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Научном већу Института за физику

Београд, 17.1.2022.

Предмет: Покретање поступка у звање истраживач сарадник

Молим Научно веће Института за физику у Београду да покрене поступак за мој избор у звање Истраживач сарадник, имајући у виду да испуњавам све критеријуме прописане од стране Министарства просвете, науке и технолошког развоја Републике Србије за стицање тог звања.

У прилогу достављам:

1. мишљење руководиоца пројекта са предлогом комисије за избор у звање;
2. стручну биографију;
3. преглед научне активности;
4. списак и копије објављених научних радова;
5. уверење о последњем уписаном семестру на докторским студијама;
6. фотокопије уверења о завршеним основним и мастер студијама;
7. уверење о положеним испитима на докторским студијама;
8. потврду о прихватању теме докторске дисертације.

С поштовањем,

Илија Иванишевић

Научном већу Института за физику у Београду

Предмет: *Мишљење руководиоца лабораторије о избору Илије Иванишевића у звање истраживач сарадник*

Илија Иванишевић , рођен 17.07.1991. године у Мостару, је уписао докторске академске студије Физичког факултета Универзитета у Београду у школској 2018/2019. години. Положио је све испите на смеру Квантна поља, честице и гравитација и успешно одбранио предлог теме докторске дисертације под насловом *Courant-ови алгеброиди у бозонској теорији струна* пред Колегијумом докторских студија. Ментор докторске дисертације је др Љубица Давидовић, виши научни сарадник Института за физику.

Илија Иванишевић је од 2018. године запослен у Групи са физику гравитације, честица и поља Института за физику где се бави проблемима везаним за теорију струна. Илија Иванишевић је до сада објавио три научна рада категорије М21. Као што се види из приложеног материјала, он задовољава све предвиђене услове у складу са Правилником о поступку, начину вредновања и квантитативном исказивању научно-истраживачких резултата истраживача Министарства просвете, науке и технолошког развоја Владе Републике Србије, за избор у звање истраживач сарадник, те предлажем Научном већу Института за физику да покрене избор Илије Иванишевића у поменуто звање. За чланове комисије предлажем следеће истраживаче:

1. др Бојан Николић, виши научни сарадник, Институт за физику

2. др Љубица Давидовић, виши научни сарадник, Институт за физику

3. проф. др Воја Радовановић, редовни професор Физичког факултета

Београд, 27.12.2021.

др Бранислав Цветковић

Руководилац Групе за физику гравитације, честица и поља

Института за физику

Биографија

Илија Иванишевић рођен је 17.07.1991. године у Мостару. Завршио је Математичку гимназију 2010. године. Дипломирао је на Физичком факултету 2014. године са просечном оценом 9.66. Наредне године на истом факултету мастерирао је са темом "Т-дуализација у закривљеном простору". Током школовања био је стипендиста града Београда (2007-2010), Министарства просвете, науке и технолошког развоја Републике Србије (2010-2013, 2015-2016) и Фонда за младе таленте (2014-2015).

Докторске студије из научне области Квантна поља, честице и гравитација уписује 2015. године. Положио је испите на докторским студијама са просечном оценом 9. Од 2018. запослен је на Институту за физику где под менторством др. Љубице Давидовић ради на изучавању теорије струна. До сада је објавио три рада у врхунским међународним часописима (М21 категорија). Пред Колегијумом докторских студија Физичког факултета одбранио је тему докторске дисертације под називом "Courant-ови алгеброиди у бозонској теорији струна".

За време студија, Илија је представљао Универзитет у Београду на међународним универзитетским дебатним такмичењима. Остварио је бројне успехе, победивши на преко десет такмичења и пласиравши се у финале Европског универзитетског дебатног првенства у Талину 2017. Говори енглески, немачки и француски језик.

Преглед научне активности

Илија Иванишевић бави се изучавањем бозонске теорије струна. Конкретно, изучавањем симетрија и њихових веза са Т-дуалношћу. За те потребе, користи се методама генералисане геометрије, у којој су тангентно и котангентно раслојење описани на јединствен начин. Илија се до сада бавио одређивањем алгебре симетрија генератора у почетној, Т-дуалној и дуплој теорији, као и у теорији инваријантној на Т-дуалност.

У првом раду, установљено је да су две заграде на генералисаном тангентном раслојењу повезане Т-дуалним трансформацијама. Показано је да алгебра набоја струја у сигма моделу даје Courant-ову заграду. Такође, установљено је да када се набоји трансформишу тако да се поља замене Т-дуалним пољима уз Т-дуалну трансформацију канонских варијабли, добија се Roytenberg-ова заграда.

У наредном раду, конструисана је екстензија генератора симетрије, тако што је генератору дифеоморфизама додат генератор локалних градијентних трансформација. У почетној теорији, показано је да Poisson-ова алгебра таквог генератора даје Courant-ову заграду деформисану пољем B, док у Т-дуалној теорији алгебарске релације генератора дају Courant-ову заграду деформисану пољем θ, које је Т-дуално пољу B. Резултати су генералисани на дуплу теорију, где је показано да алгебра генератора симетрије даје C-заграду.

Илија се такође бавио изучавањем генератора инваријантних на Т-дуалност, али тако да њихова алгебра садржи све флуксеве релевантне за теорију струна. У таквом подухвату, по први пут одређен је израз за Courant-ову заграда симултано деформисану и са B и са θ, што је објављено у трећем раду. Додатно, пронађен је ефикасан метод којим се Courant-ова заграда може деформисати било којим пољем и репрезентација такве заграде одредити из Poisson-ових заграда генератора записаног у релевантном базису.

У наставку истраживању, Илија ће се бавити одређивањем Dirac-ових структура за све за теорију струна релевантне Courant-ове алгеброиде, као и генерализацијом деформисаних Courant-ових заграда на дупли простор, настојећи да одреди деформисане C-заграде.

Списак М21 публикација

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2. Lj. Davidović, I. Ivanišević, B. Sazdović, “Courant bracket as T-dual invariant extension of Lie bracket”, JHEP **03** (2021)
3. Lj. Davidović, I. Ivanišević, B. Sazdović,”Courant bracket twisted at the same time by a 2-form B and by a bi-vector ”, EPJC **81**, 685 (2021)

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Courant bracket as T-dual invariant extension of Lie bracket

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Abstract: We consider the symmetries of a closed bosonic string, starting with the general coordinate transformations. Their generator takes vector components ξ^μ as its parameter and its Poisson bracket algebra gives rise to the Lie bracket of its parameters. We are going to extend this generator in order for it to be invariant upon self T-duality, i.e. T-duality realized in the same phase space. The new generator is a function of a $2D$ double symmetry parameter Λ, that is a direct sum of vector components ξ^μ, and 1-form components λ_μ. The Poisson bracket algebra of a new generator produces the Courant bracket in a same way that the algebra of the general coordinate transformations produces Lie bracket. In that sense, the Courant bracket is T-dual invariant extension of the Lie bracket. When the Kalb-Ramond field is introduced to the model, the generator governing both general coordinate and local gauge symmetries is constructed. It is no longer self T-dual and its algebra gives rise to the B-twisted Courant bracket, while in its self T-dual description, the relevant bracket becomes the θ-twisted Courant bracket. Next, we consider the T-duality and the symmetry parameters that depend on both the initial coordinates x^μ and T-dual coordinates y_μ. The generator of these transformations is defined as an inner product in a double space and its algebra gives rise to the C-bracket.

Keywords: Bosonic Strings, String Duality

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Contents

1 Introduction

The Courant bracket [1, 2] and various generalizations obtained by its twisting had been relevant to the string theory since its appearance in the algebra of generalized currents [3–6]. It represents the generalization of the Lie bracket on spaces of generalized vectors, understood as the direct sum of the elements of the tangent bundle and the elements of the cotangent bundle. Although the Lie bracket satisfies the Jacobi identity, the Courant bracket does not. Its Jacobiator is equal to the exterior derivative of the Nijenhuis operator.

It is well known that the commutator of two general coordinate transformations along two vector fields produces another general coordinate transformation along the vector field equal to their Lie bracket. Since the Courant bracket represents its generalization, it is worth considering how it is related to symmetries of the bosonic string σ-model.

In [7], the field theory defined on the double torus, and its symmetries for restricted parameters were considered. The double space is seen as a direct sum of the initial and T-dual phase space, and the background fields depend on both of these coordinates. The symmetry algebra is closed only for restricted parameters, defined on the same isotropic space, in which case it gives rise to the C-bracket as the T-dual invariant bracket. The C-bracket [8, 9] is the bracket that generalizes the Lie bracket on double space.

In this paper, we analyze the general classical bosonic string σ-model and algebra of its symmetries generators, where both the background fields and symmetry parameters depend only on the coordinates x^μ. We firstly consider the closed bosonic string moving

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in the background characterized solely by the metric tensor. We extend the generator of the general coordinate transformations so that it becomes invariant upon self T-duality, understood as T-duality realized in the same phase space [6]. We obtain the Courant bracket in the Poisson bracket algebra of this extended generator. The Courant bracket is therefore a self T-dual invariant extension of the Lie bracket.

Furthermore, we consider the bosonic string σ-model that includes the antisymmetric Kalb-Ramond field too. The antisymmetric field is introduced by the action of B-transformation on the generalized metric. We construct the symmetry generator and recognize that it generates both the general coordinate and the local gauge transformations [10]. In this case, the symmetry generator is not invariant upon self T-duality and it gives rise to the twisted Courant bracket. The matrix that governs this twist is exactly the matrix of B-shifts.

Next, we consider the self T-dual description of the theory, that we construct in the analogous manner, this time with the action of θ-transformation, T-dual to the B-transformation. We obtain the bracket governing the generator algebra that turns out to be the θ-twisted Courant bracket, also known as the Roytenberg bracket [4, 11]. The twisted Courant and Roytenberg brackets had been shown to be related by self T-duality [6].

Lastly, we consider the more conventional T-duality, connecting different phase spaces. We generalize our results, by demanding that the symmetry parameters depend on both the initial and T-dual coordinates. We consider the symmetry generator that is a sum of the generator of general coordinate transformations and its analogous generator in the T-dual phase space. In this case, additional constraints, similar to the ones in [7–9], have to be imposed on symmetry parameters, in order for the generator algebra to be closed. We extend the Poisson bracket relations for both initial and T-dual phase spaces and obtain the generator algebra, which produces the C-bracket. The C bracket is the generalization of the Courant bracket when parameters depend on both initial and T-dual coordinates. The invariance upon T-duality is guaranteed from the way how the bracket is obtained. If parameters do not depend on T-dual coordinates, C-bracket reduces to the Courant bracket.

2 Bosonic string moving in the background characterized by the metric field

Consider the closed bosonic string, moving in the background defined by the coordinate dependent metric field $G_{\mu\nu}(x)$, with the Kalb-Ramond field set to zero $B_{\mu\nu} = 0$ and the constant dilaton field $\Phi = const$. In the conformal gauge, the Lagrangian density is given by [12, 13]

$$\mathcal{L} = \frac{\kappa}{2}\eta^{\alpha\beta}G_{\mu\nu}(x)\partial_\alpha x^\mu \partial_\beta x^\nu \,, \tag{2.1}$$

where $x^\mu(\xi)$, $\mu = 0, 1, \ldots, D-1$ are coordinates on the D-dimensional space-time, and $\eta^{\alpha\beta}$, $\alpha, \beta = 0, 1$ is the worldsheet metric, $\epsilon^{01} = -1$ is the Levi-Civita symbol, and $\kappa = \frac{1}{2\pi\alpha'}$ with α' being the Regge slope parameter. The Legendre transformation of the Lagrangian gives the canonical Hamiltonian

$$\mathcal{H}_C = \pi_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu + \frac{\kappa}{2} x'^\mu G_{\mu\nu} x'^\nu \,, \tag{2.2}$$

where π_μ are canonical momenta conjugate to coordinates x^μ, given by

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu}(x)\dot{x}^\nu \,. \tag{2.3}$$

The Hamiltonian can be rewritten in the matrix notation

$$\mathcal{H}_C = \frac{1}{2\kappa}(X^T)^M G_{MN} X^N \,, \tag{2.4}$$

where X^M is a double canonical variable, given by

$$X^M = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu \end{pmatrix} \,, \tag{2.5}$$

and G_{MN} is the so called generalized metric, that in the absence of the Kalb-Ramond field takes the diagonal form

$$G_{MN} = \begin{pmatrix} G_{\mu\nu} & 0 \\ 0 & (G^{-1})^{\mu\nu} \end{pmatrix} \,. \tag{2.6}$$

In this paper, we firstly consider the T-duality realized without changing the phase space, which is called the self T-duality [6]. Two quantities are said to be self T-dual if they are invariant upon

$$\kappa x'^\mu \leftrightarrow \pi_\mu \,, \quad G_{\mu\nu} \leftrightarrow {}^\star G^{\mu\nu} = (G^{-1})^{\mu\nu} \,. \tag{2.7}$$

The first part of (2.7) corresponds to the T-duality interchanging the winding and momentum numbers, which are respectively obtained by integrating $\kappa x'^\mu$ and π_μ over the worldsheet space parameter σ [14]. The second part of (2.7) corresponds to swapping the background fields for the T-dual background fields. Our approach gives the same expression for the T-dual metric as the usual T-dualization procedure obtained by Buscher in the special case of zero Kalb-Ramond field [15–17].

2.1 Symmetry generator

Let us consider symmetries of the closed bosonic string. The canonical momenta π_μ generate the general coordinate transformations. The generator is given by [10]

$$\mathcal{G}_{\text{GCT}}(\xi) = \int_0^{2\pi} d\sigma \xi^\mu(x)\pi_\mu \,, \tag{2.8}$$

with ξ^μ being a symmetry parameter. The general coordinate transformations of the metric tensor are given by [7, 10]

$$\delta_\xi G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu} \,, \tag{2.9}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ. Its action on the metric field is

$$\mathcal{L}_\xi G_{\mu\nu} = D_\mu \xi_\nu + D_\nu \xi_\mu \,, \tag{2.10}$$

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where D_μ are covariant derivatives defined in a usual way

$$D_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho_{\mu\nu}\xi_\rho\,, \tag{2.11}$$

and $\Gamma^\mu_{\nu\rho} = \frac{1}{2}(G^{-1})^{\mu\sigma}(\partial_\nu G_{\rho\sigma} + \partial_\rho G_{\sigma\nu} - \partial_\sigma G_{\nu\rho})$ are Christoffel symbols. It is easy to verify, using the standard Poisson bracket relations

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu{}_\nu \delta(\sigma - \bar\sigma)\,, \tag{2.12}$$

that the Poisson bracket of these generators can be written as

$$\Big\{\mathcal{G}_{\rm GCT}(\xi_1),\, \mathcal{G}_{\rm GCT}(\xi_2)\Big\} = -\mathcal{G}_{\rm GCT}\Big([\xi_1,\xi_2]_L\Big)\,, \tag{2.13}$$

where $[\xi_1,\xi_2]_L$ is the Lie bracket. The Lie bracket is the commutator of two Lie derivatives

$$[\xi_1,\xi_2]_L = \mathcal{L}_{\xi_1}\xi_2 - \mathcal{L}_{\xi_2}\xi_1 \equiv \mathcal{L}_{\xi_3}\,, \tag{2.14}$$

which results in another Lie derivative along the vector ξ_3^μ, given by

$$\xi_3^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu\,. \tag{2.15}$$

Let us now construct the symmetry generator that is related to the generator of general coordinate transformations by self T-duality (2.7)

$$\mathcal{G}_{LG}(\lambda) = \int_0^{2\pi} d\sigma \lambda_\mu(x)\kappa x'^\mu\,, \tag{2.16}$$

where λ_μ is a gauge parameter.

The symmetry parameters ξ^μ and λ_μ are vector and 1-form components, respectively. They can be combined in a double gauge parameter, given by

$$\Lambda^M = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix}\,. \tag{2.17}$$

The double gauge parameter is a generalized vector, defined on the direct sum of elements of tangent and cotangent bundle. Combining (2.8) and (2.16), we obtain the symmetry generator that is self T-dual (2.7)

$$\mathcal{G}(\xi,\lambda) = \mathcal{G}_{\rm GCT}(\xi) + \mathcal{G}_{LG}(\lambda) = \int_0^{2\pi} d\sigma \Big[\xi^\mu \pi_\mu + \lambda_\mu \kappa x'^\mu\Big] = \int_0^{2\pi} d\sigma (\Lambda^T)^M \eta_{MN} X^N\,, \tag{2.18}$$

where η_{MN} is the $O(D,D)$ invariant metric [18], given by

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,. \tag{2.19}$$

The expression $(\Lambda^T)^M \eta_{MN} X^N$ can be recognized as the natural inner product on the space of generalized vectors

$$\langle \Lambda^M, X^N \rangle = (\Lambda^T)^M \eta_{MN} X^N\,. \tag{2.20}$$

We are interested in obtaining the algebra of this extended symmetry generator (2.18), analogous to (2.13). Using the Poisson bracket relations (2.12), we obtain

$$\begin{aligned}\Big\{\mathcal{G}(\xi_1,\lambda_1),\,\mathcal{G}(\xi_2,\lambda_2)\Big\} &= \int d\sigma\Big[\pi_\mu(\xi_2^\nu\partial_\nu\xi_1^\mu-\xi_1^\nu\partial_\nu\xi_2^\mu)+\kappa x'^\mu(\xi_2^\nu\partial_\nu\lambda_{1\mu}-\xi_1^\nu\partial_\nu\lambda_{2\mu})\Big] \\ &+\int d\sigma d\bar\sigma\kappa\Big[\lambda_{1\mu}(\sigma)\xi_2^\mu(\bar\sigma)+\lambda_{2\mu}(\bar\sigma)\xi_1^\mu(\sigma)\Big]\partial_\sigma\delta(\sigma-\bar\sigma)\,.\end{aligned} \tag{2.21}$$

In order to transform the anomalous part, we note that

$$\partial_\sigma\delta(\sigma-\bar\sigma)=\frac{1}{2}\partial_\sigma\delta(\sigma-\bar\sigma)-\frac{1}{2}\partial_{\bar\sigma}\delta(\sigma-\bar\sigma)\,, \tag{2.22}$$

and

$$f(\bar\sigma)\partial_\sigma\delta(\sigma-\bar\sigma)=f(\sigma)\partial_\sigma\delta(\sigma-\bar\sigma)+f'(\sigma)\delta(\sigma-\bar\sigma)\,. \tag{2.23}$$

Applying the previous two relations to the right hand side of (2.21), one obtains

$$\Big\{\mathcal{G}(\xi_1,\lambda_1),\,\mathcal{G}(\xi_2,\lambda_2)\Big\}=-\mathcal{G}(\xi,\lambda)\,, \tag{2.24}$$

where the resulting gauge parameters are given by

$$\begin{aligned}\xi^\mu &= \xi_1^\nu\partial_\nu\xi_2^\mu-\xi_2^\nu\partial_\nu\xi_1^\mu\,, \\ \lambda_\mu &= \xi_1^\nu(\partial_\nu\lambda_{2\mu}-\partial_\mu\lambda_{2\nu})-\xi_2^\nu(\partial_\nu\lambda_{1\mu}-\partial_\mu\lambda_{1\nu})+\frac{1}{2}\partial_\mu(\xi_1\lambda_2-\xi_2\lambda_1)\,.\end{aligned} \tag{2.25}$$

These relations define the Courant bracket $[(\xi_1,\lambda_1),(\xi_2,\lambda_2)]_C=(\xi,\lambda)$ [1, 2], allowing us to rewrite the generator algebra (2.24)

$$\Big\{\mathcal{G}(\xi_1,\lambda_1),\,\mathcal{G}(\xi_2,\lambda_2)\Big\}=-\mathcal{G}\left(\Big[(\xi_1,\lambda_1),\;(\xi_2,\lambda_2)\Big]_{\mathcal{C}}\right)\,. \tag{2.26}$$

The Courant bracket represents the self T-dual invariant extension of the Lie bracket.

In the coordinate-free notation, the Courant bracket can be written as

$$\Big[(\xi_1,\lambda_1),(\xi_2,\lambda_2)\Big]_{\mathcal{C}}=\Big([\xi_1,\xi_2]_L,\;\mathcal{L}_{\xi_1}\lambda_2-\mathcal{L}_{\xi_2}\lambda_1-\frac{1}{2}d\left(i_{\xi_1}\lambda_2-i_{\xi_2}\lambda_1\right)\Big)\,, \tag{2.27}$$

with i_ξ being the interior product along the vector field ξ, and d being the exterior derivative. The Lie derivative $\mathcal{L}_\xi$ can be written as their anticommutator

$$\mathcal{L}_\xi=i_\xi d+di_\xi\,. \tag{2.28}$$

The Courant bracket does not satisfy the Jacobi identity. Nevertheless, the Jacobiator of the Courant bracket is an exact 1-form [20]

$$\Big[(\xi_1,\lambda_1),\Big[(\xi_2,\lambda_2),(\xi_3,\lambda_3)\Big]_{\mathcal{C}}\Big]_{\mathcal{C}}+cycl.=d\varphi,\quad (d\varphi)_\mu=\partial_\mu\varphi\,. \tag{2.29}$$

However, if one makes the following change of parameters $\lambda_\mu\to\lambda_\mu+\partial_\mu\varphi$, the generator (2.18) does not change

$$\mathcal{G}(\xi,\lambda+\partial\varphi)=\mathcal{G}(\xi,\lambda)+\kappa\int_0^{2\pi}\varphi'd\sigma=\mathcal{G}(\xi,\lambda)\,, \tag{2.30}$$

since the total derivative integral vanishes for the closed string. Therefore, the deviation from Jacobi identity contributes to the trivial symmetry, and we say that the symmetry is reducible.

The theory with the metric tensor was already discussed in [19], where it was proven that the invariance under both diffeomorphisms and dual diffeomorphisms requires the introduction of the Kalb-Ramond field. In our approach, if we want to include the T-dual of the general coordinate transformation in the same theory, we obtain the local gauge transformation that constitutes a trivial symmetry, since $\delta_\lambda G_{\mu\nu} = 0$ [7, 10]. Therefore, it is necessary to include the Kalb-Ramond field, in order to have non-trivial local gauge transformations, which we do in the next section.

3 Bosonic string moving in the background characterized by the metric field and the Kalb-Ramond field

In this section, we extend the Hamiltonian so that it includes the antisymmetric Kalb-Ramond field. It is possible to obtain this Hamiltonian from the transformation of the generalized metric G_{MN} (2.6) under the so called B-transformations. The B-transformations (or B-shifts) [20] are realized by $e^{\hat{B}}$, where

$$\hat{B}^M_{\ N} = \begin{pmatrix} 0 & 0 \\ 2B_{\mu\nu} & 0 \end{pmatrix} . \tag{3.1}$$

As a result of $\hat{B}^2 = 0$, the full transformation is easily obtained

$$(e^{\hat{B}})^M_{\ N} = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} . \tag{3.2}$$

Its transpose is given by

$$((e^{\hat{B}})^T)_M^{\ N} = \begin{pmatrix} \delta^\nu_\mu & -2B_{\mu\nu} \\ 0 & \delta^\mu_\nu \end{pmatrix} , \tag{3.3}$$

from which it is easy to verify that

$$((e^{\hat{B}})^T)_M^{\ K} \, \eta_{KL} \, (e^{\hat{B}})^L_{\ N} = \eta_{MN} , \tag{3.4}$$

meaning they are the elements of the $O(D, D)$ group.

The transformation of generalized metric G_{MN} (2.6) under the B-shifts is given by

$$G_{MN} \to ((e^{\hat{B}})^T)_M^{\ K} \, G_{KQ} \, (e^{\hat{B}})^Q_{\ N} \equiv H_{MN} , \tag{3.5}$$

where H_{MN} is the generalized metric

$$H_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix} , \tag{3.6}$$

and $G^E_{\mu\nu}$ is the effective metric perceived by the open strings, given by

$$G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu} . \tag{3.7}$$

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It is straightforward to write the canonical Hamiltonian

$$\begin{aligned}\hat{\mathcal{H}}_C &= \frac{1}{2\kappa}(X^T)^M H_{MN} X^N \\ &= \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu - 2x'^\mu B_{\mu\rho}(G^{-1})^{\rho\nu}\pi_\nu \,,\end{aligned} \tag{3.8}$$

as well as the Lagrangian in the canonical form

$$\begin{aligned}\hat{\mathcal{L}}(\dot{x}, x', \pi) &= \pi_\mu \dot{x}^\mu - \hat{\mathcal{H}}_C(x', \pi) \\ &= \pi_\mu \dot{x}^\mu - \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu - \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu + 2x'^\mu B_{\mu\rho}(G^{-1})^{\rho\nu}\pi_\nu .\end{aligned} \tag{3.9}$$

On the equations of motion for π_μ, we obtain

$$\pi_\mu = \kappa G_{\mu\nu}\dot{x}^\nu - 2\kappa B_{\mu\nu}x'^\nu \,. \tag{3.10}$$

Substituting (3.10) into (3.9) we find the well known expression for bosonic string Lagrangian [12, 13]

$$\begin{aligned}&\hat{\mathcal{L}}(\dot{x}, x') = \frac{\kappa}{2}\dot{x}^\mu G_{\mu\nu}\dot{x}^\nu - \frac{\kappa}{2}x'^\mu G_{\mu\nu}x'^\nu - 2\kappa \dot{x}^\mu B_{\mu\nu}x'^\nu = \kappa \partial_+ x^\mu \Pi_{+\mu\nu}\partial_- x^\nu \,, \\ &\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu} \,,\ \partial_\pm x^\mu = \dot{x}^\mu \pm x'^\mu \,.\end{aligned} \tag{3.11}$$

It is possible to rewrite the canonical Hamiltonian (3.8) in terms of the generalized metric G_{MN}, that characterizes background with the metric only tensor. Substituting (3.5) into (3.8), we obtain

$$\hat{\mathcal{H}}_C = \frac{1}{2\kappa}(X^T)^M ((e^{\hat{B}})^T)_M^{\ K}\ G_{KL}\ (e^{\hat{B}})^L_{\ N} X^N = \frac{1}{2\kappa}(\hat{X}^T)^M\ G_{MN}\ \hat{X}^N \,, \tag{3.12}$$

where

$$\hat{X}^M = (e^{\hat{B}})^M_{\ N}\ X^N = \begin{pmatrix} \kappa x'^\mu \\ \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu \end{pmatrix} \equiv \begin{pmatrix} \kappa x'^\mu \\ i_\mu \end{pmatrix} \,, \tag{3.13}$$

with i_μ being the auxiliary current, given by

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu \,. \tag{3.14}$$

The algebra of auxiliary currents i_μ gives rise to the H-flux [6]

$$\{i_\mu(\sigma), i_\nu(\bar{\sigma})\} = -2\kappa B_{\mu\nu\rho}x'^\rho \delta(\sigma - \bar{\sigma}) \,, \tag{3.15}$$

where the structural constants are the Kalb-Ramond field strength components, given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \,. \tag{3.16}$$

3.1 Symmetry generator

Let us extend the symmetry transformations of the background fields for the theory with the non-trivial Kalb-Ramond field. The infinitesimal general coordinate transformations of the background fields are given by

$$\delta_\xi G_{\mu\nu} = \mathcal{L}_\xi G_{\mu\nu}\,,\ \delta_\xi B_{\mu\nu} = \mathcal{L}_\xi B_{\mu\nu}\,, \tag{3.17}$$

where the action of the Lie derivative $\mathcal{L}_\xi$ (2.28) on the Kalb-Ramond field is given by [10]

$$\mathcal{L}_\xi B_{\mu\nu} = \xi^\rho \partial_\rho B_{\mu\nu} + \partial_\mu \xi^\rho B_{\rho\nu} - \partial_\nu \xi^\rho B_{\rho\mu}\,, \tag{3.18}$$

while its action on the metric field is the same as in (2.10). The local gauge transformations of the background fields are [10]

$$\delta_\lambda G_{\mu\nu} = 0\,,\ \delta_\lambda B_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu\,. \tag{3.19}$$

Rewriting the symmetry generator $\mathcal{G}(\xi,\lambda)$ (2.18) in terms of the basis defined by components of $\hat{X}^M$ (3.13), one obtains

$$\begin{aligned}\mathcal{G}(\xi,\lambda) &= \int d\sigma (\Lambda^T)^M \eta_{MN} X^N = \int d\sigma (\hat{\Lambda}^T)^M ((e^{-\hat{B}})^T)_M^{\ K} \eta_{KL} (e^{-\hat{B}})^L{}_N \hat{X}^N \\ &= \int d\sigma (\hat{\Lambda}^T)^M \eta_{MN} \hat{X}^N\,,\end{aligned} \tag{3.20}$$

where (3.4) was used in the last step, and $\hat{\Lambda}^M$ is a new double gauge parameter, given by

$$\hat{\Lambda}^M = (e^{\hat{B}})^M{}_N \Lambda^N = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix} \begin{pmatrix} \xi^\nu \\ \lambda_\nu \end{pmatrix} = \begin{pmatrix} \xi^\mu \\ \lambda_\mu + 2B_{\mu\nu}\xi^\nu \end{pmatrix} \equiv \begin{pmatrix} \xi^\mu \\ \hat{\lambda}_\mu \end{pmatrix}. \tag{3.21}$$

We are going to mark the right hand side of (3.20) as a new generator

$$\mathcal{G}^{\hat{B}}(\xi,\hat{\lambda}) = \int d\sigma \Big[\xi^\mu i_\mu + \hat{\lambda}_\mu \kappa x'^\mu\Big]\,, \tag{3.22}$$

which equals the generator (2.18), when the relations between the gauge parameters (3.21) are satisfied $\mathcal{G}(\xi, \hat{\lambda} - 2B_{\mu\nu}\xi^\nu) = \mathcal{G}^{\hat{B}}(\xi,\hat{\lambda})$. The expression (3.22) exactly corresponds to the symmetry generator obtained in [10], where ξ^μ are parameters of general coordinate transformations and $\hat{\lambda}_\mu$ are parameters of local gauge transformations, that respectively correspond to transformations of the background fields (3.17) and (3.19).

