}}'''(z)\hat{\mathcal{F}}'(z)}{400u^2}\right). \tag{B.24}$$

For $n=-2,3$ we find the following quartic order equations for the generating function:

$$\begin{aligned} n&=3, & B_4(x,3) &= -2304B_4(x,3)^4 + 384B_4(x,3)^3 - 4B_4(x,3)^2 + x, \\ n&=-2, & B_4(x,-2) &= 2916B_4(x,-2)^4 + 324B_4(x,-2)^3 - 9B_4(x,-2)^2 + x. \end{aligned} \tag{B.25}$$

In these cases however, the differential equations similarly constructed do not correctly reproduce the vacuum block beyond $z \to 1$ limit. This is analogous to what happens in the case of the $\mathcal{W}_3$ vacuum block for $h=0$, where the generating function satisfies

$$n=0, \qquad B_3(x,0) = 16B_3(x,0)^3 + x. \tag{B.26}$$

It is curious that these special cases correspond to values for the ratios of the light charges for which $h < w, u$.

Let us now consider the case with $w_H \sim c \gg 1$ and the other charges parametrically smaller. For notational simplicity, we will use here the same symbol $\mathcal{F}_4(z)$. We hope that this will not create any confusion. In this case, $\mathcal{F}_4(z)$ is expanded as

$$\mathcal{F}_4(z) = \sum_{k=0}^{\infty} \left(\frac{w_H}{c}\right)^k \mathcal{F}^{(k)}(z), \tag{B.27}$$

with

$$\mathcal{F}_4^{(0)} = -2h\log(z). \tag{B.28}$$

Using the exact expression for the $\mathcal{W}_4$ block one finds that

$$\left\{\lim_{z\to 1}\left(\frac{(-1)^{k+1}\mathcal{F}_4^{(k)}(z)}{2^{k+1}3^{2k}\log(1-z)}\right)\middle|k=1,2,\dots\right\}$$
$$= w\times\left\{1, \frac{2}{45}(18n+85m), 10, \frac{2}{81}(882n+2785m), 318, \frac{44}{3645}(67158n+225635m), 13620, \dots\right\}, \tag{B.29}$$

where n, m denote the ratios of the light charges $n = \frac{h}{w}$ and $m = \frac{u}{w}$, respectively. Notice that in this case ratios of both charges appear as opposed to the previous scaling for which additional simplifications occurred that eliminated w. This may be related to the fact that a spin-3 current, having odd spin, does not appear in the OPE of two spin-4 currents.

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Conformal bootstrap and thermalization in holographic CFTs

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This dissertation is submitted for the degree of

Doctor of Philosophy

September 2021

Declaration and coauthorship

I declare that this thesis has not been submitted as an exercise for a degree at this or any other university and it is entirely my own work. I agree to deposit this thesis in the open access institutional repository of the University, or allow the library to do so on my behalf, subject to Irish Copyright Legislation and Trinity College Library conditions of use and acknowledgement. I consent to the examiners retaining a copy of the thesis beyond the examining period, should they so wish. This thesis contains, among others, results of four coauthored research papers, three with Robin Karlsson, Manuela Kulaxizi and Andrei Parnachev [12-14], and one with Robin Karlsson and Andrei Parnachev [16].

Petar Tadic
September 2021

This thesis is the result of three years of PhD studies. Parts of this thesis are based on the publications [12-14], which were published in the journal JHEP. Furthermore, parts of this thesis are based on the preprint [16]. The content of [12] can be found in Section 4. The content of [13] can be found in Section 5. The content of [14] can be found in Section 6. Section 7 is based on the preprint [16].

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Summary

This thesis covers a number of topics in conformal field theories that are supposed to have gravity duals according to the AdS/CFT correspondence. We use the conformal bootstrap in the Regge and lightcone limits as the technique for studying these theories. We also explore their thermal properties by studying the large-N conformal field theories at finite temperature.

In Section 2 we review the basic implications of conformal symmetry in quantum field theories in spacetime with the number of dimensions $d \geqslant 3$ and $d = 2$ separately. We precisely define the holographic CFTs and briefly describe the idea of conformal bootstrap as the consistency condition of all conformal field theories.

In Section 3 we introduce the "heavy-heavy-light-light" correlator in Regge and lightcone limit. We review the calculation of correlators of this type and we set up the notation for the rest of the thesis.

In Section 4 we study the heavy-heavy-light-light correlation function of the holographic CFTs in the Regge limit, based on [12]. The gravitational dual of this correlator in the Regge limit is the high energy scattering of the light probe with the fixed impact parameter in the asymptotically AdS black hole background. The Schwarzschild radius of the black hole in AdS units is proportional to the ratio of the conformal dimension of the heavy operator and the central charge. This ratio serves as a useful expansion parameter whose power counts the number of stress tensors in the multi-stress tensor operators which contribute to the four-point correlation function. In the cross-channel the four-point function is determined by the OPE coefficients and anomalous dimensions of the heavy-light double-trace operators. We explain how this data can be obtained and explicitly compute the first and second order terms in the expansion of the anomalous dimensions. We observe perfect agreement with known results in the lightcone limit, which were obtained by computing perturbative corrections to the energy eigenstates in AdS spacetimes.

In Section 5 we study the heavy-heavy-light-light correlation function in the lightcone limit, based on [13]. Near-lightcone correlators are dominated by the contributions of exchanged operators with the lowest twist. We consider the contributions of such leading lowest twist multi-stress tensor operators to the

correlator in a holographic CFT of any even dimensionality. An infinite number of such operators contribute, but their sum is described by a simple ansatz. We show that the coefficients in this ansatz can be determined recursively, thereby providing an operational procedure to compute them. This is achieved by bootstrapping the corresponding near lightcone correlator: conformal data for any minimal-twist determines that for the higher-order minimal-twist contributions and so on. To illustrate this procedure in four spacetime dimensions we determine the contributions of double- and triple-stress tensors. We compute the OPE coefficients; whenever results are available in the literature, we observe the complete agreement. We also compute the contributions of double-stress tensors in six spacetime dimensions and determine the corresponding OPE coefficients. In all cases the results are consistent with the exponentiation of the near lightcone correlator. This is similar to the situation in two spacetime dimensions for the Virasoro vacuum block.

In Section 6 we generalize the technique developed in Section 5 to include the contributions of multi stress tensor operators of arbitrary twist to the heavy-heavy-light-light correlator, based on [14]. We show how one can compute the unknown coefficients in the generalized version of the ansatz from Section 5 by the lightcone bootstrap, for the entire stress tensor sector of the correlator. Therefore, iteratively computing the OPE coefficients of multi-stress tensor operators with an increasing twist. Some parameters are not fixed by the bootstrap - they correspond to the OPE coefficients of multi-stress tensors with spin zero and two. We further show that in holographic CFTs one can use the phase shift computed in the dual gravitational theory to reduce the set of undetermined parameters to the OPE coefficients of multi-stress tensors with spin zero. Finally, we verify some of these results using the Lorentzian OPE inversion formula and comment on its regime of applicability.

Finally, in Section 7 we study the thermalization of the stress tensor sector in CFTs with a large number of degrees of freedom, based on [16]. We show that the thermalization of operators from this sector, or the equality between their expectation values in heavy states and at finite temperature, is equivalent to a universal behavior of their OPE coefficients with a pair of identical heavy scalar operators. We verify this behavior in a number of examples which include holographic and free CFTs and provide a bootstrap argument for the general case. In a free CFT we check the thermalization of multi stress tensor operators directly and also confirm the equality between the contributions of multi stress tensors to heavy-heavy-light-light correlators and to the corresponding thermal light-light two-point functions by disentangling the contributions of other light operators. Unlike multi stress tensors, we show that these light operators violate the Eigenstate Thermalization Hypothesis and do not thermalize.

Dedicated to my wife Tijana, my mother Daliborka, my father Lazar,

and my brother Milorad.

Acknowledgments

First, I would like to thank my supervisor Manuela Kulaxizi for giving me the opportunity to pursue the PhD studies in the field of holography and the AdS/CFT correspondence. It has been a great honor and pleasure to work under her guidance. I benefited greatly from countless discussions we had.

We were working together with Robin Karlsson and Andrei Parnachev and I owe them as well a huge thanks for all discussions we had at our regular meetings and beyond them. I am grateful to all of them for the open atmosphere in the group and for the mathematical and physical knowledge and intuition they unselfishly shared with me.

My thanks to Manuela and Andrei go beyond the realm of academia, as they helped me greatly in a very hard time I had during a part of my PhD studies.

I am thankful for insightful discussions with the professors at the School of Mathematics of Trinity College Dublin, such as Sergey Frolov, Tristan McLoughlin and Marius de Leeuw. I am also grateful for the many high-quality seminars, journal clubs, and events that were held at the School of Mathematics.

I would also like to thank Guy Jehu, Gim Seng Ng, Andrea Orta, Christiana Pantelidou, Raul Pereira, Ana Lucia Retore, Johannes Aspman, Elias Furrer, Riccardo Gonzo, Anton Pribytoks, Paul Ryan, Anne Spiering, Aleksandar Bukva and Ilija Burić for many discussions on various topics in physics and mathematics that have truly widen my horizons in science.

Finally, I have to thank my family, my mother Daliborka and father Lazar who thought me about the value of the proper education and always supported me in pursuing my goals and interests. I have to thank my brother Milorad for being supportive and helping me on various occasions. Last, but not least, my wife Tijana, whose love and support through the three years of a long-distance relationship between Dublin and Belgrade, as well as Oxford and Belgrade, were great motivators for me.

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1. Motivation

Constructing the theory of quantum gravity has been an open problem for many years. The lack of experimental data at energies so high that the quantum-gravitational effects are detectable implies the necessity for alternative ways for learning about quantum gravity. Modern approaches to this problem include using the dualities between gravitational theories and theories without gravitational degrees of freedom. The term "duality" means that theories have equivalent Hilbert spaces and dynamics. Their mathematical descriptions, on the other hand, can differ, for example, they can have different Lagrangians, degrees of freedom, and be situated in different spacetimes. The hope with this type of duality is that one can indirectly approach the regime where the quantum effects in gravity are important using the dual, non-gravitational description and therefore learn the general properties of quantum gravity. Knowing these properties will help the achievement of the ultimate goal, i.e. the construction of the fundamental microscopic theory of quantum gravity.

1.1. Holographic principle and AdS/CFT duality

The works of Bekenstein and Hawking [1-3] gave the first indirect hint of the existence of dualities that include gravitational theories. They showed that black holes are dynamic objects that emit thermal radiation. The entropy of a neutral, non-rotating black hole is shown to be proportional to the area of the horizon A:

$$S_{BH} = \frac{Ac^3}{4G\hbar}. \tag{1.1}$$

Colloquially, one can interpret this entropy as being proportional to the amount of information in the physical system. Naively, one might have expected that it scales with the volume of the space behind the horizon (or volume of the black hole). The fact that it scales with the area of the horizon was the first indication that the gravitational theory where the black hole is present has the degrees of freedom that fit in spacetime with one spatial dimension less. This was the first sign of the so-called holographic principle of gravitational theories [4-5].

Another important indication of the existence of dualities between gravitational and non-gravitational theories came from the work of ’t Hooft [6], where he studied non-Abelian gauge theories in the limit of a large number of colors (large-N limit). He showed that in this limit the Feynman diagrams rearrange in a such way that the expansion looks the same as the perturbative expansion in string theory with string coupling $1/N$. Since the string theories necessarily include gravity, this was strong evidence of the relation between gravitational string theories and non-gravitational gauge theories at the deep fundamental level.

The work of Brown and Henneaux [7] gave further insight into what kind of theories one should be looking at to find the dual descriptions of the gravitational theories. Namely, by studying gravity in three-dimensional Anti de-Sitter spacetime, they found the conformal symmetry algebra of conformal field theory in two-dimensional spacetime as the algebra of asymptotic symmetries in three-dimensional gravity. This was the first evidence of the relation between gravitational theories in Anti-de Sitter spacetime and the conformal field theories in spacetime with one spatial dimension less.

The first concrete instance of the duality between gravitational and non-gravitational theories was established in the work of Maldacena [8] in 1997. By studying black branes in supersymmetric string theory he proposed the famous anti-de Sitter/conformal field theory (AdS/CFT) conjecture which states that type IIB string theory with string length $l_s = \sqrt{\alpha'}$ and coupling constant g_s on $AdS_5 \times S^5$ with radius of curvature L of both five-dimensional Anti-de Sitter AdS_5 and sphere S^5, is dual to $\mathcal{N} = 4$ super-Yang-Mills (SYM) in flat four-dimensional spacetime with gauge group $SU(N)$ and gauge coupling g_{YM}. The parameters of these theories are related by the following equations

$$g_{YM}^2 = 2\pi g_s, \qquad g_{YM}^2 N = \frac{L^4}{2\alpha'^2}. \tag{1.2}$$

The $\mathcal{N} = 4$ gauge theory is conformally invariant which is the reason why it is called the "CFT side" of the duality. The gravitational part (type IIB string theory) of the duality is usually called the "AdS side", or simply the "gravity side".

The string theory reasoning behind this duality is the equivalence between the open and closed string descriptions of the Dirichlet branes (D-branes). These are the non-perturbative higher-dimensional objects present in the superstring theory. They can be viewed in the open and closed string perspectives, and which one is right depends on the value of the string coupling g_s. In terms of open strings, D-branes might be viewed as the higher-dimensional objects where the open strings end. Since the open strings must be treated as small perturbations that do not affect the gravitational background, this description is only valid when the coupling between open and closed strings g_s is small $g_s \ll 1$. Furthermore, if we neglect the massive string excitations by focusing on the low energy regime, the dynamics of the D-branes will be described in terms of the pure supersymmetric gauge theory. If there are N coincident branes, the gauge group will be[1] $SU(N)$ and the effective 't Hooft coupling will be $g_s N$. It follows that this description works for $g_s N \ll 1$. In terms of the closed strings, the D-branes can be viewed as the solitonic solutions of the low-energy limit of the string theory, i.e. the supergravity, and they represent the source of the gravitational field, therefore, they curve the spacetime around them. For the supergravity approximation to hold, there must be a scale separation between the characteristic length L of the spacetime considered and the string length $\sqrt{\alpha'}$, or in other words, $L^4/\alpha'^2 \to \infty$. In the case of N coincident branes, $L^4/\alpha'^2 \propto g_s N \gg 1$. Therefore, this description is valid in the opposite limit compared to the open string description, $g_s N \gg 1$.

One concludes that there are two, very different theories with different degrees of freedom and that even live in different spacetimes, but still describe the dynamics of the same physical objects, the D-branes, in the different limits of the effective coupling. The conjectured part of the duality is that the respective descriptions are valid even beyond the limits specified above and one can relate the parameters of these two theories by (1.2). Concretely, the candidate for the theory of quantum gravity (type IIB string theory) can be mapped to the $\mathcal{N} = 4$ SYM gauge theory without gravitational degrees of freedom. Additionally, the information about gravitational theory in five-dimensional spacetime (obtained after Kaluza-Klein reduction of type IIB string theory on five-dimensional sphere S^5) is mapped to the gauge conformal field theory in four-dimensional spacetime, following the idea from the works of Hawking and Bekenstein and therefore satisfying the holographic principle. In the rest of the thesis, we do not

[1] Actually, the gauge group is $U(N)$, but it turns out that the $U(1) \subseteq U(N)$ degrees of freedom decouple from the $SU(N)$ degrees of freedom.

use the string theory language behind the AdS/CFT and for the more detailed discussion of the duality, we refer the reader to the original papers [8-10].

The duality conjecture as stated above is the strongest version of the conjecture as it is assumed to hold for all values of parameters in both theories. Unfortunately, studying and proving the duality at a generic value of parameters is not feasible at the moment as one would have to study string theory away from the classical limit in gravity and gauge theory at an arbitrary value of coupling constant. This is the reason why the practical exploration of duality is mostly limited to the classical limit in the string theory, intending to go beyond it.

The classical limit in gravity is the limit where the string length, in units of AdS radius $\sqrt{\alpha'}/L$, and the string coupling g_s go to zero $\sqrt{\alpha'}/L \to 0$, $g_s \to 0$. This implies that the limit of classical gravity is equivalent to the limit of infinite 't Hooft coupling $\lambda = g_{YM} N^2 \to \infty$ and infinite number of colors $N \to \infty$ in $\mathcal{N} = 4$ SYM gauge theory according to eqs. (1.2). Therefore, the classical limit in gravity is dual to the limit of infinite coupling in the dual non-gravitational theory.

The idea of holographic duality is generalized using the previous example. The general statement of the AdS/CFT correspondence is that a gravitational theory in $(d+1)$-dimensional asymptotically Anti-de Sitter spacetime is dual to a certain, strongly coupled conformal field theory in d-dimensional flat spacetime. This d-dimensional flat spacetime where the CFT is present is the boundary of AdS spacetime [9-10]. The generalization of the AdS/CFT correspondence beyond the type IIB string theory and $\mathcal{N} = 4$ SYM is assumed to be independent of the string realization of both of the theories in the duality. The properties of strongly coupled conformal field theories that have the gravitational duals are conjectured in [11] and theories satisfying these are called holographic CFTs. We give the precise definition of holographic CFTs in Section 2. Most of the work in this thesis is devoted to studying the properties of holographic CFTs and their relations to the dual gravitational theories on the AdS side of the duality.

1.2. Outline

To understand the generic properties of quantum gravity using the AdS/CFT correspondence, two things are needed: first, we need to fully understand the duality at the classical level in gravity and therefore we need to establish the mappings between all variables in classical gravity with their counterparts in terms of observables in the dual CFT, and second, we need a better understanding of strongly coupled, holographic CFTs. This is the reason why we are interested in understanding better the whole class of holographic CFTs, including the parametrization of the class as well as the universal aspects of the dynamics of theories within the class.

Generally, there are two different approaches for studying holographic CFTs. The first approach is to use the classical gravitational dual of a particular CFT, such as Einstein-Hilbert gravity or the modifications thereof, and to extract the implications to the strongly coupled, holographic CFT by studying the gravity side of the duality. These implications can sometimes be generalized to the whole class of the holographic CFTs. This was the dominant approach in the early days of the AdS/CFT duality and we use it in Section 4. The other approach is to directly use the field theory techniques to solve and classify holographic CFTs and it became more dominant recently. We mostly use this approach in the rest of the thesis.

In Section 2 we give a brief introduction to the conformal field theories in general, with special attention to the holographic CFTs and the conformal bootstrap that is a technique used for studying them. In Section 3 we give a detailed introduction for studying "heavy-heavy-light-light" correlator. We also review the calculation of such correlator, we establish the notation and define Regge and lightcone limits that will be used later. In Section 4 we study the CFT version of the high energy scattering of the light probe with the fixed impact parameter in the black hole background [12]. In Section 5 and Section 6 we develop an algorithm for computing the CFT analog of the contributions of multiple gravitons to the scattering of the light probe in the black hole background [13,14]. We show that some of these contributions are universal, i.e. they are the same for all holographic CFTs, in agreement with the claim in [15]. In Section 7 we study the holographic CFTs at the finite temperature [16].

2. The dictionary of the conformal field theory

In this section we review the basics of conformal field theories in two-dimensional and higher-dimensional spacetime and we precisely define the class of holographic CFTs that have a gravitational dual according to [11]. We discuss the basics of the conformal bootstrap, as the technique used in studying the holographic CFTs. We conclude the section with the discussion of holographic CFTs at finite temperature and thermalization.

2.1. Brief introduction to the CFT

In this subsection we give a brief introduction to the kinematics of conformal field theories and establish the notation that we will be using in the rest of the thesis. We consider CFTs in spacetime with number of dimensions $d \geqslant 3$ and $d = 2$ separately.

2.1.1. CFT in spacetime with the number of dimensions $d \geqslant 3$

Conformal field theory is a quantum field theory invariant under the transformations of the conformal group. These are the diffeomorphisms that transform the metric in the following way

$$ds^2 \to ds'^2 = f(x)^2 ds^2, \tag{2.1}$$

where $f(x)$ is an arbitrary function of coordinates. These transformations preserve angles (i.e. causal structure in Lorentzian signature) but not the distances. In spacetime with d dimensions and the Lorentzian signature, these transformations make conformal group $SO(d, 2)$, while in the Euclidean signature, the conformal group is $SO(d + 1, 1)$. For now, we assume that $d \geqslant 3$, as the conformal symmetry when $d = 2$ is described differently than in the case of higher-dimensional spacetime. The conformal group in spacetime with $d \geqslant 3$ consists of transformations in Poincaré group plus the scale transformations and the special conformal transformations. Poincaré group includes translations, with generators P_μ, rotations with generators[2] M_{ij}, and boosts

[2] In our notation Greek letters $(\mu, \nu, \ldots)$ denote all spacetime coordinates, while Latin letters $(i, j, \ldots)$ denote just the spatial part of the coordinates.

with generators M_{0j}. Scale transformations, whose generator is denoted by D, act on spacetime coordinates as

$$D: \qquad x^\mu \to \lambda x^\mu, \tag{2.2}$$

while special conformal transformations, with generators denoted by K_μ, act as

$$K_\mu: \qquad x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2xb + b^2 x^2}, \tag{2.3}$$

where b^μ is an arbitrary constant vector. The full conformal algebra in spacetime with the number of dimensions $d \geqslant 3$ is given by

$$\begin{aligned}
[P_\mu, P_\nu] &= [K_\mu, K_\nu] = [M_{\mu\nu}, D] = 0, \\
[D, P_\mu] &= i P_\mu, \\
[D, K_\mu] &= -i K_\mu, \\
[K_\mu, P_\nu] &= 2i(\delta_{\mu\nu} D - M_{\mu\nu}), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\delta_{\nu\rho} M_{\mu\sigma} - \delta_{\mu\rho} M_{\nu\sigma} - \delta_{\nu\sigma} M_{\mu\rho} + \delta_{\mu\sigma} M_{\nu\rho}\right), \\
[M_{\mu\nu}, P_\rho] &= i\left(\delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu\right), \\
[M_{\mu\nu}, K_\rho] &= i\left(\delta_{\nu\rho} K_\mu - \delta_{\mu\rho} K_\nu\right),
\end{aligned} \tag{2.4}$$

where $\delta_{\mu\nu}$ is the identity matrix if we are working in the Euclidean signature and if we are working in the Lorentzian signature it should be exchanged with the Minkowski metric $\eta_{\mu\nu}$. The Euclidean space $\mathbf{R}^d$ with the metric in the spherical coordinates

$$ds^2 = dr^2 + r^2 d\Omega_{d-1}^2, \tag{2.5}$$

can be conformally mapped to the cylinder $\mathbf{R} \times S^{d-1}$, with the transformation $\tau = \log(r)$, and one obtains the metric

$$ds^2 = e^{2\tau}(d\tau^2 + d\Omega_{d-1}^2). \tag{2.6}$$

The coordinate τ is the Euclidean time on the cylinder. Dilatation operator D, that in the $\mathbf{R}^d$ shifts r coordinate $r \to \lambda r$, generates the time translations on the cylinder $\tau \to \tau + \log(\lambda)$. Therefore, the dilatation operator plays the role of the Hamiltonian on the cylinder and its eigenvalues are treated as energies. This is the physical reason why we need to demand that the eigenvalues of the dilatation operator, also known as the conformal dimensions, are bounded from below. From the algebra (2.4) it is obvious that the generators of translations P_μ increase the conformal dimension of the operator by one, while the generators of special conformal transformations K_μ lower the conformal dimension by one. We define the set of operators whose conformal dimensions can not be lowered further and we call these primary operators

$$[K_\mu, \mathcal{O}_{\Delta,s}(0)] = 0. \tag{2.7}$$

These operators are also known as the highest weight vectors that define the irreducible representation of the conformal group. They are characterized by the eigenvalues of the dilatation operator (i.e. the conformal dimensions) and their spin

$$\begin{aligned} [D, \mathcal{O}_{\Delta,s}(0)] &= i\Delta \mathcal{O}_{\Delta,s}(0), \\ [M_{\mu\nu}, \mathcal{O}_{\Delta,s}(0)] &= \hat{\mathcal{M}}_{\mu\nu} \mathcal{O}_{\Delta,s}(0), \end{aligned} \tag{2.8}$$

where $\hat{\mathcal{M}}_{\mu\nu}$ are generators of representation s of the group[3] $SO(d)$. Starting from the primary operator $\mathcal{O}_{\Delta,s}$, the entire, infinite-dimensional, irreducible representation of the conformal group is obtained by acting on primary operators $\mathcal{O}_{\Delta,s}$ with generators of translations, which are represented by derivatives in coordinate representation. The Hilbert space of the conformal field theory factorizes in a sum of irreducible representations generated from highest weight vectors, i.e. primary operators with quantum numbers (Δ, s).

The main objects of study in the conformal field theory are the correlation functions of the local operators present in the spectrum of the theory. The conformal symmetry fixes the one-point, two-point and three-point correlation functions up to the position-independent constants [17,18]. For external scalar primary operators ϕ_i with conformal dimension Δ_i, these are given by

$$\begin{aligned} &\langle \phi_i(x) \rangle = 0, \qquad \langle \phi_i(x_i)\phi_j(x_j) \rangle = \frac{\delta_{ij}}{x_{ij}^{2\Delta_i}}, \\ &\langle \phi_i(x_i)\phi_j(x_j)\phi_k(x_k) \rangle = \frac{\lambda_{ijk}}{x_{ij}^{\Delta-2\Delta_k} x_{ik}^{\Delta-2\Delta_j} x_{jk}^{\Delta-2\Delta_i}}, \end{aligned} \tag{2.9}$$

where $x_{ab} = x_a - x_b$ and $\Delta = \Delta_i + \Delta_j + \Delta_k$. The two-point functions are fixed by the normalization of the operators and we are using a basis of orthogonal

[3] s denotes the spin of these representations.

operators, hence the Kronecker delta δ_{ij}. The coefficients λ_{ijk} are called operator product expansion coefficients. Once the normalization of operators is fixed, they can not be scaled away, therefore they represent physical parameters of the theory. For spinning operators, multiple independent tensor structures are allowed by conformal symmetry in the three-point functions, so these are determined by more than one position-independent constant multiplying every allowed tensor structure.

The simplest non-trivial correlation functions whose spacetime dependence is not fixed by the conformal symmetry are the four-point correlation functions of the external scalar primary operators. In this thesis, we mostly focus on them in four-dimensional spacetime.

2.1.2. CFT in spacetime with the number of dimensions $d = 2$

The conformal symmetry in two-dimensional spacetime is described differently than the conformal symmetry in higher-dimensional spacetime. Namely, the conformal algebra in two dimensions is an infinite algebra that puts much stronger constraints on the dynamics of the theory compared to the conformal algebra in higher dimensions (2.4).

In two-dimensional spacetime with Euclidean spherical coordinates (r, ϕ) and metric given by

$$ds^2 = dr^2 + r^2 d\phi^2 = dz d\bar{z}, \tag{2.10}$$

where

$$z = re^{i\phi}, \qquad \bar{z} = re^{-i\phi}, \tag{2.11}$$

the set of all conformal transformations is equal to the set of all holomorphic and anti-holomorphic functions of complex coordinates z and $\bar{z}$

For practical purposes, we can lift the condition that z and $\bar{z}$ are the complex conjugate and treat them as the independent complex variables. Then, at the end of the calculation, if one wants to evaluate the result in the Euclidean signature, one requires $z^* = \bar{z}$. The vector space of infinitesimal conformal transformations has a basis given by $\{(\ell_n, \bar{\ell}_n)|n \in \mathbf{Z}\}$, where $\ell_n = -z^{n+1}\partial_z$ and

$\bar{\ell}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$. It is easy to check that generators ℓ_n and $\bar{\ell}_n$ create the two copies of the Witt algebras given by

$$
\begin{aligned}
[\ell_n, \ell_m] &= (n-m)\ell_{n+m}, \\
[\bar{\ell}_n, \bar{\ell}_m] &= (n-m)\bar{\ell}_{n+m}, \\
[\ell_n, \bar{\ell}_m] &= 0.
\end{aligned}
\tag{2.12}
$$

The subalgebras $\{\ell_{-1}, \ell_0, \ell_1\}$ and $\{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$ generate the globally well-defined conformal transformations on the Riemann sphere $S^2 \sim \mathbf{C} \cup \{\infty\}$. These six generators are the two-dimensional version of P_μ, K_μ and $M_{\mu\nu}$ in higher dimensions. It is obvious that generators ℓ_{-1} and $\bar{\ell}_{-1}$ generate translations in the complex plane, therefore they are the two-dimensional version of P_μ, and similarly one can check that ℓ_1 and $\bar{\ell}_1$ generates the special conformal transformations, therefore being the two-dimensional version of K_μ. Generators ℓ_0 and $\bar{\ell}_0$ are related to the dilatation operator D and the generator of rotations. It is useful to write these generators in terms of (r, ϕ) coordinates

$$
\ell_0 = -\frac{1}{2} r\partial_r + \frac{i}{2}\partial_\phi, \qquad \bar{\ell}_0 = -\frac{1}{2} r\partial_r - \frac{i}{2}\partial_\phi. \tag{2.13}
$$

Now, it is easy to see that the dilatation and the rotation generators can be expressed as

$$
D = (\ell_0 + \bar{\ell}_0) = -r\partial_r, \qquad M = (\ell_0 - \bar{\ell}_0) = i\partial_\phi. \tag{2.14}
$$

By the standard quantum-mechanical reasoning, for example, by demanding the unitarity of the theory, one concludes that the conformal algebra of the charges in two-dimensional spacetime has actually to be a central extension of the Witt algebra, also known as the Virasoro algebra and given by

$$
[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}, \tag{2.15}
$$

and similarly for $\bar{L}_n$, while $[L_n, \bar{L}_m] = 0$. Here, c is a constant that multiplies the center of the Virasoro algebra given by the identity operator, and it is called the central charge of a two-dimensional conformal field theory. The physical meaning of the two-dimensional central charge c is that it counts the number of degrees of freedom in the theory, as shown in [19].

One immediately observes that in contrast with the higher-dimensional case, the conformal algebra in two-dimensional spacetime is infinite-dimensional and therefore it gives stronger constraints on the dynamics of the theory. From (2.14) we conclude that the eigenvalues of the dilatation operator are related to the eigenvalues of the operators L_0 and $\bar{L}_0$. We use the same logic as in the higher-dimensional case to argue that the eigenvalue of the dilatation operator must be bounded from below, which now translates to the fact that the eigenvalues of L_0 and $\bar{L}_0$ are bounded from below as well. From the Virasoro algebra (2.15) it follows that operators L_n, for $n > 0$, decreases the eigenvalue of L_0, while the operators L_{-n}, for $n > 0$, increases this eigenvalue. Therefore, we define Virasoro primary operators as those whose eigenvalue of L_0 (and $\bar{L}_0$) can not be decreased further

$$[L_n, \mathcal{O}(0)] = [\bar{L}_n, \mathcal{O}(0)] = 0, \qquad n \geq 1. \tag{2.16}$$

These are characterized by quantum numbers $(h, \bar{h})$, which are eigenvalues of L_0 and $\bar{L}_0$

$$[L_0, \mathcal{O}(0)] = h\mathcal{O}(0), \quad [\bar{L}_0, \mathcal{O}(0)] = \bar{h}\mathcal{O}(0). \tag{2.17}$$

The quantum numbers $(h, \bar{h})$ can be related to the two-dimensional version of the quantum numbers (Δ, s) using (2.14), by the following relations:

$$\Delta = h + \bar{h}, \qquad s = h - \bar{h}. \tag{2.18}$$

Descendants of the Virasoro primary state are obtained by acting with L_{-n} and $\bar{L}_{-n}$, for $n \geqslant 1$, on the primary state. This way, the highest weight irreducible representation of the Virasoro algebra is generated. These representations are called Verma modules. The Hilbert space of two-dimensional CFT factorizes into a sum of the Verma modules.

Generators $\{L_{-1}, L_0, L_1\}$ and $\{\bar{L}_{-1}, \bar{L}_0, \bar{L}_1\}$ make the global subalgebra of the Virasoro algebra. One can notice that there is no central charge in the commutators of elements of the global subalgebra, similarly to the conformal algebra in higher-dimensional spacetime (2.4). If eqs. (2.16) hold only for $n = 1$ generators from the global subalgebra, we say that operator $\mathcal{O}$ is quasi-primary. Therefore, every Virasoro primary is a quasi-primary as well, but not the other way around. Each Verma module can be factorized into the sum of the highest weight representations of the global subalgebra, where the highest weight vectors are quasi-primary operators from the given Verma module.

The four-point correlation functions are not fixed by the conformal symmetry even in two dimensions. But here the Virasoro algebra grants a much better analytic control over them compared to the higher-dimensional cases, which is the reason we often use the two-dimensional CFT as the toy model for the calculation of the four-point correlators in CFT in higher-dimensional spacetime.

2.2. Holographic CFTs

In [11] it was conjectured that there are two necessary and sufficient conditions for CFTs to have a gravitational dual with local physics below the AdS scale. The first condition can be stated as:

The central charge C_T is large, $C_T \to \infty$, and the correlation functions in the theory factorize at large C_T.

The central charge C_T of the conformal field theory in spacetime with Euclidean signature and arbitrary number of dimensions, is defined via the two-point correlation function of the canonically normalized stress tensor

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \frac{C_T}{\Omega_{d-1}^2 x^{2d}} \mathcal{I}_{\mu\nu,\rho\sigma}(x), \tag{2.19}$$

where

$$\begin{aligned} \mathcal{I}_{\mu\nu,\rho\sigma}(x) &= \frac{1}{2}\left(I_{\mu\rho}(x) I_{\nu\sigma}(x) + I_{\mu\sigma}(x) I_{\nu\rho}(x)\right) - \frac{1}{d}\delta_{\mu\nu}\delta_{\rho\sigma}, \\ I_{\mu\nu} &= \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}, \qquad \Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}. \end{aligned} \tag{2.20}$$

In the two-dimensional spacetime, the central charge defined this way is related to the central charge c defined via the Virasoro algebra (2.15) with the relation $C_T = c/2$. However, one should remember that C_T in higher-dimensional CFTs does not satisfy the c-theorem and therefore it can not be treated as the number of degrees of freedom in the theory. The coordinate dependence of the two-point correlation function (2.19) is fixed by the conformal symmetry, while the coefficient C_T is a characteristic of a particular theory. The stress tensor in the holographic CFT is a CFT dual of the single graviton state on the AdS side of duality.

In the example of the duality with the type IIB string theory and $\mathcal{N} = 4$ SYM, the central charge of $\mathcal{N} = 4$ SYM scales in terms of numbers of colors N as

$$C_T \propto N^2. \tag{2.21}$$

Therefore, the fact that the classical gravity limit in type IIB string theory is obtained by taking the large number of colors ($N \to \infty$) is equivalent to the statement that the central charge of the $\mathcal{N} = 4$ SYM has to be large.

The gravitational interpretation of the central charge in holographic CFTs follows from [7]. Namely, in the case of three-dimensional gravitational theory in Anti-de Sitter spacetime, the infinite-dimensional Virasoro algebra (2.15) is found as the algebra of the asymptotic symmetry with the central charge $c = 3L/2G_N$, where G_N is the Newton's constant. Therefore, we generally interpret the central charge as the inverse Newton's constant G_N of the dual gravitational theory, or equivalently, the inverse graviton coupling constant. In the case when the CFT has a finite central charge, the effects of the finiteness of the central charge would, therefore, correspond to the graviton quantum loop corrections in the dual gravitational theory.

The second condition that the holographic CFTs have to satisfy can be stated as follows:

There is a parametrically large gap $\Delta_{\text{gap}} \to \infty$ in the spectrum of conformal dimensions of primary single-trace operators with spin greater than two.

When considering CFTs with the gauge symmetry, the physically relevant operators are the gauge-invariant operators. In the CFTs with the matter in the adjoint representation of the gauge group, the gauge-invariant operators are made of multiple traces of fundamental fields. The second condition which the holographic CFTs have to satisfy requires that the primary single-trace operators with spin greater than two have a large conformal dimension.

Multi-trace, gauge-invariant, primary operators are constructed as the products of single-trace primaries with a number of derivatives inserted in a such way that the product is still the primary operator (or in other words, commutes with K_μ). The large gap condition does not affect these operators in holographic CFTs.

In the language of dual gravity, this condition can be interpreted as the requirement that the fields with spin greater than two in gravity decouple from the rest of the degrees of freedom. Namely, the single-trace primary operators in CFT are dual of the single particle states in gravity. Mass of the field in gravity is proportional to the conformal dimension of the CFT operator. The large gap in the spectrum of conformal dimensions of single-trace primary operators with spin greater than two means that higher-spin fields in gravity have a large mass, and therefore, decouple from the rest of the degrees of freedom. In the example with type IIB string theory and $\mathcal{N} = 4$ SYM, the fact that in the classical limit in gravitational theory we have large 't Hooft coupling ($\lambda \to \infty$) in $\mathcal{N} = 4$ SYM is responsible for this decoupling of higher-spin primary single-trace operators. Namely, these operators receive the large anomalous dimensions at large coupling ($\lambda \to \infty$) and end up with large conformal dimensions, even though they have conformal dimensions of order one at the weak coupling ($\lambda \to 0$).

In the context of the string theory, requiring the large gap in the holographic CFTs accounts for the decoupling of the stringy degrees of freedom in the dual gravitational description. For the CFT with the finite gap, we would have a dual gravitational description with higher-spin, stringy modes in the spectrum.

Additionally, one should notice that having the gauge symmetry is not necessary for holographic CFTs. Without the gauge symmetry, we still use the notion of single and multiple-trace operators in the spectrum of the CFT, but in this case they differ by the large-C_T scaling of the corresponding three-point functions. In the conformal field theories where we do not assume the existence of the gauge symmetry, we also still use the "large-N" terminology by which we mean the large central charge, according to (2.21).

2.3. The conformal bootstrap

We review the important technique for studying strongly coupled CFTs, called conformal bootstrap. It was developed in [20-23]. For recent, more detailed reviews of the conformal bootstrap see [24-26].

The conformal symmetry allows us to write the convergent operator product expansion (OPE)

$$\phi_1(x_1)\phi_2(x_2) = \sum_{\text{primary } \mathcal{O}} \lambda_{12\mathcal{O}} C(x_{12}, \partial_y) \mathcal{O}(y)\Big|_{y=x_2}, \tag{2.22}$$

where the sum goes over all unit-normalized primary operators $\mathcal{O}$ (with arbitrary spin) in the spectrum, functions $C(x_{12}, \partial_y)$ are fixed by the conformal symmetry and they account for the contribution of the descendants of the primary operator $\mathcal{O}$. The operator $\mathcal{O}$ and the derivatives do not have to be evaluated at x_2, they can be evaluated at any point between x_1 and x_2, only the function C will change accordingly. The expansion converges within correlation functions in the Euclidean signature as long as x_1 is closer to x_2 than any other operators inserted at y_i, or in other words, as long as there is a sphere enclosing points x_1 and x_2 with no other external operator inserted within the sphere (see eg. [27-29]). The operator product expansion is always done first in the Euclidean signature and then it can be analytically continued to the Lorentzian signature if necessary.

Using the operator product expansion we reduce any n-point to $(n-1)$-point functions and so on until we arrive at two-point functions that are fixed by the conformal symmetry. Therefore, if we know quantum numbers (Δ, s) of all primary operators in the spectrum of the theory and all OPE coefficients λ_{ijk}, we can, in principle, compute any correlation function and the theory is solved. These data are colloquially called OPE data.

The conformal bootstrap relies on the fact that the operator product expansion is the convergent expansion, therefore it has to be associative. By requiring the associativity of the operator product expansion, also called the crossing symmetry, we put bounds on the OPE data, or sometimes even solve the theory. Let us consider the four-point correlation function of scalar primary operators ϕ_i, $\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle$. Using the following operator product expansions

$$\begin{aligned} \phi_1(x_1)\phi_2(x_2) &= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} C(x_{12}, \partial_y) \mathcal{O}(y)\Big|_{y=x_2}, \\ \phi_3(x_3)\phi_4(x_4) &= \sum_{\mathcal{O}} \lambda_{34\mathcal{O}} C(x_{34}, \partial_z) \mathcal{O}(z)\Big|_{z=x_4}, \end{aligned} \tag{2.23}$$

the four-point correlation function can be written as

$$\begin{aligned}&\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle=\\&\sum_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}C(x_{12},\partial_y)C(x_{34},\partial_z)\langle\mathcal{O}(y)\mathcal{O}(z)\rangle\Big|_{y=x_2,z=x_4}.\end{aligned}\tag{2.24}$$

On the other hand, we can choose to write the operator product expansions between ϕ_1 and ϕ_4, as well as ϕ_2 and ϕ_3

$$\begin{aligned}\phi_1(x_1)\phi_4(x_4)&=\sum_{\mathcal{O}'}\lambda_{14\mathcal{O}}C(x_{14},\partial_y)\mathcal{O}'(y)\Big|_{y=x_4},\\\phi_2(x_2)\phi_3(x_3)&=\sum_{\mathcal{O}'}\lambda_{23\mathcal{O}'}C(x_{23},\partial_z)\mathcal{O}'(z)\Big|_{z=x_3}.\end{aligned}\tag{2.25}$$

In this case, the four-point correlation function is given by

$$\begin{aligned}&\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle=\\&\sum_{\mathcal{O}'}\lambda_{14\mathcal{O}'}\lambda_{23\mathcal{O}'}C(x_{14},\partial_y)C(x_{23},\partial_z)\langle\mathcal{O}'(y)\mathcal{O}'(z)\rangle\Big|_{y=x_4,z=x_3}.\end{aligned}\tag{2.26}$$

The different ways of writing the OPE are called different channels of expansion. Now, the condition of consistency of the theory (or the associativity of the OPE or the crossing symmetry) requires that these two channels give the same result

$$\begin{aligned}&\sum_{\mathcal{O}}\lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}C(x_{12},\partial_y)C(x_{34},\partial_z)\langle\mathcal{O}(y)\mathcal{O}(z)\rangle\Big|_{y=x_2,z=x_4}\\=&\sum_{\mathcal{O}'}\lambda_{14\mathcal{O}'}\lambda_{23\mathcal{O}'}C(x_{14},\partial_y)C(x_{23},\partial_z)\langle\mathcal{O}'(y)\mathcal{O}'(z)\rangle\Big|_{y=x_4,z=x_3}.\end{aligned}\tag{2.27}$$

This equation is a non-trivial constraint on the OPE data and it is known as the bootstrap equation. It bounds the spectrum of the theory, as well as on the OPE coefficients.

One should notice that generally we do not have the contributions of the same primary operators in both channels, that is why they are denoted differently by $\mathcal{O}$ and $\mathcal{O}'$. The parts of the contributions of these operators that are fixed by conformal symmetry, $C(x_{12},\partial_y)C(x_{34},\partial_z)\langle\mathcal{O}(y)\mathcal{O}(z)\rangle|_{y=x_2,z=x_4}$ and $C(x_{14},\partial_y)C(x_{23},\partial_z)\langle\mathcal{O}'(y)\mathcal{O}'(z)\rangle|_{y=x_4,z=x_3}$, are called conformal blocks (or conformal partial waves). They are not crossing symmetric independently, but their corresponding sums have to be, which is the reason why the condition (2.27) is the non-trivial requirement on the spectrum of the theory. The explicit analytic expressions for the blocks are known in even-dimensional spacetime since the work of Dolan and Osborn [30,31]. Their expressions play a crucial role in the conformal bootstrap technique as they help to solve (2.27). The conformal bootstrap technique has brought many new results for holographic CFTs in recent years (see e.g. [11-14,32-55]).

We solve this equation by taking a particular kinematic limit which isolates the small number of operators that contribute in one channel and their contributions have to be recovered by an infinite number of operators in the other channel. Examples of such limits are the lightcone and Regge limits. Physically, the lightcone limit is a limit where one of the external scalar operators in the four-point function approaches the lightcone of one of the other external operators, while the Regge limit corresponds to special kinematics, which on the gravity side is described by the scattering of highly energetic particles whose trajectories in the bulk are approximately null.

The usefulness of the conformal bootstrap technique for solving CFTs lies in the fact that one does not have to specify the theory, for example, with its Lagrangian. The set of reasonable assumptions about the spectrum of the theory allows one to run the bootstrap program and learn more about the whole class of theories for which the assumptions work. The particular class of theories that we are interested in are the strongly coupled, holographic CFTs. The bootstrap equation is perfectly well-defined for the strongly coupled theories as it does not rely on the perturbative expansion in the coupling constant.

In the rest of the thesis we mostly consider the holographic CFTs in four-dimensional spacetime. We study the four-point correlation function of two pair-wise identical primary single-trace scalar operators $\mathcal{O}_L$ and $\mathcal{O}_H$, with conformal dimensions Δ_L and Δ_H, that scale as $\Delta_L \sim \mathcal{O}(1)$ and $\Delta_H \sim \mathcal{O}(C_T)$. Because of this scaling, operator $\mathcal{O}_L$ is called "light", while $\mathcal{O}_H$ is called "heavy". Taking into account the dual gravitational picture, $\mathcal{O}_H$ is a CFT analog of the black hole, while $\mathcal{O}_L$ represents a light probe. Instead of the usual $1/C_T$ expansion parameter in the correlators of the holographic CFTs, for the four-point correlation function that includes two heavy states $\mathcal{O}_H$ as the external operators, one has to use $\mu \sim \Delta_H/C_T$ as the expansion parameter.

Two channels of the expansion are called "T-channel", where two OPEs are written between $\mathcal{O}_L \times \mathcal{O}_L$ and $\mathcal{O}_H \times \mathcal{O}_H$, and "S-channel", where we have two OPEs $\mathcal{O}_H \times \mathcal{O}_L$ and $\mathcal{O}_L \times \mathcal{O}_H$. The important set of primary operators that contribute to the T-channel is called "the stress tensor sector" and consists of the stress tensor (which is a single-trace operator) and multi-trace operators made of the stress tensor. Operators made of k stress tensors contribute at order μ^k to the correlator. As the stress tensor is a conserved current of the theory, its OPE coefficient with two external scalar operators is fixed by the Ward identity. On the other hand, the operators with $k \geqslant 2$ are not the conserved currents, therefore, their OPE coefficients are not fixed by the conformal symmetry. The gravitational analog of the stress tensor sector contributions to the correlator is the exchange of the single graviton in the Witten diagram and Witten diagrams with multi-graviton exchanges (the graviton loops).

In the kinematical limits we consider the stress tensor sector decouples from the rest of the operators contributing in the T-channel, which allows us to solve this sector completely, i.e. to write an algorithm for computing all of the OPE coefficients of operators in the stress tensor sector with external scalar operators. The contributions of these operators in the T-channel are reproduced by the double-trace operators made of one $\mathcal{O}_L$ and one $\mathcal{O}_H$ operator in the S-channel. These double-trace operators receive an anomalous dimension and correction to the zeroth order OPE coefficients due to the stress tensor sector contributions in T-channel. Solving the stress tensor sector of the holographic CFT accounts also for finding the analytic expressions for these anomalous dimensions and corrections to the MFT OPE.

2.4. Thermalization in holographic CFTs

Due to the presence of finite temperature states (the black holes and black branes with finite Hawking temperature) on the gravity side of AdS/CFT duality, at least in the classical gravity limit, studying the thermal properties of the dual CFTs represents the essential task to understand the full scope of the duality and to be able to describe these states via their CFT counterparts.

One should notice that the conformal symmetry is broken at the finite temperature, as the one-point functions of the local operators are not set to zero by symmetry anymore. This is because the temperature introduces the dimensionful parameter in the theory which allows the thermal one-point functions to be non-zero. All other assumptions on the spectrum of the CFT, including the factorization of the Hilbert space and the operator product expansion[4], are assumed to hold when the CFT is set at finite temperature.

We have already suggested that the pure heavy scalar primary state $\mathcal{O}_H$ is the CFT equivalent of the black hole in gravity. This might seem to be in contradiction with the usual AdS/CFT dictionary which says that black holes (or black branes) in gravity correspond to a (mixed) thermal state in CFT, and the Hawking temperature of the black hole is equal to the temperature of the thermal state. Additionally, there is an obvious mismatch between the finite Bekenstein-Hawking entropy of the black hole and the zero entropy of the pure state, which could seem to completely invalidate the assumption.

However, we argue that the operators in the stress tensor sector thermalize in the pure heavy states, such that their OPE coefficients with scalar heavy operators are equal to their thermal one-point functions at the temperature that is related to the conformal dimension of the heavy operator Δ_H. This justifies the identification of the black hole with a pure heavy state in the dual holographic CFT, as long as one is interested in studying the graviton contributions to the gravitational or the CFT correlators. This also explains why the OPE coefficients of stress tensor sector computed in the thermal black hole background in gravity [15] are equal to those computed in CFT by the conformal bootstrap [13,14]. Other operators that are generically present in the spectrum of the holographic CFTs do not thermalize in this sense, and their contribution would see a difference between the mixed thermal state and the pure heavy state.

We extend the analysis of thermalization from holographic CFTs to all large-C_T (or equivalently, large-N) theories, without assuming the large gap in the spectrum. We show that thermalization works even in the opposite limit, for free large-C_T theories whose gap goes to zero.

[4] with modified convergence criteria

3. Introduction

The AdS/CFT correspondence provides a non-perturbative definition of quantum gravity in negatively curved spacetimes [8-10]. The correspondence in principle provides an opportunity to study generic properties of quantum gravity, possibly probing regimes unattainable by low-energy effective theories. Recent years have seen a development in conformal bootstrap techniques following [20-23], leading to many results for CFTs with holographic duals (see e.g. [11,32-55]). CFT methods have therefore become a powerful tool in the study of quantum gravity.

Crossing symmetry in CFTs imposes highly non-trivial constraints on the theory. The idea of conformal bootstrap is to use these constraints to put restrictions on the CFT data or, if possible, even solve the theory. One way to make use of the crossing symmetry is to isolate a small number of contributing operators in one channel, e.g. by going to a certain kinematical regime. This typically has to be reproduced by the exchange of an infinite number of operators in another channel. One such example is the lightcone limit where the separation between two operators in a four-point function is close to being null. One can then infer [56,57] the existence of double-trace operators at large spin in any CFT in dimensions $d > 2$. The Regge limit provides another opportunity to isolate the contribution of a class of operators, those of highest spin.

3.1. The conformal bootstrap in the Regge limit

In holographic CFTs the Regge limit of a four-point function, extensively studied in [58-63][5], is dominated by operators of spin two – the stress tensor and the double-trace operators (this is a consequence of the gap in the spectrum). In gravity, it reproduces a Witten diagram with graviton exchange (see e.g. [43]). The Regge limit corresponds to special kinematics, which on the gravity side is described by the scattering of highly energetic particles whose trajectories in the bulk are approximately null.

Such scattering can be described in the eikonal approximation where particles follow classical trajectories but their wavefunctions acquire a phase shift $\delta(S, L)$. The phase shift is a function of the total energy S and the impact

[5] See also [45,64-86] for other recent applications of Regge limit in CFTs.

parameter L. In the CFT language, this phase shift can be extracted from the Fourier transform of the four-point function. In [59] the phase shift extracted from the four-point function of the type $\langle \mathcal{O}_1\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\rangle$ was shown to be equal (up to a factor of $-\pi$) to the anomalous dimension of the double-trace operators $[\mathcal{O}_1\mathcal{O}_2]_{n,l} =: \mathcal{O}_1\partial^{2n}\partial_{\mu_1}\dots\partial_{\mu_l}\mathcal{O}_2 :$, at leading order in $1/N^2$. The Regge limit implies that the calculation is valid for $n, l \gg 1$. These anomalous dimensions have been subsequently verified in [49,87-95].

So far both operators $\mathcal{O}_1$ and $\mathcal{O}_2$ were assumed to have conformal dimensions of order one. In the following, we will refer to them as "light" operators and denote them by $\mathcal{O}_L$. In [55] one pair of operators (which we denote by $\mathcal{O}_H$) was taken to be "heavy", with conformal dimension Δ_H scaling as the central charge. The ratio $\mu \sim \Delta_H/C_T$ is a useful expansion parameter; its power k corresponds to the number of stress tensors in the multi-stress tensor operators exchanged in the T-channel $(\mathcal{O}_H \times \mathcal{O}_H \to (T_{\mu\nu})^k \to \mathcal{O}_L \times \mathcal{O}_L)$[6]. As explained in [55], one can define the phase shift as a Fourier transform of the $\langle \mathcal{O}_H\mathcal{O}_H\mathcal{O}_L\mathcal{O}_L\rangle$ four-point function. It is related to the time delay and angle deflection of a highly energetic particle traveling along a null geodesic in the background of an asymptotically AdS black hole. The black hole corresponds to the insertion of the heavy operator $\mathcal{O}_H$; its mass in the units of AdS radius is proportional to μ.

The phase shift $\delta(S, L)$ was computed in gravity in [55] as an infinite series expansion in μ, *i.e.*,

$$\delta(S, L) = \sum_{k=1}^{\infty} \delta^{(k)}\mu^k \,, \tag{3.1}$$

with terms subleading in $1/N^2$ suppressed. Conformal dimensions Δ and spins s of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ are given by

$$\Delta = \Delta_H + \Delta_L + 2n + l + \gamma(n, l), \qquad s = l, \tag{3.2}$$

where the anomalous dimensions $\gamma(n, l)$ admit a similar expansion in powers of μ

$$\gamma(n, l) = \sum_{k=1}^{\infty} \gamma^{(k)}_{n,l}\mu^k. \tag{3.3}$$

[6] Recently a similar limit was studied in [15].

In [55] it was also proven that

$$\gamma^{(1)}_{n,l} = -\frac{\delta^{(1)}}{\pi}, \tag{3.4}$$

where the following identifications are implied:

$$h = n + l, \qquad \bar{h} = n, \qquad S = 4h\bar{h}, \qquad e^{-2L} = \frac{\bar{h}}{h}. \tag{3.5}$$

However, it was observed that this relation does not hold for higher order terms, i.e. in general $\gamma^{(k)}_{n,l}$ is not proportional to $\delta^{(k)}$. One of the aims of Section 4 is to explain how higher order anomalous dimensions are related to higher order terms in the phase shift.

3.2. The minimal twist multi stress tensors

As reviewed before, the two-point function of the stress tensor in Conformal Field Theories is proportional to a single parameter, the central charge C_T. It generally serves as a measure of the number of degrees of freedom in the theory. In two spacetime dimensions this statement can be made precise: one can define a c-function which monotonically decreases along Renormalization Group flows and reduces to the central charge at conformal fixed points [19]. In four spacetime dimensions the situation is a bit more subtle and it is the a-coefficient in the conformal anomaly which necessarily satisfies $a_{IR} \leq a_{UV}$ [96]. Nevertheless, in any unitary conformal field theory a and C_T can only differ by a number of $\mathcal{O}(1)$ (see [97] for the original argument and [98-104] for more recent field theoretic proofs.) Hence, to consider the limit of infinite number of degrees of freedom one needs to take C_T to infinity.

In two spacetime dimensions conformal symmetry is described by the infinite-dimensional Virasoro algebra. This symmetry strongly constrains correlators, especially when combined with the $C_T \to \infty$ limit. Of particular interest is the heavy-heavy-light-light correlator, which involves two heavy operators with conformal dimension $\Delta_H \sim C_T$ and two light operators with conformal dimension $\Delta_L \sim \mathcal{O}(1)$. In this case the contribution of the identity operator and all its Virasoro descendants is known as the Virasoro vacuum block and has been calculated in several ways [40,105-110]. The Virasoro vacuum block (and finite C_T corrections to it) is instrumental in a variety of settings, such as e.g. the problem of information loss [111-116] and properties of the Renyi and entanglement entropies [117-120] (see also [121,122] for the original applications of large C_T correlators in this context).

The heavy-heavy-light-light Virasoro vacuum block exponentiates

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\rangle \sim e^{\Delta_L \mathcal{F}(\mu;z)}, \tag{3.6}$$

with $\mathcal{F}$ a known function which admits an expansion in powers of $\mu \sim \Delta_H/C_T$

$$\mathcal{F}(\mu;z) = \sum_k \mu^k \mathcal{F}^{(k)}(z). \tag{3.7}$$

One can consider contributions of various quasi-primaries made out of the stress tensor to $\mathcal{F}^{(k)}$. At $k=1$ the only such quasi-primary is the stress tensor itself, while for $k=2$ one needs to sum an infinite number of quasi-primaries quadratic in the stress tensor (double-stress operators) and labeled by spin. The situation is similar for all other values of k. It is possible to compute the OPE coefficients of the corresponding quasi-primaries, starting from the known result for the Virasoro vacuum block. Interestingly, at each order in μ, $\mathcal{F}^{(k)}$ can be written as a sum of particular terms [55][7]

$$\mathcal{F}^{(k)}(z) = \sum_{\{i_p\}} b_{i_1\ldots i_k} f_{i_1}(z)\ldots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = 2k, \tag{3.8}$$

where $f_a(z) = (1-z)^a {}_2F_1(a,a,2a,1-z)$.

It is an interesting question whether a similar structure appears when the number of spacetime dimensions d is greater than two. Unlike in two spacetime dimensions, in addition to spin, multi-stress tensor operators are also labeled by their twist. An interesting subset of multi-stress tensor operators is comprised out of those with minimal twist. These operators dominate in the lightcone limit over those of higher twist. In [124] an expression for the OPE coefficients of two scalars and minimal-twist double-stress tensor operators in $d=4$ was obtained, and the sum was performed to obtain a remarkably simple expression for the near lightcone $\mathcal{O}(\mu^2)$ term in the heavy-heavy-light-light correlator. It was

[7] Similar expressions in a slightly different context appeared in [123].

shown to have a similar form to (3.8). One may now wonder if the minimal-twist multi-stress tensor part of the correlator in higher dimensions exponentiates

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\big|_{\text{multi-stress tensors}} \sim e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{3.9}$$

and whether $\mathcal{F}(\mu;z,\bar{z})$ can be expressed as

$$\mathcal{F}(\mu;z,\bar{z}) = \sum_k \mu^k \mathcal{F}^{(k)}(z,\bar{z}), \tag{3.10}$$

with

$$\mathcal{F}^{(k)}(z,\bar{z}) = (1-\bar{z})^{k(\frac{d-2}{2})} \sum_{\{i_p\}} b_{i_1\ldots i_k} f_{i_1}(z)\ldots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = k\left(\frac{d+2}{2}\right), \tag{3.11}$$

and d an even number.

In Section 5 we investigate this further. We start by assuming that the multi-stress tensor sector of the heavy-heavy-light-light correlator in the near lightcone regime $\bar{z} \to 1$ admits an expansion in μ

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\big|_{\text{multi-stress tensors}} \sim \sum_k \mu^k \mathcal{G}^{(k)}(z,\bar{z}), \tag{3.12}$$

where each coefficient function $\mathcal{G}^{(k)}(z,\bar{z})$ takes a particular form:

$$\mathcal{G}^{(k)}(z,\bar{z}) = \frac{(1-\bar{z})^{k(\frac{d-2}{2})}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1\ldots i_k} f_{i_1}(z)\ldots f_{i_k}(z), \quad \sum_{p=1}^{k} i_p = k\left(\frac{d+2}{2}\right). \tag{3.13}$$

We subsequently use this ansatz to compute the contributions of the multi-stress tensor operators to the near lightcone correlator and extract the corresponding OPE coefficients.

For even d, the hypergeometric functions in (3.13) reduce to terms which contain at most one power of $\log(z)$ each. Their products contain multi-logs whose coefficients turn out to be rational functions of z. We use the conformal bootstrap approach initiated in [22] (for a review and references see eg. [24-26]) to relate these functions to the anomalous dimensions and OPE coefficients of the heavy-light double-trace operators in the cross channel. The ansatz (3.13) has just a few coefficients at any finite k which can be determined completely from the cross-channel data derived using the $(k-1)$th term. This is related to the fact that all the $\log^m(z)$ terms with $2 \le m \le k$ are completely determined by the anomalous dimensions and OPE coefficients at $\mathcal{O}(\mu^{k-1})$. At each step, we obtain an overconstrained system of equations solved by the same set of $a_{i_1 \dots i_k}$. This provides strong support to the ansatz (3.11). We then proceed to derive the OPE coefficients of the multi-stress tensor operators with two light scalars from our result. In practice, we complete this program to $\mathcal{O}(\mu^3)$ in $d=4$ and to $\mathcal{O}(\mu^2)$ in $d=6$. However the procedure outlined can be easily generalised to arbitrary order in μ and any even d.

In [15] the authors considered holographic CFTs dual to gravitational theories defined by the Einstein-Hilbert Lagrangian plus higher derivative terms and a scalar field minimally coupled to gravity in AdS_{d+1}. Interpreting the scalar propagator in an asymptotically AdS_{d+1} black hole background as a heavy-heavy-light-light four-point function, enabled the authors of [15] to extract the OPE coefficients of a few multi-stress tensor operators from holography (see also [125-127] for related work). Ref. [15] also argued that the OPE coefficients of the leading, minimal-twist multi-stress operators are universal – they do not depend on the higher derivative terms in the Lagrangian. Their results agree with the general expressions obtained in this section, upon substitution of the relevant quantum numbers. We do not use holography in Section 5; our major assumption is (3.13).

3.3. The full stress tensor sector of conformal correlators

In Section 6 we study the contribution of the entire stress-tensor sector to the scalar CFT correlation functions, $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$. We go beyond the limitation of considering only the minimal twist multi stress tensor operators. We investigate the stress tensor sector further by considering contributions from multi-stress tensors with non-minimal twist. Our goal is to determine the structure of the correlator to subleading orders in the lightcone limit and extract the relevant OPE coefficients. Once more, we motivate an ansatz similar to the one successfully describing the leading lightcone behavior of $\mathcal{G}(z, \bar{z})$ and show that most of the parameters in the ansatz can be fixed using lightcone bootstrap. A few parameters are, however, left undetermined and might depend on the details of the theory. They correspond to the OPE coefficients of multi-stress tensors with spin $s = 0, 2$. Our approach can be employed to study the stress-tensor sector to arbitrary orders in μ and $(1 - \bar{z})$. We completed this program for the $\mathcal{O}(\mu^2)$ subleading, subsubleading and subsubsubleading terms as well as the $\mathcal{O}(\mu^3)$ subleading and subsubleading terms.

The OPE coefficients of the minimal twist multi stress tensors can be obtained by the Lorentzian inversion formula [73,78] as shown in [128][8]. We also investigate this complementary approach to computing the OPE data of the stress tensor sector. As noted earlier, the validity of the Lorentzian inversion formula for the HHLL correlator has not been rigorously established. It is however natural to expect that it is applicable in the large-C_T and small-μ expansion, as long as a Regge bound is observed. Here we assume that the Regge behavior of the correlator is given by σ^{-k} at $\mathcal{O}(\mu^k)$ in the large-C_T limit, which is consistent with the behaviour of the scattering phase shift from a black hole (or a massive star) computed classically in AdS. We then find that whenever the Lorentzian inversion formula is applicable, *i.e.*, for operators of spin $s > k+1$ at $\mathcal{O}(\mu^k)$, OPE data extracted with both methods are in perfect agreement. However, already at order $\mathcal{O}(\mu^3)$, our ansatz combined with the crossing symmetry or Lorentzian inversion formula is more powerful than the Lorentzian inversion formula alone. For instance, while the former procedure allows us to determine the OPE coefficient of a triple-stress tensor with spin $s = 4$ and twist $\tau = 8$, this is not possible using solely the Lorentzian inversion formula.

Finally, we explore the possibility of obtaining the unknown OPE data from the gravitational description of the CFT. We use the phase shift calculation in the dual gravitational theory. The scattering phase shift – acquired by a highly energetic particle travelling in the background of the AdS black hole – was first computed in the Regge limit in Einstein gravity in [55]. To explicitly see how the presence of higher derivative gravitational terms affects the OPE data, we work in Einstein-Hilbert + Gauss-Bonnet gravity with small Gauss-Bonnet coupling λ_{GB}. To combine the gravitational results with those of the CFT in the lightcone regime, we follow the approach first discussed in [124]

[8] One should exercise caution when using the Lorentzian inversion formula in the context of the HHLL correlator as the Regge behaviour of the correlator has not been rigorously established.

and further developed in [13], which involves an analytic continuation of the lightcone results around $z = 0$ and an expansion around $z = 1$. Matching terms in the correlator obtained from the gravitational calculation to those obtained from the CFT enables us to completely fix the stress tensor sector of the HHLL correlator up to the OPE coefficients of the spin-0 multi-stress tensors which are left undetermined. Non-universality is manifest by the presence of the Gauss-Bonnet coupling in the expressions for the OPE coefficients.

3.4. Review of heavy-heavy-light-light correlator in holographic CFTs

In this section, crossing relations for a heavy-heavy-light-light correlator of pair-wise identical scalars are reviewed. We consider large-N CFTs, with $C_T \sim N^2$, with a parametrically large gap Δ_{gap} in the spectrum of single trace operators with spin $J > 2$. The object that we study is a four-point correlation function between two light scalar operators $\mathcal{O}_L$, with scaling dimension of order one, and two heavy scalar operators $\mathcal{O}_H$, with scaling dimension Δ_H of $\mathcal{O}(C_T)$. Explicitly, we expand the CFT data in the parameter μ defined in [55] as

$$\mu = \frac{4\Gamma(d+2)}{(d-1)^2\Gamma(d/2)^2}\frac{\Delta_H}{C_T}, \tag{3.14}$$

which is kept fixed as $C_T \to \infty$.

The four-point function is fixed by conformal symmetry up to a function $\mathcal{A}(u, v)$ of the cross-ratios as

$$\langle \mathcal{O}_H(x_4)\mathcal{O}_L(x_3)\mathcal{O}_L(x_2)\mathcal{O}_H(x_1)\rangle = \frac{\mathcal{A}(u,v)}{x_{14}^{2\Delta_H}x_{23}^{2\Delta_L}}, \tag{3.15}$$

where u, v are cross-ratios[9]

$$\begin{aligned} v &= z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \\ u &= (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \end{aligned} \tag{3.16}$$

and $x_{ij} = x_i - x_j$. Using conformal symmetry we can fix $x_1 = 0$, $x_3 = 1$ and $x_4 \to \infty$, with the last operator confined to a plane parameterized by $(z, \bar{z})$.

[9] Note that (u, v) are exchanged compared to the more common convention.

The main object of study is an appropriately rescaled version of (3.15)

$$G(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H} \langle \mathcal{O}_H(x_4)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle. \tag{3.17}$$

This can be expanded in the S-channel $\mathcal{O}_L(z,\bar{z}) \to \mathcal{O}_H(0)$ as

$$G(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{\mathcal{O}'} \left(-\frac{1}{2}\right)^{J'} \lambda_{\mathcal{O}_H\mathcal{O}_L\mathcal{O}'}\lambda_{\mathcal{O}_L\mathcal{O}_H\mathcal{O}'} g_{\mathcal{O}'}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}), \tag{3.18}$$

where $\Delta_{HL} = \Delta_H - \Delta_L$, λ_{ijk} are OPE coefficients and the sum runs over primaries $\mathcal{O}'$ with spin J' and corresponding conformal blocks $g'_{\mathcal{O}}$. The correlator can likewise be expanded in the T-channel $\mathcal{O}_L(z,\bar{z}) \to \mathcal{O}_L(1)$ as

$$G(z,\bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\mathcal{O}} \left(-\frac{1}{2}\right)^{J} \lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}}\lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}} g_{\mathcal{O}}^{0,0}(1-z,1-\bar{z}), \tag{3.19}$$

where we again sum over primaries $\mathcal{O}$ with spin J. The equality of (3.18) and (3.19) constitutes an example of a crossing relation, in both channels we sum over an infinite set of conformal blocks $g_{\mathcal{O}}^{\Delta_1,\Delta_2}(z,\bar{z})$. These contain the contribution from a primary $\mathcal{O}$ and all its descendants. (For recent reviews on conformal bootstrap see [24-26].) Here we have distinguished between operators $\mathcal{O}'$ and $\mathcal{O}$, in the S- and T-channel, respectively, in order to stress that generically different operators are relevant in different channels. As an example of this, in the lightcone limit in $d>2$ one finds [56,57] that the T-channel is dominated by the identity operator, while in the S-channel an infinite number of operators contribute. These are the so-called double-trace operators that exist at large spin in any $\text{CFT}_{d>2}$.

We will assume the following scaling for a non-trivial single trace operator $\mathcal{O}$ (not including the stress tensor)

$$\langle \mathcal{O}_{H,L}\mathcal{O}_{H,L}\mathcal{O}\rangle \sim \frac{1}{\sqrt{C_T}}. \tag{3.20}$$

The conformal Ward identity fixes the following 3-pt function for the stress tensor

$$\langle \mathcal{O}_{H,L}\mathcal{O}_{H,L}T_{\mu\nu}\rangle \sim \Delta_{H,L}, \tag{3.21}$$

which implies the following scaling for the exchange of the stress tensor in the T-channel

$$\frac{\langle \mathcal{O}_H \mathcal{O}_H T_{\mu\nu}\rangle \langle T_{\mu\nu} \mathcal{O}_L \mathcal{O}_L\rangle}{\langle T_{\mu\nu} T_{\mu\nu}\rangle} \sim \frac{\Delta_H \Delta_L}{C_T} \sim \mu. \tag{3.22}$$

Keeping μ small, it follows that the leading contribution in the T-channel is given by the disconnected correlator $\langle \mathcal{O}_H \mathcal{O}_H\rangle\langle \mathcal{O}_L \mathcal{O}_L\rangle$, i.e. the exchange of the identity operator. Decomposing the disconnected correlator in the S-channel, we will infer the existence of the "double-trace operators" $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ for all integers n, l, with scaling dimension $\Delta = \Delta_H + \Delta_L + 2n + l + \gamma(n,l)$ and spin l. Moreover, the OPE coefficients scale as

$$\langle \mathcal{O}_H \mathcal{O}_L [\mathcal{O}_H \mathcal{O}_L]_{n,l}\rangle \sim 1. \tag{3.23}$$

Keeping $\mu \sim \Delta_H / C_T$ fixed as $C_T \to \infty$, (3.22) implies that the CFT data of double-trace operators $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ receives perturbative corrections in μ. We therefore expand the anomalous dimensions of these double-trace operators, as well as the OPE coefficients

$$P^{(HL,HL)}_{n,l} \equiv \left(-\frac{1}{2}\right)^l \lambda_{\mathcal{O}_H \mathcal{O}_L [\mathcal{O}_H \mathcal{O}_L]_{n,l}} \lambda_{\mathcal{O}_L \mathcal{O}_H [\mathcal{O}_H \mathcal{O}_L]_{n,l}}\,, \tag{3.24}$$

in μ as

$$\begin{aligned} &\gamma(n,l) = \mu \gamma^{(1)}_{n,l} + \mu^2 \gamma^{(2)}_{n,l} + \dots \\ &P^{(HL,HL)}_{n,l} = P^{(HL,HL);\mathrm{MFT}}_{n,l}(1 + \mu P^{(HL,HL);(1)}_{n,l} + \mu^2 P^{(HL,HL);(2)}_{n,l} + \dots), \end{aligned} \tag{3.25}$$

with $\dots$ denoting higher order terms. The OPE coefficients $P^{(HL,HL);\mathrm{MFT}}_{n,l}$ are fixed and can be found in [38]:

$$\begin{aligned} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} =& \frac{(\Delta_H + 1 - d/2)_{\bar{h}} (\Delta_L + 1 - d/2)_{\bar{h}} (\Delta_H)_h (\Delta_L)_h}{\bar{h}!(h-\bar{h})!(\Delta_H + \Delta_L + \bar{h} + 1 - d)_{\bar{h}} (\Delta_H + \Delta_L + h + \bar{h} - 1)_{h-\bar{h}}} \\ &\times \frac{1}{(h - \bar{h} + d/2)_{\bar{h}} (\Delta_H + \Delta_L + h - d/2)_{\bar{h}}}, \end{aligned} \tag{3.26}$$

where $(a)_b$ is the Pochhammer symbol and the relation between (n,l) and $(h,\bar{h})$ variables is given by

$$h = n + l, \qquad \bar{h} = n. \tag{3.27}$$

In the limit $\Delta_H \gg \Delta_L, h, \bar{h}$, (3.26) simplifies

$$P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \approx C_{\Delta_L} \frac{\Gamma(\Delta_L + \bar{h} - d/2 + 1)\Gamma(\Delta_L + h)}{\bar{h}!(h-\bar{h})!(h - \bar{h} + d/2)_{\bar{h}}}, \tag{3.28}$$

where $C_{\Delta_L} = (\Gamma(\Delta_L - d/2 + 1)\Gamma(\Delta_L))^{-1}$.

3.4.1. Regge limit of HHLL correlator

After the analytic continuation $z \to ze^{-2i\pi}$ we use (σ, ρ) coordinates defined by:

$$\begin{aligned} 1 - z &= \sigma e^{\rho} \\ 1 - \bar{z} &= \sigma e^{-\rho}. \end{aligned} \tag{3.29}$$

The Regge limit corresponds to $\sigma \to 0$ with ρ kept fixed. The easiest way to understand the kinematics of the Regge limit is to consider the four-point correlation function on Euclidean cylinder with coordinates $(\tau, \hat{n})$, as in [55]:

$$\langle \mathcal{O}_H(\tau_4, \hat{n}_4) \mathcal{O}_L(\tau_3, \hat{n}_3) \mathcal{O}_L(\tau_2, \hat{n}_2) \mathcal{O}_H(\tau_1, \hat{n}_1) \rangle, \tag{3.30}$$

where the relation between the operators on the cylinder and plane (with coordinates x) is given by

$$\mathcal{O}_\Delta(x) = e^{-\tau\Delta} \mathcal{O}_\Delta(\tau, \hat{n}), \qquad x^2 = e^{2\tau}. \tag{3.31}$$

The heavy operators are evaluated at $\tau_4 \to \infty$ and $\tau_1 \to -\infty$ and via the operator-state correspondence, they correspond to heavy states. The light operators are evaluated near reference points P_2 and P_3, that are null separated after Wick rotation to Lorentzian signature $\tau = -it$. Basically, their time separation is $t_{3P} - t_{2P} = \pi$, and they are placed on the opposite sides of S^{d-1} sphere $\hat{n}_{3P} = -\hat{n}_{2P}$. Using the translational symmetry along τ direction and rotational symmetry on S^{d-1} we set $\mathcal{O}_L(\tau_3, \hat{n}_3) = \mathcal{O}_L(P_3)$. Now, time component of the other light operator τ_2 is measured with respect to P_3, and it is given by

$$\tau_2 = \tau_3 + ix^0. \tag{3.32}$$

We use $(x^0, \hat{n}_2)$ as the coordinates of the other light operator that will be Fourier transformed to get the gravitational phase shift as in [55]. It is easy to check

$$\begin{aligned} z\bar{z} &= \frac{x_2^2}{x_3^2} = e^{2(\tau_2 - \tau_3)} = e^{2ix^0}, \\ (1-z)(1-\bar{z}) &= \frac{x_{23}^2}{x_3^2} = 1 + e^{2ix^0} - 2e^{ix^0} \cos\phi, \end{aligned} \tag{3.33}$$

where ϕ is the angle between vectors $\hat{n}_{3P} = \hat{n}_3$ and $\hat{n}_2$. These conditions are satisfied by the following relations:

$$z = e^{ix^+}, \qquad \bar{z} = e^{ix^-}, \qquad x^\pm = x^0 \pm \phi. \tag{3.34}$$

In the correlation function (3.30) we start from the configuration where light operators are inserted close to each other, $x^0 \approx \phi \approx 0$, and we shift one of them $\mathcal{O}_L(x^0, \hat{n}_2)$ near the reference point P_2. In this case, $x^0 \approx \phi \approx -\pi$. Therefore, we need to shift $x^+ \to x^+ - 2\pi$, or in terms of z variable $z \to ze^{-2\pi i}$. Correlation function on the cylinder (3.30) is related to the correlation function on the plane (3.17) by the simple conformal transformation.

After the analytic continuation, in $\sigma \to 0$ limit the dominant contribution to the correlator comes from high-spin operators, which follows from the fact that the conformal blocks that scale as σ^{-J+1}. These contributions can be calculated via the means of conformal Regge theory, as explained in [63].

The conformal blocks in the S-channel transform as (see e.g. [44, 75])

$$g_{\Delta,J}(z,\bar{z}) \to \mathrm{e}^{-i\pi(\Delta-J)} g_{\Delta,J}(z,\bar{z}), \tag{3.35}$$

after the analytic continuation. In particular, for double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ with scaling dimension $\Delta = \Delta_H + \Delta_L + 2n + l + \gamma(n,l)$, the blocks transform as

$$g^{\Delta_{HL},-\Delta_{HL}}_{[\mathcal{O}_H\mathcal{O}_L]_{n,l}}(z,\bar{z}) \to e^{-i\pi(\Delta_H+\Delta_L)} e^{-i\pi\gamma(n,l)} g^{\Delta_{HL},-\Delta_{HL}}_{[\mathcal{O}_H\mathcal{O}_L]_{n,l}}(z,\bar{z}). \tag{3.36}$$

In what follows it will be convienent to do a change of variables to $h = n + l$ and $\bar{h} = n$ and to denote the block due to a heavy-light double-trace operator $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ as $g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}$. Substituting the μ expansion (3.25) in the S-channel (3.18) and performing the usual analytic continuation to $\mathcal{O}(\mu)$ leads to

$$\begin{aligned} G(z,\bar{z})|_{\mu^0} &= (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}) \\ G(z,\bar{z})|_{\mu^1} &= (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \Bigg(P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} \\ &\quad + \gamma^{(1)}_{\bar{h},h-\bar{h}} \left(\frac{1}{2}\left(\partial_h + \partial_{\bar{h}}\right) - i\pi \right) \Bigg) g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}). \end{aligned} \tag{3.37}$$

The new single trace operators that can possibly appear here would be subleading in $1/N^2$. Continuing to $\mathcal{O}(\mu^2)$, the imaginary part of the S-channel is given by

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = -i\pi(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \times$$
$$\times \Big(\gamma^{(2)}_{\bar{h},h-\bar{h}} + \gamma^{(1)}_{\bar{h},h-\bar{h}} P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} + \frac{(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2}{2}(\partial_h + \partial_{\bar{h}})\Big) g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}). \tag{3.38}$$

Moreover, the real part of the correlator at the same order is given by

$$\mathrm{Re}(G(z,\bar{z}))|_{\mu^2} = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \Big(P^{(HL,HL);(2)}_{\bar{h},h-\bar{h}}$$
$$- \frac{1}{2}\pi^2(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2 + \frac{1}{2}(\gamma^{(2)}_{h,\bar{h}} + P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}})(\partial_h + \partial_{\bar{h}})$$
$$+ \frac{1}{8}(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2(\partial_h + \partial_{\bar{h}})^2\Big) g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z})\,. \tag{3.39}$$

As we will see below, in the Regge limit the dominant contribution in the S-channel comes from double-trace operators with $h, \bar{h} \gg 1$. In this limit the OPE coefficients are given by

$$P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \approx C_{\Delta_L}(h\bar{h})^{\Delta_L-\frac{d}{2}}(h-\bar{h})^{\frac{d}{2}-1}. \tag{3.40}$$

We will further need $\lambda_{\mathcal{O}_L\mathcal{O}_L T}\lambda_{\mathcal{O}_H\mathcal{O}_H T}$ in (3.19), these are fixed by Ward Identities to be

$$P^{(HH,LL)}_{T_{\mu\nu}} = \left(-\frac{1}{2}\right)^2 \lambda_{\mathcal{O}_L\mathcal{O}_L T}\lambda_{\mathcal{O}_H\mathcal{O}_H T} = \frac{\Delta_L}{4}\left(\frac{d}{d-1}\right)^2 \frac{\Delta_H}{C_T} = \mu\frac{\Delta_L}{4}\frac{\Gamma\left(\frac{d}{2}+1\right)^2}{\Gamma\left(d+2\right)}. \tag{3.41}$$

Note that as explained in [55], an expansion in μ corresponds in the bulk to an expansion in the black hole Schwarzschild radius in AdS units. The bulk description of Regge limit follows almost intuitively from the CFT consideration. Namely, the heavy operators generate the black hole background where the light (but not massless) operators propagate on almost null geodesics. This implies that the energy of the light operators has to be large compared to the inverse impact parameter. On the more formal level, one can analyze the CFT Regge limit in the momentum space, for Mellin amplitude, where it is given by $s \to \infty$ and fixed t. This is the limit where we have high energy scattering with fixed momentum transfer which justifies the usage of the eikonal approximation in the bulk calculation.

3.4.2. Lightcone limit of the HHLL correlator

Below we review the setup of a heavy-heavy-light-light correlator with focus on its behaviour in the lightcone limit. Using conformal transformations we define the stress tensor sector of the correlator by

$$\mathcal{G}(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H} \langle \mathcal{O}_H(x_4)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\Big|_{\text{multi-stress tensors}}, \tag{3.42}$$

where z and $\bar{z}$ are the cross-ratios given by (3.16). In (3.42) the "multi-stress tensor" subscript stands to indicate the contribution of the identity and all multi-stress tensor operators.

The correlator $\mathcal{G}(z,\bar{z})$ can be expanded in the "T-channel" $\mathcal{O}_L(1) \times \mathcal{O}_L(z,\bar{z}) \to \mathcal{O}_{\tau,s}$ as[10]

$$\mathcal{G}(z,\bar{z}) = [(1-z)(1-\bar{z})]^{-\Delta_L} \sum_{\mathcal{O}_{\tau,s}} P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z}), \tag{3.43}$$

where $\tau = \Delta - s$ and s denote the twist and spin of the exchanged operator, respectively, and $g^{(0,0)}_{\tau,s}(z,\bar{z})$ the conformal block of the primary operator $\mathcal{O}_{\tau,s}$. Moreover, $P^{(HH,LL)}_{\mathcal{O}_{\tau,s}}$ are defined as

$$P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} = \left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_{\tau,s}} \lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}_{\tau,s}}, \tag{3.44}$$

where $\lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}}$ and $\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}}$ denote the respective OPE coefficients.

We will mainly be interested in the lightcone limit defined by $u \ll 1$ or equivalently $\bar{z} \to 1$. In this limit the T-channel expansion (3.43) is dominated by minimal-twist operators as follows from the behaviour of the conformal blocks

$$\mathcal{G}(u,v) \underset{u\to 0}{\approx} u^{-\Delta_L} \sum_{\mathcal{O}_{\tau,s}} P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} u^{\frac{\tau}{2}} (1-v)^{-\frac{\tau}{2}} f_{\frac{\tau}{2}+s}(v), \tag{3.45}$$

where twist τ is given just above and

$$f_{\frac{\tau}{2}+s}(v) = (1-v)^{\frac{\tau}{2}+s} {}_2F_1\Big(\frac{\tau}{2}+s, \frac{\tau}{2}+s, \tau+2s, 1-v\Big) \tag{3.46}$$

[10] For reasons of convenience, here and in the rest of the thesis we refer to $\mathcal{G}(z,\bar{z})$ as the correlator; the reader should keep in mind that $\mathcal{G}(z,\bar{z})$ is not the full correlator $G(z,\bar{z})$ but only its stress tensor sector, as defined in (3.42).

is a $SL(2;R)$ conformal block.

For any CFT in $d > 2$ the leading contribution in the lightcone limit comes from the exchange of the identity operator with twist $\tau = 0$. Another operator present in any unitary CFT is the stress tensor with twist $\tau = d - 2$. Its contribution to the correlator is completely fixed by a Ward identity and given by (3.41).

As explained in [55], the correlator admits a natural perturbative expansion in μ,

$$\mathcal{G}(z,\bar{z}) = \sum_k \mu^k \mathcal{G}^{(k)}(z,\bar{z})\,. \tag{3.47}$$

Using (3.45) and (3.41), we find the following contribution due to the stress tensor at $\mathcal{O}(\mu)$

$$\mathcal{G}^{(1)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^{\frac{d-2}{2}}(1-z)^{\frac{d+2}{2}}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{\Delta_L \Gamma(\frac{d}{2}+1)^2}{4\Gamma(d+2)}\, {}_2F_1\Big(\frac{d+2}{2}, \frac{d+2}{2}; d+2; 1-z\Big). \tag{3.48}$$

Let us study the correlator perturbatively in μ in the lightcone limit. At k-th order in that expansion we expect contributions from minimal-twist multi-stress tensor operators of the schematic form

$$[T^k]_{\tau_{k,\min},s} =: T_{\mu_1\nu_1}\dots\partial_{\lambda_1}\dots\partial_{\lambda_\ell}T_{\mu_k\nu_k} :, \tag{3.49}$$

where the minimal-twist $\tau_{k,\min}$ and spin s of these operators are given by

$$\begin{aligned} \tau_{k,\min} &= k(d-2), \\ s &= 2k+\ell \end{aligned} \tag{3.50}$$

and ℓ an even integer denoting the number of uncontracted derivatives. We moreover define the product of OPE coefficients for minimal-twist operators at order k as

$$P^{(HH,LL)}_{[T^k]_{\tau_{k,\min},s}} = \mu^k P^{(HH,LL);(k)}_{\tau_{k,\min},s}. \tag{3.51}$$

Compared to the $k = 1$ case, there exists an infinite number of minimal-twist multi-stress tensor operators for each value of $k > 1$. To obtain their contribution to the correlator in the lightcone limit, we thus have to sum over all these operators.

The correlator can likewise be expanded in the "S-channel" $\mathcal{O}_L(z,\bar{z}) \times \mathcal{O}_H(0) \to \mathcal{O}_{\tau',s'}$ as

$$\mathcal{G}(z,\bar{z}) = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{\mathcal{O}_{\tau',s'}} P^{(HL,HL)}_{\mathcal{O}_{\tau',s'}} g^{(\Delta_{HL},-\Delta_{HL})}_{\tau',s'}(z,\bar{z}). \tag{3.52}$$

where $P^{(HL,HL)}_{\mathcal{O}_{\tau',s'}}$ are the products of the corresponding OPE coefficients and $\Delta_{HL} = \Delta_H - \Delta_L$. Operators contributing in the S-channel are heavy-light double-trace operators [55,12][11] $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$. In the $\Delta_H \to \infty$ limit the $d=4$ blocks are given by

$$g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \approx \frac{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L+2n+\gamma)}}{\bar{z}-z}\left(\bar{z}^{l+1} - z^{l+1}\right), \tag{3.53}$$

and similarly in $d=6$

$$\begin{aligned} g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \approx & \frac{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L+2n+\gamma(n,l))}}{(\bar{z}-z)^3} \\ & \times \left(\bar{z}^{l+3} - \frac{l+3}{l+1}\bar{z}^{l+2}z^1 - (z \leftrightarrow \bar{z})\right). \end{aligned} \tag{3.54}$$

The anomalous dimensions $\gamma(n,l)$ and the product of the OPE coefficients of the double-trace operators $P^{(HL,HL)}_{n,l}$ admit an expansion in μ given by (3.25). The zeroth order OPE coefficients in large Δ_H limit, given by (3.28), can be subsequently expanded in large l limit ($l \gg n \sim 1$). We obtain

$$P^{(HL,HL);\mathrm{MFT}}_{n,l} \approx \frac{l^{\Delta_L-1}(\Delta_L - \frac{d}{2}+1)_n}{n!\,\Gamma(\Delta_L)}. \tag{3.55}$$

To reproduce the correct singularities manifest in the T-channel one has to sum over infinitely many heavy-light double-trace operators with $l \gg 1$. For such operators the dependence of the OPE data on the spin l for $l \gg 1$ is[12]:

$$\begin{aligned} P^{(HL,HL);(k)}_{n,l} &= \frac{P^{(k)}_n}{l^{\frac{k(d-2)}{2}}}, \\ \gamma^{(k)}_{n,l} &= \frac{\gamma^{(k)}_n}{l^{\frac{k(d-2)}{2}}}. \end{aligned} \tag{3.56}$$

3.4.3. Higher-twist multi stress tensors contributions to HHLL correlator

Here, we review the contributions of the higher-twist multi stress tensor operators to the heavy-heavy-light-light correlator in four-dimensional spacetime ($d=4$).

[11] This the analogue of light-light double-trace operators that are present in the cross channel of $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\mathcal{O}_1\rangle$, with $\mathcal{O}_1$ and $\mathcal{O}_2$ both light, in any CFT [56,57].

[12] This behavior in the large l limit is different from that of the OPE data of light-light double-trace operators [56,57].

3.4.3.1. T-channel expansion

Consider the T-channel expansion (3.43) in $d = 4$. Conformal blocks in $d = 4$ are given by [30,31]

$$g_{\tau,s}^{(0,0)}(1-z, 1-\bar{z}) = \frac{(1-z)(1-\bar{z})}{\bar{z}-z}\left(f_{\frac{\beta}{2}}(z) f_{\frac{\tau-2}{2}}(\bar{z}) - f_{\frac{\beta}{2}}(\bar{z}) f_{\frac{\tau-2}{2}}(z)\right), \quad (3.57)$$

with conformal spin, $\beta = \Delta + s$, and

$$f_a(z) = (1-z)^a {}_2F_1(a, a, 2a, 1-z). \quad (3.58)$$

In the lightcone limit, defined by $\bar{z} \to 1$ and z fixed, the leading contribution to the conformal blocks (3.57) comes from the first term in parenthesis in (3.57)

$$g_{\tau,s}^{(0,0)}(1-z, 1-\bar{z}) = (1-\bar{z})^{\frac{\tau}{2}}\left(f_{\frac{\beta}{2}}(z) + \mathcal{O}((1-\bar{z}))\right). \quad (3.59)$$

From (3.59) it is clear that the operators with the lowest twist in the T-channel dominate the correlator in the lightcone limit.

Note that the only single-trace primaries with twist equal to or lower than that of the stress tensor are scalars $\mathcal{O}$ with dimension $1 \leq \Delta_{\mathcal{O}} \leq 2$, or conserved currents with twist $\tau = 2$. In a theory without supersymmetry there is no *a priori* reason for the contributions of these operators, even if they exist, to be enhanced by a factor of Δ_H, so generically we expect them to be subleading in $C_T \to \infty$ limit.[13]

The stress tensor sector of the correlator (3.42) admits a perturbative expansion in μ given by (3.47), where the cases $k = 0$ and $k = 1$ correspond to the exchange of the identity and the stress tensor, respectively. For higher k we expect "multi-stress tensors" to contribute to $\mathcal{G}(z, \bar{z})$. Since we are interested in the four-point function of pairwise identical scalar operators, only multi-stress tensor operators with even spin give a nonvanishing contribution. At $\mathcal{O}(\mu^2)$, the contribution of these operators was explicitly calculated in [124]. Following that, it was shown in [13] how one can write the contributions of these operators

[13] Interestingly, in [129] it is conjectured that OPE coefficients $\lambda_{\phi\psi\psi}$ of operators ϕ with conformal dimension $\Delta_\phi \ll \Delta_{\text{gap}}$ and ψ with conformal dimension Δ_ψ, such that $\Delta_\phi \ll \Delta_\psi \ll C_T^{\#>0}$, scale as $\lambda_{\phi\psi\psi} \propto \frac{\Delta_\psi}{\sqrt{C_T}}$. Note however that here we are working in different regime, as $\Delta_H \propto \mathcal{O}(C_T)$.

at arbitrary order in the μ-expansion, in the lightcone limit $(1-\bar{z}) \ll 1$, using an appropriate ansatz and lightcone bootstrap. We briefly review this procedure here since the contribution from non-minimal-twist operators is obtained in a similar manner.

At $\mathcal{O}(\mu^k)$, there are infinitely many minimal-twist multi-stress tensors with twist $2k$ according to (3.50) which are distinguished by their conformal spin $\beta = \Delta + s$ given by $\beta = 6k + 4\ell$ with $\ell = 0, 1, 2, \ldots$. Inserting the leading behavior of the blocks (3.59) in (3.43) one finds

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^k}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\ell=0} P^{(k)}_{\Delta(\ell),s(\ell)} f_{\frac{\beta(\ell)}{2}}(z), \tag{3.60}$$

with

$$\mu^k P^{(k)}_{\Delta(\ell),s(\ell)} = P^{(HH,LL)}_{[T^k]_{\tau,s(\ell)}}, \tag{3.61}$$

where $\Delta(\ell) = \frac{\tau_{k,\min}+\beta}{2}$, $\tau_{k,\min} = 2k$, $s(\ell) = 2k + 2\ell$ and conformal spin $\beta = 6k + 4\ell$. Here $\underset{\bar{z}\to 1}{\approx}$ means that only the leading contribution as $\bar{z} \to 1$ is kept. It was shown in [13] that the infinite sum in (3.60) takes a particular form

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^k}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1\ldots i_k} f_{i_1}(z) \ldots f_{i_k}(z), \qquad \sum_{p=1}^{k} i_p = 3k, \tag{3.62}$$

with i_p being integers and $a_{i_1\ldots i_k}$ are coefficients that can be determined via lightcone bootstrap. Furthermore, using an identity for the product of two f_a functions (Eq. (A.1) in [124]) one can express the $\mathcal{G}^{(k)}(z,\bar{z})$ in the form of (3.60) to read off the OPE coefficients for the exchange of minimal-twist multi-stress tensors of arbitrary conformal spin.

In this section, we want to consider multi-stress tensors with non-minimal twist. These operators are obtained by contracting indices in (3.49) either between the derivatives or between the operators. At $\mathcal{O}\left(\mu^k\right)$ there exist operators $[T^k]_{\tau_{k,m},s}$ with twist

$$\tau_{k,m} = \tau_{k,\min} + 2m, \tag{3.63}$$

for any non-negative integer m. For $m \neq 0$, these operators provide subleading contributions to the correlator in the lightcone limit. To consider these subleading contributions it is convenient to expand $\mathcal{G}^{(k)}(z,\bar{z})$ from (3.47) as

$$\mathcal{G}^{(k)}(z,\bar{z}) = \sum_{m=0}^{\infty} (1-\bar{z})^{-\Delta_L+k+m} \mathcal{G}^{(k,m)}(z), \tag{3.64}$$

where $\mathcal{G}^{(k,m)}(z)$ comes from operators of twists $\tau_{k,m}$ and less.

For illustration, let us consider the case $k=2$ with $m=1$. There exist two infinite families of operators with twist $\tau_{2,1}=6$ of the schematic form

$$\begin{aligned} \mathcal{O}_{6,2\ell_1+2} &\sim : T_{\mu\kappa}\partial_{\lambda_1}\dots\partial_{\lambda_{2\ell_1}}T^{\kappa}{}_{\nu} :, \\ \mathcal{O}'_{6,2\ell_2+4} &\sim : T_{\mu\nu}\partial_{\lambda_1}\dots\partial_{\lambda_{2\ell_2}}\partial^2 T_{\rho\sigma} : . \end{aligned} \tag{3.65}$$

These two families share the same twist and spin for $\ell_1=\ell_2+1$. Hence, they are indistinguishable for $\ell_1\geq 1$ at order $1/C_T$ in the large C_T expansion. A single operator stands out; it corresponds to $\ell_1=0$ and is of the schematic form $: T_{\mu\alpha}T^{\alpha}{}_{\nu} :$. Note that $: T_{\mu\alpha}T^{\alpha}{}_{\nu} :$ has minimal conformal spin $\beta=10$, among the ones in (3.65), since $\beta_{\ell_1}=\beta_{\ell_2+1}=10+4\ell_1$, for $\ell_1\geq 1$.

Let us now move on to the case $k=2$ and $m=2$. Here, there are three infinite families $\mathcal{O}_{8,s}$, $\mathcal{O}'_{8,s}$ and $\mathcal{O}''_{8,s}$ with conformal spin $8+4\ell_1$, $12+4\ell_2$ and $16+4\ell_3$, respectively. Schematically, these families can be represented as

$$\begin{aligned} \mathcal{O}_{8,2\ell_1} &\sim : T_{\alpha\beta}\partial_{\lambda_1}\dots\partial_{\lambda_{2\ell_1}}T^{\alpha\beta} :, \\ \mathcal{O}'_{8,2\ell_2+2} &\sim : T_{\mu\alpha}\partial_{\lambda_1}\dots\partial_{\lambda_{2\ell_2}}\partial^2 T^{\alpha}{}_{\nu} :, \\ \mathcal{O}''_{8,2\ell_3+4} &\sim : T_{\mu\nu}\partial_{\lambda_1}\dots\partial_{\lambda_{2\ell_3}}(\partial^2)^2 T_{\rho\sigma} : . \end{aligned} \tag{3.66}$$

Notice once more that the infinite families are indistinguishable for conformal spin $\beta\geq 16$. Here, operators with $\beta=8,12$ stand out. The operator with $\beta=8$ is of the schematic form $: T_{\alpha\beta}T^{\alpha\beta} :$. For $\beta=12$, there are two indistinguishable operators of the schematic form $: T_{\mu\alpha}\partial^2 T^{\alpha}{}_{\nu} :$ and $: T_{\alpha\beta}\partial_\mu\partial_\nu T^{\alpha\beta} :$.

The same holds for $m\geq 3$ (and $\tau\geq 10$) since there is no other independent way to contract stress tensor indices. The discussion above generalizes straightforwardly to $\mathcal{O}(\mu^k)$ with $k+1$ number of infinite families at high enough twist.

3.4.3.2. S-channel expansion

The anomalous dimensions and the product of OPE coefficients for heavy-light double-trace operators that contribute to the S-channel admit an expansion in powers of μ given by (3.25).

We begin by briefly reviewing the calculation in the lightcone expansion, i.e. due to the multi-stress tensors in the T-channel. Inserting the blocks (3.53) in the S-channel expansion (3.52) one finds that

$$\mathcal{G}(z,\bar{z}) = \sum_{n=0}^{\infty} \frac{(z\bar{z})^n}{\bar{z}-z} \int_0^{\infty} dl P_{n,l}^{(HL,HL)} (z\bar{z})^{\frac{1}{2}\gamma_{n,l}} (\bar{z}^{l+1} - z^{l+1}), \tag{3.67}$$

where the sum was approximated by an integral over l, with $(1-\bar{z})$ being the small parameter. Namely, the difference between sum and integral is $\mathcal{O}(1-\bar{z})$ and since we are always interested in the singular terms when $1-\bar{z} \ll 1$, this difference does not affect the calculation. Expanding the OPE data in (3.67) according to (3.25) and noting that

$$(z\bar{z})^{\frac{1}{2}\gamma_{n,l}} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\gamma_{n,l} \log(z\bar{z})}{2} \right)^j, \tag{3.68}$$

it follows that terms proportional to $\log^i z$ at $\mathcal{O}(\mu^k)$, with $i = 2, 3, \ldots k$, in (3.67) are determined by OPE data at $\mathcal{O}(\mu^{k-1})$. These terms can therefore be matched with the T-channel in order to fix the coefficients in the ansatz.

In [13], the leading contribution of the OPE data of heavy-light double-trace operators as $l \to \infty$, together with the leading contribution of the conformal blocks as $\bar{z} \to 1$, was used to determine the minimal-twist contributions in the stress tensor sector of the T-channel. This section extends that analysis by considering subleading corrections in the lightcone expansion and therefore probing non-minimal-twist contributions in the T-channel. In particular, the S-channel OPE data have the following dependence on the spin l as $l \to \infty$:

$$\begin{aligned} \gamma_{n,l}^{(k)} &= \frac{1}{l^k} \sum_{p=0}^{\infty} \frac{\gamma_n^{(k,p)}}{l^p}, \\ P_{n,l}^{(HL,HL);(k)} &= \frac{1}{l^k} \sum_{p=0}^{\infty} \frac{P_n^{(HL,HL);(k,p)}}{l^p}, \end{aligned} \tag{3.69}$$

which is necessary in order to reproduce the correct power of $(1-\bar{z})$ as $\bar{z} \to 1$. This can be seen by substituting the expansion of (3.28) in the large-l limit

$$P_{n,l}^{(HL,LH);MFT} = l^{\Delta_L} \left(\frac{(\Delta_L - 1)_n}{n!\Gamma(\Delta_L) l} + \frac{(2n(\Delta_L - 2) + \Delta_L(\Delta_L - 1))(\Delta_L - 1)_n}{2(n!)\Gamma(\Delta_L) l^2} + \mathcal{O}\left(\frac{1}{l^3}\right) \right), \tag{3.70}$$

and (3.69) in (3.67) which result in integrals of the form

$$\int_0^\infty dl \bar{z}^l l^{\Delta_L - m - 1} = \frac{\Gamma(\Delta_L - m)}{(-\log \bar{z})^{\Delta_L - m}}, \tag{3.71}$$

where m is a positive integer. Expanding (3.71) for $\bar{z} \to 1$, the correct $\bar{z}$-behavior of the stress tensor sector in the T-channel is reproduced from the S-channel.

3.5. The rest of the thesis

Our first goal is to get insight into the heavy-heavy-light-light correlator at $\mathcal{O}(\mu^2)$, where there are the contributions of double-trace operators, made of two stress tensors, whose OPE coefficients, in general, are unknown, although some of these OPE coefficients are found in [15]. In Section 4 we study the heavy-heavy-light-light correlator in the Regge limit. We show how to compute the anomalous dimensions and the corrections to the MFT OPE coefficients of the double-trace operators in the S-channel at $\mathcal{O}(\mu^2)$ using the gravitational calculation of the wave function phase shift of the light probe in the black hole background.

In Section 5 we establish the complete algorithm for computing the contributions of the minimal-twist[14] subset of the stress tensor sector at arbitrary order μ^k in even-dimensional spacetime. The algorithm relies on the appropriate functional ansatz for these contributions and the lightcone conformal bootstrap that fixes the coefficients in the ansatz. When these coefficients in the ansatz are fixed, one can read off the OPE coefficients of arbitrary multi stress tensor operator with a minimal twist at the given order μ^k. We show that these operators have universal OPE coefficients in agreement with the statement in [15], which means that they are the same in the whole class of holographic CFTs.

In Section 6 we extend the algorithm to include the contributions of multi stress tensor operators with an arbitrary twist at each order in μ. We show that the only OPE coefficients that can not be determined by the conformal bootstrap technique are those of operators of spin $s = 0, 2$. These OPE coefficients can be thought of as the parameters of the theory, that are not fixed by the consistency condition and they parametrize the class of holographic CFTs.

[14] Twist τ of the operator is defined as the difference between the conformal dimension and spin, $\tau = \Delta - s$.

We show how the OPE coefficients of operators of spin two can be extracted directly from the gravitational computation of the wave function phase shift of the light probe in the black hole background.

Finally, in Section 7 we study the thermalization properties of the stress tensor sector in general large-N theories, without the large gap assumptions. We show that the stress tensor sector of the theory thermalizes in the pure heavy scalar state $\mathcal{O}_H$ with the large conformal dimension $\Delta_H \sim \mathcal{O}(C_T)$. The thermalization is manifested by the equality of the OPE coefficients of the multi stress tensor operators with two heavy operators and the thermal expectation value of stress multi stress tensor. Section 8 sums up the conclusions.

4. Black holes and conformal Regge bootstrap

4.1. Summary of the results

In this section we explain how to compute the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ order by order in μ, using the phase shift result of [55]. In particular, we show that the $\mathcal{O}(\mu^2)$ anomalous dimensions in any d are given by

$$\gamma^{(2)}_{\bar{h},h-\bar{h}} = -\frac{\delta^{(2)}}{\pi} + \frac{\gamma^{(1)}_{\bar{h},h-\bar{h}}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}_{\bar{h},h-\bar{h}}, \qquad \Delta_H \gg h, \bar{h} \gg 1. \tag{4.1}$$

Using known results for $\delta^{(1)}$ and $\delta^{(2)}$ from [55], we find an explicit expression for $\gamma^{(2)}_{\bar{h},h-\bar{h}}$ and compare it with the known results in the lightcone limit ($\Delta_H \gg h \gg \bar{h} \gg 1$). We find perfect agreement.

The rest of this section is organized as follows. In Section 4.2 we focus on four-dimensional holographic CFTs. At $\mathcal{O}(\mu)$, we use the crossing equation between the S- and T-channel to solve for the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$. The result is eq. (3.4), valid for $l, n \gg 1$. We then introduce the impact parameter representation which allows us to rewrite the S-channel expansion as a Fourier transform. We use this to relate the phase shift to the anomalous dimensions of $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ at $\mathcal{O}(\mu^2)$, thereby deriving (4.1). Using a known result for the phase shift $\delta^{(2)}$, we write down an explicit expression for $\gamma^{(2)}_{n,l}$. In the subsequent $l \gg n$ limit it reduces to the result which has been obtained in [55] in a completely different way (by computing corrections to the energies of excited states in the AdS-Schwarzschild background).

In Section 4.3 we generalize these results to any d ($d = 2$ is treated separately in Appendix A.4.). By solving the Casimir equation in the limit $\Delta_H \gg \Delta_L, l, n$, we obtain the conformal blocks for heavy-light double-trace operators in the S-channel. Using the explicit expression for the blocks together with the zeroth order OPE coefficients, we derive an impact parameter representation valid in general dimensions. Just as in the $d = 4$ case, this allows us to write the S-channel sum as a Fourier transform. Hence, we show that (4.1) holds for any d. We compute $\gamma^{(2)}_{n,l}$ in the lightcone limit and find perfect agreement with the results quoted in [55]. In addition, we find an expression for the $\mathcal{O}(\mu^2)$ corrections to the OPE coefficients.

Section 4.4 discusses various observations and mentions some open problems. Appendices contain additional technical details. The conformal bootstrap calculations are summarized in Appendix A.1, the proof of the impact parameter representation in $d = 4$ in Appendix A.2 and the proof in general dimension d in Appendix A.3. The special case of $d = 2$ is treated in Appendix A.4. Appendix A.5 discusses the fate of some boundary terms. Appendices A.6 and A.7 contain some identities which are used in Section 4.4.

4.2. Anomalous dimensions of heavy-light double-trace operators in $d = 4$

In this section we investigate the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ in $d = 4$ using conformal bootstrap. Moreover, using a four-dimensional impact parameter representation we relate the anomalous dimensions to the bulk phase shift to $\mathcal{O}(\mu^2)$. This procedure can be repeated order by order in μ to obtain the OPE data (anomalous dimensions and OPE coefficients – see also Section 4.3) to the desired order.

4.2.1. Anomalous dimensions in the Regge limit using bootstrap

The conformal blocks in $d = 4$ are given by [30]

$$g^{\Delta_{12},\Delta_{34}}_{\Delta,J}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}\left(k_{\Delta+J}(z)k_{\Delta-J-2}(\bar{z}) - (z \leftrightarrow \bar{z})\right) \tag{4.2}$$

where

$$k_\beta(z) = z^{\beta/2}{}_2F_1\left(\frac{\beta-\Delta_{12}}{2}, \frac{\beta+\Delta_{34}}{2}, \beta, z\right). \tag{4.3}$$

In the limit $\Delta_H \sim C_T \gg 1$ the hypergeometric functions in (4.2) can be substituted by the identity up to $1/\Delta_H$ corrections. Explicitly, the conformal blocks of $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ in the heavy limit are given by

$$g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}) = \frac{(z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L)}\left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right)}{z-\bar{z}}. \tag{4.4}$$

Inserting this form of the conformal blocks in (3.37) together with the OPE coefficients in the Regge limit (3.40), we approximate the sums by integrals and find the following expression at $\mathcal{O}(\mu^0)$ in the S-channel

$$G(z,\bar{z})|_{\mu^0} = \frac{C_{\Delta_L}}{z-\bar{z}}\int_0^\infty dh \int_0^h d\bar{h}(h\bar{h})^{\Delta_L-2}(h-\bar{h})\left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right). \tag{4.5}$$

The integrals are explicitly computed in Appendix A.1; the result is the disconnected correlator in the T-channel $[(1-z)(1-\bar{z})]^{-\Delta_L}$ in the Regge limit $\sigma \to 0$.

At $\mathcal{O}(\mu)$ in holographic CFTs the leading corrections in the T-channel come from the exchanges of the stress tensor and double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ ($[\mathcal{O}_H\mathcal{O}_H]_{n,l=2}$ are heavy and therefore decouple). The conformal block for the T-channel exchange of the stress tensor is found after $z \to e^{-2\pi i}z$ to be given by

$$g_{T_{\mu\nu}} = \frac{360 i\pi e^{-\rho}}{\sigma(e^{2\rho}-1)} + \ldots, \tag{4.6}$$

where ... denotes non-singular terms. The contribution from the stress tensor exchange in the T-channel is thus imaginary for real values of σ and ρ. The only imaginary term at order μ in the S-channel expansion (3.37) comes from the term proportional to $-i\pi\gamma$; it must reproduce (4.6).

In the Regge limit, we approximate the sum in the S-channel by an integral and insert the OPE coefficients from (3.40); the imaginary part at $\mathcal{O}(\mu)$ in the S-channel is thus given by

$$\begin{aligned}\mathrm{Im}(G(z,\bar{z}))|_{\mu^1} = \frac{-i\pi C_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h})\gamma^{(1)}_{\bar{h},h-\bar{h}} \\ \left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right).\end{aligned} \tag{4.7}$$

With the ansatz $\gamma^{(1)}_{\bar{h},h-\bar{h}} = c_1 h^a \bar{h}^b/(h-\bar{h})$ the integrals in (4.7) can be computed (for more details see Appendix A.1). In order to reproduce the exchange of the stress tensor, the anomalous dimensions at $\mathcal{O}(\mu)$ must be equal to

$$\begin{aligned}\gamma^{(1)}_{\bar{h},h-\bar{h}} &= -\frac{90\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}\lambda_{\mathcal{O}_L\mathcal{O}_L T_{\mu\nu}}}{\mu\Delta_L}\frac{\bar{h}^2}{h-\bar{h}} \\ &= -\frac{3\bar{h}^2}{h-\bar{h}},\end{aligned} \tag{4.8}$$

where in the second line we inserted the OPE coefficients from (3.41). With the form (4.8) not only the stress tensor exchange is reproduced, but also an infinite sum of spin-2 double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ with scaling dimension $\Delta_n = 2\Delta_L + 2 + 2n$. This is similar to what happens in the light-light case [75].

To determine the second order corrections to the anomalous dimensions we use the derivative relationship:

$$P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} = \frac{1}{2}(\partial_h + \partial_{\bar{h}}) \left(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \gamma^{(1)}_{\bar{h},h-\bar{h}} \right). \tag{4.9}$$

We will prove below (see Section 4.3.3) that this relationship is true in the limit $h, \bar{h} \gg 1$. The imaginary part at $\mathcal{O}(\mu^2)$ in the S-channel from (3.37) is then given by

$$\begin{aligned} \mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = -i\pi \int_0^\infty dh \int_0^h d\bar{h} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \Bigg(& \gamma^{(2)}_{\bar{h},h-\bar{h}} \\ & + \gamma^{(1)}_{\bar{h},h-\bar{h}} P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}} + \frac{(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2}{2} (\partial_h + \partial_{\bar{h}}) \Bigg) g_{h,\bar{h}}. \end{aligned} \tag{4.10}$$

With the help of (4.9), one can write (4.10) as

$$\begin{aligned} \mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = & -i\pi \int_0^\infty dh \int_0^h d\bar{h} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} g_{h,\bar{h}} \\ & \times \left(\gamma^{(2)}_{\bar{h},h-\bar{h}} - \frac{\gamma^{(1)}_{\bar{h},h-\bar{h}}}{2} (\partial_h + \partial_{\bar{h}}) \gamma^{(1)}_{\bar{h},h-\bar{h}} \right) + \text{total derivative}, \end{aligned} \tag{4.11}$$

where the total derivate term does not contribute (see Appendix A.5 for details). In order to fix $\gamma^{(2)}_{\bar{h},h-\bar{h}}$ completely from crossing symmetry, we would need to consider the exchange of infinitely many double-trace operators made out of the stress tensor in the T-channel. Instead, we will use an impact parameter representation to relate $\gamma^{(2)}_{\bar{h},h-\bar{h}}$ to the bulk phase shift calculated from the gravity dual in [55].

4.2.2. 4d impact parameter representation and relation to bulk phase shift

In [59] the anomalous dimensions of light-light double-trace operators in the limit $h, \bar{h} \gg 1$ were shown to be related to the bulk phase shift. An impact parameter representation for the case when one of the operators is heavy was introduced in [55], where it was also shown that the bulk phase shift and the anomalous dimensions are equal at $\mathcal{O}(\mu)$. The goal of this section is to see explicitly how the bulk phase shift and the anomalous dimensions are related to $\mathcal{O}(\mu^2)$.

The correlator (3.17) can be written in an impact parameter representation as

$$G(z,\bar{z}) = \int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}} f(h,\bar{h}), \tag{4.12}$$

with $\mathcal{I}_{h,\bar{h}}$ given by

$$\mathcal{I}_{h,\bar{h}} = (z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}) \tag{4.13}$$

and $f(h,\bar{h})$ some function that generically depends on the anomalous dimension and corrections to the OPE coefficients. In particular, for $f(h,\bar{h})=1$, (4.12) is equal to the disconnected correlator. In Appendix A.2 it is shown that $\mathcal{I}_{h,\bar{h}}$ can be equivalently written as

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta_L) \int_{M^+} \frac{d^4p}{(2\pi)^4} (-p^2)^{\Delta_L-2} e^{-ipx} (h-\bar{h}) \delta(p\cdot\bar{e}+h+\bar{h})\, \delta\left(\frac{p^2}{4}+h\bar{h}\right) \tag{4.14}$$

where M^+ is the upper Milne wedge with $\{p^2 \leq 0,\ p^0 \geq 0\}$, $C(\Delta_L)$ given by (with $d=4$)

$$C(\Delta) \equiv \frac{2^{d+1-2\Delta}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta)\Gamma(\Delta-\frac{d}{2}+1)} \tag{4.15}$$

and $\bar{e}=(1,0,0,0)$. Moreover, following [55], we will set $z=e^{ix^+}$ and $\bar{z}=e^{ix^-}$, with $x^+=t+r$ and $x^-=t-r$ in spherical coordinates.

Using the identity

$$\delta(p\cdot\bar{e}+h+\bar{h})\, \delta\left(\frac{p^2}{4}+h\bar{h}\right) = \frac{1}{|h-\bar{h}|}\left(\delta\left(\frac{p^+}{2}-h\right)\delta\left(\frac{p^-}{2}-\bar{h}\right)+(h\leftrightarrow\bar{h})\right), \tag{4.16}$$

with $p^+=p^t+p^r$, $p^-=p^t-p^r$, the integrals over $h,\bar{h}$ in (4.12) are easily computed. With the identification $h=\frac{p^+}{2}$ and $\bar{h}=\frac{p^-}{2}$ it follows that a generic term like (4.12) can be written as a Fourier transform

$$\int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}} f(h,\bar{h}) = C(\Delta_L)\int_{M^+} \frac{d^4p}{(2\pi)^4}(-p^2)^{\Delta_L-2} e^{-ipx} f\left(\frac{p^+}{2},\frac{p^-}{2}\right). \tag{4.17}$$

We thus see that the impact parameter representation allows us to rewrite the S-channel expression as a Fourier transform.

The phase shift $\delta(p)$ for a pair of operators $\mathcal{O}_H$ and $\mathcal{O}_L$, with scaling dimensions $\Delta_H/C_T \propto \mu$ and $\Delta_L/C_T \ll 1$, respectively, was defined in [55] by

$$\mathcal{B}(p) \equiv \int d^4x e^{ipx} G(x) = \mathcal{B}_0(p) e^{i\delta(p)}, \tag{4.18}$$

where $G(x)$ is given in (3.17) and $\mathcal{B}_0(p)$ denotes the Fourier transform of the disconnected correlator. As the OPE data, the phase shift admits an expansion in μ:

$$\delta(p) = \mu\delta^{(1)}(p) + \mu^2\delta^{(2)}(p) + \ldots, \tag{4.19}$$

where ... denotes higher order terms in the expansion. Expanding the exponential in (4.18) in μ we get

$$\mathcal{B}(p) = \mathcal{B}_0(p)\Big(1 + i\mu\delta^{(1)} + \mu^2(-\frac{(\delta^{(1)})^2}{2} + i\delta^{(2)}) + \ldots\Big). \tag{4.20}$$

With (4.20) the relationship between the anomalous dimensions and the bulk phase shift to $\mathcal{O}(\mu^2)$ can be established using (3.37), (3.38) and (4.17):

$$\begin{aligned} \gamma^{(1)}_{\bar{h},h-\bar{h}} &= -\frac{\delta^{(1)}}{\pi} \\ \gamma^{(2)}_{\bar{h},h-\bar{h}} &= -\frac{\delta^{(2)}}{\pi} + \frac{\gamma^{(1)}_{\bar{h},h-\bar{h}}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}_{\bar{h},h-\bar{h}}. \end{aligned} \tag{4.21}$$

The phase shift was calculated in closed form to all orders in μ for the four-dimensional case [55], with the first and second order terms given by

$$\begin{aligned} \delta^{(1)} &= \frac{3\pi}{2}\sqrt{-p^2}\frac{e^{-L}}{e^{2L}-1} \\ \delta^{(2)} &= \frac{35\pi}{8}\sqrt{-p^2}\frac{2e^{L}-e^{-L}}{(e^{2L}-1)^3}, \end{aligned} \tag{4.22}$$

where

$$-p^2 = p^+p^-, \qquad \cosh L = \frac{p^+ + p^-}{2\sqrt{-p^2}}. \tag{4.23}$$

Using (4.22) and (4.23), the $\mathcal{O}(\mu)$ corrections to the anomalous dimensions are given by $\gamma^{(1)}_{n,l} = -3n^2/l$, which agrees with (4.8). From (4.22) and (4.21), we deduce the anomalous dimensions at $\mathcal{O}(\mu^2)$:

$$\gamma^{(2)}_{n,l} = -\frac{35}{4}\frac{(2l+n)n^3}{l^3} + 9\frac{n^3}{l^2}. \tag{4.24}$$

Taking the lightcone limit ($l \gg n \gg 1$) in (4.24) we find

$$\gamma^{(2)}_{n,l} \approx -\frac{17}{2}\frac{n^3}{l^2}. \tag{4.25}$$

The anomalous dimensions in the lightcone limit (4.25) agree with eq. (6.40) in [55], which was obtained independently by considering corrections to the energy levels in the AdS-Schwarzschild background.

4.3. OPE data of heavy-light double-trace operators in generic d

In this section we will write the general form of conformal blocks for heavy-light double-trace operators in the limit $\Delta_H \sim C_T \gg 1$ and general $d > 2$. These blocks will be used to confirm the validity of the impact parameter representation in Appendix A.3. Using the impact parameter representation the OPE data will be related to the bulk phase shift. In particular, we show that (4.21) remains valid in any number of dimensions and find explicit expressions for the corrections to the OPE coefficients up to $\mathcal{O}(\mu^2)$.

4.3.1. Conformal blocks in the heavy limit

In order to find conformal blocks in general spacetime dimension d in the limit $\Delta_H \gg \Delta_L, h, \bar{h}$, we write them in the following form:

$$g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = (z\bar{z})^{\frac{\Delta_H+\Delta_L}{2}} F(z,\bar{z}), \tag{4.26}$$

where the function $F(z,\bar{z})$ does not depend on Δ_H and is symmetric with respect to the exchange $z \leftrightarrow \bar{z}$. Let us now insert the expression (4.26) into the Casimir equation and consider the leading $\mathcal{O}(\Delta_H)$ term:

$$z\frac{\partial}{\partial z}F(z,\bar{z}) + \bar{z}\frac{\partial}{\partial \bar{z}}F(z,\bar{z}) - (h+\bar{h})F(z,\bar{z}) = 0. \tag{4.27}$$

The most general solution to eq. (4.27) is:

$$F(z,\bar{z}) = z^{h+\bar{h}} f\Big(\frac{\bar{z}}{z}\Big), \tag{4.28}$$

where f is an arbitrary function that satisfies $f(\frac{1}{x}) = x^{-h-\bar{h}} f(x)$, since conformal blocks must be symmetric with respect to the exchange $z \leftrightarrow \bar{z}$.

The behavior of the conformal blocks as $z, \bar{z} \to 0$ and $z/\bar{z}$ fixed is given by [30,130]

$$g_{\Delta,l}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) \to \frac{l!}{(\frac{d}{2}-1)_l} (z\bar{z})^{\frac{\Delta}{2}} C_l^{(\frac{d}{2}-1)}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big), \tag{4.29}$$

where $\Delta = \Delta_1 + \Delta_2 + 2n + l$ and $C_q^{(p)}(x)$ are the Gegenbauer polynomials. Using (4.29), we can completely determine the function f:

$$f\Big(\frac{\bar{z}}{z}\Big) = \frac{(h-\bar{h})!}{(\frac{d}{2}-1)_{h-\bar{h}}} \Big(\frac{\bar{z}}{z}\Big)^{\frac{h+\bar{h}}{2}} C_{h-\bar{h}}^{(\frac{d}{2}-1)}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big). \tag{4.30}$$

That is, the conformal blocks in the limit of large Δ_H are given by

$$g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = \frac{(h-\bar{h})!}{(\frac{d}{2}-1)_{h-\bar{h}}}(z\bar{z})^{\frac{\Delta_H+\Delta_L+h+\bar{h}}{2}} C_{h-\bar{h}}^{(\frac{d}{2}-1)}\left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\right). \tag{4.31}$$

It is easy to explicitly check that this form of the conformal blocks agrees with the one we used in $d=4$ in the previous section.

4.3.2. Anomalous dimensions

In Appendix A.3 we prove the validity of the impact parameter representation in any d. This means that the derivation of (4.21) goes through for arbitrary d. Using known results for the bulk phase shift from [55], we thus find

$$\gamma^{(1)}_{\bar{h},h-\bar{h}} = -\frac{\bar{h}^{\frac{d}{2}}}{h^{\frac{d}{2}-1}}\frac{\Gamma(d)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)}\,{}_2F_1(\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}). \tag{4.32}$$

In the lightcone limit ($h = l \gg \bar{h} = n$) this reduces to

$$\gamma^{(1)}_{\bar{h},h-\bar{h}} \approx -\frac{\bar{h}^{\frac{d}{2}}}{h^{\frac{d}{2}-1}}\frac{\Gamma(d)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)}. \tag{4.33}$$

Similarly, using (4.21) together with eq. (2.29) and eq. (A.5) from [55], we find the $\mathcal{O}(\mu^2)$ corrections to the anomalous dimensions in the limit $h, \bar{h} \gg 1$:

$$\begin{aligned}
\gamma^{(2)}_{\bar{h},h-\bar{h}} &= -\frac{\delta^{(2)}}{\pi} + \frac{1}{2}\gamma^{(1)}_{\bar{h},h-\bar{h}}\left\{\frac{2}{h+\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}} - \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)^2}\bar{h}^{\frac{d}{2}-1}h^{\frac{d}{2}-1}\frac{(h-\bar{h})^{3-d}}{h+\bar{h}}\right\} = \\
&= -\left(\frac{\bar{h}^{d-1}}{h^{d-2}}\right)\frac{2^{2d-4}\Gamma\left(d+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(d)}\ {}_2F_1[2d-3, d-2, d, \frac{\bar{h}}{h}] + \\
&+ \frac{\bar{h}^d h^{2-d}}{(h+\bar{h})}\ \frac{4\Gamma^2(d)}{d^2\,\Gamma^4\left(\frac{d}{2}\right)}\ \left({}_2F_1[\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}]\right)^2 + \\
&+ \frac{\bar{h}^{d-1}(h-\bar{h})^{3-d}}{h+\bar{h}}\ \frac{\Gamma^2(d)}{d\,\Gamma^4\left(\frac{d}{2}\right)}\ {}_2F_1[\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}]
\end{aligned} \tag{4.34}$$

Taking further the lightcone limit ($h \gg \bar{h}$) we find that

$$\gamma^{(2)}_{\bar{h},h-\bar{h}} \approx \frac{\bar{h}^{d-1}}{h^{d-2}}\frac{2^{2d-4}}{\pi}\left(\frac{d\Gamma\left(\frac{d+1}{2}\right)^2}{\Gamma\left(\frac{d+2}{2}\right)^2} - \frac{\sqrt{\pi}\Gamma\left(d+\frac{1}{2}\right)}{\Gamma\left(d\right)}\right). \tag{4.35}$$

The result (4.35) agrees with eq. (6.42) in [55] which was obtained independently using perturbation theory in the bulk. In order to see this explicitly, one should notice the following expression for the hypergeometric function:

$$
{}_3F_2(1,-\frac{d}{2},-\frac{d}{2};1+\frac{d}{2},1+\frac{d}{2};1)=\frac{1}{2}\Big(1+\frac{\Gamma^4(1+\frac{d}{2})\Gamma(2d+1)}{\Gamma^4(d+1)}\Big). \tag{4.36}
$$

4.3.3. Corrections to the OPE coefficients

So far, we have only considered the imaginary part of the S-channel. The real part at $\mathcal{O}(\mu)$ is given by the following expression:

$$
\begin{aligned}
\mathrm{Re}(G(z,\bar{z}))|_\mu =&(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dh\int_0^h d\bar{h}P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \\
&\times\Big(P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}}+\frac{1}{2}\gamma^{(1)}_{\bar{h},h-\bar{h}}(\partial_h+\partial_{\bar{h}})\Big)g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}),
\end{aligned} \tag{4.37}
$$

which can be rewritten as:

$$
\begin{aligned}
&\mathrm{Re}(G(z,\bar{z}))|_\mu=(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dh\int_0^h d\bar{h}g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}\times \\
&\times\Big(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}}-\frac{1}{2}(\partial_h+\partial_{\bar{h}})(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}})\Big) \\
&+\text{total derivative}.
\end{aligned} \tag{4.38}
$$

The total derivative term in (4.38) can be shown to vanish as explained in Appendix A.5.

To derive a relation between the corrections to the OPE coefficients and the anomalous dimensions at $\mathcal{O}(\mu)$, let us consider the limit $h,\bar{h}\gg 1$ and substitute $\bar{h}$ by h everywhere. Using (4.32), one can deduce $\gamma^{(1)}_{\bar{h},h-\bar{h}}\propto h$. Then, it follows that $(\partial_h+\partial_{\bar{h}})(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}})\propto P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}$ and hence the second term on the right hand side of (4.38) behaves as:

$$
\begin{aligned}
\frac{(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}}{2}\int_0^{+\infty}dh\int_0^h d\bar{h}g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(\partial_h+\partial_{\bar{h}})(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}}) \\
\propto\frac{1}{\sigma^{2\Delta_L}}.
\end{aligned} \tag{4.39}
$$

On the other hand, we know that in the Regge limit the leading contribution in the T-channel at $\mathcal{O}(\mu)$ comes from the exchange of the stress tensor. The real part of its conformal block is proportional to σ^d, so the T-channel result behaves as $\frac{1}{\sigma^{2\Delta_L-d}}$. This is way less singular than (4.39). Hence (4.39) must be canceled by the first term on the right hand side of (4.38), at least in the limit $h,\bar{h}\gg 1$. That is:

$$P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}}=\frac{1}{2}(\partial_h+\partial_{\bar{h}})(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}}). \tag{4.40}$$

A similar relation holds for the OPE coefficients of light-light double-trace operators, e.g. see [11,38,53]. In that case it was observed in [49] that the relation is not exact in $(h,\bar{h})$. We expect the same to be true here. Furthermore, the real part at $\mathcal{O}(\mu^2)$ was given in (3.39) as:

$$\begin{aligned}\mathrm{Re}(G(z,\bar{z}))|_{\mu^2}=(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}&\sum_{h\geq\bar{h}\geq 0}^{\infty}P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}\Big(P^{(HL,HL);(2)}_{\bar{h},h-\bar{h}}\\&-\frac{1}{2}(\pi\gamma^{(1)}_{\bar{h},h-\bar{h}})^2+\frac{1}{2}(\gamma^{(2)}_{\bar{h},h-\bar{h}}+P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}})(\partial_h+\partial_{\bar{h}})\\&+\frac{1}{8}(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2(\partial_h+\partial_{\bar{h}})^2\Big)g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}.\end{aligned} \tag{4.41}$$

Using the impact parameter representation this can be expressed as:

$$\begin{aligned}\mathrm{Re}(G(z,\bar{z}))|_{\mu^2}=&\int_0^\infty dh\int_0^h d\bar{h}\mathcal{I}_{h,\bar{h}}\ \Big(P^{(HL,HL);(2)}_{\bar{h},h-\bar{h}}-\frac{\pi^2}{2}(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2\\&-\frac{1}{2P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}}(\partial_h+\partial_{\bar{h}})(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}(\gamma^{(2)}_{\bar{h},h-\bar{h}}+P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}}))\\&+\frac{1}{8P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}}(\partial_h+\partial_{\bar{h}})^2(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2)\Big),\end{aligned} \tag{4.42}$$

where we repeatedly integrated by parts. It follows from (4.20) and (4.17), together with $\pi\gamma^{(1)}_{\bar{h},h-\bar{h}}=-\delta^{(1)}$, that the corrections to the OPE coefficients at $\mathcal{O}(\mu^2)$ satisfy the following relationship:

$$\begin{aligned}P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}P^{(2)}_{\bar{h},h-\bar{h}}=&-\frac{1}{8}(\partial_h+\partial_{\bar{h}})^2(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}(\gamma^{(1)}_{\bar{h},h-\bar{h}})^2)\\&+\frac{1}{2}(\partial_h+\partial_{\bar{h}})(P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}}(\gamma^{(2)}_{\bar{h},h-\bar{h}}+P^{(HL,HL);(1)}_{\bar{h},h-\bar{h}}\gamma^{(1)}_{\bar{h},h-\bar{h}})).\end{aligned} \tag{4.43}$$

The arguments above are similar to the ones used in [75,59].

4.3.4. Flat space limit

In the flat space limit the relation between the scattering phase shift and the anomalous dimensions has been previously discussed in [131]. Hence, it is interesting to consider the flat space limit of eq. (4.1). This limit is achieved by taking the apparent impact parameter to be much smaller than the AdS radius. This corresponds to the small L regime or, equivalently, using $e^{-2L} = \bar{h}/h$ to the $1 \ll l \ll n \ll \Delta_H$ limit.

In this limit, according to (4.32), the behavior of $\gamma^{(1)}_{n,l}$ is given by

$$\gamma^{(1)}_{n,l} \propto n \left(\frac{n}{l}\right)^{d-3} . \tag{4.44}$$

Hence, the $\gamma^{(1)}_{\bar{h},h-\bar{h}}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}_{\bar{h},h-\bar{h}}$ term in eq. (4.1) behaves as

$$\gamma^{(1)}_{n,l}\partial_n\gamma^{(1)}_{n,l} \propto n \left(\frac{n}{l}\right)^{2d-6} . \tag{4.45}$$

Similarly, using equation (A.5) from [55], one finds that $\delta^{(2)}$ behaves as

$$\delta^{(2)} \propto n \left(\frac{n}{l}\right)^{2d-5} . \tag{4.46}$$

Since (4.45) is subleading to (4.46), in the flat space limit the anomalous dimensions are proportional to the phase shift,

$$\gamma^{(2)}_{n,l} \approx -\frac{\delta^{(2)}}{\pi}. \tag{4.47}$$

4.4. Discussion

In this section we studied a four-point function of pairwise identical scalar operators, $\mathcal{O}_H$ and $\mathcal{O}_L$, in holographic CFTs of any dimensionality. Scaling Δ_H with the central charge, the CFT data admits an expansion in the ratio $\mu \sim \Delta_H/C_T$ which we keep fixed. Using crossing symmetry and the bulk phase shift calculated in [55], we studied $\mathcal{O}(\mu^2)$ corrections to the OPE data of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ for large l and n. In particular, the relationship between the bulk phase shift and the OPE data of heavy-light double-trace operators is found using an impact parameter representation. Furthermore, this allows us in principle to determine the OPE data of $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$, for $l, n \gg 1$

to all orders in μ, i.e., to all orders in an expansion in the dual black hole Schwarzschild radius.

Scaling Δ_H with the central charge enhances the effect of stress tensor exchanges compared to the $1/C_T$ corrections due to the exchange of generic operators. At $\mathcal{O}(\mu^2)$ and higher, we therefore expect multi-stress tensor operators to contribute. The OPE coefficients for such exchanges are not known in general. They would be needed to determine corrections to the OPE data of heavy-light double-trace operators using purely CFT methods. In a recent paper [15] some of these OPE coefficients have been computed. In particular, the OPE coefficients with the multi-stress tensor operators of lowest twist have been argued to be universal (independent of the higher derivative couplings in the bulk gravitational lagrangian). It would be interesting to connect these results to the ones discussed in this section.

It is a curious fact that each term in the μ-expansion of the bulk phase shift as computed in gravity in [55] can be expressed as an infinite sum of "Regge conformal blocks" corresponding to operators of dimension $\Delta = k(d-2)+2n+2$ and spin $J = 2$. Explicitly,

$$i\,\delta^{(k)}(S,L) = \, f(k) \sum_{n=0}^{\infty} \lambda_k(n) \;\, g^R_{k(d-2)+2n+2,\,2}(S,L)\,, \tag{4.48}$$

where the coefficients $(f(k), \lambda_k(n))$ are listed in Appendix A.6 and we set $S \equiv \sqrt{-p^2}$ compared to [55]. Here $g^R_{\Delta,J}(S,L)$ denotes a "Regge conformal block", and is equal to the leading behaviour of the analytically continued T-channel conformal block in the Regge limit [132,74]

$$g^R_{\Delta,J}(S,L) = i\, c_{\Delta,J}\; S^{J-1}\; \Pi_{\Delta-1,d-1}(L) \tag{4.49}$$

defined in terms of

$$1 - z = \frac{e^L}{S}, \quad 1 - \bar{z} = \frac{e^{-L}}{S} \tag{4.50}$$

as $S \to \infty$ and L fixed. Here $c_{\Delta,J}$ are known coefficients which can be found in Appendix A.6 and $\Pi_{\Delta-1,d-1}(L)$ denotes the $(d-1)$-dimensional hyperbolic space propagator for a massive scalar of mass square $m^2 = (\Delta - 1)$.

To understand the implications of (4.48) let us focus on $k = 2$ and consider large impact parameters, a.k.a. the lightcone limit. In this case, one expects that the dominant contribution to the bulk phase shift comes from the infinite sum of the minimal twist double-trace operators built from the stress tensor, schematically denoted by $T_{\mu\nu}\partial_{\mu_1}\cdots\partial_{\mu_\ell}T_{\rho\sigma}$. (4.48) implies that this infinite sum gives rise to a contribution which can be interpreted as coming from a *single* conformal block of an "effective" operator of the same twist $\tau = 2(d-2)$, but spin $J = 2$. At finite impact parameter, one would then need to add the contributions of an infinite tower of such effective operators of twist $\tau = 2(d-2)+2n$ and spin $J = 2$, as expressed by the infinite sum in (4.48). From this point of view, the coefficients λ_n in (4.48) can be interpreted as ratios of sums of OPE coefficients of double-trace operators. It is clear that this picture appears to hold to all orders in $\left(\frac{\Delta_H}{C_T}\right)$ or equivalently, the Schwarzschild radius of the black hole.

It would be interesting to investigate whether Rindler positivity constrains the Regge behaviour of the bulk phase shift to grow at most linearly with the energy S, similarly to Section 5.2 in [74]. If this were true, one would perhaps only need to understand the origin of the λ_n to compute the bulk phase shift to arbitrary order in $\left(\frac{\Delta_H}{C_T}\right)$ purely from CFT techniques.

5. The minimal twist multi-stress tensors and conformal bootstrap

5.1. Summary of the results

In this section we argue that for a large class of CFTs (including holographic CFTs) in even d, the contribution of minimal-twist multi-stress tensors to the correlator in the lightcone limit can be written as a sum of products of certain hypergeometric functions. To be explicit, let us define functions $f_a(z)$ as

$$f_a(z) = (1-z)^a {}_2F_1(a, a, 2a; 1-z). \tag{5.1}$$

The stress tensor contribution to the correlator in the lightcone limit is given in any dimension d by

$$\mathcal{G}^{(1)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^{\frac{d-2}{2}}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{\Delta_L \Gamma(\frac{d}{2}+1)^2}{4\Gamma(d+2)} f_{\frac{d+2}{2}}(z). \tag{5.2}$$

At $\mathcal{O}(\mu^2)$ the contribution from twist-four double-stress tensor operators in $d=4$ is

$$\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L-2)}\right) \times \\ \left((\Delta_L-4)(\Delta_L-3) f_3^2(z) + \frac{15}{7}(\Delta_L-8) f_2(z) f_4(z) + \frac{40}{7}(\Delta_L+1) f_1(z) f_5(z)\right). \tag{5.3}$$

This result agrees with the expression obtained by different methods in [124].

The contribution from twist-six triple-stress tensors in the lightcone limit in $d=4$ at order $\mathcal{O}(\mu^3)$ is

$$\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(a_{117} f_1(z)^2 f_7(z) + a_{126} f_1(z) f_2(z) f_6(z) \\ + a_{135} f_1(z) f_3(z) f_5(z) + a_{225} f_2(z)^2 f_5(z) + a_{234} f_2(z) f_3(z) f_4(z) + a_{333} f_3(z)^3\Big), \tag{5.4}$$

where coefficients a_{ijk} are given by (5.26).

Furthermore, from (5.4) and (5.26), we find the OPE coefficients of twist-six triple-stress tensor operators as a finite sum (for details see Section 5.2.5). Two such OPE coefficients for twist-6 triple-stress tensors were calculated holographically in [15] and agree with our results.

The contribution from twist-eight double-stress tensors to the correlator in the lightcone limit in $d = 6$ at order $\mathcal{O}(\mu^2)$ is

$$\begin{aligned} \mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^4}{[(1-z)(1-\bar{z})]^{\Delta_L}} \times \\ & \Big(a_{13} f_1(z) f_7(z) + a_{26} f_2(z) f_6(z) + a_{35} f_3(z) f_5(z) + a_{44} f_4(z)^2 \Big), \end{aligned} \tag{5.5}$$

where a_{mn} are given by (5.49). Using (5.5) and (5.49) we find the OPE coefficients for operators of type $: T_{\mu\nu} \partial_{\lambda_1} \ldots \partial_{\lambda_{2\ell}} T_{\alpha\beta} :$ in $d = 6$ to be equal to:

$$P_{8,s}^{(HH,LL)} = \mu^2 \frac{c\Delta_L}{(\Delta_L - 3)(\Delta_L - 4)} (a_3 \Delta_L^3 + a_2 \Delta_L^2 + a_1 \Delta_L + a_0), \tag{5.6}$$

where c and a_m, given by (5.57), are functions of the total spin $s = 4 + 2\ell$.

In general we propose that the contribution from minimal-twist multi-stress tensor operators to the correlator in even d at $\mathcal{O}(\mu^k)$ in the lightcone limit takes the form

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^{k(\frac{d}{2}-1)}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1\ldots i_k} f_{i_1}(z) \ldots f_{i_k}(z), \quad \sum_{p=1}^{k} i_p = k\Big(\frac{d+2}{2}\Big), \tag{5.7}$$

where the sum goes over all sets of $\{i_p\}$ with $i_p \leq i_{p+1}$ and $a_{i_1\ldots i_k}$ coefficients that need to be fixed.[15]

We also check that the stress tensor sector of the near lightcone correlator exponentiates

$$\langle \mathcal{O}_H(x_4) \mathcal{O}_L(1) \mathcal{O}_L(z,\bar{z}) \mathcal{O}_H(0) \rangle|_{\text{multi-stress tensors}} \underset{\bar{z}\to 1}{\approx} \frac{e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}}{[(1-z)(1-\bar{z})]^{\Delta_L}}, \tag{5.8}$$

where $\mathcal{F}(\mu; z, \bar{z})$ is a rational function of Δ_L that remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$. We explicitly verify this up to $\mathcal{O}(\mu^3)$ in $d = 4$ and $\mathcal{O}(\mu^2)$ in $d = 6$.

5.1.1. Outline

The rest of Section 5 is organized as follows. In Section 5.2 we find the contribution of minimal-twist double- and triple-stress tensor operators in $d = 4$ in the lightcone limit. We show that this contribution exponentiates and we write

[15] One only needs to sum the linearly independent products of functions f_a.

an expression for the OPE coefficients of minimal-twist triple-stress tensors of spin s with scalar operators, in the form of a finite sum. In Section 5.3, we repeat this program up to $\mathcal{O}(\mu^2)$ in $d = 6$. Again we confirm exponentiation and we find a closed form expression for the OPE coefficients of minimal-twist double-stress tensors of arbitrary spin with scalar operators. We discuss our results in Section 5.4.

5.2. Multi-stress tensors in four dimensions

In this section we describe how to use crossing symmetry to fix the contribution of minimal-twist multi-stress tensors to the heavy-heavy-light-light correlator in $d = 4$ to $\mathcal{O}(\mu^3)$. The methods described generalize to other even spacetime dimensions, with the six-dimensional case to $\mathcal{O}(\mu^2)$ described in Section 5.3. In principle the same technology can also be used to determine the correlator at higher orders. Moreover, the resulting expression can be decomposed into multi-stress tensor blocks of minimal-twist, allowing us at each order to read off the OPE coefficients of minimal-twist multi-stress tensors.