Our goal is to obtain the algebra in the form

$$\Big\{\mathcal{G}^{\hat{B}}(\xi_1,\hat{\lambda}_1), \mathcal{G}^{\hat{B}}(\xi_2,\hat{\lambda}_2)\Big\} = -\mathcal{G}^{\hat{B}}(\xi,\hat{\lambda})\,, \tag{3.23}$$

where

$$\lambda_{i\mu} = \hat{\lambda}_{i\mu} - 2B_{\mu\nu}\xi_i^\nu\,,\ i=1,2\,;\quad \lambda_\mu = \hat{\lambda}_\mu - 2B_{\mu\nu}\xi^\nu\,, \tag{3.24}$$

due to (3.21). The Poisson bracket between canonical variables (2.12) remains the same after the introduction of the Kalb-Ramond field. Therefore the results from previous section, as well as mutual relations between coefficients in different bases can be used to obtain the algebra (3.23). Firstly, substituting (3.24) into the second equation in (2.25), one obtains

$$\lambda_\mu = \xi_1^\nu(\partial_\nu\hat\lambda_{2\mu} - \partial_\mu\hat\lambda_{2\nu}) - \xi_2^\nu(\partial_\nu\hat\lambda_{1\mu} - \partial_\mu\hat\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\hat\lambda_2 - \xi_2\hat\lambda_1) \\ +2B_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho - 2B_{\mu\nu}(\xi_1^\rho\partial_\rho\xi_2^\nu - \xi_2^\rho\partial_\rho\xi_1^\nu)\,. \tag{3.25}$$

Secondly, substituting the previous equation in (3.21), one obtains

$$\xi^\mu = \xi_1^\nu\partial_\nu\xi_2^\mu - \xi_2^\nu\partial_\nu\xi_1^\mu\,, \tag{3.26}$$
$$\hat\lambda_\mu = \xi_1^\nu(\partial_\nu\hat\lambda_{2\mu} - \partial_\mu\hat\lambda_{2\nu}) - \xi_2^\nu(\partial_\nu\hat\lambda_{1\mu} - \partial_\mu\hat\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\xi_1\hat\lambda_2 - \xi_2\hat\lambda_1) + 2B_{\mu\nu\rho}\xi_1^\nu\xi_2^\rho\,.$$

The above relations define the twisted Courant bracket $[(\xi_1,\hat\lambda_1),(\xi_2,\hat\lambda_2)]_{\mathcal{C}_B} = (\xi,\hat\lambda)$ [21]. This is the bracket of the symmetry transformations

$$\left\{\mathcal{G}^{\hat B}(\xi_1,\hat\lambda_1),\,\mathcal{G}^{\hat B}(\xi_2,\hat\lambda_2)\right\} = -\mathcal{G}^{\hat B}\left(\Big[(\xi_1,\hat\lambda_1),(\xi_2,\hat\lambda_2)\Big]_{\mathcal{C}_B}\right)\,, \tag{3.27}$$

in the theory defined by both metric and Kalb-Ramond field.

In the coordinate free notation, the twisted Courant bracket is given by

$$\Big[(\xi_1,\hat\lambda_1),(\xi_2,\hat\lambda_2)\Big]_{\mathcal{C}_B} = \left([\xi_1,\xi_2]_L, \mathcal{L}_{\xi_1}\hat\lambda_2 - \mathcal{L}_{\xi_2}\hat\lambda_1 - \frac{1}{2}d(i_{\xi_1}\hat\lambda_2 - i_{\xi_2}\hat\lambda_1) + H(\xi_1,\xi_2,.)\right)\,, \tag{3.28}$$

where $H(\xi_1,\xi_2,.)$ represents the contraction of the H-flux $H = dB$ (3.16) with two gauge parameters ξ_1 and ξ_2. This term is the corollary of the non-commutativity of the auxiliary currents i_μ (3.15), due to twisting of the Courant bracket with the Kalb-Ramond field. In special case when the Kalb-Ramond field B is a closed form $dB = 0$, the twisted Courant bracket (3.28) reduces to the Courant bracket (2.27). This can also be seen from the well known fact that B-shifts (3.2) are symmetries of the Courant bracket when B is a closed form [20].

4 Courant bracket twisted by $\theta^{\mu\nu}$

When both the metric and the Kalb-Ramond field are present in the theory, the expressions for T-dual fields are given by [15]

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}\,, \tag{4.1}$$

where $\theta^{\mu\nu}$ is the non-commutativity parameter for the string endpoints on a D-brane [22], given by

$$\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1})^{\mu\rho}B_{\rho\sigma}(G^{-1})^{\sigma\nu}\,. \tag{4.2}$$

We say that two quantities are self T-dual, if they are invariant under the interchange [6]

$$\pi_\mu \leftrightarrow \kappa x'^\mu\,,\ G_{\mu\nu} \leftrightarrow (G_E^{-1})^{\mu\nu}\,,\ B_{\mu\nu} \leftrightarrow \frac{\kappa}{2}\theta^{\mu\nu}\,. \tag{4.3}$$

When the Kalb-Ramond field is set to zero $B_{\mu\nu} = 0$, (4.3) reduces to the self T-duality transformation laws in the background without the B field (2.7).

From the relations (4.3), it is apparent that the introduction of Kalb-Ramond field breaks down the self T-duality invariance of the symmetry generator (3.22). To find a new self T-dual invariant generator, we will analogously to the prior construction start with the background containing only T-dual metric. The Hamiltonian in the metric only background, similar to (2.2), reads

$$ {}^\star\mathcal{H}_C = \frac{1}{2\kappa}\pi_\mu (G_E^{-1})^{\mu\nu}\pi_\nu + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu = (X^T)^M \, {}^\star G_{MN} X^N \,, \tag{4.4} $$

where ${}^\star G_{MN}$ is the T-dual generalized metric for the above Hamiltonian, given by

$$ {}^\star G_{MN} = \begin{pmatrix} G^E_{\mu\nu} & 0 \\ 0 & (G_E^{-1})^{\mu\nu} \end{pmatrix} . \tag{4.5} $$

Note that the self T-duality is realized as the joint action of the permutation of the coordinate σ-derivatives with the canonical momenta and the swapping all the fields in (2.6) for their T-duals. This is equivalent to the Buscher's procedure [15–17], when it is done in the same phase space.

In order to construct the Hamiltonian in the self T-dual description, we consider how the T-dual generalized metric (4.5) is transformed with respect to the so called θ-transformations $e^{\hat\theta}$, where

$$ \hat\theta^M{}_N = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2\, {}^\star B^{\mu\nu} \\ 0 & 0 \end{pmatrix} . \tag{4.6} $$

The full exponential $e^{\hat\theta}$ is given by

$$ (e^{\hat\theta})^M{}_N = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta_\mu^\nu \end{pmatrix} , \tag{4.7} $$

and its transpose by

$$ ((e^{\hat\theta})^T)_M{}^N = \begin{pmatrix} \delta_\mu^\nu & 0 \\ -\kappa\theta^{\mu\nu} & \delta^\mu_\nu \end{pmatrix} . \tag{4.8} $$

They are elements of the $O(D,D)$ group as well, i.e.

$$ ((e^{\hat\theta})^T)_M{}^L \, \eta_{LK} \, (e^{\hat\theta})^K{}_N = \eta_{MN} \,. \tag{4.9} $$

Under (4.7), the T-dual generalized metric (4.5) transforms in the following way

$$ {}^\star G_{MN} \to \; ((e^{\hat\theta})^T)_M{}^L \, {}^\star G_{LK} (e^{\hat\theta})^K{}_N \equiv \, {}^\star H_{MN} \,, \tag{4.10} $$

where

$$ {}^\star H_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix} , \tag{4.11} $$

which is exactly equal to the generalized metric (3.6). From it we can write the T-dual Hamiltonian

$$ \begin{aligned} {}^\star\mathcal{H}_C &= \frac{1}{2\kappa}(X^T)^M \, {}^\star H_{MN} X^N \\ &= \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu + \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu - 2x'^\mu B_{\mu\rho}(G^{-1})^{\rho\nu}\pi_\nu \equiv \hat{\mathcal{H}}_C \,. \end{aligned} \tag{4.12} $$

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The canonical Lagrangian is given by

$$\begin{aligned}{}^\star\mathcal{L}(\pi,\dot{x},x) &= \pi_\mu\dot{x}^\mu - {}^\star\mathcal{H}_C(x',\pi) \\ &= \pi_\mu\dot{x}^\mu - \frac{1}{2\kappa}\pi_\mu(G^{-1})^{\mu\nu}\pi_\nu - \frac{\kappa}{2}x'^\mu G^E_{\mu\nu}x'^\nu + 2x'^\mu B_{\mu\rho}(G^{-1})^{\rho\nu}\pi_\nu\,,\end{aligned} \tag{4.13}$$

from which one easily obtains

$$\pi_\mu = \kappa G_{\mu\nu}\dot{x}^\nu - 2\kappa B_{\mu\nu}x'^\nu\,. \tag{4.14}$$

We see that the canonical momentum remains the same, which is expected, since the self T-duality is realized in the same phase space. Substituting (4.14) into (4.13), one obtains

$${}^\star\mathcal{L}(\dot{x},x) = \frac{\kappa}{2}\dot{x}^\mu G_{\mu\nu}\dot{x}^\nu - \frac{\kappa}{2}x'^\mu G_{\mu\nu}x'^\nu - 2\kappa\dot{x}^\mu B_{\mu\nu}x'^\nu = \kappa\partial_+x^\mu\Pi_{+\mu\nu}\partial_-x^\nu\,. \tag{4.15}$$

It is obvious that both the Hamiltonian and the Lagrangian are invariant under the self T-duality.

In the same manner as in the previous section, substituting (4.10) into (4.12), we rewrite the Hamiltonian

$${}^\star\hat{\mathcal{H}}_C = \frac{1}{2\kappa}(X^T)_M^{\ L}\,((e^{\hat\theta})^T)_L^{\ K}\,{}^\star G_{KJ}\,(e^{\hat\theta})^J_{\ N}\,X^N = \frac{1}{2\kappa}\tilde{X}^M\,{}^\star G_{MN}\tilde{X}^N\,, \tag{4.16}$$

where

$$\tilde{X}^M = (e^{\hat\theta})^M_{\ N}X^N = \begin{pmatrix}\delta^\mu_\nu & \kappa\theta^{\mu\nu}\\ 0 & \delta^\nu_\mu\end{pmatrix}\begin{pmatrix}\kappa x'^\nu\\ \pi_\nu\end{pmatrix} = \begin{pmatrix}\kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu\\ \pi_\mu\end{pmatrix} \equiv \begin{pmatrix}k^\mu\\ \pi_\mu\end{pmatrix}\,, \tag{4.17}$$

and k^μ is an auxiliary current, given by

$$k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu\,. \tag{4.18}$$

The Poisson bracket algebra of these currents is obtained in [6]

$$\{k^\mu(\sigma),k^\nu(\bar\sigma)\} = -\kappa Q_\rho^{\ \mu\nu}k^\rho\delta(\sigma-\bar\sigma) - \kappa^2R^{\mu\nu\rho}\pi_\rho\delta(\sigma-\bar\sigma)\,, \tag{4.19}$$

where Q and R are non-geometric fluxes [23], given by

$$Q_\rho^{\ \mu\nu} = \partial_\rho\theta^{\mu\nu}\,,\qquad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}\,. \tag{4.20}$$

We now define a new double gauge parameter

$$\tilde\Lambda^M = (e^{\hat\theta})^M_{\ N}\Lambda^N = \begin{pmatrix}\delta^\mu_\nu & \kappa\theta^{\mu\nu}\\ 0 & \delta^\nu_\mu\end{pmatrix}\begin{pmatrix}\xi^\nu\\ \lambda_\nu\end{pmatrix} = \begin{pmatrix}\xi^\mu + \kappa\theta^{\mu\nu}\lambda_\nu\\ \lambda_\mu\end{pmatrix} \equiv \begin{pmatrix}\hat\xi^\mu\\ \lambda_\mu\end{pmatrix}\,. \tag{4.21}$$

The generator (2.18) written in terms of new gauge parameters $\mathcal{G}(\hat\xi-\kappa\theta\lambda,\lambda)\equiv\mathcal{G}^{\hat\theta}(\hat\xi,\lambda)$ is given by

$$\mathcal{G}^{\hat\theta}(\hat\xi,\lambda) = \int d\sigma\Big[\hat\xi^\mu\pi_\mu + \lambda_\mu k^\mu\Big]\,. \tag{4.22}$$

The auxiliary currents i_μ (3.14) and k^μ (4.18) are related by the self T-duality relations (4.3). Moreover, one easily demonstrates that the self T-dual image of the generator $\mathcal{G}^{\hat{B}}$ (3.22) is the generator $\mathcal{G}^{\hat{\theta}}$ (4.22).

Like in a previous case, we want to obtain the algebra in the form

$$\Big\{\mathcal{G}^{\hat{\theta}}(\hat{\xi}_1,\lambda_1),\, \mathcal{G}^{\hat{\theta}}(\hat{\xi}_2,\lambda_2)\Big\} = -\mathcal{G}^{\hat{\theta}}(\hat{\xi},\lambda)\,, \tag{4.23}$$

where from (4.21) we read the relations between the old and new gauge parameters

$$\xi_i^\mu = \hat{\xi}_i^\mu - \kappa\theta^{\mu\nu}\lambda_{i\nu}\,,\ i=1,2;\quad \xi^\mu = \hat{\xi}^\mu - \kappa\theta^{\mu\nu}\lambda_\nu\,. \tag{4.24}$$

Combining (4.24), (2.25) and (4.21), one obtains

$$\begin{aligned}
\hat{\xi}^\mu =& \hat{\xi}_1^\nu\partial_\nu\hat{\xi}_2^\mu - \hat{\xi}_2^\nu\partial_\nu\hat{\xi}_1^\mu + \\
&+\kappa\theta^{\mu\nu}\Big(\hat{\xi}_1^\rho(\partial_\nu\lambda_{2\rho}-\partial_\rho\lambda_{2\nu}) - \hat{\xi}_2^\rho(\partial_\nu\lambda_{1\rho}-\partial_\rho\lambda_{1\nu}) - \frac{1}{2}\partial_\nu(\hat{\xi}_1\lambda_2-\hat{\xi}_2\lambda_1)\Big)\\
&+\kappa\hat{\xi}_1^\nu\partial_\nu(\lambda_{2\rho}\theta^{\rho\mu}) - \kappa\hat{\xi}_2^\nu\partial_\nu(\lambda_{1\rho}\theta^{\rho\mu}) + \kappa(\lambda_{1\nu}\theta^{\nu\rho})\partial_\rho\hat{\xi}_2^\mu - \kappa(\lambda_{2\nu}\theta^{\nu\rho})\partial_\rho\hat{\xi}_1^\mu\\
&+\kappa^2R^{\mu\nu\rho}\lambda_{1\nu}\lambda_{2\rho}\,,\\
\lambda_\mu =& \hat{\xi}_1^\nu(\partial_\nu\lambda_{2\mu}-\partial_\mu\lambda_{2\nu}) - \hat{\xi}_2^\nu(\partial_\nu\lambda_{1\mu}-\partial_\mu\lambda_{1\nu}) + \frac{1}{2}\partial_\mu(\hat{\xi}_1\lambda_2-\hat{\xi}_2\lambda_1)\\
&+\kappa\theta^{\nu\rho}(\lambda_{1\nu}\partial_\rho\lambda_{2\mu}-\lambda_{2\nu}\partial_\rho\lambda_{1\mu}) + \kappa\lambda_{1\rho}\lambda_{2\nu}Q_\mu{}^{\rho\nu}\,.
\end{aligned} \tag{4.25}$$

The relations (4.25) define a bracket $[(\hat{\xi}_1,\lambda_1),(\hat{\xi}_2,\lambda_2)]_{\mathcal{C}_\theta} = (\hat{\xi},\lambda)$ that is known as the θ-twisted Courant bracket, or Roytenberg bracket. It is related by self T-duality with the twisted Courant bracket, when the relations between the fields (4.1) hold [6].

In the coordinate free notation, the θ-twisted Courant bracket is given by

$$\begin{aligned}
\Big[(\hat{\xi}_1,\lambda_1),(\hat{\xi}_2,\lambda_2)\Big]_{\mathcal{C}_\theta} =& \Big([\hat{\xi}_1,\hat{\xi}_2]_L - \kappa[\hat{\xi}_2,\lambda_1\theta]_L + \kappa[\hat{\xi}_1,\lambda_2\theta]_L + \frac{\kappa^2}{2}[\theta,\theta]_S(\lambda_1,\lambda_2,.)\\
&+\kappa\Big(\mathcal{L}_{\hat{\xi}_2}\lambda_1 - \mathcal{L}_{\hat{\xi}_1}\lambda_2 + \frac{1}{2}d(i_{\hat{\xi}_1}\lambda_2 - i_{\hat{\xi}_2}\lambda_1)\Big)\theta\\
&\mathcal{L}_{\hat{\xi}_1}\lambda_2 - \mathcal{L}_{\hat{\xi}_2}\lambda_1 - \frac{1}{2}d(i_{\hat{\xi}_1}\lambda_2 - i_{\hat{\xi}_2}\lambda_1) - \kappa[\lambda_1,\lambda_2]_\theta\Big)\,,
\end{aligned} \tag{4.26}$$

where $[\theta,\theta]_S(\lambda_1,\lambda_2,.)$ represents the Schouten-Nijenhuis bracket [24] contracted with two 1-forms, that when having bi-vectors as domain is given by

$$[\theta,\theta]_S|^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}_{\alpha\beta\gamma}\theta^{\sigma\alpha}\partial_\sigma\theta^{\beta\gamma} = 3R^{\mu\nu\rho}\,, \tag{4.27}$$

where

$$\epsilon^{\mu\nu\rho}_{\alpha\beta\gamma} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\beta & \delta^\rho_\gamma \\ \delta^\nu_\alpha & \delta^\rho_\beta & \delta^\mu_\gamma \\ \delta^\rho_\alpha & \delta^\mu_\beta & \delta^\nu_\gamma \end{vmatrix}\,, \tag{4.28}$$

and $[\lambda_1,\lambda_2]_\theta$ is the Koszul bracket [25] given by

$$[\lambda_1,\lambda_2]_\theta = \mathcal{L}_{\theta\lambda_1}\lambda_2 - \mathcal{L}_{\theta\lambda_2}\lambda_1 + d(\theta(\lambda_1,\lambda_2))\,. \tag{4.29}$$

The Koszul bracket is a generalization of the Lie bracket on the space of differential forms, while the Schouten-Nijenhuis bracket is a generalization of the Lie bracket on the space of multi-vectors.

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5 C-bracket

In this section, we will show how our results can be generalized, so that they give rise to the C-bracket [8, 9] as the T-dual invariant bracket, in the accordance with [7]. Consider that T-dual theory is defined in the T-dual phase space, characterized by T-dual coordinates y_μ and the T-dual momenta ${}^\star\pi^\mu$. They are related with the initial phase space by T-duality relations [15]

$$\pi_\mu \simeq \kappa y'_\mu \,, \qquad {}^\star\pi^\mu \simeq \kappa x'^\mu \,. \tag{5.1}$$

We can define a double phase space obtained as a sum of two canonical phase spaces. Let us introduce the double coordinate

$$X^M = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix} \,, \tag{5.2}$$

as well as the double canonical momentum

$$\Pi_M = \begin{pmatrix} \pi_\mu \\ {}^\star\pi^\mu \end{pmatrix} \,. \tag{5.3}$$

In this notation, the T-duality laws (5.1) take a form

$$\Pi_M \simeq \kappa\, \eta_{MN} X'^M \,, \tag{5.4}$$

where η_{MN} is the $O(D,D)$ metric (2.19).

5.1 Poisson brackets of canonical variables

The standard Poisson bracket algebra is assumed for both initial and T-dual phase space

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu \delta(\sigma-\bar\sigma) \,, \qquad \{y_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = \delta^\nu_\mu \delta(\sigma-\bar\sigma) \,, \tag{5.5}$$

with other bracket of canonical variables within the same phase space being zero.

For the remaining Poisson bracket relations, one must use the consistency with T-duality relations. Firstly, applying the T-dualization along all initial coordinates x^μ, i.e. the second relation of (5.1) on the Poisson bracket algebra between coordinates derivatives, one obtains

$$\{\kappa x'^\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} \simeq \{{}^\star\pi^\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} = \kappa \delta^\mu_\nu \delta'(\sigma-\bar\sigma) \,. \tag{5.6}$$

Similarly, applying the T-dualization along all T-dual coordinates y_μ, i.e. the first relation of (5.1), one obtains

$$\{\kappa x'^\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} \simeq \{\kappa x'^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \kappa \delta^\mu_\nu \delta'(\sigma-\bar\sigma) \,. \tag{5.7}$$

Hence, we conclude

$$\{\kappa x'^\mu(\sigma), \kappa y'_\nu(\bar\sigma)\} = \kappa \delta^\mu_\nu \delta'(\sigma-\bar\sigma) \,. \tag{5.8}$$

The successive integration along both σ and $\bar\sigma$ for the appropriate choice of the integration constant produces the relation [26]

$$\{\kappa x^\mu(\sigma), \kappa y_\nu(\bar\sigma)\} = -\kappa \delta^\mu_\nu \theta(\sigma-\bar\sigma) \,, \tag{5.9}$$

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where

$$\theta(\sigma) = \begin{cases} -\frac{1}{2} & \sigma = -\pi \\ 0 & -\pi < \sigma < \pi \\ \frac{1}{2} & \sigma = \pi \end{cases} . \tag{5.10}$$

Secondly, taking into the account T-duality (5.1), the Poisson bracket algebra of momenta is easily transformed into

$$\{\pi_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} \simeq \kappa\{\pi_\mu(\sigma), x'^\nu(\bar\sigma)\} = \kappa\delta^\nu_\mu\delta'(\sigma-\bar\sigma)\,, \tag{5.11}$$

when T-dualization is applied along the coordinates y_μ, and

$$\{\pi_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} \simeq \kappa\{y'_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = \kappa\delta^\nu_\mu\delta'(\sigma-\bar\sigma)\,, \tag{5.12}$$

when it is applied along the coordinates x^μ. As in the previous case, we obtain

$$\{\pi_\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = \kappa\delta^\nu_\mu\delta'(\sigma-\bar\sigma)\,. \tag{5.13}$$

In a same manner, it is easy to demonstrate that

$$\{x^\mu(\sigma), {}^\star\pi^\nu(\bar\sigma)\} = 0\,,\ \ \{y_\mu(\sigma), \pi_\nu(\bar\sigma)\} = 0\,. \tag{5.14}$$

In a double space, the above relations can be simply written as

$$\Big\{\kappa X^M(\sigma), \kappa X^N(\bar\sigma)\Big\} = -\kappa\eta^{MN}\theta(\sigma-\bar\sigma)\,, \quad \{\Pi_M(\sigma), \Pi_N(\bar\sigma)\} = \kappa\,\eta_{MN}\delta'(\sigma-\bar\sigma)\,. \tag{5.15}$$

5.2 Generator in double space

Now let us extend the generator of general coordinate transformations, so that it includes the T-dual version of that generator

$$G(\xi,\lambda) = \int d\sigma \mathcal{G}(\xi,\lambda) = \int d\sigma\Big[\xi^\mu(x,y)\pi_\mu + \lambda_\mu(x,y){}^\star\pi^\mu\Big]\,, \tag{5.16}$$

where the symmetry parameters ξ and λ depend on both initial coordinates x^μ and T-dual coordinates y_μ. The generator $\mathcal{G}(\xi,\lambda)$ can be rewritten in terms of double canonical variables as

$$\mathcal{G}(\Lambda) = \Lambda^M(x,y)\eta_{MN}\Pi^N \qquad \Longleftrightarrow \qquad \mathcal{G}_\Lambda = \langle\Lambda,\Pi\rangle\,, \tag{5.17}$$

where

$$\Lambda^M(X) = \begin{pmatrix} \xi^\mu(x^\mu,y_\mu) \\ \lambda_\mu(x^\mu,y_\mu) \end{pmatrix}\,. \tag{5.18}$$

This generator is manifestly $O(D,D)$ invariant.

We are interested in the algebra of the form

$$\{G(\Lambda_1), G(\Lambda_2)\} = -G(\Lambda)\,. \tag{5.19}$$

To obtain it, it is convenient to introduces double derivative

$$\partial_M = \begin{pmatrix} \partial_\mu \\ \tilde\partial^\mu \end{pmatrix} \qquad \left(\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\,, \quad \tilde\partial^\mu \equiv \frac{\partial}{\partial y_\mu}\right)\,, \tag{5.20}$$

so that the following Poisson bracket relations can be written

$$\Big\{\Lambda^M(\sigma),\Pi_N(\bar\sigma)\Big\}=\partial_N\Lambda^M\delta(\sigma-\bar\sigma),\Big\{\Lambda^M(\sigma),\Lambda^N(\bar\sigma)\Big\}=-\frac{1}{\kappa}\,\partial^P\Lambda_1^M\partial_P\Lambda_2^N\theta(\sigma-\bar\sigma)\,. \tag{5.21}$$

The second relation makes the situation more complicated, since it would result in the symmetry algebra not closing on another generator. However, in the accordance with [7, 9], we can consider restricted parameters on isotropic spaces, for which $\Delta=\eta^{PQ}\partial_P\partial_Q=\partial^Q\partial_Q$ anihilates all gauge parameters, as well as their products. Therefore, we write

$$\Delta\left(\Lambda_1^M\Lambda_2^N\right)=\Delta\Lambda_1^M\Lambda_2^N+2\partial_Q\Lambda_1^M\partial^Q\Lambda_2^N+\Lambda_1^M\Delta\Lambda_2^N=0\,, \tag{5.22}$$

from which one obtains

$$\partial_Q\Lambda_1^M\partial^Q\Lambda_2^N=0\,. \tag{5.23}$$

Substituting (5.23) into (5.21), we obtain

$$\Big\{\Lambda^M(\sigma),\Lambda^N(\bar\sigma)\Big\}=0\,. \tag{5.24}$$

We see that the restriction of gauge parameters to isotropic spaces is necessary for the algebra of generator (5.16) to be closed.

Now we are ready to calculate the algebra. Using the second relation of (5.15), the first relation of (5.21), and (5.24), we have

$$\{\mathcal{G}_{\Lambda_1}(\sigma),\mathcal{G}_{\Lambda_2}(\bar\sigma)\}=-\Big(\Lambda_1^N\partial_N\Lambda_2^M-\Lambda_2^N\partial_N\Lambda_1^M\Big)\Pi_M\delta(\sigma-\bar\sigma)+\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma)\,. \tag{5.25}$$

Using (2.23), the anomalous term can be rewritten as

$$\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma)=\kappa\langle\Lambda_1(\sigma),\Lambda_2(\sigma)\rangle\delta'(\sigma-\bar\sigma)+\kappa\langle\Lambda_1(\sigma),\Lambda_2'(\sigma)\rangle\delta(\sigma-\bar\sigma)\,, \tag{5.26}$$

which with the help of (2.22) can be further transformed into

$$\begin{aligned}\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma)=&\frac{\kappa}{2}\Big(\langle\Lambda_1,\Lambda_2'\rangle-\langle\Lambda_1',\Lambda_2\rangle\Big)\delta(\sigma-\bar\sigma)\\&+\frac{\kappa}{2}\Big(\langle\Lambda_1,\Lambda_2\rangle(\sigma)+\langle\Lambda_1,\Lambda_2\rangle(\bar\sigma)\Big)\delta'(\sigma-\bar\sigma)\,,\end{aligned} \tag{5.27}$$

where the dependence of σ has been omitted, where all terms depend solely on it.