The idea is to study the S-channel expansion in (3.52) in the limit $1 - \bar{z} \ll z \ll 1$. In this limit operators with $l \gg 1$ and low values of n dominate. Namely, the OPE coefficients and the anomalous dimensions of the S-channel operators can be expanded in powers of $1/l$. The leading contribution in $1 - \bar{z} \ll 1$ limit is due to the leading term in $1/l$ expansion and same is true for subleading contributions. Subsequently, the leading contribution in $z \ll 1$ limit is due to the $n = 0$ term, the subleading is due to the $n = 1$, and so on. Expanding the conformal blocks in (3.53) for small $\gamma(n, l)$ and $\bar{z} \to 1$, the blocks in $d = 4$ reduce to

$$(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \bar{z}^l p(\log z, \gamma(n,l)) \frac{z^n}{1-z}, \tag{5.9}$$

where $p(\log z, \gamma(n, l))$ is given by

$$p(\log z, \gamma(n,l)) = z^{\frac{1}{2}\gamma(n,l)} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\gamma(n,l)\log z}{2} \right)^j . \tag{5.10}$$

Inserting (5.9) into (3.52) and converting the sum into an integral, we have the following expression for the correlator in the limit $\bar{z} \to 1$

$$\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \sum_{n=0}^{\infty} \frac{z^n}{1-z} \int_0^{\infty} dl P^{(HL,HL)}_{n,l} \bar{z}^l p(\log z, \gamma(n,l)). \tag{5.11}$$

In the following we consider an expansion of (5.11) around $z = 0$. The key point is to note that by expanding the anomalous dimensions and OPE coefficients, as in (3.25), terms proportional to $z^p \log^i z$ with $i = 2, 3, \ldots, k$ and any p at $\mathcal{O}(\mu^k)$, in (5.11) are completely determined in terms of OPE data at $\mathcal{O}(\mu^{k-1})$. Moreover, using (3.56) one sees that the integral over the spin l yields[16]

$$\int_0^\infty dl l^{\Delta_L - 1 - k} \bar{z}^l = \frac{\Gamma(\Delta_L - k)}{(-\log \bar{z})^{\Delta_L - k}} \underset{\bar{z}\to 1}{\approx} \frac{\Gamma(\Delta_L - k)}{(1-\bar{z})^{\Delta_L - k}}, \tag{5.12}$$

at $\mathcal{O}(\mu^k)$ in the limit $\bar{z} \to 1$. This correctly reproduces the expected $\bar{z}$ behavior of minimal-twist multi-stress tensors in the T-channel, thus verifying (3.56). Additionally, it is easy to check that the difference between the integral and the sum of expression $\bar{z}^l l^{\Delta_L - k - 1}$ is $\mathcal{O}(1-\bar{z})$ in $1 - \bar{z} \ll 1$ limit, while both integral and sum scale as $(1-\bar{z})^{-\Delta_L + k}$ when $\bar{z} \to 1$. As we always (implicitly) assume that $\Delta_L > k$, at each order in μ we are only interested in singular terms of sum or integral, therefore, the $\mathcal{O}(1-\bar{z})$ difference will never affect the calculation.

We now make the following ansatz for the correlator

$$\mathcal{G}^{(k)}(z, \bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^k}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1 \ldots i_k} f_{i_1}(z) \ldots f_{i_k}(z), \tag{5.13}$$

where the sum goes over all sets of $\{i_p\}$ with i_p integers and $i_p \le i_{p+1}$ such that $\sum_{p=1}^k i_p = 3k$ and $a_{i_1 \ldots i_k}$ coefficients that need to be fixed. Generally $f_a(z)$ are given by

$$f_a(z) = q_{1,a}(z) + q_{2,a}(z) \log z, \tag{5.14}$$

where $q_{(1,2),a}(z)$ are rational functions and the ansatz (5.13) at $\mathcal{O}(\mu^k)$ is therefore a polynomial in $\log z$ of degree k. By crossing symmetry terms with $\log^a z$, with $2 \le a \le k$, are determined by OPE data at $\mathcal{O}(\mu^{k-1})$. This is what we will use to determine the coefficients $a_{i_1 \ldots i_p}$.

[16] At each order in μ we implicitly assume that integrals of this type converge, i.e. that $\Delta_L > k$.

5.2.1. Stress tensor

We start by determining the OPE data at $\mathcal{O}(\mu)$. This is easily obtained by matching (5.11) at $\mathcal{O}(\mu)$ with the stress tensor contribution (3.48). Explicitly, multiplying both channels by $(1-z)$ we have at $\mathcal{O}(\mu)$

$$\frac{\Delta_L f_3(z)}{120[(1-z)(1-\bar{z})]^{\Delta_L-1}} = \frac{1}{(1-\bar{z})^{\Delta_L-1}} \sum_{n=0}^{\infty} \frac{\Gamma(\Delta_L+n-1)z^n}{\Gamma(\Delta_L)n!} \times \left(P_n^{(1)} + \frac{\gamma_n^{(1)}}{2}\log z\right). \tag{5.15}$$

Expanding the LHS in (5.15) for $z \ll 1$ we find

$$\begin{aligned}\frac{\Delta_L/120}{[(1-z)(1-\bar{z})]^{\Delta_L-1}} f_3(z) = \frac{1}{(1-\bar{z})^{\Delta_L-1}}\Big(&-\frac{\Delta_L}{4}(3+\log z) \\ &- z\frac{\Delta_L}{4}(3(\Delta_L+1)+(\Delta_L+5)\log z) \\ &- z^2\frac{\Delta_L}{8}\left(3\Delta_L(\Delta_L+3)+(12+\Delta_L(\Delta_L+11))\right) \\ &+\mathcal{O}(z^3, z^3\log z)\Big),\end{aligned} \tag{5.16}$$

while the RHS is given by

$$\begin{aligned}\frac{\sum_{n=0}^{\infty}\frac{\Gamma(\Delta_L+n-1)z^n}{\Gamma(\Delta_L)n!}(P_n^{(1)}+\frac{\gamma_n^{(1)}}{2}\log z)}{(1-\bar{z})^{\Delta_L-1}} = \frac{1}{(1-\bar{z})^{\Delta_L-1}}\Big(&\frac{P_0^{(1)}+\frac{\gamma_0^{(1)}}{2}\log z}{\Delta_L-1} \\ &+ z(P_1^{(1)}+\frac{\gamma_1^{(1)}}{2}\log z) \\ &+ z^2\frac{\Delta_L}{2}(P_2^{(1)}+\frac{\gamma_2^{(1)}}{2}\log z) \\ &+\mathcal{O}(z^3, z^3\log z)\Big).\end{aligned} \tag{5.17}$$

Comparing (5.16) and (5.17) order-by-order in z one finds the following OPE data

$$\begin{aligned}\gamma_0^{(1)} &= -\frac{\Delta_L(\Delta_L-1)}{2}, \\ \gamma_1^{(1)} &= -\frac{\Delta_L(\Delta_L+5)}{2}, \\ \gamma_2^{(1)} &= -\frac{12+\Delta_L(\Delta_L+11)}{2},\end{aligned} \tag{5.18}$$

which agrees with eq. (6.10) in [55], and the OPE coefficients

$$\begin{aligned} P_0^{(1)} &= -\frac{3\Delta_L(\Delta_L-1)}{4}, \\ P_1^{(1)} &= -\frac{3\Delta_L(\Delta_L+1)}{4}, \\ P_2^{(1)} &= -\frac{3\Delta_L(\Delta_L+3)}{4}. \end{aligned} \tag{5.19}$$

It is straightforward to continue and compute the $\mathcal{O}(\mu)$ OPE data in the S-channel for any value of n.

5.2.2. Twist-four double-stress tensors

From (5.13) we infer the following expression for the contribution due to twist-four double-stress tensors to the heavy-heavy-light-light correlator in the limit $\bar{z} \to 1$:

$$\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(a_{15} f_1(z) f_5(z) + a_{24} f_2(z) f_4(z) + a_{33} f_3^2(z) \Big). \tag{5.20}$$

By expanding (5.20) further in the limit $z \ll 1$ and collecting terms that goes as $z^p \log^2 z$, we will match with known contributions obtained from (5.11).

Inserting (5.18) and (5.19) in the S-channel (5.11) fixes terms proportional to $z^p \log^2 z$ up to $\mathcal{O}(z^2 \log^2 z)$. Expanding the ansatz (5.20) and matching with the S-channel reproduces the result obtained in [124]:

$$\begin{aligned} \mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L-2)} \right) \times \\ & \Big((\Delta_L-4)(\Delta_L-3) f_3^2(z) + \frac{15}{7}(\Delta_L-8) f_2(z) f_4(z) \\ & + \frac{40}{7}(\Delta_L+1) f_1(z) f_5(z) \Big). \end{aligned} \tag{5.21}$$

Using the $\mathcal{O}(\mu)$ OPE data in the S-channel for $n > 2$ in (5.16) and (5.17) one gets an overconstrained system which is still solved by (5.21). This is a strong argument in favor of the validity of our ansatz (5.13).

We can now use (5.21) to derive the $\mathcal{O}(\mu^2)$ OPE data in the S-channel by matching terms proportional to $z^p \log^i z$ as $z \to 0$, with $i = 0, 1$, by comparing with (5.11). This is done in the same way it was done for $\mathcal{O}(\mu)$ OPE data in the S-channel. For example, one finds the following data for $n = 0, 1, 2, 3$:

$$\begin{aligned}
\gamma_0^{(2)} &= -\frac{(\Delta_L - 1)\Delta_L(4\Delta_L + 1)}{8}, \\
\gamma_1^{(2)} &= -\frac{\Delta_L(\Delta_L + 1)(4\Delta_L + 35)}{8}, \\
\gamma_2^{(2)} &= -\frac{(3 + \Delta_L)(68 + \Delta_L(69 + 4\Delta_L))}{8}, \\
\gamma_3^{(2)} &= -\frac{(5 + \Delta_L)(204 + \Delta_L(4\Delta_L + 103))}{8},
\end{aligned} \tag{5.22}$$

which agrees with Eq. (6.39) in [55], and for the OPE coefficients

$$\begin{aligned}
P_0^{(2)} &= \frac{(\Delta_L - 1)\Delta_L(-28 + \Delta_L(-145 + 27\Delta_L))}{96}, \\
P_1^{(2)} &= \frac{\Delta_L(-596 + \Delta_L(-399 + \Delta_L(-64 + 27\Delta_L)))}{96}, \\
P_2^{(2)} &= \frac{-1248 + \Delta_L(-2252 + \Delta_L(-699 + \Delta_L(44 + 27\Delta_L)))}{96}, \\
P_3^{(2)} &= \frac{-3744 + \Delta_L(-4940 + \Delta_L(-783 + \Delta_L(152 + 27\Delta_L)))}{96}.
\end{aligned} \tag{5.23}$$

It is again straightforward to extract the OPE data for any value of n.

5.2.3. Twist-six triple-stress tensors

We now consider the multi-stress tensor sector of the correlator at $\mathcal{O}(\mu^3)$ and proceed similarly to the previous section. From (5.13) we infer the following expression for the contribution due to twist-six triple-stress tensors:

$$\begin{aligned}
\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(& a_{117} f_1^2 f_7 + a_{126} f_1 f_2 f_6 + a_{135} f_1 f_3 f_5 \\
& + a_{225} f_2^2 f_5 + a_{234} f_2 f_3 f_4 + a_{333} f_3^3 \Big),
\end{aligned} \tag{5.24}$$

where $f_i = f_i(z)$ is given by (3.46).[17] Taking the limit $1 - \bar{z} \ll z \ll 1$ of (5.24), we fix the coefficients by matching with terms proportional to $z^p \log^2 z$

[17] Note that we omitted a potential term of the form $f_1 f_4^2$. This can be written in terms of f_3^3, $f_1 f_3 f_5$, $f_2^2 f_5$ and $f_2 f_3 f_4$, as follows from:

$$f_3^3(z) = \frac{20}{21} f_1(z) f_3(z) f_5(z) - \frac{27}{28} f_1(z) f_4^2(z) - \frac{20}{21} f_2^2(z) f_5(z) + \frac{55}{28} f_2(z) f_3(z) f_4(z). \tag{5.25}$$

and $z^p \log^3 z$, with $p = 0, 1, 2$ from (5.11). This requires using the OPE data of the heavy-light double-trace operators $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ for $n = 0, 1, 2$ and $l \gg 1$ to $\mathcal{O}(\mu^2)$, given in (5.18), (5.19), (5.22) and (5.23).

We find the following solution:

$$\begin{aligned}
a_{117} &= \frac{5\Delta_L(\Delta_L+1)(\Delta_L+2)}{768768(\Delta_L-2)(\Delta_L-3)}, \\
a_{126} &= \frac{5\Delta_L(5\Delta_L^2-57\Delta_L-50)}{6386688(\Delta_L-2)(\Delta_L-3)}, \\
a_{135} &= \frac{\Delta_L(2\Delta_L^2-11\Delta_L-9)}{1209600(\Delta_L-3)}, \\
a_{225} &= -\frac{\Delta_L(7\Delta_L^2-51\Delta_L-70)}{2903040(\Delta_L-2)(\Delta_L-3)}, \\
a_{234} &= \frac{\Delta_L(\Delta_L-4)(3\Delta_L^2-17\Delta_L+4)}{4838400(\Delta_L-2)(\Delta_L-3)}, \\
a_{333} &= \frac{\Delta_L(\Delta_L-4)(\Delta_L^3-16\Delta_L^2+51\Delta_L+24)}{10368000(\Delta_L-2)(\Delta_L-3)}.
\end{aligned} \tag{5.26}$$

We can also consider higher values of p and obtain an overconstrained system of equations, whose solution is still (5.26). Inserting (5.26) into (5.24), we obtain the contribution from minimal-twist triple-stress tensor operators to the heavy-heavy-light-light correlator in the lightcone limit.

Note that for $\Delta_L \to \infty$, the correlator is determined by the exponentiation of the stress tensor discussed e.g. in [124], i.e.

$$\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{1}{3!} \left(\frac{\Delta_L}{120}(1-z)^3 {}_2F_1(3,3;6;1-z) \right)^3 + \cdots , \tag{5.27}$$

which one indeed obtains by taking $\Delta_L \to \infty$ of (5.24) with (5.26). Here ellipses denote terms subleading in Δ_L.

By analytically continuing $z \to e^{-2\pi i} z$ and sending $z \to 1$, one can access the large impact parameter regime of the Regge limit. To do this we use the following property of the hypergeometric function (see e.g. [98]):

$${}_2F_1(a,a,2a,1-ze^{-2\pi i}) = {}_2F_1(a,a,2a,1-z) + 2\pi i \frac{\Gamma(2a)}{\Gamma(a)^2} {}_2F_1(a,a,1,z). \tag{5.28}$$

Using (5.28) the leading term from (5.24) with the coefficients (5.26) in the limit $1-\bar{z} \ll 1-z \ll 1$ is given by

$$\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1, z\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} \times \left(-\frac{9i\pi^3 \Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{2(\Delta_L-2)(\Delta_L-3)} \left(\frac{1-\bar{z}}{(1-z)^2}\right)^3 \right). \tag{5.29}$$

This agrees with the holographic calculation in a shockwave background at $\mathcal{O}(\mu^3)$ given by Eq. (45) in [126] based on techniques developed in [58-60,63,133].

5.2.4. *Exponentiation of leading-twist multi-stress tensors*

In $d = 2$ the heavy-heavy-light-light correlator is determined by the heavy-heavy-light-light Virasoro vacuum block. This block contains the exchange of any number of stress tensors and derivatives thereof in the T-channel [40,105,110], and therefore all multi-stress tensor contributions. This block, together with the disconnected part, exponentiates as

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\rangle = e^{\Delta_L \mathcal{F}(z)}, \tag{5.30}$$

for a known function $\mathcal{F}(z)$ independent of Δ_L. It is interesting to ask if something similar happens for the contribution of the minimal-twist multi-stress tensors in the lightcone limit of the correlator in higher dimensions. By this we mean whether the stress tensor sector of the correlator can be written as

$$\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{5.31}$$

for some function $\mathcal{F}(\mu;z,\bar{z})$ which is a rational function of Δ_L and remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$.

The $\bar{z}$ dependence implies the following form of $\mathcal{F}(\mu;z,\bar{z})$:

$$\mathcal{F}(\mu;z,\bar{z}) = \mu(1-\bar{z})\mathcal{F}^{(1)}(z) + \mu^2(1-\bar{z})^2\mathcal{F}^{(2)}(z) + \mu^3(1-\bar{z})^3\mathcal{F}^{(3)}(z) + \mathcal{O}(\mu^4). \tag{5.32}$$

At leading order we observe $\mathcal{F}^{(1)}(z) = \frac{1}{120} f_3(z)$, which is just the stress tensor contribution. At second order we find:

$$\mathcal{F}^{(2)}(z) = \frac{(12-5\Delta_L)f_3(z)^2 + \frac{15}{7}(\Delta_L-8)f_2(z)f_4(z) + \frac{40}{7}(\Delta_L+1)f_1(z)f_5(z)}{28800(\Delta_L-2)}. \tag{5.33}$$

Note that $\mathcal{F}^{(2)}(z)$ is independent of Δ_L in the limit $\Delta_L \to \infty$.

To find $\mathcal{F}^{(3)}(z)$ we parametrise it as

$$\begin{aligned}\mathcal{F}^{(3)}(z) = \Big(& b_{117} f_1^2(z) f_7(z) + b_{126} f_1(z) f_2(z) f_6(z) + b_{135} f_1(z) f_3(z) f_5(z) \\ &+ b_{225} f_2^2(z) f_5(z) + b_{234} f_2(z) f_3(z) f_4(z) + b_{333} f_3^3(z)\Big).\end{aligned} \tag{5.34}$$

It is clear that for terms which do not contain a factor of $f_3(z)$, the coefficients b_{ijk} should satisfy $b_{ijk} = a_{ijk}/\Delta_L$. This is not true for terms which contain a factor of f_3. Inserting $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ and Eq. (5.34) in (5.31), expanding in μ and matching with (5.24) yields

$$\begin{aligned}
b_{117} &= \frac{a_{117}}{\Delta_L}, \\
b_{126} &= \frac{a_{126}}{\Delta_L}, \\
b_{225} &= \frac{a_{225}}{\Delta_L}, \\
b_{135} &= -\frac{11\Delta_L^2 - 19\Delta_L - 18}{1209600(\Delta_L - 2)(\Delta_L - 3)}, \\
b_{234} &= \frac{(\Delta_L - 2)(\Delta_L + 2)}{1209600(\Delta_L - 2)(\Delta_L - 3)}, \\
b_{333} &= \frac{7\Delta_L^2 - 18\Delta_L - 24}{2592000(\Delta_L - 2)(\Delta_L - 3)}.
\end{aligned} \tag{5.35}$$

From (5.33) and (5.35), one finds that the correlator exponentiates to $\mathcal{O}(\mu^3)$ in the sense described above, i.e. $\mathcal{F}(\mu; z, \bar{z})$ is a rational function of Δ_L of $\mathcal{O}(1)$ as $\Delta_L \to \infty$.

To leading order in Δ_L, exponentiation for large Δ_L is a prediction of the AdS/CFT correspondence. The two-point function of the operator $\mathcal{O}_L$ in the state created by the heavy operator $\mathcal{O}_H$ is given in terms of the exponential of the (regularized) geodesic distance between the boundary points in the dual bulk geometry. For details on this, see e.g. [124].

5.2.5. OPE coefficients of triple-stress tensors

In this section we describe how to decompose the correlator (5.24) into an infinite sum of minimal-twist triple-stress tensor operators. In order to do this we use the following multiplication formula for hypergeometric functions [124]:

$$
\begin{aligned}
{}_2F_1(a,a;2a;w){}_2F_1(b,b;2b;w) = \sum_{m=0}^{\infty} p[a,b,m] w^{2m} \times \\
{}_2F_1[a+b+2m, a+b+2m, 2a+2b+4m, w],
\end{aligned}
\tag{5.36}
$$

where

$$
\begin{aligned}
p[a,b,m] = \frac{\Gamma(a+b+2m)}{\Gamma(a+b+2m-\frac{1}{2})}\times \\
\frac{2^{-4m}\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})\Gamma(m+\frac{1}{2})\Gamma(a+m)\Gamma(b+m)\Gamma(a+b+m-\frac{1}{2})}{\sqrt{\pi}\Gamma(a)\Gamma(b)\Gamma(m+1)\Gamma(a+m+\frac{1}{2})\Gamma(b+m+\frac{1}{2})\Gamma(a+b+m)}.
\end{aligned}
\tag{5.37}
$$

It is useful to note that by using (5.36) we can write a similar formula for the functions f_a defined in (3.46):

$$
f_a(z) f_b(z) = \sum_{m=0}^{\infty} p[a,b,m] f_{a+b+2m}(z), \tag{5.38}
$$

where $p[a,b,m]$ is defined in (5.37). It is now clear that the correlator (5.24) can be written as a double sum over functions $f_{9+2(n+m)}$. We can thus write the stress tensor sector of the correlator in the lightcone limit at $\mathcal{O}(\mu^3)$ as

$$
\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{n,m=0}^{\infty} c[m,n] f_{9+2(n+m)}(z), \tag{5.39}
$$

with

$$
\begin{aligned}
c[m,n] =& \big(a_{333} p[3,3,m] p[3,6+2m,n] + a_{117} p[1,7,m] p[1,8+2m,n] \\
&+ a_{126} p[2,6,m] p[1,8+2m,n] + a_{135} p[3,5,m] p[1,8+2m,n] \\
&+ a_{225} p[2,5,m] p[2,7+2m,n] + a_{234} p[3,4,m] p[2,7+2m,n]\big),
\end{aligned}
\tag{5.40}
$$

where coefficients a_{ijk} are fixed in (5.26).

Comparing (5.39) with (3.45) we see that the contribution at $\mathcal{O}(\mu^3)$ comes from operators of the schematic form $:T_{\alpha\beta}T_{\gamma\delta}\partial_{\rho_1}\ldots\partial_{\rho_{2\ell}}T_{\mu\nu}:$. These operators have $\frac{\tau_{3,\min}}{2}+s = 9+2\ell$, where s is total spin $s = 6+2\ell$. The corresponding OPE coefficients of such operators will be a sum of all contributions in (5.39) for which $n+m=\ell$.

Now, one can write OPE coefficients of operators of type $:T_{\alpha\beta}T_{\gamma\delta}\partial_{\rho_1}\ldots\partial_{\rho_{2\ell}}T_{\mu\nu}:$ as

$$P^{(HH,LL);(3)}_{6,6+2\ell} = \sum_{n=0}^{\ell} c[\ell-n,n]. \tag{5.41}$$

Let us write a few of the coefficients explicitly here:

$$\begin{aligned}
\mu^3 P^{(HH,LL);(3)}_{6,6} &= \mu^3\frac{\Delta_L(3024+\Delta_L(7500+\Delta_L(7310+143\Delta_L(25+7\Delta_L))))}{10378368000(\Delta_L-2)(\Delta_L-3)},\\
\mu^3 P^{(HH,LL);(3)}_{6,8} &= \mu^3\frac{\Delta_L(2688+\Delta_L(7148+\Delta_L(9029+13\Delta_L(464+231\Delta_L))))}{613476864000(\Delta_L-3)(\Delta_L-2)},\\
\mu^3 P^{(HH,LL);(3)}_{6,10} &= \mu^3\frac{\Delta_L(888+\Delta_L(2216+\Delta_L(3742+17\Delta_L(181+143\Delta_L))))}{9468531072000(\Delta_L-3)(\Delta_L-2)}.
\end{aligned} \tag{5.42}$$

We further find that $P^{(HH,LL);(3)}_{6,6}$ and $P^{(HH,LL);(3)}_{6,8}$ agree with the expression obtained holographically in [15].

5.3. Minimal-twist double-stress tensors in six dimensions

In this section we derive the contribution of minimal-twist double-stress tensors to the heavy-heavy-light-light correlator in the lightcone limit in $d=6$. The method is analogous to the four-dimensional case described in Section 5.2.

From (5.7) we make the following ansatz for the stress tensor sector in the lightcone limit:

$$\begin{aligned}
\mathcal{G}^{(2)}(z,\bar z) \underset{\bar z\to 1}{\approx} & \frac{(1-\bar z)^4}{[(1-z)(1-\bar z)]^{\Delta_L}} \times \\
& \Big(a_{17}f_1(z)f_7(z)+a_{26}f_2(z)f_6(z)+a_{35}f_3(z)f_5(z)+a_{44}f_4^2(z)\Big).
\end{aligned} \tag{5.43}$$

The S-channel conformal blocks in six dimensions in the limit $\Delta_H\to\infty$ are given by (3.54). In the lightcone limit $\bar z\to 1$ operators with $l\gg 1$ dominate and the blocks can be approximated by

$$(z\bar z)^{-\frac{1}{2}(\Delta_H+\Delta_L)} g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar z) \simeq \frac{\bar z^l z^n p(\log z,\gamma)}{(1-z)^2}, \tag{5.44}$$

with p given by (5.10). Replacing the sum in (3.52) with an integral and inserting (5.44) we have

$$\mathcal{G}^{(2)}(z,\bar z) \underset{\bar z\to 1}{\approx} \sum_{n=0}^{\infty}\frac{z^n}{(1-z)^2}\int_0^\infty dl P^{(HL,HL)}_{n,l}\bar z^l p(\log z,\gamma). \tag{5.45}$$

As in $d=4$ one finds that terms proportional to $\log^i z$ with $i=2,3,\ldots,k$ at $\mathcal{O}(\mu^k)$, are determined by the OPE data at $\mathcal{O}(\mu^{k-1})$.

At $\mathcal{O}(\mu)$ we can use the known contribution from the stress tensor exchange (3.48) to derive the anomalous dimensions $\gamma_n^{(1)}$ and the OPE coefficients $P_n^{(1)}$ just as it was done in four dimensions. This is done by matching (5.45) order by order in the small z expansion. Using (3.56) one can integrate over spin. E.g. for $n=0,1,2,3$:

$$\begin{aligned}
\gamma_0^{(1)} &= -\frac{(\Delta_L-2)(\Delta_L-1)\Delta_L}{2}, \\
\gamma_1^{(1)} &= -\frac{(\Delta_L-1)\Delta_L(\Delta_L+10)}{2}, \\
\gamma_2^{(1)} &= -\frac{\Delta_L(\Delta_L+2)(\Delta_L+19)}{2}, \\
\gamma_3^{(1)} &= -\frac{(\Delta_L+4)(\Delta_L(\Delta_L+29)+30)}{2},
\end{aligned} \tag{5.46}$$

These anomalous dimensions agree with eq. (6.10) in [55]. Similarly, we obtain the following OPE coefficients:

$$\begin{aligned}
P_0^{(1)} &= -\frac{11(\Delta_L-2)(\Delta_L-1)\Delta_L}{12}, \\
P_1^{(1)} &= -\frac{(\Delta_L-1)\Delta_L(11\Delta_L+38)}{12}, \\
P_2^{(1)} &= -\frac{\Delta_L(22+\Delta_L(87+11\Delta_L))}{12}, \\
P_3^{(1)} &= -\frac{\Delta_L(202+\Delta_L(147+11\Delta_L))}{12}.
\end{aligned} \tag{5.47}$$

It is straightforward to continue to higher values of n.

Plugging (5.46) into (5.45) in the limit $1-\bar{z} \ll z \ll 1$ one finds the following contribution to the terms proportional to $\frac{z^p \log^2 z}{(1-\bar{z})^{\Delta_L-4}}$ at $\mathcal{O}(\mu^2)$:

$$\begin{aligned}
p=0: &\frac{\Delta_L^2(\Delta_L-1)(\Delta_L-2)}{32(\Delta_L-3)(\Delta_L-4)}, \\
p=1: &\frac{\Delta_L^2(\Delta_L-1)(\Delta_L+6)(\Delta_L+16)}{32(\Delta_L-3)(\Delta_L-4)} \\
p=2: &\frac{\Delta_L^2(\Delta_L^4+46\Delta_L^3+599\Delta_L^2+1898\Delta_L+1056)}{64(\Delta_L-3)(\Delta_L-4)}, \\
p=3: &\frac{\Delta_L^7+72\Delta_L^6+1651\Delta_L^5+13344\Delta_L^4+40180\Delta_L^3+41952{\Delta_L}^2+14400\Delta_L}{192(\Delta_L-3)(\Delta_L-4)}.
\end{aligned} \tag{5.48}$$

It is now straightforward to expand the ansatz (5.43) in the limit $1-\bar{z} \ll z \ll 1$, collect terms that behave as $z^p \log^2 z$ and compare them to the S-channel (5.48). This determines the coefficients:

$$
\begin{aligned}
a_{17} &= \frac{\Delta_L(\Delta_L+1)(\Delta_L+2)}{64064(\Delta_L-3)(\Delta_L-4)}, \\
a_{26} &= \frac{\Delta_L(-18+(-12+\Delta_L)\Delta_L)}{133056(\Delta_L-3)(\Delta_L-4)}, \\
a_{35} &= \frac{\Delta_L(\Delta_L-6)(\Delta_L-15)}{302400(\Delta_L-3)(\Delta_L-4)}, \\
a_{44} &= \frac{\Delta_L(\Delta_L-5)(\Delta_L-6)}{627200(\Delta_L-3)}.
\end{aligned}
\tag{5.49}
$$

One can consider higher values of p; eq. (5.49) is still the solution of the corresponding overconstrained system.

The double-stress tensor contribution to the correlator in the lightcone limit $\bar{z} \to 1$ is therefore given by

$$
\begin{aligned}
\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^4}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{\Delta_L}{(\Delta_L-3)(\Delta_L-4)} \left(\frac{1}{627200}\right) \\
\times \Big(& (\Delta_L-4)(\Delta_L-5)(\Delta_L-6) f_4^2(z) + \frac{56(\Delta_L-6)(\Delta_L-15)}{27} f_3(z) f_5(z) \\
& + \frac{1400(\Delta_L(\Delta_L-12)-18)}{297} f_2(z) f_6(z) \\
& + \frac{1400(\Delta_L+1)(\Delta_L+2)}{143} f_1(z) f_7(z) \Big).
\end{aligned}
\tag{5.50}
$$

Using (5.50) one can deduce the second order OPE data in the S-channel. The anomalous dimensions at this order can then be compared to the holographic calculations in [55] to reveal perfect agreement.

5.3.1. Exponentiation of minimal-twist multi-stress tensors in six dimensions

It is interesting to study whether the stress tensor sector of the correlator exponentiates in the lightcone limit

$$
\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{5.51}
$$

with $\mathcal{F}(\mu; z, \bar{z})$ a rational function of Δ_L that is of $\mathcal{O}(1)$ as $\Delta_L \to \infty$. In the lightcone limit $\mathcal{F}(\mu; z, \bar{z})$ admits an expansion

$$\mathcal{F}(\mu; z, \bar{z}) = \mu(1-\bar{z})^2 \mathcal{F}^{(1)}(z) + \mu^2 (1-\bar{z})^4 \mathcal{F}^{(2)}(z) + \mathcal{O}(\mu^3). \tag{5.52}$$

At $\mathcal{O}(\mu)$ one finds $\mathcal{F}^{(1)}(z) = \frac{\Gamma(\frac{6}{2}+1)^2}{4\Gamma(6+2)} f_4(z)$ from the stress tensor contribution. Using (5.50) we find

$$\mathcal{F}^{(2)}(z) = b_{17} f_1(z) f_7(z) + b_{26} f_2(z) f_6(z) + b_{35} f_3(z) f_5(z) + b_{44} f_4^2(z) \tag{5.53}$$

with

$$\begin{aligned} b_{17} &= \frac{a_{17}}{\Delta_L}, \\ b_{26} &= \frac{a_{26}}{\Delta_L}, \\ b_{35} &= \frac{a_{35}}{\Delta_L}, \\ b_{44} &= -\frac{4\Delta_L^2 - 31\Delta_L + 60}{313600(\Delta_L - 3)(\Delta_L - 4)}. \end{aligned} \tag{5.54}$$

From (5.54) we indeed see that the stress tensor sector of the correlator exponentiates at least to $\mathcal{O}(\mu^2)$ in $d = 6$.

5.3.2. OPE coefficients of minimal-twist double-stress tensors

In this section we decompose the stress tensor sector of the correlator (5.43) into a sum over minimal-twist double-stress tensors. The discussion follows that of Section 5.2.5.

Applying (5.38) to (5.50), we find that $a+b+2\ell = 8+2\ell$ which is $\frac{\tau_{2,\min}}{2}+s+2\ell$, with $\tau_{2,\min} = 8$ and $s = 4$ being the twist and spin of the simplest minimal-twist double-stress tensor operator $:T_{\mu\nu}T_{\rho\lambda}:$. Non-zero value of ℓ thus gives the contribution from operators of higher spin of the form $:T_{\mu\nu}\partial_{\rho_1}\ldots\partial_{\rho_{2\ell}}T_{\delta\lambda}:$, where no indices are contracted and only even spin operators contribute to the OPE between identical scalars.

It is now straightforward to write down the OPE coefficients for minimal-twist double-stress tensors in six dimensions. E.g. one finds for the lowest-spin operators the following OPE coefficients:

$$\begin{aligned} \mu^2 P_{8,4}^{(HH,LL);(2)} &= \mu^2 \frac{\Delta_L(600 + \Delta_L(1394 + \Delta_L(677 + 429\Delta_L)))}{269068800(\Delta_L - 3)(\Delta_L - 4)}, \\ \mu^2 P_{8,6}^{(HH,LL);(2)} &= \mu^2 \frac{\Delta_L(30 + \Delta_L(187 + \Delta_L(-120 + 143\Delta_L)))}{3430627200(\Delta_L - 3)(\Delta_L - 4)}, \\ \mu^2 P_{8,8}^{(HH,LL);(2)} &= \mu^2 \frac{\Delta_L(60 + \Delta_L(1382 + \Delta_L(-1857 + 1105\Delta_L)))}{657033721344(\Delta_L - 3)(\Delta_L - 4)}. \end{aligned} \tag{5.55}$$

For general spin we have ($s = 4 + 2\ell$)

$$P_{8,s}^{(HH,LL)} = \mu^2 \frac{c\Delta_L}{(\Delta_L - 3)(\Delta_L - 4)} (a_3\Delta_L^3 + a_2\Delta_L^2 + a_1\Delta_L + a_0) \tag{5.56}$$

where

$$\begin{aligned}
c &= \frac{2^{-9-2s}\sqrt{\pi}s(s+2)\Gamma(s-1)}{(s-3)(s+4)(s+6)(s+8)(s+10)\Gamma(s+\frac{7}{2})}, \\
a_3 &= (s-2)s(s+2)(s+5)(s+7)(s+9), \\
a_2 &= -3(2880 + s(s+7)(-276 + s(s+7)(-56 + s(s+7)))), \\
a_1 &= 2(25920 + s(s+7)(3276 + s(s+7)(-80 + s(s+7)))), \\
a_0 &= 675 \times 2^7.
\end{aligned} \tag{5.57}$$

5.4. Discussion

In this section we consider the minimal-twist multi-stress tensor contributions to the heavy-heavy-light-light correlator of scalars in large C_T CFTs in even spacetime dimensions. We provide strong evidence for the conjecture that all such contributions are described by the ansatz (5.7) and determine the coefficients by performing a bootstrap procedure. In practice this is completed for twist-four double-stress tensors and twist-six triple-stress tensors in four dimensions as well as twist-eight double-stress tensors in six dimensions. In principle it is straightforward to use our technology to determine the coefficients $a_{i_1\ldots i_k}$ to arbitrarily high order in μ; this must be related to the universality of the minimal-twist OPE coefficients.

In two dimensions the heavy-heavy-light-light Virasoro vacuum block exponentiates [see eq. (3.6)], with $\mathcal{F}(\mu; z)$ independent of Δ_L. In higher dimensions we observe a similar exponentiation with $\mathcal{F}(\mu; z, \bar{z})$ a rational function of Δ_L that remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$. It would be interesting to see whether it is possible to write down a closed-form recursion formula for $\mathcal{F}(\mu; z, \bar{z})$. Solving such a recursion formula would give a higher-dimensional analogue of the two-dimensional Virasoro vacuum block.

An immediate technical question concerns CFTs in odd spacetime dimensions. We could not immediately generalize our results in this context – the ansatz in eq. (5.7) fails in odd dimensions. However, the heavy-light conformal blocks are known [12], so a similar approach should be feasible.

It would be interesting to study the regime of applicability of our results. We have not used holography; our main assumption is the ansatz (3.13), known to be true for holographic CFTs to $\mathcal{O}(\mu^2)$ in $d = 4$ [124]. Yet, our general expressions for the OPE coefficients agree with the OPE coefficients computed in some holographic examples [15]. What happens once one goes beyond holographic CFTs - will our ansatz need to be modified by the inclusion of terms suppressed by the gap or the central charge? We leave these questions for subsequent investigations.

Another interesting direction concerns the study of the bulk scattering phase-shift in the presence of a black hole background. In the context of higher dimensional CFTs, this problem was first considered in [55] where the gravitational expression was given to all orders in μ and the CFT computation was performed to $\mathcal{O}(\mu)$. Subsequently, $\mathcal{O}(\mu^2)$ was discussed in [12]. In [126] the $\mathcal{O}(\mu)$ contribution was exponentiated to yield the scattering phase shift in the presence of a shock-wave geometry. A CFT computation of the phase shift to all orders in μ is still lacking. This would in principle involve understanding Regge theory beyond the leading order. It will be interesting to see whether the results of this article could be helpful in this regard.

6. Stress tensor sector of conformal correlators

6.1. Summary of the results

In this section, we show that the stress tensor sector of the HHLL correlator in $d = 4$ can be written in terms of products of $f_a(z)$ functions defined as

$$f_a(z) = (1-z)^a {}_2F_1(a, a, 2a, 1-z). \tag{6.1}$$

The stress tensor sector of the HHLL correlator can be expanded in powers of μ and then in powers of $(1-\bar{z})$ as

$$\begin{aligned} \mathcal{G}(z,\bar{z}) = \sum_{k=0}^{\infty} \mu^k \mathcal{G}^{(k)}(z,\bar{z}) &= \frac{1}{((1-z)(1-\bar{z}))^{\Delta_L}} \\ &+ \sum_{k=1}^{\infty}\sum_{m=0}^{\infty} \mu^k (1-\bar{z})^{-\Delta_L+k+m} \mathcal{G}^{(k,m)}(z), \end{aligned} \tag{6.2}$$

where we have explicitly separated the contribution of the identity operator.[18] We explain how one can write $\mathcal{G}^{(k,m)}(z)$ for arbitrary k and m.

We write an ansatz for each $\mathcal{G}^{(k,m)}(z)$ with a few unknown coefficients and fix all, but a handful of them, via lightcone bootstrap. The undetermined coefficients correspond to the OPE coefficients of spin-0 and spin-2 exchanged operators. We further show that in holographic CFTs one can use the phase shift computed in the dual gravitational theory to reduce the set of undetermined parameters to the OPE coefficients of multi-stress tensors with spin zero.

Operators of non-minimal twist give a subleading contribution in the light-cone limit, $1-\bar{z} \ll 1$, which can be expressed as a sum of products of the functions $f_a(z)$ (times an appropriate power of $(1-\bar{z})$). This form is similar to the contribution of minimal-twist multi-stress tensor operators considered in [13]. While our method can be used to address the contribution of operators of arbitrary twist, here we focus on determining the specific contributions of operators with twist $\tau = 6, 8, 10$, at $\mathcal{O}(\mu^2)$ and $\tau = 8, 10$, at $\mathcal{O}(\mu^3)$.

At $\mathcal{O}(\mu)$, the only operator that contributes to the stress tensor sector of the correlator is the stress tensor and its contribution is completely fixed by

[18] The contribution of the identity operator is denoted with $k = 0$, schematically $(T_{\mu\nu})^0 = \mathbf{1}$.

conformal symmetry. In $d=4$ its exact (to all orders in $\bar{z}$) contribution is given by

$$\mathcal{G}^{(1)}(z,\bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L-1}} \frac{\Delta_L}{120(\bar{z}-z)} \Big(f_3(z) - f_3(\bar{z}) \Big). \tag{6.3}$$

At $\mathcal{O}(\mu^2)$, the leading contribution in the lightcone limit, due to twist-four double-stress tensors, was evaluated in [124]

$$\begin{aligned} \mathcal{G}^{(2,0)}(z) =& \frac{1}{(1-z)^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L-2)} \right) \times \\ & \Big((\Delta_L-4)(\Delta_L-3) f_3^2(z) + \frac{15}{7}(\Delta_L-8) f_2(z) f_4(z) \\ & + \frac{40}{7}(\Delta_L+1) f_1(z) f_5(z) \Big). \end{aligned} \tag{6.4}$$

We show that the subleading contribution in the lightcone limit, due to twist-four and twist-six double-stress tensors, is given by

$$\begin{aligned} \mathcal{G}^{(2,1)}(z) = \frac{1}{(1-z)^{\Delta_L}} \Big(\left(\frac{3-z}{2(1-z)} \right) \left(a_{33} f_3(z)^2 + a_{24} f_2(z) f_4(z) + a_{15} f_1(z) f_5(z) \right) \\ + \left(b_{14} f_1(z) f_4(z) + c_{16} f_1(z) f_6(z) + c_{25} f_2(z) f_5(z) + c_{34} f_3(z) f_4(z) \right) \Big), \end{aligned} \tag{6.5}$$

with coefficients a_{mn} and c_{mn} given in (6.30). The coefficient b_{14} is non-universal and generically depends on the details of the theory. It corresponds to the OPE coefficient of twist-six double-stress tensor with spin $s=2$

$$b_{14} = P^{(2)}_{8,2}, \tag{6.6}$$

obtained holographically in [15] and here, via the gravitational phase-shift calculation in (6.107).

The subsubleading contribution in the lightcone limit, due to twist-four, six and eight double-stress tensor operators, is

$$\begin{aligned} \mathcal{G}^{(2,2)}(z) = \frac{1}{(1-z)^{\Delta_L}} \Bigg(\left(\frac{z(2z-7)+11}{6(z-1)^2} \right) (a_{33} f_3^2 + a_{24} f_2 f_4 + a_{15} f_1 f_5) \\ + \left(\frac{2-z}{1-z} \right) (b_{14} f_1 f_4 + c_{16} f_1 f_6 + c_{25} f_2 f_6 + c_{34} f_3 f_4) + (d_{17} f_1 f_7 + d_{26} f_2 f_6 \\ + d_{35} f_3 f_5 + d_{44} f_4^2 + e_{15} f_1 f_5 + g_{13} f_1 f_3) \Bigg), \end{aligned} \tag{6.7}$$

with coefficients d_{mn} given in (6.35). By f_a we mean $f_a(z)$ which we will use for brevity. The coefficients g_{13} and e_{15} are theory dependent and are related to the OPE coefficients of twist-eight double-stress tensors with spin $s = 0, 2$ by

$$
\begin{aligned}
g_{13} &= P^{(2)}_{8,0}, \\
e_{15} &= P^{(2)}_{10,2} - \frac{5}{252} P^{(2)}_{8,0}.
\end{aligned}
\tag{6.8}
$$

These coefficients were also obtained by a gravitational computation in [15]. Here we have used the calculation of the phase shift in the dual gravitational theory to determine the OPE coefficient of the spin-2 operator, $P^{(2)}_{10,2}$, in (6.110).

The subsubsubleading contribution in the lightcone limit, due to double-stress tensors with twists $\tau = 4, 6, 8, 10$, is given by

$$
\begin{aligned}
\mathcal{G}^{(2,3)}(z) = \frac{1}{(1-z)^{\Delta_L}} \Bigg(&\left(\frac{z((13-3z)z-23)+25}{12(1-z)^3} \right) (a_{33} f_3^2 + a_{24} f_2 f_4 + a_{15} f_1 f_5) \\
&+ \left(\frac{1}{(1-z)^2} + \frac{1}{1-z} + \frac{9}{10} \right) (b_{14} f_1 f_4 + c_{16} f_1 f_6 + c_{25} f_2 f_5 + c_{34} f_3 f_4) \\
&+ \left(\frac{1}{1-z} + \frac{3}{2} \right) (d_{17} f_1 f_7 + d_{26} f_2 f_6 + d_{35} f_3 f_5 + d_{44} f_4^2 + e_{15} f_1 f_5 + g_{13} f_1 f_3 +) \\
&+ g_{13} f_3 + (h_{18} f_1 f_8 + h_{27} f_2 f_7 + h_{36} f_3 f_6 + h_{45} f_4 f_5 + j_{16} f_1 f_6 + i_{14} f_1 f_4) \Bigg),
\end{aligned}
\tag{6.9}
$$

with h_{mn} given in (6.41). The non-universal coefficients here are i_{14} and j_{16} which are related to the OPE coefficients of twist-ten double-stress tensor operators with spin $s = 0, 2$

$$
\begin{aligned}
i_{14} &= P^{(2)}_{10,0}, \\
j_{16} &= P^{(2)}_{12,2} - \frac{2}{99} P^{(2)}_{10,0}.
\end{aligned}
\tag{6.10}
$$

The OPE coefficient $P^{(2)}_{12,2}$ is determined in (6.111) using the phase shift calculation in the dual gravitational theory. Non-universality is manifest through dependence on the Gauss-Bonnet coupling.

Using the results above, we also extract the OPE coefficients $P^{(2)}_{\Delta,s}$ of double-stress tensors of given twist. For $\tau = 6$:

$$
\begin{aligned}
P^{(2)}_{10+2\ell,4+2\ell} = &\frac{\sqrt{\pi} 2^{-4\ell-17} \Gamma(2n+7)}{(\ell+4)(\ell+5)(\ell+6)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{13}{2}\right)} \\
&\times \frac{\Delta_L}{(\Delta_L-3)(\Delta_L-2)} (a_{1,\ell} \Delta_L^3 + b_{1,\ell} \Delta_L^2 + c_{1,\ell} \Delta_L + d_{1,\ell}),
\end{aligned}
\tag{6.11}
$$

where $a_{1,\ell}$, $b_{1,\ell}$, $c_{1,\ell}$, $d_{1,\ell}$ can be found in (6.33). For $\tau = 8$:

$$\begin{aligned} P^{(2)}_{12+2\ell,4+2\ell} = & \frac{\sqrt{\pi}\Delta_L 2^{-4\ell-19}\Gamma(2\ell+7)}{3(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)(\ell+4)(\ell+5)} \\ & \times \frac{a_{2,\ell}\Delta_L^4 + b_{2,\ell}\Delta_L^3 + c_{2,\ell}\Delta_L^2 + d_{2,\ell}\Delta_L + e_{2,\ell}}{(\ell+6)(\ell+7)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{15}{2}\right)}, \end{aligned} \tag{6.12}$$

with $a_{2,\ell}$, $b_{2,\ell}$, $c_{2,\ell}$, $d_{2,\ell}$ and $e_{2,\ell}$ given in (6.38). Similarly for $\tau = 10$:

$$\begin{aligned} P^{(2)}_{14+2\ell,4+2\ell} = & \frac{\sqrt{\pi}2^{-4\ell-22}\Gamma(2\ell+9)}{5(2\ell+1)(2\ell+3)(2\ell+5)(2\ell+7)\Gamma\left(2\ell+\frac{17}{2}\right)} \\ & \times \frac{\Delta_L(\Delta_L+1)(a_{3,\ell}\Delta_L^4 + b_{3,\ell}\Delta_L^3 + c_{3,\ell}\Delta_L^2 + d_{3,\ell}\Delta_L + e_{3,\ell})}{(\ell+5)(\ell+6)(\ell+7)(\ell+8)(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \end{aligned} \tag{6.13}$$

with $a_{3,\ell}$, $b_{3,\ell}$, $c_{3,\ell}$, $d_{3,\ell}$ and $e_{3,\ell}$ expressed in terms of Δ_L in (6.44). Note that in all of these formulas $\ell \geq 0$ and, therefore, the OPE coefficients of operators with spin $s = 0, 2$ are not included here. It appears that at $\mathcal{O}(\mu^2)$, the OPE coefficients of all operators with spin $s \geq 4$ are universal in the sense that they only depend on Δ_L and C_T. On the other hand, the OPE coefficients of double-stress tensors with $s = 0, 2$ are non-universal.

At $\mathcal{O}(\mu^3)$, the leading contribution of twist-six triple-stress tensors in the lightcone limit, was computed in [13]

$$\begin{aligned} \mathcal{G}^{(3,0)}(z) = & \frac{1}{(1-z)^{\Delta_L}}\Big(a_{117}f_1(z)^2 f_7(z) + a_{126}f_1(z)f_2(z)f_6(z) \\ & + a_{135}f_1(z)f_3(z)f_5(z) + a_{225}f_2(z)^2 f_5(z) + a_{234}f_2(z)f_3(z)f_4(z) + a_{333}f_3(z)^3\Big), \end{aligned} \tag{6.14}$$

where the coefficients a_{ijk} can be found in (6.46).

The subleading contribution to the correlator is due to twist-eight and twist-six triple-stress tensors

$$\begin{aligned} \mathcal{G}^{(3,1)}(z) = & \frac{1}{(1-z)^{\Delta_L}}\Bigg(\left(\frac{2-z}{1-z}\right)(a_{117}f_1^2 f_7 + a_{126}f_1 f_2 f_6 + a_{135}f_1 f_3 f_5 \\ & + a_{225}f_2^2 f_5 + a_{234}f_2 f_3 f_4 + a_{333}f_3^3) + (b_{116}f_6 f_1^2 + c_{118}f_8 f_1^2 + c_{145}f_4 f_5 f_1 \\ & + c_{127}f_2 f_7 f_1 + c_{244}f_2 f_4^2 + c_{334}f_3^2 f_4 + c_{235}f_2 f_3 f_5 + c_{226}f_2^2 f_6)\Bigg), \end{aligned} \tag{6.15}$$

with b_{ijk} and c_{ijk} given in (B.2.1). Terms proportional to a_{ijk} come from the subleading contribution due to the minimal-twist triple-stress tensors in (6.14). Note that all of these coefficients are non-universal, since they depend on b_{14} from the $\mathcal{O}(\mu^2)$ result. Accordingly, no OPE coefficients of non-minimal-twist triple-stress tensors are universal.

A similar story holds for the subsubleading contribution to the correlator at $\mathcal{O}(\mu^3)$. This is due to multi-stress tensors with twist six, eight and ten and takes the following form

$$\begin{aligned}\mathcal{G}^{(3,2)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(&\left(\frac{144z^2-448z+464}{160(z-1)^2}\right)(a_{117}f_1^2f_7 + a_{126}f_1f_2f_6 \\ &+ a_{135}f_1f_3f_5 + a_{225}f_2^2f_5 + a_{234}f_2f_3f_4 + a_{333}f_3^3) + \left(\frac{1}{1-z}+\frac{3}{2}\right)(b_{116}f_6f_1^2 \\ &+ c_{118}f_8f_1^2 + c_{145}f_4f_5f_1 + c_{127}f_2f_7f_1 + c_{244}f_2f_4^2 + c_{334}f_3^2f_4 + c_{235}f_2f_3f_5 \\ &+ c_{226}f_2^2f_6) + (d_{117}f_1^2f_7 + e_{115}f_1^2f_5 + g_{119}f_1^2f_9 + g_{128}f_1f_2f_8 + g_{155}f_1f_5^2 \\ &+ g_{227}f_2^2f_7 + g_{236}f_2f_3f_6 + g_{245}f_2f_4f_5 + g_{335}f_3^2f_5 + g_{344}f_3f_4^2)\Bigg),\end{aligned} \tag{6.16}$$

with d_{117} and g_{ijk} in (B.3.1)-(B.3.3) and e_{115} in (6.115).

We further explain how one can write an ansatz for the correlator at arbitrary order in μ and the lightcone expansion. All unknown coefficients in the ansatz, except those that correspond to OPE coefficients of spin-0 and spin-2 operators, can be fixed by means of the lightcone bootstrap. We further show that in holographic CFTs one can use the phase shift computed in the dual gravitational theory to reduce the set of undetermined parameters to the OPE coefficients of multi-stress tensors with spin zero. Our results for these OPE coefficients precisely match those in [15] whenever available in the latter.

The OPE coefficients of multi-stress tensors can also be calculated using the Lorentzian inversion formula as in [128]. In order to determine for which operators the formula can be applied, one should consider the behavior of the correlation function in the Regge limit. The Regge behavior of the correlator at $\mathcal{O}(\mu^k)$ is $1/\sigma^k$, implying that the Lorentzian inversion formula can be used to extract the OPE coefficients of the operators with spin $s > k+1$. Accordingly, already at $\mathcal{O}(\mu^3)$, fixing the relevant OPE coefficients by combining an ansatz with the lightcone bootstrap allows one to determine more OPE data compared to those obtained with the sole use of the Lorentzian inversion formula. We explicitly check that it is not possible to extract the OPE coefficient of a triple-stress tensor with spin $s = 4$ and twist $\tau = 8$ using the Lorentzian inversion formula. Note, however, that this coefficient is completely determined in this article (where an ansatz is additionally employed).

6.1.1. Outline

This section is organized as follows. In Section 6.2 we analyze the stress tensor sector of the correlator at $\mathcal{O}(\mu^2)$, where we compute the subleading, subsubleading and subsubsubleading contributions in the lightcone expansion. We also compute the OPE coefficients of double-stress tensors with twist $\tau = 6, 8, 10$ and spin $s > 2$. In Section 6.3, we analyze the stress tensor sector of the correlator at $\mathcal{O}(\mu^3)$, where we explicitly calculate the subleading and subsubleading contributions in the lightcone expansion. In Section 6.4, we investigate the Gauss-Bonnet dual gravitational theory and give additional evidence for the universality of the OPE coefficients of minimal-twist multi-stress tensors using the phase shift calculation. Furthermore, we calculate the OPE coefficients of double- and triple-stress tensors with spin $s = 2$ (up to undetermined spin zero data). In Section 6.5, we show how one can use the Lorentzian inversion formula in order to extract the OPE coefficients of double-stress tensors with twist $\tau = 4, 6$. We discuss our results in Section 6.6. Appendix B.1 contains certain relations that products of f_a functions satisfy, while Appendices B.2 and B.3 contain explicit expressions for the coefficients which determine the correlator in subleading and subsubleading lightcone order at $\mathcal{O}(\mu^3)$. Several OPE coefficients of twist-eight triple-stress tensors are listed in Appendix B.4. In Appendix B.5 we clarify the relationship between the scattering phase shift as defined in [55] and the deflection angle and finally, in Appendix B.6 we explicitly write some of the S-channel anomalous dimensions at $\mathcal{O}(\mu^2)$ and we investigate their relation with the phase shift.

6.2. Double-stress tensors in four dimensions

In this section, we analyze the stress tensor sector of the HHLL correlator at $\mathcal{O}(\mu^2)$ in $d = 4$. The operators that contribute at this order in the T-channel are the double-stress tensors. Here, we investigate the subleading contributions that are coming from families of operators with nonminimal twist, specifically, $\tau_{2,1} = 6$, $\tau_{2,2} = 8$ and $\tau_{2,3} = 10$, according to (3.63).

The dominant contribution in the lightcone limit at $\mathcal{O}(\mu^2)$ was calculated in [124]. It comes from the operators with minimal twist $\tau_{2,\text{min}} = 4$ and they are of the schematic form $: T_{\mu\nu}\partial_{\alpha_1}\ldots\partial_{\alpha_{2\ell}}T_{\rho\sigma} :$. These operators have conformal dimension $\Delta = 8 + 2\ell$ and spin $s = 4 + 2\ell$. The result is [124]

$$\mathcal{G}^{(2,0)}(z) = \frac{1}{(1-z)^{\Delta_L}}\left(\frac{\Delta_L}{28800(\Delta_L-2)}\right)\times \\ \Big((\Delta_L-4)(\Delta_L-3)f_3^2(z) + \frac{15}{7}(\Delta_L-8)f_2(z)f_4(z) + \frac{40}{7}(\Delta_L+1)f_1(z)f_5(z)\Big), \tag{6.17}$$

where $f_a(z) = (1-z)^a {}_2F_1(a, a, 2a, 1-z)$.

6.2.1. Twist-six double-stress tensors

Twist-six double-stress tensors contribute at $\mathcal{O}(\mu^2)$ and at subleading order in the lightcone expansion $\sim (1-\bar{z})^{-\Delta_L+3}$ as $\bar{z} \to 1$. As shown in this section, this contribution again takes a particular form with a few undetermined coefficients which, except for a single one, can be fixed using lightcone bootstrap. The undetermined data is shown to correspond to a single OPE coefficient due to the exchange of the twist-six and spin-two double-stress tensor $: T_\mu{}^\rho T_{\rho\nu} :$.

We will now motivative an ansatz for the subleading contribution to the stress tensor sector at $\mathcal{O}(\mu^2)$. Let us focus first on corrections due to the leading lightcone contribution of twist-four double-stress tensors. These corrections originate from subleading terms in the lightcone expansion of the conformal blocks in (3.59). Note however that they are purely kinematical and do not contain any new data. Explicitly, the subleading corrections to the blocks of twist-four double-stress tensors are given by

$$g^{(0,0)}_{4,s}(1-z,1-\bar{z}) \underset{\bar{z}\to 1}{\approx} (1-\bar{z})^2\left(1+(1-\bar{z})\left(\frac{3-z}{2(1-z)}\right)+\mathcal{O}\left((1-\bar{z})^2\right)\right)f_{\frac{\beta}{2}}(z) \\ -(1-\bar{z})^{s+3}\left(1+(1-\bar{z})\left(\frac{s+2}{2}+\frac{1}{1-z}\right)+\mathcal{O}((1-\bar{z})^2)\right)f_1(z). \tag{6.18}$$

Since we are interested in the subleading contribution, i.e. terms that behave as $(1-\bar{z})^3$ as $\bar{z} \to 1$ in (6.18), only the first line in (6.18) needs to be considered. (Note that $s \geq 4$ for minimal-twist double-stress tensors.)

Next, consider the contribution of twist-six double-stress tensors. Recall that the form of the minimal-twist double-stress tensors' contribution to (6.17) can be motivated by decomposing products of the type $f_a(z)f_b(z)$ in terms of the lightcone conformal blocks. This decomposition is explicitly given by [124]:

$$f_a(z)f_b(z) = \sum_{\ell=0}^{\infty} p(a,b,\ell) f_{a+b+2\ell}(z), \tag{6.19}$$

where

$$\begin{aligned} p(a,b,\ell) = & \frac{\Gamma(a+b+2\ell)}{\Gamma\left(a+b+2\ell-\frac{1}{2}\right)} \times \\ & \frac{2^{-4\ell}\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(b+\frac{1}{2}\right)\Gamma\left(\ell+\frac{1}{2}\right)\Gamma(a+\ell)\Gamma(b+\ell)\Gamma\left(a+b+\ell-\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(a)\Gamma(b)\Gamma(\ell+1)\Gamma\left(a+\ell+\frac{1}{2}\right)\Gamma\left(b+\ell+\frac{1}{2}\right)\Gamma(a+b+\ell)}. \end{aligned} \tag{6.20}$$

Using the leading behavior of the conformal blocks (6.18) in the lightcone limit, it was found that $a+b+2\ell$ should be identified with $\frac{\beta}{2} = \frac{\Delta+s}{2}$. In order to reproduce twist-six double-stress tensors of the form $:T_{\mu\nu}\partial^2\partial_{\alpha_1}\ldots\partial_{\alpha_{2\ell}}T_{\rho\sigma}:$ we should therefore consider products f_af_b with $a+b=7$. Likewise, to take into account operators of the form $:T_{\mu\beta}\partial_{\alpha_1}\ldots\partial_{\alpha_{2\ell}}T^{\beta}{}_{\nu}:$ we include products f_af_b with $a+b=5$.

From the arguments above, we make the following ansatz for the subleading correction in the lightcone expansion due to double-stress tensors:

$$\begin{aligned} \mathcal{G}^{(2,1)}(z) = & \frac{1}{(1-z)^{\Delta_L}}\Bigg(\left(\frac{3-z}{2(1-z)}\right)(a_{33}f_3(z)^2 + a_{24}f_2(z)f_4(z) \\ & + a_{15}f_1(z)f_5(z)) + (b_{14}f_1(z)f_4(z) + b_{23}f_2(z)f_3(z) + c_{16}f_1(z)f_6(z) \\ & + c_{25}f_2(z)f_5(z) + c_{34}f_3(z)f_4(z))\Bigg), \end{aligned} \tag{6.21}$$

where b_{ij}, c_{ij} are coefficients that will be determined using lightcone bootstrap and encode the contribution from twist-six double-stress tensors. Once b_{ij} and c_{ij} are determined, one can use the decomposition in (6.19) to read off the OPE coefficients of twist-six double-stress tensors with any given spin. Moreover, a_{ij} in (6.21) are coefficients that can be read off from the minimal-twist contribution in (6.17) and do therefore not contain any new information.

We proceed with the S-channel calculation to fix the unknown coefficients in (6.21). Let us first mention that the products of $f_a(z)$ functions in the second line of (6.21) are not linearly independent as one can see from (A.1), so we set $b_{23} = 0$. Moreover, the coefficients a_{ij} must be the same as in (6.17). We will momentarily keep them undetermined to have an extra consistency check of our calculation.

In the S-channel we have double-trace operators of the form $:\mathcal{O}_H \partial^{2n} \partial^l \mathcal{O}_L:$ with conformal dimension $\Delta = \Delta_H + \Delta_L + 2n + l + \gamma_{n,l}$. The relevant anomalous dimensions $\gamma_{n,l}$ and OPE coefficients are given in (3.25) and (3.69) ($k = 2$ in this case). In the lightcone limit, the dominant contribution comes from operators with large spin l, $l \gg n$. The zeroth order OPE coefficients are given by (3.70). The conformal blocks of these operators in the limit $1 - \bar{z} \ll z \ll 1$ are

$$g_{n,l}^{(\Delta_{HL}, -\Delta_{HL})}(z, \bar{z}) \approx \frac{(z\bar{z})^{\frac{\Delta_H + \Delta_L + \gamma(n,l)}{2}}}{\bar{z} - z} z^n \bar{z}^{l+n+1}. \tag{6.22}$$

We first need to fix the OPE data at $\mathcal{O}(\mu)$. Coefficients $\gamma_n^{(1,p)}$ and $P_n^{(1,p)}$ can be determined for every p and n by matching the S-channel correlator with the correlator in the T-channel at $\mathcal{O}(\mu)$. This is just the stress tensor block times its OPE coefficient and it is known for arbitrary z and $\bar{z}$. As we saw earlier

$$(\bar{z} - z)\mathcal{G}^{(1)}(z, \bar{z}) = \frac{1}{[(1 - z)(1 - \bar{z})]^{\Delta_L - 1}} \frac{\Delta_L}{120} \left(f_3(z) - f_3(\bar{z})\right). \tag{6.23}$$

Expanding (6.23) near $\bar{z} \to 1$ leads to

$$(\bar{z} - z)\mathcal{G}^{(1)}(z, \bar{z}) = \frac{-1 + \bar{z}}{((1 - z)(1 - \bar{z}))^{\Delta_L}} \left(\Delta_L \left(\frac{3}{4}(1 + z) + \frac{1 + z(z + 4)}{4(1 - z)} \log(z) \right) \right.$$
$$\left. \sum_{p=1}^{\infty} \frac{\Delta_L (p - 2)(p - 1)(1 - z)}{4p(p + 1)\,(p + 2)} (1 - \bar{z})^p \right). \tag{6.24}$$

On the other hand, we expand the integrand of (3.67) up to the $\mathcal{O}(\mu)$, integrate this expansion over l, and then expand in the lightcone limit $\bar{z} \to 1$ to obtain a result of the form

$$(\bar{z} - z)\mathcal{G}^{(1)}(z, \bar{z}) = \frac{1}{(1 - \bar{z})^{\Delta_L - 1}} \sum_{p=0}^{\infty} \left(\sum_{n=0}^{\infty} r_{n,p}(z) z^n (1 - \bar{z})^p \right). \tag{6.25}$$

The functions $r_{n,p}(z)$ can be explicitly calculated. Here $r_{n,0}(z)$, $r_{n,1}(z)$ and $r_{n,2}(z)$ are given by

$$\begin{aligned} r_{n,0}(z) &= \frac{\Gamma(\Delta_L+n-1)}{2\Gamma(\Delta_L)\Gamma(n+1)}\left(2P_n^{(1,0)}+\log(z)\gamma_n^{(1,0)}\right), \\ r_{n,1}(z) &= \frac{\Gamma(\Delta_L+n-1)}{2\Gamma(\Delta_L)\Gamma(n+1)(\Delta_L-2)}\Big(2(P_n^{(1,0)}+P_n^{(1,1)})-(\Delta_L-2)\gamma_n^{(1,0)} \\ &\quad +\log(z)(\gamma_n^{(1,0)}+\gamma_n^{(1,1)})\Big), \\ r_{n,2}(z) &= \frac{\Gamma(\Delta_L+n-1)}{2(\Delta_L-2)(\Delta_L-3)\Gamma(\Delta_L)\Gamma(n+1)}\Big(2(\Delta_L+n-1)P_n^{(1,0)} \\ &\quad +2(\Delta_L+n)P_n^{(1,1)}+2P_n^{(1,2)}-\frac{1}{2}(\Delta_L-3)(\Delta_L\gamma_n^{(1,0)}+2\gamma_n^{(1,1)}) \\ &\quad +\log(z)((\Delta_L+n-1)\gamma_n^{(1,0)}+(\Delta_L+n)\gamma_n^{(1,1)}+\gamma_n^{(1,2)})\Big). \end{aligned} \tag{6.26}$$

Similarly, one can calculate any $r_{n,p}(z)$ for arbitrary p. In each $r_{n,p}(z)$ the z-dependence enters only through a single logarithmic term as in (6.26). In order to extract the OPE data we match (6.24) and (6.25) and obtain the following relations

$$\begin{aligned} \sum_{n=0}^{\infty} z^n r_{n,0}(z) &= -\frac{\Delta_L}{(1-z)^{\Delta_L}}\left(\frac{3}{4}(1+z)+\frac{1+z(z+4)}{4(1-z)}\log(z)\right), \\ \sum_{n=0}^{\infty} z^n r_{n,p}(z) &= -\frac{\Delta_L}{(1-z)^{\Delta_L}}\frac{(p-2)(p-1)(1-z)}{4p(p+1)(p+2)}, \end{aligned} \tag{6.27}$$

for $p \geq 1$. To solve these equations, we start from the first line, expand the right-hand side in $z \to 0$ limit and match term by term on both sides. From terms with $\log(z)$ we extract the $\gamma_n^{(1,0)}$ and from terms without $\log(z)$, we extract the $P_n^{(1,0)}$. We move on to $p = 1$ case, where we again expand the right-hand side of the second line in (6.27) in $z \to 0$ limit. Using $\gamma_n^{(1,0)}$ and $P_n^{(1,0)}$, we extract $\gamma_n^{(1,1)}$ and $P_n^{(1,1)}$. Straightforwardly, one can continue this process and extract OPE data for any value of p.

By proceeding with this calculation to high enough values and p one can notice that there is a simple expression for $\gamma_n^{(1,p)}$ given by

$$\gamma_n^{(1,p)} = (-1)^{p+1}\left(\frac{1}{2}(\Delta_L-1)\Delta_L+3n^2-3(1-\Delta_L)n\right), \tag{6.28}$$

for all $p \geq 0$ and $n \geq 0$. Note that for $p = 0$ this expression agrees with the one in [128]. There is no similar expression for $P_n^{(1,p)}$ so we list results for first p-s:

$$
\begin{aligned}
P_n^{(1,0)} =& -\frac{3}{4}(\Delta_L - 1)\Delta_L - \frac{3\Delta_L n}{2}, \\
P_n^{(1,1)} =& 3(n-1)n - \frac{1}{4}\Delta_L \left(\Delta_L \left(\Delta_L + 6n - 6\right) + 6(n-4)n + 5\right), \\
P_n^{(1,2)} =& \frac{1}{8}(\Delta_L(\Delta_L(\Delta_L^2 + 8n\Delta_L + 6n(3n-1) - 13) \\
& + 2(n(3n(2n-5) - 25) + 6)) - 12n(2n^2 + n - 3)), \\
P_n^{(1,3)} =& \frac{1}{120}(180n(n(3-(n-3)n) + 5) - 234)\Delta_L + 3n(n^3 + n^2 - 2) \\
& + \frac{1}{120}\Delta_L^2(-\Delta_L(\Delta_L\left(11\Delta_L + 90n - 20\right) + 90n(3n-1) + 55) \\
& + 90(3-4n)n^2 + 280).
\end{aligned}
\tag{6.29}
$$

After the calculation of the OPE data at $\mathcal{O}(\mu)$, one can fix the coefficients in the ansatz (6.21) by expanding the integrand of (3.67) up to $\mathcal{O}(\mu^2)$ and then integrating the obtained expression over l. The result of the integration is expanded near $\bar{z} \to 1$ and we collect the term that behaves as $(1-\bar{z})^{-\Delta_L+3}$. It depends on z, n and OPE data $P_n^{(k,p)}$ and $\gamma_n^{(k,p)}$ for $k = 1, 2$ and $p = 0, 1$, but we are interested only in the part of this term that contains $\log^2(z)$. This part only depends on OPE data at $\mathcal{O}(\mu)$, so it will be completely determined. We collect terms that behave as $(1-\bar{z})^{-\Delta_L+3}\log^2(z)z^m$. By expanding the ansatz (6.21) near $z \to 0$ we can collect terms that behave as $\log^2(z)z^m$ and by matching these to the ones calculated through S-channel, we obtain a system of linear equations for the coefficients in the ansatz. This system will be over-determined by taking m to be large enough. Solving it for $m \leq 20$, we obtain

$$
\begin{aligned}
a_{33} &= \frac{(\Delta_L-4)(\Delta_L-3)\Delta_L}{28800(\Delta_L-2)},\\
a_{24} &= \frac{(\Delta_L-8)\Delta_L}{13440(\Delta_L-2)},\\
a_{15} &= \frac{\Delta_L(\Delta_L+1)}{5040(\Delta_L-2)},\\
c_{16} &= \frac{25}{396}b_{14}+\frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(83-7\Delta_L\right)+158\right)+108\right)}{3193344\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
c_{25} &= -\frac{1}{12}b_{14}+\frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L+19\right)-146\right)-108\right)}{1451520\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
c_{34} &= \frac{\left(\Delta_L-4\right)\Delta_L\left(11\left(\Delta_L-4\right)\Delta_L-27\right)}{2419200\left(\Delta_L-3\right)\left(\Delta_L-2\right)}.
\end{aligned}
\tag{6.30}
$$

As expected, the coefficients a_{mn} are identical to those in (6.17). We are left with one undetermined coefficient. This is perhaps not surprising since we know from [15] that the OPE coefficients of the subleading twist multi-stress tensor operators are not universal. This non-universality is introduced in our correlator through coefficient b_{14}. One can check that after inserting (6.30) to (6.21) the term that multiplies the unknown coefficient b_{14} corresponds to the lightcone limit of the conformal block of the operator with dimension $\Delta = 8$ and spin $s = 2$. We thus conclude that b_{14} is the OPE coefficient of $: T_{\mu\alpha}T^{\alpha}{}_{\nu} :$,

$$
b_{14} = P^{(2)}_{8,2}. \tag{6.31}
$$

Now, using (6.19) we can write the T-channel OPE coefficients for the remaining double-stress tensor operators with twist $\tau_{2,1} = 6$ and conformal spin $\Delta + s \geq 14$. Explicitly, these are found to be given by

$$
\begin{aligned}
P^{(2)}_{10+2\ell,4+2\ell} &= \frac{\sqrt{\pi}2^{-4\ell-17}\Gamma(2\ell+7)}{(\ell+4)(\ell+5)(\ell+6)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{13}{2}\right)}\\
&\times\frac{\Delta_L}{(\Delta_L-3)(\Delta_L-2)}(a_{1,\ell}\Delta_L^3+b_{1,\ell}\Delta_L^2+c_{1,\ell}\Delta_L+d_{1,\ell}),
\end{aligned}
\tag{6.32}
$$

where

$$
\begin{aligned}
a_{1,\ell} &= (\ell+2)(2\ell+9)(\ell(2\ell+13)+9),\\
b_{1,\ell} &= 144-2\ell(2\ell+13)(\ell(2\ell+13)+12),\\
c_{1,\ell} &= \ell(2\ell+13)(\ell(2\ell+13)+33)+558,\\
d_{1,\ell} &= 216.
\end{aligned}
\tag{6.33}
$$

Here $\ell \geq 0$ and $P^{(2)}_{\Delta,s}$ is the sum of OPE coefficients of all operators with conformal dimension Δ and spin s. There is no way to distinguish operators with the same quantum numbers Δ and s at this level in the large C_T expansion. This type of degeneracy occurs for each conformal spin greater than 10 for twist $\tau_{2,1} = 6$. Also, perfect agreement between (6.32) and all the OPE coefficients of double-stress tensor operators of twist $\tau_{2,1} = 6$ and spin $s > 2$ calculated in [15] is observed. Note that $P^{(2)}_{8,2}$ can not be found from (6.32) by setting $\ell = -1$, this would not agree with the result in [15]. In Section 6.5 we rederive (6.32) using the Lorentzian inversion formula.