Next, we write

$$\kappa\Lambda'^M=\kappa\,X'^N\partial_N\Lambda^M\,, \tag{5.28}$$

and with the help of (5.4)

$$\kappa\Lambda'^M\simeq\eta^{NR}\Pi_R\partial_N\Lambda^M\,. \tag{5.29}$$

The full anomalous term can now be written as

$$\begin{aligned}\kappa\langle\Lambda_1(\sigma),\Lambda_2(\bar\sigma)\rangle\delta'(\sigma-\bar\sigma)=&\frac{1}{2}\eta_{PQ}\,\eta^{MN}\Big(\Lambda_1^P\partial_N\Lambda_2^Q-\Lambda_2^P\partial_N\Lambda_1^Q\Big)\Pi_M\delta(\sigma-\bar\sigma)\\&+\frac{\kappa}{2}\Big(\langle\Lambda_1,\Lambda_2\rangle(\sigma)+\langle\Lambda_1,\Lambda_2\rangle(\bar\sigma)\Big)\delta'(\sigma-\bar\sigma)\,.\end{aligned} \tag{5.30}$$

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The second line of the previous equation disappears after the integration with respect to σ and $\bar{\sigma}$.

Consequently,

$$\Big\{G_{\Lambda_1}(\sigma), G_{\Lambda_2}(\bar{\sigma})\Big\} = -\Big[\Lambda_1^N \partial_N \Lambda_2^M - \Lambda_2^N \partial_N \Lambda_1^M \\ -\frac{1}{2}\eta_{PQ}\,\eta^{MN}\Big(\Lambda_1^P \partial_N \Lambda_2^Q - \Lambda_2^P \partial_N \Lambda_1^Q\Big)\Big]\Pi_M \delta(\sigma - \bar{\sigma})\,. \tag{5.31}$$

We recognize that we can write the relation (5.19) as

$$\{G(\Lambda_1), G(\Lambda_2)\} = -G\Big([\Lambda_1, \Lambda_2]_C\Big)\,, \tag{5.32}$$

where $[\Lambda_1, \Lambda_2]_C$ is the C-bracket, given by

$$[\Lambda_1, \Lambda_2]_C{}^M = \Lambda_1^N \partial_N \Lambda_2^M - \Lambda_2^N \partial_N \Lambda_1^M - \frac{1}{2}\Big(\Lambda_1^N \partial^M \Lambda_{2N} - \Lambda_2^N \partial^M \Lambda_{1N}\Big)\,. \tag{5.33}$$

The C-bracket was firstly obtained in [8, 9] as the generalization of the Lie derivative in the double space. For ${}^\star\pi^\mu = 0$, and $y = 0$ the double phase space reduces to the initial one, while the generator (5.17) reduces to the generator of general coordinate transformations (2.8), which gives rise to the Lie bracket.

We could have obtained C-bracket within the framework of self T-duality as well, by demanding that the parameters depend on both x and y, substituting ${}^\star\pi^\mu = \kappa x'^\mu$ in (5.17)

$$\mathcal{G}(\xi, \lambda) = \xi^\mu(x, y)\pi_\mu + \kappa\lambda_\mu(x, y)x'^\mu\,. \tag{5.34}$$

If we additionally demand that the symmetry parameters do not depend on the T-dual coordinates y_μ, this generator turns out to be exactly the Courant bracket generator (3.20). It is in the accordance with [7] that the C-bracket reduces to the Courant bracket, in case when there is no dependence on y.

6 Conclusion

In this paper, we firstly considered the bosonic string moving in the background defined solely by the metric tensor, in which the generalized metric G_{MN} has a simple diagonal form (2.6). The general coordinate transformations are generated by canonical momenta π_μ, parametrised with vector components ξ^μ. We have extended this generator, so that it is self T-dual, adding the symmetry generated by coordinate σ-derivative x'^μ, that are T-dual to the canonical momenta π_μ (2.7). The extended generator of both of these symmetries is a function of a double gauge parameter Λ^M (2.17). The latter is a generalized vector, i.e. an element of a space obtained from a direct sum of vectors and 1-forms. The symmetry generator $\mathcal{G}(\Lambda) = \mathcal{G}(\xi, \lambda)$ of both of aforementioned symmetries was expressed as the standard $O(D, D)$ inner product of two generalized vectors (2.18). The Poisson bracket between the extended generators $\mathcal{G}(\Lambda_1)$ and $\mathcal{G}(\Lambda_2)$ resulted up to a sign in the generator $\mathcal{G}(\Lambda)$, with its argument being equal to the Courant bracket of the double gauge parameters

$\Lambda = [\Lambda_1, \Lambda_2]_\mathcal{C}$. As this is analogous to an appearance of the Lie bracket in the algebra of general coordinate transformations generators, we concluded that the Courant bracket is the self T-dual extension of the Lie bracket.

Afterwards, we added the Kalb-Ramond field $B_{\mu\nu}$ to the background, transforming the diagonal generalized metric G_{MN} acting by the B-transformation $e^{\hat{B}}$ (3.2). The standard generalized metric for bosonic string H_{MN} was obtained (3.6), as well as the well known expressions for the Hamiltonian (3.8) and the Lagrangian (3.11). We noted that it is possible to express the Hamiltonian in terms of the diagonal generalized metric G_{MN}, on the expense of transforming the double canonical variable X^M by the B-shift. This newly obtained canonical variable $\hat{X}$ was suitable for rewriting the symmetry generator $\mathcal{G}$ as $\mathcal{G}^{\hat{B}}$, which is no longer self T-dual. This is the generator of both general coordinate, and local gauge transformations. The Poisson bracket algebra of this new generator was calculated and as an argument of the resulting generator the Courant bracket twisted by the Kalb-Ramond field was obtained. It deviates from the Courant bracket by the term related to the H-flux, which is the term that breaks down the self T-duality invariance.

We considered the self T-dual description of the bosonic string σ-model. Analogously as in the first description, the complete Hamiltonian was constructed starting from the background characterized only by the T-dual metric ${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}$. We applied the θ-transformations $e^{\hat{\theta}}$ (4.7), T-dual to B-shifts, and obtained the same canonical Hamiltonian. Similarly to the previous case, the action of θ-transformation on the double canonical variable was chosen for an appropriate basis. In this basis, the symmetry generator dependent upon some new gauge parameters was constructed and its algebra gave rise to the θ-twisted Courant bracket. This bracket is characterized by the presence of terms related to non-geometric Q and R fluxes.

It would be interesting to obtain the bracket that includes all of the fluxes, while remaining invariant upon the self T-duality. The natural candidate for this is the Courant bracket twisted by both the Kalb-Ramond field and the non-commutativity parameter. This could be done by the matrix $e^{\breve{B}}$, where

$$\breve{B} = \hat{B} + \hat{\theta} = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 2B_{\mu\nu} & 0 \end{pmatrix} . \tag{6.1}$$

This transformation is not trivial, as the square of the matrix $\breve{B}$ is not zero. Nevertheless, the transformation is also an element of the $O(D, D)$ group, and it remains an interesting idea for future research [27].

Lastly, we considered the symmetry generator in the double phase space that is a sum of the initial and T-dual phase space. The generator of general coordinate transformations is extended so that it includes the analogous generator in the T-dual phase space, generated by T-dual momenta ${}^\star\pi^\mu$. Both symmetry parameters were taken to depend on both the initial and T-dual coordinates, in which case the C-bracket is obtained as the bracket of the algebra of those generators. The C bracket has already been established as the T-dual invariant bracket [7–9], from the gauge algebra in the double space. We obtain its Poisson bracket representation, using the T-duality relations between canonical variables of different, mutually T-dual, phase spaces. These T-duality relations gave rise to the non-trivial Poisson bracket between the initial and T-dual momenta, which makes a crucial step in obtaining C-bracket.

We conclude that both Courant and C-bracket are T-dual invariant extension of the Lie bracket. The former is their extension in the initial phase space, that governs both the local gauge and general coordinate transformations. The latter is the extension of Lie bracket in the double phase space, that is a direct sum of the initial and T-dual phase space. Though the algebra of the generators that gives rise to the Courant bracket always closes, the algebra of generators in a double phase space that produces C-bracket only closes on a restricted parameters on an isotropic space. If all variables are independent of T-dual coordinates y_μ, the C-bracket reduces to the Courant bracket, which confirms results from our paper.

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Courant bracket twisted both by a 2-form B and by a bi-vector θ

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Abstract We obtain the Courant bracket twisted simultaneously by a 2-form B and a bi-vector θ by calculating the Poisson bracket algebra of the symmetry generator in the basis obtained acting with the relevant twisting matrix. It is the extension of the Courant bracket that contains well known Schouten–Nijenhuis and Koszul bracket, as well as some new star brackets. We give interpretation to the star brackets as projections on isotropic subspaces.

1 Introduction

The Courant bracket [1,2] represents the generalization of the Lie bracket on spaces of generalized vectors, understood as the direct sum of the elements of the tangent bundle and the elements of the cotangent bundle. It was obtained in the algebra of generalized currents firstly in [3]. Generalized currents are arbitrary functionals of the fields, parametrized by a pair of vector field and covector field on the target space. Although the Lie bracket satisfies the Jacobi identity, the Courant bracket does not.

In bosonic string theory, the Courant bracket is governing both local gauge and general coordinate transformations, invariant upon T-duality [4,5]. It is a special case of the more general C-bracket [6,7]. The C-bracket is obtained as the T-dual invariant bracket of the symmetry generator algebra, when the symmetry parameters depend both on the initial and T-dual coordinates. It reduces to the Courant bracket once when parameters depend solely on the coordinates from the initial theory.

It is possible to obtain the twisted Courant bracket, when the self T-dual generator algebra is considered in the basis obtained from the action of the appropriate $O(D, D)$ transformation [8]. The Courant bracket is usually twisted by a 2-form B, giving rise to what is known as the twisted Courant bracket [9], and by a bi-vector θ, giving rise to the θ-twisted Courant bracket [10]. In [3,8,11,12], the former bracket was obtained in the generalized currents algebra, and it was shown to be related to the latter by self T-duality [13], when the T-dual of the B field is the bi-vector θ.

The B-twisted Courant bracket contains H flux, while the θ-twisted Courant bracket contains non-geometric Q and R fluxes. The fluxes are known to play a crucial role in the compactification of additional dimensions in string theory [14]. Non-geometric fluxes can be used to stabilize moduli. In this paper, we are interested in obtaining the Poisson bracket representation of the twisted Courant brackets that contain all fluxes from the generators algebra. Though it is possible to obtain various twists of the C-bracket as well [15], we do not deal with them in this paper.

The realization of all fluxes using the generalized geometry was already considered, see [16] for a comprehensive review. In [17], one considers the generalized tetrads originating from the generalized metric of the string Hamiltonian. As the Lie algebra of tetrads originating from the initial metric defines the geometric flux, it is suggested that all the other fluxes can be extracted from the Courant bracket of the generalized tetrads. Different examples of $O(D, D)$ and $O(D) \times O(D)$ transformations of generalized tetrads lead to the Courant bracket algebras with different fluxes as its structure constants.

In [18], one considers the standard Lie algebroid defined with the Lie bracket and the identity map as an anchor on the tangent bundle, as well as the Lie algebroid with the Koszul bracket and the bi-vector θ as an anchor on the cotangent bundle. The tetrad basis in these Lie algebroids is suitable for defining the geometric f and non-geometric Q fluxes. It was shown that by twisting both of these Lie algebroids by H-flux one can construct the Courant algebroid, which gives rise to all of the fluxes in the Courant bracket algebra.

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Unlike previous approaches where generalized fluxes were defined using the Courant bracket algebra, in a current paper we obtain them in the Poisson bracket algebra of the symmetry generator.

Firstly, we consider the symmetry generator of local gauge and global coordinate transformations, defined as a standard inner product in the generalized tangent bundle of a double gauge parameter and a double canonical variable. The $O(D, D)$ group transforms the double canonical variable into some other basis, in terms of which the symmetry generator can be expressed. We demonstrate how the Poisson bracket algebra of this generator can be used to obtain twist of the Courant bracket by any such transformation. We give a brief summary of how $e^{\hat{B}}$ and $e^{\hat{\theta}}$ produce respectively the B-twisted and θ-twisted Courant bracket in the Poisson bracket algebra of generators [8].

Secondly, we consider the matrix $e^{\breve{B}}$ used for twisting the Courant bracket simultaneously by a 2-form and a bi-vector. The argument $\breve{B}$ is defined simply as a sum of the arguments $\hat{B}$ and $\hat{\theta}$. Unlike $\hat{B}$ or $\hat{\theta}$, the square of $\breve{B}$ is not zero. The full Taylor series gives rise to the hyperbolic functions of the parameter depending on the contraction of the 2-form with the bi-vector $\alpha^{\mu}_{\ \nu} = 2\kappa\theta^{\mu\rho}B_{\rho\nu}$. We represent the symmetry generator in the basis obtained acting with the twisting matrix $e^{\breve{B}}$ on the double canonical variable. This generator is manifestly self T-dual and its algebra closes on the Courant bracket twisted by both B and θ.

Instead of computing the $B - \theta$ twisted Courant bracket directly, we introduce the change of basis in which we define some auxiliary generators, in order to simplify the calculations. This change of basis is also realized by the action of an element of the $O(D, D)$ group. The structure constants appearing in the Poisson bracket algebra have exactly the same form as the generalized fluxes obtained in other papers [16–18]. The expressions for fluxes is given in terms of new auxiliary fields $\mathring{B}$ and $\mathring{\theta}$, both being the function of α^{μ}.

The algebra of these new auxiliary generators closes on another bracket, that we call $\mathring{C}$-twisted Courant bracket. We obtain its full Poisson bracket representation, and express it in terms of generalized fluxes. We proceed with rewriting it in the coordinate free notation, where many terms are recognized as the well known brackets, such as the Koszul or Schouten–Nijenhuis bracket, but some new brackets, that we call star brackets, also appear. These star brackets as a domain take the direct sum of tangent and cotangent bundle, and as a result give the graph of the bi-vector $\mathring{\theta}$ in the cotangent bundle, i.e. the sub-bundle for which the vector and 1-form components are related as $\xi^{\mu} = \kappa\mathring{\theta}^{\mu\nu}\lambda_{\nu}$. We show that they can be defined in terms of the projections on isotropic subspaces acting on different twists of the Courant bracket.

Lastly, we return to the previous basis and obtain the full expression for the Courant bracket twisted by both B and θ. It has a similar form as $\mathring{C}$-twisted Courant bracket, but in this case the other brackets contained within it are also twisted. The Courant bracket twisted by both B and θ and the one twisted by $\mathring{C}$ are directly related by a $O(D, D)$ transformation represented with the block diagonal matrix.

2 The bosonic string essentials

The canonical Hamiltonian for closed bosonic string, moving in the D-dimensional space-time with background characterized by the metric field $G_{\mu\nu}$ and the antisymmetric Kalb–Ramond field $B_{\mu\nu}$ is given by [19,20]

$$\mathcal{H}_{C} = \frac{1}{2\kappa}\pi_{\mu}(G^{-1})^{\mu\nu}\pi_{\nu} + \frac{\kappa}{2}x'^{\mu}G^{E}_{\mu\nu}x'^{\nu} - 2x'^{\mu}B_{\mu\rho}(G^{-1})^{\rho\nu}\pi_{\nu}, \tag{2.1}$$

where π_{μ} are canonical momenta conjugate to coordinates x^{μ}, and

$$G^{E}_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu} \tag{2.2}$$

is the effective metric. The Hamiltonian can be rewritten in the matrix notation

$$\mathcal{H}_{C} = \frac{1}{2\kappa}(X^{T})^{M}H_{MN}X^{N}, \tag{2.3}$$

where X^{M} is a double canonical variable given by

$$X^{M} = \begin{pmatrix}\kappa x'^{\mu} \\ \pi_{\mu}\end{pmatrix}, \tag{2.4}$$

and H_{MN} is the so called generalized metric, given by

$$H_{MN} = \begin{pmatrix} G^{E}_{\mu\nu} & -2B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}B_{\rho\nu} & (G^{-1})^{\mu\nu}\end{pmatrix}, \tag{2.5}$$

with $M, N \in \{0, 1\}$. In the context of generalized geometry [21], the double canonical variable X^{M} represents the generalized vector. The generalized vectors are $2D$ structures that combine both vector and 1-form components in a single entity.

The standard T-duality [22,23] laws for background fields have been obtained by Buscher [24]

$${}^{\star}G^{\mu\nu} = (G_{E}^{-1})^{\mu\nu}, \quad {}^{\star}B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}, \tag{2.6}$$

where $(G_E^{-1})^{\mu\nu}$ is the inverse of the effective metric (2.2), and $\theta^{\mu\nu}$ is the non-commutativity parameter, given by

$$\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1})^{\mu\rho} B_{\rho\sigma} (G^{-1})^{\sigma\nu}. \tag{2.7}$$

The T-duality can be realized without changing the phase space, which is called the self T-duality [13]. It has the same transformation rules for the background fields like T-duality (2.6), with additionally interchanging the coordinate σ-derivatives $\kappa x'^\mu$ with canonical momenta π_μ

$$\kappa x'^\mu \cong \pi_\mu. \tag{2.8}$$

Since momenta and winding numbers correspond to σ integral of respectively π_μ and $\kappa x'^\mu$, we see that the self T-duality, just like the standard T-duality, swaps momenta and winding numbers.

2.1 Symmetry generator

We consider the symmetry generator that at the same time governs the general coordinate transformations, parametrized by ξ^μ, and the local gauge transformations, parametrized by λ_μ. The generator is given by [25]

$$G(\xi, \lambda) = \int_0^{2\pi} d\sigma \mathcal{G}(\xi, \lambda) = \int_0^{2\pi} d\sigma \Big[\xi^\mu \pi_\mu + \lambda_\mu \kappa x'^\mu\Big]. \tag{2.9}$$

It has been shown that the general coordinate transformations and the local gauge transformations are related by self T-duality [25], meaning that this generator is self T-dual. If one makes the following change of parameters $\lambda_\mu \to \lambda_\mu + \partial_\mu \varphi$, the generator (2.9) does not change

$$G(\xi, \lambda + \partial\varphi) = G(\xi, \lambda) + \kappa \int_0^{2\pi} \varphi' d\sigma = G(\xi, \lambda), \tag{2.10}$$

since the total derivative integral vanishes for the closed string. Therefore, the symmetry is reducible.

Let us introduce the double gauge parameter Λ^M, as the generalized vector, given by

$$\Lambda^M = \begin{pmatrix} \xi^\mu \\ \lambda_\mu \end{pmatrix}, \tag{2.11}$$

where ξ^μ represent the vector components, and λ_μ represent the 1-form components. The space of generalized vectors is endowed with the natural inner product

$$\begin{aligned}\langle \Lambda_1, \Lambda_2 \rangle = (\Lambda_1^T)^M \eta_{MN} \Lambda_2^N \Leftrightarrow \langle (\xi_1, \lambda_1), (\xi_2, \lambda_2) \rangle \\ = i_{\xi_1}\lambda_2 + i_{\xi_2}\lambda_1 = \xi_1^\mu \lambda_{2\mu} + \xi_2^\mu \lambda_{1\mu},\end{aligned} \tag{2.12}$$

where i_ξ is the interior product along the vector field ξ, and η_{MN} is $O(D, D)$ metric, given by

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.13}$$

Now it is possible to rewrite the generator (2.9) as

$$G(\Lambda) = \int d\sigma \langle \Lambda, X \rangle. \tag{2.14}$$

In [8], the Poisson bracket algebra of generator (2.9) was obtained in the form

$$\Big\{G(\Lambda_1),\ G(\Lambda_2)\Big\} = -G\Big([\Lambda_1, \Lambda_2]_{\mathcal{C}}\Big), \tag{2.15}$$

where the standard Poisson bracket relations between coordinates and canonical momenta were assumed

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu \delta(\sigma - \bar\sigma). \tag{2.16}$$

The bracket $[\Lambda_1, \Lambda_2]_{\mathcal{C}}$ is the Courant bracket [1], defined by

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}} = \Lambda \Leftrightarrow [(\xi_1, \lambda_1), (\xi_2, \lambda_2)]_{\mathcal{C}} = (\xi, \lambda), \tag{2.17}$$

where

$$\xi^\mu = \xi_1^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu,$$

and

$$\begin{aligned}\lambda_\mu = \xi_1^\nu(\partial_\nu \lambda_{2\mu} - \partial_\mu \lambda_{2\nu}) - \xi_2^\nu(\partial_\nu \lambda_{1\mu} - \partial_\mu \lambda_{1\nu}) \\ + \frac{1}{2}\partial_\mu(\xi_1\lambda_2 - \xi_2\lambda_1).\end{aligned} \tag{2.18}$$

It is the generalization of the Lie bracket on spaces of generalized vectors.

3 $O(D, D)$ group

Consider the orthogonal transformation $\mathcal{O}$, i.e. the transformation that preserves the inner product (2.12)

$$\langle \mathcal{O}\Lambda_1, \mathcal{O}\Lambda_2 \rangle = \langle \Lambda_1, \Lambda_2 \rangle \Leftrightarrow (\mathcal{O}\Lambda_1)^T \eta (\mathcal{O}\Lambda_2) = \Lambda_1^T \eta \Lambda_2, \tag{3.19}$$

which is satisfied for the condition

$$\mathcal{O}^T \eta \mathcal{O} = \eta. \tag{3.20}$$

There is a solution for the above equation in the form $\mathcal{O} = e^T$, see Sec. 2.1 of [21], where

$$T = \begin{pmatrix} A & \theta \\ B & -A^T \end{pmatrix}, \tag{3.21}$$

with $\theta : T^\star M \to TM$ and $B : TM \to T^\star M$ being anti-symmetric, and $A : TM \to TM$ being the endomorphism. In general case, B and θ can be independent for $\mathcal{O}$ to satisfy condition (3.20).

Consider now the action of some element of $O(D, D)$ on the double coordinate X (2.4) and the double gauge parameter Λ (2.11)

$$\hat{X}^M = \mathcal{O}^M_{\ N} X^N, \ \hat{\Lambda}^M = \mathcal{O}^M_{\ N} \Lambda^N, \tag{3.22}$$

and note that the relation (2.15) can be written as

$$\int d\sigma \Big\{ \langle \Lambda_1, X \rangle, \langle \Lambda_2, X \rangle \Big\} = -\int d\sigma \langle [\Lambda_1, \Lambda_2]_{\mathcal{C}}, X \rangle, \tag{3.23}$$

and using (3.19) and (3.22) as

$$\int d\sigma \Big\{ \langle \hat{\Lambda}_1, \hat{X} \rangle, \langle \hat{\Lambda}_2, \hat{X} \rangle \Big\} = -\int d\sigma \langle [\Lambda_1, \Lambda_2]_{\mathcal{C}}, X \rangle = -\int d\sigma \langle [\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T}, \hat{X} \rangle, \tag{3.24}$$

where we expressed the right hand side in terms of some new bracket $[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T}$. Moreover, using (3.19) and (3.22), the right hand side of (3.23) can be written as

$$\langle [\Lambda_1, \Lambda_2]_{\mathcal{C}}, X \rangle = \langle [\mathcal{O}^{-1}\hat{\Lambda}_1, \mathcal{O}^{-1}\hat{\Lambda}_2]_{\mathcal{C}}, \mathcal{O}^{-1}\hat{X} \rangle = \langle \mathcal{O}[\mathcal{O}^{-1}\hat{\Lambda}_1, \mathcal{O}^{-1}\hat{\Lambda}_2]_{\mathcal{C}}, \hat{X} \rangle. \tag{3.25}$$

Using (3.24) and (3.25), one obtains

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_{\mathcal{C}_T} = \mathcal{O}[\mathcal{O}^{-1}\hat{\Lambda}_1, \mathcal{O}^{-1}\hat{\Lambda}_2]_{\mathcal{C}} = e^T [e^{-T}\hat{\Lambda}_1, e^{-T}\hat{\Lambda}_2]_{\mathcal{C}}. \tag{3.26}$$

This is a definition of a T-twisted Courant bracket. Throughout this paper, we use the notation where $[,]_{\mathcal{C}}$ is the Courant bracket, while when $\mathcal{C}$ has an additional index, it represents the twist of the Courant bracket by the indexed field, e.g. $[,]_{\mathcal{C}_B}$ is the Courant bracket twisted by B.

In a special case, when $A = 0$, $\theta = 0$, the bracket (3.26) becomes the Courant bracket twisted by a 2-form B [9]

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_B} = e^{\hat{B}} [e^{-\hat{B}}\Lambda_1, e^{-\hat{B}}\Lambda_2]_{\mathcal{C}}, \tag{3.27}$$

where $e^{\hat{B}}$ is the twisting matrix, given by

$$e^{\hat{B}} = \begin{pmatrix} \delta^\mu_\nu & 0 \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix}, \quad \hat{B}^M_{\ N} = \begin{pmatrix} 0 & 0 \\ 2B_{\mu\nu} & 0 \end{pmatrix}. \tag{3.28}$$

This bracket has been obtained in the algebra of generalized currents [11,13].

In case of $A = 0$, $B = 0$, the bracket (3.26) becomes the Courant bracket twisted by a bi-vector θ

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_\theta} = e^{\hat{\theta}} [e^{-\hat{\theta}}\Lambda_1, e^{-\hat{\theta}}\Lambda_2]_{\mathcal{C}}, \tag{3.29}$$

where $e^{\hat{\theta}}$ is the twisting matrix, given by

$$e^{\hat{\theta}} = \begin{pmatrix} \delta^\mu_\nu & \kappa\theta^{\mu\nu} \\ 0 & \delta^\nu_\mu \end{pmatrix}, \quad \hat{\theta}^M_{\ N} = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 0 & 0 \end{pmatrix}. \tag{3.30}$$

The B-twisted Courant bracket (3.27) and θ-twisted Courant bracket (3.29) are related by self T-duality [13]. It is easy to demonstrate that both $e^{\hat{B}}$ and $e^{\hat{\theta}}$ satisfy the condition (3.20).

We can now deduce a simple algorithm for finding the Courant bracket twisted by an arbitrary $O(D, D)$ transformation. One rewrites the double symmetry generator $G(\xi, \lambda)$ in the basis obtained by the action of the matrix e^T on the double coordinate (2.4). Then, the Poisson bracket algebra between these generators gives rise to the appropriate twist of the Courant bracket. In this paper, we apply this algorithm to obtain the Courant bracket twisted by both B and θ.

4 Twisting matrix

The transformations $e^{\hat{B}}$ and $e^{\hat{\theta}}$ do not commute. That is why we define the transformations that simultaneously twists the Courant bracket by B and θ as $e^{\breve{B}}$, where

$$\breve{B} = \hat{B} + \hat{\theta} = \begin{pmatrix} 0 & \kappa\theta^{\mu\nu} \\ 2B_{\mu\nu} & 0 \end{pmatrix}. \tag{4.1}$$

The Courant bracket twisted at the same time both by a 2-form B and by a bi-vector θ is given by

$$[\Lambda_1, \Lambda_2]_{\mathcal{C}_{B\theta}} = e^{\breve{B}} [e^{-\breve{B}}\Lambda_1, e^{-\breve{B}}\Lambda_2]_{\mathcal{C}}. \tag{4.2}$$

The full expression for $e^{\breve{B}}$ can be obtained from the well known Taylor series expansion of exponential function

$$e^{\breve{B}} = \sum_{n=0}^{\infty} \frac{\breve{B}^n}{n!}. \tag{4.3}$$

The square of the matrix $\breve{B}$ is easily obtained

$$\breve{B}^2 = 2\begin{pmatrix} \kappa(\theta B)^\mu{}_\nu & 0 \\ 0 & \kappa(B\theta)_\mu{}^\nu \end{pmatrix}, \tag{4.4}$$

as well as its cube

$$\breve{B}^3 = 2\begin{pmatrix} 0 & \kappa^2(\theta B\theta)^{\mu\nu} \\ 2\kappa(B\theta B)_{\mu\nu} & 0 \end{pmatrix}. \tag{4.5}$$

The higher degree of $\breve{B}$ are given by

$$\breve{B}^{2n} = \begin{pmatrix} (\alpha^n)^\mu{}_\nu & 0 \\ 0 & ((\alpha^T)^n)_\mu{}^\nu \end{pmatrix}, \tag{4.6}$$

for even degrees, and for odd degrees by

$$\breve{B}^{2n+1} = \begin{pmatrix} 0 & \kappa(\alpha^n\theta)^{\mu\nu} \\ 2(B\alpha^n)_{\mu\nu} & 0 \end{pmatrix}, \tag{4.7}$$

where we have marked

$$\alpha^\mu{}_\nu = 2\kappa\theta^{\mu\rho}B_{\rho\nu}. \tag{4.8}$$

Substituting (4.6) and (4.7) into (4.3), we obtain the twisting matrix

$$e^{\breve{B}} = \begin{pmatrix} \left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n)!}\right)^\mu{}_\nu & \kappa\left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\right)^\mu{}_\rho \theta^{\rho\nu} \\ 2B_{\mu\rho}\left(\sum_{n=0}^\infty \frac{\alpha^n}{(2n+1)!}\right)^\rho{}_\nu & \left(\sum_{n=0}^\infty \frac{(\alpha^T)^n}{(2n)!}\right)_\mu{}^\nu \end{pmatrix}. \tag{4.9}$$

Taking into the account the Taylor's expansion of hyperbolic functions

$$cosh(x) = \sum_{n=0}^\infty \frac{x^{2k}}{(2k)!}, \quad sinh(x) = \sum_{n=0}^\infty \frac{x^{2k+1}}{(2k+1)!}, \tag{4.10}$$

the twisting matrix (4.9) can be rewritten as

$$e^{\breve{B}} = \begin{pmatrix} \mathcal{C}^\mu{}_\nu & \kappa\mathcal{S}^\mu{}_\rho\theta^{\rho\nu} \\ 2B_{\mu\rho}\mathcal{S}^\rho{}_\nu & (\mathcal{C}^T)_\mu{}^\nu \end{pmatrix}, \tag{4.11}$$

with $\mathcal{S}^\mu{}_\nu = \left(\frac{\sinh\sqrt{\alpha}}{\sqrt{\alpha}}\right)^\mu{}_\nu$ and $\mathcal{C}^\mu{}_\nu = \left(\cosh\sqrt{\alpha}\right)^\mu{}_\nu$. Its determinant is given by

$$\det(e^{\breve{B}}) = e^{Tr(\breve{B})} = 1, \tag{4.12}$$

and the straightforward calculations show that its inverse is given by

$$e^{-\breve{B}} = \begin{pmatrix} \mathcal{C}^\mu{}_\nu & -\kappa\mathcal{S}^\mu{}_\rho\theta^{\rho\nu} \\ -2B_{\mu\rho}\mathcal{S}^\rho{}_\nu & (\mathcal{C}^T)_\mu{}^\nu \end{pmatrix}. \tag{4.13}$$

One easily obtains the relation

$$(e^{\breve{B}})^T\ \eta\ e^{\breve{B}} = \eta, \tag{4.14}$$

therefore the transformation (4.11) is indeed an element of $O(D, D)$.