6.2.2. Twist-eight double-stress tensors

We follow the same logic as in the previous section in order to write the subsubleading part of the stress tensor sector of the HHLL correlator in the lightcone limit at $\mathcal{O}(\mu^2)$. This part scales as $(1-\bar{z})^{-\Delta_L+4}$. Here, we include contributions coming from operators with twist $\tau_{2,2} = 8$. These operators can be grouped in three families and they are schematically written as $:T_{\mu\nu}(\partial^2)^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T_{\rho\sigma}:$ with $\Delta = 12 + 2\ell$ and $s = 4 + 2\ell$, $:T_{\mu\beta}\partial^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\beta}{}_{\nu}:$ with $\Delta = 10 + 2\ell$ and $s = 2 + 2\ell$ and finally $:T_{\beta\gamma}\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\beta\gamma}:$ with $\Delta = 8 + 2\ell$ and $s = 2\ell$. Subtleties with regard to the contributions of the different families are discussed in Section 3.4.3.1.

Once more, we need to include the contributions of lower twist operators, i.e. by expanding their conformal blocks as $\bar{z} \to 1$ up to order $(1-\bar{z})^4$ and collect the additional z dependence. Accordingly, we write the following ansatz

$$\begin{aligned}\mathcal{G}^{(2,2)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(&\left(\frac{z(2z-7)+11}{6(z-1)^2}\right)(a_{33}f_3^2 + a_{24}f_2f_4 + a_{15}f_1f_5)\\ &+\left(\frac{2-z}{1-z}\right)(b_{14}f_1f_4 + c_{16}f_1f_6 + c_{25}f_2f_6 + c_{34}f_3f_4)\\ &+(d_{17}f_1f_7 + d_{26}f_2f_6 + d_{35}f_3f_5 + d_{44}f_4^2 + e_{15}f_1f_5 + e_{24}f_2f_4\\ &+e_{33}f_3^2 + g_{13}f_1f_3 + g_{22}f_2^2)\Bigg),\end{aligned} \tag{6.34}$$

where f_a means $f_a(z)$. Coefficients a_{mn} and c_{mn} are already calculated, while b_{14} is undetermined from the bootstrap. The linear dependence between certain products of $f_a(z)$ functions (for more details see Appendix B.1, in particular (A.2)) allows us to set three coefficients to zero, e.g., $g_{22} = 0$, $e_{33} = 0$ and $e_{24} = 0$.

To fix the unknown coefficients in (6.34) we match terms that behave as $(1-\bar{z})^{-\Delta_L+4} z^m \log^2 z$ from the S-channel calculation of the correlator to terms with the same behavior in (6.34) for small z. For the S-channel calculation, we need the OPE data at $\mathcal{O}(\mu)$ up to $p = 2$, given by (6.28) and (6.29). We obtain an over-constrained system of linear equations, whose solution is

$$\begin{aligned}
d_{17} &= \frac{9e_{15}}{143} + \frac{5g_{13}}{4004} + \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(232-17\Delta_L\right)+1009\right)+1908\right)+1008\right)}{115315200\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
d_{26} &= -\frac{e_{15}}{12} + \frac{5g_{13}}{1386} - \frac{\Delta_L\left(\Delta_L\left(\left(\Delta_L-7\right)\Delta_L\left(11\Delta_L-179\right)+3636\right)+2736\right)}{119750400\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
d_{35} &= -\frac{g_{13}}{180} + \frac{\Delta_L\left(\Delta_L\left(\left(\Delta_L-7\right)\Delta_L\left(37\Delta_L-13\right)+1332\right)+3312\right)}{108864000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},\\
d_{44} &= \frac{\left(\Delta_L-6\right)\Delta_L\left(\Delta_L+2\right)}{9408000\left(\Delta_L-2\right)}.
\end{aligned} \tag{6.35}$$

The undetermined coefficients g_{13} and e_{15} are related to the T-channel OPE coefficients $P^{(2)}_{8,0}$ and $P^{(2)}_{10,2}$ by the following relations

$$\begin{aligned}
g_{13} &= P^{(2)}_{8,0},\\
e_{15} &= P^{(2)}_{10,2} - \frac{5}{252} P^{(2)}_{8,0}.
\end{aligned} \tag{6.36}$$

Here $P^{(2)}_{8,0}$ is the T-channel OPE coefficient of the operator of the schematic form $:T_{\alpha\beta}T^{\alpha\beta}:$, while $P^{(2)}_{10,2}$ is related to the OPE coefficients of the operators $:T_{\alpha\beta}\partial_{\mu_1}\partial_{\mu_2}T^{\alpha\beta}:$ and $:T_{\mu\alpha}\partial^2 T^{\alpha}{}_{\nu}:$ which have the same quantum numbers Δ and s and are thus indistinguishable at this order in large C_T expansion. After inserting (6.36) and (6.35) into (6.34) one can check that both $P^{(2)}_{8,0}$ and $P^{(2)}_{10,2}$ will be multiplied by the relevant lightcone conformal blocks.

Exactly as in the previous section, we can now extract the OPE coefficients $P^{(2)}_{\Delta,s}$ for operators with twist $\tau_{2,2} = 8$ and $\Delta = 12 + 2\ell$, $s = 4 + 2\ell$, for $\ell \geq 0$[19]

$$\begin{aligned}
P^{(2)}_{12+2\ell,4+2\ell} =& \frac{\sqrt{\pi}\Delta_L 2^{-4\ell-19}\Gamma(2\ell+7)}{3(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)(\ell+4)(\ell+5)}\\
&\times \frac{a_{2,\ell}\Delta_L^4 + b_{2,\ell}\Delta_L^3 + c_{2,\ell}\Delta_L^2 + d_{2,\ell}\Delta_L + e_{2,\ell}}{(\ell+6)(\ell+7)(2\ell+1)(2\ell+3)(2\ell+5)\Gamma\left(2\ell+\frac{15}{2}\right)},
\end{aligned} \tag{6.37}$$

[19] For each $\Delta = 12 + 2\ell$ and $s = 4 + 2\ell$ with $\ell \geq 0$ there is a triple degeneracy, because all three families of operators with twist $\tau_{2,2} = 8$ will be mixed.

where

$$
\begin{aligned}
a_{2,\ell} &= \ell(2\ell+15)(\ell(2\ell+15)(\ell(2\ell+15)+59)+1084)+6012,\\
b_{2,\ell} &= 14004-2\ell(2\ell+15)(\ell(2\ell+15)(\ell(2\ell+15)+32)-131),\\
c_{2,\ell} &= \ell(2\ell+15)(\ell(2\ell+15)(\ell(2\ell+15)+113)+4594)+60984,\\
d_{2,\ell} &= 216(11\ell(2\ell+15)+302),\\
e_{2,\ell} &= 864(\ell(2\ell+15)+34).
\end{aligned} \tag{6.38}
$$

It is quite remarkable that these OPE coefficients are fixed purely by the bootstrap.

6.2.3. Twist-ten double-stress tensors

Now we want to go one step further and analyze the subsubsubleading contribution to the stress tensor sector of the HHLL correlator. This contribution scales as $(1-\bar{z})^{-\Delta_L+5}$ in the lightcone limit. We have to take in to account the double-stress tensor operators of twist $\tau_{2,3}=10$ in order to calculate this contribution. These operators can again be grouped in three families of the schematic form $:T_{\mu\nu}(\partial^2)^3\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T_{\rho\sigma}:$ with $\Delta=14+2\ell$ and $s=4+2\ell$, $:T_{\mu\beta}(\partial^2)^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\beta}{}_{\nu}:$ with $\Delta=12+2\ell$ and $s=2+2\ell$ and finally $:T_{\beta\gamma}\partial^2\partial_{\alpha_1}\dots\partial_{\alpha_{2\ell}}T^{\beta\gamma}:$ with $\Delta=10+2\ell$ and $s=2\ell$.

In order to include contributions from lower twist operators we have to expand their conformal blocks up to $(1-\bar{z})^5$ for $\bar{z}\to 1$. The ansatz takes the following form

$$
\begin{aligned}
\mathcal{G}^{(2,3)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(&\left(\frac{z((13-3z)z-23)+25}{12(1-z)^3}\right)(a_{33}f_3^2+a_{24}f_2f_4+a_{15}f_1f_5)\\
&+\left(\frac{1}{(1-z)^2}+\frac{1}{1-z}+\frac{9}{10}\right)(b_{14}f_1f_4+c_{16}f_1f_6+c_{25}f_2f_5+c_{34}f_3f_4)\\
&+\left(\frac{1}{1-z}+\frac{3}{2}\right)(d_{17}f_1f_7+d_{26}f_2f_6+d_{35}f_3f_5+d_{44}f_4^2+e_{15}f_1f_5\\
&+g_{13}f_1f_3)-g_{13}f_3+(h_{18}f_1f_8+h_{27}f_2f_7+h_{36}f_3f_6+h_{45}f_4f_5\\
&+j_{16}f_1f_6+j_{25}f_2f_5+j_{34}f_3f_4+i_{14}f_1f_4+i_{23}f_2f_3)\Bigg),
\end{aligned} \tag{6.39}
$$

with h_{mn}, j_{mn} and i_{mn}, coefficients that we need to determine, and with b_{14}, e_{15} and g_{13} undetermined from the bootstrap. The term $g_{13}f_3(z)$ in the next-to-last line of the previous equation has its origin in the correction to the conformal block of operator $:T_{\alpha\beta}T^{\alpha\beta}:$. This operator has $\beta = \tau_{2,2} = 8$ which implies that both lines in the following expansion of the conformal block

$$\begin{aligned} g_{8,0}^{(0,0)}(1-z,1-\bar{z}) = (1-\bar{z})^4 &\left(1 + (1-\bar{z})\left(\frac{3}{2} + \frac{1}{1-z}\right) + \mathcal{O}\left((1-\bar{z})^2\right)\right) f_4(z) \\ - (1-\bar{z})^5 &\left(1 + (1-\bar{z})\left(2 + \frac{1}{1-z}\right) + \mathcal{O}((1-\bar{z})^2)\right) f_3(z) \end{aligned} \tag{6.40}$$

contribute. The contribution from the first line of (6.40) is included in the third line of (6.39), while we had to explicitly add the contribution from the second line. Using (A.1) and (A.3) we set $i_{23} = 0$, $j_{34} = 0$ and $j_{25} = 0$.

From the S-channel calculation, we collect the terms in the correlator which behave as $(1-\bar{z})^{-\Delta_L+5}\log^2(z)z^m$ and are fixed in terms of OPE data at $\mathcal{O}(\mu)$ for $p \leq 3$. By expanding (6.39) near $z \to 0$ we obtain terms with the same behavior as linear functions of unknown coefficients and by matching them with the terms from the S-channel, we determine the unknown coefficients. These are

$$\begin{aligned} h_{18} =& -\frac{\Delta_L(\Delta_L+1)(\Delta_L(\Delta_L(\Delta_L(47\Delta_L-721)-5182)-15204)-13680)}{4942080000(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)} \\ &+ \frac{49i_{14}}{38610} + \frac{49j_{16}}{780}, \\ h_{27} =& -\frac{\Delta_L(\Delta_L+1)(\Delta_L(\Delta_L(\Delta_L(8\Delta_L-229)+1097)+7224)+10080)}{1383782400(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)} \\ &+ \frac{5i_{14}}{1404} - \frac{j_{16}}{12}, \\ h_{36} =& \frac{\Delta_L(\Delta_L+1)(\Delta_L(\Delta_L(\Delta_L(34\Delta_L-137)-1829)+5712)+23040)}{2661120000(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)} \\ &- \frac{i_{14}}{180}, \\ h_{45} =& \frac{(\Delta_L-6)\Delta_L(\Delta_L+1)(\Delta_L+2)}{62720000(\Delta_L-3)(\Delta_L-2)}. \end{aligned} \tag{6.41}$$

Our approach does not allow us to determine the coefficients j_{16} and i_{14}. These are related to the T-channel OPE coefficients of operators with twist $\tau_{2,3} = 10$ and minimal conformal spin by

$$\begin{aligned} i_{14} &= P_{10,0}^{(2)}, \\ j_{16} &= P_{12,2}^{(2)} - \frac{2}{99}P_{10,0}^{(2)}. \end{aligned} \tag{6.42}$$

Notice that, despite the fact that the h_{mn} depend on the undetermined OPE data, we are able to extract all the OPE coefficients of double-stress tensors with twist $\tau_{2,3} = 10$ and conformal spin $\Delta + s \geq 18$. Explicitly, they are given by:

$$
\begin{aligned}
P^{(2)}_{14+2\ell,4+2\ell} = & \frac{\sqrt{\pi}2^{-4\ell-22}\Gamma(2\ell+9)}{5(2\ell+1)(2\ell+3)(2\ell+5)(2\ell+7)\Gamma\left(2\ell+\frac{17}{2}\right)} \\
& \times \frac{\Delta_L(\Delta_L+1)(a_{3,\ell}\Delta_L^4 + b_{3,\ell}\Delta_L^3 + c_{3,\ell}\Delta_L^2 + d_{3,\ell}\Delta_L + e_{3,\ell})}{(\ell+5)(\ell+6)(\ell+7)(\ell+8)(\Delta_L-5)(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)},
\end{aligned}
\tag{6.43}
$$

where

$$
\begin{aligned}
a_{3,\ell} =& \ell(2\ell+17)(\ell(2\ell+17)(\ell(2\ell+17)+70)+1513)+9756, \\
b_{3,\ell} =& 38232 - 2(\ell-1)\ell(2\ell+17)(2\ell+19)(\ell(2\ell+17)+44), \\
c_{3,\ell} =& 196164 + \ell(17+2\ell(11647+\ell(17+2\ell)(196+\ell(17+2\ell)))), \\
d_{3,\ell} =& 504(647+19\ell(17+2\ell)), \\
e_{3,\ell} =& 4320(53+\ell(17+2\ell)).
\end{aligned}
\tag{6.44}
$$

We expect that a similar picture is true for all subleading twist double-stress tensor operators. At $\mathcal{O}(\mu^2)$, the ansatz for $\mathcal{G}^{(2,m)}(z)$ will naturally include products of the type $f_a(z)f_b(z)$, such that $a+b = 6+m$, together with $f_1(z)f_{3+m}(z)$ and $f_1(z)f_{1+m}(z)$. The coefficients of the latter two will be left undetermined from the lightcone bootstrap at every order in the lightcone expansion. Such coefficients will be related to the non-universal OPE coefficients of double-stress tensors with spin $s = 0, 2$ for a given twist. On the other hand, the coefficients of the products $f_a(z)f_b(z)$, with $a+b = 6+m$, once determined, will allow us to extract the OPE coefficients of all double-stress tensors with conformal spin $\beta \geq 12 + 2m$. We expect them to be universal, despite the fact that the coefficients of the products $f_a(z)f_b(z)$, with $a+b = 6+m$, will be plagued by the ambiguities present in the determination of the OPE coefficients of operators spin $s = 0, 2$ – just as herein.

6.3. Triple-stress tensors in four dimensions

In this section, we consider the stress tensor sector of the HHLL correlator at $\mathcal{O}(\mu^3)$ in $d = 4$. The operators which contribute in the T-channel are triple-stress tensors. Since we are interested in the lightcone limit $1 - \bar{z} \ll 1$, we consider contributions of operators with low twist. Triple-stress tensors with minimal twist can be written in the schematic form $: T_{\mu\nu} T_{\rho\sigma} \partial_{\alpha_1} \ldots \partial_{\alpha_{2\ell}} T_{\eta\xi} :$. These operators have twist $\tau_{3,\min} = 6$ and their contribution to the HHLL correlator in the lightcone limit was found in [13]:

$$
\begin{aligned}
\mathcal{G}^{(3,0)}(z) = \frac{1}{(1-z)^{\Delta_L}} \Big(& a_{117} f_1(z)^2 f_7(z) + a_{126} f_1(z) f_2(z) f_6(z) \\
+ a_{135} f_1(z) f_3(z) f_5(z) + a_{225} f_2(z)^2 f_5(z) & + a_{234} f_2(z) f_3(z) f_4(z) + a_{333} f_3(z)^3 \Big),
\end{aligned}
\tag{6.45}
$$

where the coefficients a_{ikl} are

$$
\begin{aligned}
a_{117} &= \frac{5\Delta_L(\Delta_L+1)(\Delta_L+2)}{768768(\Delta_L-2)(\Delta_L-3)}, \\
a_{126} &= \frac{5\Delta_L(5\Delta_L^2-57\Delta_L-50)}{6386688(\Delta_L-2)(\Delta_L-3)}, \\
a_{135} &= \frac{\Delta_L(2\Delta_L^2-11\Delta_L-9)}{1209600(\Delta_L-3)}, \\
a_{225} &= -\frac{\Delta_L(7\Delta_L^2-51\Delta_L-70)}{2903040(\Delta_L-2)(\Delta_L-3)}, \\
a_{234} &= \frac{\Delta_L(\Delta_L-4)(3\Delta_L^2-17\Delta_L+4)}{4838400(\Delta_L-2)(\Delta_L-3)}, \\
a_{333} &= \frac{\Delta_L(\Delta_L-4)(\Delta_L^3-16\Delta_L^2+51\Delta_L+24)}{10368000(\Delta_L-2)(\Delta_L-3)}.
\end{aligned}
\tag{6.46}
$$

6.3.1. Twist-eight triple-stress tensors

We now consider the subleading contributions at $\mathcal{O}(\mu^3)$ coming from triple-stress tensor operators with twist $\tau_{3,1} = 8$. There are two families of such operators, these can be schematically written as $: T_{\mu\nu} T_{\rho\alpha} \partial_{\alpha_1} \ldots \partial_{\alpha_{2\ell}} T^{\alpha}{}_{\xi} :$ with $\Delta = 12+2\ell$ and spin $s = 4+2\ell$ and $: T_{\mu\nu} T_{\rho\sigma} \partial^2 \partial_{\alpha_1} \ldots \partial_{\alpha_{2\ell}} T_{\eta\xi} :$ with $\Delta = 14+2\ell$ and spin $s = 6+2\ell$. The conformal spins of these families are $\beta = 16+4\ell$ and $\beta = 20+4\ell$, respectively, so we expect products of three $f_a(z)$ functions such that their indices add up to 8 and 10. The contribution to the correlator of these operators scales as $(1-\bar{z})^{-\Delta_L+4}$ for $\bar{z} \to 1$. This implies that one needs to include the contribution from the minimal twist triple-stress tensor operators (due to corrections to their conformal blocks).

Our ansatz takes the form

$$\mathcal{G}^{(3,1)}(z) = \frac{1}{(1-z)^{\Delta_L}} \Bigg(\left(\frac{2-z}{1-z}\right) (a_{117} f_1^2 f_7 + a_{126} f_1 f_2 f_6 + a_{135} f_1 f_3 f_5 \\ + a_{225} f_2^2 f_5 + a_{234} f_2 f_3 f_4 + a_{333} f_3^3) + (b_{116} f_6 f_1^2 + b_{134} f_3 f_4 f_1 \\ + b_{125} f_2 f_5 f_1 + b_{233} f_2 f_3^2 + b_{224} f_2^2 f_4 + c_{118} f_8 f_1^2 + c_{145} f_4 f_5 f_1 \\ + c_{136} f_3 f_6 f_1 + c_{127} f_2 f_7 f_1 + c_{244} f_2 f_4^2 + c_{334} f_3^2 f_4 + c_{235} f_2 f_3 f_5 \\ + c_{226} f_2^2 f_6) \Bigg), \tag{6.47}$$

where a_{jkl} are given in (6.46). The linear dependence between products of three f_a functions, with explicit relations given in Appendix B.1, allows us to set the following coefficients to zero

$$b_{125} = b_{134} = b_{224} = b_{233} = c_{136} = 0. \tag{6.48}$$

To fix the coefficients b_{116} and c_{jkl} we perform an S-channel calculation up to $\mathcal{O}(\mu^3)$. The relevant terms now scale as $(1-\bar{z})^{-\Delta_L+4} \log^3(z) z^m$ and $(1-\bar{z})^{-\Delta_L+4} \log^2(z) z^m$ when $\bar{z} \to 1$ and $z \to 0$.

We fix the S-channel OPE data at $\mathcal{O}(\mu^2)$ using the results of the previous section, specifically eqs. (6.21), (6.34) and (6.39). Since the OPE coefficients of double-stress operators of spin 0 and 2 are left undetermined, the S-channel OPE data is fixed in terms of these. Concretely, $\gamma_n^{(2,0)}$ and $P_n^{(2,0)}$ are completely determined since the leading-twist OPE coefficients are known and universal, while $\gamma_n^{(2,1)}$ and $P_n^{(2,1)}$ depend on b_{14}, $\gamma_n^{(2,2)}$ and $P_n^{(2,2)}$ depend on b_{14}, g_{13} and e_{15} and so on.[20]

We were able to fix all the unknown coefficients in the ansatz (6.47) using bootstrap. Crucially, there are no spin $s = 0, 2$ operators that contribute at this level. Here, we list two of the coefficients while all others can be found in Appendix B.2.

[20] Explicit expressions for the S-channel OPE data are too cumbersome to quote here.

$$
\begin{aligned}
b_{116} =&\frac{-\Delta_L\left(\Delta_L+3\right)\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(1001\Delta_L+387\right)-4326\right)+13828\right)+5040\right)}{10378368000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\\
&+\frac{b_{14}\left(\Delta_L\left(143\Delta_L+427\right)+540\right)}{17160\left(\Delta_L-4\right)},\\
c_{118} =&7\left(\Delta_L+3\right)\times\\
&\frac{604800 b_{14}\left(\Delta_L^2-5\Delta_L+6\right)+\Delta_L\left(-21\Delta_L^3+229\Delta_L^2+414\Delta_L+284\right)}{856627200\left(\Delta_L^3-9\Delta_L^2+26\Delta_L-24\right)}.
\end{aligned}
\tag{6.49}
$$

Notice that they depend on b_{14}. This is because the anomalous dimensions at $\mathcal{O}(\mu^2)$, $\gamma_n^{(2,2)}$ depend on it. Moreover, no OPE coefficient of triple-stress tensors with twist $\tau_{3,1}=10$ is universal since all of them depend on b_{14}. These OPE coefficients can be written in the form of a finite sum, similarly to what happens for the OPE coefficients of leading twist triple-stress tensor, given in [13]. We define $i_1(r,q)$ and $i_2(r,q)$ as

$$
i_1(r,q)=b_{116}p(1,1,r)p(2r+2,6,q),
\tag{6.50}
$$

and

$$
\begin{aligned}
i_2(r,q)=&\,c_{118}p(1,1,r)p(2r+2,8,q)+c_{127}p(1,2,r)p(2r+3,7,q)\\
&+c_{145}p(1,4,r)p(2r+5,5,q)+c_{226}p(2,2,r)p(2r+4,6,q)\\
&+c_{235}p(2,3,r)p(2r+5,5,q)+c_{244}p(2,4,r)p(2r+6,4,q)\\
&+c_{334}p(3,3,r)p(2r+6,4,q),
\end{aligned}
\tag{6.51}
$$

where $p(a,b,\ell)$ are given by (6.20). The OPE coefficients can be written as

$$
P^{(3)}_{14+2\ell,6+2\ell}=\sum_{r=0}^{\ell+1} i_1(r,\ell+1-r)+\sum_{r=0}^{\ell} i_2(r,\ell-r),
\tag{6.52}
$$

for $k\geq 0$, while $P^{(3)}_{12,4}=i_1(0,0)=b_{116}$. We give the explicit expressions for some OPE coefficients in Appendix B.4.

6.3.2. Twist-ten triple-stress tensors

Here, we consider the contribution of triple-stress tensor operators of twist $\tau_{3,2}=10$. These operators can be divided in three families of the schematic form

$: T_{\mu\nu}T_{\alpha\beta}\partial_{\mu_1}\dots\partial_{\mu_{2\ell}}(\partial^2)^2 T_{\rho\sigma}:$ with conformal dimension $\Delta = 16 + 2\ell$ and spin $s = 6 + 2\ell$, $: T_{\mu\nu}T_{\alpha\beta}\partial_{\mu_1}\dots\partial_{\mu_{2\ell}}\partial^2 T^{\beta}{}_{\rho}:$ with $\Delta = 14 + 2\ell$ and $s = 4 + 2\ell$ and finally $: T_{\mu\alpha}T_{\nu\beta}\partial_{\mu_1}\dots\partial_{\mu_{2\ell}}T^{\alpha\beta}:$ with $\Delta = 12 + 2\ell$ and $s = 2 + 2\ell$. One can see that in the last family an operator of spin $s = 2$ is included.

An appropriate ansatz in this case is

$$\mathcal{G}^{(3,2)}(z) = \frac{1}{(1-z)^{\Delta_L}}\Bigg(\left(\frac{144z^2 - 448z + 464}{160(z-1)^2}\right)(a_{117}f_1^2 f_7 + a_{126}f_1 f_2 f_6 + a_{135}f_1 f_3 f_5 + a_{225}f_2^2 f_5 + a_{234}f_2 f_3 f_4 + a_{333}f_3^3) + \left(\frac{1}{1-z} + \frac{3}{2}\right)(b_{116}f_6 f_1^2 + c_{118}f_8 f_1^2 + c_{145}f_4 f_5 f_1 + c_{127}f_2 f_7 f_1 + c_{244}f_2 f_4^2 + c_{334}f_3^2 f_4 + c_{235}f_2 f_3 f_5 + c_{226}f_2^2 f_6) + (d_{117}f_1^2 f_7 + e_{115}f_1^2 f_5 + g_{119}f_1^2 f_9 + g_{128}f_1 f_2 f_8 + g_{155}f_1 f_5^2 + g_{227}f_2^2 f_7 + g_{236}f_2 f_3 f_6 + g_{245}f_2 f_4 f_5 + g_{335}f_3^2 f_5 + g_{344}f_3 f_4^2)\Bigg), \tag{6.53}$$

where $f_a = f_a(z)$ and we have included only the linearly independent products of these functions.

The lightcone bootstrap fixes all coefficients except e_{115}. One can check that this is exactly the OPE coefficient $P^{(3)}_{12,2}$ of the spin-2 operator $: T_{\mu\alpha}T_{\nu\beta}T^{\alpha\beta}:$ with $\Delta = 12$ and spin $s = 2$

$$e_{115} = P^{(3)}_{12,2}. \tag{6.54}$$

All other coefficients can be found in Appendix B.2. Notice that all coefficients depend on b_{14}, g_{13} and e_{15} because the S-channel OPE data at $\mathcal{O}(\mu^2)$ depend on them.

Again, we write the OPE coefficients for all triple-stress tensor operators with twist $\tau_{3,2} = 10$ and $\beta \geq 18$ in the form of a finite sum. We define $j_1(r,q)$, $j_2(r,q)$ and $j_3(r,q)$ as

$$j_1(r,q) = e_{115}p(1,1,r)p(2r+2,5,q), \tag{6.55}$$

$$j_2(r,q) = d_{117}p(1,1,r)p(2r+2,7,q) \tag{6.56}$$

and

$$
\begin{aligned}
j_3(r,q) = {}& g_{119}p(1,1,r)p(2r+2,9,q) + g_{128}p(1,2,r)p(2r+3,8,q) \\
& + g_{155}p(1,5,r)p(2r+6,5,q) + g_{227}p(2,2,r)p(2r+4,7,q) \\
& + g_{236}p(2,3,r)p(2r+5,6,q) + g_{245}p(2,4,r)p(2r+6,5,q) \\
& + g_{335}p(3,3,r)p(2r+6,5,q) + g_{344}p(3,4,r)p(2r+7,4,q),
\end{aligned}
\tag{6.57}
$$

where $p(a,b,\ell)$ is given by (6.20). The OPE coefficients can now be written as

$$
P^{(3)}_{16+2\ell,6+2\ell} = \sum_{r=0}^{\ell+2} j_1(r,\ell+2-r) + \sum_{r=0}^{\ell+1} j_2(r,\ell+1-r) + \sum_{r=0}^{\ell} j(r,\ell-r), \tag{6.58}
$$

for $\ell \geq 0$, while

$$
P^{(3)}_{14,4} = j_1(0,1) + j_1(1,0) + j_2(0,0). \tag{6.59}
$$

Finally, we conclude that the stress tensor sector of the HHLL correlator to all orders in μ and in the lightcone expansion will take a similar form in terms of products of f_a functions. One should be able to completely fix the coefficients, except for terms that correspond to the OPE coefficients of multi-stress tensor operators with spin $s = 0, 2$, using the lightcone bootstrap.

6.4. Holographic phase shift and multi-stress tensors

In this section, we demonstrate how to calculate the T-channel OPE coefficients of spin-2 operators (up to undetermined spin-0 data) which are left undetermined after the lightcone bootstrap, using a gravitational calculation of the scattering phase shift. We are interested in the scattering phase shift – or eikonal phase – resulting from the eikonal resummation of graviton exchanges when a fast particle is scattered by a black hole[21]. Seeking to explore the universality properties of the undetermined OPE coefficients of the previous section, we perform the calculation in Gauss-Bonnet gravity extending the results of [55] to this case. We argue that the phase shift in the large impact parameter limit is independent of higher-derivative corrections to the dual gravitational lagrangian. This is consistent with the universality of the minimal-twist multi-stress tensor sector in the dual CFT. On the other hand, we observe that the

[21] For CFT approach to the Regge scattering of scalar particles in pure AdS see [58-63,126,134].

subleading OPE data of spin-2 multi-stress tensors depend explicitly on the Gauss-Bonnet coupling $\lambda_{\rm GB}$.

The computation involves performing an inverse Fourier transform of the exponential of the phase shift in the large impact parameter expansion, to obtain the HHLL correlator in position space[22]. This is done following the approach of [135]. Comparison with the expressions for the HHLL correlator in the lightcone limit requires analytically continuing the results of Sections 6.2 and 6.3 and taking the limit $z \to 1$. Identifying terms in the HHLL four-point function with the same large impact parameter and $z \to 1$ behavior allows us to extract the spin-2 OPE coefficients of the double- and triple-stress tensor operators (up to undetermined spin zero data).

6.4.1. Universality of the phase shift in the large impact parameter limit

In this subsection, we consider Gauss-Bonnet gravity in $(d+1)$–dimensions and argue that the phase shift obtained by a highly energetic particle traveling in a spherical AdS-Schwarzschild background is independent of the Gauss-Bonnet coupling $\lambda_{\rm GB}$ in the large impact parameter limit.

The action of Gauss-Bonnet gravity in $(d+1)$-dimensional spacetime is

$$S = \frac{1}{16\pi G}\int d^{d+1}\sqrt{-g}\Big(R + \frac{d(d-1)}{\ell^2} + \frac{\tilde{\lambda}_{\rm GB}}{(d-2)(d-3)}(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\Big), \tag{6.60}$$

where the coupling parameter $\tilde{\lambda}_{\rm GB}$ is measured in units of the cosmological constant ℓ: $\tilde{\lambda}_{\rm GB} = \lambda_{\rm GB}\ell^2$, with $\lambda_{\rm GB}$ being a dimensionless coefficient. The AdS-Schwarzschild black hole metric which is a solution of the Gauss-Bonnet theory is given by [136,137]:

$$ds^2 = -r_{AdS}^2 f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2, \tag{6.61}$$

where

$$f(r) = 1 + \frac{r^2}{2\lambda_{\rm GB}}\left(1 - \sqrt{1 - 4\lambda_{\rm GB}(1 - \frac{\tilde{\mu}}{r^d})}\right), \tag{6.62}$$

[22] Recall that the exponential of the phase shift corresponds to the Regge limit of HHLL four-point function in momentum space [55].

with

$$\tilde{\mu} = \frac{16\pi GM}{(d-1)\Omega_{d-1}\ell^{d-2}}, \qquad \mu = \frac{\tilde{\mu}}{r_{AdS}^{d-2}\sqrt{1-4\lambda_{\rm GB}}}, \tag{6.63}$$

and

$$r_{AdS} = \left(\frac{1}{2}(1+\sqrt{1-4\lambda_{\rm GB}})\right)^{1/2} \tag{6.64}$$

where Ω_{d-1} is the surface area of a $(d-1)$-dimensional unit sphere embedded in d-dimensional Euclidean space. The metric is normalized such that the speed of light is equal to 1 at the boundary (i.e. $g_{tt}/g_{\phi\phi} \to 1$ as $r \to \infty$) and all dimensionful parameters are measured in units of ℓ. The product (ℓr_{AdS}) is the radius of the asymptotic Anti-de Sitter space.

The two conserved charges along the geodesics, p^t and p^ϕ, are

$$\begin{aligned} p^t &= r_{AdS}^2 f(r)\frac{dt}{d\lambda}, \\ p^\phi &= r^2\frac{d\phi}{d\lambda}. \end{aligned} \tag{6.65}$$

where λ denotes an affine parameter. Null geodesics are described by the following equation,

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{(p^\phi)^2}{2r^2}f(r) = \frac{1}{2}\frac{(p^t)^2}{r_{AdS}^2}. \tag{6.66}$$

similarly to Einstein gravity.

A light particle, starting from the boundary, traversing the bulk and reemerging on the boundary experiences a time delay and a path deflection given by :

$$\begin{aligned} \Delta t &= 2\int_{r_0}^{\infty} \frac{dr}{r_{AdS} f(r)\sqrt{1-\alpha^2\frac{r_{AdS}^2}{r^2}f(r)}}, \\ \Delta\phi &= 2\alpha\, r_{AdS}\int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{1-\alpha^2\frac{r_{AdS}^2}{r^2}f(r)}}, \end{aligned} \tag{6.67}$$

where $\alpha = p^\phi/p^t$ and r_0 the impact parameter determined by $\frac{dr}{d\lambda}|_{r(\lambda)=r_0} = 0$, *i.e.*,

$$1 - \alpha^2\frac{r_{AdS}^2}{r_0^2}f(r_0) = 0. \tag{6.68}$$

Defining the phase shift as $\delta = -p\cdot\Delta x = p^t\Delta t - p^\phi\Delta\phi$, we find that

$$\delta = 2\frac{p^t}{r_{AdS}}\int_{r_0}^{\infty}\frac{dr}{f(r)}\sqrt{1-\alpha^2\frac{r_{AdS}^2}{r^2}f(r)}. \tag{6.69}$$

Just as in [55], we are interested in expanding the phase shift order by order in μ. It is easy to see that in terms of CFT data μ can be expressed as

$$\mu = \frac{4}{(d-1)^2} \frac{\Gamma(d+2)}{\Gamma(d/2)^2} \frac{\Delta_H}{C_T}, \tag{6.70}$$

which is consistent with (3.14). Here C_T is the central charge of the dual conformal theory [138]:

$$C_T = \frac{\pi^{\frac{d}{2}-1}}{2(d-1)} \frac{\Gamma(d+2)}{\Gamma(d/2)^3 G} (r_{AdS}\ell)^{d-1} \sqrt{1-4\lambda_{\rm GB}}, \tag{6.71}$$

and $\Delta_H = M\ell r_{AdS}$.

In order to calculate the phase shift, we introduce a new variable y, given by $y = \frac{r_0}{r}$. Using this variable (6.69) can be written as:

$$\delta = 2\frac{p^t r_0}{r_{AdS}} \int_0^1 \frac{dy}{y^2 f(\frac{r_0}{y})} \left(1 - \alpha^2 \frac{r_{AdS}^2 y^2}{{r_0}^2} f(\frac{r_0}{y})\right)^{1/2}. \tag{6.72}$$

Expanding the phase shift

$$\delta = \sum_{k=0}^{\infty} \mu^k \delta^{(k)}, \tag{6.73}$$

and solving (6.68) perturbatively in μ reads

$$r_0 = b - \frac{b^{3-d}}{2r_{AdS}^{2-d}}\mu + \frac{b^{3-2d}}{8r_{AdS}^{4-2d}} \left(b^2(3-2d) + \frac{4\lambda_{\rm GB}}{\sqrt{1-4\lambda_{\rm GB}}}\right)\mu^2 + \mathcal{O}(\mu^3). \tag{6.74}$$

Generically, we get an expansion of the form

$$r_0 = b + \sum_{k=1}^{\infty} a_k \mu^k, \tag{6.75}$$

where the a_k, which depend on b, in the large impact parameter limit $(b \to \infty)$ behave as

$$a_k \propto b \left(\frac{r_{AdS}}{b}\right)^{k(d-2)}. \tag{6.76}$$

Notice that there is no explicit $\lambda_{\rm GB}$ dependence in the leading term[23], since the metric (6.61) approaches the one in pure GR.

[23] Except the overall dependence on r_{AdS}.

To study the leading behavior of the phase shift for large impact parameters it is convenient to define a function $g(x)$ as

$$g(x) = r_{AdS}^2 \frac{f(x)}{x^2}, \tag{6.77}$$

with f given by (6.62), and denote the integrand of (6.72) by $h\left(g\left(\frac{r_0}{y}\right)\right)$, with

$$h(x) = \frac{1}{x}\sqrt{1-\alpha^2 x}, \tag{6.78}$$

to express (6.72) as

$$\delta = 2p^t \left(\frac{r_{AdS}}{r_0}\right) \int_0^1 h\left(g\left(\frac{r_0}{y}\right)\right) dy. \tag{6.79}$$

In practice, to calculate the phase shift in the large impact parameter limit, we first expand the integrand of (6.79) in powers of μ, perform the integration with respect to y, and then expand the result in powers of b. The b-dependence of $\delta^{(k)}$ is therefore fixed before the integration and the integral just determines the overall numerical factor (assuming that it is convergent).

We can immediately see that $g\left(\frac{r_0}{y}\right)$ depends on μ explicitly and implicitly through $r_0(\mu)$ in (6.74). In order to make this clear we write $g\left(\frac{r_0}{y}, \mu\right)$ instead of just $g\left(\frac{r_0}{y}\right)$. Defining $g^{(n,m)}\left(\frac{b}{y}, 0\right)$ as

$$g^{(n,m)}\left(\frac{b}{y}, 0\right) = \frac{\partial^n \partial^m}{\partial r_0^n \partial \mu^m} g\left(\frac{r_0}{y}, \mu\right)\Big|_{r_0=b,\mu=0}. \tag{6.80}$$

allows us to write the following expansion for $h\left(g\left(\frac{r_0}{y}, \mu\right)\right)$:

$$\begin{aligned} h\left(g\left(r_0/y, \mu\right)\right) =& h(g(b/y,0)) + \mu h'(g(b/y,0))\left(g^{(0,1)}(b/y,0) + a_1 g^{(1,0)}(b/y,0)\right) \\ &+ \frac{\mu^2}{2} h''(g(b/y,0))\left(g^{(0,1)}(b/y,0) + a_1 g^{(1,0)}(b/y,0)\right)^2 \\ &+ \frac{\mu^2}{2} h'(g(b/y,0))\Big(g^{(0,2)}(b/y,0) + 2a_2 g^{(1,0)}(b/y,0) \\ &+ 2a_1 g^{(1,1)}(b/y,0) + a_1^2 g^{(2,0)}(b/y,0)\Big) + \mathcal{O}(\mu^3), \end{aligned} \tag{6.81}$$

where a_k are the coefficients appearing in (6.75). It is clear that at each order in the μ-expansion we will have a sum of products composed from derivatives of $h(x)$ and sums of the form

$$\sum_{\{k_i:\sum_{i=1}^{p} k_i \leqslant n\}} a_{k_1} a_{k_2} \ldots a_{k_p} g^{(p,n-\sum_{i=1}^{p} k_i)}(b/y,0)\,. \tag{6.82}$$

Notice first that $g(b/y,0)$, $g^{(m,0)}(b/y,0)$ and $g^{(m,1)}(b/y,0)$ do not depend on $\lambda_{\rm GB}$ as can be seen from (6.77). The same is true for $h^{(n)}(g(b/y,0))$ for any n as follows from (6.78). On the contrary, $g^{(m,n)}(b/y,0)$ with $n \geq 2$ depend explicitly on $\lambda_{\rm GB}$. It is then evident that any dependence on $\lambda_{\rm GB}$ will come from terms like the ones in parenthesis in (6.81) which are of the type (6.82). We will now show that all the terms in such sums which contain $\lambda_{\rm GB}$, are subleading in the large impact parameter limit.

Recall that $a_k \propto b^{1-k(d-2)}$ for $k \geq 1$. Using (6.77) one can check that $g^{(m,n)}(b/y,0) \propto b^{-m-nd}$ for $n > 0$ and $g^{(m,0)}(b/y,0) \propto b^{-m-2}$. We thus need to spearately consider two cases: products of the form $a_{k_1} a_{k_2} \ldots a_{k_p} g^{(p,n-q)}(b/y,0)$, with $q = \sum_{i=1}^{p} k_i$ and $q < n$ and products of the form $a_{k_1} a_{k_2} \ldots a_{k_p} g^{(p,0)}(b/y,0)$ for which $q = n$.

The former behave as

$$a_{k_1} a_{k_2} \ldots a_{k_p} g^{(p,n-q)}(b/y,0) \propto \frac{1}{b^{nd-2q}}. \tag{6.83}$$

Clearly, the leading behavior in the large impact parameter regime corresponds in this case to $q = n-1$, recall, however, that $g^{(p,1)}$ does not depend on $\lambda_{\rm GB}$. The behavior of the latter terms is

$$a_{k_1} a_{k_2} \ldots a_{k_p} g^{(p,0)}(b/y,0) \propto \frac{1}{b^{nd-2(n-1)}}. \tag{6.84}$$

which is again independent of $\lambda_{\rm GB}$. The conclusion is that the leading behavior in the large impact parameter regime comes from terms containing $g^{(p,0)}(b/y,0)$ and $g^{(p,1)}(b/y,0)$ that do not contain $\lambda_{\rm GB}$.

One can extend these considerations straightforwardly to any gravitational theory that contains a spherical black hole with a metric given by

$$ds^2 = -(1 + r^2 \tilde{f}(r))dt^2 + \frac{dr^2}{1 + r^2 \tilde{h}(r)} + r^2 d\Omega_{d-1}^2 \tag{6.85}$$

where the functions $\tilde{f}(r)$ and $\tilde{h}(r)$ admit an expansion of the following form in the large r limit:

$$\begin{aligned}\tilde{f}(r) &= 1 - \sum_{n=0}^{\infty} \frac{\tilde{f}_{nd}}{r^{nd}} = 1 - \frac{\tilde{f}_0}{r^d} - \frac{\tilde{f}_d}{r^{2d}} - \dots \\ \tilde{h}(r) &= 1 - \sum_{n=0}^{\infty} \frac{\tilde{h}_{nd}}{r^{nd}} = 1 - \frac{\tilde{h}_0}{r^d} - \frac{\tilde{h}_d}{r^{2d}} - \dots,\end{aligned} \tag{6.86}$$

for some constants $\tilde{f}_{nd}$ and $\tilde{h}_{nd}$ (these are the spherical black hole metrics considered in eqs. (5.1) and (5.10) in [15]).

6.4.2. Spin-2 multi-stress tensor OPE data from the gravitational phase shift

The gravitational phase shift in a black hole background is related to the lightcone HHLL four-point function discussed extensively in this article. In the following, we will exploit the precise relationship between the two to extract the OPE data of multi-stress tensor operators of spin-2 in the dual conformal field theory (modulo spin zero data). While the explicit procedure can be worked out for arbitrary multi-stress tensors, we will herein focus on double and triple-stress tensor operators, which control the $\mathcal{O}(\mu^2)$ and $\mathcal{O}(\mu^3)$ lightcone behavior of the HHLL correlation function.

6.4.2.3. The phase shift in Gauss-Bonnet gravity to $\mathcal{O}(\mu^3)$.

In this section, we focus on the gravity side and determine the phase shift order by order in μ up to $\mathcal{O}(\mu^3)$ relevant for this article. Starting from $\mathcal{O}(\mu^0)$ we consider the following expression

$$\delta^{(0)} = 2b\, p^t\, r_{AdS} \sqrt{1-\alpha^2} \int_0^1 \frac{\sqrt{1-y^2}}{b^2 + r_{AdS}^2 y^2} dy. \tag{6.87}$$

Evaluating this integral and using the following notation $p^{\pm} = p^t \pm p^{\phi}$, $-p^2 = p^+ p^-$, leads to

$$\delta^{(0)} = \pi p^-. \tag{6.88}$$

This is of course none other but the "phase shift" in pure AdS space.

At $\mathcal{O}(\mu)$ the result is the same as in [55], where Einstein gravity was considered,

$$\delta^{(1)} = \sqrt{-p^2}\left(\frac{b}{r_{AdS}}\right)^{1-d}\left(\frac{d-1}{2}\right)B\Big[\frac{d-1}{2},\frac{3}{2}\Big]{}_2F_1\Big(1,\frac{d-1}{2},\frac{d}{2}+1,-\frac{r_{AdS}^2}{b^2}\Big). \tag{6.89}$$

At this order, the phase shift depends only on the single graviton exchange, which is unaffected by the higher derivative terms in the gravitational action. According to the holographic dictionary, the exchange of a single graviton is related to the exchange of a single stress tensor in the T-channel. The corresponding OPE coefficient is fixed by the Ward identity, so it does not depend on the details of the theory.

We now consider the phase shift at higher orders in μ. For convenience herein all results are presented in $d=4$. At $\mathcal{O}(\mu^2)$, using the technique presented in the previous subsection, we find that:

$$\begin{aligned}\delta^{(2)} =& \frac{7\pi}{8}\sqrt{-p^2}\Bigg(5\frac{b}{r_{AdS}}(\sqrt{1+\frac{r_{AdS}^2}{b^2}}-1)-\frac{5}{2}\frac{r_{AdS}}{b}+\frac{5}{4}\frac{r_{AdS}^3}{b^3}\\ &+\frac{\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\Big(4\frac{b}{r_{AdS}}(\sqrt{1+\frac{r_{AdS}^2}{b^2}}-1)-2\frac{r_{AdS}}{b}+\frac{1}{2}\frac{r_{AdS}^3}{b^3}\\ &-\frac{1}{4}\frac{r_{AdS}^5}{b^5}\Big)\Bigg).\end{aligned} \tag{6.90}$$

In the lightcone limit ($b\to\infty$) this reduces to

$$\delta^{(2)} \underset{b\to\infty}{\approx} \frac{35\pi\sqrt{-p^2}r_{AdS}^5}{128b^5}-\frac{35\pi\sqrt{-p^2}r_{AdS}^7}{1024b^7}\left(5+\frac{4\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\right)+\dots. \tag{6.91}$$

We explicitly see that the leading contribution does not depend on $\lambda_{\rm GB}$, while the subleading does.

Let us denote $\delta_{\rm GR}^{(2)}$ to be equal to (6.90) when $\lambda_{\rm GB}=0$,

$$\delta_{\rm GR}^{(2)} = \frac{35\pi r_{AdS}^5\sqrt{-p^2}}{128b^5}{}_2F_1(1,\frac{5}{2},4,-\frac{r_{AdS}^2}{b^2}), \tag{6.92}$$

which is the pure Einstein gravity result for the phase shift at $\mathcal{O}(\mu^2)$. Then $\delta^{(2)}$ can be written as

$$\delta^{(2)} = \delta_{\rm GR}^{(2)}\left(1+\frac{4\lambda_{\rm GB}}{5r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\right)-\frac{7\pi\sqrt{-p^2}\lambda_{\rm GB}}{32r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\left(\frac{r_{AdS}}{b}\right)^5. \tag{6.93}$$

The phase shift at $\mathcal{O}(\mu^3)$ is given by

$$
\begin{aligned}
\delta^{(3)} =&\delta^{(3)}_{GR}\left(1+\frac{12\lambda_{\text{GB}}}{7r_{AdS}^2\sqrt{1-4\lambda_{\text{GB}}}}+\frac{16\lambda_{\text{GB}}^2}{21r_{AdS}^4(1-4\lambda_{\text{GB}})}\right)\\
&-\sqrt{-p^2}\left(\frac{r_{AdS}}{b}\right)^7\left(\frac{495\pi\lambda_{\text{GB}}}{512r_{AdS}^2\sqrt{1-4\lambda_{\text{GB}}}}+\frac{55\pi\lambda_{\text{GB}}^2}{128r_{AdS}^4(1-4\lambda_{\text{GB}})}\right)\\
&+\sqrt{-p^2}\left(\frac{r_{AdS}}{b}\right)^9\frac{77\pi\lambda_{\text{GB}}^2}{256r_{AdS}^4(1-4\lambda_{\text{GB}})},
\end{aligned}
\tag{6.94}
$$

where

$$
\delta^{(3)}_{GR}=\frac{231r_{AdS}^7}{16b^7}\sqrt{-p^2}B\left(\frac{7}{2},\frac{3}{2}\right){}_2F_1(1,\frac{7}{2},5,-\frac{r_{AdS}^2}{b^2}). \tag{6.95}
$$

By expanding (6.94) in the large impact parameter limit, one again explicitly sees that the leading term does not depend on λ_{GB}.

6.4.2.4. Inverse Fourier transform of the phase shift at $\mathcal{O}(\mu^2)$.

To make contact with the position space HHLL correlation function, one needs to perform a Fourier transform of the phase shift. According to [55], the HHLL four-point function in the Regge limit $\sqrt{-p^2}\gg 1$ is given by

$$
\tilde{\mathcal{G}}(x)=\int\frac{d^dp}{(2\pi)^d}e^{ipx}\mathcal{B}(p), \tag{6.96}
$$

where $\tilde{\mathcal{G}}(x)=\langle\mathcal{O}_H(x_1)\mathcal{O}_L(x_2)\mathcal{O}_L(x_3)\mathcal{O}_H(x_4)\rangle_{\text{Regge limit}}$ and $\mathcal{B}(p)=\mathcal{B}_0(p)e^{i\delta}$. The factor $\mathcal{B}_0(p)$ reproduces the disconnected correlator and it is given by

$$
\mathcal{B}_0(p)=C(\Delta_L)\theta(p^0)\theta(-p^2)e^{i\pi\Delta_L}(-p^2)^{\Delta_L-\frac{d}{2}}, \tag{6.97}
$$

with normalization

$$
C(\Delta_L)=\frac{2^{d+1-2\Delta_L}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta_L)\Gamma(\Delta_L-\frac{d}{2}+1)}. \tag{6.98}
$$

We expand the integrand of (6.96) in powers of μ using (6.73), explicitly

$$
\begin{aligned}
\mathcal{B}(p)=\mathcal{B}_0(p)\Bigg(&1+\mu i\delta^{(1)}+\mu^2\left(i\delta^{(2)}-\frac{1}{2}\delta^{(1)^2}\right)\\
&+\mu^3\left(i\delta^{(3)}-\delta^{(1)}\delta^{(2)}-\frac{i}{6}\delta^{(1)^3}\right)+\mathcal{O}(\mu^4)\Bigg).
\end{aligned}
\tag{6.99}
$$

This generates an expansion for $\tilde{\mathcal{G}}(x)$ from (6.96) as

$$\tilde{\mathcal{G}}(x) = \sum_{k=0}^{\infty} \mu^k \tilde{\mathcal{G}}^{(k)}(x). \tag{6.100}$$

Let us start by studying the correlator at $\mathcal{O}(\mu^2)$. The imaginary part of the correlator in the Regge limit at this order comes from $i\delta^{(2)}$ in (6.99) while the real part comes from $-\frac{1}{2}{\delta^{(1)}}^2$.

Consider first the imaginary part. To perform the inverse Fourier transform it is convenient to first expand $\delta^{(2)}$ as follows:

$$\begin{aligned}\delta^{(2)} = 7\pi^2\sqrt{-p^2}\Bigg(&\frac{5}{2}\Pi_{5,3}(L) + \left(\frac{15}{4} - \frac{5\lambda_{\text{GB}}}{r_{AdS}^2\sqrt{1-4\lambda_{\text{GB}}}}\right)\Pi_{7,3}(L) \\ &+ \left(5 - \frac{16\lambda_{\text{GB}}}{r_{AdS}^2\sqrt{1-4\lambda_{\text{GB}}}}\right)\Pi_{9,3}(L) + \dots\Bigg).\end{aligned} \tag{6.101}$$

In (6.101) $b/r_{AdS} = \sinh(L)$ and

$$\Pi_{\Delta-1;d-1}(x) = \frac{\pi^{1-\frac{d}{2}}\Gamma(\Delta-1)}{2\Gamma(\Delta-\frac{d-2}{2})}\, e^{-(\Delta-1)x}\ {}_2F_1(\frac{d}{2}-1, \Delta-1, \Delta-\frac{d-2}{2}, e^{-2x})\,, \tag{6.102}$$

the three-dimensional hyperbolic space propagator of a massive particle with mass square equal to $(\Delta-1)^2$. The dots in (6.101) stand for terms with hyperbolic space propagators with $\Delta > 10$. We can now perform the inverse Fourier transform of (6.101) with the help of eqs. (3.23) in [55] and (3.4) in [135].

The term which contains $\Pi_{5,3}(L)$ includes (after the inverse Fourier transform) the contribution of double-stress tensors with minimal twist $\tau = 4$. As we have already shown it does not depend on λ_{GB}, which we can also explicitly see in (6.101). The next term, that contains $\Pi_{7,3}(L)$, includes the contribution from the double-stress tensor operators of twist $\tau_{2,1} = 6$. We can use this term to fix the coefficient b_{14} which was left undetermined in (6.21). Similar reasoning applies to all the higher-order terms in the large impact parameter expansion of (6.101). Namely, the term proportional to $\Pi_{2m+1,3}(L)$ is related to double-stress tensor operators of twist $\tau = 2m$.

Performing the inverse Fourier transform following [135] leads to

$$
\begin{aligned}
&i\text{Im}\left(\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)\right) = \int \frac{d^4p}{(2\pi)^4} e^{ipx}\mathcal{B}_0(p) i\delta^{(2)} = \frac{2i}{\Gamma(\Delta_L)\Gamma(\Delta_L-1)\sigma^{2\Delta_L+1}} \\
&\times \Big(a_1 \Pi_{5,3}(\rho)\Gamma(\Delta_L-2)\Gamma(\Delta_L+2) + b_1\Pi_{7,3}(\rho)\Gamma(\Delta_L-3)\Gamma(\Delta_L+3) \\
&+ c_1\Pi_{9,3}(\rho)\Gamma(\Delta_L-4)\Gamma(\Delta_L+4) + \ldots\Big) + \ldots,
\end{aligned}
\tag{6.103}
$$

where $a_1 = \frac{35}{2}\pi^2$, $b_1 = 7\pi^2\left(\frac{15}{4} - \frac{5\lambda_{\text{GB}}}{r_{AdS}^2\sqrt{1-4\lambda_{\text{GB}}}}\right)$ and $c_1 = 7\pi^2\left(5 - \frac{16\lambda_{\text{GB}}}{r_{AdS}^2\sqrt{1-4\lambda_{\text{GB}}}}\right)$. The ellipses outside the parenthesis in (6.103) denote contributions due to double-trace operators in the T-channel that are not important for studying the stress tensor sector. The position space coordinates σ and ρ are defined as

$$
z = 1-\sigma e^{\rho}, \qquad \bar{z} = 1-\sigma e^{-\rho}. \tag{6.104}
$$

after the analytic continuation $z \to ze^{-2i\pi}$. Once more, notice that the dominant contribution in the large impact parameter regime, $\rho\to\infty$, comes from the factor $\Pi_{5,3}(\rho)$ in (6.103) which exactly matches the imaginary part of the correlator (6.17) in [55].

6.4.2.5. Comparison with HHLL correlation function in the lightcone limit at $\mathcal{O}(\mu^2)$.

A few simple steps are required before we can finally relate (6.103) with the results of Section 6.2 and determine the OPE coefficients of the spin-2 double-stress tensor operators. As explained in [55], one has to analytically continue $\mathcal{G}^{(2,1)}$, $\mathcal{G}^{(2,2)}$ and $\mathcal{G}^{(2,3)}$ (defined in Section 6.1) around the origin by taking $z\to ze^{-2i\pi}$ and expand the result in the vicinity of $\sigma\to 0$. The relevant term, which corresponds to the imaginary part of the correlator (6.21) as $\sigma\to 0$, reads:

$$
\begin{aligned}
i\text{Im}\left((\sigma e^{-\rho})^{3-\Delta_L}\mathcal{G}^{(2,1)}(1-\sigma e^{\rho})\right) =& 7i\pi\frac{e^{-7\rho}}{\sigma^{2\Delta_L+1}}\Big(12600 b_{14} \\
&+ \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(123-7\Delta_L\right)+78\right)-12\right)}{16\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\Big).
\end{aligned}
\tag{6.105}
$$

Comparing this with the subleading term of (6.103) as $\rho \to \infty$, *i.e.*,

$$i\mathrm{Im}\left(\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)\right)|_{e^{-7\rho}} = \\ -\frac{35i\pi e^{-7\rho}\Delta_L\left(\Delta_L+1\right)\left(8\lambda_{\mathrm{GB}}+\Delta_L\left(4\lambda_{\mathrm{GB}}-5\sqrt{1-4\lambda_{\mathrm{GB}}}r_{AdS}^2\right)\right)}{4\sigma^{2\Delta_L+1}\sqrt{1-4\lambda_{\mathrm{GB}}}r_{AdS}^2\left(\Delta_L^2-5\Delta_L+6\right)}+\dots, \tag{6.106}$$

with the ellipses again denoting double-trace operators, allows one to obtain the following expression for the unknown parameter b_{14}:

$$\begin{aligned} b_{14} = P^{(2)}_{8,2} =& \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(7\Delta_L-23\right)+22\right)+12\right)}{201600\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\ &- \frac{\lambda_{\mathrm{GB}}\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)}{2520\sqrt{1-4\lambda_{\mathrm{GB}}}r_{AdS}^2\left(\Delta_L-3\right)\left(\Delta_L-2\right)}. \end{aligned} \tag{6.107}$$

Note that this precisely matches the OPE coefficient of the double trace operator of conformal dimension $\Delta = 8$ and $s = 2$ calculated in [15] from gravity by other means. As expected, the OPE coefficient in (6.107) explicitly depends on λ_{GB}.

Let us now go one step further and fix $P^{(2)}_{10,2}$ contributing to $\mathcal{G}^{(2,2)}(z)$ through (6.36). Analytically continuing (6.34) and taking the limit $\sigma \to 0$, yields

$$i\mathrm{Im}\left(\left(\sigma e^{-\rho}\right)^{4-\Delta_L}\mathcal{G}^{(2,2)}(1-\sigma e^{\rho})\right) = i\frac{49}{400}\frac{\pi e^{-9\rho}}{\sigma^{2\Delta_L+1}}\left(720000b_{14}+11404800\frac{P^{(2)}_{10,2}}{\mu^2}\right. \\ \left.+\frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(6327-362\Delta_L\right)+749\right)+12888\right)+12288\right)}{7\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\right). \tag{6.108}$$

For reasons that will be explained later, we only consider here the imaginary part of the subsubleading term in the correlator. To extract the OPE data we need to compare (6.108) with the subsubleading contribution in the large impact parameter limit of (6.103), which is

$$i\mathrm{Im}\left(\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)\right)|_{e^{-9\rho}} = i\frac{7}{4}\frac{\pi e^{-9\rho}}{\sigma^{2\Delta_L+1}}\left(\frac{10\Delta_L\left(\Delta_L+1\right)}{\Delta_L-2}\right. \\ \left.-\frac{7\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)\left(16\lambda_{\mathrm{GB}}+\Delta_L\left(12\lambda_{\mathrm{GB}}-5\sqrt{1-4\lambda_{\mathrm{GB}}}r_{AdS}^2\right)\right)}{\sqrt{1-4\lambda_{\mathrm{GB}}}r_{AdS}^2\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\right). \tag{6.109}$$

Substituting (6.107) in (6.108) and matching to (6.109) enables us to determine the OPE coefficient $P^{(2)}_{10,2}$,

$$
\begin{aligned}
P^{(2)}_{10,2} =& \frac{\Delta_L\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(187\Delta_L-552\right)+901\right)+1012\right)+912\right)}{79833600\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&-\frac{\lambda_{\mathrm{GB}}\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)\left(\Delta_L+3\right)}{12474\sqrt{1-4\lambda_{\mathrm{GB}}}r^2_{AdS}\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}.
\end{aligned}
\tag{6.110}
$$

This precisely matches the one calculated in [15].

Similarly, one can match the CFT expression for $\mathrm{Im}\left((\sigma e^{-\rho})^{5-\Delta_L}\mathcal{G}^{(2,3)}(1-\sigma e^{\rho})\right)$ in (6.39), to its gravitational counterpart $\mathrm{Im}\left(\mathcal{G}^{(2)}(x)\right)|_{e^{-11\rho}}$, by expanding (6.101) and (6.103) up to $\mathcal{O}(e^{-11\rho})$. This allows one to additionally determine $P^{(2)}_{12,2}$ in (6.42)

$$
\begin{aligned}
P^{(2)}_{12,2} = &-\frac{5\lambda_{\mathrm{GB}}\Delta_L\left(\Delta_L+1\right)\left(\Delta_L+2\right)\left(\Delta_L+3\right)\left(\Delta_L+4\right)}{453024\sqrt{1-4\lambda_{\mathrm{GB}}}r^2_{AdS}\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}+ \\
&\frac{\Delta_L\left(\Delta_L+1\right)\left(\Delta_L\left(\Delta_L\left(\Delta_L\left(6721\Delta_L-15603\right)+46474\right)+100828\right)+143760\right)}{44396352000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}.
\end{aligned}
\tag{6.111}
$$

Notice that we did not use the real part of $\tilde{\mathcal{G}}^{(2)}(\sigma,\rho)$, which comes from the term $-\frac{1}{2}{\delta^{(1)}}^2$ in (6.99) and behaves as $\sigma^{-2\Delta_L-2}$ for $\sigma\to 0$. This term matches the corresponding term with the same σ behavior in the correlator. It does not give us any new information, because it is independent of the OPE coefficients of operators with spin $s=0,2$.

6.4.2.6. Extracting OPE data from the gravitational phase shift at $\mathcal{O}(\mu^3)$.

Let us now consider the $\mathcal{O}(\mu^3)$ terms in the correlator. Focusing on the gravity side, we start by performing an inverse Fourier transform. (6.99) instructs us to consider three terms $i\delta^{(3)}$, $\delta^{(1)}\delta^{(2)}$ and $i(\delta^{(1)})^3$, which give rise to terms that behave as $\sigma^{-2\Delta_L-1}$, $\sigma^{-2\Delta_L-2}$ and $\sigma^{-2\Delta_L-3}$, respectively. Performing the relevant computations, we observe that $\delta^{(1)}\delta^{(2)}$ and $i(\delta^{(1)})^3$ do not provide additional information because the corresponding terms in the correlators are already fixed by bootstrap (these terms simply give us an extra consistency check). Focusing on the inverse Fourier transform of $i\delta^{(3)}$, we expand (6.94) in terms of the hyperbolic space propagators, $\Pi_{m,3}(L)$,

$$
\delta^{(3)} = \sqrt{-p^2}\Big(a_2\Pi_{7,3}(L)+b_2\Pi_{9,3}(L)+c_2\Pi_{11,3}(L)+\dots\Big), \tag{6.112}
$$

where

$$a_2 = \frac{1155}{8}\pi^2,$$
$$b_2 = 231\pi^2\left(-\frac{3\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}+2\right), \tag{6.113}$$
$$c_2 = \frac{231\pi^2}{8}\left(\frac{32\lambda_{\rm GB}^2}{r_{AdS}^4(1-4\lambda_{\rm GB})}-\frac{120\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}+35\right),$$

which leads to

$$i{\rm Im}\left(\tilde{\mathcal{G}}^{(3)}(\sigma,\rho)\right)\Big|_{\frac{1}{\sigma^{2\Delta_L+1}}} = \int \frac{d^4p}{(2\pi)^4}e^{ipx}\mathcal{B}_0(p)i\delta^{(3)} = \frac{2i}{\Gamma(\Delta_L)\Gamma(\Delta_L-1)\sigma^{2\Delta_L+1}}$$
$$\times\Big(a_2\Pi_{7,3}(\rho)\Gamma(\Delta_L-3)\Gamma(\Delta_L+3)+b_2\Pi_{9,3}(\rho)\Gamma(\Delta_L-4)\Gamma(\Delta_L+4)$$
$$+c_2\Pi_{11,3}(\rho)\Gamma(\Delta_L-5)\Gamma(\Delta_L+5)+\dots\Big)+\text{double traces}, \tag{6.114}$$

The leading and subleading contributions in the large impact parameter limit $\rho\to\infty$ come from $\Pi_{7,3}(\rho)$ and $\Pi_{9,3}(\rho)$ and behave as $\frac{i\pi e^{-7\rho}}{\sigma^{2\Delta_L+1}}$ and $\frac{i\pi e^{-9\rho}}{\sigma^{2\Delta_L+1}}$, respectively. They are precisely matched by the relevant terms in (6.45) in the vicinity of $\sigma\to 0$ after analytic continuation [135]. This is another sanity check of the procedure described herein, since these terms do not incorporate contributions from spin-2 operators.

To extract further OPE data, we proceed to match the subsubleading correction of (6.114) in the large impact parameter limit to the term in (6.53) which behaves as $\sim\frac{i\pi e^{-11\rho}}{\sigma^{2\Delta_L+1}}$. This allows us to determine the coefficient $e_{115}=P^{(3)}_{12,2}$ in (6.53) which corresponds to the OPE coefficient of the triple-stress tensors of spin $s=2$ with conformal dimension $\Delta=12$:

$$e_{115} = -\frac{117\Delta_L^6-439\Delta_L^5+407\Delta_L^4+859\Delta_L^3+202\Delta_L^2+696\Delta_L}{172972800(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)}$$
$$-\frac{\lambda_{\rm GB}(143\Delta_L^6-231\Delta_L^5-3597\Delta_L^4-9489\Delta_L^3-11186\Delta_L^2-4920\Delta_L)}{43243200r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)}$$
$$+\frac{\lambda_{\rm GB}^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{24024r_{AdS}^4(1-4\lambda_{\rm GB})(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)}$$
$$+P^{(2)}_{8,0}\frac{76+\frac{400}{\Delta_L-5}+11\Delta_L}{1320}. \tag{6.115}$$

Notice that e_{115} is not completely determined by the above procedure since the spin-0 OPE data, $P^{(2)}_{8,0}$, is not fixed. Summarising, we conclude that we are able to fix all coefficients in the ansatz except those that correspond to the OPE coefficients of operators of spin-0. However, using the expression for $P^{(2)}_{8,0}$ found in [15] one finds

$$\begin{aligned}
&P^{(3)}_{12,2} = \\
&\quad \frac{1001\Delta_L^7 - 6864\Delta_L^6 + 12615\Delta_L^5 - 3980\Delta_L^4 - 6156\Delta_L^3 - 11736\Delta_L^2 - 1440\Delta_L}{3459456000(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\
&\quad - \frac{\lambda_{\mathrm{GB}}(143\Delta_L^6 - 206\Delta_L^5 - 1631\Delta_L^4 - 3622\Delta_L^3 - 3540\Delta_L^2 - 1200\Delta_L)}{28828800 r_{AdS}^2\sqrt{1-4\lambda_{\mathrm{GB}}}(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)} \\
&\quad + \frac{\lambda_{\mathrm{GB}}^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{24024 r_{AdS}^4(1-4\lambda_{\mathrm{GB}})(\Delta_L-2)(\Delta_L-3)(\Delta_L-4)(\Delta_L-5)}.
\end{aligned} \tag{6.116}$$

6.5. Lorentzian inversion formula

It was recently shown in [128] that one can obtain the OPE coefficients of minimal twist double and triple-stress tensors using the Lorentzian inversion formula. Here, we review this method and show how it can be generalized to extract the OPE coefficients of twist-six double-stress tensors. In principle, it can also be generalized to multi-stress tensors of arbitrarily high twist.

6.5.1. Twist-four double-stress tensors

Consider the correlation function

$$(w\bar{w})^{-\Delta_L}\hat{\mathcal{G}}(w,\bar{w}) = \langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(w,\bar{w})\mathcal{O}_L(0)\rangle. \tag{6.117}$$

The Lorentzian inversion formula is given by [73,78]

$$\begin{aligned}
c(\tau,\beta) = \frac{1+(-1)^{\frac{\beta-\tau}{2}}}{2}\kappa_\beta \int_0^1 & dw d\bar{w}\mu^{(0,0)}(w,\bar{w}) \\
&\times g^{(0,0)}_{-\tau+2(d-1),\frac{\beta+\tau}{2}-d+1}(w,\bar{w})\mathrm{dDisc}[\hat{\mathcal{G}}(w,\bar{w})],
\end{aligned} \tag{6.118}$$

where

$$\mu^{(0,0)}(w,\bar{w}) = \frac{|w-\bar{w}|^{d-2}}{(w\bar{w})^d}, \tag{6.119}$$

$$\kappa_\beta = \frac{\Gamma(\frac{\beta}{2})^4}{2\pi^2\Gamma(\beta)\Gamma(\beta-1)}, \tag{6.120}$$

where $\tau = \Delta - s$ and $\beta = \Delta + s$. Here $g^{(0,0)}_{\tau,s}$ is a conformal block given with $\Delta \to s + d - 1$ and $s \to \Delta - d + 1$ and in $d = 4$ is given by (3.57). Moreover, dDisc denotes the double-discontinuity of $\hat{\mathcal{G}}(w, \bar{w})$ in (6.117), which is equal to the correlator of a double commutator, and it is given by

$$\mathrm{dDisc}[\hat{\mathcal{G}}(w,\bar{w})] = \hat{\mathcal{G}}(w,\bar{w}) - \frac{1}{2}\hat{\mathcal{G}}^{\circlearrowleft}(w,\bar{w}) - \frac{1}{2}\hat{\mathcal{G}}^{\circlearrowright}(w,\bar{w})\,. \tag{6.121}$$

Here $\hat{\mathcal{G}}^{\circlearrowleft}$ and $\hat{\mathcal{G}}^{\circlearrowright}$ correspond to the same correlator analytically continued in two different ways around $w = 1$, namely $(1-w) \to (1-w)e^{\pm 2\pi i}$. The OPE data, $P_{\frac{\tau'+\beta}{2},\frac{\beta-\tau'}{2}}$, can be extracted from $c(\tau, \beta)$ via[24]

$$P_{\frac{\tau'+\beta}{2},\frac{\beta-\tau'}{2}} = -\mathrm{Res}_{\tau=\tau'}c(\tau,\beta), \tag{6.122}$$

where τ' and β denote the twist and conformal spin of operators in the physical spectrum of the theory exchanged in the channel $\mathcal{O}_L \times \mathcal{O}_L \to \mathcal{O}_{\tau',J'} \to \mathcal{O}_H \times \mathcal{O}_H$.