It is worth pointing out characteristics of the matrix $\alpha^\mu{}_\nu$. It is easy to show that $\alpha^\mu{}_\rho\theta^{\rho\nu} = \theta^{\mu\rho}(\alpha^T)_\rho{}^\nu$ and $B_{\mu\rho}\alpha^\rho{}_\nu = (\alpha^T)_\mu{}^\rho B_{\rho\nu}$, which is further generalized to

$$(f(\alpha))^\mu{}_\rho\theta^{\rho\nu} = \theta^{\mu\rho}(f(\alpha^T))_\rho{}^\nu, \quad B_{\mu\rho}(f(\alpha))^\rho{}_\nu = (f(\alpha^T))_\mu{}^\rho B_{\rho\nu}, \tag{4.15}$$

for any analytical function $f(\alpha)$. Moreover, the well known hyperbolic identity $\cosh(x)^2 - \sinh(x)^2 = 1$ can also be expressed in terms of newly defined tensors

$$(\mathcal{C}^2)^\mu{}_\nu - \alpha^\mu{}_\rho(\mathcal{S}^2)^\rho{}_\nu = \delta^\mu_\nu. \tag{4.16}$$

Lastly, the self T-duality relates the matrix α to its transpose $\alpha \cong \alpha^T$, due to (2.6). Consequently, we write the following self T-duality relations

$$\mathcal{C} \cong \mathcal{C}^T, \quad \mathcal{S} \cong \mathcal{S}^T. \tag{4.17}$$

5 Symmetry generator in an appropriate basis

The direct computation of the bracket (4.2) would be difficult, given the form of the matrix $e^{\breve{B}}$. Therefore, we use the indirect computation of the bracket, by computing the Poisson bracket algebra of the symmetry generator (2.9), rewritten in the appropriate basis. As elaborated at the end of the Chapter 3, this basis is obtained by the action of the matrix (4.11) on the double coordinate (2.4)

$$\breve{X}^M = (e^{\breve{B}})^M{}_N\ X^N = \begin{pmatrix} \breve{k}^\mu \\ \breve{\imath}_\mu \end{pmatrix}, \tag{5.18}$$

where

$$\breve{k}^\mu = \kappa\mathcal{C}^\mu{}_\nu x'^\nu + \kappa(\mathcal{S}\theta)^{\mu\nu}\pi_\nu, \\ \breve{\imath}_\mu = 2(B\mathcal{S})_{\mu\nu}x'^\nu + (\mathcal{C}^T)_\mu{}^\nu\pi_\nu, \tag{5.19}$$

are new currents. Applying (2.6), (2.8) and (4.17) to currents $\breve{k}^\mu$ and $\breve{\imath}_\mu$ we obtain $\breve{\imath}_\mu$ and $\breve{k}^\mu$ respectively, meaning that these currents are directly related by self T-duality. Multiplying the Eq. (5.18) with the matrix (4.13), we obtain the relations inverse to (5.19)

$$\kappa x'^\mu = \mathcal{C}^\mu{}_\nu\breve{k}^\nu - \kappa(\mathcal{S}\theta)^{\mu\nu}\breve{\imath}_\nu, \\ \pi_\mu = -2(B\mathcal{S})_{\mu\nu}\breve{k}^\nu + (\mathcal{C}^T)_\mu{}^\nu\breve{\imath}_\nu. \tag{5.20}$$

Applying the transformation (4.11) to a double gauge parameter (2.11), we obtain new gauge parameters

$$\breve{\Lambda}^M = \begin{pmatrix} \breve{\xi}^\mu \\ \breve{\lambda}_\mu \end{pmatrix} = (e^{\breve{B}})^M{}_N\, \Lambda^N = \begin{pmatrix} \mathcal{C}^\mu{}_\nu \xi^\nu + \kappa(\mathcal{S}\theta)^{\mu\nu}\lambda_\nu \\ 2(B\mathcal{S})_{\mu\nu}\xi^\nu + (\mathcal{C}^T)_\mu{}^\nu\lambda_\nu \end{pmatrix}. \tag{5.21}$$

The symmetry generator (2.9) rewritten in a new basis $\mathcal{G}(\mathcal{C}\xi + \kappa\mathcal{S}\theta\lambda, 2(B\mathcal{S})\xi + \mathcal{C}^T\lambda) \equiv \breve{\mathcal{G}}(\breve{\xi}, \breve{\lambda})$ is given by

$$\breve{G}(\breve{\Lambda}) = \int d\sigma \langle \breve{\Lambda}, \breve{X}\rangle \Leftrightarrow \breve{G}(\breve{\xi}, \breve{\lambda}) = \int d\sigma \Big[\breve{\xi}^\mu \breve{\imath}_\mu + \breve{\lambda}_\mu \breve{k}^\mu\Big]. \tag{5.22}$$

Substituting (5.18) and (5.21) into (5.22), the symmetry generator in the initial canonical basis (2.9) is obtained. Due to mutual self T-duality between basis currents (5.19), this generator is invariant upon self T-duality.

Rewriting the Eq. (2.15) in terms of new gauge parameters (5.21) in the basis of auxiliary currents (5.19), the Courant bracket twisted by both a 2-form $B_{\mu\nu}$ and by a bi-vector $\theta^{\mu\nu}$ is obtained in the new generator (5.22) algebra

$$\Big\{\breve{G}(\breve{\Lambda}_1),\ \breve{G}(\breve{\Lambda}_2)\Big\} = -\breve{G}\Big([\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathcal{C}_{B\theta}}\Big). \tag{5.23}$$

5.1 Auxiliary generator

Let us define a new auxiliary basis, so that both the matrices $\mathcal{C}$ and $\mathcal{S}$ are absorbed in some new fields, giving rise to the generator algebra that is much more readable. When the algebra in this basis is obtained, simple change of variables back to the initial ones will provide us with the bracket in need.

Multiplying the second equation of (5.19) with the matrix $\mathcal{C}^{-1}$, we obtain

$$\breve{\imath}_\nu (\mathcal{C}^{-1})^\nu{}_\mu = \pi_\mu + 2\kappa(B\mathcal{S}\mathcal{C}^{-1})_{\mu\nu} x'^\nu, \tag{5.24}$$

where we have used $(B\mathcal{S})_{\nu\rho}(\mathcal{C}^{-1})^\nu{}_\mu = -(B\mathcal{S}\mathcal{C}^{-1})_{\rho\mu} = (B\mathcal{S}\mathcal{C}^{-1})_{\mu\rho}$, due to tensor $B\mathcal{S}$ being antisymmetric, and properties (4.15). We will mark the result as a new auxiliary current, given by

$$\mathring{\imath}_\mu = \pi_\mu + 2\kappa \mathring{B}_{\mu\nu} x'^\nu, \tag{5.25}$$

where $\mathring{B}$ is an auxiliary B-field, given by

$$\mathring{B}_{\mu\nu} = B_{\mu\rho}\mathcal{S}^\rho{}_\sigma (\mathcal{C}^{-1})^\sigma{}_\nu. \tag{5.26}$$

On the other hand, multiplying the first equation of (5.19) with the matrix $\mathcal{C}$, we obtain

$$\mathcal{C}^\mu{}_\nu \breve{k}^\nu = (\mathcal{C}^2)^\mu{}_\nu \kappa x'^\nu + \kappa(\mathcal{C}\mathcal{S}\theta)^{\mu\nu}\pi_\nu. \tag{5.27}$$

Substituting (4.16) in the previous equation, and keeping in mind that $\mathcal{C}$, $\mathcal{S}$ and θ commute (4.15), we obtain

$$\mathcal{C}^\mu{}_\nu \breve{k}^\nu = \kappa x'^\mu + \kappa(\mathcal{C}\mathcal{S}\theta)^{\rho\nu}(\pi_\nu + 2\kappa(B\mathcal{S}\mathcal{C}^{-1})_{\nu\sigma} x'^\sigma). \tag{5.28}$$

Using (5.25), the results are marked as a new auxiliary current

$$\mathring{k}^\mu = \kappa x'^\mu + \kappa\mathring{\theta}^{\mu\nu}\mathring{\imath}_\nu, \tag{5.29}$$

where $\mathring{\theta}$ is given by

$$\mathring{\theta}^{\mu\nu} = \mathcal{C}^\mu{}_\rho \mathcal{S}^\rho{}_\sigma \theta^{\sigma\nu}. \tag{5.30}$$

There is no explicit dependence on either $\mathcal{C}$ nor $\mathcal{S}$ in redefined auxiliary currents, rather only on canonical variables and new background fields. From (5.29), it is easy to express the coordinate σ-derivative in the basis of new auxiliary currents

$$\kappa x'^\mu = \mathring{k}^\mu - \kappa\mathring{\theta}^{\mu\nu}\mathring{\imath}_\nu. \tag{5.31}$$

The first equation of (5.19) could have been multiplied with $\mathcal{C}$, instead of $\mathcal{C}^{-1}$, given that the latter would also produce a current that would not explicitly depend on $\mathcal{C}$. However, the expression for coordinate σ-derivative $\kappa x'^\mu$ would explicitly depend on $\mathcal{C}^2$ in that case, while with our choice of basis it does not (5.31).

Substituting (5.24) and (5.28) in the expression for the generator (5.22), we obtain

$$\breve{G}(\breve{\xi}, \breve{\lambda}) = \int d\sigma \Big[\breve{\lambda}_\mu (\mathcal{C}^{-1})^\mu{}_\nu \mathring{k}^\nu + \breve{\xi}^\mu (\mathcal{C}^T)_\mu{}^\nu \mathring{\imath}_\nu\Big], \tag{5.32}$$

from which it is easily seen that the generator (5.22) is equal to an auxiliary generator

$$\mathring{G}(\mathring{\Lambda}) = \int d\sigma \langle \mathring{X}, \mathring{\Lambda}\rangle \Leftrightarrow \mathring{G}(\mathring{\xi}, \mathring{\lambda}) = \int d\sigma \Big[\mathring{\lambda}_\mu \mathring{k}^\mu + \mathring{\xi}^\mu \mathring{\imath}_\mu\Big], \tag{5.33}$$

provided that

$$\mathring{\Lambda}^M = \begin{pmatrix} \mathring{\xi}^\mu \\ \mathring{\lambda}_\mu \end{pmatrix}, \quad \mathring{\lambda}_\mu = \breve{\lambda}_\nu (\mathcal{C}^{-1})^\nu{}_\mu, \quad \mathring{\xi}^\mu = \mathcal{C}^\mu{}_\nu \breve{\xi}^\nu, \tag{5.34}$$

and

$$\mathring{X}^M = \begin{pmatrix} \mathring{k}^\mu \\ \mathring{\imath}_\mu \end{pmatrix}. \tag{5.35}$$

Once that the algebra of (5.33) is known, the algebra of generator (5.22) can be easily obtained using (5.34).

The change of basis to the one suitable for the auxiliary generator (5.33) corresponds to the transformation

$$A^M_{\ N}=\begin{pmatrix}(\mathcal{C})^\mu_{\ \nu} & 0\\ 0 & ((\mathcal{C}^{-1})^T)_\mu^{\ \nu}\end{pmatrix},\ \mathring{\Lambda}^M=A^M_{\ N}\,\check{\Lambda}^N,\ \mathring{X}^M=A^M_{\ N}\,\check{X}^N, \tag{5.36}$$

that can be rewritten as

$$\mathring{X}^M=(Ae^{\check{B}})^M_{\ N}\,X^N,\quad \mathring{\Lambda}^M=(Ae^{\check{B}})^M_{\ N}\,\Lambda^N, \tag{5.37}$$

where (5.18) and (5.21) were used. It is easy to show that the transformation $A^M_{\ N}$, and consequentially $(Ae^{\check{B}})^M_{\ N}$, is the element of $O(D,D)$ group

$$A^T\eta\,A=\eta,\ (Ae^{\check{B}})^T\,\eta\,(Ae^{\check{B}})=\eta, \tag{5.38}$$

which means that there is $\mathring{C}$, for which [21]

$$e^{\mathring{C}}=Ae^{\check{B}}. \tag{5.39}$$

The generator (5.33) gives rise to algebra that closes on $\mathring{C}$-twisted Courant bracket

$$\left\{\mathring{G}(\mathring{\Lambda}_1),\mathring{G}(\mathring{\Lambda}_1)\right\}=-\mathring{G}\Big([\mathring{\Lambda}_1,\mathring{\Lambda}_2]_{\mathcal{C}_{\mathring{C}}}\Big), \tag{5.40}$$

where the $\mathring{C}$-twisted Courant bracket is defined by

$$[\mathring{\Lambda}_1,\mathring{\Lambda}_2]_{\mathcal{C}_{\mathring{C}}}=e^{\mathring{C}}[e^{-\mathring{C}}\mathring{\Lambda}_1,e^{-\mathring{C}}\mathring{\Lambda}_2]_{\mathcal{C}}. \tag{5.41}$$

In the next chapter, we will obtain this bracket by direct computation of the generators Poisson bracket algebra.

Lastly, let us briefly comment on reducibility conditions for the $\mathring{C}$-twisted Courant bracket. Since we are working with the closed strings, the total derivatives vanishes when integrated out over the worldsheet. Using (5.31), we obtain

$$\int d\sigma\kappa\varphi'=\int d\sigma\kappa x'^\mu\partial_\mu\varphi=\int d\sigma\Big(\mathring{k}^\mu\partial_\mu\varphi+\kappa\mathring{\imath}_\mu\mathring{\theta}^{\mu\nu}\partial_\nu\varphi\Big)=0, \tag{5.42}$$

for any parameter λ. Hence, the generator (5.33) remains invariant under the following change of parameters

$$\mathring{\xi}^\mu\to\mathring{\xi}^\mu+\kappa\mathring{\theta}^{\mu\nu}\partial_\nu\varphi,\quad \mathring{\lambda}_\mu\to\mathring{\lambda}_\mu+\partial_\mu\varphi. \tag{5.43}$$

These are reducibility conditions (2.10) in the basis spanned by $\mathring{k}^\mu$ and $\mathring{\imath}_\mu$.

6 Courant bracket twisted by $\mathring{C}$ from the generator algebra

In order to obtain the Poisson bracket algebra for the generator (5.33), let us firstly calculate the algebra of basis vectors, using the standard Poisson bracket relations (2.16). The auxiliary currents $\mathring{\imath}_\mu$ algebra is

$$\{\mathring{\imath}_\mu(\sigma),\mathring{\imath}_\nu(\bar{\sigma})\}=-2\mathring{B}_{\mu\nu\rho}\mathring{k}^\rho\delta(\sigma-\bar{\sigma})-\mathring{\mathcal{F}}_{\mu\nu}^{\ \ \rho}\,\mathring{\imath}_\rho\delta(\sigma-\bar{\sigma}), \tag{6.1}$$

where $\mathring{B}_{\mu\nu\rho}$ is the generalized H-flux, given by

$$\mathring{B}_{\mu\nu\rho}=\partial_\mu\mathring{B}_{\nu\rho}+\partial_\nu\mathring{B}_{\rho\mu}+\partial_\rho\mathring{B}_{\mu\nu}, \tag{6.2}$$

and $\mathring{\mathcal{F}}^\rho_{\mu\nu}$ is the generalized f-flux, given by

$$\mathring{\mathcal{F}}_{\mu\nu}^{\ \ \rho}=-2\kappa\,\mathring{B}_{\mu\nu\sigma}\mathring{\theta}^{\sigma\rho}. \tag{6.3}$$

The algebra of currents $\mathring{k}^\mu$ is given by

$$\{\mathring{k}^\mu(\sigma),\mathring{k}^\nu(\bar{\sigma})\}=-\kappa\,\mathring{\mathcal{Q}}_\rho^{\ \mu\nu}\mathring{k}^\rho\delta(\sigma-\bar{\sigma})-\kappa^2\mathring{\mathcal{R}}^{\mu\nu\rho}\mathring{\imath}_\rho\delta(\sigma-\bar{\sigma}), \tag{6.4}$$

where

$$\mathring{\mathcal{Q}}_\mu^{\ \nu\rho}=\mathring{Q}_\mu^{\ \nu\rho}+2\kappa\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\rho\tau}\mathring{B}_{\mu\sigma\tau},\quad \mathring{Q}_\mu^{\ \nu\rho}=\partial_\mu\mathring{\theta}^{\nu\rho} \tag{6.5}$$

and

$$\begin{aligned}\mathring{\mathcal{R}}^{\mu\nu\rho}&=\mathring{R}^{\mu\nu\rho}+2\kappa\mathring{\theta}^{\mu\lambda}\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\rho\tau}\mathring{B}_{\lambda\sigma\tau},\\ \mathring{R}^{\mu\nu\rho}&=\mathring{\theta}^{\mu\sigma}\partial_\sigma\mathring{\theta}^{\nu\rho}+\mathring{\theta}^{\nu\sigma}\partial_\sigma\mathring{\theta}^{\rho\mu}+\mathring{\theta}^{\rho\sigma}\partial_\sigma\mathring{\theta}^{\mu\nu}.\end{aligned} \tag{6.6}$$

The terms in (6.4) containing both $\mathring{\theta}$ and $\mathring{B}$ are the consequence of non-commutativity of auxiliary currents $\mathring{\imath}_\mu$. The remaining algebra of currents $\mathring{k}^\mu$ and $\mathring{\imath}_\mu$ can be as easily obtained

$$\begin{aligned}\{\mathring{\imath}_\mu(\sigma),\mathring{k}^\nu(\bar{\sigma})\}&=\kappa\delta^\nu_\mu\delta'(\sigma-\bar{\sigma})\\ &\quad+\mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu}\,\mathring{k}^\rho\delta(\sigma-\bar{\sigma})-\kappa\,\mathring{\mathcal{Q}}_\mu^{\ \nu\rho}\mathring{\imath}_\rho\delta(\sigma-\bar{\sigma}).\end{aligned} \tag{6.7}$$

The basic algebra relations can be summarized in a single algebra relation where the structure constants contain all generalized fluxes

$$\{\mathring{X}^M,\mathring{X}^N\}=-\mathring{F}^{MN}_{\ \ \ P}\,\mathring{X}^P\delta(\sigma-\bar{\sigma})+\kappa\eta^{MN}\delta'(\sigma-\bar{\sigma}), \tag{6.8}$$

with

$$F^{MN\rho}=\begin{pmatrix}\kappa^2\mathring{\mathcal{R}}^{\mu\nu\rho} & -\kappa\mathring{\mathcal{Q}}_\nu^{\ \mu\rho}\\ \kappa\mathring{\mathcal{Q}}_\mu^{\ \nu\rho} & \mathring{\mathcal{F}}_{\mu\nu}^{\ \ \rho}\end{pmatrix},$$
$$F^{MN}{}_\rho=\begin{pmatrix}\kappa\mathring{\mathcal{Q}}_\rho^{\ \mu\nu} & \mathring{\mathcal{F}}_{\nu\rho}^{\ \ \mu}\\ -\mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu} & 2\mathring{B}_{\mu\nu\rho}\end{pmatrix}. \tag{6.9}$$

The form of the generalized fluxes is the same as the ones already obtained using the tetrad formalism [16–18]. In our approach, the generalized fluxes are obtained in the Poisson bracket algebra, only from the fact that the generalized canonical variable X^M is transformed with an element of the $O(D,D)$ group that twists the Courant bracket both by B and θ at the same time. Consequentially, the fluxes obtained in this paper are functions of some new effective fields, $\mathring{B}_{\mu\nu}$ (5.26) and $\mathring{\theta}^{\mu\nu}$ (5.30).

We now proceed to obtain the full bracket. Let us rewrite the generator (5.33) algebra

$$\begin{aligned}&\Big\{\mathring{G}(\mathring{\xi}_1,\mathring{\lambda}_1)(\sigma),\ \mathring{G}(\mathring{\xi}_2,\mathring{\lambda}_2)(\bar\sigma)\Big\}\\ &=\int d\sigma d\bar\sigma\Big[\Big\{\mathring{\xi}_1^\mu(\sigma)\mathring{\imath}_\mu(\sigma),\mathring{\xi}_2^\nu(\bar\sigma)\mathring{\imath}_\nu(\bar\sigma)\Big\}\\ &+\Big\{\mathring{\lambda}_{1\mu}(\sigma)\mathring{k}^\mu(\sigma),\mathring{\lambda}_{2\nu}(\bar\sigma)\mathring{k}^\nu(\bar\sigma)\Big\}\\ &+\Big\{\mathring{\xi}_1^\mu(\sigma)\mathring{\imath}_\mu(\sigma),\mathring{\lambda}_{2\nu}(\bar\sigma)\mathring{k}^\nu(\bar\sigma)\Big\}\\ &+\Big\{\mathring{\lambda}_{1\mu}(\sigma)\mathring{k}^\mu(\sigma),\mathring{\xi}_2^\nu(\bar\sigma)\mathring{\imath}_\nu(\bar\sigma)\Big\}\Big].\end{aligned} \tag{6.10}$$

The first term of (6.10) is obtained, using (6.1)

$$\begin{aligned}&\int d\sigma d\bar\sigma\Big\{\mathring{\xi}_1^\mu(\sigma)\mathring{\imath}_\mu(\sigma),\mathring{\xi}_2^\nu(\bar\sigma)\mathring{\imath}_\nu(\bar\sigma)\Big\}\\ &=\int d\sigma\Big[\mathring{\imath}_\mu\Big(\mathring{\xi}_2^\nu\partial_\nu\mathring{\xi}_1^\mu-\mathring{\xi}_1^\nu\partial_\nu\mathring{\xi}_2^\mu-\mathring{\mathcal{F}}_{\nu\rho}^{\ \ \mu}\,\mathring{\xi}_1^\nu\mathring{\xi}_2^\rho\Big)\\ &-2\mathring{B}_{\mu\nu\rho}\mathring{k}^\mu\mathring{\xi}_1^\nu\mathring{\xi}_2^\rho\Big].\end{aligned} \tag{6.11}$$

The second term is obtained, using (6.4)

$$\begin{aligned}&\int d\sigma d\bar\sigma\Big\{\mathring{\lambda}_{1\mu}(\sigma)\mathring{k}^\mu(\sigma),\mathring{\lambda}_{2\nu}(\bar\sigma)\mathring{k}^\nu(\bar\sigma)\Big\}\\ &=\int d\sigma\Big[\mathring{k}^\mu\Big(\kappa\mathring{\theta}^{\nu\rho}(\mathring{\lambda}_{2\nu}\partial_\rho\mathring{\lambda}_{1\mu}-\mathring{\lambda}_{1\nu}\partial_\rho\mathring{\lambda}_{2\mu})-\kappa\mathring{\mathcal{Q}}_\mu^{\ \nu\rho}\\ &\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}\Big)-\mathring{\imath}_\mu\kappa^2\mathring{\mathcal{R}}^{\mu\nu\rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}\Big].\end{aligned} \tag{6.12}$$

The remaining terms are antisymmetric with respect to $1\leftrightarrow 2,\ \sigma\leftrightarrow\bar\sigma$ interchange. Therefore, it is sufficient to calculate only the first term in the last line of (6.10)

$$\begin{aligned}&\int d\sigma d\bar\sigma\Big\{\mathring{\xi}_1^\mu(\sigma)\mathring{\imath}_\mu(\sigma),\mathring{\lambda}_{2\nu}(\bar\sigma)\mathring{k}^\nu(\bar\sigma)\Big\}\\ &=\int d\sigma\Big[\mathring{k}^\mu\Big(-\mathring{\xi}_1^\nu\partial_\nu\mathring{\lambda}_{2\mu}-\mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu}\,\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu}\Big)\\ &+\mathring{\imath}_\mu\Big(\kappa(\mathring{\lambda}_{2\nu}\mathring{\theta}^{\nu\rho})\partial_\rho\mathring{\xi}_1^\mu-\kappa\mathring{\mathcal{Q}}_\rho^{\ \nu\mu}\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu}\Big)\Big]\\ &+\int d\sigma d\bar\sigma\kappa\mathring{\xi}_1^\nu(\sigma)\mathring{\lambda}_{2\nu}(\bar\sigma)\partial_\sigma\delta(\sigma-\bar\sigma).\end{aligned} \tag{6.13}$$

In order to transform the anomalous part, we note that

$$\partial_\sigma\delta(\sigma-\bar\sigma)=\frac{1}{2}\partial_\sigma\delta(\sigma-\bar\sigma)-\frac{1}{2}\partial_{\bar\sigma}\delta(\sigma-\bar\sigma), \tag{6.14}$$

and

$$f(\bar\sigma)\partial_\sigma\delta(\sigma-\bar\sigma)=f(\sigma)\partial_\sigma\delta(\sigma-\bar\sigma)+f'(\sigma)\delta(\sigma-\bar\sigma). \tag{6.15}$$

Applying (6.14) and (6.15) to the last row of (6.13), we obtain

$$\begin{aligned}&\int d\sigma d\bar\sigma\kappa\mathring{\xi}_1^\nu(\sigma)\mathring{\lambda}_{2\nu}(\bar\sigma)\partial_\sigma\delta(\sigma-\bar\sigma)\\ &=\frac{1}{2}\int d\sigma\kappa x'^\mu\Big(\mathring{\xi}_1^\nu\partial_\mu\mathring{\lambda}_{2\nu}-\partial_\mu\mathring{\xi}_1^\nu\mathring{\lambda}_{2\nu}\Big)\\ &+\frac{\kappa}{2}\int d\sigma d\bar\sigma\Big(\mathring{\xi}_1^\nu(\sigma)\mathring{\lambda}_{2\nu}(\sigma)\partial_\sigma\delta(\sigma-\bar\sigma)\\ &-\mathring{\xi}_1^\nu(\bar\sigma)\mathring{\lambda}_{2\nu}(\bar\sigma)\partial_{\bar\sigma}\delta(\sigma-\bar\sigma)\Big)\\ &=\frac{1}{2}\int d\sigma\Big[\mathring{k}^\mu\Big(\mathring{\xi}_1^\nu\partial_\mu\mathring{\lambda}_{2\nu}-\partial_\mu\mathring{\xi}_1^\nu\mathring{\lambda}_{2\nu}\Big)\\ &+\mathring{\imath}_\mu\kappa\mathring{\theta}^{\mu\rho}\Big(\mathring{\xi}_1^\nu\partial_\rho\mathring{\lambda}_{2\nu}-\partial_\rho\mathring{\xi}_1^\nu\mathring{\lambda}_{2\nu}\Big)\Big],\end{aligned} \tag{6.16}$$

where (5.31) was used, as well as antisymmetry of $\mathring{\theta}$. Substituting (6.16) to (6.13), we obtain

$$\begin{aligned}&\int d\sigma d\bar\sigma\Big\{\mathring{\xi}_1^\mu(\sigma)\mathring{\imath}_\mu(\sigma),\mathring{\lambda}_{2\nu}(\bar\sigma)\mathring{k}^\nu(\bar\sigma)\Big\}\\ &=\int d\sigma\Big[\mathring{k}^\mu\Big(\mathring{\xi}_1^\nu(\partial_\mu\mathring{\lambda}_{2\nu}-\partial_\nu\mathring{\lambda}_{2\mu})\\ &-\frac{1}{2}\partial_\mu(\mathring{\xi}_1\mathring{\lambda}_2)-\mathring{\mathcal{F}}_{\mu\rho}^{\ \ \nu}\,\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu}\Big)\\ &+\mathring{\imath}_\mu\Big(\kappa(\mathring{\lambda}_{2\nu}\mathring{\theta}^{\nu\rho})\partial_\rho\mathring{\xi}_1^\mu+\kappa\mathring{\theta}^{\mu\rho}\Big(\mathring{\xi}_1^\nu\partial_\rho\mathring{\lambda}_{2\nu}-\frac{1}{2}\partial_\rho(\mathring{\xi}_1\mathring{\lambda}_2)\Big)\\ &-\kappa\mathring{\mathcal{Q}}_\rho^{\ \nu\mu}\mathring{\xi}_1^\rho\mathring{\lambda}_{2\nu}\Big)\Big].\end{aligned} \tag{6.17}$$

Substituting (6.11), (6.12) and (6.17) into (6.10), we write the full algebra of generator in the form

$$\begin{aligned}&\Big\{\mathring{G}(\mathring{\Lambda}_1),\ \mathring{G}(\mathring{\Lambda}_2)\Big\}\\ &=-\mathring{G}(\mathring{\Lambda})\Leftrightarrow\Big\{\mathring{G}(\mathring{\xi}_1,\mathring{\lambda}_1),\ \mathring{G}(\mathring{\xi}_2,\mathring{\lambda}_2)\Big\}=-\mathring{G}(\mathring{\xi},\mathring{\lambda}),\end{aligned} \tag{6.18}$$

where

$$\mathring{\xi}^\mu = \mathring{\xi}_1^\nu \partial_\nu \mathring{\xi}_2^\mu - \mathring{\xi}_2^\nu \partial_\nu \mathring{\xi}_1^\mu - \kappa \mathring{\theta}^{\mu\rho} \times \Big(\mathring{\xi}_1^\nu \partial_\rho \mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\nu \partial_\rho \mathring{\lambda}_{1\nu} - \frac{1}{2}\partial_\rho (\mathring{\xi}_1 \mathring{\lambda}_2 - \mathring{\xi}_2 \mathring{\lambda}_1) \Big) + \kappa \mathring{\theta}^{\nu\rho} (\mathring{\lambda}_{1\nu} \partial_\rho \mathring{\xi}_2^\mu - \mathring{\lambda}_{2\nu} \partial_\rho \mathring{\xi}_1^\mu) + \kappa^2 \mathring{\mathcal{R}}^{\mu\nu\rho} \mathring{\lambda}_{1\nu} \mathring{\lambda}_{2\rho} + \mathring{\mathcal{F}}_{\rho\sigma}^{\ \ \mu} \mathring{\xi}_1^\rho \mathring{\xi}_2^\sigma + \kappa \mathring{\mathcal{Q}}_\rho^{\ \nu\mu} (\mathring{\xi}_1^\rho \mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\rho \mathring{\lambda}_{1\nu}), \tag{6.19}$$

and

$$\mathring{\lambda}_\mu = \mathring{\xi}_1^\nu (\partial_\nu \mathring{\lambda}_{2\mu} - \partial_\mu \mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu (\partial_\nu \mathring{\lambda}_{1\mu} - \partial_\mu \mathring{\lambda}_{1\nu}) + \frac{1}{2}\partial_\mu (\mathring{\xi}_1 \mathring{\lambda}_2 - \mathring{\xi}_2 \mathring{\lambda}_1) + \kappa \mathring{\theta}^{\nu\rho} (\mathring{\lambda}_{1\nu} \partial_\rho \mathring{\lambda}_{2\mu} - \mathring{\lambda}_{2\nu} \partial_\rho \mathring{\lambda}_{1\mu}) + 2\mathring{B}_{\mu\nu\rho} \mathring{\xi}_1^\nu \mathring{\xi}_2^\rho + \kappa \mathring{\mathcal{Q}}_\mu^{\ \nu\rho} \mathring{\lambda}_{1\nu} \mathring{\lambda}_{2\rho} + \mathring{\mathcal{F}}_{\mu\sigma}^{\ \ \nu} \times (\mathring{\xi}_1^\sigma \mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\sigma \mathring{\lambda}_{1\nu}). \tag{6.20}$$

It is possible to rewrite the previous two equations, if we note the relations between the generalized fluxes

$$\mathring{\mathcal{R}}^{\mu\nu\rho} = \mathring{R}^{\mu\nu\rho} + \mathring{\theta}^{\mu\sigma} \mathring{\theta}^{\nu\tau} \mathring{\mathcal{F}}_{\sigma\tau}^{\ \ \rho}, \quad \mathring{\mathcal{Q}}_\mu^{\ \nu\rho} = \mathring{Q}_\mu^{\ \nu\rho} + \mathring{\theta}^{\nu\sigma} \mathring{\mathcal{F}}_{\mu\sigma}^{\ \ \rho}. \tag{6.21}$$

Now we have

$$\mathring{\xi}^\mu = \mathring{\xi}_1^\nu \partial_\nu \mathring{\xi}_2^\mu - \mathring{\xi}_2^\nu \partial_\nu \mathring{\xi}_1^\mu + \kappa \mathring{\theta}^{\mu\rho} \Big(\mathring{\xi}_1^\nu (\partial_\nu \mathring{\lambda}_{2\rho} - \partial_\rho \mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu (\partial_\nu \mathring{\lambda}_{1\rho} - \partial_\rho \mathring{\lambda}_{1\nu}) + \frac{1}{2}\partial_\rho (\mathring{\xi}_1 \mathring{\lambda}_2 - \mathring{\xi}_2 \mathring{\lambda}_1) \Big) + \kappa \mathring{\xi}_1^\rho \partial_\rho (\mathring{\lambda}_{2\nu} \mathring{\theta}^{\nu\mu}) - \kappa (\mathring{\lambda}_{2\nu} \mathring{\theta}^{\nu\rho}) \partial_\rho \mathring{\xi}_1^\mu - \kappa \mathring{\xi}_2^\rho \partial_\rho (\mathring{\lambda}_{1\nu} \mathring{\theta}^{\nu\mu}) + \kappa (\mathring{\lambda}_{1\nu} \mathring{\theta}^{\nu\rho}) \partial_\rho \mathring{\xi}_2^\mu + \kappa^2 \mathring{R}^{\mu\nu\rho} \mathring{\lambda}_{1\nu} \mathring{\lambda}_{2\rho} + \mathring{\mathcal{F}}_{\rho\sigma}^{\ \ \mu} \mathring{\xi}_1^\rho \mathring{\xi}_2^\sigma + \kappa \mathring{\theta}^{\mu\sigma} \mathring{\mathcal{F}}_{\sigma\rho}^{\ \ \nu} (\mathring{\xi}_1^\rho \mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\rho \mathring{\lambda}_{1\nu}) + \kappa^2 \mathring{\theta}^{\mu\sigma} \mathring{\theta}^{\nu\tau} \mathring{\mathcal{F}}_{\sigma\tau}^{\ \ \rho} \mathring{\lambda}_{1\nu} \mathring{\lambda}_{2\rho}, \tag{6.22}$$

and

$$\mathring{\lambda}_\mu = \mathring{\xi}_1^\nu (\partial_\nu \mathring{\lambda}_{2\mu} - \partial_\mu \mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu (\partial_\nu \mathring{\lambda}_{1\mu} - \partial_\mu \mathring{\lambda}_{1\nu}) + \frac{1}{2}\partial_\mu (\mathring{\xi}_1 \mathring{\lambda}_2 - \mathring{\xi}_2 \mathring{\lambda}_1) + \kappa \mathring{\theta}^{\nu\rho} (\mathring{\lambda}_{1\nu} \partial_\rho \mathring{\lambda}_{2\mu} - \mathring{\lambda}_{2\nu} \partial_\rho \mathring{\lambda}_{1\mu}) + \kappa \mathring{\mathcal{Q}}_\mu^{\ \nu\rho} \mathring{\lambda}_{1\nu} \mathring{\lambda}_{2\rho} + 2\mathring{B}_{\mu\nu\rho} \mathring{\xi}_1^\nu \mathring{\xi}_2^\rho + \mathring{\mathcal{F}}_{\mu\sigma}^{\ \ \nu} \times (\mathring{\xi}_1^\sigma \mathring{\lambda}_{2\nu} - \mathring{\xi}_2^\sigma \mathring{\lambda}_{1\nu}) + \kappa \mathring{\theta}^{\nu\sigma} \mathring{\mathcal{F}}_{\mu\sigma}^{\ \ \rho} \mathring{\lambda}_{1\nu} \mathring{\lambda}_{2\rho}, \tag{6.23}$$

where the partial integration was used in the equation (6.22).

The relation (6.18) defines the $\check{C}$-twisted Courant bracket

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]_{\mathcal{C}_{\check{C}}} = \mathring{\Lambda} \Leftrightarrow [(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]_{\mathcal{C}_{\check{C}}} = (\mathring{\xi}, \mathring{\lambda}), \tag{6.24}$$

that gives the same bracket as (5.41). Both (6.19)–(6.20) and (6.22)–(6.23) are the products of $\check{C}$-twisted Courant bracket. The former shows explicitly how the gauge parameters depend on the generalized fluxes. In the latter, similarities between the expressions for two parameters is easier to see.

6.1 Special cases and relations to other brackets

Even though the non-commutativity parameter θ and the Kalb Ramond field B are not mutually independent, while obtaining the bracket (6.24) the relation between these fields (2.7) was not used. Therefore, the results stand even if a bi-vector and a 2-form used for twisting are mutually independent. This will turn out to be convenient to analyze the origin of terms appearing in the Courant bracket twisted by $\check{C}$.

Primarily, consider the case of zero bi-vector $\theta^{\mu\nu} = 0$ with the 2-form $B_{\mu\nu}$ arbitrary. Consequently, the parameter α (4.8) is zero, while the hyperbolic functions $\mathcal{C}$ and $\mathcal{S}$ are identity matrices. Therefore, the auxiliary fields (5.26) and (5.30) simplify in a following way

$$\mathring{B}_{\mu\nu} \to B_{\mu\nu} \quad \mathring{\theta}^{\mu\nu} \to 0, \tag{6.25}$$

and the twisting matrix $e^{\check{B}}$ (4.11) becomes the matrix $e^{\hat{B}}$ (3.28). The expressions (6.19) and (6.20) respectively reduce to

$$\mathring{\xi}^\mu = \mathring{\xi}_1^\nu \partial_\nu \mathring{\xi}_2^\mu - \mathring{\xi}_2^\nu \partial_\nu \mathring{\xi}_1^\mu, \tag{6.26}$$

and

$$\mathring{\lambda}_\mu = \mathring{\xi}_1^\nu (\partial_\nu \mathring{\lambda}_{2\mu} - \partial_\mu \mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu (\partial_\nu \mathring{\lambda}_{1\mu} - \partial_\mu \mathring{\lambda}_{1\nu}) + \frac{1}{2}\partial_\mu (\mathring{\xi}_1 \mathring{\lambda}_2 - \mathring{\xi}_2 \mathring{\lambda}_1) + 2B_{\mu\nu\rho} \mathring{\xi}_1^\nu \mathring{\xi}_2^\rho, \tag{6.27}$$

where $B_{\mu\nu\rho}$ is the Kalb–Ramond field strength, given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \tag{6.28}$$

The equations (6.26) and (6.27) define exactly the B-twisted Courant bracket (3.27) [9].

Secondarily, consider the case of zero 2-form $B_{\mu\nu} = 0$ and the bi-vector $\theta^{\mu\nu}$ arbitrary. Similarly, $\alpha = 0$ and $\mathcal{C}$ and $\mathcal{S}$ are identity matrices. The auxiliary fields $\mathring{B}_{\mu\nu}$ and $\mathring{\theta}^{\mu\nu}$ are given by

$$\mathring{B}_{\mu\nu} \to 0 \quad \mathring{\theta}^{\mu\nu} \to \theta^{\mu\nu}. \tag{6.29}$$

The twisting matrix $e^{\check{B}}$ becomes the matrix of θ-transformations $e^{\hat{\theta}}$ (3.30). The gauge parameters (6.19) and (6.20) are respectively given by

$$\begin{aligned}\mathring{\xi}^\mu &= \mathring{\xi}_1^\nu \partial_\nu \mathring{\xi}_2^\mu - \mathring{\xi}_2^\nu \partial_\nu \mathring{\xi}_1^\mu \\ &+ \kappa\theta^{\mu\rho}\Big(\mathring{\xi}_1^\nu(\partial_\nu \mathring{\lambda}_{2\rho} - \partial_\rho \mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu \mathring{\lambda}_{1\rho} - \partial_\rho \mathring{\lambda}_{1\nu}) \\ &+ \frac{1}{2}\partial_\rho(\mathring{\xi}_1 \mathring{\lambda}_2 - \mathring{\xi}_2 \mathring{\lambda}_1)\Big) \\ &+ \kappa \mathring{\xi}_1^\nu \partial_\nu(\mathring{\lambda}_{2\rho}\theta^{\rho\mu}) - \kappa \mathring{\xi}_2^\nu \partial_\nu(\mathring{\lambda}_{1\rho}\theta^{\rho\mu}) \\ &+ \kappa(\mathring{\lambda}_{1\nu}\theta^{\nu\rho})\partial_\rho \mathring{\xi}_2^\mu - \kappa(\mathring{\lambda}_{2\nu}\theta^{\nu\rho})\partial_\rho \mathring{\xi}_1^\mu \\ &+ \kappa^2 R^{\mu\nu\rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}, \end{aligned} \tag{6.30}$$

and

$$\begin{aligned}\mathring{\lambda}_\mu &= \mathring{\xi}_1^\nu(\partial_\nu \mathring{\lambda}_{2\mu} - \partial_\mu \mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu \mathring{\lambda}_{1\mu} - \partial_\mu \mathring{\lambda}_{1\nu}) \\ &+ \frac{1}{2}\partial_\mu(\mathring{\xi}_1 \mathring{\lambda}_2 - \mathring{\xi}_2 \mathring{\lambda}_1) \\ &+ \kappa\theta^{\nu\rho}(\mathring{\lambda}_{1\nu}\partial_\rho \mathring{\lambda}_{2\mu} - \mathring{\lambda}_{2\nu}\partial_\rho \mathring{\lambda}_{1\mu}) + \kappa \mathring{\lambda}_{1\rho}\mathring{\lambda}_{2\nu} Q_\mu{}^{\rho\nu}, \end{aligned} \tag{6.31}$$

where by $Q_\mu{}^{\nu\rho}$ and $R^{\mu\nu\rho}$ we have marked the non-geometric fluxes, given by

$$Q_\mu{}^{\nu\rho} = \partial_\mu \theta^{\nu\rho}, \;\; R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma \theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma \theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma \theta^{\mu\nu}. \tag{6.32}$$

The bracket defined by these relations is θ-twisted Courant bracket (3.29) [8] and it features the non-geometric fluxes only.

Let us comment on terms in the obtained expressions for gauge parameters (6.22) and (6.23). The first line of (6.22) appears in the Courant bracket and in all brackets that can be obtained from its twisting by either a 2-form or a bi-vector. The next two lines correspond to the terms appearing in the θ-twisted Courant bracket (6.30). The other terms do not appear in either B- or θ-twisted Courant bracket.

Similarly, the first line of (6.20) appears in the Courant bracket (2.18) and in all other brackets obtained from its twisting, while the terms in the second line appear exclusively in the θ twisted Courant bracket (6.27). The first term in the last line appear in the B-twisted Courant bracket (6.31), while the rest are some new terms. We see that all the terms that do not appear in neither of two brackets are the terms containing $\mathring{\mathcal{F}}$ flux.

6.2 Coordinate free notation

In order to obtain the formulation of the $\mathring{C}$-twisted Courant bracket in the coordinate free notation, independent of the local coordinate system that is used on the manifold, let us firstly provide definitions for a couple of well know brackets and derivatives.

The Lie derivative along the vector field ξ is given by

$$\mathcal{L}_{\mathring{\xi}} = i_{\mathring{\xi}} d + d i_{\mathring{\xi}}, \tag{6.33}$$

with $i_{\mathring{\xi}}$ being the interior product along the vector field $\mathring{\xi}$ and d being the exterior derivative. Using the Lie derivative one easily defines the Lie bracket

$$[\mathring{\xi}_1, \mathring{\xi}_2]_L = \mathcal{L}_{\mathring{\xi}_1}\mathring{\xi}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\xi}_1. \tag{6.34}$$

The generalization of the Lie bracket on a space of 1-forms is a well known Koszul bracket [26]

$$[\mathring{\lambda}_1, \mathring{\lambda}_2]_\theta = \mathcal{L}_{\mathring{\theta}\mathring{\lambda}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\theta}\mathring{\lambda}_2}\mathring{\lambda}_1 + d(\mathring{\theta}(\mathring{\lambda}_1, \mathring{\lambda}_2)). \tag{6.35}$$

The expressions (6.19) and (6.20) in the coordinate free notation are given by

$$\begin{aligned}\mathring{\xi} &= [\mathring{\xi}_1, \mathring{\xi}_2]_L - [\mathring{\xi}_2, \mathring{\lambda}_1\kappa\mathring{\theta}]_L + [\mathring{\xi}_1, \mathring{\lambda}_2\kappa\mathring{\theta}]_L \\ &-\Big(\mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1 - \frac{1}{2}d(i_{\mathring{\xi}_1}\mathring{\lambda}_2 - i_{\mathring{\xi}_2}\mathring{\lambda}_1)\Big)\kappa\mathring{\theta} \\ &+\mathring{\mathcal{F}}(\mathring{\xi}_1, \mathring{\xi}_2, .) - \kappa\mathring{\theta}\mathring{\mathcal{F}}(\mathring{\lambda}_1, ., \mathring{\xi}_2) + \kappa\mathring{\theta}\mathring{\mathcal{F}}(\mathring{\lambda}_2, ., \mathring{\xi}_1) \\ &+\mathring{\mathcal{R}}(\mathring{\lambda}_1, \mathring{\lambda}_2, .), \end{aligned} \tag{6.36}$$

and

$$\begin{aligned}\mathring{\lambda} &= \mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1 - \frac{1}{2}d(i_{\mathring{\xi}_1}\mathring{\lambda}_2 - i_{\mathring{\xi}_2}\mathring{\lambda}_1) - [\mathring{\lambda}_1, \mathring{\lambda}_2]_{\kappa\mathring{\theta}} \\ &+ \mathring{H}(\mathring{\xi}_1, \mathring{\xi}_2, .) - \mathring{\mathcal{F}}(\mathring{\lambda}_1, ., \mathring{\xi}_2) + \mathring{\mathcal{F}}(\mathring{\lambda}_2, ., \mathring{\xi}_1) \\ &+ \kappa\mathring{\theta}\mathring{\mathcal{F}}(\mathring{\lambda}_1, \mathring{\lambda}_2, .), \end{aligned} \tag{6.37}$$

where

$$\mathring{H} = 2d\mathring{B}. \tag{6.38}$$

We have marked the geometric H flux as $\mathring{H}$, so that it is distinguished from the 2-form $\mathring{B}$. In the local basis, the full term containing H-flux is given by

$$\mathring{H}(\mathring{\xi}_1, \mathring{\xi}_2, .)\Big|_\mu = 2\mathring{B}_{\mu\nu\rho}\, \mathring{\xi}_1^\nu \mathring{\xi}_2^\rho. \tag{6.39}$$

Similarly are defined the terms containing $\mathring{\mathcal{F}}$ flux

$$\mathring{\mathcal{F}}(\mathring{\xi}_1, \mathring{\xi}_2, .)\Big|^\mu = \mathring{\mathcal{F}}_{\nu\rho}^{\ \mu}\, \mathring{\xi}_1^\nu \mathring{\xi}_2^\rho, \tag{6.40}$$

and the non-geometric $\mathring{\mathcal{R}}$ flux

$$\mathring{\mathcal{R}}(\mathring{\lambda}_1, \mathring{\lambda}_2, .)\Big|^\mu = \mathring{\mathcal{R}}^{\mu\nu\rho}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}, \tag{6.41}$$

as well as

$$\mathring{\theta}\mathring{\mathcal{F}}(\mathring{\lambda}_1, ., \mathring{\xi}_2)\Big|^\mu = \mathring{\theta}^{\nu\sigma}\mathring{\mathcal{F}}_{\sigma\rho}^{\ \mu}\, \mathring{\lambda}_{1\nu}\mathring{\xi}_2^\rho. \tag{6.42}$$

It is possible to rewrite the coordinate free notation in terms of the $\mathring{H}$-flux and $\mathring{\theta}$ bi-vector only. The geometric $\mathring{\mathcal{F}}$ flux is just the contraction of the $\mathring{H}$-flux with a bi-vector

$$\mathring{\mathcal{F}} = \kappa\mathring{\theta}\ \mathring{H}. \tag{6.43}$$

The non-geometric $\mathring{\mathcal{R}}$ flux can be rewritten as

$$\mathring{\mathcal{R}} = \frac{1}{2}[\mathring{\theta},\mathring{\theta}]_S + \wedge^3(\kappa\mathring{\theta})\mathring{H}, \tag{6.44}$$

where $\wedge$ is the wedge product, and by $[\mathring{\theta},\mathring{\theta}]_S$ we have marked the Schouten–Nijenhuis bracket [27], given by

$$[\mathring{\theta},\mathring{\theta}]_S\Big|^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}_{\alpha\beta\gamma}\mathring{\theta}^{\sigma\alpha}\partial_\sigma\mathring{\theta}^{\beta\gamma} = 2\mathring{R}^{\mu\nu\rho}, \tag{6.45}$$

where

$$\epsilon^{\mu\nu\rho}_{\alpha\beta\gamma} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\beta & \delta^\rho_\gamma \\ \delta^\nu_\alpha & \delta^\rho_\beta & \delta^\mu_\gamma \\ \delta^\rho_\alpha & \delta^\mu_\beta & \delta^\nu_\gamma \end{vmatrix}. \tag{6.46}$$

Expressing both $\mathring{\mathcal{F}}$ and $\mathring{\mathcal{R}}$ fluxes in terms of the bi-vector $\mathring{\theta}$ and 3-form $\mathring{H}$, we obtain

$$\begin{aligned}\mathring{\xi} &= [\mathring{\xi}_1,\mathring{\xi}_2]_L - [\mathring{\xi}_2,\mathring{\lambda}_1\kappa\mathring{\theta}]_L + [\mathring{\xi}_1,\mathring{\lambda}_2\kappa\mathring{\theta}]_L \\ &\quad -\Big(\mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1 - \frac{1}{2}d(i_{\mathring{\xi}_1}\mathring{\lambda}_2 - i_{\mathring{\xi}_2}\mathring{\lambda}_1)\Big)\kappa\mathring{\theta} \\ &\quad +\frac{\kappa^2}{2}[\mathring{\theta},\mathring{\theta}]_S(\mathring{\lambda}_1,\mathring{\lambda}_2,.) \\ &\quad +\kappa\mathring{\theta}\mathring{H}(.,\mathring{\xi}_1,\mathring{\xi}_2) - \wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2) \\ &\quad + \wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_2,.,\mathring{\xi}_1) + \wedge^3\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,\mathring{\lambda}_2,.),\end{aligned} \tag{6.47}$$

and

$$\begin{aligned}\mathring{\lambda} &= \mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1 - \frac{1}{2}d(i_{\mathring{\xi}_1}\mathring{\lambda}_2 - i_{\mathring{\xi}_2}\mathring{\lambda}_1) - [\mathring{\lambda}_1,\mathring{\lambda}_2]_{\kappa\mathring{\theta}} \\ &\quad +\mathring{H}(\mathring{\xi}_1,\mathring{\xi}_2,.) - \kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2) \\ &\quad +\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2) + \wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,\mathring{\lambda}_2,.).\end{aligned} \tag{6.48}$$

The term $\kappa\mathring{\theta}\mathring{H}(.,\mathring{\xi}_1,\mathring{\xi}_2)$ is the wedge product of a bi-vector with a 3-form, contracted with two vectors, given by

$$\Big(\kappa\mathring{\theta}\mathring{H}(.,\mathring{\xi}_1,\mathring{\xi}_2)\Big)^\mu = 2\kappa\mathring{\theta}^{\mu\nu}\mathring{B}_{\nu\rho\sigma}\mathring{\xi}_1^\rho\mathring{\xi}_2^\sigma, \tag{6.49}$$

and $\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2)$ is similarly defined, with the 1-form contracted instead of one vector field

$$\Big(\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2)\Big)_\mu = 2\kappa\mathring{\theta}^{\nu\rho}\mathring{B}_{\rho\mu\sigma}\mathring{\lambda}_{1\nu}\mathring{\xi}_2^\sigma. \tag{6.50}$$

The terms like $\wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2)$ are the wedge product of two bi-vectors with a 3-form, contracted with the 1-form $\mathring{\lambda}_1$ and the vector $\mathring{\xi}_2$

$$\Big(\wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,.,\mathring{\xi}_2)\Big)^\mu = 2\kappa^2\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\mu\rho}\ \mathring{B}_{\sigma\rho\tau}\mathring{\lambda}_{1\nu}\mathring{\xi}_2^\tau, \tag{6.51}$$

and similarly when contraction is done with two forms

$$\Big(\wedge^2\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,\mathring{\lambda}_2,.)\Big)_\mu = 2\kappa^2\mathring{\theta}^{\tau\rho}\mathring{\theta}^{\nu\sigma}\ \mathring{B}_{\rho\sigma\mu}\mathring{\lambda}_{1\tau}\mathring{\lambda}_{2\nu}. \tag{6.52}$$

Lastly, the term $\wedge^3\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,\mathring{\lambda}_2,.)$ is obtained by taking a wedge product of three bi-vectors with a 3-form and than contracting it with two 1-forms. It is given by

$$\Big(\wedge^3\kappa\mathring{\theta}\mathring{H}(\mathring{\lambda}_1,\mathring{\lambda}_2,.)\Big)^\mu = 2\kappa^3\mathring{\theta}^{\nu\sigma}\mathring{\theta}^{\rho\tau}\mathring{\theta}^{\mu\lambda}\mathring{B}_{\sigma\tau\lambda}\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}, \tag{6.53}$$

7 Star brackets

The expressions for gauge parameters (6.36) and (6.37) produce some well known bracket, such as Lie bracket and Koszul bracket. The remaining terms can be combined so that they are expressed by some new brackets, acting on pairs of generalized vectors. It turns out that these brackets produce a generalized vector, where the vector part $\mathring{\xi}^\mu$ and the 1-form part $\mathring{\lambda}_\mu$ are related by $\mathring{\xi}^\mu = \kappa\mathring{\theta}^{\mu\nu}\mathring{\lambda}_\nu$, effectively resulting in the graphs in the generalized cotangent bundle $T^\star M$ of the bi-vector $\mathring{\theta}$, i.e. $\xi = \kappa\theta(.,\lambda)$. The star brackets can be interpreted in terms of projections on isotropic subspaces.