We would like to apply the Lorentzian inversion formula to the HHLL correlator to extract the OPE data of the double-stress tensors. To this end, we will use information of the correlator from the channel where $\mathcal{O}_H\mathcal{O}_L$ merge. The function $\hat{\mathcal{G}}(z, \bar{z})$ can be obtained from $\mathcal{G}(z, \bar{z})$ via

$$\hat{\mathcal{G}}(w,\bar{w}) = (w\bar{w})^{\Delta_L}\mathcal{G}(1-w, 1-\bar{w}). \tag{6.123}$$

To apply the Lorentzian inversion formula we first need to calculate $\mathcal{G}(z, \bar{z})$ using the S-channel operator product expansion (3.52). First, let us start with the leading contribution of $\mathcal{G}(z, \bar{z})$ in the lightcone limit $\bar{z} \to 1$ at $\mathcal{O}(\mu^2)$. These give the leading contributions when $\bar{w} \to 0$ in $G(w, \bar{w})$. After the integration with respect to $\bar{w}$ in (6.118), these contributions fix the position of the pole and residue of $c(\tau, \beta)$ that corresponds to lowest-twist double-stress tensors. Subleading contributions in $\bar{z} \to 1$ (or $\bar{w} \to 0$) only create new poles, without changing the residue of existing ones, therefore, they do not affect the OPE

[24] In principle there is an extra term in this relation when $\tau - d = 0, 1, 2, \ldots$ [73], however, it vanishes in the cases considered.

coefficients of lowest-twist operators. The leading contribution in the $(1-\bar{z})$-expansion comes from the leading contribution of the $1/l$-expansion of the S-channel OPE data. Only the term proportional to $\log^2(z)$ contributes to the double-discontinuity and we denote it by $\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)}$. The number in the superscript denotes the power of μ in which we are working. Substituting in to (3.67) equations (3.25), (3.70), (5.10) and (3.69), we find that

$$\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)} = \log^2(z\bar{z}) \int_0^\infty dl \sum_{n=0}^\infty \frac{(z\bar{z})^n l^{\Delta_L-3}\left(z^{l+1}-\bar{z}^{l+1}\right)\Gamma\left(n+\Delta_L-1\right)}{8(z-\bar{z})\Gamma(n+1)\Gamma\left(\Delta_L-1\right)\Gamma\left(\Delta_L\right)} \times \left(\left(\gamma_n^{(1,0)}\right)^2 + \mathcal{O}\left(\frac{1}{l}\right)\right). \tag{6.124}$$

In the lightcone limit, the dominant contribution to this expression comes from operators with large spin $l \gg 1$, we can, therefore, approximate the sum over l by an integral. Note that only $\mathcal{O}(\mu)$ OPE data, *i.e.*, $\gamma_n^{(1,0)}$, appears in (6.124). Using (6.28) we evaluate (6.124) and collect the leading term as $\bar{z}\to 1$,

$$\begin{aligned}&\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)} = \log^2(z)\frac{(1-\bar{z})^{2-\Delta_L}(1-z)^{-\Delta_L-4}}{32\left(\Delta_L-2\right)}\times \\ &\Delta_L\left(\Delta_L\left((z(z+4)+1)^2\Delta_L + z(z(54-(z-28)z)+28)-1\right)+72z^2\right) \\ &+\mathcal{O}\left((1-\bar{z})^{3-\Delta_L}\right).\end{aligned} \tag{6.125}$$

With the help of (6.123) one obtains

$$\begin{aligned}&\hat{\mathcal{G}}^{(2)}(w,\bar{w})\big|_{\log^2(1-w)} = \frac{\Delta_L\bar{w}^2\log^2(1-w)}{32w^4(\Delta_L-2)}\times \\ &\left(\Delta_L\left(((w-6)w+6)^2\Delta_L - w(w(w(w+24)-132)+216)+108\right)+72(w-1)^2\right) \\ &+\mathcal{O}(\bar{w}^3),\end{aligned} \tag{6.126}$$

which agrees with (4.12) in [128]. Now, it is easy to see that

$$\begin{aligned}&\mathrm{dDisc}[\hat{\mathcal{G}}^{(2)}(w,\bar{w})] = \frac{\pi\bar{w}^2\Delta_L}{8w^4(\Delta_L-2)}\times \\ &\left(\Delta_L\left(((w-6)w+6)^2\Delta_L - w(w(w(w+24)-132)+216)+108\right)\right. \\ &\left.+72(w-1)^2\right)+\mathcal{O}(\bar{w}^3).\end{aligned} \tag{6.127}$$

To compute the integral (6.118) we substitute

$$\mu^{(0,0)}(w,\bar{w}) = \frac{1}{w^2\bar{w}^4} + \mathcal{O}\left(\frac{1}{\bar{w}^3}\right), \tag{6.128}$$

$$g^{(0,0)}_{-\tau+2(d-1),\frac{\tau+\beta}{2}-d+1}(w,\bar{w}) = \bar{w}^{3-\frac{\tau}{2}}\left(f_{\frac{\beta}{2}}(1-w)+\mathcal{O}(\bar{w})\right), \tag{6.129}$$

valid in the lightcone limit $\bar{w}\to 0$ (or $\bar{z}\to 1$), and set $(-1)^{\frac{\beta-\tau}{2}}=1$ since only even-spin operators contribute. Combining the above we arrive at the following expression for $c(\tau,\beta)$

$$\begin{aligned} c_0(\tau,\beta) = &-\frac{\sqrt{\pi}2^{-\beta+1}\Delta_L\Gamma\left(\frac{\beta}{2}\right)}{(\tau-4)(\beta-10)(\beta-6)(\beta-2)\beta(\beta+4)}\times \\ &\left(\frac{384\left(\Delta_L-7\right)\Delta_L+4608}{(\beta+8)\left(\Delta_L-2\right)\Gamma\left(\frac{1}{2}(\beta-1)\right)}+\right. \\ &\left.+\frac{(\beta-2)\beta\Delta_L\left((\beta-2)\beta\left(\Delta_L-1\right)-56\Delta_L+200\right)}{(\beta+8)\left(\Delta_L-2\right)\Gamma\left(\frac{1}{2}(\beta-1)\right)}\right), \end{aligned} \tag{6.130}$$

where the subscript denotes that this result is obtained in the leading order of the lightcone expansion. The OPE coefficients of the minimal-twist double-stress tensors are given by

$$P^{(2)}_{\frac{\beta}{2}+2,\frac{\beta}{2}-2} = -\mathrm{Res}_{\tau=4}c_0(\tau,\beta), \tag{6.131}$$

where $\beta=12+4\ell$, $\ell\geq 0$, and are in precise agreement with (1.6) in [124] and (4.15) in [128].

6.5.2. Twist-six double-stress tensors

Here we use the same method to obtain the OPE coefficients of double-stress tensors with twist $\tau_{2,1}=6$. We first need to compute the subleading contribution in the lightcone limit to eqs. (6.127), (6.128) and (6.129). Specifically, the integration measure

$$\mu^{(0,0)}(w,\bar{w}) = \frac{1}{w^2\bar{w}^4}-\frac{2}{w^3\bar{w}^3}+\mathcal{O}\left(\bar{w}^{-2}\right), \tag{6.132}$$

and the conformal block,

$$g^{(0,0)}_{-\tau+2(d-1),\frac{\tau+\beta}{2}-d+1}(w,\bar{w}) = \bar{w}^{3-\frac{\tau}{2}}f_{\frac{\beta}{2}}(1-w)\left(1+\bar{w}\left(1-\frac{\tau}{4}+\frac{1}{w}\right)+\mathcal{O}(\bar{w}^2)\right), \tag{6.133}$$

were obtained from the explicit expressions given in (6.119) and (3.57).

To evaluate the subleading term in $\mathrm{dDisc}[\hat{\mathcal{G}}^{(2)}(w,\bar{w})]$ we reconsider the S-channel computation. Similarly to the case of leading twist, only the part of the correlator with $\log^2(z)$ contributes to the discontinuity. However, we now have to include the subleading corrections in the $1/l$-expansion of the S-channel OPE data. With the help of (3.67), (3.25), (5.10), (3.69) and (3.70) one finds that

$$\mathcal{G}^{(2)}(z,\bar{z})\big|_{\log^2(z)} = \frac{\log^2(z\bar{z})}{16(z-\bar{z})\Gamma(\Delta_L)\Gamma(\Delta_L-1)}\sum_{n=0}^{\infty}(z\bar{z})^n\frac{\Gamma(\Delta_L-1+n)}{\Gamma(n+1)}$$
$$\int_0^{\infty} dl l^{\Delta_L-6}\left(z^{l+1}-\bar{z}^{l+1}\right)\left(2(l-2n)+\Delta_L\left(\Delta_L+2n-1\right)\right)\left(l\gamma_n^{(1,0)}+\gamma_n^{(1,1)}\right)^2$$
$$+\mathcal{O}\left(l^{\Delta_L-7}\right). \tag{6.134}$$

To proceed, one evaluates (6.134) using (6.28) and collects the leading and subleading contributions as $\bar{z}\to 1$, which behave as $(1-\bar{z})^{2-\Delta_L}$ and $(1-\bar{z})^{3-\Delta_L}$ respectively. Using (6.123) it is then simple to obtain $\hat{\mathcal{G}}^{(2)}(w,\bar{w})\big|_{\log^2(1-w)}$ up to $\mathcal{O}(\bar{w}^4)$ and evaluate its double-discontinuity:

$$\mathrm{dDisc}[\hat{\mathcal{G}}^{(2)}(w,\bar{w})] = -\frac{\pi^2\bar{w}^2\Delta_L}{8w^5\left(\Delta_L-3\right)\left(\Delta_L-2\right)}\Big(-3w^5\Delta_L-72w^4\Delta_L$$
$$+324w^3\Delta_L-504w^2\Delta_L+252w\Delta_L+216w^3-432w^2+216w+4w^5\Delta_L^2$$
$$-12w^4\Delta_L^2+12w^3\Delta_L^2-36w\Delta_L^3-w^5\Delta_L^3+12w^4\Delta_L^3-48w^3\Delta_L^3+72w^2\Delta_L^3$$
$$+\bar{w}(-144\Delta_L+612w\Delta_L+216w^3-432w^2+216w-w^5\Delta_L-52w^4\Delta_L$$
$$+324w^3\Delta_L-744w^2\Delta_L+540w\Delta_L^2-216\Delta_L^2-72\Delta_L^3+w^5\Delta_L^2-18w^4\Delta_L^2$$
$$+156w^3\Delta_L^2-456w^2\Delta_L^2+144w\Delta_L^3-2w^4\Delta_L^3+24w^3\Delta_L^3-96w^2\Delta_L^3)\Big)$$
$$+\mathcal{O}(\bar{w}^4). \tag{6.135}$$

Substituting (6.132), (6.133) and (6.135) in (6.118) and integrating leads to an analytic expression for $c(\tau,\beta)$. The relevant part of this expression – the one with non-zero residue at $\tau = 6$ – turns out to be:

$$
\begin{aligned}
c_1(\tau,\beta) = &-\frac{2^{4-\beta}\sqrt{\pi}\Gamma\left(\frac{\beta}{2}\right)\Delta_L}{(\beta-12)(\beta-8)(\beta-4)(\tau-10)(\tau-8)(\tau-6)(\tau-4)} \\
&\times\left(\frac{\beta^4\Delta_L-4\beta^3\Delta_L-68\beta^2\Delta_L-960\beta\Delta_L^2+144\beta\Delta_L-14976\Delta_L^2}{(\beta+2)(\beta+6)(\beta+10)\Gamma\left(\frac{\beta-1}{2}\right)(\Delta_L-3)(\Delta_L-2)}\right. \\
&+\frac{\beta^4\Delta_L^3-2\beta^4\Delta_L^2-4\beta^3\Delta_L^3+8\beta^3\Delta_L^2-116\beta^2\Delta_L^3+472\beta^2\Delta_L^2}{(\beta+2)(\beta+6)(\beta+10)\Gamma\left(\frac{\beta-1}{2}\right)(\Delta_L-3)(\Delta_L-2)} \\
&\left.+\frac{240\beta\Delta_L^3+2304\Delta_L^3+19584\Delta_L+13824}{(\beta+2)(\beta+6)(\beta+10)\Gamma\left(\frac{\beta-1}{2}\right)(\Delta_L-3)(\Delta_L-2)}\right)+\ldots,
\end{aligned}
\tag{6.136}
$$

where the ellipsis stands for the terms with zero residue at $\tau = 6$ and 1 in the subscript denotes that this expression is obtained in the subleading order of the lightcone expansion.

It is now straightforward to read off the OPE coefficients of double-stress tensors with twist $\tau_{2,1} = 6$ from

$$
P^{(2)}_{\frac{\beta}{2}+3,\frac{\beta}{2}-3} = -\mathrm{Res}_{\tau=6}c_1(\tau,\beta). \tag{6.137}
$$

For $\beta = 14 + 4\ell$, eq. (6.32) is reproduced. It is already stated in Section 6.2 that this formula does not reproduce the right OPE coefficient $P^{(2)}_{8,2}$ for $\ell = -1$. Thus, we explicitly see that the Lorentzian inversion formula does not allow us to obtain the OPE data of spin-2 double-stress tensors with twist $\tau = 6$.

In general, to determine for which operators at $\mathcal{O}(\mu^k)$ the Lorentzian inversion formula can be applied, one has to consider the behavior of the correlator in the Regge limit. At $\mathcal{O}(\mu^k)$ the correlator in the Regge limit behaves like $1/\sigma^{2\Delta_L+k}$. Therefore, the Lorentzian inversion formula correctly produces the OPE coefficients of multi-stress tensor operators with spin $s > k + 1$. Accordingly, already at order $\mathcal{O}(\mu^3)$, fixing the OPE coefficients by combining an ansatz for the correlator with the crossing symmetry (or Lorentzian inversion formula) appears more powerful than the Lorentzian inversion formula alone. Namely, we were able to fix the OPE coefficients of spin-4 operators and the one with twist $\tau = 8$ is given by (D.1), while using the Lorentzian inversion formula one can only fix the OPE coefficients of operators with spin $s > 4$.

In this section, we consider the stress tensor sector of a four-point function of pairwise identical scalars in a class of CFTs with a large central charge. It is completely determined by the OPE coefficients of multi-stress tensor operators, which can be read off the result for a heavy-heavy-light-light correlator. The stress tensor sector of the HHLL correlator is naturally expanded perturbatively in $\mu \sim \frac{\Delta_H}{C_T}$, where Δ_H is the scaling dimension of the heavy operator. The power of μ counts the number of stress tensors within the exchanged multi-stress tensor operators. By further expanding the HHLL stress tensor sector in the lightcone limit, the multi-stress tensor operators can be organized into sectors of different twists. Similarly to the minimal-twist sector, combining an appropriate ansatz with the lightcone bootstrap, we show that the contribution from the non-minimal twist multi-stress tensors is almost completely determined. Unlike the minimal twist case, a few coefficients are not fixed by the bootstrap – these correspond to the OPE coefficients of multi-stress tensors with spin $s = 0, 2$.

An extra check is provided by applying the Lorentzian OPE inversion formula (see [128] for an earlier application of the inversion formula in this context). It gives the same results but has less predictive power than the ansatz.

The OPE coefficients for double-stress tensors are particularly simple and we provide closed-form expressions for those with twist $\tau = 4, 6, 8, 10$ and any spin greater than 2. All of these OPE coefficients are completely fixed by the bootstrap. This is related to their independence of the higher-derivative terms in the dual bulk gravitational Lagrangian. The OPE coefficients for double-stress tensors with spin $s = 0, 2$ are not fixed by the bootstrap and do depend on such higher derivative terms. It is interesting that at the level of double-stress tensors, only the OPE coefficients with spin $s = 0, 2$ are not fixed by the bootstrap (non-universal). On the other hand, all non-minimal twist triple-stress tensor OPE coefficients are non-universal[25] .

Assuming a holographic dual, we show that the OPE coefficients for spin-2 multi-stress tensors can be determined by studying the large impact parameter

[25] Here we use universality and "fixed by the bootstrap" terms interchangeably. However, it remains to be determined what is the universality class and whether it the same as the set of unitary holographic theories.

regime of the Regge limit, following [55,12,135] (modulo the spin zero OPE data). This is done explicitly in Einstein Hilbert+Gauss-Bonnet gravity. Some of these OPE coefficients are known [15] and agree with our results.

It would be interesting if one could compute the spin zero and spin two multi stress tensor OPE coefficients with CFT techniques. Perhaps the conglomeration approach first discussed in [38] or the more recent work [139,140] will be useful in this direction.

The regime of applicability of the ansatz (and the exact meaning of universality) used in this section remains unsettled (the ansatz seems to work in holographic CFTs, but does it also apply for other CFTs with a large central charge?). This question appears already in the leading twist case studied in [13]. To address this issue, it would be interesting to investigate the OPE coefficients of multi-stress tensors in CFTs with a large central charge, but not necessarily holographic. A related question is the existence of an infinite-dimensional algebra responsible for the form of the near-lightcone correlator. In two dimensions the relevant algebra is simply the Virasoro algebra. The Virasoro vacuum block has been computed in several ways [40,105-108,110,141]. Recently an algebraic way of reproducing the near lightcone contribution of the stress tensor was discussed in [142] – it would be interesting to investigate this further.

Returning to holographic theories, one interesting question would be to understand the critical behavior of geodesics in the vicinity of the circular light orbit, recently studied in [143], from the CFT point of view. This corresponds to the situation where the deflection angle is very large. The deflection angle φ in asymptotically flat Schwarzschild geometries is supposed to be related to the eikonal phase δ via

$$2\sin\frac{\varphi}{2}=-\frac{1}{E}\frac{\partial\delta}{\partial b} \tag{6.138}$$

where E is the incoming particle energy and b is the impact parameter (see e.g. [144] for a recent discussion). This agrees with eq. (B.5.1) for small deflection angles, but deviations might occur for large deflection angles. It would be interesting to investigate this further.

7. Thermalization in large-N CFTs

7.1. Introduction and summary

Holography [8-10] provides us with a useful tool to study d-dimensional CFTs at large central charge C_T, especially when combined with modern CFT techniques (see e.g. [24-26] for reviews). One of the basic objects in this setup is a Witten diagram with a single graviton exchange which contributes to four-point functions. It can be decomposed into the conformal blocks of the stress-tensor and of the double-trace operators made out of external fields [43].

When a pair of the external operators denoted by $\mathcal{O}_H$ is taken to be heavy, with the conformal dimension $\Delta_H \sim C_T$, and the other pair denoted by $\mathcal{O}_L$ stays light, the resulting heavy-heavy-light-light (HHLL) correlator describes a light probe interacting with a heavy state. In this case, operators which are comprised out of many stress tensors (multi stress tensor operators) contribute, together with the multi-trace operators involving $\mathcal{O}_L$. As we review below, the OPE coefficients of the scalar operators with a (unit-normalized) multi stress tensor operator $T^k_{\tau,s}$, which contains k stress tensors and has twist τ and spin s, scale like $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^k_{\tau,s}} \sim \Delta^k / C_T^{k/2}$ for large Δ.

The contribution of a given multi stress tensor operator to the HHLL four-point function $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$ can be compared to the contribution of the same operator to the corresponding two-point function at finite temperature[26] β^{-1}, $\langle \mathcal{O}_L \mathcal{O}_L \rangle_\beta$. In this section we argue that they are the same in generic large-C_T CFTs. As we explain later, this means that OPE coefficients of $T^k_{\tau,s}$ with the two heavy operators $\mathcal{O}_H$, $\langle \mathcal{O}_H T^k_{\tau,s} \mathcal{O}_H \rangle$, are equal to their finite temperature expectation values, $\langle T^k_{\tau,s} \rangle_\beta$. The relation between the inverse temperature β and the conformal dimension Δ_H is set by considering the stress tensor ($k = 1, \tau = d-2, s = 2$), but the equality between the thermal expectation values and the OPE coefficients for all other multi stress tensor operators is a nontrivial statement. We call it "the thermalization of the stress tensor sector" [27]. It is

[26] See [29,35,82,145-156] for some previous work on finite temperature conformal field theories in $d > 2$.

[27] We show this explicitly for certain primary heavy operators $\mathcal{O}_H$ in free CFTs. We also observe that other light operators do not satisfy the thermalization property that the stress tensor sector enjoys.

directly related to the Eigenstate Thermalization Hypothesis (ETH) [157-161], as we review below. Hence, we argue that all multi stress tensor operators in the large-C_T CFTs satisfy the ETH. In $d = 2$ the ETH and thermalization have been studied in e.g. [105,108,111-117,162-181].

Here we want to address the $d > 2$ case. In holographic theories CFT and bootstrap techniques provide a lot of data which indicates that the thermalization of the stress tensor sector happens [12-15,54,55,124-126,128,135,182]. Some of the OPE coefficients in holographic CFTs were computed using two-point functions in a black hole background [15] – these are thermal correlators according to the standard holographic dictionary. It is also worth noting that the leading Δ behavior of the OPE coefficients in holographic models does not depend on the coefficients of the higher derivative terms in the bulk lagrangian [14] (this should not be confused with the universality of the OPE coefficients of the minimal-twist multi stress tensors [15]). Such a universality follows from the thermalization of the stress tensor sector as we discuss below.

A natural question is whether the thermalization of the stress tensor sector is just a property of holographic CFTs or if it holds more generally. In this section we argue for the latter scenario. We compute the OPE coefficients (and the thermal expectation values) for a number of multi stress tensor operators in a free CFT and observe thermalization as well as universality of OPE coefficients. We also provide a bootstrap argument for all CFTs with a large central charge.

The rest of the section is organized as follows. In Section 7.2, we begin by considering the thermalization of multi stress tensor operators $T^k_{\tau,s}$. The heavy state we consider is created by a scalar operator $\mathcal{O}_H$ with dimension $\Delta_H \sim C_T$ and by thermalization of a multi stress tensor operator we mean[28]

$$\langle \mathcal{O}_H | T^k_{\tau,s} | \mathcal{O}_H \rangle \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{\tau,s}} \Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \langle T^k_{\tau,s} \rangle_\beta, \tag{7.1}$$

where the heavy state $|\mathcal{O}_H\rangle$ on the sphere of unit radius is created by the operator $\mathcal{O}_H$, $\lambda_{\mathcal{O}_H \mathcal{O}_H T^k_{\tau,s}}$ are the OPE coefficients of $T^k_{\tau,s}$ in the $\mathcal{O}_H \times \mathcal{O}_H$ OPE and $|_{\Delta_H^k / C_T^{k/2}}$ means we keep only leading terms that scale like $\Delta_H^k / C_T^{k/2} \sim C_T^{k/2}$.

28 Here we are suppressing the tensor structure. Note that all terms scale like $C_T^{k/2}$ which is consistent with $T^k_{\tau,s}$ being unit-normalized.

In (7.1) $\langle T^k_{\tau,s}\rangle_\beta$ is the one-point function on the sphere at finite temperature β^{-1}. Note that the OPE coefficients involving the stress tensor are fixed by the Ward identity, and hence eq. (7.1) for the stress tensor establishes a relation between the temperature β^{-1} and Δ_H. By the large-C_T factorization[29], the thermal one-point functions of multi stress tensors can be related to the thermal one-point function of the stress tensor itself. Explicitly,

$$\langle T^k_{\tau,s}\rangle_\beta = c^k_{\tau,s}(\langle T^1_{d-2,2}\rangle_\beta)^k = c^k_{\tau,s}(\lambda_{\mathcal{O}_H\mathcal{O}_H T^1_{d-2,2}})^k, \tag{7.2}$$

where $c^k_{\tau,s}$ are theory-independent coefficients that appear because of the index structure in $\langle T^k_{\tau,s}\rangle_\beta$. In the second equality in (7.2) we used (7.1) for the stress tensor. Note that (7.1) and (7.2) imply that the leading Δ_H behavior of the multi stress tensor OPE coefficients is universal, i.e. it does not depend on the theory[30]. We provide a bootstrap argument for this universality in all large-C_T theories. Also note that (7.2) is written for multi-trace operators $T^k_{\tau,s}$ which do not contain derivatives, but the presence of derivatives does not affect the statement of universality.

In Section 7.3, we check the universality by computing a number of the multi stress tensor OPE coefficients in a free $SU(N)$ adjoint scalar theory in $d = 4$ dimensions. We compare the leading Δ_H behavior in the free theory with results from holography/bootstrap and find perfect agreement in all cases listed below. After fixing the coefficients for the stress tensor case in Section 7.3.1, we look at the first nontrivial case, $T^2_{4,4}$ in Section 7.3.2. Section 7.3.3 is devoted to the double stress tensor with two derivatives, $T^2_{4,6}$. This is an operator whose finite temperature expectation value vanishes in the large volume limit (on the plane), but is finite on the sphere. In Section 7.3.4 we consider minimal twist multi stress tensors of the type $T^k_{2k,2k}$. Section 7.3.5 is devoted to multi stress tensors with non-minimal twist, $T^2_{6,2}$ and $T^2_{8,0}$.

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[29] See [183] for a general discussion of large-N factorization and [184,185] and [35] for the discussion in the context of gauge theories and CFTs respectively. The factorization holds in adjoint models in the 't Hooft limit at finite temperature, but there are counterexamples, like e.g. a direct product of low-C_T CFTs. However the factorization of multi stress tensors would still apply in these models.

[30] This amounts to the large-C_T factorization of correlators $\langle\mathcal{O}_H|T_{\mu\nu}\ldots T_{\alpha\beta}|\mathcal{O}_H\rangle$ in heavy states.

In Section 7.4, we verify that (7.1) holds in the free adjoint scalar theory for a variety of operators. In this section we again consider $d = 4$, but in addition, take the infinite volume limit. This is for technical reasons – it is easier to compute a finite temperature expectation value on the plane than on the sphere. We spell out the index structure in (7.1) in detail and go over all the examples discussed in the previous section. In addition, we discuss some triple stress tensor operators.

We continue in Section 7.5 by studying thermal two-point functions in the free adjoint scalar model in $d = 4$. By decomposing the correlator into thermal blocks we read off the product of thermal one-point functions and the OPE coefficients for several operators of low dimension and observe agreement with the results of Sections 7.3 and 7.4. Due to the presence of multiple operators with the same dimension and spin, we have to solve a mixing problem to find which operators contribute to the thermal two-point function.

In Section 7.6 we explain the relation between our results and the Eigenstate Thermalization Hypothesis. We observe that unlike multi stress tensors, other light operators explicitly violate the Eigenstate Thermalization Hypothesis and do not thermalize. We end with a discussion in Section 7.7.

Appendices C.1, C.2, and C.3 contain explicit calculations of OPE coefficients while in Appendices C.4 and C.5 thermal one-point functions are calculated. In Appendix C.6 we review the statement that the thermal one-point functions of multi-trace operators with derivatives vanish on $S^1 \times \mathbf{R}^{d-1}$. In Appendix C.7 we study a free scalar in two dimensions and calculate thermal two-point functions of certain quasi-primary operators. In Appendix C.8 we consider a free scalar vector model in four dimensions. Appendix C.9 discusses the factorization of multi-trace operators in the large volume limit.

7.2. Thermalization and universality

In the following we consider large-C_T CFTs on a $(d-1)$-dimensional sphere of radius R, which we set to unity for most of this section. As reviewed in [14], the stress tensor sector of conformal four-point functions consists of the contributions of the stress tensor and all its composites (multi stress tensors). The HHLL correlators we consider involve two heavy operators inserted at $x_E^0 = \pm\infty$ and two light operators inserted on the Euclidean cylinder, with angular separation φ and time separation x_E^0. The correlator in a heavy state (the HHLL correlator on the cylinder) is related to the correlator on the plane by a conformal transformation

$$\langle \mathcal{O}_H|\mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)|\mathcal{O}_H\rangle = \lim_{x_4\to\infty} x_4^{2\Delta_H}(z\bar{z})^{-\Delta/2}\langle \mathcal{O}_H(x_4)\mathcal{O}(1)\mathcal{O}(z,\bar{z})\mathcal{O}_H(0)\rangle, \tag{7.3}$$

where the cross-ratios $(z,\bar{z})$ on the plane are related to the coordinates (x_E^0,φ) via

$$z = e^{-x_E^0 - i\varphi}, \qquad \bar{z} = e^{-x_E^0 + i\varphi}. \tag{7.4}$$

The stress tensor sector of the HHLL correlator is given by

$$\mathcal{G}(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H}\langle \mathcal{O}_H(x_4)\mathcal{O}(1)\mathcal{O}(z,\bar{z})\mathcal{O}_H(0)\rangle\Big|_{\text{multi stress tensors}} \tag{7.5}$$

and can be expanded in conformal blocks

$$\mathcal{G}(z,\bar{z}) = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta}}\sum_{T^k_{\tau,s}} P^{(HH,LL)}_{T^k_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z}), \tag{7.6}$$

where τ, s, k label the twist, spin, and multiplicity of multi stress tensors. We are interested in the double scaling limit where the central charge and the dimension of $\mathcal{O}_H$ are large, $C_T, \Delta_H \to \infty$ with their ratio $\mu \propto \Delta_H/C_T$ fixed. In this limit the products of the OPE coefficients which appear in (7.6) are given by

$$P^{(HH,LL)}_{T^k_{\tau,s}} = \left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}}\lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}\Bigg|_{\left(\frac{\Delta_H}{C_T}\right)^k}, \tag{7.7}$$

where we only keep the leading, $\left(\frac{\Delta_H}{\sqrt{C_T}}\right)^k$ term in the OPE coefficients $\lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}$, but retain all terms in the OPE coefficients of the light operators $\lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}}$. The contribution of the conformal family of a multi stress operator $T^k_{\tau,s}$ to the HHLL correlator is therefore

$$\langle \mathcal{O}_H|\mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)|\mathcal{O}_H\rangle|_{T^k_{\tau,s}} = \frac{P^{(HH,LL)}_{T^k_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z})}{[\sqrt{z\bar{z}}(1-z)(1-\bar{z})]^{\Delta}}. \tag{7.8}$$

We now consider these CFTs at finite temperature β^{-1}. To isolate the contribution of the conformal family associated with $T^k_{\tau,s}$, we can write the thermal correlator as

$$\begin{aligned}
\langle \mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)\rangle_\beta &= \frac{1}{Z(\beta)}\sum_i e^{-\beta\Delta_i}\langle \mathcal{O}_i|\mathcal{O}(x_E^0,\varphi)\mathcal{O}(0)|\mathcal{O}_i\rangle \\
=&\frac{1}{[\sqrt{z\bar{z}}(1-z)(1-\bar{z})]^\Delta}\sum_{T^k_{\tau,s}}\left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}\mathcal{O}T^k_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z})\ \langle T^k_{\tau,s}\rangle_\beta \\
&+\ldots,
\end{aligned} \tag{7.9}$$

where

$$\langle T^k_{\tau,s}\rangle_\beta = \frac{1}{Z(\beta)}\sum_i e^{-\beta\Delta_i}\lambda_{\mathcal{O}_i\mathcal{O}_i T^k_{\tau,s}} \tag{7.10}$$

is the finite temperature one-point function on the sphere of the $T^k_{\tau,s}$ operator and the dots denote contributions from other operators. In (7.10) $Z(\beta)$ is the partition function and the sum runs over all operators, including descendants[31]. Note that

$$\langle T^k_{\tau,s}\rangle_\beta = \beta^{-(\tau+s)} f^k_{\tau,s}(\beta). \tag{7.11}$$

Here and below the indices are suppressed (see e.g. [151] for the explicit form) and $f^k_{\tau,s}(\beta)\sim C_T^{k/2}$ is a theory-dependent nontrivial function of β which approaches a constant $f^k_{\tau,s}(0)$ in the large volume ($\beta\to 0$) limit.

Consider the thermalization of the stress tensor sector:

$$\langle \mathcal{O}_H|T^k_{\tau,s}|\mathcal{O}_H\rangle\Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}\Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = \langle T^k_{\tau,s}\rangle_\beta. \tag{7.12}$$

Note that $T^k_{\tau,s}$ is unit-normalized, so all terms in (7.12) scale like $C_T^{k/2}$. Eq. (7.12) implies the equality between (7.8) and the corresponding term in (7.9). Note that the left-hand side of (7.12) is a function of the energy density while the right-hand side is a function of temperature. The relationship is fixed by considering the stress tensor case: the corresponding function $f^1_{d-2,2}(\beta)$ is determined by the free energy on the sphere (see Section 7.6).

[31] The corresponding conformal blocks can be obtained in the usual way by applying the quadratic conformal Casimir and solving the resulting differential equation [31].

In the following, we will first discuss the case where the multi stress operators $T^k_{\tau,s}$ do not have any derivatives inserted, and then show that the derivatives do not change the conclusions. Assuming large-C_T factorization, the leading C_T behavior of $\langle T^k_{\tau,s}\rangle_\beta$ on the sphere is determined by that of the stress tensor. Schematically,

$$\langle T^k_{\tau,s}\rangle_\beta = c^k_{\tau,s}\ (\langle T^1_{d-2,2}\rangle_\beta)^k + \dots, \tag{7.13}$$

where $c^k_{\tau,s}$ are numerical coefficients, which depend on k, τ, s, but are independent of the details of the theory and the dots stand for terms subleading in C_T^{-1}. By combining (7.13) and (7.12), one can formulate a universality condition

$$\lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{\tau,s}}\Big|_{\frac{\Delta_H^k}{C_T^{k/2}}} = c^k_{\tau,s}\ (\lambda_{\mathcal{O}_H\mathcal{O}_H T^1_{d-2,2}})^k = c^k_{\tau,s}\left(\frac{d}{1-d}\right)^k \frac{\Delta_H^k}{C_T^{\frac{k}{2}}}, \tag{7.14}$$

where the last equality follows from the stress tensor Ward identity for the three-point function which fixes $\lambda_{\mathcal{O}_H\mathcal{O}_H T^1_{d-2,2}}$ ($T^1_{d-2,2}$ here is unit-normalized). In other words, thermalization and large-C_T factorization imply that the leading $\Delta^k/C_T^{k/2}$ behavior of the multi stress tensor OPE coefficients is completely fixed and given by (7.14) in all large-C_T CFTs.

In the paragraph above we considered multi stress tensor operators that did not contain any derivatives in them. However, the story largely remains the same when the derivatives are included, as long as their number does not scale with C_T. Indeed, the three-point function involving the stress-tensor with added derivatives, $\partial_\alpha \dots \partial_\beta T_{\mu\nu}$ still behaves like $\lambda_{\mathcal{O}_H\mathcal{O}_H\partial_\alpha\dots\partial_\beta T_{\mu\nu}} \simeq \Delta_H/\sqrt{C_T}$ up to a theory-independent coefficient. Hence, (7.14) still holds, provided thermalization and large-C_T factorization hold on the sphere.

Note that due to conformal invariance, correlators on the sphere depend on R only through the ratio β/R. Moreover, in the large volume limit, factors of R need to drop out of (7.8) and (7.9) to have a well defined limit. To see this we use that $(1-z)\to 0$ and $(1-\bar{z})\to 0$ when $R\to\infty$ and the conformal blocks behave as (see e.g. [24])

$$\begin{aligned} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z}) &\sim \mathcal{N}_{d,s}[(1-z)(1-\bar{z})]^{\frac{\tau+s}{2}} C_s^{(d/2-1)}\Big(\frac{(1-z)+(1-\bar{z})}{2\sqrt{(1-z)(1-\bar{z})}}\Big) \\ &\sim \mathcal{N}_{d,s}\frac{|x|^{\tau+s}}{R^{\tau+s}} C_s^{(d/2-1)}\Big(\frac{x_E^0}{|x|}\Big), \end{aligned} \tag{7.15}$$

where $|x| = \sqrt{(x_E^0)^2 + \mathbf{x}^2}$, $C_s^{(d/2-1)}(\frac{x_E^0}{|x|})$ is a Gegenbauer polynomial and $\mathcal{N}_{d,s} = \frac{s!}{(d/2-1)_s}$. Including the factor $[(1-z)(1-\bar{z})]^{-\Delta}$ from (7.8) in (7.15) this agrees with the thermal block on $S^1 \times \mathbf{R}^{d-1}$ in [82]. Now from the thermalization of the stress tensor we will find in the large volume limit that

$$\frac{\Delta_H}{C_T} \propto \left(\frac{R}{\beta}\right)^d, \tag{7.16}$$

and from (7.14) and (7.15) it follows that

$$g_{\tau,s}^{(0,0)}(1-z, 1-\bar{z})\lambda_{\mathcal{O}\mathcal{O}T_{\tau,s}^k}\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\tau,s}^k}\Big|_{\frac{\Delta_H^k}{C_T^k}} \propto R^{dk-(\tau+s)}\beta^{-dk}. \tag{7.17}$$

The dimension of multi stress tensors $T_{\tau,s}^k$ is given by $\tau + s = dk + n$ where $n = 0, 2, \ldots$. Therefore, the only multi stress tensors that contribute in the large volume limit have dimensions dk. Restoring R in (7.8)-(7.9) and inserting (7.17) one finds that R drops out in the large volume limit. The correct dependence $\beta^{-(\tau+s)}$ from (7.11) in the $R \to \infty$ limit is also recovered in (7.8) using (7.17). The multi stress tensor operators that contribute in the large volume limit are therefore of the schematic form $T_{\mu_1\nu_1}T_{\mu_2\nu_2}\cdots T_{\mu_k\nu_k}$ with arbitrarily many contractions and no derivatives.

In holographic theories thermalization and the Wilson line prescription for the correlator allows one to compute the universal part of the OPE coefficients (see [124,186] for explicit computations in the $d = 4$ case). It is also easy to check explicitly that the universality (7.14) holds for holographic theories with a Gauss-Bonnet gravitational coupling added. While the statement was shown to be true for the leading twist OPE coefficients in [15], it was not immediately obvious for multi stress tensors of non-minimal twist. Some such OPE coefficients were computed in [15,14]. (See e.g. eqs. (5.48), (5.51), (5.52), (5.57) and (D.1)-(D.5) in [14]). Indeed, the leading $\Delta^k/C_T^{k/2}$ behavior of these OPE coefficients is independent of the Gauss-Bonnet coupling.

What about a general large-C_T theory? We first consider the OPE coefficients of double-stress tensors. To this end, consider the four point function[32] $\langle \mathcal{O}T_{\mu\nu}T_{\rho\sigma}\mathcal{O}\rangle$ where $\mathcal{O}$ is a scalar operator with scaling dimension Δ.

[32] This correlator for finite Δ was recently considered in holographic CFTs with $\Delta_{\text{gap}} \gg 1$ and $\Delta \ll \Delta_{\text{gap}}$ in [54].

In the direct channel $\mathcal{O} \times \mathcal{O} \to \mathcal{O}' \to T_{\mu\nu} \times T_{\rho\sigma}$ for finite Δ and large C_T, the leading contribution in the large-C_T limit comes from the identity operator $\mathcal{O} \times \mathcal{O} \to \mathbf{1} \to T_{\mu\nu} \times T_{\rho\sigma}$. The subleading contributions in the direct-channel are due to single trace operators as well as double trace operators made out of the external operators of the schematic form $T^2_{\tau,s}$ and $[\mathcal{O}\mathcal{O}]_{n,l} =: \mathcal{O}\partial^{2n}\partial_1 \ldots \partial_l \mathcal{O} :$. The exchange of the identity operator is reproduced in the cross-channel $\mathcal{O} \times T_{\mu\nu} \to [\mathcal{O}T_{\alpha\beta}]_{n,l} \to \mathcal{O} \times T_{\rho\sigma}$ by mixed double-trace operators $[\mathcal{O}T_{\alpha\beta}]_{n,l}$ with OPE coefficients fixed by the MFT [38,56-57]. The subleading contributions in $1/C_T$ are then due to corrections to the anomalous dimension and OPE coefficients of $[\mathcal{O}T_{\alpha\beta}]_{n,l}$ and single trace operators in the $\mathcal{O} \times T_{\mu\nu}$ OPE. An important example of the latter is the exchange of the single trace operator $\mathcal{O}$, whose contribution is universally fixed by the stress tensor Ward identity to be $(\lambda_{\mathcal{O}T^1_{d-2,2}\mathcal{O}})^2 \propto \Delta^2/C_T$ times the conformal block. This gives a universal contribution to $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$ as was also noted in [54].

We now want to consider the case where $\Delta \sim C_T$ and study the OPE coefficients of the double-stress tensor operators in the $\mathcal{O} \times \mathcal{O}$ OPE. Firstly, note that the contribution from $T^2_{\tau,s}$ to the four-point function expanded in the direct channel is proportional to $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}} \lambda_{TTT^2_{\tau,s}}$. The OPE coefficients $\lambda_{TTT^2_{\tau,s}}$ are fixed by the MFT and are independent of Δ and therefore the dependence on the scaling dimension comes solely from the OPE coefficients $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$. In the cross-channel, we analyze two kinds of contributions: from the exchanged operator $\mathcal{O}$ and from all other operators $\mathcal{O}' \neq \mathcal{O}$. From the operator $\mathcal{O}$ we get a universal contribution to the OPE coefficients in the direct channel $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$, that we denote by $\lambda^{(1)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$. This contribution is universal since it only depends on $(\lambda_{\mathcal{O}T^1_{d-2,2}\mathcal{O}})^2 \propto \Delta^2/C_T$ in the cross-channel, which is fixed by the Ward identity. The contributions from other operators $\mathcal{O}'$ to the same OPE coefficient will be denoted by $\lambda^{(2)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$, such that $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}} = \lambda^{(1)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}} + \lambda^{(2)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$. Note that it also follows from the stress tensor Ward identity that the only scalar primary that appears in the cross-channel is $\mathcal{O}$. The operator $\mathcal{O}'$ therefore necessarily has spin $s \neq 0$.

To prove universality we need to show that $\lambda^{(2)}_{\mathcal{O}\mathcal{O}T^2_{\tau,s}} \ll \Delta^2/C_T$ in limit $1 \ll \Delta \propto C_T$ by studying the Δ dependence of the OPE coefficients $\lambda_{\mathcal{O}T^1_{d-2,2}\mathcal{O}'}$ in the cross-channel. For operators $\mathcal{O}'$, such that $\Delta_{\mathcal{O}'} \ll \Delta$, we expect that these OPE coefficients are heavily suppressed. It would be interesting to understand if one could put a general bound on the contribution of these operators in the cross-channel in any large-C_T theory. On the other hand, assuming thermalization, the OPE coefficients due to operators $\mathcal{O}'$ such that $\Delta_{\mathcal{O}'} \sim \Delta$ have been calculated in [154]. The obtained results are in agreement with our expectation, namely, these OPE coefficients are suppressed in $1 \ll \Delta \propto C_T$ limit. Additionally, in the cross-channel we have double-trace operators $[\mathcal{O}T_{\alpha\beta}]_{n,l}$, whose OPE is fixed by the MFT and it does not get Δ-enhanced.

One can iteratively extend the argument given here to multi stress tensors operators (with $k > 2$) by considering multi stress tensors as external operators. For example, to argue the universality of $\lambda_{\mathcal{O}\mathcal{O}T^3_{\tau,s}}$ one may consider $\langle \mathcal{O}T^1_{d-2,2}T^2_{\tau,s}\mathcal{O}\rangle$. The bootstrap argument above can be applied again by using the fact that OPE coefficients $\lambda_{\mathcal{O}\mathcal{O}T^2_{\tau,s}}$ are universal, and the OPE coefficients $\lambda_{\mathcal{O}T^2_{\tau,s}\mathcal{O}'}$ are again expected to be subleading.

7.3. OPE coefficients in the free adjoint scalar model

In this section we consider a four-dimensional theory of a free scalar in the adjoint representation of $SU(N)$, see [187-192] for related work. The relation between N and the central charge C_T in this theory is [18]

$$C_T = \frac{4}{3}(N^2 - 1), \tag{7.18}$$

and we consider the large-N (large-C_T) limit. The propagator for the scalar field $\phi^i{}_j$ is given by

$$\langle \phi^i{}_j(x)\phi^k{}_l(y)\rangle = \left(\delta^i{}_l\delta^k{}_j - \frac{1}{N}\delta^i{}_j\delta^k{}_l\right)\frac{1}{|x-y|^2}. \tag{7.19}$$

A single trace scalar operator with dimension Δ is given by

$$\mathcal{O}_\Delta(x) = \frac{1}{\sqrt{\Delta}N^{\frac{\Delta}{2}}} : Tr(\phi^\Delta) : (x), \tag{7.20}$$

where $: \ldots :$ denotes the oscillator normal ordering and the normalization is fixed by

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(y)\rangle = \frac{1}{|x-y|^{2\Delta}}. \tag{7.21}$$

The CFT data that we compute in this section are the OPE coefficients of multi stress tensors in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. Assuming we can take $\Delta \to \Delta_H \sim C_T$, the large-$\Delta$ limit of these OPE coefficients is shown to be universal. One may worry that for $\Delta_H \sim C_T$ we can no longer trust the planar expansion, but, as we show in Appendix C.3, the large-Δ limit of the planar result yields the correct expression even for $\Delta_H \sim C_T$.

7.3.1. Stress tensor

The stress tensor operator is given by

$$T_{\mu\nu}(x) = \frac{1}{3\sqrt{C_T}} : Tr\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\phi\partial_\mu\partial_\nu\phi - (\text{trace})\right) : (x), \tag{7.22}$$

with the normalization

$$\langle T^{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{1}{|x|^8}\left(I^{(\mu}{}_\rho(x)I^{\nu)}{}_\sigma(x) - (\text{traces})\right), \tag{7.23}$$

where $I^\mu{}_\nu(x) := \delta^\mu{}_\nu - \frac{2x^\mu x_\nu}{|x|^2}$. The OPE coefficient is fixed by the stress tensor Ward identity to be

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^1_{2,2}} = -\frac{4\Delta}{3\sqrt{C_T}}. \tag{7.24}$$

It is also useful to find (7.24) using Wick contractions since an analogous calculation will be necessary for multi stress tensors. We do this explicitly in Appendix C.1.

7.3.2. Double-stress tensor with minimal twist

In this section we study the minimal-twist composite operator made out of two stress tensors

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : (x) - (\text{traces}), \tag{7.25}$$

with the normalization

$$\langle (T^2)^{\mu\nu\rho\sigma}(x)(T^2)_{\kappa\lambda\delta\omega}(0)\rangle = \frac{1}{|x|^{16}}\left(I^{(\mu}{}_\kappa I^\nu{}_\lambda I^\rho{}_\delta I^{\sigma)}{}_\omega - (\text{traces})\right). \tag{7.26}$$

Consider the following three-point function

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}}{|x_{12}|^{2\Delta-4}|x_{13}|^4|x_{23}|^4}\left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})\right), \tag{7.27}$$

where $Z^\mu = \frac{x^\mu_{13}}{|x_{13}|^2} - \frac{x^\mu_{12}}{|x_{12}|^2}$. It is shown in Appendix C.1 that the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}$ is given at leading order in the large-C_T limit by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} = \frac{8\sqrt{2}\Delta(\Delta-1)}{9C_T}. \tag{7.28}$$

Evaluating $P^{(HH,LL)}_{T^2_{4,4}}$ defined by (7.7) in the large-Δ limit[33], we obtain

$$
\begin{aligned}
P^{(HH,LL)}_{T^2_{4,4}} &= \left(-\frac{1}{2}\right)^4 \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} \bigg|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\
&= \frac{8}{81}\frac{\Delta_H^2}{C_T^2}\left(\Delta^2 + \mathcal{O}\left(\Delta\right)\right) = \mu^2\left(\frac{\Delta^2}{28800} + \mathcal{O}(\Delta)\right),
\end{aligned}
\tag{7.29}
$$

where we use the following relation

$$
\mu = \frac{160}{3}\frac{\Delta_H}{C_T}. \tag{7.30}
$$

The result (7.29) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [15] and bootstrap in [124,13].

7.3.3. Double-stress tensor with minimal twist and spin $s = 6$

We consider double-stess tensor operator with two (uncontracted) derivatives inserted

$$
\begin{aligned}
(T^2)_{\mu\nu\rho\sigma\eta\kappa}(x) = \frac{1}{2\sqrt{182}} : \Big(T_{(\mu\nu}\partial_\rho\partial_\sigma T_{\eta\kappa)}(x) - \frac{7}{6}\left(\partial_{(\rho}T_{\mu\nu}\right)\left(\partial_\sigma T_{\eta\kappa)}\right)(x) \\
- (\text{traces})\Big) : .
\end{aligned}
\tag{7.31}
$$

Using the conformal algebra eq. (C.2), it is straightforward to check that this operator is primary. It is unit-normalized such that

$$
\langle (T^2)^{\mu\nu\rho\sigma\eta\kappa}(x)(T^2)_{\alpha\beta\gamma\delta\xi\epsilon}(0)\rangle = \frac{1}{|x|^{20}}\left(I^{(\mu}{}_\alpha I^\nu{}_\beta I^\rho{}_\gamma I^\sigma{}_\delta I^\eta{}_\xi I^{\kappa)}{}_\epsilon - (\text{traces})\right). \tag{7.32}
$$

By a calculation similar to those summarized in Appendix C.1, we observe that the OPE coefficient of $(T^2)_{\mu\nu\rho\sigma\eta\kappa}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE is given at leading order in the large-C_T limit by

$$
\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,6}} = \frac{8}{3}\sqrt{\frac{2}{91}}\frac{\Delta(\Delta-1)}{C_T}. \tag{7.33}
$$

[33] By the large-Δ limit, we strictly speaking mean $1 \ll \Delta \ll C_T$. However in this section we often extrapolate this to the $\Delta \sim C_T$ regime.

Evaluating $P^{(HH,LL)}_{T^2_{4,6}}$, defined by (7.7), in the large-Δ limit, we obtain

$$\begin{aligned} P^{(HH,LL)}_{T^2_{4,6}} &= \left(-\frac{1}{2}\right)^6 \lambda_{\mathcal{O}_H \mathcal{O}_H T^2_{4,6}} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^2_{4,6}} \Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{2}{819} \frac{\Delta_H^2}{C_T^2} \left(\Delta^2 + \mathcal{O}(\Delta)\right) = \mu^2 \left(\frac{\Delta^2}{1164800} + \mathcal{O}(\Delta)\right). \end{aligned} \tag{7.34}$$

The result (7.34) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [15] and bootstrap in [124,13].

7.3.4. Minimal-twist multi stress tensors

We now consider multi stress tensors $T^k_{2k,2k}$. Just like the double stress tensor ($k = 2$), we show that these have universal OPE coefficients in the large-Δ limit for any k.

Consider the unit-normalized minimal-twist multi stress tensor operator given by

$$(T^k)_{\mu_1\mu_2\ldots\mu_{2k}}(x) = \frac{1}{\sqrt{k!}} : T_{(\mu_1\mu_2} T_{\mu_3\mu_4} \cdots T_{\mu_{2k-1}\mu_{2k})} : (x) - (\text{traces}). \tag{7.35}$$

The OPE coefficient of $(T^k)_{\mu_1\mu_2\ldots\mu_{2k}}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE, in the large-C_T limit is given by[34]

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^k_{2k,2k}} = \left(-\frac{4}{3}\right)^k \frac{1}{\sqrt{k!} C_T^{k/2}} \frac{\Gamma(\Delta+1)}{\Gamma(\Delta-k+1)}. \tag{7.36}$$

First, we write $P^{(HH,LL)}_{T^3_{6,6}}$, defined by (7.7), in the large-Δ limit. We obtain this OPE coefficient from (7.36) for $k = 3$,

$$\begin{aligned} P^{(HH,LL)}_{T^3_{6,6}} &= \left(-\frac{1}{2}\right)^6 \lambda_{\mathcal{O}_H \mathcal{O}_H T^3_{6,6}} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^3_{6,6}} \Big|_{\left(\frac{\Delta_H}{C_T}\right)^3} \\ &= \frac{32}{2187} \frac{\Delta_H^3}{C_T^3} \left(\Delta^3 + \mathcal{O}(\Delta^2)\right) = \mu^3 \left(\frac{\Delta^3}{10368000} + \mathcal{O}(\Delta^2)\right). \end{aligned} \tag{7.37}$$

The result (7.37) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [15] and bootstrap in [13].

[34] See Appendix C.1 for detailed computations of similar OPE coefficients.

Additionally, we consider the OPE coefficient $P^{(HH,LL)}_{T^k_{2k,2k}}$ in the large-Δ limit for general k,

$$\begin{aligned}P^{(HH,LL)}_{T^k_{2k,2k}} &= \left(-\frac{1}{2}\right)^{2k} \lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{2k,2k}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^k_{2k,2k}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^k} \\ &= \frac{1}{k!}\left(\frac{2}{3}\right)^{2k} \frac{\Delta_H^k}{C_T^k}\left(\Delta^k + \mathcal{O}\left(\Delta^{k-1}\right)\right) = \mu^k\left(\frac{\Delta^k}{120^k k!} + \mathcal{O}(\Delta^{k-1})\right).\end{aligned} \tag{7.38}$$

If we consider the limit $1-\bar{z} \ll 1-z \ll 1$, such that $\mu(1-\bar{z})(1-z)^3$ is held fixed, only operators $T^k_{2k,2k}$ contribute to the heavy-heavy-light-light four-point function given by eq. (7.5). The conformal blocks of $T^k_{2k,2k}$ in this limit are given by

$$g^{(0,0)}_{2k,2k}(1-z,1-\bar{z}) \approx (1-\bar{z})^k(1-z)^{3k}, \tag{7.39}$$

and we can sum all contributions in eq. (7.6) explicitly to obtain

$$\mathcal{G}(z,\bar{z}) \approx \frac{1}{((1-z)(1-\bar{z}))^{\Delta}} e^{\frac{\mu\Delta}{120}(1-\bar{z})(1-z)^3}. \tag{7.40}$$

Notice that the term in the exponential is precisely the stress-tensor conformal block in the limit $1-\bar{z} \ll 1-z \ll 1$ times its OPE coefficient. Therefore, the OPE coefficients (7.36) imply the exponentiation of stress-tensor conformal block. We conclude that these OPE coefficients are the same as the ones computed using holography and bootstrap in the limit of large Δ.

7.3.5. Double-stress tensors with non-minimal twist

So far we have shown that the minimal-twist multi stress tensor OPE coefficients are universal in the limit of large Δ. In this subsection, we extend this to show that the simplest non-minimal twist double-stress tensors also have universal OPE coefficients at large Δ.

The subleading twist double-stress tensor with twist $\tau = 6$ is of the schematic form $:T^{\mu}{}_{\alpha}T^{\alpha\nu}:$ and has dimension $\Delta = 8$ and spin $s = 2$. It is given by

$$(T^2)_{\mu\nu}(x) = \frac{1}{\sqrt{2}} :T_{\mu\alpha}T^{\alpha}{}_{\nu}:(x) - (\text{trace}). \tag{7.41}$$

The normalization in (7.41) is again chosen such that $(T^2)_{\mu\nu}$ is unit-normalized, see Appendix C.2 for details.

The OPE coefficient of $(T^2)_{\mu\nu}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE is found from the three-point function in the large-C_T limit, for details see Appendix C.2,

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle = \frac{4\sqrt{2}\Delta(\Delta-1)}{9C_T}\frac{Z^\mu Z^\nu - (\text{trace})}{|x_{12}|^{2\Delta-6}|x_{13}|^6|x_{23}|^6}, \tag{7.42}$$

from which we read off the OPE coefficient

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{6,2}} = \frac{4\sqrt{2}\Delta(\Delta-1)}{9C_T}. \tag{7.43}$$

Evaluating $P^{(HH,LL)}_{T^2_{6,2}}$, defined by (7.7), in the large-Δ limit, we obtain

$$\begin{aligned} P^{(HH,LL)}_{T^2_{6,2}} &= \left(-\frac{1}{2}\right)^2 \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{6,2}}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{6,2}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{8}{81}\frac{{\Delta_H}^2}{C_T^2}\left(\Delta^2 + \mathcal{O}\left(\Delta\right)\right) = \mu^2\left(\frac{\Delta^2}{28800} + \mathcal{O}(\Delta)\right). \end{aligned} \tag{7.44}$$

The result (7.44) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [15] and bootstrap in [14].

We further consider the scalar double-stress tensor with $\Delta = 8$ and spin $s = 0$ which is given by

$$(T^2)(x) = \frac{1}{3\sqrt{2}} : T_{\mu\nu}T^{\mu\nu} : (x). \tag{7.45}$$

The three point function $\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle$ is found in Appendix C.2 to be

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle = \frac{2\sqrt{2}\Delta(\Delta-1)}{9C_T}\frac{1}{|x_{12}|^{2\Delta-8}|x_{13}|^8|x_{23}|^8}, \tag{7.46}$$

from which we read off the OPE coefficient

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{8,0}} = \frac{2\sqrt{2}\Delta(\Delta-1)}{9C_T}. \tag{7.47}$$

We write $P^{(HH,LL)}_{T^2_{8,0}}$ in the large-Δ limit

$$\begin{aligned} P^{(HH,LL)}_{T^2_{8,0}} &= \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{8,0}}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{8,0}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^2} \\ &= \frac{8}{81}\frac{{\Delta_H}^2}{C_T^2}\left(\Delta^2 + \mathcal{O}(\Delta)\right) = \mu^2\left(\frac{\Delta^2}{28800} + \mathcal{O}(\Delta)\right). \end{aligned} \tag{7.48}$$

The result (7.48) agrees with the leading behavior of the corresponding OPE coefficients computed using holography in [15] and bootstrap in [14].

7.4. Thermal one-point functions in the free adjoint scalar model

In this section we explicitly show that multi stress tensor operators thermalize in the free theory by calculating the thermal one-point function of some of these operators on $S^1 \times \mathbf{R}^3$. One-point functions of primary symmetric traceless operators at finite temperature are fixed by symmetry up to a dimensionless coefficient $b_{\mathcal{O}}$ (see e.g. [35,82])

$$\langle \mathcal{O}_{\mu_1 \cdots \mu_{s_{\mathcal{O}}}} \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \Big(e_{\mu_1} \cdots e_{\mu_{s_{\mathcal{O}}}} - (\text{traces}) \Big) . \tag{7.49}$$

Here e_μ is a unit vector along the thermal circle.

To compare the thermal one-point functions and OPE coefficients from the previous section, we need to derive a relation between $\frac{\Delta_H}{C_T}$ and the temperature[35] β^{-1}. Here $\Delta_H \sim N^2$ refers to the scaling dimension of a heavy operator $\mathcal{O}_H$ with OPE coefficients given by the large-Δ limit of those obtained in Section 7.3. One can relate the inverse temperature β to the parameter $\mu = \frac{160}{3}\frac{\Delta_H}{C_T}$ using the Stefan-Boltzmann's law $E/\text{vol}(S^3) = N^2\pi^2/30\beta^4$. The energy of the state E is related to its conformal dimension Δ via $E = \Delta/R$. One can then use $\text{vol}(S^3) = 2\pi^2 R^3$ and the relation between N and C_T given by (7.18), to find

$$\mu = \frac{160}{3}\frac{\Delta_H}{C_T} = \frac{160}{3} E \frac{R}{C_T} = \frac{8}{3}\Big(\frac{\pi R}{\beta}\Big)^4 . \tag{7.50}$$

7.4.1. Stress tensor

The thermal one-point function for the stress tensor $T^1_{2,2} = T_{\mu\nu}$ is calculated in Appendix C.4 where we find that $b_{T^1_{2,2}}$ is given by

$$b_{T^1_{2,2}} = -\frac{2\pi^4 N}{15\sqrt{3}} . \tag{7.51}$$

Using (7.50) and (7.51) one arrives at

$$b_{T^1_{2,2}} \beta^{-4} = \lambda_{\mathcal{O}_H \mathcal{O}_H T^1_{2,2}} . \tag{7.52}$$

7.4.2. Double-stress tensor with minimal twist

In this section we calculate the thermal one-point function of the double-stress tensor operator with $\tau = 4$ and spin $s = 4$. The operator is written explicitly in

[35] See also Section 7.6 and Appendix C.4 for alternative derivations.

(7.25). The leading contribution to the thermal one-point function of $(T^2)_{\mu\nu\rho\sigma}$ follows from the large-N factorization and is given by

$$\begin{aligned}\langle (T^2)_{\mu\nu\rho\sigma}\rangle_\beta &= \frac{1}{\sqrt{2}}\langle T_{(\mu\nu}\rangle_\beta \langle T_{\rho\sigma)}\rangle_\beta - (\text{traces}) \\ &= \frac{2\sqrt{2}\pi^8 N^2}{675\beta^8}\left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right).\end{aligned} \tag{7.53}$$

Using the relation (7.50) and the OPE coefficient (7.28), we observe the thermalization of this operator,

$$b_{T^2_{4,4}}\beta^{-8} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}}\Big|_{\frac{\Delta_H^2}{C_T}}. \tag{7.54}$$

7.4.3. Minimal-twist multi stress tensors

Consider now multi stress tensors $T^k_{2k,2k}$ with twist $\tau = 2k$ and spin $s = 2k$. We show that these operators thermalize for any k by calculating their thermal one-point functions:

$$\langle (T^k)_{\mu_1\mu_2\ldots\mu_{2k}}\rangle_\beta = \frac{b_{T^k_{2k,2k}}}{\beta^{4k}}(e_{\mu_1}e_{\mu_2}\cdots e_{\mu_{2k}} - (\text{traces})), \tag{7.55}$$

where the leading behavior of $b_{T^k_{2k,2k}}$ follows from the large-N factorization:

$$b_{T^k_{2k,2k}} = \frac{1}{\sqrt{k!}}(b_{T^1_{2,2}})^k = \frac{(-\frac{2}{5})^k N^k \pi^{4k}}{3^{\frac{3k}{2}}\sqrt{k!}}. \tag{7.56}$$

Eqs. (7.50) and (7.56) may be combined to yield

$$b_{T^k_{2k,2k}}\beta^{-4k} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^k_{2k,2k}}\Big|_{\frac{\Delta_H^k}{C_T^{k/2}}}. \tag{7.57}$$

7.4.4. Double-stress tensors with non-minimal twist

The subleading twist double-stress tensor is of the schematic form $:T^\mu{}_\alpha T^{\alpha\nu}:$ and has twist $\tau = 6$ and spin $s = 2$. The explicit form can be found in (7.41). The leading term in the thermal one-point function is given by

$$\begin{aligned}\langle (T^2)^{\mu\nu}\rangle_\beta &= \frac{1}{\sqrt{2}}\langle T^{\mu\alpha}\rangle_\beta \langle T^\nu{}_\alpha\rangle_\beta - (\text{trace}) \\ &= \frac{b^2_{T^1_{2,2}}}{2\sqrt{2}\beta^8}(e^\mu e^\nu - \frac{1}{4}\delta^{\mu\nu}) \\ &= \frac{\sqrt{2}N^2\pi^8}{675\beta^8}(e^\mu e^\nu - \frac{1}{4}\delta^{\mu\nu}),\end{aligned} \tag{7.58}$$

therefore,

$$b_{T^2_{6,2}} = \frac{\sqrt{2}N^2\pi^8}{675}. \tag{7.59}$$

Taking the large-Δ limit of the OPE coefficient in (7.43) and substituting (7.50), we observe thermalization,

$$b_{T^2_{6,2}}\beta^{-8} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{6,2}}\Big|_{\frac{\Delta_H^2}{C_T}}. \tag{7.60}$$

We further consider the scalar double-stress tensor with $\tau = 8$ and $s = 0$ which is given by (7.45). The thermal one-point function for this operator is

$$\begin{aligned}\langle (T^2)\rangle_\beta &= \frac{1}{3\sqrt{2}}\langle T_{\mu\nu}\rangle_\beta\langle T^{\mu\nu}\rangle_\beta \\ &= \frac{1}{3\sqrt{2}}\frac{3}{4}b^2_{T^1_{2,2}}\beta^{-8} = \frac{\pi^8 N^2}{675\sqrt{2}\beta^8},\end{aligned} \tag{7.61}$$

where the factor of $\frac{3}{4}$ in the first line comes from the index contractions. Hence,

$$b_{T^2_{8,0}} = \frac{\pi^8 N^2}{675\sqrt{2}}. \tag{7.62}$$

Using (7.62), (7.47) and (7.50), we again observe thermalization,

$$b_{T^2_{8,0}}\beta^{-8} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{8,0}}\Big|_{\frac{\Delta_H^2}{C_T}}. \tag{7.63}$$

7.4.5. Triple-stress tensors with non-minimal twist

We consider the triple stress tensors with $\tau = 8, s = 4$ and $\tau = 10, s = 2$. The unit-normalized triple stress tensor with $\tau = 8$ can be written as

$$(T^3)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{3}}\left(: T_{(\mu\nu}T_{\rho|\alpha|}T^{\alpha}{}_{\sigma)} : (x) - (\text{traces})\right), \tag{7.64}$$

where $|\alpha|$ denotes that index α is excluded from the symmetrization. The thermal one-point function follows from large-N factorization

$$\begin{aligned}\langle (T^3)_{\mu\nu\rho\sigma}\rangle_\beta &= \frac{1}{\sqrt{3}}\left(\langle T_{(\mu\nu}\rangle_\beta\langle T_{\rho|\alpha|}\rangle_\beta\langle T^{\alpha}{}_{\sigma)}\rangle_\beta - (\text{traces})\right) \\ &= \frac{1}{2\sqrt{3}}\frac{b^3_{T^1_{2,2}}}{\beta^{12}}\left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right) \\ &= -\frac{4\pi^{12}N^3}{30375\beta^{12}}\left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces})\right),\end{aligned} \tag{7.65}$$

therefore,

$$b_{T^3_{8,4}} = -\frac{4\pi^{12}N^3}{30375}. \tag{7.66}$$

The OPE coefficient of the operator with same quantum numbers ($\Delta = 12$, $s = 4$) is calculated holographically and is given by (D.1) in [14]. In the large-Δ limit it can be written as

$$P^{(HH,LL)}_{T^3_{8,4}} = \left(-\frac{1}{2}\right)^4 \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{8,4}} \lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{8,4}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^3} = \frac{64}{2187}\frac{\Delta_H^3\Delta^3}{C_T^3} + \mathcal{O}(\Delta^2). \tag{7.67}$$

Now, one can easily read-off $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{8,4}}$ in the large-Δ limit

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{8,4}} = -\frac{32\Delta^3}{27\sqrt{3}{C_T}^{3/2}} + \mathcal{O}(\Delta^2) = -\frac{4\Delta^3}{9N^3} + \mathcal{O}(\Delta^2), \tag{7.68}$$

where we use the relation between central charge C_T and N given by (7.18). Using (7.50) one can obtain

$$b_{T^3_{8,4}}\beta^{-12} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{8,4}}\Big|_{\frac{\Delta_H^3}{C_T^{3/2}}}. \tag{7.69}$$

We also consider the triple stress tensors with quantum numbers $\Delta = 12$ and $s = 2$. There are two linearly independent such operators that schematically can be written as $:T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu}:$ and $:T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu}:$. We write the following linear combinations of these operators

$$(T^3)_{\mu\nu}(x) = \frac{1}{10\sqrt{2}}\left(:T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu}:(x) + 4 :T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu}:(x) - (\text{trace})\right), \tag{7.70}$$

$$(\tilde{T}^3)_{\mu\nu}(x) = \frac{7}{20}\left(:T_{\alpha\beta}T^{\alpha\beta}T_{\mu\nu}:(x) - \frac{12}{7} :T_{\mu\alpha}T^{\alpha\beta}T_{\beta\nu}:(x) - (\text{trace})\right). \tag{7.71}$$

Both $(T^3)_{\mu\nu}$ and $(\tilde{T}^3)_{\mu\nu}$ are unit-normalized and their overlap vanishes in the large-N limit

$$\langle (T^3)_{\mu\nu}(x)(\tilde{T}^3)^{\rho\sigma}(y)\rangle = \mathcal{O}(1/N^2). \tag{7.72}$$

The thermal one-point functions of these operators, obtained by large-N factorization, in the large-N limit are given by

$$\begin{aligned}\langle (T^3)_{\mu\nu}\rangle_\beta &= -\sqrt{\frac{2}{3}}\frac{N^3\pi^{12}}{10125\beta^{12}}\left(e_\mu e_\nu - (\text{trace})\right),\\ \langle (\tilde{T}^3)_{\mu\nu}\rangle_\beta &= \mathcal{O}(N),\end{aligned} \tag{7.73}$$

therefore,

$$
\begin{aligned}
b_{T^3_{10,2}} &= -\sqrt{\frac{2}{3}}\frac{N^3\pi^{12}}{10125},\\
b_{\tilde{T}^3_{10,2}} &= 0.
\end{aligned}
\tag{7.74}
$$

The holographic OPE coefficient of the operator with the same quantum numbers ($\Delta = 12$, $s = 2$), with external scalar operators is given by (5.57) in [14]. In the large-Δ limit it can be written as

$$
P^{(HH,LL)}_{T^3_{10,2}} = \left(-\frac{1}{2}\right)^2 \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{10,2}}\lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{10,2}}\Big|_{\left(\frac{\Delta_H}{C_T}\right)^3} = \frac{32}{729}\frac{\Delta_H^3\Delta^3}{C_T^3} + \mathcal{O}(\Delta^2).
\tag{7.75}
$$

We can read-off $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{10,2}}$:

$$
\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^3_{10,2}} = -\frac{8\sqrt{2}}{27}\frac{\Delta^3}{C_T^{3/2}} + \mathcal{O}(\Delta^2) = -\frac{\sqrt{2}}{3\sqrt{3}}\frac{\Delta^3}{N^3} + \mathcal{O}(\Delta^2).
\tag{7.76}
$$

Again, using (7.50), one can confirm that this operator thermalizes

$$
b_{T^3_{10,2}}\beta^{-12} = \lambda_{\mathcal{O}_H\mathcal{O}_H T^3_{10,2}}\Big|_{\frac{\Delta_H^3}{C_T^{3/2}}}.
\tag{7.77}
$$

7.5. Thermal two-point function and block decomposition

In this section we study the thermal two-point function $\langle\mathcal{O}_\Delta\mathcal{O}_\Delta\rangle_\beta$ and decompose it in thermal blocks. We determine the contributions of a few low-lying operators, including the stress tensor $T^1_{2,2}$ and the double stress tensor $T^2_{4,4}$. They exactly match the corresponding OPE coefficients and thermal expectation values computed in previous sections. Due to the presence of multiple operators with equal scaling dimension and spin, there is a mixing problem which we solve explicitly in a few cases. Related appendices include Appendix C.6, where we review the statement that the thermal one-point functions of multi-trace operators with derivatives vanish on $S^1 \times \mathbf{R}^{d-1}$ and Appendix C.7, where we consider two-dimensional thermal two-point functions. In Appendix C.8 we do a similar analysis for the vector model in four dimensions.