7.1 θ-star bracket

Let us firstly consider the second line of (6.22) and the first line of (6.23). When combined, they define a bracket acting on a pair of generalized vectors

$$[\mathring{\Lambda}_1,\mathring{\Lambda}_2]^\star_{\mathring{\theta}} = \mathring{\Lambda}^\star \Leftrightarrow [(\mathring{\xi}_1,\mathring{\lambda}_1),(\mathring{\xi}_2,\mathring{\lambda}_2)]^\star_{\mathring{\theta}} = (\mathring{\xi}_\star,\mathring{\lambda}^\star), \tag{7.1}$$

where

$$\begin{aligned}\mathring{\xi}^\mu_\star &= \kappa\mathring{\theta}^{\mu\rho}\Big(\mathring{\xi}_1^\nu(\partial_\nu\mathring{\lambda}_{2\rho} - \partial_\rho\mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu\mathring{\lambda}_{1\rho} - \partial_\rho\mathring{\lambda}_{1\nu}) \\ &\quad + \frac{1}{2}\partial_\rho(\mathring{\xi}_1\mathring{\lambda}_2 - \mathring{\xi}_2\mathring{\lambda}_1)\Big),\end{aligned} \tag{7.2}$$

and

$$\begin{aligned}\mathring{\lambda}^\star_\mu &= \mathring{\xi}_1^\nu(\partial_\nu\mathring{\lambda}_{2\mu} - \partial_\mu\mathring{\lambda}_{2\nu}) - \mathring{\xi}_2^\nu(\partial_\nu\mathring{\lambda}_{1\mu} - \partial_\mu\mathring{\lambda}_{1\nu}) \\ &\quad +\frac{1}{2}\partial_\mu(\mathring{\xi}_1\mathring{\lambda}_2 - \mathring{\xi}_2\mathring{\lambda}_1),\end{aligned} \tag{7.3}$$

from which one easily reads the relation

$$\mathring{\xi}^\mu_\star = \kappa\mathring{\theta}^{\mu\rho}\mathring{\lambda}^\star_\rho. \tag{7.4}$$

In a coordinate free notation, this bracket can be written as

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]^\star_{\mathring{\theta}} = [(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]^\star_{\mathring{\theta}} = \Big(\kappa\mathring{\theta}\Big(., \mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1\Big), \mathcal{L}_{\mathring{\xi}_1}\mathring{\lambda}_2 - \mathcal{L}_{\mathring{\xi}_2}\mathring{\lambda}_1\Big). \tag{7.5}$$

7.2 $B\theta$-star bracket

The remaining terms contain geometric $\mathring{H}$ and $\mathring{\mathcal{F}}$ fluxes. Note that they are the only terms that depend on the new effective Kalb–Ramond field $\mathring{B}$. Firstly, we mark the last line of (6.23) as

$$\mathring{\lambda}^*_\mu = 2\mathring{B}_{\mu\nu\rho}\mathring{\xi}^\nu_1\mathring{\xi}^\rho_2 + \mathring{\mathcal{F}}_{\mu\sigma}{}^\nu\left(\mathring{\xi}^\sigma_1\mathring{\lambda}_{2\nu} - \mathring{\xi}^\sigma_2\mathring{\lambda}_{1\nu}\right) + \kappa\mathring{\theta}^{\nu\sigma}\mathring{\mathcal{F}}_{\mu\sigma}{}^\rho\mathring{\lambda}_{1\nu}\mathring{\lambda}_{2\rho}. \tag{7.6}$$

Secondly, using the definition of $\mathring{\mathcal{F}}$ (6.3) and the fact that $\mathring{\theta}$ is antisymmetric, the last line of (6.22) can be rewritten as

$$\begin{aligned}\mathring{\xi}^\mu_* &= 2\kappa\mathring{\theta}^{\mu\nu}\mathring{B}_{\nu\rho\sigma}\mathring{\xi}^\rho_1\mathring{\xi}^\sigma_2 + \kappa\mathring{\theta}^{\mu\sigma}\mathring{\mathcal{F}}_{\sigma\rho}{}^\nu\left(\mathring{\xi}^\rho_1\mathring{\lambda}_{2\nu} - \mathring{\xi}^\rho_2\mathring{\lambda}_{1\nu}\right) \\ &\quad + \kappa^2\mathring{\theta}^{\mu\nu}\mathring{\theta}^{\tau\sigma}\mathring{\mathcal{F}}_{\nu\sigma}{}^\rho\mathring{\lambda}_{1\tau}\mathring{\lambda}_{2\rho} \\ &= \kappa\mathring{\theta}^{\mu\nu}\mathring{\lambda}^*_\nu.\end{aligned} \tag{7.7}$$

Now relations (7.6) and (7.7) define the $B\theta$-star bracket by

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]^*_{\mathring{B}\mathring{\theta}} = \mathring{\Lambda}^* \Leftrightarrow [(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]^*_{\mathring{B}\mathring{\theta}} = (\mathring{\xi}_*, \mathring{\lambda}^*), \tag{7.8}$$

We can write the full bracket (6.24) as

$$\begin{aligned}&[(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]_{\mathcal{C}_{\mathring{\mathcal{C}}}} \\ &= \Big([\mathring{\xi}_1, \mathring{\xi}_2]_L - [\mathring{\xi}_2, \mathring{\lambda}_1\kappa\mathring{\theta}]_L + [\mathring{\xi}_1, \mathring{\lambda}_2\kappa\mathring{\theta}]_L \\ &\quad + \frac{\kappa^2}{2}[\mathring{\theta}, \mathring{\theta}]_S(\mathring{\lambda}_1, \mathring{\lambda}_2, .), -[\mathring{\lambda}_1, \mathring{\lambda}_2]_{\kappa\mathring{\theta}}\Big) \\ &\quad + [(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]^*_{\mathring{B},\mathring{\theta}} + [(\mathring{\xi}_1, \mathring{\lambda}_1), (\mathring{\xi}_2, \mathring{\lambda}_2)]^\star_{\mathring{\theta}}.\end{aligned} \tag{7.9}$$

7.3 Isotropic subspaces

In order to give an interpretation to newly obtained starred brackets, it is convenient to consider isotropic subspaces. A subspace L is isotropic if the inner product (2.12) of any two generalized vectors from that sub-bundle is zero

$$\langle \Lambda_1, \Lambda_2\rangle = 0, \quad \Lambda_1, \Lambda_2 \in L. \tag{7.10}$$

From (2.12), one easily finds that

$$\xi^\mu_i = \kappa\,\theta^{\mu\nu}\lambda_{i\nu}. \quad (i=1,2) \quad \theta^{\mu\nu} = -\theta^{\nu\mu}, \tag{7.11}$$

for any bi-vector θ, and

$$\lambda_{i\mu} = 2B_{\mu\nu}\xi^\mu_i. \quad (i=1,2) \quad B_{\mu\nu} = -B_{\nu\mu}, \tag{7.12}$$

for any 2-form B satisfy the condition (7.10).

Furthermore, it is straightforward to introduce projections on these isotropic subspaces by

$$\mathcal{I}^\theta(\Lambda^M) = \mathcal{I}^\theta(\xi^\mu, \lambda_\mu) = (\kappa\,\theta^{\mu\nu}\lambda_\nu, \lambda_\mu), \tag{7.13}$$

and

$$\mathcal{I}_B(\Lambda^M) = \mathcal{I}_B(\xi^\mu, \lambda_\mu) = (\xi^\mu, 2B_{\mu\nu}\xi^\nu). \tag{7.14}$$

Now it is easy to give an interpretation to star brackets. The θ-star bracket (7.1) can be defined as the projection of the Courant bracket (3.29) on the isotropic subspace (7.13)

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]^\star_{\mathring{\theta}} = \mathcal{I}^{\mathring{\theta}}\Big([\mathring{\Lambda}_1, \mathring{\Lambda}_2]_{\mathcal{C}}\Big). \tag{7.15}$$

Similarly, note that all the terms in (6.37) that do not appear in the θ-twisted Courant bracket, contribute exactly to the $B\theta$-star bracket. From that, it is easy to obtain the definition of the $B\theta$-star bracket (7.8)

$$[\mathring{\Lambda}_1, \mathring{\Lambda}_2]^*_{\mathring{B}\mathring{\theta}} = \mathcal{I}^{\mathring{\theta}}\Big([\mathring{\Lambda}_1, \mathring{\Lambda}_2]_{\mathcal{C}_{\mathring{\mathcal{C}}}}\Big) - \mathcal{I}^{\mathring{\theta}}\Big([\mathring{\Lambda}_1, \mathring{\Lambda}_2]_{\mathcal{C}_{\mathring{\theta}}}\Big). \tag{7.16}$$

8 Courant bracket twisted by B and θ

Now it is possible to write down the expression for the Courant bracket twisted by B and θ (4.2), using the expression for $\mathring{C}$-twisted Courant bracket

$$[\breve{\Lambda}_1, \breve{\Lambda}_2]_{\mathcal{C}_{B\theta}} = A^{-1}[A\breve{\Lambda}_1, A\breve{\Lambda}_2]_{\mathcal{C}_{\mathring{\mathcal{C}}}}, \tag{8.1}$$

where A is defined in (5.36). Substituting (8.1) into (6.36), we obtain

$$\begin{aligned}\breve{\xi} &= \mathcal{C}^{-1}[\mathcal{C}\breve{\xi}_1, \mathcal{C}\breve{\xi}_2]_L \\ &\quad - \mathcal{C}^{-1}[\mathcal{C}\breve{\xi}_2, \breve{\lambda}_1\kappa\mathcal{C}^{-1}\mathring{\theta}]_L + \mathcal{C}^{-1}[\mathcal{C}\breve{\xi}_1, \breve{\lambda}_2\kappa\mathcal{C}^{-1}\mathring{\theta}]_L \\ &\quad - \Big(\mathcal{L}_{\mathcal{C}\breve{\xi}_1}(\breve{\lambda}_2\mathcal{C}^{-1}) - \mathcal{L}_{\mathcal{C}\breve{\xi}_2}(\breve{\lambda}_1\mathcal{C}^{-1}) \\ &\quad - \frac{1}{2}d(i_{\breve{\xi}_1}\breve{\lambda}_2 - i_{\breve{\xi}_2}\breve{\lambda}_1)\Big)\kappa\mathring{\theta}\mathcal{C}^{-1} \\ &\quad + \frac{\kappa^2}{2}\mathcal{C}^{-1}[\mathring{\theta}, \mathring{\theta}]_S(\breve{\lambda}_1\mathcal{C}^{-1}, \breve{\lambda}_2\mathcal{C}^{-1}, .) \\ &\quad + \kappa\mathcal{C}^{-1}\mathring{\theta}\mathring{H}(., \mathcal{C}\breve{\xi}_1, \mathcal{C}\breve{\xi}_2) \\ &\quad - \mathcal{C}^{-1}\wedge^2\kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_1\mathcal{C}^{-1}, ., \mathcal{C}\breve{\xi}_2) \\ &\quad + \mathcal{C}^{-1}\wedge^2\kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_2\mathcal{C}^{-1}, ., \mathcal{C}\breve{\xi}_1) \\ &\quad + \mathcal{C}^{-1}\wedge^3\kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_1\mathcal{C}^{-1}, \breve{\lambda}_2\mathcal{C}^{-1}, .),\end{aligned} \tag{8.2}$$

and similarly, substituting (8.1) into (6.37), we obtain

$$\begin{aligned}\breve{\lambda} = &\Big(\mathcal{L}_{\mathcal{C}\breve{\xi}_1}(\breve{\lambda}_2\mathcal{C}^{-1}) - \mathcal{L}_{\mathcal{C}\breve{\xi}_2}(\breve{\lambda}_1\mathcal{C}^{-1}) - \frac{1}{2}d(i_{\breve{\xi}_1}\breve{\lambda}_2 - i_{\breve{\xi}_2}\breve{\lambda}_1)\Big)\mathcal{C}\\ &+\mathring{H}(\mathcal{C}\breve{\xi}_1, \mathcal{C}\breve{\xi}_2, .)\mathcal{C}\\ &-[\breve{\lambda}_1\mathcal{C}^{-1}, \breve{\lambda}_2\mathcal{C}^{-1}]_{\kappa\mathring{\theta}}\mathcal{C} - \kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_1\mathcal{C}^{-1}, ., \mathcal{C}\breve{\xi}_2)\mathcal{C}\\ &+\kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_2\mathcal{C}^{-1}, ., \mathcal{C}\breve{\xi}_1)\mathcal{C}\\ &+\wedge^2\kappa\mathring{\theta}\mathring{H}(\breve{\lambda}_1\mathcal{C}^{-1}, \breve{\lambda}_2\mathcal{C}^{-1}, .)\mathcal{C},\end{aligned} \tag{8.3}$$

where $\mathcal{C}^\mu{}_\nu = \Big(\cosh\sqrt{\alpha}\Big)^\mu{}_\nu$ and $\breve{\Lambda} = (\breve{\xi}, \breve{\lambda})$ (5.21). This is somewhat a cumbersome expression, making it difficult to work with. To simplify it, with the accordance of our convention, we define the twisted Lie bracket by

$$[\breve{\xi}_1, \breve{\xi}_2]_{L_\mathcal{C}} = \mathcal{C}^{-1}[\mathcal{C}\breve{\xi}_1, \mathcal{C}\breve{\xi}_2]_L, \tag{8.4}$$

as well as the twisted Schouten–Nijenhuis bracket

$$\Big([\breve{\theta}, \breve{\theta}]_{S_\mathcal{C}}\Big)^{\mu\nu\rho} = (\mathcal{C}^{-1})^\mu{}_\sigma(\mathcal{C}^{-1})^\nu{}_\lambda(\mathcal{C}^{-1})^\rho{}_\tau\Big([\mathcal{C}\breve{\theta}, \mathcal{C}\breve{\theta}]_S\Big)^{\sigma\lambda\tau}, \tag{8.5}$$

and twisted Koszul bracket

$$[\breve{\lambda}_1, \breve{\lambda}_2]_{\theta_\mathcal{C}} = (\mathcal{C}^T)^{-1}[\mathcal{C}^T\breve{\lambda}_1, \mathcal{C}^T\breve{\lambda}_2]_{\theta\mathcal{C}}, \tag{8.6}$$

where the transpose of the matrix is necessary because the Koszul bracket acts on 1-forms. Now, the first three terms of (8.2) can be written as

$$[\breve{\xi}_1, \breve{\xi}_2]_{L_\mathcal{C}} - [\breve{\xi}_2, \breve{\lambda}_1\kappa\mathcal{C}^{-1}\breve{\theta}]_{L_\mathcal{C}} + [\breve{\xi}_1, \breve{\lambda}_2\kappa\mathcal{C}^{-1}\breve{\theta}]_{L_\mathcal{C}}, \tag{8.7}$$

where

$$\breve{\theta}^{\mu\nu} = (\mathcal{C}^{-1})^\mu{}_\rho\mathring{\theta}^{\rho\nu} = \mathcal{S}^\mu{}_\rho\theta^{\rho\nu}. \tag{8.8}$$

The second line of (8.2) and the first line of (8.3) originating from $\mathring{\theta}$ star bracket (7.1) can be easily combined into

$$[(\mathcal{C}\breve{\xi}_1, \breve{\lambda}_1\mathcal{C}^{-1}), (\mathcal{C}\breve{\xi}_2, \breve{\lambda}_2\mathcal{C}^{-1})]^\star_{\mathcal{C}^{-1}\breve{\theta}}\,\mathcal{C}. \tag{8.9}$$

The terms originating from $\mathring{B}\mathring{\theta}$ star bracket (7.8) are combined into

$$[(\breve{\xi}_1, \breve{\lambda}_1), (\breve{\xi}_2, \breve{\lambda}_2]^*_{\breve{B},\mathcal{C}^{-1}\breve{\theta}}, \tag{8.10}$$

where

$$\begin{aligned}\breve{B}_{\mu\nu\rho} = \mathring{B}_{\alpha\beta\gamma}\mathcal{C}^\alpha{}_\mu\mathcal{C}^\beta{}_\nu\mathcal{C}^\gamma{}_\rho = &\Big(\partial_\alpha(B\mathcal{S}\mathcal{C}^{-1})_{\beta\gamma}\\ &+\partial_\beta(B\mathcal{S}\mathcal{C}^{-1})_{\gamma\alpha} + \partial_\gamma(B\mathcal{S}\mathcal{C}^{-1})_{\alpha\beta}\Big)\mathcal{C}^\alpha{}_\mu\mathcal{C}^\beta{}_\nu\mathcal{C}^\gamma{}_\rho.\end{aligned} \tag{8.11}$$

The expressions for the Courant bracket twisted by both B and θ can be written in a form

$$\begin{aligned}&[(\breve{\xi}_1, \breve{\lambda}_1), (\breve{\xi}_2, \breve{\lambda}_2)]_{\mathcal{C}_{B\theta}}\\ &= \Big([\breve{\xi}_1, \breve{\xi}_2]_{L_\mathcal{C}} - [\breve{\xi}_2, \breve{\lambda}_1\kappa\mathcal{C}^{-1}\breve{\theta}]_{L_\mathcal{C}} + [\breve{\xi}_1, \breve{\lambda}_2\kappa\mathcal{C}^{-1}\breve{\theta}]_{L_\mathcal{C}}\\ &\quad+\frac{\kappa^2}{2}[\breve{\theta}, \breve{\theta}]_{S_\mathcal{C}}(\breve{\lambda}_1, \breve{\lambda}_2, .), -[\breve{\lambda}_1, \breve{\lambda}_2]_{\theta_\mathcal{C}}\Big)\\ &\quad+[(\mathcal{C}\breve{\xi}_1, \breve{\lambda}_1\mathcal{C}^{-1}), (\mathcal{C}\breve{\xi}_2, \breve{\lambda}_2\mathcal{C}^{-1})]^\star_{\mathcal{C}^{-1}\breve{\theta}}\,\mathcal{C}\\ &\quad+[(\breve{\xi}_1, \breve{\lambda}_1), (\breve{\xi}_2, \breve{\lambda}_2]^*_{\breve{B},\mathcal{C}^{-1}\breve{\theta}}.\end{aligned} \tag{8.12}$$

When the Courant bracket is twisted by both B and θ, it results in a bracket similar to $\mathring{C}$-twisted Courant bracket, where Lie brackets, Schouten Nijenhuis bracket and Koszul bracket are all twisted as well.

9 Conclusion

We examined various twists of the Courant bracket, that appear in the Poisson bracket algebra of symmetry generators written in a suitable basis, obtained acting on the double canonical variable (2.4) by the appropriate elements of $O(D, D)$ group. In this paper, we considered the transformations that twists the Courant bracket simultaneously by a 2-form B and a bi-vector θ. When these fields are mutually T-dual, the generator obtained by this transformation is invariant upon self T-duality.

We obtained the matrix elements of this transformation, that we denoted $e^{\breve{B}}$ (4.11), expressed in terms of the hyperbolic functions of a parameter α (4.8). In order to avoid working with such a complicated expression, we considered another $O(D, D)$ transformation A (5.36) and introduced a new generator, written in a basis of auxiliary currents $\mathring{\imath}_\mu$ and $\mathring{k}^\mu$. The Poisson bracket algebra of a new generator was obtained and it gave rise to the $\mathring{C}$-twisted Courant bracket, which contains all of the fluxes.

The generalized fluxes were obtained using different methods [10–12,16–18]. In our approach, we started by an $O(D, D)$ transformation that twists the Courant bracket simultaneously by a 2-form B and bi-vector θ, making it manifestly self T-dual. We obtained the expressions for all fluxes, written in terms of the effective fields

$$\begin{aligned}\mathring{B}_{\mu\nu} &= B_{\mu\rho}\Big(\frac{\tanh\sqrt{2\kappa\theta B}}{\sqrt{2\kappa\theta B}}\Big)^\rho{}_\nu,\\ \mathring{\theta}^{\mu\nu} &= \Big(\frac{\sinh 2\sqrt{2\kappa\theta B}}{2\sqrt{2\kappa\theta B}}\Big)^\mu{}_\sigma\theta^{\sigma\nu}.\end{aligned} \tag{9.1}$$

The fluxes, as a function of these effective fields, appear naturally in the Poisson bracket algebra of such generators.

Similar bracket was obtained in the algebra of generalized currents in [11,12] and is sometimes referred to as the Roytenberg bracket [10]. In that approach, phase space has been changed, so that the momentum algebra gives rise to the H-flux, after which the generalized currents were defined in terms of the open string fields. The bracket obtained this way corresponds to the Courant bracket that was firstly twisted by B field, and then by a bi-vector θ. The matrix of that twist is given by

$$e^R = e^{\hat{\theta}} e^{\hat{B}} = \begin{pmatrix} \delta^\mu_\nu + \alpha^\mu{}_\nu & \kappa\theta^{\mu\nu} \\ 2B_{\mu\nu} & \delta^\nu_\mu \end{pmatrix}. \tag{9.2}$$

In our approach, we obtained the transformations that twists the Courant bracket at the same time by B and θ, resulting in a $\mathring{C}$-twisted Courant bracket. As a consequence, the $\mathring{C}$-twisted Courant bracket is defined in terms of auxiliary fields $\mathring{B}$ (5.26) and $\mathring{\theta}$ (5.30), that are themselves function of α. This is not the case in [11,12]. The Roytenberg bracket calculated therein can be also obtained following our approach by twisting with the matrix

$$e^C = A e^{\breve{B}} = \begin{pmatrix} \mathcal{C}^2 & \kappa(\mathcal{C}\mathcal{S}\theta) \\ 2B\mathcal{C}\mathcal{S} & 1 \end{pmatrix}, \tag{9.3}$$

demanding that the background fields are infinitesimal $B \sim \epsilon$, $\theta \sim \epsilon$ and keeping the terms up to ϵ^2. With these conditions, e^C (9.3) becomes exactly e^R (9.2), and the bracket becomes the Roytenberg bracket.

Analyzing the $\mathring{C}$-twisted Courant bracket, we recognized that certain terms can be seen as new brackets on the space of generalized vectors, that we named star brackets. We demonstrated that they are closely related to projections on isotropic spaces. It is well established that the Courant bracket does not satisfy the Jacobi identity in general case. The sub-bundles on which the Jacobi identity is satisfied are known as Dirac structures, which as a necessary condition need to be subsets of isotropic spaces. Therefore, the star brackets might provide future insights into integrability conditions for the $\mathring{C}$-twisted Courant bracket [28].

In the end, we obtained the Courant bracket twisted at the same time by B and θ by considering the generator in the basis spanned by $\breve{\imath}$ and $\breve{k}$, equivalent to undoing A transformation, used to simplify calculations. With the introduction of new fields $\breve{B}_{\mu\nu}$ and $\breve{\theta}^{\mu\nu}$, this bracket has a similar form as $\mathring{C}$-twisted Courant bracket, whereby the Lie, Schouten–Nijenhuis and Koszul brackets became their twisted counterparts.

It has already been established that B-twisted and θ-twisted Courant brackets appear in the generator algebra defined in bases related by self T-duality [13]. When the Courant bracket is twisted by both B and θ, it is self T-dual, and as such, represent the self T-dual extension of the Lie bracket that includes all fluxes. It has been already shown [8] how the Hamiltonian can be obtained acting with B-transformations on diagonal generalized metric. The same method could be replicated with the twisting matrix $e^{\breve{B}}$, that would give rise to a different Hamiltonian, whose further analysis can provide interesting insights in the role that the Courant bracket twisted by both B and θ plays in understanding T-duality.

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Courant bracket found out to be T-dual to Roytenberg bracket

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Abstract Bosonic string moving in coordinate dependent background fields is considered. We calculate the generalized currents Poisson bracket algebra and find that it gives rise to the Courant bracket, twisted by a 2-form $2B_{\mu\nu}$. Furthermore, we consider the T-dual generalized currents and obtain their Poisson bracket algebra. It gives rise to the Roytenberg bracket, equivalent to the Courant bracket twisted by a bi-vector $\Pi^{\mu\nu}$, in case of $\Pi^{\mu\nu} = 2^\star B^{\mu\nu} = \kappa\theta^{\mu\nu}$. We conclude that the twisted Courant and Roytenberg brackets are T-dual, when the quantities used for their deformations are mutually T-dual.

1 Introduction

Non-geometric backgrounds [1–3] include various dualities. Duality symmetry is a way to show the equivalence between two apparently different theories. Specifically, T-duality [4,5] is a symmetry between two theories corresponding to different geometries and topologies. It was firstly noticed as the spectrum equivalence of the bosonic closed string with one dimension compactified to a radius R, with the bosonic closed string with one dimension compactified to a radius α'/R.

The Courant bracket [6,7] is the generalization of the Lie bracket so that it includes both vectors and 1-forms. It is a fundamental structure of the generalized complex geometry. Vectors and 1-forms are treated on equal footing in the generalized complex structures. Many for string theory relevant geometries, such as complex, symplectic and Kähler geometry, are integrated into the framework of generalized complex structures. Moreover, the generalized complex geometry provides a framework for a unified description of diffeomorphisms and gauge transformations of the Kalb–Ramond field. Hitchin was the first one to introduce the generalized Calabi–Yau manifolds, that unified the concept of a Calabi–Yau manifold with the one of a symplectic manifold [8]. Gualtieri in his PhD thesis contributed further to the mathematical development of generalized complex geometry [9].

In generalized complex geometry, closure under the Courant bracket represents the integrability condition, in a same way that closure under the Lie bracket represents the integrability condition of almost complex structures. Moreover, the Courant bracket governs the gauge transformation in the double field string theory [10].

The Roytenberg bracket is the generalization of the Courant bracket, so that it includes a bi-vector. It was firstly introduced by Roytenberg [11]. In [12], the σ-model with both 2-form and a bi-vector was considered. The Poisson bracket algebra of the generalized currents was obtained. It has been observed that, while the current algebra is anomalous, the algebra of charges is closed and gives rise to the Roytenberg bracket. In [13], the Roytenberg bracket was obtained by lifting the topological sector of the first order action for the NS string σ-model to three dimensions. In [14], the higher order Roytenberg bracket is realized, by twisting by a p-vector.

In this paper, we consider the closed bosonic string moving in the coordinate dependent background fields. Generalized currents are defined as linear combinations of worldsheet basis vectors with arbitrary coordinate dependent coefficients, and their Poisson bracket algebra is calculated. We follow the work of [15], that analyzed the most general currents of the general σ model, where it has been shown that the algebra of most general currents gives rise to the Courant bracket, twisted by the Kalb–Ramond field. Moreover, we consider the self T-duality, that is to say T-duality realized in the same phase space. The self T-duality interchanges momenta with coordinate derivatives, as well as the background fields with their T-dual background fields.

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Another set of generalized currents, T-dual to the aforementioned ones, are constructed and their algebra obtained. We find that their algebra gives rise to the Roytenberg bracket obtained by twisting the Courant bracket by the T-dual of the Kalb–Ramond field. Hence, we show that the twisted Courant bracket is T-dual to the corresponding Roytenberg one, obtaining the relation that connects the mathematically relevant structures with the T-duality.

2 Hamiltonian of the bosonic string

Consider the closed bosonic string in the nontrivial background defined by the symmetric metric tensor field $G_{\mu\nu}$ and the Kalb-Ramond antisymmetric tensor field $B_{\mu\nu}$, as well as the constant dilaton field $\Phi = const$. In the conformal gauge, the propagation is described by the action [16,17]

$$S = \int_\Sigma d^2\xi \mathcal{L} = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu}(x) + \epsilon^{\alpha\beta} B_{\mu\nu}(x)\right] \partial_\alpha x^\mu \partial_\beta x^\nu, \tag{2.1}$$

where integration goes over a two-dimensional world-sheet Σ parametrized by $\xi^\alpha(\xi^0 = \tau, \xi^1 = \sigma)$ with the worldsheet metric $\eta^{\alpha\beta}$. Coordinates of the D-dimensional space-time are $x^\mu(\xi),\ \mu = 0, 1, \dots, D-1, \epsilon^{01} = -1$ and $\kappa = \frac{1}{2\pi\alpha'}$.

It is convenient to rewrite the action (2.1) using the light-cone coordinates $\xi^\pm = \xi^0 \pm \xi^1$ and derivatives $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$ as

$$S = \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu}(x) \partial_- x^\nu, \tag{2.2}$$

where

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x). \tag{2.3}$$

The canonical momenta are given by

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu}(x)\dot{x}^\nu - 2\kappa B_{\mu\nu}(x) x'^\nu. \tag{2.4}$$

The Hamiltonian is obtained in a usual way,

$$\mathcal{H}_C = \pi_\mu \dot{x}^\mu - \mathcal{L} = \frac{1}{2\kappa}\pi_\mu (G^{-1})^{\mu\nu}\pi_\nu - 2x'^\mu B_{\mu\nu}(G^{-1})^{\nu\rho}\pi_\rho + \frac{\kappa}{2} x'^\mu G^E_{\mu\nu} x'^\nu, \tag{2.5}$$

where

$$G^E_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu} \tag{2.6}$$

is the effective metric.

Energy-momentum tensor components can be written as

$$T_\pm = \mp \frac{1}{4\kappa}(G^{-1})^{\mu\nu} j_{\pm\mu} j_{\pm\nu}, \tag{2.7}$$

where the currents $j_{\pm\mu}$ are given by

$$j_{\pm\mu}(x) = \pi_\mu + 2\kappa \Pi_{\pm\mu\nu}(x) x'^\nu. \tag{2.8}$$

In terms of the energy-momentum tensor components (2.7), the Hamiltonian is given by

$$\mathcal{H}_C = T_- - T_+ = \frac{1}{4\kappa}(G^{-1})^{\mu\nu}\left[j_{+\mu} j_{+\nu} + j_{-\mu} j_{-\nu}\right]. \tag{2.9}$$

In this paper, we are interested in these currents, currents T-dual to them, their generalizations, as well as their Poisson bracket algebra. Before that, let us present a short overview of T-duality.

2.1 Lagrangian approach to T-duality

In the Lagrangian approach to T-duality, the Buscher procedure of T-dualization has been developed [18–21]. It provides us with the procedure of transforming coordinates from one theory to the coordinates from its T-dual theory, when there is a global Abelian isometry of coordinates along which T-dualization is applied. The T-dualization rules for coordinates are given by [22,23]

$$\partial_\pm x^\mu \cong -\kappa \Theta^{\mu\nu}_\pm \partial_\pm y_\nu, \quad \partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu, \tag{2.10}$$

where we have introduced the T-dual coordinate y_μ and new fields $\Theta^{\mu\nu}_\pm$, defined by

$$\Theta^{\mu\nu}_\pm = -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}, \tag{2.11}$$

where $\theta^{\mu\nu}$ is the non-commutativity parameter, that first appeared in the context of open string coordinates non-commutativity in the presence of non-zero Kalb Ramond field [24], given by

$$\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1} B G^{-1})^{\mu\nu}, \tag{2.12}$$

where $(G_E^{-1})^{\mu\nu}$ is the inverse of the effective metric defined in (2.6). It is straightforward to verify that $\Theta^{\mu\nu}_\pm$ fields are inverse to $\Pi_{\mp\mu\nu}$ fields

$$\Theta^{\mu\rho}_\pm \Pi_{\mp\rho\nu} = \frac{1}{2\kappa}\delta^\mu_\nu. \tag{2.13}$$

Successive application of the T-dualization (2.10) is involutive

$$\partial_\pm x^\mu \cong -\kappa\Theta_\pm^{\mu\nu}\partial_\pm y_\nu \cong 2\kappa\Theta_\pm^{\mu\nu}\Pi_{\mp\nu\rho}\partial_\pm x^\rho = \partial_\pm x^\mu, \tag{2.14}$$

where in the last step we have used (2.13).