7.5.1. Thermal two-point function of a single trace scalar operator

The correlator at finite temperature β^{-1} in the free theory can be calculated by Wick contractions using the propagators on $S^1 \times \mathbf{R}^3$. Explicitly, the two-point function at finite temperature is given by[36]

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \tilde{g}(x_E^0, |\mathbf{x}|)^\Delta + \frac{\pi^4 \Delta(\Delta-2)}{9\beta^4}\tilde{g}(x_E^0, |\mathbf{x}|)^{\Delta-2} + \dots, \tag{7.78}$$

where

$$\begin{aligned} \tilde{g}(x_E^0, |\mathbf{x}|) &= \sum_{m=-\infty}^{\infty} \frac{1}{(x_E^0 + m\beta)^2 + \mathbf{x}^2} \\ &= \frac{\pi}{2\beta|\mathbf{x}|}\Big[\mathrm{Coth}\Big(\frac{\pi}{\beta}(|\mathbf{x}| - ix_E^0)\Big) + \mathrm{Coth}\Big(\frac{\pi}{\beta}(|\mathbf{x}| + ix_E^0)\Big)\Big]. \end{aligned} \tag{7.79}$$

The dots in (7.78) contain contributions due to further self-contractions which will not be important below[37]. Taking the $\beta \to \infty$ limit of (7.78) we can read off the decomposition of the two-point function in terms of thermal conformal blocks on $S^1 \times \mathbf{R}^3$ with coordinates $x = (x_E^0, \mathbf{x})$.

Following [82], if $|x| = \sqrt{(x_E^0)^2 + \mathbf{x}^2} \leq \beta$ the two-point function can be evaluated using the OPE:

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \sum_{\mathcal{O}} \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}} |x|^{\tau-2\Delta} x_{\mu_1}\cdots x_{\mu_{s_\mathcal{O}}} \langle \mathcal{O}^{\mu_1\cdots\mu_{s_\mathcal{O}}}\rangle_\beta, \tag{7.80}$$

where $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}}$ is the OPE coefficient, τ and $s_\mathcal{O}$ is the twist and spin of $\mathcal{O}$, respectively. Using (7.49) together with (7.80), the two-point function on $S^1 \times \mathbf{R}^3$ can be organized in the following way [82]:

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \sum_{\mathcal{O}_{\tau,s} \in \mathcal{O}_\Delta \times \mathcal{O}_\Delta} \frac{a_{\mathcal{O}_{\tau,s}}}{\beta^{\Delta_\mathcal{O}}} \frac{1}{|x|^{2\Delta-\tau+s}} C_s^{(1)}\left(\frac{x_E^0}{|x|}\right), \tag{7.81}$$

---

[36] Here and below we assume that $\Delta > 4$. We further drop the disconnected term $\langle \mathcal{O}_\Delta\rangle_\beta^2 \sim N^2$.

[37] These terms will be proportional to $\beta^{-2a}\tilde{g}(x_E^0, |\mathbf{x}|)^{\Delta-a}$, with $a \geq 4$. When decomposed into thermal blocks, these will not affect the operators with dimension $\Delta < 8$ or $\Delta = 8$ with non-zero spin $s \neq 0$.

where we sum over primary operators $\mathcal{O}_{\tau,s}$, with twist τ and spin s, appearing in the OPE $\mathcal{O}_\Delta \times \mathcal{O}_\Delta \sim \mathcal{O}_{\tau,s} + \ldots$. In (7.81) $C_s^{(1)}(x_E^0/|x|)$ is a Gegenbauer polynomial which, together with a factor of $|x|^{-2\Delta+\tau-s}$, forms a thermal conformal block in $d=4$ dimensions and the coefficients $a_{\mathcal{O}_{\tau,s}}$ are given by

$$a_{\mathcal{O}_{\tau,s}} = \left(\frac{1}{2}\right)^s \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{\tau,s}} b_{\mathcal{O}_{\tau,s}}. \tag{7.82}$$

Expanding (7.78) for $\beta \to \infty$ one finds:

$$\begin{aligned}\langle \mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta =& \frac{1}{|x|^{2\Delta}}\Big[1 + \frac{\pi^2\Delta}{3\beta^2}|x|^2 \\ &+ \frac{\pi^4\Delta}{90\beta^4}|x|^2(3\mathbf{x}^2(5\Delta-9) + (15\Delta-19)(x_E^0)^2) + \mathcal{O}(\beta^{-6})\Big].\end{aligned} \tag{7.83}$$

From the expansion (7.83), we can read off the coefficients $a_{\tau',s'} := \sum_{\mathcal{O}_{\tau',s'}} a_{\mathcal{O}_{\tau',s'}}$ where we sum over all operators with twist τ' and spin s':

$$\begin{aligned} a_{2,0} &= \frac{\pi^2\Delta}{3}, \\ a_{4,0} &= \frac{\pi^4\Delta(3\Delta-5)}{18}, \\ a_{2,2} &= \frac{\pi^4\Delta}{45}. \end{aligned} \tag{7.84}$$

For future reference, expanding (7.78) to $\mathcal{O}(\frac{1}{\beta^8})$ one finds

$$\begin{aligned} a_{2,4} &= \frac{2\pi^6\Delta}{945}, \\ a_{4,4} &= \frac{\pi^8\Delta(\Delta-1)}{1050}. \end{aligned} \tag{7.85}$$

Note that due to the mixing of operators with the same twist and spin, $a_{\tau,s}$ generically contains the contribution from multiple operators. In the following section we calculate the OPE coefficients and thermal one-point functions of operators which are not multi stress tensors but contribute to (7.84) and (7.85).

7.5.2. CFT data of scalar operators with dimensions two and four

We explicitly calculate the thermal one-point functions $\langle \mathcal{O}\rangle_\beta = b_{\mathcal{O}}\beta^{-\Delta_{\mathcal{O}}}$ and OPE coefficients $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}}$ for scalar operators $\mathcal{O}$ with twist $\tau' = 2$ and $\tau' = 4$

using Wick contractions. This is done to find which operators contribute to the thermal two-point function and to resolve a mixing problem.

For $\tau' = 2$ there is only one such operator, the single trace operator $\mathcal{O}_2(x) = \frac{1}{\sqrt{2}N} : Tr(\phi^2) : (x)$ given in (7.20). The OPE coefficient is found by considering the three-point correlator

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_2(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}. \tag{7.86}$$

The three-point function is calculated in Appendix C.1, in the large-N limit, and it is given by

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_2(x_3)\rangle = \frac{\sqrt{2}\Delta}{N}\frac{1}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \tag{7.87}$$

and therefore $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2} = \frac{\sqrt{2}\Delta}{N}$ to leading order in $1/N$. To calculate the thermal one-point function $\propto \langle Tr(\phi^2)\rangle_\beta$, we include self-contractions, i.e. contractions of fundamental fields within the same composite operator separated by a distance $m\beta$ along the thermal circle for $m \neq 0$ and integer. Explicitly, the one-point function of $\mathcal{O}_2$ is given by

$$\langle \mathcal{O}_2(x)\rangle_\beta = \frac{1}{\sqrt{2}N}\sum_{m\neq 0}\frac{N^2}{(m\beta)^2} = \frac{\pi^2 N}{3\sqrt{2}\beta^2}, \tag{7.88}$$

therefore,

$$b_{\mathcal{O}_2} = \frac{\pi^2 N}{3\sqrt{2}}. \tag{7.89}$$

The contribution to the thermal two-point function $a_{\mathcal{O}_2}$ is found using (7.87) and (7.89)

$$a_{2,0} = b_{\mathcal{O}_2}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2} = \frac{\pi^2\Delta}{3}. \tag{7.90}$$

This agrees with (7.84) which was obtained from the thermal two-point function.

We now continue with scalar operators of twist four. There are two such linearly independent operators appearing in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. In order to construct an orthonormal basis, consider the following single and double trace operators:

$$\begin{aligned} \mathcal{O}_4(x) &= \frac{1}{2N^2} : Tr(\phi^4) : (x), \\ \mathcal{O}_{4,\mathrm{DT}}(x) &= \frac{1}{2\sqrt{2}N^2} : Tr(\phi^2)Tr(\phi^2) : (x). \end{aligned} \tag{7.91}$$

We further construct the operator $\tilde{\mathcal{O}}_4$ that has vanishing overlap with $\mathcal{O}_{4,\mathrm{DT}}(x)$ as follows:

$$\tilde{\mathcal{O}}_4 = \mathcal{N}\Big[\mathcal{O}_4 - c_{\mathcal{O}_4\mathcal{O}_{4,\mathrm{DT}}}\mathcal{O}_{4,\mathrm{DT}}\Big], \tag{7.92}$$

with $\mathcal{N}$ a normalization constant and $c_{\mathcal{O}_4\mathcal{O}_{4,\mathrm{DT}}}$ is the overlap defined by

$$\langle \mathcal{O}_4(x)\mathcal{O}_{4,\mathrm{DT}}(y)\rangle = \frac{c_{\mathcal{O}_4\mathcal{O}_{4,\mathrm{DT}}}}{|x-y|^8}. \tag{7.93}$$

Explicit calculation gives $c_{\mathcal{O}_4\mathcal{O}_{4,\mathrm{DT}}} = \frac{2\sqrt{2}}{N}$ and $\mathcal{N} = \frac{1}{\sqrt{2}}$ in the large-N limit, and the scalar dimension four operator orthogonal to the double trace operator $\mathcal{O}_{4,\mathrm{DT}}$ is therefore

$$\tilde{\mathcal{O}}_4 = \frac{1}{\sqrt{2}}\Big[\mathcal{O}_4 - \frac{2\sqrt{2}}{N}\mathcal{O}_{4,\mathrm{DT}}\Big]. \tag{7.94}$$

Note that even though the second term in (7.92) is suppressed by $1/N$, it can still contribute to the thermal two-point function due to the scaling of OPE coefficients and one-point function of a k-trace operator $\mathcal{O}^{(k)}$:

$$\begin{aligned} b_{\mathcal{O}^{(k)}} &\sim N^k, \\ \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}^{(k)}} &\sim \frac{1}{N^k}, \end{aligned} \tag{7.95}$$

in the limit $N \to \infty$.

The one-point function and the OPE coefficient for $\mathcal{O}_4$ is found analogously to that of $\mathcal{O}_2$ in the large-N limit

$$\begin{aligned} b_{\mathcal{O}_4} &= \frac{\pi^4 N}{9}, \\ \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_4} &= \frac{2\Delta}{N}. \end{aligned} \tag{7.96}$$

Consider now the double trace operator given in (7.91). The one-point function factorizes in the large-N limit:

$$\begin{aligned} \langle \mathcal{O}_{4,\mathrm{DT}}(x)\rangle_\beta &= \frac{1}{\sqrt{2}}(\langle \mathcal{O}_2(x)\rangle_\beta)^2 \\ &= \frac{\pi^4 N^2}{18\sqrt{2}\beta^4}. \end{aligned} \tag{7.97}$$

Likewise, the OPE coefficient can be computed in the large-N limit (see Appendix C.1)

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{4,\mathrm{DT}}} = \frac{\sqrt{2}\Delta(3\Delta-5)}{N^2}. \tag{7.98}$$

Consider now the thermal one-point function of $\tilde{\mathcal{O}}_4$ in (7.94)

$$\begin{aligned} \langle \tilde{\mathcal{O}}_4 \rangle_\beta =& \frac{1}{\sqrt{2}\beta^4}\left[b_{\mathcal{O}_4} - \frac{2\sqrt{2}}{N} b_{\mathcal{O}_{4,\mathrm{DT}}} \right] \\ =& \mathcal{O}(N^{-1}), \end{aligned} \tag{7.99}$$

where we have used (7.96) and (7.97). Since the corresponding OPE coefficient is suppressed by N^{-1}, it follows that the only scalar operator with dimension four contributing to the thermal two-point function is the double trace operator $\mathcal{O}_{4,\mathrm{DT}}$. From the OPE coefficient and thermal one-point function of this double trace operator, using (7.97) and (7.98), we find the following contribution to the thermal two-point function

$$a_{4,0} = \frac{\pi^4 \Delta (3\Delta - 5)}{18}, \tag{7.100}$$

which agrees with (7.84).

7.5.3. CFT data of single-trace operator with twist two and spin four

The primary single trace operator $\Xi = \mathcal{O}_{2,4}$ with twist $\tau = 2$ and spin $s = 4$ is given by

$$\begin{aligned} \Xi_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{35}N} :Tr\big(&\phi(\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16(\partial_{(\mu}\phi)(\partial_\nu\partial_\rho\partial_{\sigma)}\phi) \\ &+ 18(\partial_{(\mu}\partial_\nu\phi)(\partial_\rho\partial_{\sigma)}\phi) - (\text{traces})\big) : (x). \end{aligned} \tag{7.101}$$

The relative coefficients follow from requiring that the operator is a primary, see Appendix C.5 for details.

The thermal one-point function of this operator is found from Wick contractions in the large-N limit to be

$$\langle \Xi_{\mu\nu\rho\sigma} \rangle_\beta = \frac{8\pi^6 N}{27\sqrt{35}\beta^6} \left(e_\mu e_\nu e_\rho e_\sigma - (\text{traces}) \right). \tag{7.102}$$

Moreover, the OPE coefficient in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE can again be calculated using Wick contractions similarly to how it was done for $T^2_{4,4}$ in Appendix C.1. By explicit calculation one finds

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\Xi_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{4\Delta}{\sqrt{35}N} \frac{Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})}{|x_{12}|^{2\Delta-2}|x_{13}|^2|x_{23}|^2}, \tag{7.103}$$

and therefore the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}}$ is given by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}} = \frac{4\Delta}{\sqrt{35}N}. \tag{7.104}$$

Now, it is easy to check that

$$\frac{1}{2^4}\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_{2,4}} b_{\mathcal{O}_{2,4}} = \frac{2\pi^6\Delta}{945}, \tag{7.105}$$

which agrees with $a_{2,4}$ in (7.85).

7.5.4. CFT data of double-trace operators with twist and spin equal to four

To find the full contribution to the thermal two-point function from the operators with $\tau = 4$ and $s = 4$ we need to take into account the contribution of all operators with these quantum numbers and solve a mixing problem. In addition to the double-stress tensor operator with these quantum numbers, the other double trace primary operator which contributes is given by

$$\begin{aligned}\mathcal{O}^{\mathrm{DT}}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70}N^2} &: Tr(\phi^2)\Big(Tr(\phi\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16Tr(\partial_{(\mu}\phi\partial_\nu\partial_\rho\partial_{\sigma)}\phi) \\ &+18Tr(\partial_{(\mu}\partial_\nu\phi\partial_\rho\partial_{\sigma)}\phi)(x) - (\mathrm{traces})\Big) : (x),\end{aligned} \tag{7.106}$$

where the operator is unit-normalized. Notice that this is the double trace operator obtained by taking the normal ordered product of two single trace operators, the scalar operator with dimension 2 and the single trace spin-4 operator with dimension 6. There are more double trace operators with these quantum numbers which are, however, not simply products of single trace operators. These do not contribute to the thermal two-point function to leading order in $\frac{1}{N^2}$ (see Appendix C.6).

Note that it follows from large-N factorization that the overlap of this operator with $(T^2)_{\mu\nu\rho\sigma}$ is suppressed by powers of $\frac{1}{N}$; since both of these are double trace operators and obey the scaling (7.95), to study the thermal two-point function to leading order in N^2, one can therefore neglect this overlap.

The thermal one-point function of $\mathcal{O}^{\mathrm{DT}}_{\mu\nu\rho\sigma}$ follows from the large-N factorization and we find that

$$b_{\mathcal{O}^{\mathrm{DT}}_{4,4}} = \sqrt{\frac{2}{35}}\frac{4\pi^8N^2}{81}, \tag{7.107}$$

where we used the thermal one-point functions for each single trace operator given by (7.88) and (7.102). The OPE coefficient is calculated in Appendix C.1,

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}^{\mathrm{DT}}_{4,4}} = \sqrt{\frac{2}{35}} \frac{4\Delta(\Delta-1)}{N^2}. \tag{7.108}$$

Using the thermal one point function and the OPE coefficient in (7.107) and (7.108) respectively, it is found that it the operator $\mathcal{O}^{\mathrm{DT}}_{\mu\nu\rho\sigma}$ gives the following contribution to the thermal two point function:

$$a_{\mathcal{O}^{\mathrm{DT}}_{4,4}} = \left(\frac{1}{2}\right)^4 b_{\mathcal{O}^{\mathrm{DT}}_{4,4}} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}^{\mathrm{DT}}_{4,4}} = \frac{2\pi^8 \Delta(\Delta-1)}{2835}. \tag{7.109}$$

The total contribution from $T^2_{4,4}$ together with that of $\mathcal{O}^{\mathrm{DT}}_{4,4}$, using (7.28), (7.53) and (7.109), is

$$a_{4,4} = (a_{T^2_{4,4}} + a_{\mathcal{O}^{\mathrm{DT}}_{4,4}}) = \frac{\pi^8 \Delta(\Delta-1)}{1050}. \tag{7.110}$$

This agrees with $a_{4,4}$ in (7.85).

7.6. Comparison with the eigenstate thermalization hypothesis

In this section we discuss the relation of our results to the eigenstate thermalization hypothesis (ETH). We argue that the stress tensor sector of the free $SU(N)$ adjoint scalar theory in $d = 4$ satisfies the ETH to leading order in $C_T \sim N^2 \gg 1$. We explain the equivalence of the micro-canonical and canonical ensemble when $\Delta_H \sim C_T$ in large-C_T theories. In this regime, the diagonal part of the ETH is (up to exponentially suppressed terms which we do not consider), equivalent to thermalization. Note that in two dimensions the Virasoro descendants of the identity satisfy the ETH (see e.g. [165] for a recent discussion).

We begin by showing the equivalence between the micro-canonical and the canonical ensemble on $S^1_\beta \times S^{d-1}$ when $\Delta_H \sim C_T \gg 1$. See [29,148] for a similar discussion at infinite volume as well as [116] in the two-dimensional case. The expectation value in the micro-canonical ensemble of an operator $\mathcal{O}$, which we take to be a scalar for simplicity, at energy $E = \Delta_H/R$ is given by

$$\langle \mathcal{O} \rangle^{(\mathrm{micro})}_E = \frac{1}{N(E)} \sum_{\tilde{\mathcal{O}}} \langle \tilde{\mathcal{O}} | \mathcal{O} | \tilde{\mathcal{O}} \rangle, \tag{7.111}$$

where we sum over states $|\tilde{\mathcal{O}}\rangle$ with energy $(E, E+\delta E)$ and $N(E)$ is the number of states in this interval. On the other hand, consider the partition function at inverse temperature β given by

$$Z(\beta) = \sum_{\tilde{\mathcal{O}}} e^{-\frac{\beta\tilde{\Delta}}{R}} = \int d\tilde{\Delta}\rho(\tilde{\Delta})e^{-\frac{\beta\tilde{\Delta}}{R}}, \tag{7.112}$$

where we sum over all states in the theory. In the second line in (7.112) we have approximated the sum of delta-functions by a continuous function $\rho(\tilde{\Delta})$. Expectation values in the canonical ensemble is then computed by[38]

$$\langle\mathcal{O}\rangle_\beta = Z(\beta)^{-1}\int d\tilde{\Delta}\rho(\tilde{\Delta})\langle\mathcal{O}\rangle_E^{(\text{micro})}e^{-\frac{\beta\tilde{\Delta}}{R}}. \tag{7.113}$$

Consider the partition function in (7.112) with a free energy $F = -\beta^{-1}\log Z(\beta)$. By an inverse Laplace transform of (7.112) we find the density of states

$$\rho(\Delta_H) = \frac{1}{2\pi i R}\int d\beta' e^{\beta'(\frac{\Delta_H}{R} - F(\beta'))}. \tag{7.114}$$

For $\Delta_H \sim C_T$ and a large free energy[39] $F \sim C_T$, we can evaluate (7.114) using a saddlepoint approximation with the saddle at β given by

$$\frac{\Delta_H}{R} = \partial_{\beta'}(\beta' F)|_\beta. \tag{7.115}$$

Consider now the thermal expectation value in (7.113), multiplying both sides by $Z(\beta)$ and doing an inverse Laplace transform evaluated at $\Delta_H \sim C_T$ we find

$$\rho(\Delta_H)\langle\mathcal{O}\rangle_{\Delta_H/R}^{(\text{micro})} = \frac{1}{2\pi i R}\int d\beta' \langle\mathcal{O}\rangle_{\beta'} e^{\beta'(\frac{\Delta_H}{R} - F(\beta'))}. \tag{7.116}$$

For $F \sim C_T \gg 1$ we again use a saddlepoint approximation to evaluate (7.116) with the saddle at β determined by (7.115), assuming $\langle\mathcal{O}\rangle_{\beta'}$ does not grow exponentially with C_T. The RHS of (7.116) is therefore the thermal expectation value $\langle\mathcal{O}\rangle_\beta$ multiplied by the saddlepoint approximation of the density of states in (7.114). It then follows that

$$\langle\mathcal{O}\rangle_{\Delta_H/R}^{(\text{micro})} \approx \langle\mathcal{O}\rangle_\beta, \tag{7.117}$$

[38] It was argued in [148] that the existence of the thermodynamic limit implies that we only need to sum over operators with low spin.

[39] We consider a CFT in a high temperature phase.

with β determined by (7.115). In particular, in the infinite volume limit $R \to \infty$, the free energy is given by[40]

$$F = \frac{b_{T^{(\text{can})}_{\mu\nu}} S_d R^{d-1}}{d\beta^d}, \tag{7.118}$$

where $S_d = Vol(S^{d-1}) = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$. Inserting (7.118) in (7.115) we find [29]

$$\frac{\beta}{R} = \left(\frac{-(d-1) b_{T^{(\text{can})}_{\mu\nu}} S_d}{d\Delta_H}\right)^{\frac{1}{d}}. \tag{7.119}$$

We can use (7.117) to see the thermalization of the stress tensor. The free energy is related to the expectation value of the stress tensor $T^{(\text{can})}_{\mu\nu}$ [25]

$$\langle T^{(\text{can})}_{00}\rangle_\beta = \frac{1}{S_d R^{d-1}} \partial_\beta(-\beta F(\beta)). \tag{7.120}$$

On the other hand, the expectation value of $T^{(\text{can})}_{00}$ in a heavy state $|\mathcal{O}_H\rangle$ is fixed by the Ward identity to be

$$\langle \mathcal{O}_H | T^{(\text{can})}_{00} | \mathcal{O}_H\rangle = -\frac{\Delta_H}{S_d R^d}. \tag{7.121}$$

Multiplying (7.115) with $(S_d R^{d-1})^{-1}$ and comparing with (7.120)-(7.121) we find that

$$\langle \mathcal{O}_H | T^{(\text{can})}_{00} | \mathcal{O}_H\rangle = \langle T^{(\text{can})}_{00}\rangle_\beta. \tag{7.122}$$

This shows the thermalization of the stress tensor in heavy states where $F \sim \Delta_H \sim C_T$ in large-C_T theories. Note that this follows from (7.117) since we can replace the micro-canonical expectation value at $E = \Delta_H/R$, on the LHS, with the expectation value in any single heavy state with dimension Δ_H due to the Ward identity, independent of the heavy state. Put differently, the stress tensor satisfies the ETH as we will review below.

We now consider the eigenstate thermalization hypothesis for CFTs at finite temperature on the sphere S^{d-1} of radius R. The diagonal part of the ETH is given by

$$\langle \mathcal{O}_H | \mathcal{O}_{\tau,s} | \mathcal{O}_H\rangle = \langle \mathcal{O}_{\tau,s}\rangle^{(\text{micro})}_E + \mathcal{O}\left(e^{-S(E)}\right), \tag{7.123}$$

[40] Here we denote the canonically normalized stress tensor by $T^{(\text{can})}_{\mu\nu}$, whose two-point function is given by $\langle T^{(\text{can})\mu\nu}(x) T^{(\text{can})}_{\rho\sigma}(y)\rangle = \frac{C_T}{S_d^2}(I^\mu{}_{(\rho} I^\nu{}_{\sigma)} - (\text{trace}))$.

where $\mathcal{O}_H$ and $\mathcal{O}_{\tau,s}$ are local primary operators and $\langle \mathcal{O}_{\tau,s} \rangle_E^{(\text{micro})}$ is the expectation value of $\mathcal{O}_{\tau,s}$ in the micro-canonical ensemble at energy $E = \frac{\Delta_H}{R}$. Here we assume that the operator $\mathcal{O}_H$ is a heavy scalar operator with large conformal dimension $\Delta_H \propto C_T \gg 1$. The operator $\mathcal{O}_{\tau,s}$ on the other hand can have non-zero spin.[41]. In (7.123), $e^{S(E)}$ is the density of states at energy $E = \Delta_H/R$. As shown in (7.117), in the limit $\Delta_H \sim C_T \gg 1$, the micro-canonical ensemble is equivalent to the canonical ensemble at inverse temperature β determined by (7.115). It then follows from (7.123) that the diagonal part of the ETH can written in terms of OPE coefficients and thermal one-point functions:

$$\frac{\lambda_{\mathcal{O}_H \mathcal{O}_H \mathcal{O}_{\tau,s}}}{R^{\tau+s}} = \frac{b_{\mathcal{O}_{\tau,s}} f_{\mathcal{O}_{\tau,s}}(\beta/R)}{\beta^{\tau+s}} + \mathcal{O}\left(e^{-S(E)}\right), \tag{7.124}$$

where $f_{\mathcal{O}_{\tau,s}}$ also appears in (7.11). This is equivalent to the statement of thermalization discussed in the rest of the section.

In this section we observed that the multi stress tensor operators satisfy (7.124). One can also ask if (7.124) holds for any operator in the specific heavy state we considered. By comparing eqs. (7.87) and (7.88) using (7.50), one can check that operator $\mathcal{O}_2 = \frac{1}{\sqrt{2}N} : Tr(\phi^2) :$ does not satisfy (7.124). Since this is a free theory, it is not a surprise that the ETH is not satisfied by all operators in the spectrum which is seen explicitly in this case.

7.7. Discussion

In this section we argued that multi stress tensor operators $T^k_{\tau,s}$ in CFTs with a large central charge C_T thermalize: their expectation values in heavy states are the same as their thermal expectation values. This is equivalent to the statement that multi stress tensor operators in higher-dimensional CFTs satisfy the diagonal part of the ETH in the thermodynamic limit. The analogous statement in the $d = 2$ case is that the Virasoro descendants of the identity satisfy the ETH condition in the large-C_T limit.

We observed that the operator $\mathcal{O}_2 = \frac{1}{\sqrt{2}N} : Tr(\phi^2) :$ does not satisfy the ETH. This is seen by comparing eqs. (7.87) and (7.88) using (7.50). While this operator does not thermalize in the heavy states we considered, the OPE coefficient averaged over all operators with $\Delta_H \sim C_T$ is expected to be proportional

[41] The tensor structure in (7.123) is suppressed.

to the thermal one-point function. The averaged OPE coefficients should therefore scale like $\sim \sqrt{\Delta_H}$ compared to $\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_2} \sim \Delta_H/\sqrt{C_T}$ for the heavy states we considered. It would be interesting to exhibit heavy operators that produce the former scaling.

We provided a bootstrap argument in favor of the thermalization of multi stress tensor operators. One should be able to refine it to give an explicit form for leading behavior of the multi stress tensor OPE coefficients – we leave it for future work. The holographic/bootstrap OPE coefficients for the leading twist double stress tensor operators can be found in e.g. [124] – they are nontrivial functions of the spin. As explained in [124,13], the leading Δ behavior of the minimal-twist double- and triple-stress tensor OPE coefficients is consistent with the exponentiation of the near lightcone stress tensor conformal block. One can go beyond the leading twist multi stress tensors. In holographic HHLL correlators each term of the type $(\Delta\mu)^k \sim (\Delta\Delta_H/C_T)^k$ comes from the exponentiation of the stress-tensor block – this follows from the Wilson line calculation of the correlator in the AdS-Schwarzschild background [193,124,186].

In this section we argue that this behavior is universal, and is not just confined to holographic theories. Hence, one can formulate another statement equivalent to the thermalization of multi stress tensor operators. Namely, scalar correlators of pairwise identical operators of dimensions $\Delta_{1,2}$ in large-C_T theories in the limit $\Delta_{1,2} \gg 1$, $\Delta_1\Delta_2/C_T$ fixed are given by the exponentials of the stress tensor conformal block[42]. This is similar to what happens in two-dimensional CFTs.

Note that the universality of the OPE coefficients is naively in tension with the results of [54], where finite gap (Δ_{gap}) corrections to the multi stress tensor OPE coefficients were considered. In particular, for double stress tensors, such corrections behave like $\Delta^3/\Delta_{\text{gap}}$ which is clearly at odds with the universality statement. Of course, the results of [54] are obtained in the limit $\Delta \ll \Delta_{\text{gap}}$, while in this section we consider the opposite regime $\Delta \gg \Delta_{\text{gap}}$.

One may also wonder what happens with the universality of the OPE coefficients beyond leading order in Δ. In particular, in [186], it was shown that

[42] See [105] for previous work on the eikonalization of the multi stress tensor OPE coefficients at large spin.

the bootstrap result for the HHLL correlator exactly matches the holographic Wilson line calculation (in the double scaling limit where only the minimal twist multi stress tensor operators contribute). This corresponds to including terms beyond the exponential of the stress tensor block – one needs to compute the HHLL correlator, take a logarithm of the result, divide by Δ, and then take the large-Δ limit. The result is sensitive to terms subleading in the large-Δ limit of the multi stress tensor OPE coefficients. In four spacetime dimensions the result in [186] is given by an elliptic integral – is it applicable beyond holography?

In [124] terms subleading in Δ were shown to be important for the computation of the phase shift. The simplest nontrivial case in two spacetime dimensions is the operator Λ_4 which is a level four Virasoro descendant of the identity (see e.g.[194]). One could also get it by using the CFT normal ordering and imposing the quasi-primary condition [195]. Consider now the case of minimal twist (twist four) operators in four dimensions. How do we determine the analog of Λ_4? There is no Virasoro algebra now.

Presumably, one can reconstruct the analog of Λ_4 in four spacetime dimensions by considering a CFT normal ordered product of stress tensors, and adding a single trace term to ensure that the resulting operator is a primary and is orthogonal to the stress tensor itself. Note that the CFT normal ordering differs from the oscillator normal ordering in a free theory by the addition of a single trace operator, as reviewed in Appendix C.7. This procedure can then be generalized to other multi-trace operators. We leave it for future work.

It is also helpful to imagine what happens in a theory like $\mathcal{N} = 4$ Super Yang-Mills, where there is a marginal line connecting the weak and the strong coupling (the latter admits a holographic description). Presumably, as the coupling is turned on, only one operator remains light (with dimension eight and spin four), while others get anomalous dimensions. It would be interesting to see this explicitly even to the leading nontrivial order in the 't Hooft coupling. It would also be interesting to see how the corresponding OPE coefficient interpolates between its free and strong coupling values.

Using crossing symmetry, we argued that the universality of multi stress tensor OPE coefficients is related to the OPE coefficients $\lambda_{\mathcal{O}_H T_{\mu\nu} \mathcal{O}'}$, with $\mathcal{O}' \neq \mathcal{O}_H$ being either heavy or light, present in the cross-channel expansion. Such OPE coefficients with at least one operator being heavy were recently studied in

[154,196]. It would be interesting to further study the connection of our results to this work.

Another interesting question concerns the fate of the double trace operators of the schematic form $[\mathcal{O}_\Delta \mathcal{O}_\Delta]_{n,l}$. Consider the $d=4$ case in the large volume limit and $n,l=0$, for simplicity. We expect that the corresponding OPE coefficients in the free theory behave like $\lambda_{\mathcal{O}_H \mathcal{O}_H [\mathcal{O}_\Delta \mathcal{O}_\Delta]_{0,0}} \propto \Delta_H^2/C_T \propto C_T \mu^2$,[43] while their thermal one-point functions behave like $\langle [\mathcal{O}_\Delta \mathcal{O}_\Delta]_{0,0} \rangle_\beta \propto C_T \beta^{-2\Delta}$. Comparing the two results with the help of (7.50) one observes that such operators do not thermalize in the free theory for generic Δ. The situation is more nontrivial in holography where we do not know the large μ behavior of the OPE coefficients[44]. As pointed out in [15], the contribution of double-trace operators to thermal two-point functions is different from that of multi stress tensors. The latter is only sensitive to the behavior of the metric near the boundary, but the former knows about the full black hole metric. This seems to indicate that the thermalization of the double trace operators in holographic theories is also unlikely[45].

It is a natural question how generic are the heavy states for which the stress tensor sector thermalizes. The our results seem to suggest that such thermalization is more generic than the thermalization of other light operators[46]. Other interesting questions include generalizations to the case of finite but large central charge and to non-conformal quantum field theories.

[43] This scaling is obtained by computing the OPE coefficient $\lambda_{\mathcal{O}_H \mathcal{O}_H [\mathcal{O}_\Delta \mathcal{O}_\Delta]_{0,0}}$ for $1 \ll \Delta_H \ll C_T$ and extrapolating it to the $\Delta_H \sim C_T$ regime.

[44] Note that the large-N scaling in holography is different. Both the OPE coefficients and the thermal expectation values behave like C_T^0 as opposed to $C_T \sim N^2$.

[45] A simple way to decouple such operators is to take the large-Δ limit.

[46] A closely related question of finding "typical" states where the stress tensor sector thermalizes in the large volume limit in $d=2$ was recently discussed in [179]. There it was observed that such states are Virasoro descendants when the central charge is finite.

8. Discussion and conclusions

We started this research to learn more about the whole class of holographic conformal field theories and their implications for the dual gravity. The conformal bootstrap in the Regge and lightcone limit allowed us to gain a better insight into the four-point correlation functions at higher orders of the inverse central charge. We specifically focus on the stress tensor sector of the conformal field theory that consists of the stress tensor itself and the multi-trace primary operators made from the stress tensor. The importance of this sector lies in the fact that it is necessarily present in all holographic CFTs that have a dual theory with dynamical gravity. The gravitational analog of multi stress tensor operators that contribute to the correlators in CFTs are the graviton loops in the corresponding Witten diagrams. In the four-point correlation functions where two operators $\mathcal{O}_H$ have the large conformal dimension Δ_H, the contributions of the stress tensor sector get enhanced by factors of Δ_H compared to the contributions of other, generic, multi-trace operators, and therefore decouple from them. That is the reason why the heavy-heavy-light-light correlation function is the perfect setup for studying the stress tensor sector contributions.

We confirm that the OPE coefficients of the multi stress tensor operators with a minimal twist at each order in μ are universal, i.e. they are the same in all holographic CFTs, as claimed in [15]. Introducing the finite gap in the theory breaks this universality, as shown in [54]. One can still ask is it possible to add up all contributions of the minimal twist subset of the stress tensor sector and obtain the correlator exact in μ. As the contributions of these operators dominate in the lightcone limit, one could hope to obtain the exact non-perturbative correlator in this limit by adding up all of them. This result would be the higher-dimensional analog of the Virasoro vacuum block in two-dimensional CFT. So far, it has not been established how to write this non-perturbative correlator and we leave this for future work. The general idea is that there exists some Virasoro-like symmetry algebra generated by the components of the stress tensor that emerges near the lightcone limit and protects the OPE coefficients of the minimal twist multi stress tensors in the holographic CFTs [127,142,197-199]. If we had the closed algebra of this type, computing the contributions of its irreducible representations to the correlator would greatly simplify the problem.

The important property of the OPE coefficients of multi-trace operators in the stress tensor sector with external scalars is that they have poles at the integer values of Δ_L. These poles are the consequence of the mixing of multi stress tensors with double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l}$ when Δ_L are integers. It would be interesting to find a way to solve this mixing problem and obtain finite contributions for integer Δ_L. According to the statement in [15], to calculate the OPE coefficients of these double trace operators using the dual gravity approach, one would have to analytically solve the equation of motion of the light probe in the black hole background all the way to the horizon, therefore, the perturbative $1/r$ calculation that works for the stress tensor sector is not applicable here. One comes to a similar conclusion when trying to compute the OPE coefficients of $[\mathcal{O}_L\mathcal{O}_L]_{n,l}$ operators using the conformal bootstrap technique. Namely, in this case one would have to solve the bootstrap equation non-perturbatively in large spin, for which the non-perturbative results for the anomalous dimensions of the operators in the S-channel are needed. Therefore, we leave this task for future work as well.

The multi stress tensor operators whose OPE coefficients are not fixed by the conformal bootstrap are those with spin $s = 0, 2$. As they are not fixed by the consistency conditions of the underlying CFT these OPE coefficients can be viewed as the parameters in the class of holographic CFTs. They depend on the other parameters of the holographic CFT, besides Δ_L and μ, which is the reason we say they are not universal. They introduce these parameters in the correlation function and then the bootstrap forces the other OPE coefficients of multi stress tensor operators with the non-minimal twist and higher spin to depend on them. The universality of the OPE coefficients of the minimal twist multi stress tensors at all orders in μ can be explained by the fact that there are no multi stress tensor operators with a minimal twist and spin $s = 0, 2$ that would introduce additional parameters in the OPE coefficients of the minimal twist subset of the stress tensor sector.

One could ask if there is some other consistency condition in the CFT that could fix or bound the coefficients of multi stress tensors with spin $s = 0, 2$, for example, obtained from the bootstrap of the higher-point correlation functions or correlation functions of spinning operators. We leave these questions for future research. We showed how one can fix the OPE coefficients of spin-2 multi stress tensor operators using the wave function phase shift calculation in the gravity dual. In our case, this is done for Gauss-Bonnet gravity dual and the results obtained match with those from [15] whenever available in the latter. The only way, known so far, to fix the OPE coefficients of the spin zero multi stress tensors through the gravity dual is the one developed in [15].

One obvious question to ask is which part of the formalism developed for the stress tensor sector would apply to the case when we have additional conserved currents in the holographic CFT, for example, $U(1)$ current J_μ. In this case, one would have to consider the contributions of the stress tensor sector together with the contributions of the conserved current sector and multi-trace operators created from both single-traces $T_{\mu\nu}$ and J_μ. We plan to tackle this problem in the near future.

Finally, by studying the thermalization properties of the stress tensor sector, we demonstrated that it satisfies the diagonal part of the eigenstate thermalization hypothesis in the thermodynamic limit of large-N CFTs. This justifies using the pure heavy state $\mathcal{O}_H$ as the CFT analog of the black hole when we are interested in the graviton contributions to the correlation functions in holographic CFTs. We also showed that the other operators generically present in the large-N CFTs do not thermalize in this sense. Here, one could again ask what happens in the large-N theory with additional conserved charges, whether the conserved current sector thermalizes the same way the stress tensor sector does. Additionally, it would be interesting to check what happens with the off-diagonal elements of the eigenstate thermalization hypothesis for both the stress tensor sector and the conserved current sector.

Appendix A.1. Details on the conformal bootstrap

Below we review some of the details of the confomal bootstrap calculations. Explicitly, we will show that exchanges of heavy-light double-trace operators in the S-channel reproduce the disconnected correlator at $\mathcal{O}(\mu^0)$ and the stress tensor exchange at $\mathcal{O}(\mu)$.

A.1.1. Solving the crossing equation to $\mathcal{O}(\mu)$ in $d = 4$

We start with the leading $\mathcal{O}(\mu^0)$ term in the S-channel that should reproduce the disconnected propagator in the T-channel. This is given in $d = 4$ by

$$G(z,\bar{z})|_{\mu^0} = \frac{C_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L - 2}(h-\bar{h})(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}). \quad \text{(A.1.1)}$$

Let us look at the following piece of (A.1.1):

$$\begin{aligned}
&-\int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h}) z^{\bar{h}} \bar{z}^{h+1} = \\
&-\int_0^\infty d\bar{h} \int_{\bar{h}}^\infty dh (h\bar{h})^{\Delta_L-2}(h-\bar{h}) z^{\bar{h}} \bar{z}^{h+1} = \\
&\frac{\bar{z}}{z} \int_0^\infty dh \int_h^\infty d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h}) z^{h+1} \bar{z}^{\bar{h}} \, .
\end{aligned} \quad \text{(A.1.2)}$$

Setting $\bar{z}/z = 1$ to leading order in the Regge limit, we find that the S-channel expression reproduces the disconnected correlator:

$$\begin{aligned}
G(z,\bar{z})|_{\mu^0} &= \frac{zC_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^\infty d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h}) z^h \bar{z}^{\bar{h}} \\
&= \frac{zC_{\Delta_L}}{z-\bar{z}} \frac{(\log\bar{z} - \log z)}{(\log z \log\bar{z})^{\Delta_L}} \Gamma(\Delta_L)\Gamma(\Delta_L - 1) \simeq \frac{1}{(1-z)^{\Delta_L}(1-\bar{z})^{\Delta_L}} \, .
\end{aligned} \quad \text{(A.1.3)}$$

Notice that to arrive in the last equality we expanded $(z, \bar{z})$ around unity and substituted $C_{\Delta_L} = (\Gamma(\Delta_L)\Gamma(\Delta_L - 1))^{-1}$.

Consider now the imaginary part at $\mathcal{O}(\mu)$ in the S-channel. For convenience we define

$$I^{(d=4)} \equiv \mathrm{Im}(G(z,\bar{z}))|_\mu \, , \quad \text{(A.1.4)}$$

which is then equal to:

$$\begin{aligned}
I^{(d=4)} &= \frac{-i\pi C_{\Delta_L}}{\sigma(e^{-\rho} - e^\rho)} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h}) \gamma^{(1)}_{\bar{h},h-\bar{h}} \\
&\times \Big((1-\sigma e^\rho)^{h+1}(1-\sigma e^{-\rho})^{\bar{h}} - (h \leftrightarrow \bar{h}) \Big) \, .
\end{aligned} \quad \text{(A.1.5)}$$

Notice that we used the variables (σ,ρ) defined as $z=1-\sigma e^{\rho}$ and $\bar{z}=1-\sigma e^{-\rho}$.

Consider the following ansatz for $\gamma^{(1)}_{\bar{h},h-\bar{h}}=\frac{ch^a\bar{h}^b}{h-\bar{h}}$, where (a,b,c) are numbers to be determined by the crossing equation. Substituting into (A.1.5) and collecting the leading singularity σ^{-k} as $\sigma\to 0$ with $k=2\Delta_L+a+b-1$ leads to

$$
\begin{aligned}
I^{(d=4)}|_{\sigma^{-k}} =& \frac{-ic\pi C_{\Delta_L}}{(e^{-\rho}-e^{\rho})}\Big(\Gamma(\Delta_L+a-1)\Gamma(\Delta_L+b-1)(e^{(b-a)\rho}-e^{(a-b)\rho})+\\
&+\frac{\Gamma(2\Delta_L+a+b-2)}{(\Delta_L+a-1)e^{(2\Delta_L+a+b-2)\rho}}\times\\
&{}_2F_1(\Delta_L+a-1,2\Delta_L+a+b-2,\Delta_L+a,-e^{-2\rho})\\
&-\frac{\Gamma(2\Delta_L+a+b-2)}{(\Delta_L+a-1)e^{-(2\Delta_L+a+b-2)\rho}}\times\\
&{}_2F_1(\Delta_L+a-1,2\Delta_L+a+b-2,\Delta_L+a,-e^{2\rho})\Big).
\end{aligned}
\tag{A.1.6}
$$

Note that in order to do these integrals we need $\Delta_L+a>1$ and $\Delta_L+b>1$. Using the following identity of the hypergeometric function

$$
\begin{aligned}
{}_2F_1(a,b,c,x) =& \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}{}_2F_1(a,a-c+1,a-b+1,\frac{1}{x})\\
&+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}{}_2F_1(b,b-c+1,-a+b+1,\frac{1}{x}),
\end{aligned}
\tag{A.1.7}
$$

the third line in (A.1.6) can be simplified and we are left with

$$
\begin{aligned}
I^{(d=4)}|_{\sigma^{-k}} =& \frac{ic\pi C_{\Delta_L}}{(e^{2\rho}-1)}\Big(-\Gamma(\Delta_L+a-1)\Gamma(\Delta_L+b-1)e^{(a-b+1)\rho}\\
&+\frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+a-1}e^{-(2\Delta_L+a+b-3)\rho}\times\\
&{}_2F_1(\Delta_L+a-1,2\Delta_L+a+b-2,\Delta_L+a,-e^{-2\rho})\\
&+\frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+b-1}e^{-(2\Delta_L+a+b-3)\rho}\times\\
&{}_2F_1(\Delta_L+b-1,2\Delta_L+a+b-2,\Delta_L+b,-e^{-2\rho})\Big).
\end{aligned}
\tag{A.1.8}
$$

On the other hand, the Regge limit in the T-channel is dominated by operators of maximal spin. In a holographic CFT, we have $J=2$. If we further take the lightcone limit, $\rho\gg 1$, the dominant contribution is due to the stress tensor exchange and behaves as $\sigma^{-1}e^{-(d-1)\rho}$. To reproduce this behavior from the S-channel, we must set $a = 0$ and $b = 2$ and make an appropriate choice for the overall constant c. Substituting the designated values of (a, b, c) revals that the first term in (A.1.8) precisely matches the T-channel stress tensor contribution, which in the Regge limit (after analytic continuation) behaves like:

$$g_{\Delta,J} \propto \frac{1}{\sigma^{J-1}} \frac{e^{-(\Delta-3)\rho}}{(e^{2\rho}-1)} + \dots, \tag{A.1.9}$$

with $\Delta = d$ and $J = 2$. Furthermore, the remaining terms correspond to the exchange of operators with spin 2 and dimension $2\Delta_L + 2 + 2n$; these are the double-trace operators $[\mathcal{O}_L \mathcal{O}_L]_{n,l=2}$.

A.1.2. Integrating the S-channel result at $\mathcal{O}(\mu^2)$ in $d = 4$

Below we describe how to use the results for the anomalous dimensions at $\mathcal{O}(\mu^2)$ in order to recover the imaginary part of the correlator to the same order. Using the obtained expressions for the anomalous dimensions (4.8) and (4.24), we note that the integrand in (4.11) can be written as

$$\begin{aligned} &P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} \left(\gamma^{(2)}_{\bar{h},h-\bar{h}} - \frac{\gamma^{(1)}_{\bar{h},h-\bar{h}}}{2} (\partial_h + \partial_{\bar{h}}) \gamma^{(1)}_{\bar{h},h-\bar{h}} \right) \\ &= -\frac{35\bar{h}^3(2h-\bar{h})}{4(h-\bar{h})^3} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} = -\frac{35 h^{\Delta_L-3} \bar{h}^{\Delta_L+1}}{2\Gamma(\Delta_L-1)\Gamma(\Delta_L)} \sum_{n=0}^{\infty} \left(\frac{\bar{h}}{h}\right)^n (1+\frac{n}{2}). \end{aligned} \tag{A.1.10}$$

Therefore we see that (4.11) can be written as an infinite sum of integrals of the same form that appeared at $\mathcal{O}(\mu)$ in (A.1.5). It then follows that the full S-channel result can be integrated in order to obtain the correlator in position space. Especially, the lightcone result is obtained by setting $k = 0$ in (A.1.10) and taking $\rho \to \infty$ which gives

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = \frac{i35\pi\Delta_L(\Delta_L+1)}{2(\Delta_L-2)} \frac{e^{-3\rho}}{\sigma^{2\Delta_L+1}(e^{2\rho}-1)} + \dots, \tag{A.1.11}$$

with $\dots$ denoting terms that are subleading in the lightcone limit. The result (A.1.11) has a form consistent with the contribution of an operator with spin-2 and $\Delta = 6$. The full result (beyond the lightcone limit) further contains an infinite number of operators with spin-2 of dimension $\Delta = 6 + 2n$ and $\Delta = 2\Delta_L + 2n + 2$.

A.1.3. Solving the crossing equation to $\mathcal{O}(\mu)$ in $d=2$

Here we review the calculations needed for the $d=2$ case explained in Appendix D. To $\mathcal{O}(\mu^0)$ the S-channel (3.37) is given by

$$G(z,\bar{z})|_{\mu^0} = \frac{1}{\Gamma(\Delta_L)^2}\int_0^\infty \int_0^h d\bar{h}(h\bar{h})^{\Delta_L-1}(z^h\bar{z}^{\bar{h}} + (z\leftrightarrow\bar{z})). \qquad \text{(A.1.12)}$$

The integrand in (A.1.12) is symmetric w.r.t. $h\leftrightarrow\bar{h}$ and can thus be rewritten as

$$G(z,\bar{z})|_{\mu^0} = \frac{1}{\Gamma(\Delta_L)^2}\int_0^\infty \int_0^\infty d\bar{h}(h\bar{h})^{\Delta_L-1}z^h\bar{z}^{\bar{h}}, \qquad \text{(A.1.13)}$$

which can easily be seen to reproduce the disconnected correlator $[(1-z)(1-\bar{z})]^{-\Delta_L}$ in the Regge limit.

As in the previous subsection we proceed to consider the imaginary part of the correlator in the S-channel expansion to $\mathcal{O}(\mu)$. Using a similar notation,

$$I^{(d=2)} \equiv \mathrm{Im}(G(z,\bar{z}))|_{\mu}\,, \qquad \text{(A.1.14)}$$

combined with the ansatz $\gamma^{(1)}_{\bar{h},h-\bar{h}} = c\, h^a\bar{h}^b$, allows us to write:

$$I^{(d=2)} = -\frac{ic\,\pi}{\Gamma(\Delta_L)^2}\int_0^\infty \int_0^h d\bar{h}(h\bar{h})^{\Delta_L-1}h^a\bar{h}^b(z^h\bar{z}^{\bar{h}} + (z\leftrightarrow\bar{z})). \qquad \text{(A.1.15)}$$

The integrals in (A.1.15) can be easily performed given that $a+\Delta_L>0$ and $b+\Delta_L>0$. Changing variables to $z=1-\sigma e^{\rho}$, $\bar{z}=1-\sigma e^{-\rho}$ and collecting the most singular term σ^{-k}, with $k=2\Delta_L+a+b$, leads to

$$\begin{aligned} I^{(d=2)}|_{\sigma^{-k}} &= \frac{ic\pi}{\Gamma(\Delta_L)^2}\Big(\Gamma(a+\Delta_L)\Gamma(b+\Delta_L)(-e^{\rho(b-a)}-e^{\rho(a-b)}) \\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{-\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L,a+b+2\Delta_L,1+a+\Delta_L,-e^{-2\rho}) \\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L,a+b+2\Delta_L,1+a+\Delta_L,-e^{2\rho})\Big). \end{aligned} \qquad \text{(A.1.16)}$$

Using again (A.1.7) we express (A.1.16) as follows

$$\begin{aligned} I^{(d=2)}|_{\sigma^{-k}} &= \frac{ic\pi}{\Gamma(\Delta_L)^2}\Big(-\Gamma(a+\Delta_L)\Gamma(b+\Delta_L)e^{\rho(a-b)} \\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{-\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L,a+b+2\Delta_L,1+a+\Delta_L,-e^{-2\rho}) \\ &- \frac{\Gamma(a+b+2\Delta_L)e^{-(a+b+2\Delta_L)\rho}}{b+\Delta_L}{}_2F_1(b+\Delta_L,a+b+2\Delta_L,1+b+\Delta_L,-e^{-2\rho})\Big). \end{aligned} \qquad \text{(A.1.17)}$$

In matching (A.1.17) with the T-channel expansion, following the same logic as in the previous subsection we deduce that $a = 0$ and $b = 1$ and fix c. The first line in (A.1.17) then reproduces the exchange of the stress tensor in the T-channel. The other two lines match the contribution of double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ with dimension $\Delta = 2\Delta_L + 2n + 2$ and spin 2 in the T-channel expansion.

Appendix A.2. Details on the impact parameter representation in $d = 4$

Here we will see how the impact parameter representation in four dimensions leads to the expression for the disconnected correlator in the Regge limit, in terms of the integral over $h, \bar{h}$.

The objective of this section is to explicitly see that the disconnected contribution of the correlator in the Regge limit

$$\frac{1}{((1-z)(1-\bar{z}))^{\Delta}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)} \int_0^{\infty} dh \int_0^{h} d\bar{h} (h\bar{h})^{\Delta-2} \frac{h-\bar{h}}{z-\bar{z}} \left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right), \tag{A.2.1}$$

can be equivalently written as

$$\int_0^{\infty} dh \int_0^{h} d\bar{h}\, \mathcal{I}_{h,\bar{h}}\,, \tag{A.2.2}$$

with

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta) \int_{M^+} \frac{d^4p}{(2\pi)^4} (-p^2)^{\Delta-2} e^{-ipx} (h-\bar{h}) \delta(p\cdot\bar{e} + h + \bar{h})\, \delta\left(\frac{p^2}{4} + h\bar{h}\right). \tag{A.2.3}$$

where M^+ is the upper Milne wedge with $\{p^2 \leq 0,\ p^0 \geq 0\}$ and

$$C(\Delta) \equiv \frac{2^{d+1-2\Delta}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta)\Gamma(\Delta-\frac{d}{2}+1)}\,, \tag{A.2.4}$$

with d the dimensionality of the spacetime, here $d = 4$.

In practice, we need to perform the integral over p in (A.2.3). To do so, we will use spherical polar coordinates and write:

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3} \int_{-\infty}^{\infty} dp^0 \int_0^{\infty} dp^r\, (p^r)^2 \int_{-1}^{1} d(\cos\theta)\; (-p^2)^{\Delta-2}\, \theta(p^0)\theta(-p^2)\times$$
$$e^{ip^0x^0}\, e^{-irp^r\cos\theta} \left(\delta\left(\frac{p^0+p^r}{2}-h\right)\delta\left(\frac{p^0-p^r}{2}-\bar{h}\right)+h\leftrightarrow\bar{h}\right). \tag{A.2.5}$$

The overall factor of (2π) is simply the result of the integration with respect to the angular variable ϕ. Next we perform the integral over $\cos\theta$:

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3} \int_{-\infty}^{\infty} dp^0 \int_0^{\infty} dp^r\, (p^r)^2\, (-p^2)^{\Delta-2}\, e^{ip^0x^0} \left(\frac{e^{-irp^r}-e^{irp^r}}{-irp^r}\right)\times$$
$$\theta(p^0)\theta(-p^2)\,(\delta\,\delta), \tag{A.2.6}$$

where we set

$$(\delta\,\delta) \equiv \delta\left(\frac{p^0+p^r}{2}-h\right)\delta\left(\frac{p^0-p^r}{2}-\bar{h}\right)+(h\leftrightarrow\bar{h}). \tag{A.2.7}$$

Notice that

$$\int_0^{\infty} dp^r\, \frac{p^r}{ir}\,(-p^2)^{\Delta-2}\, e^{irp^r}\,(\delta\,\delta) - \int_0^{\infty} dp^r\, \frac{p^r}{ir}\,(-p^2)^{\Delta-2}\, e^{-irp^r}\,(\delta\,\delta) =$$
$$= \int_{-\infty}^{\infty} dp^r\, \frac{p^r}{ir}\,(-p^2)^{\Delta-2}\, e^{irp^r}\,(\delta\,\delta)\,. \tag{A.2.8}$$

Hence we can write (A.2.6) as follows

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp^+\,dp^-}{2}\, \frac{p^+-p^-}{i(x^+-x^-)}\,(-p^2)^{\Delta-2}\, e^{\frac{i}{2}(p^+x^-+p^-x^+)}\times$$
$$\theta(p^+)\theta(p^-)\,(\delta\,\delta)\,. \tag{A.2.9}$$

Performing the last two integrals is trivial due to the delta-functions. The result is

$$\mathcal{I}_{h,\bar{h}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)}\, \frac{h-\bar{h}}{i(x^+-x^-)}\,(h\bar{h})^{\Delta-2}\,(e^{ihx^+}e^{i\bar{h}x^-}-e^{i\bar{h}x^+}e^{ihx^-})\,, \tag{A.2.10}$$

which allows us to write (A.2.2) as follows:

$$\int_0^{\infty} dh \int_0^{h} d\bar{h}\,\mathcal{I}_{h,\bar{h}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)} \int_0^{\infty} dh \int_0^{h} d\bar{h} \frac{h-\bar{h}}{i(x^+-x^-)}\,(h\bar{h})^{\Delta-2}\times$$
$$(z^h\bar{z}^{\bar{h}}-z^{\bar{h}}\bar{z}^h)\,. \tag{A.2.11}$$

Here we also used the identification ($z = e^{ix^+}, \bar{z} = e^{ix^-}$).

Observe that (A.2.11) is equal to (A.2.1) in the Regge limit, where

$$\frac{z}{z-\bar{z}} \simeq \frac{1}{i(x^+ - x^-)}, \quad \frac{\bar{z}}{z-\bar{z}} \simeq \frac{1}{i(x^+ - x^-)}. \tag{A.2.12}$$

However, when considering next order corrections in (x^+, x^-) the impact parameter represention may require corrections. Below we show that these are irrelevant for the questions we are interested in.

A.2.1. Exact Fourier transform

Here we will compute the Fourier transform for the S-channel expression with the identification ($z = e^{ix^+}, \bar{z} = e^{ix^-}$) and show that the leading order results in the Regge limit given in the previous section do not miss any important contributions.

The generic term in the S-channel which we would like to Fourier transform looks like:

$$\int dh\, d\bar{h}\, g(x^+, x^-)\tilde{f}(h, \bar{h})\,, \tag{A.2.13}$$

where

$$g(x^+, x^-) = \frac{e^{i(1+h)x^+} e^{i\bar{h}x^-} - e^{i\bar{h}x^+} e^{i(h+1)x^-}}{(e^{ix^+} - e^{ix^-})}\,, \tag{A.2.14}$$

and

$$\tilde{f}(h, \bar{h}) = i\pi(h\bar{h})^{\Delta-2}(h - \bar{h})f(h, \bar{h})\,, \tag{A.2.15}$$

where $f(h, \bar{h})$ stands for all the contributions in the S-channel to a given order.

The Fourier transform is:

$$\int d^4x\, e^{ipx} \int dh\, d\bar{h}\, g(x^+, x^-)\tilde{f}(h, \bar{h}) = \int dh\, d\bar{h} \tilde{f}(h, \bar{h}) \int d^4x\, e^{ipx} g(x^+, x^-)\,, \tag{A.2.16}$$

where we simply reversed the order of integration. Our focus in what follows will be the integral:

$$I \equiv \int d^4x\, e^{ipx} g(x^+, x^-)\,. \tag{A.2.17}$$

Since $x^+ = t + r$ and $x^- = t - r$, it is convenient to use spherical polar coordinates to perform the integration. The angular integration over ϕ gives us an overall factor of (2π) as the integrand is independent of ϕ. Next we perform the integration over the other angular variable. Similar to what was discussed in the previous section,

$$\int_{-1}^{1} d(\cos\theta)\, e^{ip^r r\cos\theta} = \frac{e^{irp^r} - e^{-irp^r}}{irp^r}\,. \tag{A.2.18}$$

Combining the above we can write:

$$I = 2\pi \int_{-\infty}^{\infty} dt e^{-itp^t} \int_{0}^{\infty} drr\, \frac{e^{irp^r} - e^{-irp^r}}{ip^r}\, g(t,r)\,. \tag{A.2.19}$$

It is easy to see that $g(t,r) = g(t,-r)$ and as a result:

$$\int_{0}^{\infty} dr\, re^{-irp^r}\, g(t,r) = -\int_{-\infty}^{0} dr\, re^{irp^r}\, g(t,r)\,, \tag{A.2.20}$$

which allows us to write the integral as:

$$I = 2\pi \int_{-\infty}^{\infty} \frac{dx^+dx^-}{2} e^{ip\cdot x} \frac{x^+ - x^-}{i(p^+ - p^-)} g(x^+, x^-)\,. \tag{A.2.21}$$

Here $e^{ip\cdot x} = e^{-\frac{i}{2}(p^+x^- + p^-x^+)}$ and the above integral can be thought of as a two-dimensional Fourier transform.

To proceed we need the explicit form of $g(x^+, x^-)$ which we write as

$$g(x^+, x^-) = \frac{e^{ihx^+} e^{i\bar{h}x^-}}{1 - e^{-i(x^+ - x^-)}} + (x^+ \leftrightarrow x^-) \tag{A.2.22}$$

and then expand the denominator in the Regge limit

$$\frac{1}{1 - e^{-i(x^+ - x^-)}} = \frac{1}{i(x^+ - x^-)} \left(1 - \frac{i}{2}(x^+ - x^-) + \cdots\right)\,. \tag{A.2.23}$$

Substituting into (A.2.21) leads to:

$$I = 2\pi \frac{1}{(-p^+ + p^-)} \int \frac{dx^+dx^-}{2} e^{ip\cdot x} \Bigg(e^{ihx^+} e^{i\bar{h}x^-} \left(1 - \frac{i}{2}(x^+ - x^-) + \cdots\right) + (x^+ \leftrightarrow x^-)\Bigg)\,. \tag{A.2.24}$$

Let us compute the integral term by term. The leading term in the Regge limit yields the standard delta functions:

$$
\begin{aligned}
I_0 &= 2^2\pi^3 \,\frac{1}{p^- - p^+}\delta(\frac{p^+}{2} - \bar{h})\delta(\frac{p^-}{2} - h) + (p^+ \leftrightarrow p^-) = \\
&= 2\pi^3 \,\frac{1}{h-\bar{h}} \left\{\delta(\frac{p^+}{2} - \bar{h})\delta(\frac{p^-}{2} - h) + (p^+ \leftrightarrow p^-)\right\} = \\
&= 2\pi^3 \,\frac{1}{h-\bar{h}}\delta(p\cdot\bar{e} + h + \bar{h})\delta(\frac{p^2}{4} + h\bar{h})\,.
\end{aligned}
\tag{A.2.25}
$$

The subleading terms on the other hand produce the same result except that the delta functions are replaced with derivatives of themselves with respect to $p^r = \frac{p^+ - p^-}{2}$.

Let us now consider the full result which up to an overall numerical coefficient can be written as:

$$
\int dh\, d\bar{h}\, \tilde{f}(h,\bar{h}) \left(1 - \frac{\partial}{\partial p^r} + \cdots\right) \delta(p\cdot\bar{e} + h + \bar{h})\delta(\frac{p^2}{4} + h\bar{h})\,. \tag{A.2.26}
$$

To evaluate the terms with derivatives of the delta function we need to integrate by parts. Now recall that we are interested in the imaginary piece of the S-channel whose leading behaviour is $\sim \sqrt{-p^2}$ (this dependence is hidden in what we called $\tilde{f}$). It is obvious that the derivatives will produce subleading terms which we are not interested in.

What about the other pieces in the S-channel which are not imaginary? To $\mathcal{O}(\mu^2)$ in this case, we know that the leading behaviour grows like $\sim (\sqrt{-p^2})^2$, so by differentiation, a term of the order $\sqrt{-p^2}$ may be produced. However, it is clear that this term will never contribute to the *imaginary* term of the S-channel (note that the coefficient in the first term in the parenthesis in (A.2.26) is real). We thus deduce that the subleading terms in (A.2.24) are irrelevant for our study.

Appendix A.3. Impact parameter representation in general spacetime dimension d

Here we want to prove the following equation for general spacetime dimension d:

$$\mathcal{I}_{h,\bar{h}} = (z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(HL,HL);\text{MFT}}_{\bar{h},h-\bar{h}} g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}), \tag{A.3.1}$$

using the form of conformal blocks given in (4.31). We start with the definition of $\mathcal{I}_{h,\bar{h}}$ that is given as:

$$\mathcal{I}_{h,\bar{h}} = C(\Delta_L) \int_{M^+} \frac{d^d p}{(2\pi)^d} (-p^2)^{\Delta_L - \frac{d}{2}} e^{-ipx} (h-\bar{h}) \delta(p\cdot\bar{e} + h + \bar{h}) \delta(\frac{p^2}{4} + h\bar{h}), \tag{A.3.2}$$

where:

$$C(\Delta_L) \equiv \frac{2^{d+1-2\Delta_L} \pi^{1+\frac{d}{2}}}{\Gamma(\Delta_L)\Gamma(\Delta_L - \frac{d}{2} + 1)} \,. \tag{A.3.3}$$

Using spherical coordinates we write (A.3.2) as:

$$\begin{aligned}\mathcal{I}_{h,\bar{h}} = C(\Delta_L) &\int_{-\infty}^{\infty} dp^t e^{ip^t t} \int_0^{\infty} dp^r (p^r)^{d-2} \int_{S_{d-2}} \sin^{d-3}\phi_1 d\phi_1 \, d\Omega_{d-3} \\ &\times e^{-ip^r r\cos\phi_1} (-p^2)^{\Delta_L - \frac{d}{2}} \theta(-p^2)\theta(p^t) \Big\{ \delta\left(\frac{p^t + p^r}{2} - h\right) \delta\left(\frac{p^t - p^r}{2} - \bar{h}\right) \\ &+ (h \leftrightarrow \bar{h}) \Big\} \,,\end{aligned} \tag{A.3.4}$$

where $\Omega_{d-3} = \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d-2}{2})}$ denotes the area of the unit $(d-3)$-dimensional hypersphere.

Notice now that

$$\int_0^{\pi} \sin^{d-3}\phi_1 e^{-ip^r r\cos\phi_1} d\phi_1 = \sqrt{\pi}\Gamma(\frac{d}{2} - 1) {}_0F_1(\frac{d-1}{2}; -\frac{1}{4}(p^r)^2 r^2) \,. \tag{A.3.5}$$

Substituting (A.3.5) back in to (A.3.4), one is left with integrals with respect to p^t and p^r only. These integrals are trivial due to the presence of delta functions.[47] When these integrals are calculated, the expression for $\mathcal{I}_{h,\bar{h}}$ is given as:

$$\begin{aligned}\mathcal{I}_{h,\bar{h}} = \frac{2^{3-d}\sqrt{\pi}}{\Gamma(\Delta_L)\Gamma(\Delta_L - \frac{d}{2} + 1)} & e^{it(h+\bar{h})} (h-\bar{h})^{d-2} (h\bar{h})^{\Delta_L - \frac{d}{2}} \times \\ & {}_0F_{1R}(\frac{d-1}{2}; -\frac{1}{4}(h-\bar{h})^2 r^2),\end{aligned} \tag{A.3.6}$$

[47] One only needs to remember that $h \geq \bar{h} \geq 0$.

where ${}_0F_{1R}(a,x) = \Gamma(a)^{-1}{}_0F_1(a,x)$. Relations between coordinates t and r with x^+ and x^- are given as: $x^+ = t + r$ and $x^- = t - r$.

On the other hand, using the explicit form for conformal blocks (4.31) and OPE coefficients in the Regge limit (3.40) one finds that:

$$(z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h-\bar{h}} g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}) = \\ = \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(\Delta_L)\Gamma(\Delta_L-\frac{d}{2}+1)}(h\bar{h})^{\Delta_L+\frac{d}{2}}(h-\bar{h})(z\bar{z})^{\frac{h+\bar{h}}{2}} C^{(\frac{d}{2}-1)}_{h-\bar{h}}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big). \tag{A.3.7}$$

Using the relations between coordinates r, t and $z, \bar{z}$ it is easy to see that $(z\bar{z})^{\frac{h+\bar{h}}{2}} = e^{it(h+\bar{h})}$. Next, one can use the relation between Gegenbauer polynomials and hypergeometric functions:

$$C^{(\alpha)}_n(z) = \frac{(2\alpha)_n}{n!}{}_2F_1(-n, 2\alpha+n, \alpha+\frac{1}{2}; \frac{1-z}{2}), \tag{A.3.8}$$

which for $h - \bar{h} = l \gg 1$ gives:

$$C^{(\frac{d}{2}-1)}_l\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big) = \frac{l^{d-3}}{\Gamma(d-2)}{}_2F_1(-l, l+d-2, \frac{d-1}{2}; \frac{1}{2}-\frac{1}{2}(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}})). \tag{A.3.9}$$

With the help of the following properties of hypergeometric functions:

$$ {}_2F_1(a,b,c;z) = (1-z)^{-b}{}_2F_1(c-a,b,c;\frac{z}{z-1}), \\ \lim_{m,n\to\infty}{}_2F_1(m,n,b;\frac{z}{mn}) = {}_0F_1(b;z). \tag{A.3.10}$$

Using these, together with the assumption that in the Regge limit the values of x^+l and x^-l are fixed constants: $x^+l = a_1$ and $x^-l = a_2$ while $l \to \infty$, one can easily see[48] that (A.3.6) reproduces (A.3.1). This confirms the validity of the impact parameter representation.

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[48] By noting that:

$$\Gamma(x-\frac{1}{2}) = 2^{2-2x}\sqrt{\pi}\frac{\Gamma(2x-1)}{\Gamma(x)}. \tag{A.3.11}$$

Appendix A.4. Anomalous dimensions of heavy-light double-trace operators in $d = 2$

The OPE data of the heavy-light double trace operators in $d = 2$ dimensions can be directly obtained from the heavy-light Virasoro vacuum block [40,105]. For completeness, in this appendix we investigate the anomalous dimension of $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ in $d = 2$ following the discussion in Section 4.2. As in $d = 4$, we introduce an impact parameter representation following [55]. We calculate the anomalous dimensions to $\mathcal{O}(\mu)$ by solving the crossing equation and then use the impact parameter representation to relate them to the bulk phase shift. We find a precise agreement between the two. Using the bulk phase shift we furthermore determine the anomalous dimension to second order in μ. Much of the discussion follows closely the four-dimensional case and will be briefer.

A.4.1. Anomalous dimensions in the Regge limit using bootstrap

The conformal blocks in two dimension are given by [30,24]

$$g_{\Delta,J}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) = k_{\Delta+J}(z)k_{\Delta-J}(\bar{z}) + (z \leftrightarrow \bar{z}), \qquad \text{(A.4.1)}$$

where $k_\beta(z)$ was defined in (4.3). Similar to the four dimensional case, the blocks for heavy-light double-trace operators simplify in the heavy limit ($\Delta_H \sim C_T$)

$$g_{[\mathcal{O}_H\mathcal{O}_L]_{h,\bar{h}}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = (z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L)}\left(z^h\bar{z}^{\bar{h}} + (z \leftrightarrow \bar{z})\right). \qquad \text{(A.4.2)}$$

Inserting this form of the conformal blocks in (3.37) together with the OPE coefficients in the Regge limit (3.40) and approximating the sums with integrals, one can due to symmetry extend the region of integration and it is easily found that the disconnected correlator in the T-channel is reproduced.

Similar to the four-dimensional case the stress tensor dominates at order μ in the T-channel. The block of the stress tensor after analytic continuation in the Regge limit is given by

$$g_{T_{\mu\nu}} = \frac{24i\pi e^{-\rho}}{\sigma} + \dots, \qquad \text{(A.4.3)}$$

where $\dots$ denote non-singular terms. As in the four-dimensional case, this has to be reproduced in the S-channel by the term in (3.37) proportional to $-i\pi\gamma^{(1)}$.

With the conformal blocks (A.4.2), the imaginary part in the S-channel to $\mathcal{O}(\mu)$ is given by

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu} = -i\pi C_{\Delta_L} \int_0^{\infty} dh \int_0^{h} d\bar{h} (h\bar{h})^{\Delta_L - 1} \gamma^{(1)}_{\bar{h},h-\bar{h}} \left(z^h \bar{z}^{\bar{h}} + z^{\bar{h}} \bar{z}^{h} \right). \tag{A.4.4}$$

Using the ansatz $\gamma^{(1)}_{\bar{h},h-\bar{h}} = c_1 h^a \bar{h}^b$ we find that the T-channel contribution is reproduced for $a = 0$ and $b = 1$ (see Appendix A.2 for details). We thus find using (3.41)

$$\gamma^{(1)}_{\bar{h},h-\bar{h}} = -\frac{6\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}} \lambda_{\mathcal{O}_L\mathcal{O}_L T_{\mu\nu}}}{\mu \Delta_L} \bar{h} = -\bar{h}. \tag{A.4.5}$$

To $\mathcal{O}(\mu^2)$ we can use (4.11) to find the following contribution to the purely imaginary terms in the S-channel

$$\begin{aligned} \mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = -i\pi C_{\Delta_L} \int_0^{\infty} dh \int_0^{h} d\bar{h} (h\bar{h})^{\Delta_L - 1} \left(\gamma^{(2)}_{\bar{h},h-\bar{h}} - \frac{c_1^2 \bar{h}}{2} \right) \times \\ (z^h \bar{z}^{\bar{h}} + z^{\bar{h}} \bar{z}^{h}). \end{aligned} \tag{A.4.6}$$

A.4.2. 2d impact parameter representation and relation to bulk phase shift

Similar to the four-dimensional case we introduce an impact parameter representation in order to relate the anomalous dimension with the bulk phase shift. The impact parameter representation in $d = 2$ is given by

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta_L) \int_{M^+} d^2p (-p^2)^{\Delta - 1} e^{-ipx} (h - \bar{h}) \delta(p \cdot \bar{e} + h + \bar{h}) \, \delta\left(\frac{p^2}{4} + h\bar{h} \right), \tag{A.4.7}$$

with straightforward generalization of the $d = 4$ case explained above. This is again chosen such that when the impact parameter represetation is integrated over $h, \bar{h}$ the disconnected correlator is reproduced:

$$\int_0^{\infty} dh \int_0^{h} \mathcal{I}_{h,\bar{h}} = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}}. \tag{A.4.8}$$

The discussion of the phase shift is completely analogous to the four-dimensional case, as in (4.21) we find the following relation between the bulk phase shift and the anomalous dimension to second order in μ

$$\begin{aligned} \gamma^{(1)}_{\bar{h},h-\bar{h}} &= -\frac{\delta^{(1)}}{\pi} \\ \tilde{\gamma}^{(2)}_{\bar{h},h-\bar{h}} - \frac{c_1^2 p^-}{4} &= -\frac{\delta^{(2)}}{\pi}. \end{aligned} \tag{A.4.9}$$

In [55] the phase shift in $d = 2$ was found to be

$$\begin{aligned}\delta^{(1)} &= \frac{\pi}{2}\sqrt{-p^2}e^{-L} \\ \delta^{(2)} &= \frac{3\pi}{8}\sqrt{-p^2}e^{-L}.\end{aligned} \tag{A.4.10}$$

Using the identification $p^+ = 2h$ and $p^- = 2\bar{h}$ together with (4.23) we find for the anomalous dimension in the Regge limit

$$\begin{aligned}\gamma^{(1)}_{\bar{h},h-\bar{h}} &= -\bar{h} \\ \gamma^{(2)}_{\bar{h},h-\bar{h}} &= -\frac{1}{4}\bar{h}.\end{aligned} \tag{A.4.11}$$

We thus see that the first order result agrees with that obtained from bootstrap (A.4.5). Furthermore, the second order correction agrees also in $d = 2$ with the result (6.40) in [55].

Appendix A.5. Discussion of the boundary term integrals

There are a few integrals containing total derivative terms that we have ignored throughout this section and we analyze more carefully here. Let us start with a total derivative term which shows up in the real part of the correlator at $\mathcal{O}(\mu)$. It is given by[49]:

$$I_1 = \frac{1}{2}(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \int_0^{+\infty} dl \Big[P^{(HL,HL);\text{MFT}}_{n,l} \gamma^{(1)}_{n,l} g^{\Delta_{HL},-\Delta_{HL}}_{n+l,n}(z,\bar{z})\Big]^{n\to\infty}_{n=0}. \tag{A.5.1}$$

Let us focus on the integrand: $\Big[P^{(HL,HL);\text{MFT}}_{n,l} \gamma^{(1)}_{n,l} g^{\Delta_{HL},-\Delta_{HL}}_{n+l,n}(z,\bar{z})\Big]^{n\to\infty}_{n=0}$. When $n = 0$, the expression within the brackets trivially vanishes. On the other hand, when $n \to \infty$, it takes the form $n^{2\Delta_L-2}(z\bar{z})^n \times f(l)$, where f is some function of l only. We are instructed here to take the limit $n \to \infty$ independently of all other limits (recall that the Regge limit is taken after the integration). For generic values $0 < (z, \bar{z}) < 1$ it is clear that $\lim_{n\to\infty} \Big[P^{(HL,HL);\text{MFT}}_{n,l} \gamma^{(1)}_{n,l} g^{\Delta_{HL},-\Delta_{HL}}_{n+l,n}(z,\bar{z})\Big] = \lim_{n\to\infty} n^{2\Delta_L-2}(z\bar{z})^n \times f(l) \to$

[49] We are again using variables n and l, one can notice that $n = \bar{h}$ and $l = h - \bar{h}$. It is trivial to prove that $\partial_n = \partial_h + \partial_{\bar{h}}$.

0. In other words, the expression $\left[P_{n,l}^{(HL,HL);\text{MFT}}\gamma_{n,l}^{(1)}g_{n+l,n}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z})\right]_{n=0}^{n\to\infty} \to$ 0, and we conclude that the integral (A.5.1) does not contribute to the S-channel expansion of the correlator.

There are a few more integrals of similar kind that appear at $\mathcal{O}(\mu^2)$. We will analyse one of them here:

$$I_2 = \frac{-i\pi}{2}(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty} dl\left[P_{n,l}^{(HL,HL);\text{MFT}}(\gamma_{n,l}^{(1)})^2 g_{n+l,n}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z})\right]_{n=0}^{n\to\infty}. \tag{A.5.2}$$

The same logic can be applied here. Again, the value of the expression in brackets at $n=0$ is trivially zero, while for large n it behaves like: $n^{2\Delta_L+d-4}(z\bar{z})^n\tilde{f}(l)$. As long as $(z,\bar{z})<1$, this vanishes exponentially in the limit $n\to\infty$. One concludes therefore that the integral (A.5.2) vanishes. The same logic is valid for all other integrals of similar total derivative terms that appear at $\mathcal{O}(\mu^2)$.

Appendix A.6. An identity for the bulk phase shift.

The aim is to elaborate on the results of [55] for the bulk phase shift in a black hole background as computed in gravity. Firstly, let us note the following identity involving hypergeometric functions:

$$\sum_{n=0}^{\infty} a(n)x^n\, {}_2F_1[\tau_0+2n+1,\frac{d}{2}-1,\tau_0+2n-\frac{d}{2}+3,x] = {}_2F_1[\tau_0+1,\frac{\tau_0}{2},\frac{\tau_0}{2}+2,x]$$
$$a(n) = \frac{2^{2n}}{n!}\frac{\tau_0+2}{\tau_0+2+2n}\frac{(\frac{\tau_0}{2}+1-\frac{d}{2})_n\left(\frac{\tau_0+1}{2}\right)_n}{(\tau_0+n+2-\frac{d}{2})_n}, \quad \tau_0\neq 0\,. \tag{A.6.1}$$

Given that both sides of the equality can be expressed as an infinite series expansion around $x=0$, one simply needs to show that the expansion coefficients match to all orders in x. This is proven in Appendix G.

Consider now the case $\tau_0 = k(d-2)$ where $k\in N^\star$. Setting $x\equiv e^{-2L}$ and multiplying both sides with $e^{-[k(d-2)+1]L}$ yields:

$$\Pi_{k(d-2)+1,k(d-2)+1}(L) = \sum_{n=0}^{\infty}\beta_n\Pi_{k(d-2)+2n+1,d-1}(L)$$
$$\beta(n) \equiv \pi^{\frac{(1-k)(d-2)}{2}}\frac{a(n)}{(k(d-2)+1)_n}\frac{\Gamma\left[k(d-2)-\frac{d}{2}+2n+3\right]}{\Gamma\left[\frac{k(d-2)}{2}+2\right]}\,. \tag{A.6.2}$$

The left hand side represents the hyperbolic space propagator for a scalar field of squared mass equal to $k(d-2)+1$ in a hyperbolic space of dimensionality $k(d-2)+1$ and is proportional to the k-th order expression for the bulk phase shift computed in gravity in [55], where

$$\delta^{(k)}(S,L) = \frac{1}{k!}\frac{2\Gamma\left(\frac{dk+1}{2}\right)}{\Gamma\left(\frac{k(d-2)+1}{2}\right)}\frac{\pi^{1+\frac{k(d-2)}{2}}}{\Gamma\left(\frac{k(d-2)}{2}+1\right)} S\,\Pi_{k(d-2)+1,k(d-2)+1}(L)\,. \quad \text{(A.6.3)}$$

On the other hand, the right-hand side of (A.6.2) expresses the k-th order term of the bulk phase shift as an infinite sum of $(d-1)$-dimensional hyperbolic space propagators for fields with mass-squared equal to $m^2 = k(d-2)+1+2n$.

It can be shown [132,74] that the analytically continued T-channel scalar conformal block in the Regge limit behaves like:

$$g_{\Delta,J}(\sigma,\rho) = i\,c_{\Delta,J}\frac{\Pi_{\Delta-1,d-1}(\rho)}{\sigma^{J-1}}\,, \quad \text{(A.6.4)}$$

where

$$c_{\Delta,J} = \frac{4^{\Delta+J-1}\Gamma\left(\frac{\Delta+J-1}{2}\right)\Gamma\left(\frac{\Delta+J+1}{2}\right)}{\Gamma(\frac{\Delta+J}{2})^2}\frac{2\Gamma\left(\Delta-\frac{d}{2}+1\right)}{\pi^{1-\frac{d}{2}}\Gamma\left(\Delta-1\right)}\,. \quad \text{(A.6.5)}$$

Here $\Pi_{\Delta-1,d-1}$ denotes as usual the $(d-1)$-dimensional hyperbolic space propagator for a massive scalar of mass-squared $m^2 = (\Delta-1)$.

It follows that the k-th order term in the μ-expansion of the bulk phase shift in a black hole background can be expressed as an infinite sum of conformal blocks corresponding to operators of twist $\tau = \tau_0(k)+2n = k(d-2)+2n$ and spin $J=2$ in the Regge limit. In other words, we can write:

$$\begin{aligned} i\,\delta^{(k)}(S,L) &= f(k)\sum_{n=0}^{\infty}\lambda_k(n)\; g^R_{\tau_0(k)+2n+2,2}(S,L) \\ \lambda_k(n) &= a(n)\,\frac{2^{-4n}\left[\left(\frac{\tau_0(k)+4}{2}\right)_n\right]^2}{\left(\frac{\tau_0(k)+3}{2}\right)_n\left(\frac{\tau_0(k)+5}{2}\right)_n}, \quad \tau_0(k) = k(d-2) \end{aligned} \quad \text{(A.6.6)}$$

where

$$f(k) \equiv \frac{\sqrt{\pi}}{64}\;\frac{1}{2^{k(d-2)}\;k!}\;\frac{\Gamma\left(\frac{kd+1}{2}\right)\Gamma\left(\frac{k(d-2)+4}{2}\right)}{\Gamma\left(\frac{k(d-2)+5}{2}\right)\Gamma\left(\frac{k(d-2)+3}{2}\right)}\,, \quad \text{(A.6.7)}$$

and

$$g^R_{\Delta,J}(S,L) = ic_{\Delta,J}\,S^{J-1}\,\Pi_{\Delta-1,d-1}(L)\,. \quad \text{(A.6.8)}$$

Appendix A.7. An identity for hypergeometric functions.

Here we will show that for $q \neq 0$,

$$\sum_{n=0}^{\infty} a(n)x^n\, {}_2F_1[q+2n+1, \frac{d}{2}-1, q+2n-\frac{d}{2}+3, x] = {}_2F_1[q+1, \frac{q}{2}, \frac{q}{2}+2, x]$$

$$a(n) = \frac{2^{2n}}{n!}\frac{q+2}{q+2+2n}\frac{(\frac{q}{2}+1-\frac{d}{2})_n \left(\frac{q+1}{2}\right)_n}{(q+n+2-\frac{d}{2})_n}, \quad q \neq 0\,. \tag{A.7.1}$$

Given that both sides of the equality can be expressed as an infinite series expansion around $x=0$, one simply needs to show that the expansion coefficients match to all orders in x. Let us first set:

$$\begin{aligned} b(n,m) &\equiv \frac{1}{m!}\frac{(q+1+2n)_m \left(\frac{d}{2}-1\right)_m}{\left(q-\frac{d}{2}+2n+3\right)_m} \\ c(\ell) &\equiv \frac{1}{\ell!}\frac{(q+1)_\ell \left(\frac{q}{2}\right)_\ell}{\left(\frac{q}{2}+2\right)_\ell} = \frac{(q+1)_\ell}{\ell!}\frac{q(q+2)}{(q+2\ell)(q+2\ell+2)}\,, \end{aligned} \tag{A.7.2}$$

such that:

$$\begin{aligned} {}_2F_1[q+2n+1, \frac{d}{2}-1, q+2n-\frac{d}{2}+3, x] &= \sum_{m=0}^{\infty} b(n,m)x^m, \\ {}_2F_1[q+1, \frac{q}{2}, \frac{q}{2}+2, x] &= \sum_{\ell=0}^{\infty} c(\ell)x^\ell. \end{aligned} \tag{A.7.3}$$

It is easy to check that the coefficients of the first few powers of x precisely match. Indeed, e.g.,

$$\begin{aligned} a(0)b(0,0) - c(0) &= 0 \\ a(1)b(1,0) + a(0)b(0,1) - c(1) &= 0 \\ a(2)b(2,0) + a(1)b(1,1) + a(0)b(0,2) - c(2) &= 0. \end{aligned} \tag{A.7.4}$$

To show that the above identity is true for all powers of x we must show that:

$$\sum_{k=0}^{\ell} a(k)b(k, \ell-k) = c(\ell)\,, \tag{A.7.5}$$

for all $\ell \in N$. The left-hand side of (A.7.5) can be easily summed to yield:

$$\sum_{k=0}^{\ell} a(k)b(k,\ell-k) = \frac{1}{\ell!}\frac{\Gamma[q+1+\ell]}{\Gamma[q]}\frac{(q+2)}{(q+2\ell)(2+2\ell+q)}\,, \tag{A.7.6}$$

which can be trivially shown to be equal to $c(\ell)$.

Appendix B.1. Linear relations between products of $f_a(z)$ functions

Here we list some linear relations between products of the $f_a(z)$ functions used in the main text.

$$f_1(z)f_4(z) + \frac{1}{15}f_3(z)f_4(z) - \frac{4}{63}f_2(z)f_5(z) - f_2(z)f_3(z) = 0, \tag{B.1.1}$$

$$\begin{aligned}
&\frac{308}{25}f_2^2(z) - \frac{308}{25}f_1(z)f_3(z) + \frac{5929}{375}f_3^2(z) - \frac{2673}{2500}f_4^2(z) - \frac{396}{25}f_1(z)f_5(z)\\
&+ f_2(z)f_6(z) = 0,\\
&245f_2^2(z) - 245f_1(z)f_3(z) - \frac{7}{12}f_3^2(z) - \frac{81}{80}f_4^2(z) + f_3(z)f_5(z) = 0,\\
&\frac{140}{9}f_2^2(z) - \frac{140}{9}f_1(z)f_3(z) - \frac{28}{27}f_3^2(z) + f_2(z)f_4(z) = 0,
\end{aligned} \tag{B.1.2}$$

$$\begin{aligned}
&\frac{3991680}{16000}f_2(z)f_3(z) - \frac{99}{125}f_4(z)f_3(z) + f_6(z)f_3(z) - \frac{6237}{25}f_1(z)f_4(z)\\
&- \frac{891}{875}f_4(z)f_5(z) = 0,
\end{aligned} \tag{B.1.3}$$

$$\begin{aligned}
&f_2(z)f_7(z) + \frac{7007}{500}f_2(z)f_3(z) + \frac{39611}{2500}f_4(z)f_3(z) - \frac{7007}{500}f_1(z)f_4(z)\\
&- \frac{4719}{4375}f_4(z)f_5(z) - \frac{143}{9}f_1(z)f_6(z) = 0,
\end{aligned}$$

$$\begin{aligned}
&-\frac{1}{15}f_6(z)f_2(z)^2 + \frac{297}{4375}f_4(z)^2f_2(z) + f_1(z)f_5(z)f_2(z) + \frac{44}{625}f_3(z)f_5(z)f_2(z)\\
&+ \frac{9}{143}f_1(z)f_7(z)f_2(z) - \frac{44}{625}f_3(z)^2f_4(z) - \frac{297}{4375}f_1(z)f_4(z)f_5(z)\\
&- f_1(z)f_1(z)f_6(z) = 0,
\end{aligned} \tag{B.1.4}$$

$$\begin{aligned}
&-f_6(z)f_1(z)^2 + f_3(z)f_4(z)f_1(z) - \frac{297}{4375}f_4(z)f_5(z)f_1(z) + \frac{9}{143}f_2(z)f_7(z)f_1(z)\\
&+ \frac{9}{2500}f_2(z)f_4(z)^2 - \frac{7}{1875}f_3(z)^2f_4(z) + \frac{7}{1875}f_2(z)f_3(z)f_5(z)\\
&- \frac{7}{1980}f_2(z)^2f_6(z) = 0,
\end{aligned} \tag{B.1.5}$$

$$\begin{aligned}
&-f_6(z)f_1(z)^2 + \frac{9}{143}f_2(z)f_7(z)f_1(z) - \frac{297}{4375}f_4(z)f_5(z)f_1(z) + \frac{297}{4375}f_2(z)f_4(z)^2\\
&+ f_2(z)^2f_4(z) - \frac{44}{625}f_3(z)^2f_4(z) + \frac{7}{1875}f_2(z)f_3(z)f_5(z) - \frac{7}{1980}f_2(z)^2f_6(z)\\
&+ \frac{297}{4375}f_2(z)f_4(z)^2 - \frac{7}{1980}f_2(z)^2f_6(z) = 0,
\end{aligned} \tag{B.1.6}$$

$$
\begin{aligned}
&- f_6(z)f_1(z)^2 + \frac{9}{143}f_2(z)f_7(z)f_1(z) - \frac{297}{4375}f_4(z)f_5(z)f_1(z) + f_2(z)f_3(z)^2 \\
&+ \frac{9}{2500}f_2(z)f_4(z)^2 - \frac{44}{625}f_3(z)^2f_4(z) + \frac{2647}{39375}f_2(z)f_3(z)f_5(z) \\
&- \frac{7}{1980}f_2(z)^2f_6(z) = 0,
\end{aligned} \tag{B.1.7}
$$

$$
\begin{aligned}
-f_6(z)f_2(z)^2 + \frac{891}{875}f_4(z)^2f_2(z) + \frac{132}{125}f_3(z)f_5(z)f_2(z) - \frac{132}{125}f_3(z)^2f_4(z) \\
- \frac{891}{875}f_1(z)f_4(z)f_5(z) + f_1(z)f_3(z)f_6(z) = 0,
\end{aligned} \tag{B.1.8}
$$

Appendix B.2. Coefficients in $\mathcal{G}^{(3,1)}(z)$

Here we list the coefficients in $\mathcal{G}^{(3,1)}(z)$:

$$
\begin{aligned}
&b_{116} = -\frac{\Delta_L(\Delta_L+3)(\Delta_L(\Delta_L(\Delta_L(1001\Delta_L+387)-4326)+13828)+5040)}{10378368000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)} \\
&+\frac{b_{14}(\Delta_L(143\Delta_L+427)+540)}{17160(\Delta_L-4)}, \\
&c_{118} = 7(\Delta_L+3)\times \\
&\frac{\left(604800b_{14}\left(\Delta_L^2-5\Delta_L+6\right)+\Delta_L\left(-21\Delta_L^3+229\Delta_L^2+414\Delta_L+284\right)\right)}{856627200(\Delta_L^3-9\Delta_L^2+26\Delta_L-24)}, \\
&c_{127} = \frac{\Delta_L(\Delta_L(\Delta_L(\Delta_L(\Delta_L(14\Delta_L-15)+6040)-36125)-75814)-49620)}{2306304000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)} \\
&-\frac{3b_{14}(\Delta_L(2\Delta_L+3)+135)}{11440(\Delta_L-4)}, \\
&c_{145} = \frac{3b_{14}(\Delta_L(257\Delta_L-2227)+510)}{700000(\Delta_L-4)}+\Delta_L\times \\
&\frac{(\Delta_L(\Delta_L(\Delta_L((32680-1183\Delta_L)\Delta_L-183605)+34900)+570808)+436440)}{47040000000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \\
&c_{226} = \frac{b_{14}(\Delta_L(22\Delta_L-267)+960)}{39600(\Delta_L-4)}+\Delta_L\times \\
&\frac{(\Delta_L(\Delta_L(\Delta_L((40020-1337\Delta_L)\Delta_L-274845)+96350)+2323212)+1910160)}{71850240000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \\
&c_{235} = \frac{b_{14}((10283-1153\Delta_L)\Delta_L-5790)}{900000(\Delta_L-4)} \\
&+\frac{\Delta_L\left(51463\Delta_L^5-846480\Delta_L^4+1320405\Delta_L^3\right)}{1632960000000(\Delta_L^3-9\Delta_L^2+26\Delta_L-24)} \\
&+\frac{\Delta_L\left(22381100\Delta_L^2-46886088\Delta_L-46446840\right)}{1632960000000(\Delta_L^3-9\Delta_L^2+26\Delta_L-24)}, \\
&c_{244} = \frac{9b_{14}(\Delta_L(71-11\Delta_L)+270)}{175000(\Delta_L-4)}+\Delta_L\times \\
&\frac{(\Delta_L(\Delta_L(\Delta_L(\Delta_L(1337\Delta_L-32145)+160095)+19525)-266712)-182160)}{70560000000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}, \\
&c_{334} = \frac{b_{14}(\Delta_L(11\Delta_L-71)-270)}{18750(\Delta_L-4)}+\Delta_L\times \\
&\frac{(\Delta_L(\Delta_L(\Delta_L(\Delta_L(509\Delta_L-1515)+83415)-808325)+823116)+902880)}{90720000000(\Delta_L-4)(\Delta_L-3)(\Delta_L-2)}.
\end{aligned}
\tag{B.2.1}
$$

Appendix B.3. Coefficients in $\mathcal{G}^{(3,2)}(z)$

Here we list the coefficients in $\mathcal{G}^{(3,2)}(z)$:

$$
\begin{aligned}
g_{119} &= \frac{g_{13}\left(7\Delta_L\left(128-77\Delta_L\right)+6720\right)}{16409250\left(\Delta_L-5\right)} \\
&+ \frac{49b_{14}\left(\Delta_L\left(\Delta_L\left(170-11\Delta_L\right)+981\right)+1620\right)}{16409250\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
&+ \frac{196e_{115}}{49725} + \frac{539\Delta_L^7-15386\Delta_L^6+54215\Delta_L^5+951510\Delta_L^4+2911426\Delta_L^3}{472586400000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+ \frac{98e_{15}\left(\Delta_L+4\right)}{16575\left(\Delta_L-5\right)} + \frac{3737076\Delta_L^2+1779120\Delta_L}{472586400000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
g_{128} &= -\frac{7g_{13}\left(\Delta_L\left(4\Delta_L-469\right)+930\right)}{12355200\left(\Delta_L-5\right)} \\
&- \frac{7b_{14}\left(\Delta_L\left(22\Delta_L^2-64\Delta_L+4197\right)+11745\right)}{6177600\left(\Delta_L-5\right)\left(\Delta_L-4\right)} + \\
&\frac{462\Delta_L^7-24203\Delta_L^6+1044630\Delta_L^5-3466005\Delta_L^4-24181012\Delta_L^3-39855972\Delta_L^2}{1779148800000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&- \frac{49e_{15}\left(\Delta_L\left(\Delta_L+2\right)+102\right)}{93600\left(\Delta_L-5\right)} \\
&- \frac{61201\Delta_L}{4942080000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
g_{155} &= \frac{11e_{15}\left(\Delta_L\left(278\Delta_L-2789\right)+126\right)}{2756250\left(\Delta_L-5\right)} \\
&+ \frac{11g_{13}\left(\Delta_L\left(2279\Delta_L-7400\right)-8370\right)}{231525000\left(\Delta_L-5\right)} \\
&- \frac{3146e_{115}}{275625} + \frac{b_{14}\left(12063\Delta_L^3-88048\Delta_L^2-131165\Delta_L+196110\right)}{77175000\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
&+ \frac{-244401285\Delta_L^4+853023786\Delta_L^3+2178372216\Delta_L^2+1399907880\Delta_L}{233377200000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+ \frac{-1406986\Delta_L^7+28367309\Delta_L^6-123035140\Delta_L^5}{233377200000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
g_{227} &= \frac{e_{15}\left(\Delta_L\left(52\Delta_L-751\right)+3234\right)}{93600\left(\Delta_L-5\right)} \\
&- \frac{e_{115}}{240} + \frac{g_{13}\left(\Delta_L\left(1051\Delta_L-12370\right)-52530\right)}{86486400\left(\Delta_L-5\right)} \\
&+ \frac{b_{14}\left(\Delta_L\left(\Delta_L\left(3131\Delta_L-33896\right)-62985\right)+1236870\right)}{86486400\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
&+ \frac{-213549\Delta_L^7+6031106\Delta_L^6-23990385\Delta_L^5-205647690\Delta_L^4}{87178291200000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+ \frac{853227874\Delta_L^3+2135805744\Delta_L^2+1445776920\Delta_L}{87178291200000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},
\end{aligned}
\tag{B.3.1}
$$

$$
\begin{aligned}
g_{245} = & -\frac{99e_{15}\left(\Delta_L\left(83\Delta_L-754\right)-1064\right)}{4900000\left(\Delta_L-5\right)} \\
& +\frac{g_{13}\left(73\Delta_L\left(275-274\Delta_L\right)+170060\right)}{137200000\left(\Delta_L-5\right)} \\
& +\frac{5577e_{115}}{245000}+\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(79801-14981\Delta_L\right)+410980\right)-55320\right)}{68600000\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
& +\frac{1300313\Delta_L^7-22489422\Delta_L^6+63989995\Delta_L^5+399569530\Delta_L^4}{138297600000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
& +\frac{-690996588\Delta_L^3-2276065528\Delta_L^2-1491467040\Delta_L}{138297600000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
g_{335} = & \frac{1144e_{115}}{5315625}+\frac{g_{13}\left(\Delta_L\left(6426275-894839\Delta_L\right)+685170\right)}{17860500000\left(\Delta_L-5\right)} \\
& -\frac{11e_{15}\left(\Delta_L\left(11143\Delta_L-143659\right)+451206\right)}{212625000\left(\Delta_L-5\right)} \\
& -\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(446853\Delta_L-4788638\right)+4992635\right)+44234910\right)}{5953500000\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
& +\frac{43544683\Delta_L^7-877022702\Delta_L^6+4877336920\Delta_L^5-1356232020\Delta_L^4}{9001692000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
& +\frac{-28767381333\Delta_L^3-34411007748\Delta_L^2-12217009140\Delta_L}{9001692000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \\
g_{344} = & \frac{11e_{15}\left(\Delta_L\left(278\Delta_L-2789\right)+126\right)}{2625000\left(\Delta_L-5\right)} \\
& +\frac{g_{13}\left(\Delta_L\left(17194\Delta_L-10525\right)-249570\right)}{220500000\left(\Delta_L-5\right)} \\
& -\frac{1573e_{115}}{131250}+\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(9438\Delta_L-48673\right)-325415\right)+511110\right)}{73500000\left(\Delta_L-5\right)\left(\Delta_L-4\right)} \\
& +\frac{-1593347\Delta_L^7+27045868\Delta_L^6-6670280\Delta_L^5-1193221320\Delta_L^4}{444528000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
& +\frac{1878076947\Delta_L^3+5698801932\Delta_L^2+3877115760\Delta_L}{444528000000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}.
\end{aligned}
\tag{B.3.2}
$$

$$d_{117} = -\frac{9}{220}e_{115} + \frac{84+\Delta_L(53+13\Delta_L)}{1560(\Delta_L-5)}e_{15}$$
$$+\frac{13\Delta_L\left(209\Delta_L+409\right)+8340}{7207200\left(\Delta_L-5\right)}g_{13}$$
$$-\frac{4641\Delta_L^7+22727\Delta_L^6+44901\Delta_L^5+67569\Delta_L^4+519742\Delta_L^3}{290594304000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \quad \text{(B.3.3)}$$
$$-\frac{828876\Delta_L^2+333648\Delta_L}{290594304000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}$$
$$+\frac{\Delta_L\left(\Delta_L\left(5317\Delta_L+18140\right)+68763\right)+69660}{7207200\left(\Delta_L-5\right)\left(\Delta_L-4\right)}b_{14}.$$

$$g_{236} = \frac{e_{15}\left(\left(15074-1223\Delta_L\right)\Delta_L-39816\right)}{6804000\left(\Delta_L-5\right)}$$
$$+\frac{g_{13}\left(\Delta_L\left(186926\Delta_L-1951295\right)+5891220\right)}{6286896000\left(\Delta_L-5\right)}$$
$$+\frac{143e_{115}}{340200}+\frac{b_{14}\left(\Delta_L\left(\Delta_L\left(23001\Delta_L-469741\right)+3383740\right)-7782480\right)}{1047816000\left(\Delta_L-5\right)\left(\Delta_L-4\right)}$$
$$-\frac{9324749\Delta_L^7-433851406\Delta_L^6+5233472135\Delta_L^5-21967190310\Delta_L^4}{6337191168000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}$$
$$-\frac{10644674676\Delta_L^3+72859312056\Delta_L^2+65903302080\Delta_L}{6337191168000000\left(\Delta_L-5\right)\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \quad \text{(B.3.4)}$$

Appendix B.4. OPE coefficients of twist-eight triple-stress tensors

Here we list a few OPE coefficients of twist-eight triple-stress tensors which are found using (6.52):

$$P^{(3)}_{12,4} = \frac{P^{(2)}_{8,2}\left(\Delta_L\left(143\Delta_L+427\right)+540\right)}{17160\left(\Delta_L-4\right)}$$
$$-\frac{1001\Delta_L^6+3390\Delta_L^5-3165\Delta_L^4+850\Delta_L^3+46524\Delta_L^2+15120\Delta_L}{10378368000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \quad \text{(B.4.1)}$$

$$P^{(3)}_{14,6} = \frac{9P^{(2)}_{8,2}\left(\Delta_L\left(13\Delta_L+11\right)+12\right)}{544544\left(\Delta_L-4\right)}$$
$$+\frac{7917\Delta_L^6+38174\Delta_L^5+140795\Delta_L^4+266390\Delta_L^3+253908\Delta_L^2+97776\Delta_L}{548900352000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}, \quad \text{(B.4.2)}$$

$$
\begin{aligned}
P^{(3)}_{16,8} = &\frac{5P^{(2)}_{8,2}\left(\Delta_L\left(17\Delta_L+2\right)+6\right)}{9876048\left(\Delta_L-4\right)} \\
&+\frac{362593\Delta_L^6+881129\Delta_L^5+2782307\Delta_L^4}{438022480896000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&\frac{4155839\Delta_L^3+3518084\Delta_L^2+1198176\Delta_L}{438022480896000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)},
\end{aligned} \tag{B.4.3}
$$

$$
\begin{aligned}
P^{(3)}_{18,10} = &\frac{P^{(2)}_{8,2}\left(\Delta_L\left(323\Delta_L-77\right)+54\right)}{823727520\left(\Delta_L-4\right)} \\
&+\frac{17413253\Delta_L^6+23717684\Delta_L^5+79039447\Delta_L^4}{377794389772800000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)} \\
&+\frac{92754344\Delta_L^3+73231064\Delta_L^2+22535496\Delta_L}{377794389772800000\left(\Delta_L-4\right)\left(\Delta_L-3\right)\left(\Delta_L-2\right)}.
\end{aligned} \tag{B.4.4}
$$

Assuming Einstein-Hilbert + Gauss-Bonnet gravity in the bulk, the OPE coefficient $P^{(2)}_{8,2}$ was derived in (6.107) and can be inserted in (B.4.1)-(B.4.4).

Appendix B.5. Derivation of the deflection angle from the phase shift.

Here we simply show that the bulk phase shift, defined as $\delta = p^t(\Delta t) - p^\phi(\Delta\phi)$ in [55] is consistent with the standard equation relating the eikonal phase and the scattering angle

$$
\frac{\partial \delta}{\partial b} = -p^t\,\Delta\phi \tag{B.5.1}
$$

obtained with the use of the stationary phase approximation for small scattering angles. Our discussion is focused on asymptotically flat space. In this case, the formulas in classical gravity which provide the deflection angle and the time delay are:

$$
\begin{aligned}
\Delta t &= 2\int_{r_0}^{\infty} \frac{dr}{f\sqrt{1-\frac{b^2 f}{r^2}}} \\
\Delta\phi &= 2b\int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{1-\frac{b^2 f}{r^2}}}\,.
\end{aligned} \tag{B.5.2}
$$

They can be obtained from eq. (2.9) in [55] with the substitution $\frac{p^\phi}{p^t} = b$ (and the appropriate definition of the blackening factor $f(r)$). Note that the equation for the turning point of the geodesic, r_0, reduces in Schwarzchild geometry to:

$$
1 - \frac{b^2}{r^2 f(r_0)} = 0 \tag{B.5.3}
$$

Defining the bulk phase shift via $\delta = p^t(\Delta t) - p^\phi(\Delta\phi)$, leads to

$$\delta = p^t(\Delta t) - p^\phi(\Delta\phi) = p^t\left(\Delta t - b\Delta\phi\right) = 2p^t \int_{r_0}^{\infty} \frac{dr}{f}\sqrt{1-\frac{b^2 f}{r^2}} \tag{B.5.4}$$

Differentiating the bulk phase shift with respect to the impact parameter yields:

$$\frac{\partial\delta}{\partial b} = -2p^t\, b\int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{1-\frac{b^2 f}{r^2}}} - 2p^t\frac{1}{f(r_0)}\sqrt{1-\frac{b^2 f(r_0)}{r_0^2}} = -p^t(\Delta\phi)\,, \tag{B.5.5}$$

where to arrive at the last equality we used the equation satisfied by the turning point r_0. Hence,

$$\Delta\phi = -\frac{1}{p^t}\frac{\partial\delta}{\partial b}\,. \tag{B.5.6}$$

Finally note that assuming the classical relation $J \equiv p_\phi = b\,p^t$, the deflection angle can also be computed through

$$\Delta\phi = -\frac{\partial\delta}{\partial J}\,. \tag{B.5.7}$$

Appendix B.6. Anomalous dimensions and phase shift at $\mathcal{O}(\mu^2)$

We give explicit expressions for $\gamma_n^{(2,0)}$, $\gamma_n^{(2,1)}$ and $\gamma_n^{(2,2)}$ from (3.69)

$$\begin{aligned}\gamma_n^{(2,0)} =& -\frac{1}{8}\left(\Delta_L-1\right)\Delta_L\left(4\Delta_L+1\right) - \frac{51}{4}n^2\left(\Delta_L-1\right)\\ &+\frac{1}{4}n\left(3\left(11-7\Delta_L\right)\Delta_L-17\right) - \frac{17}{2}n^3,\end{aligned} \tag{B.6.1}$$

$$\begin{aligned}\gamma_n^{(2,1)} =& \frac{1}{8\sqrt{1-4\lambda_{\rm GB}}\,r_{AdS}^2}\Big(\lambda_{\rm GB}(4\Delta_L^4+8\Delta_L^3-4\Delta_L^2-8\Delta_L+560n^3\Delta_L\\ &+360n^2\Delta_L^2-600n^2\Delta_L+80n\Delta_L^3-120n\Delta_L^2+200n\Delta_L+280n^4\\ &-560n^3+440n^2-160n)+r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}(-\Delta_L^4+6\Delta_L^3-5\Delta_L^2\\ &-140n^3\Delta_L-90n^2\Delta_L^2+354n^2\Delta_L-20n\Delta_L^3+114n\Delta_L^2\\ &-182n\Delta_L-70n^4+276n^3-314n^2+108n)\Big),\end{aligned} \tag{B.6.2}$$

$$\gamma_n^{(2,2)} = \frac{1}{8\sqrt{1-4\lambda_{\rm GB}}r_{AdS}^2}\Big(\lambda_{\rm GB}(16\Delta_L^3 - 16\Delta_L + 840n^4\Delta_L + 720n^3\Delta_L^2$$
$$- 2880n^3\Delta_L + 240n^2\Delta_L^3 - 1440n^2\Delta_L^2 + 3720n^2\Delta_L + 24n\Delta_L^4 - 192n\Delta_L^3$$
$$+ 888n\Delta_L^2 - 1536n\Delta_L + 336n^5 - 1680n^4 + 3440n^3 - 3120n^2 + 1024n)$$
$$+ r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}(3\Delta_L^4 - 10\Delta_L^3 + 6\Delta_L^2 + \Delta_L + 420n^3\Delta_L + 270n^2\Delta_L^2$$
$$- 876n^2\Delta_L + 60n\Delta_L^3 - 264n\Delta_L^2 + 420n\Delta_L + 210n^4 - 704n^3 + 756n^2$$
$$- 262n)\Big), \tag{B.6.3}$$

where we use the expression for $P_{8,0}^{(2)}$, found in [15], to fix $\gamma_n^{(2,2)}$. If one considers limit $1 \ll l, n \ll \Delta_H$ one gets

$$\gamma_{n,l}^{(2)} \underset{l,n\to\infty}{\approx} -\frac{17n^3}{2l^2} - \frac{35n^4}{4l^3}\left(1 - \frac{4\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\right) + \frac{42\lambda_{\rm GB}n^5}{\sqrt{1-4\lambda_{\rm GB}}l^4 r_{AdS}^2} + \dots, \tag{B.6.4}$$

where ... denote terms that come from $\gamma_n^{(2,m)}$ for $m > 2$ and they have higher powers of l (and n) as well as terms that are subleading in the given limit and behave as $\mathcal{O}(1)$.

By using the following relations from [55,12]

$$\sinh(L) = \frac{b}{r_{AdS}}, \quad \cosh(L) = \frac{p^+ + p^-}{2\sqrt{-p^2}}, \tag{B.6.5}$$

with

$$-p^2 = p^+p^-, \quad p^+ = 2h, \quad p^- = 2\bar{h}, \tag{B.6.6}$$

where

$$h = n + l, \quad \bar{h} = n, \tag{B.6.7}$$

one obtains $\delta^{(2)}$ from (6.93) in terms of the S-channel variables n and l

$$\delta^{(2)} = \frac{7\pi n^3}{4l^5}\left(10l^3 + 5l^2 n - \frac{4n(5l^2 + 6ln + 2n^2)\lambda_{\rm GB}}{r_{AdS}^2\sqrt{1-4\lambda_{\rm GB}}}\right). \tag{B.6.8}$$

From (6.28) and (3.69) one concludes that the leading behavior in the large-l and large-n limit ($1 \ll n, l \ll \Delta_H$) of $\gamma_{n,l}^{(1)}$ is

$$\gamma_{n,l}^{(1)} \underset{l,n\to\infty}{\approx} -\frac{3n^2}{l} + \mathcal{O}(1). \tag{B.6.9}$$

Now, one can evaluate (1.5) from [12] using (B.6.8) and (B.6.9)

$$\begin{aligned}\gamma^{(2)}_{n,l} &\underset{l,n\to\infty}{\approx} -\frac{\delta^{(2)}}{\pi} + \frac{1}{2}\gamma^{(1)}_{n,l}\partial_n\gamma^{(1)}_{n,l} \\ &\underset{l,n\to\infty}{\approx} -\frac{17n^3}{2l^2} - \frac{35n^4}{4l^3}\left(1 - \frac{4\lambda_{\mathrm{GB}}}{r^2_{AdS}\sqrt{1-4\lambda_{\mathrm{GB}}}}\right) + \frac{42\lambda_{\mathrm{GB}}n^5}{\sqrt{1-4\lambda_{\mathrm{GB}}}l^4r^2_{AdS}} \\ &\quad + \frac{14\lambda_{\mathrm{GB}}n^6}{\sqrt{1-4\lambda_{\mathrm{GB}}}l^5r^2_{AdS}} + \mathcal{O}(1).\end{aligned} \tag{B.6.10}$$

We see that first three terms in (B.6.10) precisely matches with terms in (B.6.4), which explicitly confirms the validity of relation (1.5) in [12]. One would expect that term $\frac{14\lambda_{\mathrm{GB}}n^6}{\sqrt{1-4\lambda_{\mathrm{GB}}}l^5r^2_{AdS}}$ is due to $\frac{\gamma_n^{(2,3)}}{l^5}$ in (3.69), while all other $\frac{\gamma_n^{(2,k)}}{l^{2+k}}$, for $k > 3$, should behave as $\mathcal{O}(1)$ in $1 \ll n, l \ll \Delta_H$ limit for (1.5) from [12] to be true.

Appendix C.1. OPE coefficients from Wick contractions

In this appendix we go through the calculations needed for finding the OPE coefficients of various operators using Wick contractions. This mainly amounts to counting the number of contractions leading to a planar diagram. For simplicity, the figures are shown for external operators with $\Delta = 4$ while we write down the result for general Δ as this is needed for the main body of the section.

To begin with, since we consider a large-N matrix theory, it is convenient to use the double-line notation for fundamental field propagators. In Fig. 1 the two-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : \rangle$ is visualised.

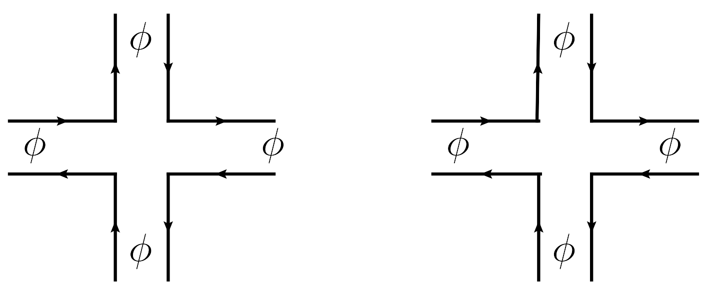

Fig. 1: The two-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : \rangle$ before any contractions.

In Fig. 2, the planar diagram is shown for $\Delta = 4$ and there are Δ number of such contractions giving a planar diagram

$$P_{\langle :Tr(\phi^\Delta)::Tr(\phi^\Delta):\rangle} = \Delta, \tag{C.1.1}$$

where the $P_{\langle ... \rangle}$ denotes the number of planar diagrams for $\langle ... \rangle$.

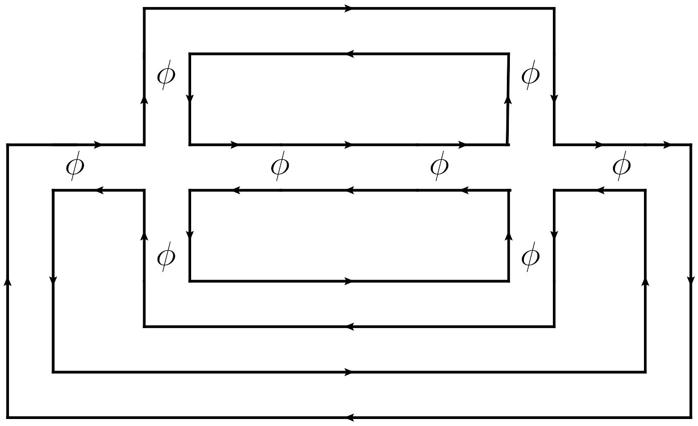

Fig. 2: The two-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : \rangle$ completely contracted.

We further need the OPE coefficient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_2}$. This is shown in Fig. 3 for $\Delta = 4$ and there are 2Δ possibilities for step (1), Δ number of possibilites for step (2) after which everything is fixed assuming that the diagram is planar. This gives

$$P_{\langle :Tr(\phi^\Delta)::Tr(\phi^\Delta)::Tr(\phi^2):\rangle} = 2\Delta^2. \tag{C.1.2}$$

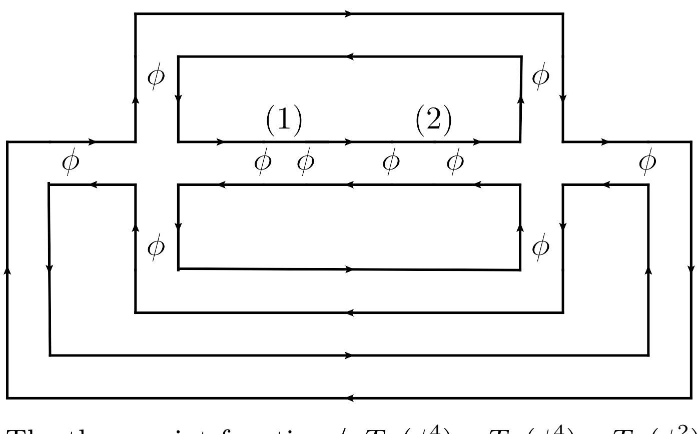

Fig. 3: The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2) : \rangle$ completely contracted.

In Fig. 4 the three-point function $\langle : Tr(\phi^\Delta) :: Tr(\phi^\Delta) :: Tr(\phi^4) : \rangle$ for $\Delta = 4$ is shown. For the first contraction (1) there are 2Δ possibilites, for the second contraction there are Δ and for step (3) there are two possibilites. This gives overall

$$P_{\langle :Tr(\phi^\Delta)::Tr(\phi^\Delta)::Tr(\phi^4):\rangle} = 4\Delta^2. \tag{C.1.3}$$

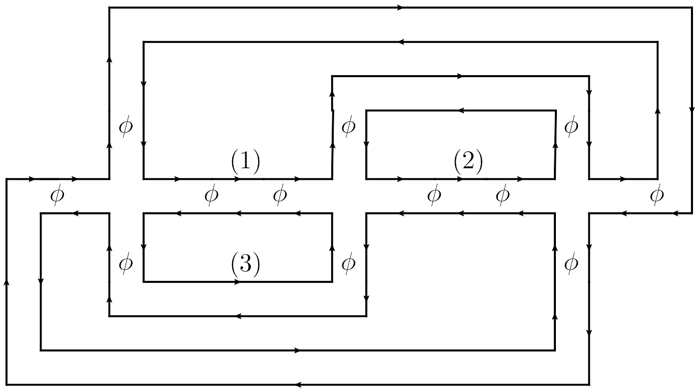

Fig. 4: The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^4) : \rangle$ completely contracted.

In Fig. 5 and Fig. 6, the three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) : Tr(\phi^2)Tr(\phi^2) : \rangle$ is shown. The reason for there being two different types of diagrams is because each trace term in the double trace operator $: Tr(\phi^2)Tr(\phi^2) :$ can either be contracted with the same $: Tr(\phi^4) :$ (Fig. 5, type B), or to both (Fig. 6, type A).

Consider first the type of diagrams in Fig. 5. For the first contraction there are 2Δ such terms and the second contraction gives another factor of 2. Contraction (3) and (4) contributes factors of Δ and 2 respectively. What remains is equivalent to the two-point function $\langle : Tr(\phi^{\Delta-2}) :: Tr(\phi^{\Delta-2}) : \rangle$ which further give a factor of $(\Delta - 2)$ and therefore there are $8\Delta^2(\Delta - 2)$ contractions of type B in Fig. 5.

Continuing with Fig. 6, the first contraction gives a factor of 2Δ, the second contraction Δ and the third one a factor of $2(\Delta - 1)$. What remains is then fixed by imposing that the diagram is planar. The type A diagrams in Fig. 6 therefore further contributes $4\Delta^2(\Delta - 1)$ planar diagrams to $\langle : Tr(\phi^\Delta) :: Tr(\phi^\Delta) : Tr(\phi^2)Tr(\phi^2) : \rangle$. It is therefore found that

$$P_{\langle :Tr(\phi^\Delta)::Tr(\phi^\Delta):Tr(\phi^2)Tr(\phi^2):\rangle} = 4\Delta^2(3\Delta - 5). \tag{C.1.4}$$

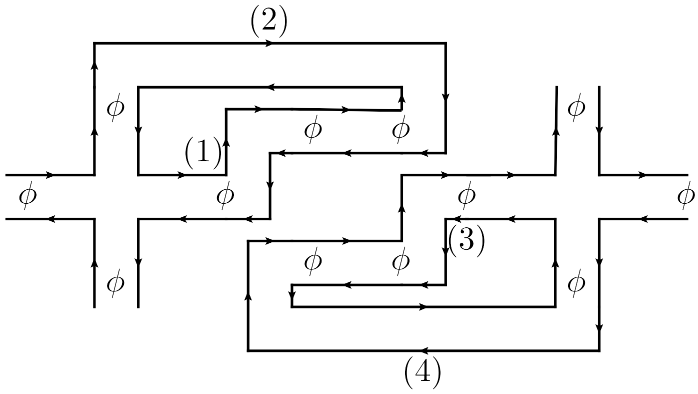

Fig. 5: The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2)Tr(\phi^2) : \rangle$. There are two such types of contractions that give planar diagrams, here it shown when each $: Tr(\phi^2) :$ connect to a separate $: Tr(\phi^4) :$.

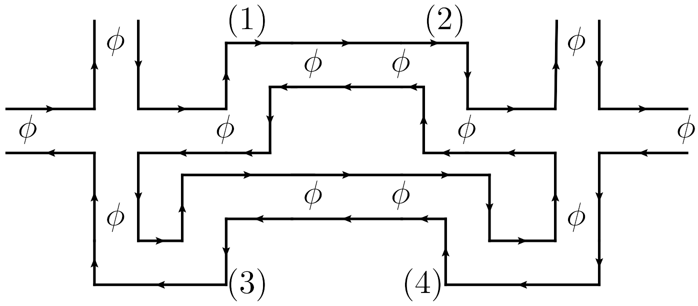

Fig. 6: The three-point function $\langle : Tr(\phi^4) :: Tr(\phi^4) :: Tr(\phi^2)Tr(\phi^2) : \rangle$. There are two such types of contractions that give planar diagrams, here it shown when each $: Tr(\phi^2) :$ connect to both $: Tr(\phi^4) :$ operators.

Consider now the stress tensor OPE coffiecient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}}$ where

$$T_{\mu\nu}(x) = \frac{1}{2\sqrt{3}N} : Tr\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\phi\partial_\mu\partial_\nu\phi - (\text{trace})\right) : (x) \tag{C.1.5}$$

and the three-point function $\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}\rangle$:

$$\langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)T_{\mu\nu}(x_3)\rangle = \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}} \frac{Z_\mu Z_\nu - \text{traces}}{|x_{12}|^{2\Delta-2}|x_{23}|^2|x_{13}|^2}, \tag{C.1.6}$$

where $Z_\mu = \frac{x_{13\mu}}{|x_{13}|^2} - \frac{x_{12\mu}}{|x_{12}|^2}$. From the definition of $T_{\mu\nu}$ in (C.1.5) it is clear that the only term that contributes to term $x_{13\mu}x_{13\nu}$ comes from the second term in (C.1.5) that is of the form $\propto Tr(\phi\partial_\mu\partial_\nu\phi)$. Up to the derivatives, the diagram will look like those visualised in Fig. 3. The number of diagrams is half of that given in (C.1.2) since we restrict to terms proportional to $x_{13\mu}x_{13\nu}$:

$$P_{\langle \mathcal{O}_\Delta \mathcal{O}_\Delta T_{\mu\nu}\rangle|_{x_{13\mu}x_{13\nu}}} = \Delta^2, \tag{C.1.7}$$

from which we reproduce (7.24).

Now we want to find the OPE coefficient $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}$ for the double-stress tensor $T^2_{4,4}$. This is done similarly to the way the stress tensor OPE coefficient was found. First, the operator $(T^2)_{\mu\nu\rho\sigma}$ was given in (7.25) to be

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : (x) - (\text{traces}) \tag{C.1.8}$$

and the three-point function $\langle\mathcal{O}_\Delta\mathcal{O}_\Delta(T^2)_{\mu\nu\rho\sigma}\rangle$ is fixed by conformal symmetry to be

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}}{|x_{12}|^{2\Delta-4}|x_{13}|^4|x_{23}|^4}\left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})\right). \tag{C.1.9}$$

Consider the term in (C.1.9) proportional to $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$. This will be due to the term in $(T^2)_{\mu\nu\rho\sigma}$ of the form $Tr(\phi\partial_{(\mu}\partial_\nu\phi)Tr(\phi\partial_\rho\partial_{\sigma)}\phi)$. Using this we find that

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}} = \frac{1}{\Delta N^\Delta}\frac{1}{\sqrt{2}}\left(\frac{-1}{4\sqrt{3}N}\right)^2 8^2N^\Delta \times \frac{P_{\langle\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}}}}{|x_{12}|^{2(\Delta-2)}|x_{23}|^4|x_{13}|^{12}}. \tag{C.1.10}$$

The number of contractions giving a planar diagram, $P_{\langle\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}}}$, come from diagrams of the form given in Fig. 6. Since we are considering the term proportional $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$, the number of such diagrams are reduced compared to scalar double trace operator. Instead the first contraction, (1) in Fig. 6, give a factor of Δ, the second contraction, (2), a factor of $(\Delta-1)$, the third contraction (3) gives a further factor Δ after which everything is fixed by imposing that the diagram is planar. We therefore find that

$$P_{\langle\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}}} = \Delta^2(\Delta-1), \tag{C.1.11}$$

and inserting this in (C.1.10) gives

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} = \frac{2\sqrt{2}\Delta(\Delta-1)}{3N^2}, \tag{C.1.12}$$

and therefore reproduces (7.28).

Similar to the double-stress tensor, consider the dimension-eight spin-four double trace operator

$$\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70}N^2} : Tr(\phi^2)\Big(Tr(\phi\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16Tr(\partial_{(\mu}\phi\partial_\nu\partial_\rho\partial_{\sigma)}\phi) + 18Tr(\partial_{(\mu}\partial_\nu\phi\partial_\rho\partial_{\sigma)}\phi)(x) - (\text{traces})\Big) : (x). \tag{C.1.13}$$

The three-point function $\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}(x_3)\rangle$ is given by

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}(x_3)\rangle = \frac{\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}}}{|x_{12}|^{2\Delta-4}|x_{13}|^4|x_{23}|^4}\left(Z_\mu Z_\nu Z_\rho Z_\sigma - (\text{traces})\right). \tag{C.1.14}$$

By again considering terms in (C.1.14) proportional to $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$ we find that each term in (C.1.13) will contribute planar diagram of the type in Fig. 5, while only the term $\sim Tr(\phi\partial^4\phi)$ also give a contribution of the type in Fig. 6. Considering first the terms coming from the diagram in Fig. 5, one finds that this contribution vanishes. The remaining contribution to the term (C.1.14) proportional to $x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}$ comes from the first term in (C.1.13) and the planar diagram pictured in Fig. 6; there are $2\Delta^2(\Delta-1)$ contractions giving such a planar diagram leading to

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}(x_3)\rangle|_{x_{13\mu}x_{13\nu}x_{13\rho}x_{13\sigma}} = \frac{1}{\Delta N^\Delta}\frac{384}{96\sqrt{70}N^2}N^\Delta \times\frac{2\Delta^2(\Delta-1)}{|x_{12}|^{2(\Delta-2)}|x_{23}|^4|x_{13}|^{12}}, \tag{C.1.15}$$

where the 384 in the numerator come from the derivatives. This gives the OPE coefficient:

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}^{\rm DT}_{\mu\nu\rho\sigma}} = \sqrt{\frac{2}{35}}\frac{4\Delta(\Delta-1)}{N^2} + \mathcal{O}(N^{-4}). \tag{C.1.16}$$

Appendix C.2. Subleading twist double-stress tensors

In this Appendix we study the subleading twist double-stress tensors, both with dimension 8 and spin $s = 0, 2$ denoted (T^2) and $(T^2)^{\mu\nu}$ respectively. The calculations needed to find the OPE coefficient in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE are reviewed as well as the normalization of $(T^2)^{\mu\nu}$.

The $(T^2)^{\mu\nu}$ was defined in (7.41) which we repeat here:

$$(T^2)^{\mu\nu}(x) = \frac{1}{\sqrt{2}} : T^{\mu}{}_{\alpha} T^{\alpha\nu} : (x) - \frac{\delta^{\mu\nu}}{4\sqrt{2}} : T^{\beta}{}_{\alpha} T^{\alpha}{}_{\beta} : (x). \tag{C.2.1}$$

The operator $(T^2)^{\mu\nu}$ can be seen to be unit-normalized to leading order in N:

$$\begin{aligned} \langle (T^2)^{\mu\nu}(x_1)(T^2)_{\rho\sigma}(x_2)\rangle =& \frac{1}{\sqrt{2}} \langle T^{\mu\alpha}(x_1) T_{\rho\beta}(x_2)\rangle \langle T^{\nu}{}_{\alpha}(x_1) T^{\beta}{}_{\sigma}\rangle \\ &+ (\rho \longleftrightarrow \sigma) - (\text{traces}) + \mathcal{O}(N^{-2}). \end{aligned} \tag{C.2.2}$$

Using the two-point function of the stress tensor in (7.23) and $I^{\mu}{}_{\alpha} I^{\alpha}{}_{\rho} = \delta^{\mu}{}_{\rho}$ one finds

$$\langle (T^2)^{\mu\nu}(x_1)(T^2)_{\rho\sigma}(x_2)\rangle = \frac{1}{|x|^{16}} \left(I^{(\mu}{}_{\rho} I^{\nu)}{}_{\sigma} - (\text{traces}) \right), \tag{C.2.3}$$

from which it is seen that $(T^2)^{\mu\nu}$ is unit-normalised.

We now want to find the OPE coefficient of $(T^2)^{\mu\nu}$ in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE. It can be found from the basic objects $I^{(1)}_{\mu\nu\rho\sigma}$, $I^{(2)}_{\mu\nu\rho\sigma}$ and $I^{(3)}_{\mu\nu\rho\sigma}$ which we calculate below.

We first consider a similar quantity $J^{(1)\mu\nu\rho\sigma}$:

$$\begin{aligned} J^{(1)\mu\nu\rho\sigma} &= \langle : Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\partial_\mu\phi\partial_\nu\phi) Tr(\partial_\rho\phi\partial_\sigma\phi) : (x_3)\rangle \\ = & \frac{2^4 N^\Delta}{|x_{13}|^8 |x_{23}|^8 |x_{12}|^{2\Delta-4}} \times \Big[(2\Delta)^2(\Delta-2)(x_{13}^\mu x_{13}^\nu x_{23}^\rho x_{23}^\sigma + x_{23}^\mu x_{23}^\nu x_{13}^\rho x_{13}^\sigma) + \\ & \Delta^2(\Delta-1)(x_{13}^\mu x_{23}^\nu (x_{13}^\rho x_{23}^\sigma + x_{23}^\rho x_{13}^\sigma) + x_{23}^\mu x_{13}^\nu (x_{13}^\rho x_{23}^\sigma + x_{23}^\rho x_{13}^\sigma)) \Big]. \end{aligned} \tag{C.2.4}$$

Definining $X_{13}^{\mu\nu} = \frac{1}{|x_{13}|^4}(-\delta^{\mu\nu} + 4\frac{x_{13}^\mu x_{13}^\nu}{|x_{13}|^2})$ we then study $J^{(2)\mu\nu\rho\sigma}$:

$$\begin{aligned} J^{(2)\mu\nu\rho\sigma} &= \langle : Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\phi\partial_\mu\partial_\nu\phi) Tr(\phi\partial_\rho\partial_\sigma\phi) : (x_3)\rangle \\ =& \frac{N^\Delta}{|x_{12}|^{2\Delta-4}} \Big[\Delta^2(\Delta-1)2^2 \Big(X_{13}^{\mu\nu} \frac{1}{|x_{23}|^2} X_{13}^{\rho\sigma} \frac{1}{|x_{23}|^2} + X_{13}^{\mu\nu} \frac{1}{|x_{23}|^2} X_{23}^{\rho\sigma} \frac{1}{|x_{13}|^2} \Big) \\ & + ((2\Delta)^2(\Delta-2)) 2^2 X_{13}^{\mu\nu} \frac{1}{|x_{13}|^2} X_{23}^{\rho\sigma} \frac{1}{|x_{23}|^2} \\ & + (13) \longleftrightarrow (23) \Big]. \end{aligned} \tag{C.2.5}$$

And lastly $J^{(3)}{}^{\mu\nu\rho\sigma}$:

$$\begin{aligned}J^{(3)}{}^{\mu\nu\rho\sigma} =& \langle : Tr(\phi^\Delta) : (x_1) : Tr(\phi^\Delta) : (x_2) :: Tr(\phi\partial_\mu\partial_\nu\phi)Tr(\partial_\rho\phi\partial_\sigma\phi) : (x_3)\rangle \\ =& \frac{N^\Delta}{|x_{12}|^{2\Delta-4}}\Big[((2\Delta)^2(\Delta-2))2^3X_{13}^{\mu\nu}\frac{1}{|x_{13}|^2}\frac{x_{23}^\rho x_{23}^\sigma}{|x_{23}|^8}+ \\ &+\Delta^2(\Delta-1)2^3X_{13}^{\mu\nu}\frac{1}{|x_{23}|^2}\frac{x_{13}^\rho x_{23}^\sigma + x_{23}^\rho x_{13}^\sigma}{|x_{13}|^4|x_{23}|^4} \\ &+(13)\longleftrightarrow(23)\Big].\end{aligned} \tag{C.2.6}$$

We further need to make (C.2.4)-(C.2.6) traceless in the pairs (μ,ν) and (ρ,σ) and therefore define $I^{(i)}{}^{\mu\nu\rho\sigma}$ as

$$I^{(i)}{}^{\mu\nu\rho\sigma} = J^{(i)}{}^{\mu\nu\rho\sigma} - \frac{\delta^{\mu\nu}}{4}J^{(i)}{}^{\alpha}{}_{\alpha}{}^{\rho\sigma} - \frac{\delta^{\rho\sigma}}{4}J^{(i)}{}^{\mu\nu\alpha}{}_{\alpha} + \frac{\delta^{\mu\nu}\delta^{\rho\sigma}}{16}J^{(i)}{}^{\alpha}{}_{\alpha}{}^{\gamma}{}_{\gamma}. \tag{C.2.7}$$

From (C.2.4)-(C.2.6), the three-point function $\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle$ is given by

$$\begin{aligned}\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}\rangle =& \frac{1}{12\sqrt{2}\Delta N^{\Delta+2}}(I^{(1)}{}^{(\mu|\alpha}{}_{\alpha}{}^{|\nu)} - I^{(3)}{}^{(\mu|\alpha}{}_{\alpha}{}^{|\nu)} \\ &+\frac{1}{4}I^{(2)}{}^{(\mu|\alpha}{}_{\alpha}{}^{|\nu)} - (\text{trace})).\end{aligned} \tag{C.2.8}$$

Explicitly we find that

$$\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)^{\mu\nu}(x_3)\rangle = \frac{\sqrt{2}\Delta(\Delta-1)}{3N^2}\frac{Z^\mu Z^\nu - (\text{trace})}{|x_{12}|^{2\Delta-6}|x_{13}|^6|x_{23}|^6} + \mathcal{O}(N^{-4}). \tag{C.2.9}$$

Consider now the scalar operator (T^2) defined by

$$(T^2)(x) = \frac{1}{36\sqrt{2}N^2} : T_{\mu\nu}T^{\mu\nu} : (x). \tag{C.2.10}$$

The three-point function $\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle$ can be found using $I^{(i)}$ defined in (C.2.7) as follows

$$\begin{aligned}\langle\mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)(T^2)(x_3)\rangle =& \frac{1}{36\sqrt{2}\Delta N^{2+\Delta}}(I^{(1)}{}^{\mu\nu}{}_{\mu\nu} - I^{(3)}{}^{\mu\nu}{}_{\mu\nu} + \frac{1}{4}I^{(2)}{}^{\mu\nu}{}_{\mu\nu}) \\ +\mathcal{O}(N^{-4}) =& \frac{\Delta(\Delta-1)}{3\sqrt{2}N^2}\frac{1}{|x_{12}|^{2\Delta-8}|x_{13}|^8|x_{23}|^8} + \mathcal{O}(N^{-4}).\end{aligned} \tag{C.2.11}$$

Appendix C.3. Single trace operator with dimension $\Delta \sim C_T$

In this appendix we study the single trace scalar operator $\mathcal{O}_{\Delta_H}$ given by

$$\mathcal{O}_H(x) = \frac{1}{\sqrt{\mathcal{N}_{\Delta_H}}} : Tr(\phi^{\Delta_H}) : (x), \tag{C.3.1}$$

with $\Delta_H \sim C_T$ and $\mathcal{N}_{\Delta_H}$ a normalization constant[50]. When calculating the normalization constant $\mathcal{N}_{\Delta_H}$ as well as the three-point functions $\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\mathcal{O}(x_3)\rangle$, non-planar diagrams generically gets enhanced by powers of Δ_H and therefore invalidates the naive planar expansion. The goal of this appendix is to show that

$$\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\hat{\mathcal{O}}(x_3)\rangle = \langle \mathcal{O}_\Delta(x_1)\mathcal{O}_\Delta(x_2)\hat{\mathcal{O}}(x_3)\rangle|_{\Delta=\Delta_H}, \tag{C.3.2}$$

where $\hat{\mathcal{O}}$ is either $: Tr(\phi^2) :$ or, more importantly, minimal-twist multi stress tensors with any spin. Moreover, note that the LHS in (C.3.2) is in principle exact in $C_T \sim N^2$ while the RHS is obtained by keeping only planar diagrams with $\Delta \ll C_T$ and then setting $\Delta = \Delta_H$ in the end.

The propagator for the field ϕ was given in (7.19) by

$$\langle \phi^i{}_j(x)\phi^k{}_l(y)\rangle = \left(\delta^i{}_l\delta^k{}_j - \frac{1}{N}\delta^i{}_j\delta^k{}_l\right)\frac{1}{|x-y|^2}. \tag{C.3.3}$$

Consider now the three-point function $\langle : Tr(\phi^{\Delta_H}) : (x_1) : Tr(\phi^{\Delta_H}) : (x_2) : Tr(\phi^2) : (x_3)\rangle$. Due to the normal ordering, one ϕ field in $: Tr(\phi^2) : (x_3)$ need to be contracted with $: Tr(\phi^{\Delta_H}) : (x_1) :$ and the other one with $: Tr(\phi^{\Delta_H}) : (x_2) :$. Note that for this contraction the second term in (C.3.3) give a contribution proportional to $Tr(\phi(x_3)) = 0$. It is therefore seen that

$$\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) :: Tr(\phi_3^2) :\rangle = 2\Delta_H\langle : Tr(\phi_3\phi_1^{\Delta_H-1}) :: Tr(\phi_2^{\Delta_H}) :\rangle, \tag{C.3.4}$$

where we introduced the notation $\phi_i = \phi(x_i)$ and dropped the $|x_{ij}|^{-2}$ coming from (C.3.3). The position dependence is easily restored in the end. Now it is seen that the RHS of (C.3.4) is proportional to the two-point function[51] of $\mathcal{O}_H$ and we therefore find that

$$\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) :: Tr(\phi_3^2) :\rangle = 2\Delta_H\mathcal{N}_{\Delta_H}, \tag{C.3.5}$$

[50] Mixing with other operators with $\Delta \sim C_T$ is not important for this discussion.

[51] Up to the position dependence.

which is exact to all orders in C_T. Including the normalization factor of $\mathcal{O}_H$ in (C.3.1) and $\mathcal{O}_2$ from (7.20) we find that

$$\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)\mathcal{O}_2(x_3)\rangle = \frac{\sqrt{2}\Delta_H}{N}\frac{1}{|x_{12}|^{2\Delta_H-2}|x_{13}|^2|x_{23}|^2} + \mathcal{O}(N^{-3}). \quad \text{(C.3.6)}$$

By comparing (C.3.6) with (7.87) we find that

$$\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_2} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}|_{\Delta=\Delta_H}. \quad \text{(C.3.7)}$$

Note that in (C.3.6) the normalization of $\mathcal{O}_H$ cancels the contribution from non-planar diagrams in limit $\Delta_H \sim C_T$. For $\Delta = 2$ in (7.20), it is trivial to compute the normalization exact in N to get the correction to $\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\mathcal{O}_2}$ in (C.3.6).

Consider now the stress tensor operator defined in (7.22) and the three-point function $\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)T_{\mu\nu}(x_3)\rangle$. This is fixed by the Ward identity but is an instructive example before considering more general multi stress tensors. In the same way as the OPE coefficient was found in the $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ OPE, due to the tensor structure being fixed by conformal symmetry, we consider the term proportional to $x_{13}^\mu x_{13}^\nu$ in the three-point function. This comes from the $-\frac{1}{6\sqrt{C_T}}Tr(\phi\partial_\mu\partial_\nu\phi)$ term in the stress tensor when $\partial_\mu\partial_\nu\phi$ is contracted with one of the Δ_H number of $\phi(x_1)$ fields. Doing this contraction we therefore see that

$$\begin{aligned}&\langle : Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) :: Tr(\phi_3\partial_\mu\partial_\nu\phi_3) :\rangle|_{x_{13}^\mu x_{13}^\nu} = \\ &8\Delta_H\langle : Tr(\phi_3\phi_1^{\Delta_H-1}) :: Tr(\phi_2^{\Delta_H}) :\rangle,\end{aligned} \quad \text{(C.3.8)}$$

where the factor 8 comes from the derivatives and we again suppress the space-time dependence. The RHS of (C.3.8) is also proportional to the normalization constant of $\mathcal{O}_H$. Including the normalization factor of the stress tensor in (7.22) and that of $\mathcal{O}_H$ in (C.3.1), the three-point function $\langle \mathcal{O}_H\mathcal{O}_H T_{\mu\nu}\rangle$ can be obtained from (C.3.8) from which we read off the OPE coefficient

$$\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}} = -\frac{4\Delta_H}{3\sqrt{C_T}}. \quad \text{(C.3.9)}$$

This agrees with (7.24).