Applying the T-dualization laws (2.10) to the action (2.2), we obtain the T-dual action

$$^\star S = \int d^2\xi\, ^\star\mathcal{L} = \frac{\kappa^2}{2}\int d^2\xi\,\partial_+ y_\mu \Theta_-^{\mu\nu}\partial_- y_\nu. \tag{2.15}$$

Expressing the action (2.15) in the form of the initial action (2.2), we obtain

$$^\star\Pi_+^{\mu\nu} = \frac{\kappa}{2}\Theta_-^{\mu\nu}, \tag{2.16}$$

which allows us to read the T-dual background fields

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad ^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}. \tag{2.17}$$

These relations correspond exactly to the expressions for T-dual background fields obtained by Buscher [18–21] in case of the existence of Abelian group of isometries along coordinates along which we perform T-duality.

2.2 Hamiltonian formulation of T-duality

Let us rewrite the T-dualization laws (2.10) in terms of phase space variables. Firstly, we need the expression for the T-dual canonical momentum. It is given by

$$^\star\pi^\mu = \frac{\partial\, ^\star\mathcal{L}}{\partial \dot{y}_\mu} = \kappa(G_E^{-1})^{\mu\nu}\dot{y}_\nu - \kappa^2\theta^{\mu\nu}y'_\nu. \tag{2.18}$$

Secondly, let us rewrite equations (2.10), separating the part that changes the sign from the part that does not. For the coordinates of the initial theory, we obtain

$$\dot{x}^\mu \cong -\kappa\theta^{\mu\nu}\dot{y}_\nu + (G_E^{-1})^{\mu\nu}y'_\nu, \quad x'^\mu \cong (G_E^{-1})^{\mu\nu}\dot{y}_\nu - \kappa\theta^{\mu\nu}y'_\nu, \tag{2.19}$$

and for the coordinates of the T-dual theory, we obtain

$$\dot{y}_\mu \cong -2B_{\mu\nu}\dot{x}^\nu + G_{\mu\nu}x'^\nu, \quad y'_\mu \cong G_{\mu\nu}\dot{x}^\nu - 2B_{\mu\nu}x'^\nu. \tag{2.20}$$

Comparing the second relation of (2.19) with (2.4), as well as the second relation of (2.20) with (2.18), we obtain the T-dualization laws (2.10) formulated in terms of the phase space variables

$$\kappa x'^\mu \cong {}^\star\pi^\mu, \quad \pi_\mu \cong \kappa y'_\mu. \tag{2.21}$$

When coordinate σ-derivatives and canonical momenta are integrated over the worldsheet space parameter σ, the winding numbers and momenta are respectively obtained [25]. Hence, we see that the T-dualization transforms the momenta of the initial theory into the winding numbers in its T-dual theory, and vice versa.

The T-duality can be considered as the canonical transformation generated by the type I functional [26,27]

$$F = \kappa\int d\sigma\, x^\mu y'_\mu, \tag{2.22}$$

which gives rise to momenta

$$\pi_\mu = \frac{\delta F}{\delta x^\mu} = \kappa y'_\mu, \quad ^\star\pi^\mu = \frac{-\delta F}{\delta y_\mu} = \kappa x'^\mu. \tag{2.23}$$

This is exactly the relation (2.21). The T-duality does not change the Hamiltonian, since the generating function (2.22) does not depend explicitly on time $\mathcal{H}_C \rightarrow \mathcal{H}_C + \frac{\partial F}{\partial t} = \mathcal{H}_C$.

In order to obtain the T-dual Hamiltonian, we apply relations (2.21)–(2.5), and obtain

$$^\star\mathcal{H}_C = \frac{1}{2\kappa}{}^\star\pi^\mu\, G^E_{\mu\nu}\, {}^\star\pi^\nu - 2\, {}^\star\pi^\mu(BG^{-1})_\mu{}^\nu y'_\nu + \frac{\kappa}{2}y'_\mu(G^{-1})^{\mu\nu}y'_\nu. \tag{2.24}$$

Expressing the T-dual Hamiltonian in the form of the initial one (2.5), as

$$^\star\mathcal{H}_C = \frac{1}{2\kappa}{}^\star\pi^\mu\, {}^\star G^{-1}_{\mu\nu}\, {}^\star\pi^\nu - 2y'_\mu({}^\star B\, {}^\star G^{-1})^\mu{}_\nu\, {}^\star\pi^\nu + \frac{\kappa}{2}y'_\mu\, {}^\star G_E^{\mu\nu}y'_\nu. \tag{2.25}$$

We are able to read once again the expressions for the T-dual background fields (2.17).

Given that we were able to write the Hamiltonian in terms of currents $j_{\pm\mu}$, we would like to write the T-dual Hamiltonian (2.25) in terms of T-dual currents. By analogy with the initial theory (2.7), we write the T-dual energy momentum tensor components as

$$^\star T_\pm = \mp\frac{1}{4\kappa}{}^\star G^{-1}_{\mu\nu}\, {}^\star j_\pm^\mu\, {}^\star j_\pm^\nu, \tag{2.26}$$

where ${}^\star j_\pm^\mu$ are T-dual currents, given by

$$^\star j_\pm^\mu = {}^\star\pi^\mu + 2\kappa\, {}^\star\Pi_\pm^{\mu\nu}y'_\nu. \tag{2.27}$$

The T-dual Hamiltonian is then given by

$$^\star\mathcal{H}_C = {}^\star T_- - {}^\star T_+ = \frac{1}{4\kappa}{}^\star G^{-1}_{\mu\nu}\left({}^\star j_+^\mu\, {}^\star j_+^\nu + {}^\star j_-^\mu\, {}^\star j_-^\nu\right), \tag{2.28}$$

We can check that substituting (2.27) into (2.28), the T-dual Hamiltonian in the form (2.25) is obtained. Therefore,

$$\mathcal{H}_C \cong {}^\star\mathcal{H}_C, \quad T_\pm \cong {}^\star T_\pm. \tag{2.29}$$

2.3 T-dual currents

Let us consider the transformation of the currents under T-duality. Applying (2.21)–(2.8), we obtain

$$j_{\pm\mu} \cong \kappa y'_\mu + 2\Pi_{\pm\mu\nu}{}^\star\pi^\nu = 2\Pi_{\pm\mu\nu}{}^\star j_\pm^\nu, \tag{2.30}$$

where we have used (2.13). Similarly, the T-dualization applied on the T-dual currents is as easily obtained

$${}^\star j_\pm^\mu \cong \kappa x'^\mu + \kappa\Theta_\mp^{\mu\nu}\pi_\nu = \kappa\Theta_\mp^{\mu\nu} j_{\pm\nu}. \tag{2.31}$$

The successive application of T-dualization on any current returns exactly that current.

Although the initial and T-dual theories are equivalent (2.29), the currents $j_{\pm\mu}$ and ${}^\star j_\pm^\mu$ do not transform exactly one into another by the T-dualization laws (2.21). There are couple of ways to see the nature of this fact. Firstly, the current $j_{\pm\mu}$ has the lower indices, while ${}^\star j_\pm^\mu$ has the upper indices.

Secondly, substituting (2.30) into (2.7), we obtain the T-dual transformation of the energy momentum tensor

$$T_\pm \cong \pm\frac{1}{\kappa}{}^\star j_\pm^\mu(\Pi_\mp G^{-1}\Pi_\pm)_{\mu\nu}{}^\star j_\pm^\nu$$
$$= \mp\frac{1}{4\kappa}{}^\star j_\pm^\mu G^E_{\mu\nu}{}^\star j_\pm^\nu = {}^\star T_\pm, \tag{2.32}$$

where in the second step we used (2.3) and (2.6). The direct transformation of currents under T-duality $j_{\pm\mu} \cong {}^\star j_\pm^\mu$ would violate invariance of the energy momentum tensor. The effective metric $G^E_{\mu\nu}$ in the expression for T-dual energy momentum tensor is obtained from $-\Pi_\mp G^{-1}\Pi_\pm = \frac{1}{4}G_E$, which is only possible due to the non-trivial T-duality relation between currents (2.30).

Lastly, let us rewrite the expressions for currents in terms of coordinates, by substituting (2.4) into (2.8) and (2.18) into (2.27)

$$j_{\pm\mu} = \kappa G_{\mu\nu}\partial_\pm x^\nu, \quad {}^\star j_\pm^\mu = \kappa(G_E^{-1})^{\mu\nu}\partial_\pm y_\nu. \tag{2.33}$$

Hence, in the same way that coordinates $\partial_\pm x^\mu$ do not transform into T-dual coordinates $\partial_\pm y_\mu$ under (2.10), in the same way the currents $j_{\pm\mu}$ do not transform into T-dual currents ${}^\star j_\pm^\mu$. The transformation of variables under T-duality (2.21) is presented in the Table 1.

Lastly, let us define for future convenience the right hand side of (2.31), as a new current $l_\pm^\mu$

$$l_\pm^\mu = \kappa\Theta_\mp^{\mu\nu} j_{\pm\nu} = \kappa x'^\mu + \kappa\Theta_\mp^{\mu\nu}\pi_\nu. \tag{2.34}$$

Table 1 Transformations under the T-dualization

Initial theory		T-dual theory
π_μ	$\cong$	$\kappa y'_\mu$
$\kappa x'^\mu$	$\cong$	${}^\star\pi^\mu$
$j_{\pm\mu}$	$\cong$	$2\Pi_{\pm\mu\nu}{}^\star j_\pm^\nu$
$\kappa\Theta_\mp^{\mu\nu} j_{\pm\nu}$	$\cong$	${}^\star j_\pm^\mu$

In the next chapter, we will see how we can avoid working in two phase spaces, and the currents $l_\pm^\mu$ will have an important role throughout the rest of the paper.

2.4 Self T-duality

So far we considered the case when two mutually T-dual theories are defined in two different phase spaces, that we have marked by $\{x^\mu, \pi_\mu\}$, and $\{y_\mu, {}^\star\pi^\mu\}$. It is in fact possible to realize T-duality in the same phase space, that we will call self T-duality.

To realize self T-duality, let us rewrite the second relation of (2.19), using (2.17)

$$\kappa x'^\mu \cong \kappa\, {}^\star G^{\mu\nu}\dot{y}_\nu - 2\kappa\, {}^\star B^{\mu\nu} y'_\nu. \tag{2.35}$$

Comparing it with the expression for momenta (2.4), we conclude that the exchange of coordinate with its T-dual $x^\mu \leftrightarrow y_\mu$ is equivalent to

$$\pi_\mu \leftrightarrow \kappa x'^\mu, \quad B_{\mu\nu} \leftrightarrow {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu},$$
$$G_{\mu\nu} \leftrightarrow {}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}. \tag{2.36}$$

These are transformation rules for what we call self T-duality. Note that unlike in (2.21), here the background fields are transformed, as well.

The self T-duality gives the same expressions for T-dual background fields (2.17) as in case of Buscher procedure. It swaps the winding numbers with momenta as well, therefore preserving all features of T-duality, with the only difference being that it is realized in the same phase space.

The two currents $j_{\pm\mu}$ and $l_\pm^\mu$ transform into each other under the self T-duality (2.36)

$$j_{\pm\mu} = \pi_\mu + 2\kappa\Pi_{\pm\mu\nu}x'^\nu \leftrightarrow \kappa x'^\mu + \kappa\Theta_\mp^{\mu\nu}\pi_\nu = l_\pm^\mu. \tag{2.37}$$

On the other hand, under (2.36) the energy-momentum tensor is invariant

$$T_\pm = \mp\frac{1}{4\kappa}(G^{-1})^{\mu\nu} j_{\pm\mu} j_{\pm\nu} \leftrightarrow \mp\frac{1}{4\kappa}G^E_{\mu\nu} l_\pm^\mu l_\pm^\nu = T_\pm. \tag{2.38}$$

With the help of (2.9), we see that the Hamiltonian does not change under (2.36). Nevertheless, the Hamiltonian can be expressed in terms of new currents $l_\pm^\mu$

Table 2 Transformations under the self T-duality

Initial theory		Self T-dual theory
π_μ	$\leftrightarrow$	$\kappa x'_\mu$
$\kappa x'^\mu$	$\leftrightarrow$	π_μ
$B_{\mu\nu}$	$\leftrightarrow$	$\frac{\kappa}{2}\theta^{\mu\nu}$
$G_{\mu\nu}$	$\leftrightarrow$	$(G_E^{-1})^{\mu\nu}$
$j_{\pm\mu}$	$\leftrightarrow$	$l_\pm^\mu$

$$\mathcal{H}_C = \frac{1}{4\kappa} G^E_{\mu\nu}\left(l_+^\mu l_+^\nu + l_-^\mu l_-^\nu\right), \tag{2.39}$$

but with the effective metric instead of the inverse metric. Substituting (2.6) and (2.34) into the previous equation, we obtain the initial form of the Hamiltonian (2.9).

It is important to point out that although the energy-momentum tensor components $T_\pm$ and the Hamiltonian $\mathcal{H}_C$ remain invariant under the self T-duality, the currents $j_{\pm\mu}$ and $l_\pm^\mu$ do not. Therefore although both currents $j_{\pm\mu}$ and $l_\pm^\mu$ are defined in terms of the initial theory variables, they have to change under self T-duality, due to the invariance of energy momentum tensor components (2.38). We summarize its transformation rules in the Table 2. Our next goal is to generalize these two currents and obtain the algebra of their generalizations.

3 Generalized currents in a new basis

In this chapter, we will construct two types of generalized currents. Generalized currents are arbitrary functionals of the fields, parametrized by a pair of vector field and covector field on the target space, treating both vectors and 1-forms on equal footing [9]. The convenient bases in which these generalized currents are defined are components of currents $j_{\pm\mu}$ and $l_\pm^\mu$.

Firstly, we will generalize the currents $j_{\pm\mu}$. From (2.8) we extract its τ and σ components

$$j_{0\mu} = \frac{j_{+\mu}+j_{-\mu}}{2} = \pi_\mu + 2\kappa B_{\mu\nu}(x)x'^\nu,$$
$$j_{1\mu} = \frac{j_{+\mu}-j_{-\mu}}{2} = \kappa G_{\mu\nu}(x)x'^\nu. \tag{3.1}$$

We will mark

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}(x)x'^\nu, \tag{3.2}$$

as a new, auxiliary current. Therefore, $\{\kappa x'^\mu, i_\mu\}$ is a new convenient basis on the world-sheet. We can now write currents (2.8) in this basis as

$$j_{\pm\mu} = i_\mu \pm \kappa G_{\mu\nu}x'^\nu. \tag{3.3}$$

In the same way as in [15], we define the generalized currents in the new basis, as the linear combination of both coordinate $\sigma-$derivatives and auxiliary currents

$$J_{C(u,a)} = u^\mu(x)i_\mu + a_\mu(x)\kappa x'^\mu, \tag{3.4}$$

where $u^\mu(x)$ and $a_\mu(x)$ are the arbitrary coefficients. The charges of these currents are

$$Q_{C(u,a)} = \int d\sigma\, J_{C(u,a)}. \tag{3.5}$$

The charges exhibit additional symmetry. In order to see that, let us firstly rewrite the integral of the total derivative of an arbitrary function λ

$$\int_0^{2\pi} d\sigma(\lambda)' = \int_0^{2\pi} d\sigma\partial_\mu\lambda x'^\mu = 0, \tag{3.6}$$

which goes to zero for closed strings. From this fact, we obtain the reducibility relations for the charges

$$Q_{C(u,a+\partial\lambda)} = Q_{C(u,a)}. \tag{3.7}$$

The expression of the form (3.4) is particularly interesting, since it gives rise to many for string theory relevant structures. Firstly, for the special case of coefficients relation $a_\mu = \pm G_{\mu\nu}u^\nu$, we obtain

$$J_{C(u,\pm Gu)} = u^\mu j_{\pm\mu}. \tag{3.8}$$

Hence, the currents (2.8) indeed can be obtained from the generalized currents (3.4). On the other hand, for special case $a_\mu = -2B_{\mu\nu}u^\nu$, we obtain

$$J_{C(u,-2Bu)} = u^\mu\pi_\mu, \tag{3.9}$$

as well as for $u^\mu = 0$, we obtain

$$J_{C(0,a)} = a_\mu\kappa x'^\nu. \tag{3.10}$$

We see that the general current algebra for the appropriate coefficients reduces to non-commutativity relations of both coordinates and momenta.

We are also interested in another type of generalized current, that in analogous way generalizes $l_\pm^\mu$, in the basis related to its τ and σ components

$$l_0^\mu = \frac{l_+^\mu + l_-^\mu}{2} = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu, \quad l_1^\mu = \frac{l_+^\mu - l_-^\mu}{2} = (G_E^{-1})^{\mu\nu}\pi_\nu. \tag{3.11}$$

The second set of generalized currents is defined by

$$J_{R(v,b)} = v^\mu(x)\pi_\mu + b_\mu(x)k^\mu, \tag{3.12}$$

where $v^\mu(x)$ and $b_\mu(x)$ are the arbitrary coefficients, and we have introduced another auxiliary current by

$$k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu. \tag{3.13}$$

Their charges are

$$Q_{R(v,b)} = \int d\sigma J_{R(v,b)}. \tag{3.14}$$

Similarly as in (3.7), these charges also exhibit additional symmetry. In order to see that, let us write the total derivative integral (3.6), using (3.13), in terms of new basis vectors

$$\int_0^{2\pi} d\sigma\kappa\partial_\mu\lambda x'^\mu = \int_0^{2\pi} d\sigma\partial_\mu\lambda(k^\mu - \kappa\theta^{\mu\nu}\pi_\nu). \tag{3.15}$$

As a result, we obtain the non-uniqueness of the charges

$$Q_{R(v+\kappa\theta\partial\lambda, b+\partial\lambda)} = Q_{R(v,b)}. \tag{3.16}$$

In a special case of $v^\mu = \pm(G_E^{-1})^{\mu\nu}b_\nu$, the generalized current (3.12) reduces to the current (2.30)

$$J_{R(\pm G_E^{-1}b,b)} = b_\mu l_\pm^\mu, \tag{3.17}$$

thus justifying calling it generalized current. Momenta π_μ and auxiliary currents k^μ can also be as easily obtained from it.

The two new bases transform into each other under (2.36):

$$i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu \leftrightarrow \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu = k^\mu, \;\; \pi_\mu \leftrightarrow \kappa x'^\mu. \tag{3.18}$$

Therefore, the generalized currents are defined in the mutually T-dual bases, and their respective algebras are also going to be mutually T-dual.

At the end of this chapter, let us obtain the relations for coefficients when two generalized currents are equal. This will enable us to obtain the algebra of currents $J_{R(v,b)}$, provided that we have the algebra of $J_{C(u,a)}$, and vice versa. Let us start with rewriting the expressions for both generalized currents in the basis $\{\pi_\mu, x'^\mu\}$. Substituting the expression (3.2) into (3.4) we obtain

$$J_{C(u,a)} = u^\mu\pi_\mu + \kappa(a_\mu - 2B_{\mu\nu}u^\nu)x'^\mu, \tag{3.19}$$

while substituting the expression (3.13) into (3.12) we obtain

$$J_{R(v,b)} = (v^\mu - \kappa\theta^{\mu\nu}b_\nu)\pi_\mu + \kappa b_\mu x'^\mu. \tag{3.20}$$

Comparing (3.19)–(3.20), we see that generalized currents are equal when coefficients satisfy following relations

$$u^\mu = v^\mu - \kappa\theta^{\mu\nu}b_\nu,$$
$$a_\mu = 2B_{\mu\nu}v^\nu + (GG_E^{-1})_\mu{}^\nu b_\nu. \tag{3.21}$$

The above relations can be easily inversed. We obtain

$$v^\mu = (G_E^{-1}G)^\mu{}_\nu u^\nu + \kappa\theta^{\mu\nu}a_\nu,$$
$$b_\mu = a_\mu - 2B_{\mu\nu}u^\nu. \tag{3.22}$$

4 Courant bracket

We are interested in calculating the Poisson bracket algebra of the most general currents $J_{C(u,a)}$, defined in (3.4), as well as of their charges $Q_{C(u,a)}$, defined in (3.5). We will start with the generators i_μ and x'^μ algebra, that we calculate using the standard Poisson bracket relations

$$\{x^\mu(\sigma,\tau), \pi_\nu(\bar\sigma,\tau)\} = \delta^\mu_\nu\delta(\sigma-\bar\sigma),$$
$$\{x^\mu(\sigma,\tau), x^\nu(\bar\sigma,\tau)\} = 0,$$
$$\{\pi_\mu(\sigma,\tau), \pi_\nu(\bar\sigma,\tau)\} = 0. \tag{4.1}$$

In the accordance with [15], we will obtain that the algebra of generalized charges (3.5) gives rise to the twisted Courant bracket [6].

We obtain the algebra of generators (3.2)

$$\{i_\mu(\sigma), i_\nu(\bar\sigma)\} = -2\kappa B_{\mu\nu\rho}x'^\rho\delta(\sigma-\bar\sigma), \tag{4.2}$$

where the structural constants are the Kalb–Ramond field strength components, given by

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}. \tag{4.3}$$

The rest of the generators algebra is given by

$$\{i_\mu(\sigma), \kappa x'^\nu(\bar\sigma)\} = \kappa\delta_\mu^\nu\partial_\sigma\delta(\sigma-\bar\sigma), \quad \{\kappa x'^\mu(\sigma), \kappa x'^\nu(\bar\sigma)\} = 0. \tag{4.4}$$

The Poisson bracket of the most general currents (3.4) is obtained using (4.2) and (4.4). It reads

$$\begin{aligned}\{J_{C(u,a)}(\sigma), J_{C(v,b)}(\bar\sigma)\} &= (v^\nu\partial_\nu u^\mu - u^\nu\partial_\nu v^\mu)i_\mu\delta(\sigma-\bar\sigma) - 2\kappa B_{\mu\nu\rho}x'^\mu u^\nu v^\rho\delta(\sigma-\bar\sigma) \\ &- \kappa\left((\partial_\mu a_\nu - \partial_\nu a_\mu)v^\nu - (\partial_\mu b_\nu - \partial_\nu b_\mu)u^\nu\right)x'^\mu\delta(\sigma-\bar\sigma) \\ &+ \kappa\left(u^\mu(\sigma)b_\mu(\sigma) + v^\mu(\bar\sigma)a_\mu(\bar\sigma)\right)\partial_\sigma\delta(\sigma-\bar\sigma).\end{aligned} \tag{4.5}$$

We can modify the anomalous part in the following manner

$$\left(u^\mu(\sigma)b_\mu(\sigma) + v^\mu(\bar\sigma)a_\mu(\bar\sigma)\right)\partial_\sigma\delta(\sigma-\bar\sigma)$$

$$= \frac{1}{2}\left((ub)(\sigma) + (va)(\bar{\sigma})\right)\partial_\sigma\delta(\sigma-\bar{\sigma})$$
$$-\frac{1}{2}(ub)(\sigma)\partial_{\bar{\sigma}}\delta(\sigma-\bar{\sigma}) + \frac{1}{2}(va)(\bar{\sigma})\partial_\sigma\delta(\sigma-\bar{\sigma})$$
$$= \frac{1}{2}\left((ub)(\sigma) + (ub)(\bar{\sigma}) + va(\sigma) + (va)(\bar{\sigma})\right)\partial_\sigma\delta(\sigma-\bar{\sigma})$$
$$+\frac{1}{2}\partial_\mu(va-ub)x'^\mu\delta(\sigma-\bar{\sigma}), \tag{4.6}$$

where we have used the notation $(ub)(\sigma) = u^\mu(\sigma)b_\mu(\sigma)$, and the relation $f(\bar{\sigma})\partial_\sigma\delta(\sigma-\bar{\sigma}) = f'(\sigma)\delta(\sigma-\bar{\sigma}) + f(\sigma)\partial_\sigma\delta(\sigma-\bar{\sigma})$ in the last step. Substituting the previous equation in (4.5) we obtain

$$\{J_{C(u,a)}(\sigma), J_{C(v,b)}(\bar{\sigma})\}$$
$$= -J_{C(\bar{w},\bar{c})}(\sigma)\delta(\sigma-\bar{\sigma})$$
$$+\frac{\kappa}{2}\left((ub)(\sigma) + (ub)(\bar{\sigma}) + (va)(\sigma)\right.$$
$$\left.+(va)(\bar{\sigma})\right)\partial_\sigma\delta(\sigma-\bar{\sigma}), \tag{4.7}$$

where the coefficients in the resulting current are

$$\bar{w}^\mu = u^\nu\partial_\nu v^\mu - v^\nu\partial_\nu u^\mu, \tag{4.8}$$

and

$$\bar{c}_\mu = 2B_{\mu\nu\rho}u^\nu v^\rho + (\partial_\mu a_\nu - \partial_\nu a_\mu)v^\nu - (\partial_\mu b_\nu - \partial_\nu b_\mu)u^\nu$$
$$+\frac{1}{2}\partial_\mu(ub-va). \tag{4.9}$$

The minus sign in front of the $J_{C(\bar{w},\bar{c})}$ is included for the future convenience. We see that $\bar{w}^\mu$ does not depend on background fields, while the coefficient $\bar{c}_\mu$ does, because of the H-flux term $B_{\mu\nu\rho}$.

The relation (4.7) defines the bracket, that acts on a pair of two ordered pairs consisting of a vector and a 1-form, that as a result has another ordered pair, that we can write like

$$[(u,a),(v,b)]_C = (\bar{w},\bar{c}). \tag{4.10}$$

The bracket that we have obtained is the twisted Courant bracket [6]. The Courant bracket represents the generalization of the Lie bracket on spaces that contain both vectors and 1-forms. As a result, it gives an ordered pair of a vector $w = w^\mu\partial_\mu$ and a 1-form $c = c_\mu dx^\mu$.

Let us confirm the equivalence between the twisted Courant bracket and the bracket that we have obtained in (4.10). The coordinate free expression for the twisted Courant bracket is given by

$$[(u,a),(v,b)]_C = \Big([u,v]_L, \mathcal{L}_u b - \mathcal{L}_v a - \frac{1}{2}d(i_u b - i_v a)$$
$$+H(u,v,.)\Big) \equiv (w,c), \tag{4.11}$$

where $[u,v]_L$ is the Lie bracket and $H(u,v,.)$ is a 1-form obtained by contracting a three form. The Lie derivative $\mathcal{L}_u$ is defined in a usual way $\mathcal{L}_u = i_u d + d i_u$, where d is the exterior derivative and i_u the interior derivative. Their action on 1-forms is given by $da = \partial_\mu a_\nu dx^\mu dx^\nu$ and $i_u a = u^\mu a_\mu$.

The Lie bracket is given by

$$[u,v]_L\,|^\mu = u^\nu\partial_\nu v^\mu - v^\nu\partial_\nu u^\mu. \tag{4.12}$$

Using the definition of Lie derivative, we furthermore obtain

$$\left.\left(\mathcal{L}_u b - \mathcal{L}_v a - \frac{1}{2}d(i_u b - i_v a)\right)\right|_\mu$$
$$= u^\nu(\partial_\nu b_\mu - \partial_\mu b_\nu) - v^\nu(\partial_\nu a_\mu - \partial_\mu a_\nu)$$
$$+\frac{1}{2}\partial_\mu(ub-va). \tag{4.13}$$

As for the last term in (4.11), it is given by

$$H(u,v,.)|_\mu = 2B_{\mu\nu\rho}u^\nu v^\rho. \tag{4.14}$$

The expression for the generalized current corresponding to the Courant bracket is obtained by substituting (4.12), (4.13) and (4.14) in (4.11)

$$[(u,a),(v,b)]_C = (w,c), \tag{4.15}$$

where w^μ and c_μ are exactly the same as $\bar{w}^\mu$ and $\bar{c}_\mu$ defined in (4.8) and (4.9), respectively. Therefore, we see that the bracket defined in (4.10) is indeed the twisted Courant bracket.

Besides the current algebra, we are interested in the algebra of charges (3.5). The anomalous term is canceled when integrated. For example, consider the first term in anomaly

$$\int d\sigma d\bar{\sigma}(ub)(\sigma)\partial_\sigma\delta(\sigma-\bar{\sigma})$$
$$= -\int d\bar{\sigma}\partial_{\bar{\sigma}}\int d\sigma(ub)(\sigma)\delta(\sigma-\bar{\sigma})$$
$$= -\int d\bar{\sigma}\partial_{\bar{\sigma}}(ub(\bar{\sigma})) = 0, \tag{4.16}$$

since we are working with the closed strings. The other terms cancel in a similar manner. Integrating the generalized currents (4.7) over σ and $\bar{\sigma}$ we obtain

$$\{Q_{C(u,a)}, Q_{C(v,b)}\} = -Q_{C[(u,a),(v,b)]_C}. \tag{4.17}$$

We see that the algebra of charges is anomaly free. The relation (4.17) was firstly obtained in [15] for the general case of the Hamiltonian formulation of string σ-model, in which momenta and coordinates satisfy the same Poisson bracket relations as auxiliary currents and coordinates in our theory.