We now want to show that is true for minimal-twist multi stress tensors with any spin. For simplicity, consider the double-stress tensor with spin 4 defined in (7.25)

$$(T^2)_{\mu\nu\rho\sigma}(x) = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : (x) - (\text{traces}). \quad \text{(C.3.10)}$$

Similarly to the calculation of the three-point function with the stress tensor, we can obtain the three-point function $\langle\mathcal{O}_H(x_1)\mathcal{O}_H(x_2)(T^2)_{\mu\nu\rho\sigma}(x_3)\rangle$ by considering the term proportional to $x_{13}^\mu x_{13}^\nu x_{13}^\rho x_{13}^\sigma$. This will be due to the term $\frac{1}{\sqrt{2}6^2C_T}Tr(\phi\partial_\mu\partial_\nu\phi)Tr(\phi\partial_\rho\partial_\sigma\phi)$ when contracting $\partial_\mu\partial_\nu\phi$ with some $\phi(x_1)$ and likewise contracting $\partial_\rho\partial_\sigma\phi$ with some other $\phi(x_1)$. The number of such contractions is given by $\Delta_H(\Delta_H-1)$ and we find that

$$\begin{aligned}\langle: Tr(\phi_1^{\Delta_H}) :: Tr(\phi_2^{\Delta_H}) :: Tr(\phi_3\partial_\mu\partial_\nu\phi_3)Tr(\phi_3\partial_\rho\partial_\sigma\phi_3) :\rangle|_{x_{13}^\mu x_{13}^\nu x_{13}^\rho x_{13}^\sigma} \\ = 8^2\Delta_H(\Delta_H-1)\langle: Tr(\phi_3^2\phi_1^{\Delta_H-2}) :: Tr(\phi_2^{\Delta_H}) :\rangle,\end{aligned} \tag{C.3.11}$$

where the factor of 8^2 again is due to acting with the derivatives and note that the position of the ϕ_3 fields in in the last line is not important. It is again seen that the RHS of (C.3.11) is proportional to the normalization constant of $\mathcal{O}_H$. Including the normalization in (7.25) and (C.3.1) we find the three-point function $\langle\mathcal{O}_H\mathcal{O}_H(T^2)_{\mu\nu\rho\sigma}\rangle$ and read off the OPE coefficient:

$$\lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}} = \frac{8\sqrt{2}\Delta_H(\Delta_H-1)}{9C_T} + \mathcal{O}(C_T^{-3/2}). \tag{C.3.12}$$

which is seen to agree with (7.28) when setting $\Delta_H = \Delta$. Note that the corrections in (C.3.12) are solely due to corrections in the normalization of $T^2_{4,4}$ and therefore $\lambda_{\mathcal{O}_H\mathcal{O}_H T^2_{4,4}} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}}$ to all orders in C_T. These arguments generalize straightforwardly to minimal-twist multi stress tensor with any spin such that the results are the same as those obtained in the planar limit for $\Delta \ll C_T^2$ in Section 7.3 by setting $\Delta_H = \Delta$. The only correction in C_T is then due to the normalization of the multi stress tensor.

The same argument applies to any scalar primary multi-trace operator $\mathcal{O}_\Delta$, without any derivatives, with OPE coefficients given by (C.3.6), (C.3.9) and (C.3.12).

Appendix C.4. Stress tensor thermal one-point function

In order to calculate thermal one-point functions in the free adjoint scalar model we use the fact that the thermal correlation function is related to the zero-temperature case by summing over images. Consider now the thermal one-point function of the stress tensor. Generally, the one-point function of a spin-s symmetric traceless operator with dimension $\Delta_{\mathcal{O}}$ on $S^1 \times \mathbf{R}^{d-1}$ is given by [82]

$$\langle \mathcal{O}^{\mu_1 \dots \mu_s}(x) \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} (e^{\mu_1} \dots e^{\mu_s} - \text{(traces)}), \tag{C.4.1}$$

where e^{μ_1} is a unit-vector along the thermal circle. Consider first the canonically normalized stress tensor given by $T^{(\text{can})}_{\mu\nu} = \frac{1}{3S_d}(Tr(\partial_\mu \phi \partial_\mu \phi) - \frac{1}{2} Tr(\phi \partial_\mu \partial_\nu \phi) - \text{(traces)})$. In order to find the one-point function, use the following:

$$\langle Tr(\partial^{(x)}_\mu \phi(x) \partial^{(y)}_\nu \phi(y)) \rangle = \frac{2(N^2-1)}{|x-y|^4} (\delta_{\mu\nu} - 4(y-x)_\mu (y-x)_\nu \frac{1}{|x-y|^2}) \tag{C.4.2}$$

and

$$\langle Tr(\partial^{(x)}_\mu \partial^{(x)}_\nu \phi(x) \phi(y)) \rangle = \frac{2(N^2-1)}{|x-y|^4} (-\delta_{\mu\nu} + 4(y-x)_\mu (y-x)_\nu \frac{1}{|x-y|^2}). \tag{C.4.3}$$

To get the thermal correlator, we use (C.4.2) and (C.4.3) with x, y along the thermal circle separated by a distance $m\beta$, with m integer, and sum over $m \neq 0$. The reason for summing over $m \neq 0$ is the normal ordering of the operators. Namely, at $T = 0$, the normal ordering means that the correlation functions are computed without the self contractions, which would give the divergent contributions to the correlator. Removing the self-contractions in the correlation functions at $T = 0$ is the same as removing the divergent term with $m = 0$ when computing the thermal expectation value. The relevant terms for calculating the one-point functions in terms of fundamental fields are therefore

$$\begin{aligned} \langle Tr(\partial_\mu \phi \partial_\nu \phi) \rangle_{\beta,m} &= -\frac{8(N^2-1)}{(m\beta)^4} e^\mu e^\nu + \frac{2(N^2-1)}{(m\beta)^4} \delta_{\mu\nu}, \\ \langle Tr(\partial_\mu \partial_\nu \phi \phi) \rangle_{\beta,m} &= \frac{8(N^2-1)}{(m\beta)^4} e^\mu e^\nu - \frac{2(N^2-1)}{(m\beta)^4} \delta_{\mu\nu}, \end{aligned} \tag{C.4.4}$$

where we note that only the first term in each equation in (C.4.4) contribute to the non-trace term in (C.4.1).

We therefore find for the stress tensor one-point function:

$$\begin{aligned} \langle T^{(\text{can})}_{\mu\nu} \rangle_\beta &= \frac{1}{3S_d} (\langle Tr(\partial_\mu \phi \partial_\nu \phi) \rangle_\beta - \frac{1}{2} \langle Tr(\partial_\mu \partial_\nu \phi \phi) \rangle_\beta - \text{trace}) \\ &= \frac{-12(N^2-1)}{3S_d} \frac{2\zeta(4)}{\beta^4} (e_\mu e_\nu - \text{(trace)}), \end{aligned} \tag{C.4.5}$$

where the $2\zeta(4)$ comes from summing over images and we therefore have

$$b_{T^{(\mathrm{can})}_{\mu\nu}} = -\frac{4(N^2-1)}{S_d} 2\zeta(4) = -\frac{4\pi^4}{45 S_d}(N^2-1). \tag{C.4.6}$$

This agrees with $f = \frac{b_{T^{(\mathrm{can})}_{\mu\nu}}}{d}$ in eq. (2.17) in [82] for (N^2-1) free scalar fields. This also agrees with $a_{2,2} = \frac{\pi^4 \Delta}{45}$ found from the two-point thermal correlator using:

$$a_{2,2} = \frac{\pi^4 \Delta}{45} = \left(\frac{1}{2}\right)^2 \frac{\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^{(\mathrm{can})}} b_{T^{(\mathrm{can})}_{\mu\nu}}}{\frac{C_T}{S_d^2}}, \tag{C.4.7}$$

using $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T^{(\mathrm{can})}} = -\frac{4\Delta}{3 S_d}$ in this normalization and $C_T = \frac{4}{3}(N^2-1)$. This is simply related to the one-point function for the unit-normalized stress tensor by (to leading order in N)

$$\begin{aligned} b_{T_{\mu\nu}} &= \frac{b_{T^{(\mathrm{can})}_{\mu\nu}}}{\frac{\sqrt{C_T}}{S_d}} \\ &\approx -\frac{2\pi^4 N}{15\sqrt{3}}. \end{aligned} \tag{C.4.8}$$

Let us now consider the thermalization of the stress tensor, keeping all the index structures. To compare the thermal two-point function with the heavy-heavy-light-light correlator, we want to relate the dimension of the heavy operator, Δ_H, to the inverse temperature β. Consider the expectation value of the stress tensor in a heavy state created by $\mathcal{O}_H$ on the cylinder $\mathbf{R} \times S^3$

$$\langle \mathcal{O}_H | T^{\mu\nu}(x^0_{E,2}, \hat{n}) | \mathcal{O}_H \rangle_{\mathrm{cyl}} = \lim_{x_3 \to \infty} |x_3|^{2\Delta_H} |x_2|^4 \lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}} \frac{Z^\mu Z^\nu - \frac{1}{4}\delta^{\mu\nu} Z^\rho Z_\rho}{|x_{13}|^{2\Delta_H - 2} |x_{23}|^2 |x_{12}|^2}, \tag{C.4.9}$$

where the RHS is found by a conformal transformation to the plane with $Z^\mu = \left(\frac{x^\mu_{12}}{|x_{12}|^2} + \frac{x^\mu_{23}}{|x_{23}|^2}\right)$. When $x_1 = 0$ and $x_3 \to \infty$, it is seen that $Z^\mu = -\frac{x^\mu_2}{|x_2|^2}$ and (C.4.9) only depends on $\hat{x}^\mu = \frac{x^\mu_{21}}{|x_{21}|} = \hat{r}$, where $\hat{r}$ is a radial unit vector. In radial quantization it follows that

$$\langle \mathcal{O}_H | T^{\mu\nu}(x^0_{E,2}, \hat{n}) | \mathcal{O}_H \rangle_{\mathrm{cyl}} = \frac{\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}}}{R^4} (\hat{e}_\mu \hat{e}_\nu - \frac{1}{4}\delta_{\mu\nu}) \tag{C.4.10}$$

where we reintroduced the radius of the sphere R, $\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}}$ is the OPE coefficient of $T_{\mu\nu}$ in the $\mathcal{O}_H \times \mathcal{O}_H$ OPE and $\hat{e}_\mu = (1, 0, 0, 0)$.

The thermal one-point function of an operator $\mathcal{O}_{\tau,s}$, with twist τ and spin s, on $S^1 \times S^3$ is fixed by conformal symmetry [82]

$$\langle \mathcal{O}_{\tau,s}(x) \rangle_\beta = \frac{b_{\mathcal{O}_{\tau,s}} f_{\mathcal{O}_{\tau,s}}(\frac{\beta}{R})}{\beta^{\tau+s}} (e^{\mu_1} \cdots e^{\mu_s} - (\text{traces})), \tag{C.4.11}$$

where $f_{\mathcal{O}_{\tau,s}}(0) = 1$ and $e^\mu = (1, 0, 0, 0)$.

We assume thermalization of the stress tensor in the heavy state:

$$\langle \mathcal{O}_H | T_{\mu\nu}(x) | \mathcal{O}_H \rangle = \langle T_{\mu\nu}(x) \rangle_\beta \tag{C.4.12}$$

where $\langle T^{\mu\nu}(x) \rangle_\beta$ is the thermal one-point function at inverse temperature β evaluated on $S^1 \times S^3$, with R being the radius of S^3. Using (C.4.10)-(C.4.12) we find

$$\frac{\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}}}{R^4} = \frac{b_{T_{\mu\nu}} f_{T_{\mu\nu}}(\frac{\beta}{R})}{\beta^4}. \tag{C.4.13}$$

Using (C.4.13) for $R \to \infty$ in the free adjoint scalar theory, together with the one-point function $b_{T_{\mu\nu}} = -\frac{2\pi^4 N}{15\sqrt{3}}$ and the OPE coefficient $\lambda_{\mathcal{O}_H \mathcal{O}_H T_{\mu\nu}} = -\frac{4\Delta_H}{3\sqrt{C_T}}$, one finds the following relation between $\mu = \frac{160\Delta_H}{3C_T}$ and the inverse temperature β:

$$\mu = \frac{8}{3} \left(\frac{\pi R}{\beta} \right)^4. \tag{C.4.14}$$

This agrees with (7.50).

Appendix C.5. Dimension-six spin-four single trace operator

We want to calculate the contribution of the single trace operator with $\tau = 2$ and $s = 4$. The unit-normalised $\mathcal{O}_{2,4}$ operator is given by[52]

$$\begin{aligned} \Xi_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{35}N} :Tr\big(&\phi(\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi) - 16(\partial_{(\mu}\phi)(\partial_\nu\partial_\rho\partial_{\sigma)}\phi) \\ &+ 18(\partial_{(\mu}\partial_\nu\phi)(\partial_\rho\partial_{\sigma)}\phi) - (\text{traces})\big) : (x). \end{aligned} \tag{C.5.1}$$

The relative coefficients are fixed by demanding that it is a primary operator $[K_\alpha, \Xi_{\mu\nu\rho\sigma}] = 0$. Explictily, this is done using the conformal algebra

$$\begin{aligned} [K_\mu, P_\nu] &= 2i(\eta_{\mu\nu} D - M_{\mu\nu}), \\ [M_{\mu\nu}, P_\rho] &= -i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu), \end{aligned} \tag{C.5.2}$$

[52] We denote this operator either as $\mathcal{O}_{2,4}$ or $\Xi_{\mu\nu\rho\sigma}$ depending whether we want to explicitly list the indices or not.

and the action on the fundamental field ϕ

$$\begin{aligned} P_\mu \phi(0) &= -i\partial_\mu \phi(0), \\ D\phi(0) &= i\phi(0). \end{aligned} \tag{C.5.3}$$

The relevant commutators in order to fix $\Xi_{\mu\nu\rho\sigma}$ are

$$\begin{aligned}
[K_\alpha, P_\mu \phi] =& -2\eta_{\alpha\mu}\phi, \\
[K_\alpha, P_\mu P_\nu \phi] =& -4\eta_{\alpha\mu} P_\nu \phi - 4\eta_{\alpha\nu} P_\mu \phi + 2\eta_{\mu\nu} P_\alpha \phi, \\
[K_\alpha, P_\mu P_\nu P_\rho \phi] =& -6\eta_{\alpha\mu} P_\nu P_\rho \phi - 6\eta_{\alpha\nu} P_\mu P_\rho \phi - 6\eta_{\alpha\rho} P_\nu P_\mu \phi \\
&+ 2\eta_{\mu\nu} P_\rho P_\alpha \phi + 2\eta_{\rho\nu} P_\mu P_\alpha \phi + 2\eta_{\mu\rho} P_\nu P_\alpha \phi, \\
[K_\alpha, P_\mu P_\nu P_\rho P_\sigma \phi] =& -8\eta_{\alpha\mu} P_\nu P_\rho P_\sigma \phi - 8\eta_{\alpha\nu} P_\mu P_\rho P_\sigma \phi - 8\eta_{\alpha\rho} P_\nu P_\mu P_\sigma \phi \\
&+ 2\eta_{\mu\nu} P_\rho P_\sigma P_\alpha \phi + 2\eta_{\mu\rho} P_\nu P_\sigma P_\alpha \phi + 2\eta_{\mu\sigma} P_\rho P_\nu P_\alpha \phi \\
&+ 2\eta_{\nu\rho} P_\mu P_\sigma P_\alpha \phi + 2\eta_{\nu\sigma} P_\mu P_\rho P_\alpha \phi + 2\eta_{\rho\sigma} P_\mu P_\nu P_\alpha \phi \\
&- 8\eta_{\alpha\sigma} P_\nu P_\rho P_\mu \phi,
\end{aligned} \tag{C.5.4}$$

which can also be found in e.g. Appendix F in [34].

The thermal one-point function of this operator is found from Wick contractions to be

$$\langle \Xi_{\mu\nu\rho\sigma} \rangle_\beta = \frac{8(\pi T)^6 N}{27\sqrt{35}} \left(e_\mu e_\nu e_\rho e_\sigma - \text{(traces)} \right). \tag{C.5.5}$$

Moreover, the three-point function with operators $\mathcal{O}_\Delta(x) = \frac{1}{\sqrt{\Delta N^\Delta}} : Tr\left(\phi^\Delta\right) : (x)$ can again be calculated using Wick contractions similarly to how it was done for $T^2_{\mu\nu\rho\sigma}$ in Appendix A. By explicit calculation one finds

$$\langle \mathcal{O}_\Delta(x_1) \mathcal{O}_\Delta(x_2) \Xi_{\mu\nu\rho\sigma}(x_3) \rangle = \frac{4\Delta}{\sqrt{35}N} \frac{Z_\mu Z_\nu Z_\rho Z_\sigma - \text{(traces)}}{|x_{12}|^{2\Delta-2} |x_{13}|^2 |x_{23}|^2}, \tag{C.5.6}$$

and therefore the OPE coefficient $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}}$ is given by

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}} = \frac{4\Delta}{\sqrt{35}N}. \tag{C.5.7}$$

Now, it is easy to check that

$$\frac{1}{2^4} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \mathcal{O}_{2,4}} b_{\mathcal{O}_{2,4}} = \frac{2\pi^6 \Delta}{945}, \tag{C.5.8}$$

which agrees with $a_{2,4}$ in (7.85).

Appendix C.6. Thermal one-point functions of multi-trace operators in the large-N limit

In (7.110), it was shown that $a_{4,4}$ was due to double trace operators which were normal ordered products of single trace operators without any derivatives. There are, however, other double trace operators that have the same quantum numbers and are schematically represented as $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$. Concretely, the double trace operators with twist and spin four besides $(T^2)_{\mu\nu\rho\sigma}$ and $(\mathcal{O}^{\mathrm{DT}})_{\mu\nu\rho\sigma}$ are $[\mathcal{O}_2\mathcal{O}_2]_{0,4}$ and $[\mathcal{O}_2 T_{\mu\nu}]_{0,2}$. We argue that the thermal one-point functions of these operators are subleading in the large-N limit when evaluated on the plane.

Consider the thermal one-point function of a double trace operator $[\mathcal{O}_a\mathcal{O}_b]_{n,l} = \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b + \ldots$, where $\mathcal{O}_a$ and $\mathcal{O}_b$ are single trace primary operators and dots represent terms where derivatives acts on $\mathcal{O}_a$ as well, in order to make $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$ a primary operator. The term in the thermal one-point function that behaves as N^k (N^2 for double trace operators) comes from contracting the fundamental field within each trace separately. Therefore we have

$$\langle \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta \approx \langle \mathcal{O}_a\rangle_\beta\langle \partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta + \mathcal{O}(1), \tag{C.6.1}$$

which is simply due to large-N factorization. As $\partial^{2n}\partial^l\mathcal{O}_b$ is a descendant of $\mathcal{O}_b$, it is easy to explicitly show that $\langle\partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta = 0$ for $n \neq 0$ or $l \neq 0$, from which it follows that

$$\langle \mathcal{O}_a\partial^{2n}\partial^l\mathcal{O}_b\rangle_\beta = \mathcal{O}(1). \tag{C.6.2}$$

Similar reasoning holds for all terms in $[\mathcal{O}_a\mathcal{O}_b]_{n,l}$, so we conclude for $n \neq 0$ or $l \neq 0$ that

$$\langle [\mathcal{O}_a\mathcal{O}_b]_{n,l}\rangle_\beta = \mathcal{O}(1). \tag{C.6.3}$$

It is easy to generalise (n and/or l non-zero)

$$\langle [\mathcal{O}_{a_1}\ldots\mathcal{O}_{a_k}]_{n,l}\rangle_\beta = \mathcal{O}(N^{k-2}). \tag{C.6.4}$$

Using the canonical scaling for the OPE coefficients (7.95) it is found that these multi-trace operators give a suppressed contribution to the thermal two point function in the large-N limit:

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta[\mathcal{O}_{a_1}\ldots\mathcal{O}_{a_k}]_{n,l}}\langle [\mathcal{O}_{a_1}\ldots\mathcal{O}_{a_k}]_{n,l}\rangle_\beta = \mathcal{O}\Big(\frac{1}{N^2}\Big). \tag{C.6.5}$$

The conclusion is that these operators with $n \neq 0$ or $l \neq 0$ do not contribute to the thermal two-point functions to leading order in N. Note that for $n = l = 0$, the operator is just $: \mathcal{O}_{a_1}\mathcal{O}_{a_2} \dots \mathcal{O}_{a_k} :$ and it does contribute to the thermal 2pt function since

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta [\mathcal{O}_{a_1} \dots \mathcal{O}_{a_k}]_{n=0,l=0}} \langle [\mathcal{O}_{a_1} \dots \mathcal{O}_{a_k}]_{n=0,l=0} \rangle_\beta = \mathcal{O}(1). \tag{C.6.6}$$

From (C.6.5) it is seen that multi stress tensor operators of the schematic form $[T^k]_{n,l}$ with either n or l, or both, being non-zero will not contribute to the thermal correlator to leading order in N on the plane.

Appendix C.7. Free boson in two dimensions

In this appendix we discuss free scalars in two dimensions. We first consider a single scalar and then the case of the $SU(N)$ adjoint scalar. We compute two-point functions of a particular class of quasi-primary operators at finite temperature $1/\beta$. These two-point functions are not determined by the conformal symmetry, because the quasi-primary operators do not transform covariantly from the plane to the cylinder. They transform covariantly only with respect to the global conformal transformations. The only operators that have the non-zero thermal one-point functions are the Virasoro descendants of the vacuum and therefore, only these operators contribute to the thermal two-point function of the quasi-primary operators[53]. Virasoro descendants of the vacuum have different OPE coefficients with external quasi-primary operators compared with the case when primary external operators are considered.[54]

C.7.1. Review free boson in two dimensions

We consider single free boson $\phi(z)$ in two dimensions. The stress tensor can be written in terms of Virasoro modes as

$$T(z) = \sqrt{2} \sum_n z^{-n-2} L_n. \tag{C.7.1}$$

[53] We check this explicitly up to the $\mathcal{O}(1/\beta^4)$.

[54] Deviation from the Virasoro vacuum block in the Regge limit of four-point HHLL correlator is observed in [200] as well.

This stress tensor is unit-normalized

$$\langle T(z)T(w)\rangle = \frac{1}{(z-w)^4}. \tag{C.7.2}$$

The fundamental field can be expressed as Laurent series

$$\partial\phi(z) = \sum_{n=-\infty}^{+\infty} z^{-n-1}\alpha_n, \tag{C.7.3}$$

where oscillators α_n obey the following algebra

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}. \tag{C.7.4}$$

They act on the vacuum as

$$\alpha_n|0\rangle = 0, \qquad n \geq 0. \tag{C.7.5}$$

The two-point function of the fundamental fields is given by

$$\langle\partial\phi(z)\partial\phi(w)\rangle = \frac{1}{(z-w)^2}. \tag{C.7.6}$$

The unit-normalized stress tensor can be expressed in terms of the fundamental field as

$$T(z) = \frac{1}{\sqrt{2}} :\partial\phi\partial\phi:(z) = \frac{1}{\sqrt{2}}\sum_{m,n} z^{-m-n-2} :\alpha_m\alpha_n:, \tag{C.7.7}$$

where $:ab:$ denotes product of operators a and b with the corresponding free theory oscillators being normally ordered such that the operators annihilating the vacuum are put at the rightmost position. Then, it follows

$$L_n = \frac{1}{2}\sum_m :\alpha_{n-m}\alpha_m: = \frac{1}{2}\left(\sum_{m\geq 0}\alpha_{n-m}\alpha_m + \sum_{m<0}\alpha_m\alpha_{n-m}\right). \tag{C.7.8}$$

C.7.2. Thermal two-point function of quasi-primary operator

We are interested in computing the thermal two-point function of quasi-primary operators at temperature $1/\beta$. Quasi-primary operators $\mathcal{O}(z)$ are defined as $[L_1, \mathcal{O}(z)] = 0$, or equivalently, in therms of their asymptotic in-states $\mathcal{O}(0)|0\rangle = |\mathcal{O}\rangle$, as $L_1|\mathcal{O}\rangle = 0$. We denote the quantum numbers of quasi-primary operators that correspond to eigenvalues of L_0 and $\bar{L}_0$ by $(h,\bar{h})$. We consider the following unit-normalized quasi-primary operator with quantum numbers $(h,0)$

$$\mathcal{O}_h(z) = \frac{1}{\sqrt{h!}} : (\partial\phi)^h : (z) = \frac{1}{\sqrt{h!}} \sum_{m_1,m_2,\ldots,m_h} z^{-\sum_{i=1}^{h} m_i - h} : \alpha_{m_1} \ldots \alpha_{m_h} :, \tag{C.7.9}$$

which is properly defined when h is a positive integer. Its asymptotic in-state is given by

$$|\mathcal{O}_h\rangle = \mathcal{O}_h(0)|0\rangle = \frac{1}{\sqrt{h!}}(\alpha_{-1})^h|0\rangle. \tag{C.7.10}$$

One can check that this operator is a quasi-primary but not a Virasoro primary.

The thermal two-point function of this operator for even h is given by

$$\begin{aligned}\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = &\sum_{n=0}^{\frac{1}{2}(h-2)} \frac{h!}{4^n(h-2n)!}\left(\frac{2\zeta(2)}{\beta^2}\right)^{2n}\left(\sum_{m=-\infty}^{\infty}\frac{1}{(z+m\beta)^2}\right)^{h-2n} \\ &+ \frac{2^h\pi}{\Gamma\left(\frac{1}{2}-\frac{h}{2}\right)^2\Gamma(h+1)}\left(\frac{2\zeta(2)}{\beta^2}\right)^h.\end{aligned} \tag{C.7.11}$$

This expression is obtained by writing all possible Wick contractions between fundamental fields $\partial\phi$, including those that belong to same operator $\mathcal{O}_h$, that we call self-contractions. Fundamental fields are separated along the thermal circle in all Wick contractions. Factors $\left(\frac{2\zeta(2)}{\beta^2}\right)$ are due to the self-contractions,

$$\sum_{m=-\infty, m\neq 0}^{\infty} \frac{1}{\beta^2 m^2} = \left(\frac{2\zeta(2)}{\beta^2}\right). \tag{C.7.12}$$

The sum over n comes from doing n self-contractions within each of the external operators. Term $\frac{h!}{4^n(h-2n)!}$ counts the number of Wick contractions with n self-contractions for each external operator, including $1/\sqrt{h!}$ normalization factors. The term in the second line of (C.7.11) is due to the case when we take $n = h/2$ self-contractions in both external operators, i.e. it represents the disconnected contribution.

Since the state $\mathcal{O}_h$ is quasi-primary, it transforms properly only with respect to the global conformal transformation. These are just the Möbius transformations in two-dimensional spacetime $z \to \frac{az+b}{cz+d}$, with $ad - bc = 1$. On the other hand, the usual way to calculate the thermal two-point function of primary operators in two dimensions is to do a conformal transformation from the plane to the cylinder with radius β, $z \to \frac{\beta}{2\pi}\log(z)$. This transformation is clearly not one of the Möbius transformations and that is why we can not use this method to compute the thermal two-point functions of quasi-primary operators.

Expanding (C.7.11) for $T = \frac{1}{\beta} \to 0$ one finds

$$z^{2h}\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = 1 + \frac{h}{3}\frac{(\pi z)^2}{\beta^2} + \frac{h(h-\frac{1}{5})}{12}\frac{(\pi z)^4}{\beta^4} + \mathcal{O}\left(\frac{1}{\beta^6}\right). \qquad \text{(C.7.13)}$$

C.7.3. Quasi-primaries, OPE coefficients, and thermal one-point functions

In expansion (C.7.13), terms $\mathcal{O}(z^{h_1})$ are due to the quasi-primary operator with quantum numbers $(h_1, 0)$ in the operator product expansion $\mathcal{O}_h \times \mathcal{O}_h$. Identity in the expansion is due to the identity operator. We show that the second term on the RHS is due to the stress tensor. The quantum numbers of stress tensor $T(z)$ are $(2,0)$. First, we evaluate the thermal one-point function of the stress tensor

$$\langle T\rangle_\beta = \frac{1}{\sqrt{2}}\sum_{m=-\infty, m\neq 0}^{\infty}\frac{1}{\beta^2 m^2} = \frac{\pi^2}{3\sqrt{2}\beta^2}. \qquad \text{(C.7.14)}$$

This is obtained by the Wick contractions of fundamental fields in the stress tensor, that are separated along the thermal circle. The same result can be obtained by the transform of the stress tensor from the plane to the cylinder using the Schwarzian derivative.

We define the OPE coefficient of unit-normalized operator $\mathcal{O}$, with quantum numbers $(h_\mathcal{O}, 0)$, with two $\mathcal{O}_h$ operators as

$$\langle \mathcal{O}_h(z_1)\mathcal{O}_h(z_2)\mathcal{O}(z_3)\rangle = \frac{\lambda_{\mathcal{O}_h\mathcal{O}_h\mathcal{O}}}{(z_1-z_3)^{h_\mathcal{O}}(z_2-z_3)^{h_\mathcal{O}}(z_1-z_2)^{2h-h_\mathcal{O}}}. \qquad \text{(C.7.15)}$$

Next, we evaluate its OPE coefficient of the stress tensor with $\mathcal{O}_h$ by doing the Wick contractions between fundamental fields

$$\langle \mathcal{O}_h(z_1)\mathcal{O}_h(z_2)T(z_3)\rangle = \sqrt{2}h\frac{1}{(z_1-z_3)^2(z_2-z_3)^2(z_1-z_2)^{2(h-1)}}, \qquad \text{(C.7.16)}$$

therefore $\lambda_{\mathcal{O}_h\mathcal{O}_h T} = \sqrt{2}h$. This OPE coefficient is fixed by the Ward identity. Now, it follows

$$z^2\lambda_{\mathcal{O}_h\mathcal{O}_h T}\langle T\rangle_\beta = \frac{h}{3}\frac{(\pi z)^2}{\beta^2}, \qquad \text{(C.7.17)}$$

which reproduces the second term on the RHS of (C.7.13).

We are now interested in the contributions of quasi-primary operators with quantum numbers $(4,0)$. There are only two linearly independent operators with these quantum numbers given by[55]

$$
\begin{aligned}
:TT:(z) &= \frac{1}{\sqrt{24}} :(\partial\phi)^4:(z) = \frac{1}{\sqrt{24}} \sum_{a,b,c,d} z^{-a-b-c-d-4} :\alpha_a\alpha_b\alpha_c\alpha_d:, && \text{(C.7.18)}\\
\Lambda_4(z) &= \sqrt{\frac{10}{27}}\Big(\sum_{m,n=-\infty}^{\infty} z^{-m-n-4} * L_m L_n * \\
&\qquad - \frac{3}{10}\sum_{m=-\infty}^{\infty} z^{-m-4}(m+2)(m+3)L_m\Big), && \text{(C.7.19)}
\end{aligned}
$$

where $*ab*$ denotes the product where the relevant Virasoro generators are normally ordered. It should be noted that the operator $\Lambda_4(z)$ is Virasoro descendant of unity, while $:TT:(z)$ is not. The relevant asymptotic in-states are given by

$$
\begin{aligned}
|:TT:\rangle &=: TT:(0)|0\rangle = \frac{1}{\sqrt{24}}(\alpha_{-1})^4|0\rangle,\\
|\Lambda_4\rangle &= \Lambda_4(0)|0\rangle = \sqrt{\frac{10}{27}}\left(L_{-2}^2 - \frac{3}{5}L_{-4}\right)|0\rangle.
\end{aligned} \qquad \text{(C.7.20)}
$$

In terms of oscillators, $|\Lambda_4\rangle$ state can be represented as

$$
|\Lambda_4\rangle = \sqrt{\frac{10}{27}}\left(\frac{1}{4}(\alpha_{-1})^4 + \frac{2}{5}\alpha_{-1}\alpha_{-3} - \frac{3}{10}(\alpha_{-2})^2\right)|0\rangle. \qquad \text{(C.7.21)}
$$

From eqs. (C.7.20) and (C.7.21) one can see that $|:TT:\rangle$ and $|\Lambda_4\rangle$ are the only quasi-primary states with quantum numbers $(4,0)$. Namely, all such states have to be linear combinations of the following states

$$
\alpha_{-4}|0\rangle, \qquad \alpha_{-3}\alpha_{-1}|0\rangle, \qquad \alpha_{-2}^2|0\rangle, \qquad \alpha_{-2}\alpha_{-1}^2|0\rangle, \qquad \alpha_{-1}^4|0\rangle, \qquad \text{(C.7.22)}
$$

because

$$
L_0\left(\prod_{i=1}^{N}\alpha_{-k_i}\right)|0\rangle = \left(\sum_{i=1}^{N}k_i\right)\left(\prod_{i=1}^{N}\alpha_{-k_i}\right)|0\rangle, \qquad \text{(C.7.23)}
$$

[55] Both of them are unit-normalized.

where $k_i > 0$. It is straightforward to check

$$\begin{aligned} L_1\alpha_{-4}|0\rangle &= 4\alpha_{-3}|0\rangle, \\ L_1\alpha_{-3}\alpha_{-1}|0\rangle &= 3\alpha_{-2}\alpha_{-1}|0\rangle, \\ L_1\alpha_{-2}^2|0\rangle &= 4\alpha_{-2}\alpha_{-1}|0\rangle, \\ L_1\alpha_{-2}\alpha_{-1}^2|0\rangle &= 2\alpha_{-1}^3|0\rangle, \\ L_1\alpha_{-1}^4|0\rangle &= 0. \end{aligned} \tag{C.7.24}$$

It follows that $\alpha_{-1}^4|0\rangle$ is already quasi-primary and one can make only one more as $\alpha_{-3}\alpha_{-1}|0\rangle - \frac{3}{4}\alpha_{-2}\alpha_{-2}|0\rangle$.[56] $|:TT:\rangle$ and $|\Lambda_4\rangle$ are just the linear combination of these two states with overall normalization.

Now, one can calculate the overlap of $|:TT:\rangle$ and $|\Lambda_4\rangle$ states as

$$\langle 0|\Lambda_4(0):TT:(0)|0\rangle = \frac{\sqrt{5}}{3}. \tag{C.7.25}$$

The state orthogonal to $|\Lambda_4\rangle$ can be written as

$$|\tilde{\Lambda}_4\rangle = \frac{3}{2}\left(:TT:(0) - \frac{\sqrt{5}}{3}\Lambda_4(0)\right)|0\rangle. \tag{C.7.26}$$

Using (C.7.20) and (C.7.21), it can be written in terms of free theory oscillators.

We compute the OPE coefficients of $:TT:$ and Λ_4 with two $\mathcal{O}_h$ operators. We express all states in terms of free theory oscillators and use algebra (C.7.4) to find

$$\lambda_{\mathcal{O}_h\mathcal{O}_h:TT:} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|:TT:\rangle = \frac{\sqrt{6}}{2}h(h-1), \tag{C.7.27}$$

$$\lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|\Lambda_4\rangle = \sqrt{\frac{5}{6}}h\left(h-\frac{1}{5}\right), \tag{C.7.28}$$

$$\lambda_{\mathcal{O}_h\mathcal{O}_h\tilde{\Lambda}_4} = \langle\mathcal{O}_h|\mathcal{O}_h(1)|\tilde{\Lambda}_4\rangle = \frac{2}{\sqrt{6}}h\left(h-2\right). \tag{C.7.29}$$

Now, we evaluate the thermal one-point functions of Λ_4 and $\tilde{\Lambda}_4$. From (3.4) in [179] we have

$$\langle *T^2*\rangle_\beta = \frac{3\pi^4}{20\beta^4}, \tag{C.7.30}$$

[56] These states are not unit-normalized.

which is the thermal one-point function of the first term on the RHS of (C.7.19). The second term can be written as $-\frac{3}{10}\sum_{m=-\infty}^{\infty} z^{-m-4}(m+2)(m+3)L_m = -\frac{3}{10\sqrt{2}}\partial^2 T(z)$. It is clear that it will not affect the thermal one-point function of $\Lambda_4(z)$, as $\langle \partial^2 T\rangle_\beta = 0$.

Therefore, from (C.7.19), we have

$$\langle \Lambda_4\rangle_\beta = \sqrt{\frac{10}{27}}\langle *T^2*\rangle_\beta = \frac{\pi^4}{2\sqrt{30}\beta^4}. \tag{C.7.31}$$

Now, it follows

$$z^4\langle \Lambda_4\rangle_\beta \lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \frac{\pi^4 z^4}{12\beta^4}h\left(h-\frac{1}{5}\right), \tag{C.7.32}$$

which is the third therm at the RHS of (C.7.13). On the other hand, we can evaluate the thermal one-point function of $:TT:(z)$ operator by Wick contractions of fundamental fields separated along the thermal circle

$$\langle :TT:\rangle_\beta = \frac{\pi^4}{6\sqrt{6}\beta^4}. \tag{C.7.33}$$

Using (C.7.26), it is straightforward to confirm that $\langle\tilde{\Lambda}_4\rangle_\beta = 0$. Therefore, as we expected, operator $\tilde{\Lambda}_4$ does not contribute to the thermal two-point function of $\mathcal{O}_h$ operators, even thought it is present in the operator product expansion $\mathcal{O}_h \times \mathcal{O}_h$.

This is a general property of two-dimensional CFTs, that only the operators in the Virasoro vacuum module have non-zero expectation value on the cylinder.

C.7.4. Free adjoint scalar model in two dimensions

In this subsection we study a large-c theory. Consider the free adjoint $SU(N)$ scalar in 2d with

$$\partial\phi(z)^a{}_b = \sum_m z^{-m-1}(\alpha_m)^a{}_b \tag{C.7.34}$$

with

$$[(\alpha_m)^a{}_b, (\alpha_n)^c{}_d] = m\delta_{m+n}\Big(\delta^a{}_d\delta^c{}_b - \frac{1}{N}\delta^a{}_b\delta^c{}_d\Big). \tag{C.7.35}$$

The thermal two point of the quasi-primary operator $\mathcal{O}_h = \frac{1}{\sqrt{hN^h}} : Tr((\partial\phi)^h):$ follows immediately from the result in four dimensions upon replacing the propagator of fundamental fields. We find that

$$\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = g_{2d}(z)^h + \frac{\pi^4 h(h-2)}{9\beta^4}g_{2d}(z)^{h-2} + \ldots, \tag{C.7.36}$$

where

$$\begin{aligned} g_{2d}(z) &= \sum_{m=-\infty}^{\infty} \frac{1}{(z+m\beta)^2} \\ &= \Big(\frac{\pi}{\beta \sin(\pi z/\beta)}\Big)^2. \end{aligned} \tag{C.7.37}$$

Expanding (C.7.36) for $\beta \to \infty$ we find

$$\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle_\beta = z^{-2h}\Big[1 + \frac{\pi^2 h}{3\beta^2}z^2 + \frac{\pi^4 h(15h-19)}{90\beta^4}z^4 + \mathcal{O}(\beta^{-6})\Big]. \tag{C.7.38}$$

Consider first the normalized stress tensor which is given by

$$T = \frac{1}{\sqrt{2}N} : Tr(\partial\phi\partial\phi) :, \tag{C.7.39}$$

with $c = N^2$ so that $\langle T(z)T(0)\rangle = \frac{1}{z^4}$. By calculating the OPE coefficient with $\mathcal{O}_h$ and the thermal one-point function of T, one finds that these are the same as those for the scalar $Tr(\phi^2)$ operator in four dimensions so that $\langle T\rangle_\beta = \frac{\pi^2 N}{3\sqrt{2}\beta^2}$ and $\lambda_{\mathcal{O}_h\mathcal{O}_h T} = \frac{\sqrt{2}h}{N}$, and the product reproduces the weight two term in (C.7.38):

$$\langle T\rangle_\beta \lambda_{\mathcal{O}_h\mathcal{O}_h T} = \frac{\pi^2 h}{3\beta^2}. \tag{C.7.40}$$

Consider now $*TT*$ defined by

$$*TT*(0) = \lim_{z\to 0} T(z)T(0) - (\text{sing. terms}). \tag{C.7.41}$$

The OPE of the stress tensor in (C.7.39) can be found in the free theory by first performing Wick contractions

$$\begin{aligned} T(z)T(0) &= \frac{1}{2N^2} : Tr(\partial\phi(z)\partial\phi(z)) :: Tr(\partial\phi(0)\partial\phi(0)) : \\ &=: TT : (0) + \ldots + \frac{2}{N^2 z^2} : Tr(\partial\phi(z)\partial\phi(0)) : + \frac{1}{z^4}, \end{aligned} \tag{C.7.42}$$

and expanding the second term in (C.7.42) for $z \to 0$ we find

$$\begin{aligned} T(z)T(0) =&: TT : (0) + \ldots + \frac{2}{N^2 z^2} : Tr(\partial\phi(0)\partial\phi(0)) : \\ &+ \frac{2}{N^2 z} : Tr(\partial^2\phi(0)\partial\phi(0)) : + \frac{1}{N^2} : Tr(\partial^3\phi(0)\partial\phi(0)) : + \ldots \\ &+ \frac{1}{z^4}, \end{aligned} \tag{C.7.43}$$

where the dots refer to higher order terms in z. Inserting the OPE (C.7.43) in (C.7.41) we find that

$$*TT*(0) =: TT:(0) + \frac{1}{N^2} : Tr(\partial^3\phi(0)\partial\phi(0)) : . \tag{C.7.44}$$

Consider the state $*TT*(0)|0\rangle$, which is given in terms of oscillator modes by

$$*TT*(0)|0\rangle = \frac{1}{2N^2} Tr(\alpha_{-1}^2)Tr(\alpha_{-1}^2)|0\rangle + 2\frac{1}{N^2}Tr(\alpha_{-3}\alpha_{-1})|0\rangle. \tag{C.7.45}$$

Now $Tr(\alpha_{-1}^m)|0\rangle$ is a quasi-primary while $Tr(\alpha_{-3}\alpha_{-1})|0\rangle$ is not. One way to make it a quasi-primary is to simply remove the second term in (C.7.45) and then we get a quasi-primary state which is just $:TT:|0\rangle$. Another option is to remove a descendant of the stress tensor to construct $|\Lambda_4\rangle$. To do the latter we need to remove the descendant of the stress tensor with weight 4 given by $\partial^2 T$

$$\partial^2 T = \frac{\sqrt{2}}{N} : Tr(\partial^3\phi\partial\phi) : + \frac{\sqrt{2}}{N} : Tr(\partial^2\phi\partial^2\phi) : . \tag{C.7.46}$$

Acting on the vacuum we find

$$\partial^2 T(0)|0\rangle = \frac{2\sqrt{2}}{N} Tr(\alpha_{-3}\alpha_{-1})|0\rangle + \frac{\sqrt{2}}{N}Tr(\alpha_{-2}^2)|0\rangle. \tag{C.7.47}$$

Consider now $L_1 = \frac{\sqrt{2}}{N}(Tr(\alpha_{-1}\alpha_2)+Tr(\alpha_{-2}\alpha_3+\ldots))$ which acts as $L_1 Tr(\alpha_{-2}^2)|0\rangle = \frac{4\sqrt{2}}{N}Tr(\alpha_{-1}\alpha_{-2})|0\rangle$ and as $L_1 Tr(\alpha_{-3}\alpha_{-1})|0\rangle = \frac{3\sqrt{2}}{N}Tr(\alpha_{-1}\alpha_{-2})|0\rangle$. We can therefore construct a quasi-primary state annihilated by L_1: $Tr(\alpha_{-3}\alpha_{-1})|0\rangle - \frac{3}{4}Tr(\alpha_{-2}^2)|0\rangle$. The quasi-primary $|\Lambda_4\rangle$ is then given by:

$$\begin{aligned} |\Lambda_4\rangle &= \frac{1}{\sqrt{2}}\Big(*TT*(0)|0\rangle - \frac{3}{5\sqrt{2}N}\partial^2 T(0)|0\rangle\Big) \\ &= \frac{1}{2\sqrt{2}N^2}\Big(Tr(\alpha_{-1}^2)Tr(\alpha_{-1}^2)|0\rangle - \frac{6}{5}Tr(\alpha_{-2}^2)|0\rangle + \frac{8}{5}Tr(\alpha_{-1}\alpha_{-3})|0\rangle\Big) \end{aligned} \tag{C.7.48}$$

There are two more weight 4 single trace quasi-primary operators given by

$$\begin{aligned} \mathcal{O}^{(1)} &= \frac{1}{2N^2}Tr((\partial\phi)^4) \\ \mathcal{O}^{(2)} &= \frac{n_{\mathcal{O}^{(2)}}}{N}(Tr(\partial^3\phi\partial\phi) - \frac{3}{2}Tr(\partial^2\phi\partial^2\phi)), \\ &= \frac{n_{\mathcal{O}^{(2)}}}{N}(\frac{1}{2}\partial^2 Tr(\partial\phi\partial\phi) - \frac{5}{2}Tr(\partial^2\phi\partial^2\phi)), \end{aligned} \tag{C.7.49}$$

where $n_{\mathcal{O}^{(2)}}$ is some N-independent normalization constant. The state $|\Lambda_4\rangle$ can be written in terms of $:TT:(0)|0\rangle + a\mathcal{O}_2(0)|0\rangle$ in the following way

$$|\Lambda_4\rangle = \frac{1}{\sqrt{2}}\Big(:TT:(0)|0\rangle + \frac{2}{5Nn_{\mathcal{O}^{(2)}}}\mathcal{O}^{(2)}|0\rangle\Big). \tag{C.7.50}$$

The OPE coefficient for $:TT:$ is up to a normalization the same as the scalar dimension 4 double trace operator in 4d and is given by

$$\begin{aligned}\langle \mathcal{O}_h\mathcal{O}_h :TT:\rangle &= \frac{1}{hN^h}\frac{1}{2N^2}4h^2(3h-5)N^h\frac{1}{z_{13}^4z_{23}^4z_{12}^{2h-4}}\\ &= \frac{1}{N^2}2h(3h-5)\frac{1}{z_{13}^4z_{23}^4z_{12}^{2h-4}},\end{aligned} \tag{C.7.51}$$

where $4h^2(3h-5)$ come from the number of contractions giving planar diagrams. Consider now the OPE coefficient for $\mathcal{O}^{(2)}$. One finds

$$\begin{aligned}\langle \mathcal{O}_h\mathcal{O}_h\mathcal{O}^{(2)}\rangle &= \frac{n_{\mathcal{O}^{(2)}}N^h}{hN^{h+1}z_{13}^4z_{23}^4z_{12}^{2h-2}}\Big((-2)(-3)h^2(z_{13}^2+z_{23}^2)\\ &\quad -\frac{3}{2}2h^2(-2)^2)z_{13}z_{23}\Big) = \frac{6hn_{\mathcal{O}^{(2)}}}{Nz_{13}^4z_{23}^4z_{12}^{2h-4}}.\end{aligned} \tag{C.7.52}$$

Using (C.7.51), (C.7.52) and (C.7.50) we find the OPE coefficient for $|\Lambda_4\rangle$

$$\langle \mathcal{O}_h\mathcal{O}_h\Lambda_4\rangle = \frac{\sqrt{2}h(15h-19)}{5N^2}. \tag{C.7.53}$$

Note that the h dependence matches that of the weight 4 term in the two-point function (C.7.38). Additionally, the OPE coefficient given by (C.7.53) can not be extrapolated to the limit when $h \sim C_T$, as in this limit the planar expansion used for calculating (C.7.53) breaks down. For this reason, we can not test the thermalization of Λ_4 in heavy state $\mathcal{O}_{h_H}$. Let us consider the thermal one-point function which is given by

$$\langle \Lambda_4\rangle_\beta = \Big[\frac{1}{\sqrt{2}}b_T^2 + \mathcal{O}(1)\Big] = \frac{\pi^4N^2}{18\sqrt{2}\beta^4}, \tag{C.7.54}$$

where the term $\propto \frac{1}{N}\langle \mathcal{O}^{(2)}\rangle_\beta$ is subleading since it is single trace. We find that

$$\langle \Lambda_4\rangle_\beta \lambda_{\mathcal{O}_h\mathcal{O}_h\Lambda_4} = \frac{\pi^4h(15h-19)}{90\beta^4}, \tag{C.7.55}$$

which agrees with the weight 4 term in (C.7.38).

Note that it is explicitly seen that one can write Λ_4 either as $*TT*$ +(desc. of T) or as $:TT: + \frac{1}{N}\mathcal{O}_{ST}$ with $\mathcal{O}_{ST}$ a quasi-primary single trace operator. In this case the single trace operator which one needs to add to $:TT:$ to get Λ_4 can be written as a sum of descendants $\mathcal{O}^{(2)} \propto \partial^2 T - \frac{5}{\sqrt{2}} Tr(\partial^2\phi\partial^2\phi)$. Explicitly, we have

$$\begin{aligned}|\Lambda_4\rangle &= \frac{1}{\sqrt{2}}\Big[*TT*(0) - \frac{3}{5\sqrt{2}N}\partial^2 T(0)\Big]|0\rangle \\ &= \frac{1}{\sqrt{2}}\Big[:TT:(0) + \frac{2}{5Nn_{\mathcal{O}^{(2)}}}\mathcal{O}^{(2)}\Big]|0\rangle.\end{aligned} \tag{C.7.56}$$

As we saw above, using the second line in (C.7.56) it is straightforward to calculate correlation functions using Wick contractions to see that Λ_4 gives the full weight four contributions to the thermal two-point function for large-N theories.

Now, we consider the following quasi-primary operator

$$\mathcal{O}_\Delta(z,\bar{z}) = \frac{\sqrt{2}}{\sqrt{\Delta}N^{\Delta/2}} : Tr\left((\partial\phi\bar{\partial}\bar{\phi})^{\frac{\Delta}{2}}\right) : (z,\bar{z}), \tag{C.7.57}$$

where we denote the anti-holomorphic part of the free field by $\bar{\phi} = \bar{\phi}(\bar{z})$. The thermal two-point function of this operator, up to the terms subleading in large-N expansion, is given by

$$\begin{aligned}\langle\mathcal{O}_\Delta(z,\bar{z})\mathcal{O}_\Delta(0,0)\rangle_\beta &= \frac{\pi^{2\Delta}}{\beta^{2\Delta}\sin^\Delta\left(\frac{\pi z}{\beta}\right)\sin^\Delta\left(\frac{\pi\bar{z}}{\beta}\right)} \\ &= \frac{1}{(z\bar{z})^\Delta}\Big(1 + \frac{\pi^2\Delta(z^2+\bar{z}^2)}{6\beta^2} + \frac{\pi^4\Delta(5\Delta+2)}{360\beta^4}(z^4+\bar{z}^4) \\ &+ \frac{\pi^4\Delta^2}{36\beta^4}z^2\bar{z}^2 + \mathcal{O}\left(\frac{1}{\beta^6}\right)\Big).\end{aligned} \tag{C.7.58}$$

One can easily check that the OPE coefficients of stress tensor T and its anti-holomorphic partner $\bar{T}$ with $\mathcal{O}_\Delta$ are given by

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T} = \lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta\bar{T}} = \frac{\Delta}{\sqrt{2}N}, \tag{C.7.59}$$

while their thermal one-point function are given by

$$\langle T\rangle_\beta = \langle\bar{T}\rangle_\beta = \frac{\pi^2 N}{3\sqrt{2}\beta^2}. \tag{C.7.60}$$

It is easy to check that terms proportional to β^{-2} in (C.7.58) are contributions of T and $\bar{T}$ operators

$$\langle T\rangle_\beta \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta T} z^2 + \langle \bar{T}\rangle_\beta \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \bar{T}} \bar{z}^2 = \frac{\pi^2 \Delta (z^2+\bar{z}^2)}{6\beta^2}. \tag{C.7.61}$$

We compute the OPE coefficient of operators Λ_4, defined by (C.7.48), and its anti-holomorphic partner $\bar{\Lambda}_4$ with $\mathcal{O}_\Delta$ and obtain

$$\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \Lambda_4} = \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \bar{\Lambda}_4} = \frac{\Delta(5\Delta+2)}{10\sqrt{2}N^2}, \tag{C.7.62}$$

which agrees with (C.26) in [124]. Its thermal one-point function (which is the same as $\langle \bar{\Lambda}_4\rangle_\beta$) is given by (C.7.54). Another operator that contributes to thermal two-point function (C.7.58) is $:T\bar{T}:$. Its OPE coefficient with $\mathcal{O}_\Delta$ and thermal one-point function are given by

$$\begin{aligned} \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta :T\bar{T}:} &= \frac{\Delta^2}{2N^2} \\ \langle :T\bar{T}:\rangle_\beta &= \frac{\pi^4 N^2}{18\beta^4}. \end{aligned} \tag{C.7.63}$$

Again, it is easy to check

$$\begin{aligned} &\langle \Lambda_4\rangle_\beta \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \Lambda_4} z^4 + \langle \bar{\Lambda}_4\rangle_\beta \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \bar{\Lambda}_4} \bar{z}^4 + \langle :T\bar{T}:\rangle_\beta \lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta :T\bar{T}:} z^2\bar{z}^2 = \\ &= \frac{\pi^4 \Delta(5\Delta+2)}{360\beta^4}(z^4+\bar{z}^4) + \frac{\pi^4\Delta^2}{36\beta^4} z^2\bar{z}^2, \end{aligned} \tag{C.7.64}$$

which matches with the corresponding terms in (C.7.58).

The OPE coefficients $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \Lambda_4}$, $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta \bar{\Lambda}_4}$, and $\lambda_{\mathcal{O}_\Delta \mathcal{O}_\Delta :T\bar{T}:}$ can be extrapolated to the limit $\Delta \sim N^2$, by the same logic as in Appendix C. Then, we can explicitly check the thermalization property of Λ_4, $\bar{\Lambda}_4$, and $:T\bar{T}:$. To establish a relation between the inverse temperature β and the conformal dimension Δ_H of heavy state $\mathcal{O}_H = \mathcal{O}_{\Delta\sim N^2}$, we assume the thermalization of stress tensor

$$\langle T\rangle_\beta = \lambda_{\mathcal{O}_H \mathcal{O}_H T}, \tag{C.7.65}$$

which implies

$$\frac{\Delta_H}{N^2} = \frac{\pi^2}{3\beta^2}. \tag{C.7.66}$$

Using this relation, it is easy to show

$$\begin{aligned}\langle\Lambda_4\rangle_\beta &= \lambda_{\mathcal{O}_H\mathcal{O}_H\Lambda_4}\Big|_{\frac{\Delta_H^2}{N^2}},\\ \langle\bar\Lambda_4\rangle_\beta &= \lambda_{\mathcal{O}_H\mathcal{O}_H\bar\Lambda_4}\Big|_{\frac{\Delta_H^2}{N^2}},\\ \langle: T\bar T :\rangle_\beta &= \lambda_{\mathcal{O}_H\mathcal{O}_H:T\bar T:}\Big|_{\frac{\Delta_H^2}{N^2}}.\end{aligned} \tag{C.7.67}$$

This means that operators Λ_4, $\bar\Lambda_4$, and $: T\bar T :$ thermalize in the quasi-primary state $\mathcal{O}_H$ similarly to the thermalization in a Virasoro primary states in large-c theory, that was analyzed in [165].

Appendix C.8. Vector model

In this section we study the free scalar vector model at large-N. Consider the scalar operator

$$\mathcal{O}_\Delta = \frac{1}{\sqrt{\mathcal{N}(\Delta)}} : (\varphi^i\varphi^i)^{\frac{\Delta}{2}} : (x), \tag{C.8.1}$$

where $\mathcal{N}(\Delta)$ is a normalization constant which to leading order in N is given by

$$\mathcal{N}(\Delta) \approx (\Delta)!!N^{\frac{\Delta}{2}}. \tag{C.8.2}$$

The thermal two-point function is given by

$$\langle\mathcal{O}_\Delta(x)\mathcal{O}_\Delta(0)\rangle_\beta = \tilde g(x_E^0,|\mathbf{x}|)^\Delta + \left(\frac{\Delta}{2}\right)^2\frac{1}{\Delta}\tilde g(x_E^0,|\mathbf{x}|)^{\Delta-2} + \dots, \tag{C.8.3}$$

where

$$\begin{aligned}\tilde g(x_E^0,|\mathbf{x}|) &= \sum_{m=-\infty}^{\infty}\frac{1}{(x_E^0+m\beta)^2+\mathbf{x}^2}\\ &= \frac{\pi}{2\beta|\mathbf{x}|}\left[\mathrm{Coth}\Big(\frac{\pi}{\beta}(|\mathbf{x}|-ix_E^0)\Big)+\mathrm{Coth}\Big(\frac{\pi}{\beta}(|\mathbf{x}|+ix_E^0)\Big)\right].\end{aligned} \tag{C.8.4}$$

The thermal $a_{\tau,J}$ coefficients $a_{2,2}$ and $a_{4,4}$ are the same as in the adjoint model (this is so since the second term in (C.8.3) does not affect these):

$$\begin{aligned}a_{2,2} &= \frac{\pi^4\Delta}{45},\\ a_{4,4} &= \frac{\pi^8\Delta(\Delta-1)}{1050}.\end{aligned} \tag{C.8.5}$$

The unit-normalized stress tensor is given by

$$T_{\mu\nu}(x) = \frac{1}{3\sqrt{C_T}} : \left(\partial_\mu\varphi^i\partial_\nu\varphi^i - \frac{1}{2}\varphi^i\partial_\mu\partial_\nu\varphi^i - (\text{trace})\right) : (x), \tag{C.8.6}$$

where $C_T = \frac{4}{3}N$. The OPE coefficient of the stress tensor is again found by Wick contractions to be

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T_{\mu\nu}} = -\frac{4\Delta}{3\sqrt{C_T}}, \tag{C.8.7}$$

in agreement with the stress tensor Ward identity. The double-stress tensor is given by

$$T^2_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{2}} : T_{(\mu\nu}T_{\rho\sigma)} : -(\text{traces}), \tag{C.8.8}$$

and the OPE coefficient is calculated precisely as for the adjoint model and we find

$$\lambda_{\mathcal{O}_\Delta\mathcal{O}_\Delta T^2_{4,4}} = \frac{8\sqrt{2}}{9C_T}\Delta(\Delta-1). \tag{C.8.9}$$

There is another double-trace operator with twist 4 and spin 4 and takes the same form $:\mathcal{O}_2\mathcal{O}_{2,4}:$ as for the adjoint model

$$\begin{aligned}\mathcal{O}^{\text{DT}}_{\mu\nu\rho\sigma}(x) = \frac{1}{96\sqrt{70}N} : \varphi^i\varphi^i\Big(&\varphi^j\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\varphi^j - 16\partial_{(\mu}\varphi^j\partial_\nu\partial_\rho\partial_{\sigma)}\varphi^j \\ &+18\partial_{(\mu}\partial_\nu\varphi^j\partial_\rho\partial_{\sigma)}\varphi^j - (\text{traces})\Big) : (x).\end{aligned} \tag{C.8.10}$$

The OPE coefficient and the thermal one-point function yields the same result as for the corresponding operator in the adjoint model[57]. It then follows that the $a_{4,4}$ extracted from (C.8.3) is reproduced by the sum of the double stress tensor and (C.8.10).

Appendix C.9. Factorization of thermal correlators

In this appendix we argue for the factorization of thermal expectation values of multi-trace operators in large-C_T theories on $S^1 \times \mathbf{R}^{d-1}$. Consider the thermal two-point function of a scalar operator $\mathcal{O}$ with dimension Δ:

$$\langle\mathcal{O}(x)\mathcal{O}(0)\rangle_\beta = \langle\mathcal{O}\rangle_\beta\langle\mathcal{O}\rangle_\beta + \langle\mathcal{O}(x)\mathcal{O}(0)\rangle_{\beta,c}, \tag{C.9.1}$$

[57] Note that this is not true for all operators but is in line with the fact that $a_{4,4}$ is unaffected by the second term in (C.8.3).

where the second term consist of the connected part of the correlator. Note that the disconnected term in (C.9.1) is independent of the position x. On the other hand we can evaluate (C.9.1) using the OPE on the plane which takes the form

$$\mathcal{O}(x)\mathcal{O}(0) = \frac{1}{|x|^{2\Delta}} + \sum_{n,l} \lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{n,l}} x^{2n+l} [\mathcal{O}\mathcal{O}]_{n,l} + \dots, \tag{C.9.2}$$

when written in terms of primaries and the dots refer to terms surpressed in the large-C_T limit. Note that $\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{n,l}}$ are the MFT OPE coefficient which are of order 1. The term in (C.9.2) that is independent of x is due to the $n = l = 0$ term in (C.9.2) and inserting the OPE on the LHS of (C.9.2), we find that

$$\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{0,0}} \langle [\mathcal{O}\mathcal{O}]_{0,0} \rangle_\beta = \langle \mathcal{O} \rangle_\beta^2. \tag{C.9.3}$$

When $[\mathcal{O}\mathcal{O}]_{0,0}$ is unit-normalized the OPE coefficient is given by $\lambda_{\mathcal{O}\mathcal{O}[\mathcal{O}\mathcal{O}]_{0,0}} = \sqrt{2}$ and it follows that

$$\langle [\mathcal{O}\mathcal{O}]_{0,0} \rangle_\beta = \frac{1}{\sqrt{2}} \langle \mathcal{O} \rangle_\beta^2. \tag{C.9.4}$$

We therefore see the that the thermal one-point function of the double-trace operator factorizes on the plane. We expect a similar argument to hold for multi stress tensors.

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Република Србија
АГЕНЦИЈА ЗА КВАЛИФИКАЦИЈЕ
Београд, Мајке Јевросиме 51

Број: 612-03-27/2023-03
12.01.2023.године
МЛ

На основу члана 38, члана 20. став 1. тачка 7 и члана 5. став 1. тачка 10. Закона о националном оквиру квалификација Републике Србије („Сл. гласник РС", бр. 27/18, 6/20 и 129/2021 др. Закон"), члана 131. став 1. Закона о високом образовању („Сл. гласник РС", бр. 88/17, 27/18 – др. закон и 73/18), и члана 136. став 1. Закона о општем управном поступку („Службени гласник РС", бр. 18/16 и 95/18 – аутентично тумачење), решавајући по захтеву Петра Тадића из Краљева, Република Србија, за признавање високошколске исправе издате у Ирској, ради запошљавања,

директор Агенције за квалификације доноси

РЕШЕЊЕ

1. **Диплома издата 08.04.2022. године од стране Тринити колеџа (Trinity College), Даблин, Ирска, на име Петар Тадић, рођен 29.07.1994. године у Никшићу, Црна Гора, о завршеним докторским академским студијама високог образовања у трогодишњем трајању, докторска дисертација: "Conformal bootstrap and thermalization in holographic CFTs", звање/квалификација: Doctor in Philosophy/доктор наука (на основу превода овлашћеног судског тумача за енглески језик), признаје се као диплома докторских академских студија трећег степена високог образовања (180 ЕСПБ), у оквиру образовно-научног поља: Природно-математичке науке, научна односно стручна област: Математичке науке, која одговара нивоу 8. НОКС-а, ради запошљавања.**
2. **Ово решење омогућава имаоцу општи приступ тржишту рада у Републици Србији, али га не ослобађа од испуњавања посебних услова за бављење професијама које су регулисане законом или другим прописом.**
3. **Превод звања/квалификације из тачке 1. диспозитива овог решења које је са оригиналне стране јавне исправе превео овлашћени судски тумач за енглески језик, не представља стручни, академски, научни односно уметнички назив који у складу са чланом 12. ставом 1. тачка 9. Закона о високом образовању, утврђује Национални савет за високо образовање.**

Образложење

Агенцији за квалификације обратио се Петар Тадић из Краљева, Република Србија, захтевом од 10.01.2023. године за признавање дипломе Тринити колеџа, Даблин, Ирска, докторске академске студије високог образовања у трогодишњем трајању, докторска дисертација: "Conformal bootstrap and thermalization in holographic CFTs", звање/квалификација: Doctor in Philosophy/доктор наука, ради запошљавања.

Уз захтев, подносилац захтева доставио је:

1) оверену копију дипломе издате 08.04.2022. године од стране Тринити колеџа, Даблин, Ирска, докторска дисертација: "Conformal bootstrap and thermalization in holographic CFTs", звање/квалификација: Doctor in Philosophy/доктор наука;
2) оверени превод дипломе на српски језик;
3) докторску дисертацију;
4) апстракт докторске дисертације;
5) списак објављених научних радова;
6) дипломе о претходно стеченом високом образовању
7) радну биографију;
8) пријавни формулар;
9) доказ о уплати таксе за професионално признавање.

Одредбом члана 136. став 1. Закона о општем управном поступку прописано је да се решењем одлучује о праву, обавези или правном интересу странке.

Одредбом члана 38. став 1. Закона о националном оквиру квалификација Републике Србије прописано је да захтев за професионално признавање заинтересовано лице подноси Агенцији. Ставом 2. наведеног члана прописано је да професионално признавање врши ENIC/NARIC центар, као организациони део Агенције, по претходно извршеном вредновању страног студијског програма, у складу са овим и законом који уређује високо образовање. Ставом 3. наведеног члана прописано је да решење о професионалном признавању посебно садржи: назив, врсту, степен и трајање (обим) студијског програма, односно квалификације, који је наведен у страној високошколској исправи – на изворном језику и у преводу на српски језик и научну, уметничку, односно стручну област у оквиру које је остварен студијски програм, односно врсту и ниво квалификације у Републици и ниво НОКС–а којем квалификација одговара. Ставом 4. наведеног члана прописано је да директор агенције доноси решење о професионалном признавању у року од 90 дана од дана пријема уредног захтева. Ставом 5. наведеног члана прописано је да решење из става 4.овог члана не ослобађа имаоца од испуњавања посебних услова за обављање одређене професије прописане посебним законом. Ставом 6. наведеног члана прописано је да је решење о професионалном признавању коначно. Ставом 7. наведеног члана прописано је да уколико није другачије прописано, на поступак професионалног признавања примењује се закон којим се уређује општи управни поступак. Ставом 8. наведеног члана прописано је да решење о професионалном признавању има значај јавне исправе. Ставом 9. наведеног члана прописано је да ближе услове у погледу поступка професионалног признавања прописује министар надлежан за послове образовања.

Одредбом члана 20. став 1. тачка 7. Закона о националном оквиру квалификација Републике Србије, директор Агенције за потребе давања стручног мишљења у поступку вредновања страног студијског програма именује комисију од најмање три рецензента са листе рецензената коју утврђује Национални савет за високо образовање, у складу са законом који уређује високо образовање и овим законом.

Одредбом члана 131. став 1. Закона о високом образовању, прописано је да се вредновање страног студијског програма врши на основу врсте и нивоа постигнутих компетенција стечених завршетком студијског програма, узимајући у обзир систем образовања у земљи у којој је високошколска исправа стечена, услове уписа, права која проистичу из стране високошколске исправе у земљи у којој је стечена и друге релевантне чињенице,без разматрања формалних обележја и структуре студијског програма.

Одредбом члана 5. став 1. тачка 10. Закона о националном оквиру квалификација Републике Србије, прописано је да се осми ниво (ниво 8), стиче завршавањем докторских академских студија обима 180 ЕСПБ бодова (уз претходно завршене интегрисане академске, односно мастер академске студије).

Одлучујући о захтеву подносиоца, а након прибављеног мишљења комисије која је извршила вредновање страног студијског програма на основу врсте и нивоа постигнутих компетенција стечених завршетком студијског програма, узимајући у обзир систем образовања у земљи у којој је високошколска исправа стечена, услове уписа, права која проистичу из стране високошколске исправе у земљи у којој је стечена и друге релевантне чињенице,без разматрања формалних обележја и структуре студијског програма, одлучено је да се диплома Тринити колеџа, Даблин, Ирска, може признати као диплома докторских академских студија трећег степена високог образовања (180 ЕСПБ), која одговара нивоу 8. НОКС-а.

Са напред наведених разлога директор Агенције је нашао да су у конкретном случају испуњени претходно наведени сви законом прописани услови да се призна диплома Тринити колеџа, Даблин, Ирска, као диплома докторских академских студија трећег степена високог образовања (180 ЕСПБ), у оквиру образовно-научног поља: Природно-математичке науке, научна односно стручна област: Математичке науке, која одговара нивоу 8. НОКС-а, ради запошљавања.

Накнада за решење по захтеву се наплаћује на основу члана 2. став 3. Правилника о висини накнаде за трошкове поступка признавања страних школских исправа и признавање страних високошколских исправа у сврху запошљавања и о висини накнада за трошкове поступка давања одобрења другој организацији за стицање статуса јавно признатог организатора активности образовања одраслих (''Службени гласник РС'', бр. 1/2020) плаћена је и поништена.

Сходно претходно наведеном, донета је одлука као у диспозитиву решења.

Упутство о правном средству: Ово решење је коначно у управном поступку и против истог може се покренути управни спор. Тужба се подноси Управном суду у року од 30 дана од дана пријема овог решења.

Решење доставити:

- Петар Тадић, лично преузимање;
- Архиви

ДИРЕКТОР

проф. др Часлав Митровић

Ms. Ref. No.: [redacted]
Title: [redacted]
Authors: [redacted]
Nuclear Physics, Section B

Dear Tadic,

Given your expertise, we would like to invite you to review the above-mentioned manuscript that has been submitted for publication in Nuclear Physics, Section B.

To view the article, please click on [redacted]

If you are willing to review this manuscript, please click on the link below:
[redacted]

If you are unable to review this manuscript, please click on the link below. We would appreciate receiving suggestions for alternative reviewers:
[redacted]

Alternatively, you may register your response by accessing the Editorial Manager for Nuclear Physics, Section B as a REVIEWER using the login credentials below:
https://www.editorialmanager.com/npb/
Your username is: PetarTadic

If you need to retrieve password details, please go to: http://ees.elsevier.com/npb/automail_query.asp

Please bear in mind to select the 'Reviewer Login' button.

Please return your report within about two weeks, which will start from the date you let us know of your acceptance to review it.

If it is not possible for you to act as a reviewer for this article within this period, we would appreciate it if you could inform us as soon as possible.

As a reviewer you are entitled to complimentary access to Scopus and ScienceDirect for 30 days. Full instructions and details will be provided upon accepting this invitation to review.

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Thank you in advance for your kind help.

Yours sincerely,

Clay Cordova
Editor, Nuclear Physics, Section B

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Preprint: [redacted]
Authors: [redacted]
Title: [redacted]

Dear Petar Tadic,

In view of your expertise in this area, I would appreciate it if you could review the submission [redacted].

If available, the author's cover letter can be accessed by clicking on the 'open' button in the relevant section of/or as an attachment to the article status page.

The criteria for acceptance of a preprint by JHEP are high scientific quality, originality and relevance. If you consider this preprint acceptable for publication, please comment on how it meets these criteria.

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In order to let me know your intentions, please use the "ACCEPT/DECLINE ASSIGNMENT" tool within 3 days. In the hope that you will accept this assignment, we ask you to please bear in mind that your report is expected within 4 weeks from today.

AFTER you have accepted this assignment, a report form to be filled out and sent to me will be available on your pages (button "SEND REPORT"), as well as an e-mail tool for communicating with me in case you need assistance. Please make sure you always use this for sending messages about this preprint so that a record can be kept.

In the spirit that scientists' work should be compensated we have instituted a token fee to be paid to our referees. For further information see:

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I would like to emphasize that routine work, with no true challenge and little new results, should not be encouraged, whatever the area. "Not even wrong papers" have no place in JHEP. So please pay particular attention to the paper's thoroughness, significance, potential impact and scope.

Thank you in advance for your help and best regards,

Andreas Karch
JHEP Editor

The Galileo Galilei Institute For Theoretical Physics

Centro Nazionale di Studi Avanzati dell'Istituto Nazionale di Fisica Nucleare

Arcetri, Firenze

Event at Galileo Galilei Institute

Mini Workshop

XVI Avogadro Meeting

Online Event

Dec 21, 2020 - Dec 22, 2020

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Thermalization of the Stress Tensor Sector

In d-dimensional CFTs with a large number of degrees of freedom an important set of primary operators is comprised out of the stress tensor and its products, multi stress tensors. We argue that the thermal expectation value of all such operators on the (d-1)-sphere equals their expectation value in a heavy state, therefore, the stress tensor sector satisfies the Eigenstate Thermalization Hypothesis. This is equivalent to the universal leading behavior of the OPE coefficients of multi stress tensors with a pair of identical scalar operators in the limit of large conformal dimensions of the scalars. We verify this in a number of examples which include holographic and free CFTs. We check the thermalization of the stress tensor sector directly by computing thermal expectation values of multi stress tensor operators in a free CFT.

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