Let us check whether the algebra (4.7) is consistent with the known results for the Poisson bracket algebra of the currents $j_{\pm\mu}$ [28]

$$\{j_{\pm\mu}(\sigma), j_{\pm\nu}(\bar{\sigma})\} = \pm 2\kappa\Gamma_{\mu,\nu\rho}x'^\rho\delta(\sigma-\bar{\sigma})$$

$$
\begin{aligned}
&\quad -2\kappa B_{\mu\nu\rho}x'^{\rho}\delta(\sigma-\bar\sigma)\\
&\quad \pm 2\kappa G_{\mu\nu}\delta'(\sigma-\bar\sigma),\\
\{j_{\pm\mu}(\sigma), j_{\mp\nu}(\bar\sigma)\} &= \pm 2\kappa\Gamma_{\mu,\nu\rho}x'^{\rho}\delta(\sigma-\bar\sigma)\\
&\quad -2\kappa B_{\mu\nu\rho}x'^{\rho}\delta(\sigma-\bar\sigma),
\end{aligned}
\tag{4.18}
$$

where $\Gamma_{\mu,\nu\rho} = \frac{1}{2}(\partial_\nu G_{\rho\mu} + \partial_\rho G_{\nu\mu} - \partial_\mu G_{\nu\rho})$ are Christoffel symbols. If we substitute $a_\mu = \pm G_{\mu\nu}u^\nu$ and $b_\mu = \pm G_{\mu\nu}v^\nu$ for constants u^μ and v^μ in (4.7), with the help of (3.8) we obtain

$$
\begin{aligned}
&\{u^\mu j_{\pm\mu}(\sigma), v^\nu j_{\pm\nu}(\bar\sigma)\}\\
&= u^\mu v^\nu\left(-2\kappa B_{\mu\nu\rho} \pm \kappa(\partial_\nu G_{\rho\mu} - \partial_\mu G_{\nu\rho})\right)x'^\rho\delta(\sigma-\bar\sigma)\\
&\quad \pm \kappa u^\mu v^\nu(G_{\mu\nu}(\sigma) + G_{\mu\nu}(\bar\sigma))\partial_\sigma\delta(\sigma-\bar\sigma)\\
&= u^\mu v^\nu\left(-2\kappa B_{\mu\nu\rho} \pm \kappa(\partial_\nu G_{\rho\mu} + \partial_\rho G_{\mu\nu} - \partial_\mu G_{\nu\rho})\right)x'^\rho\delta(\sigma-\bar\sigma)\\
&\quad \pm 2\kappa u^\mu v^\nu G_{\mu\nu}(\sigma)\partial_\sigma\delta(\sigma-\bar\sigma)\\
&= u^\mu v^\nu\{j_{\pm\mu}, j_{\pm\nu}\}.
\end{aligned}
\tag{4.19}
$$

The consistency with the second relation in (4.18) can be as easily obtained.

5 Roytenberg bracket

The Roytenberg bracket appeared as a result of the current algebra firstly in [12], where the author twisted the Poisson structure by trading the 2-form $B_{\mu\nu}$ with the bi-vector $\Pi^{\mu\nu}$. In this paper, we firstly calculate the Poisson bracket algebra for the generalized currents $J_{R(v,b)}$ (3.12), in order to calculate the T-dual Poisson structure of the twisted Courant bracket.

While the currents (3.12) have the same form as the currents giving the Roytenberg bracket in [12], in [12] the momenta are redefined so that they are equal to the auxiliary currents i_μ (3.2) in our paper. As a result of this difference, the currents $J_{R(v,b)}$ and $J_{C(u,a)}$ are related by self T-duality, which is not the case for corresponding currents in [12]. Therefore, we will show that the Courant bracket twisted by a 2-form $2B_{\mu\nu}$ is T-dual to the Roytenberg bracket, obtained by twisting the Courant bracket by a bi-vector $\kappa\theta^{\mu\nu}$. When the fluxes are turned off, both of them reduce to the untwisted Courant bracket, that is T-dual to itself.

We will start with the algebra of auxiliary currents k^μ (3.13). Using (4.1), we obtain

$$
\begin{aligned}
\{k^\mu(\sigma), k^\nu(\bar\sigma)\} = &-\kappa\partial_\rho\theta^{\mu\nu}x'^\rho\delta(\sigma-\bar\sigma) - \kappa^2(\theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho}\\
&-\theta^{\nu\sigma}\partial_\sigma\theta^{\mu\rho})\pi_\rho\delta(\sigma-\bar\sigma),
\end{aligned}
\tag{5.1}
$$

where $\theta^{\mu\nu}$ is the non-commutativity parameter (2.12). From (3.13) we express the coordinate in terms of algebra generators and obtain

$$
\{k^\mu(\sigma), k^\nu(\bar\sigma)\} = -\kappa Q_\rho^{\ \mu\nu}k^\rho\delta(\sigma-\bar\sigma) - \kappa^2 R^{\mu\nu\rho}\pi_\rho\delta(\sigma-\bar\sigma),
\tag{5.2}
$$

where we expressed the structure constants as fluxes

$$
Q_\rho^{\ \mu\nu} = \partial_\rho\theta^{\mu\nu}, \quad R^{\mu\nu\rho} = \theta^{\mu\sigma}\partial_\sigma\theta^{\nu\rho} + \theta^{\nu\sigma}\partial_\sigma\theta^{\rho\mu} + \theta^{\rho\sigma}\partial_\sigma\theta^{\mu\nu}.
\tag{5.3}
$$

These are the non-geometric fluxes [29]. They were firstly obtained by applying the Buscher rules [18–21] on the three-torus with non-trivial Kalb-Ramond field strength (4.3). After the T-duality transformations are applied along two isometry directions, one obtains the space that is locally geometric, but globally non-geometric. The flux for this background is $Q_\rho^{\ \mu\nu}$. After the T-duality transformation is applied along all directions, one obtains the space that is neither locally, nor globally geometric, characterized with the $R^{\mu\nu\rho}$ flux. When considering a generalized T-dualization, the R flux is obtained when performing T-dualization over the arbitrary coordinate on which the background fields depend [30].

The rest of the generators algebra is calculated in a similar way

$$
\begin{aligned}
&\{k^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \kappa\delta^\mu_\nu\partial_\sigma\delta(\sigma-\bar\sigma) + \kappa Q_\nu^{\ \mu\rho}\pi_\rho\delta(\sigma-\bar\sigma),\\
&\{\pi_\mu(\sigma), \pi_\nu(\bar\sigma)\} = 0.
\end{aligned}
\tag{5.4}
$$

We obtain the Poisson bracket of the most general currents $J_{R(u,a)}$, using (5.2) and (5.4). It reads

$$
\begin{aligned}
&\{J_{R(u,a)}(\sigma), J_{R(v,b)}(\bar\sigma)\}\\
&= (v^\nu\partial_\nu u^\mu - u^\nu\partial_\nu v^\mu)\pi_\mu\delta(\sigma-\bar\sigma) - \kappa^2 R^{\mu\nu\rho}\pi_\mu a_\nu b_\rho\delta(\sigma-\bar\sigma)\\
&\quad -\kappa\left(\theta^{\nu\rho}\partial_\rho v^\mu a_\nu - v^\rho\partial_\nu a_\rho\theta^{\nu\mu} - \partial_\nu\theta^{\rho\mu}v^\nu a_\rho\right)\pi_\mu\delta(\sigma-\bar\sigma)\\
&\quad -\kappa\left(u^\rho\partial_\nu b_\rho\theta^{\nu\mu} + \kappa u^\rho\partial_\rho\theta^{\nu\mu}b_\nu - \kappa\theta^{\nu\rho}\partial_\rho u^\mu b_\nu\right)\pi_\mu\delta(\sigma-\bar\sigma)\\
&\quad +\left(u^\nu(\partial_\mu b_\nu - \partial_\nu b_\mu) - v^\nu(\partial_\mu a_\nu - \partial_\nu a_\mu)\right)k^\mu\delta(\sigma-\bar\sigma)\\
&\quad -\kappa\left(a_\rho b_\nu\partial_\mu\theta^{\rho\nu} - \theta^{\nu\rho}(\partial_\rho a_\mu b_\nu - \partial_\rho b_\mu a_\nu)\right)k^\mu\delta(\sigma-\bar\sigma)\\
&\quad +\kappa\left(u^\mu(\sigma)b_\mu(\sigma) + v^\mu(\bar\sigma)a_\mu(\bar\sigma)\right)\partial_\sigma\delta(\sigma-\bar\sigma).
\end{aligned}
\tag{5.5}
$$

Using (4.6) and (3.13) we can transform the anomaly in the following way

$$
\begin{aligned}
&\kappa\left((ub)(\sigma) + (va)(\bar\sigma)\right)\partial_\sigma\delta(\sigma-\bar\sigma)\\
&= \frac{\kappa}{2}((ub)(\sigma) + (ub)(\bar\sigma) + (va)(\sigma) + (va)(\bar\sigma))\partial_\sigma\delta(\sigma-\bar\sigma)\\
&\quad + \frac{1}{2}\partial_\mu(va - ub)(\sigma)(k^\mu - \theta^{\mu\rho}\pi_\rho)\delta(\sigma-\bar\sigma).
\end{aligned}
\tag{5.6}
$$

Substituting the last equation in (5.5), we obtain

$$
\begin{aligned}
&\{J_{R(u,a)}(\sigma), J_{R(v,b)}(\bar\sigma)\}\\
&\quad = -J_{R(\bar w,\bar c)}(\sigma)\delta(\sigma-\bar\sigma)\\
&\qquad + \frac{\kappa}{2}\left((ub)(\sigma) + (ub)(\bar\sigma) + (va)(\sigma)\right.\\
&\qquad \left.+ (va)(\bar\sigma)\right)\partial_\sigma\delta(\sigma-\bar\sigma),
\end{aligned}
\tag{5.7}
$$

where

$$
\begin{aligned}
\bar w^\mu = &\ u^\nu\partial_\nu v^\mu - v^\nu\partial_\nu u^\mu + \kappa\theta^{\nu\rho}\partial_\rho v^\mu a_\nu\\
&- \kappa v^\rho\partial_\nu a_\rho\theta^{\nu\mu} - \kappa Q_\nu^{\ \rho\mu}v^\nu a_\rho
\end{aligned}
$$

$$+\kappa u^\rho \partial_\nu b_\rho \theta^{\nu\mu} + \kappa u^\rho Q_\rho^{\ \nu\mu} b_\nu - \kappa\theta^{\nu\rho}\partial_\rho u^\mu b_\nu - \frac{\kappa}{2}\theta^{\mu\nu}\partial_\nu(va - ub) + \kappa^2 R^{\mu\nu\rho} a_\nu b_\rho, \tag{5.8}$$

and

$$\bar{c}_\mu = v^\nu(\partial_\mu a_\nu - \partial_\nu a_\mu) - u^\nu(\partial_\mu b_\nu - \partial_\nu b_\mu) - \frac{1}{2}\partial_\mu(va - ub) + \kappa a_\rho b_\nu Q_\mu^{\ \rho\nu} - \kappa\theta^{\nu\rho}(\partial_\rho a_\mu b_\nu - \partial_\rho b_\mu a_\nu), \tag{5.9}$$

where we have substituted Q and R fluxes (5.3). Unlike the coefficients in the previous case, here both coefficients depend on backgrounds, due to the presence of fluxes.

As expected, algebra is not closed due to the anomalous part. This Poisson bracket defines a new bracket

$$[(u, a), (v, b)]_R = (\bar{w}, \bar{c}), \tag{5.10}$$

which is equal to the Roytenberg bracket [11]. In case of only R and Q flux present in the generators algebra (5.3), the Roytenberg bracket is given by

$$[(u, a), (v, b)]_R = \Big([u, v]_L - [v, a\Pi]_L + [u, b\Pi]_L + \frac{1}{2}[\Pi, \Pi]_S(a, b, .) - \Big(\mathcal{L}_v a - \mathcal{L}_u b + \frac{1}{2}d(i_u b - i_v a)\Big)\Pi, + \mathcal{L}_u b - \mathcal{L}_v a - \frac{1}{2}d(i_u b - i_v a) - [a, b]_\Pi\Big), \tag{5.11}$$

where $\Pi = \Pi^{\mu\nu}\partial_\mu\partial_\nu$ is the bi-vector. The expression $[\Pi, \Pi]_S(a, b, .)$ represents the Schouten–Nijenhuis bracket [31] contracted with two 1-forms and $[a, b]_\Pi$ is the Koszul bracket [32] given by

$$[a, b]_\Pi = \mathcal{L}_{a\Pi} b - \mathcal{L}_{b\Pi} a + d(\Pi(a, b)). \tag{5.12}$$

The Koszul bracket is a generalization of the Lie bracket on the space of differential forms, while the Schouten-Nijenhuis bracket is a generalization of the Lie bracket on the space of multi-vectors.

The terms in (5.11) that we have not calculated yet can be written, using (4.13), as

$$\Big((\mathcal{L}_v a - \mathcal{L}_u b + \frac{1}{2}d(i_u b - i_v a))\Pi\Big)\Big|^\mu = \Big(u^\nu(\partial_\nu b_\rho - \partial_\rho b_\nu) - v^\nu(\partial_\nu a_\rho - \partial_\rho a_\nu) + \frac{1}{2}\partial_\rho(ub - va)\Big)\Pi^{\rho\mu}. \tag{5.13}$$

The Koszul bracket (5.12) can be further transformed in a following way

$$[a, b]_\Pi|_\mu = \Pi^{\rho\nu}(b_\rho\partial_\nu a_\mu - a_\rho\partial_\nu b_\mu) + \partial_\nu\Pi^{\nu\rho}a_\rho b_\mu, \tag{5.14}$$

while the remaining terms linear in Π become

$$([-v, a\Pi]_L + [u, b\Pi]_L)|^\mu = v^\nu(\partial_\nu a_\rho\Pi^{\mu\rho} + a_\rho\partial_\nu\Pi^{\mu\rho}) + a_\rho\Pi^{\rho\nu}\partial_\nu v^\mu - u^\nu(\partial_\nu b_\rho\Pi^{\mu\rho} + b_\rho\partial_\nu\Pi^{\mu\rho}) - b_\rho\Pi^{\rho\nu}\partial_\nu u^\mu. \tag{5.15}$$

Lastly, we write the expression for the Schouten-Nijenhuis bracket for bi-vectors

$$[\Pi, \Pi]_S|^{\mu\nu\rho} = \epsilon^{\mu\nu\rho}_{\alpha\beta\gamma}\Pi^{\sigma\alpha}\partial_\sigma\Pi^{\beta\gamma}, \tag{5.16}$$

where

$$\epsilon^{\mu\nu\rho}_{\alpha\beta\gamma} = \begin{vmatrix} \delta^\mu_\alpha & \delta^\nu_\beta & \delta^\rho_\gamma \\ \delta^\nu_\alpha & \delta^\rho_\beta & \delta^\mu_\gamma \\ \delta^\rho_\alpha & \delta^\mu_\beta & \delta^\nu_\gamma \end{vmatrix}. \tag{5.17}$$

Thus, we get

$$([\Pi, \Pi]_S(a, b, .))|^\mu = 2R^{\mu\nu\rho}a_\nu b_\rho, \tag{5.18}$$

where $R^{\mu\nu\rho}$ is the flux defined in (5.3).

Combining the previously obtained terms, we obtain the expression for the generalized current corresponding to the Roytenberg bracket twisted by the non-commutativity parameter as a bi-vector

$$[(u, a), (v, b)]_R = (w, c), \tag{5.19}$$

where w^μ and c_μ are equal to $\bar{w}^\mu$ and $\bar{c}_\mu$, defined in (5.8) and (5.9), respectively, provided that $\Pi^{\mu\nu} = \kappa\theta^{\mu\nu}$.

Integrating the previous equation over σ and $\bar{\sigma}$, we see that charges satisfy

$$\{Q_{R(u,a)}, Q_{R(v,b)}\} = -Q_{R[(u,a),(v,b)]_R}. \tag{5.20}$$

The bases in which these generalized currents have been defined are mutually T-dual (2.36). This means that the generalized currents also transform into each other

$$J_{C(u,a)} \leftrightarrow J_{R(v,b)}, \tag{5.21}$$

provided that we swap also coefficients $u^\mu \leftrightarrow b_\mu$, $a_\mu \leftrightarrow v^\mu$. We say that two types of brackets, one obtained by twisting the Courant bracket by a 2-form $2B_{\mu\nu}$, another obtained by twisting the Courant bracket by a bi-vector $\Pi^{\mu\nu}$, are mutually T-dual, as long as the aforementioned 2-form $B_{\mu\nu}$ is T-dual to the bi-vector $\Pi^{\mu\nu}$.

In [33] it has been proposed that T-duality can be understood as the isomorphism φ between two Courant algebroids [7,9]. The relations connecting coefficients of two sets of generalized currents (3.21) can in fact be rewritten as

$$\varphi(u, a) = (u - \kappa\theta a, 2Bu + (G_E^{-1}G)a), \tag{5.22}$$

which can be interpreted as the isomorphism $\varphi(u,a) = (v,b)$ between two Courant algebroids with the trivial bundles over a point and with the twisted Courant and Roytenberg brackets as brackets that act on the Cartesian product of sections of these bundles, as well as the natural inner product $\langle .,. \rangle$ between generalized vectors, given by

$$\langle (u,a),(v,b)\rangle = \frac{1}{2}(ub + va). \tag{5.23}$$

In order for φ to be the isomorphism that corresponds to T-duality, it has to satisfy the following conditions:

$$\begin{aligned} \langle \varphi(u,a),\varphi(v,b)\rangle &= \langle (u,a),(v,b)\rangle,\\ [\varphi(u,a),\varphi(v,b)]_C &= \varphi\left([(u,a),(v,b)]_R\right). \end{aligned} \tag{5.24}$$

To prove that the first condition is satisfied, using (5.22), we obtain

$$\begin{aligned} &\langle \varphi(u,a),\varphi(v,b)\rangle\\ &\quad= \langle (u-\kappa\theta a, 2Bu + (G_E^{-1}G)a), (v-\kappa\theta b, 2Bv\\ &\qquad+(G_E^{-1}G)b)\rangle\\ &\quad= \frac{1}{2}\Big(2B_{\mu\nu}u^\mu v^\nu + 2\kappa(B\theta)_\mu{}^\nu v^\mu a_\nu + (G_E^{-1}G)^\mu{}_\nu b_\mu u^\nu\\ &\qquad-\kappa(G_E^{-1}G)^\mu{}_\nu\theta^{\nu\rho}b_\mu a_\rho\\ &\qquad+2B_{\mu\nu}v^\mu u^\nu + 2\kappa(B\theta)_\mu{}^\nu u^\mu b_\nu + (G_E^{-1}G)^\mu{}_\nu a_\mu v^\nu\\ &\qquad-\kappa(G_E^{-1}G)^\mu{}_\nu\theta^{\nu\rho}a_\mu b_\rho\Big)\\ &\quad= \frac{1}{2}(u^\mu b_\nu + v^\mu a_\nu)\left((G_E^{-1}G)^\nu{}_\mu + 2\kappa(\theta B)^\nu{}_\mu\right)\\ &\quad= \langle (u,a),(v,b)\rangle, \end{aligned} \tag{5.25}$$

where we have used the fact that $B_{\mu\nu}$ and $(G_E^{-1}G\theta)^{\mu\nu}$ are both antisymmetric, as well as

$$(G_E^{-1}G)^\mu{}_\nu + 2\kappa(\theta B)^\mu{}_\nu = \delta^\mu_\nu, \tag{5.26}$$

which is the identity easily obtained from (2.6) and (2.12). As for the second relation of (5.24), it can be shown by writing the relation (4.17) for φ-transformed coefficients

$$\{\mathcal{Q}_{C\varphi(u,a)}, \mathcal{Q}_{C\varphi(v,b)}\} = -\mathcal{Q}_{C[\varphi(u,a),\varphi(v,b)]_C}. \tag{5.27}$$

On the other hand, due to $\mathcal{Q}_{C\varphi(u,a)} = \mathcal{Q}_{R(u,a)}$, the terms on the right-hand sides of (5.27) and (5.20) are equal. By equating them, one obtains

$$\mathcal{Q}_{R[(u,a),(v,b)]_R} = \mathcal{Q}_{C[\varphi(u,a),\varphi(v,b)]_C}. \tag{5.28}$$

Lastly, using (5.22), we write the above relation in the form

$$\mathcal{Q}_{C\varphi([(u,a),(v,b)]_R)} = \mathcal{Q}_{C[\varphi(u,a),\varphi(v,b)]_C}, \tag{5.29}$$

from which the second condition of (5.24) is easily read.

Therefore, we have shown that the relations connecting two types of generalized currents (5.22) define the isomorphism between two Courant algebroids, characterized by twisted Courant and Roytenberg bracket, that according to [33] is interpreted as T-duality.

6 Conclusion

In this paper, we used the T-dualization rules (2.10) for coordinates in the Lagrangian approach, and (2.21) for the canonical variables in the Hamiltonian approach. The relation for T-dual background fields (2.17) stands in both approaches. These relations between the fields provide correct relations between the Courant and Roytenberg bracket.

The T-dualization rules we used, correspond exactly to Buscher's rules obtained in its original procedure [18–21] when there is an Abelian group of isometries of coordinates x^a along which one T-dualizes: $B(x^a) = B(x^a + b^a)$, $G(x^a) = G(x^a + b^a)$. In the Buscher procedure the symmetry is gauged and the new action is obtained. Integrating out the gauge fields from that action, one obtains the T-dual Lagrangian. From that, the T-dual transformation law between the T-dual coordinate σ-derivatives and the canonical momenta of the initial theory can be obtained $\kappa y'_\mu \cong \pi_\mu$. This is exactly the relation (2.21) in our paper.

The most interesting case is when we try to perform the T-dualization along non-isometry directions x^a, such that background fields do depend on them. Then we should apply the generalized Buscher's procedure, developed in [25,34]. In this case, the expression for the T-dual background fields (2.17) remain the same but the argument of the T-dual background fields is not simply the T-dual variable y_a. It is the line integral V that is a function of the world-sheet gauge fields v_+^a and v_-^a, namely $V^a[v_+, v_-] \equiv \int_P d\xi^\alpha v_\alpha^a = \int_P (d\xi^+ v_+^a + d\xi^- v_-^a)$. The expressions for gauge fields can be obtained by varying the Lagrangian with respect to gauge fields and the expression for the argument of background fields has the form $V^a = -\kappa\,\theta^{ab} y_b + G_E^{-1ab}\,\tilde{y}_b$, where $\tilde{y}_a$ is a double of T-dual variable y_a, which satisfy relations $\dot{\tilde{y}}_a = y'_a$ and $\tilde{y}'_a = \dot{y}_a$. Let us point out that in such case the T-dual theory becomes locally non-geometric because the argument of the background fields is the line integral.

For example, in case of the weakly curved background [25] the initial theory is geometric and T-dual theory is non-geometric. In the initial theory the generalized current algebra gives rise to the twisted Courant bracket. However, in the T-dual theory, the presence of double variable $\tilde{y}_a$, makes the calculation of T-dual current algebra much more complicated. It is hard to believe that such a bracket or its corresponding self T-duality version will be equivalent to the

Roytenberg one. Therefore, in case of non-geometric theories, one might expect some new form of brackets.

Next, we introduced the T-duality in the same phase space, that we call self T-duality. It interchanges the momenta and coordinate σ-derivatives, as well as the background fields with the T-dual ones. The Hamiltonian was expressed in terms of currents $j_{\pm\mu}$ and metric tensor $G_{\mu\nu}$, as well as in terms of its T-dual currents $l_\pm^\mu$ and T-dual metric tensor ${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}$. We considered two types of generalized currents, $J_{C(u,a)}$ and $J_{R(v,b)}$, that generalize currents $j_{\pm\mu}$ and $l_\pm^\mu$ respectively. The suitable basis for the current $J_{C(u,a)}$ consists of coordinate σ-derivatives x'^μ and the auxiliary currents $i_\mu = \pi_\mu + 2\kappa B_{\mu\nu}x'^\nu$, and for the current $J_{R(v,b)}$, it consists of momenta π_μ and auxiliary currents $k^\mu = \kappa x'^\mu + \kappa\theta^{\mu\nu}\pi_\nu$. These bases transform into each other under the self T-duality (2.36).

In this paper, we obtained two types of brackets, extracted from the generalized current Poisson bracket algebra. We have shown that one of them is equal to the twisted Courant bracket, while the other equals the Roytenberg bracket. The former can be obtained by twisting the Courant bracket by a 2-form, in our paper $2B_{\mu\nu}$, resulting in the appearance of $H-$flux in generators algebra. The latter bracket can be obtained by twisting the Courant bracket by a bi-vector $\Pi^{\mu\nu}$, resulting in the appearance of $Q-$ and $R-$fluxes, but not $H-$flux, in generators algebra. Since bases in which generalized currents are defined are mutually T-dual, we conclude that the brackets are mutually T-dual, when the bi-vector $\Pi^{\mu\nu}$ equals to the non-commutativity parameter $\kappa\theta^{\mu\nu}$.

We find these results important in itself. Both the Courant and the Roytenberg bracket are well understood mathematical structures. Relation between them and T-duality has a potential to help understand the T-duality better. Moreover, by analyzing characteristics of these brackets we can examine how certain aspects of the mutually T-dual theories relate to each other.

Suppose we turn off all the fluxes. That is equivalent to setting $B_{\mu\nu} = 0$ and $\Pi^{\mu\nu} = 0$, which reduce the auxiliary currents to canonical momentum and coordinate σ derivative: $i_\mu \to \pi_\mu$ and $k^\mu \to \kappa x'^\mu$. The generalized currents now reduce to $J_{C(u,a)} = u^\mu\pi_\mu + a_\mu\kappa x'^\mu$ and $J_{R(v,b)} = v^\mu\pi_\mu + \kappa b_\mu x'^\mu$. It is easy to verify that these currents remain invariant under exchange of momenta and winding numbers, provided that we also change the coefficients in the particular way $J_{C(u,a)} \leftrightarrow \tilde{u}_\mu\kappa x'^\mu + \tilde{a}^\mu\pi_\mu = J_{C(\tilde{a},\tilde{u})}$. Therefore, we conclude that these currents are T-dual to themselves. They give rise to the Courant bracket, the untwisted one, which does not contain any fluxes.

It is interesting that both charges $Q_{C(u,a)}$ and $Q_{R(v,b)}$ can be expressed as the self T-dual symmetry generators in the form

$$\mathcal{G} = \int d\sigma \left[\xi^\mu\pi_\mu + \widetilde{\Lambda}_\mu\kappa x'^\mu\right]. \tag{6.1}$$

It is easy to show that if we define the new gauge parameter $\Lambda_\mu = \widetilde{\Lambda}_\mu + 2B_{\mu\nu}\xi^\nu$, the generators (6.1) are charges $Q_{C(\xi,\Lambda)}$; if we define $\tilde{\xi}^\mu = \xi^\mu + \kappa\theta^{\mu\nu}\widetilde{\Lambda}_\nu$ the generators are charges $Q_{R(\tilde{\xi},\widetilde{\Lambda})}$. Momenta π_μ are generators of general coordinate transformations and x'^μ generators of local gauge transformations $\delta_{\widetilde{\Lambda}}B_{\mu\nu} = \partial_\mu\widetilde{\Lambda}_\nu - \partial_\nu\widetilde{\Lambda}_\mu$, while ξ^μ and $\widetilde{\Lambda}_\mu$ are their corresponding parameters. These generators were studied in [28], where it was shown that general coordinate transformations are T-dual to gauge transformations.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and no experimental data has been listed.]

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Република Србија
Универзитет у Београду
Физички факултет
Д.Бр.2015/8016
Датум: 04.11.2021. године

На основу члана 161 Закона о општем управном поступку и службене евиденције издаје се

УВЕРЕЊЕ

Иванишевић (Зоран) Илија, бр. индекса 2015/8016, рођен 17.07.1991. године, Мостар, Федерација Босне и Херцеговине, Босна и Херцеговина, уписан школске 2021/2022. године, у статусу: самофинансирање; тип студија: докторске академске студије; студијски програм: Физика.

Према Статуту факултета студије трају (број година): три.
Рок за завршетак студија: у двоструком трајању студија.

Ово се уверење може употребити за регулисање војне обавезе, издавање визе, права на дечији додатак, породичне пензије, инвалидског додатка, добијања здравствене књижице, легитимације за повлашћену вожњу и стипендије.

Овлашћено лице факултета

УНИВЕРЗИТЕТ У БЕОГРАДУ
ФИЗИЧКИ ФАКУЛТЕТ
Број 2542014
Београд, 20. 10. 2014. године

На основу члана 161. Закона о општем управном поступку и члана 4. Правилника о садржају и облику образаца јавних исправа које издају више школе, факултети и универзитети, по захтеву, Иванишевић (Зоран) Илије издаје се следеће

У В Е Р Е Њ Е

ИВАНИШЕВИЋ (ЗОРАН) ИЛИЈА рођен- а 17. 07. 1991. године у Мостару, Босна и Херцеговина, уписан-а школске 2010/2011. године на четворогодишње основне академске студије, Студијска група **ФИЗИКА,** смер: **Теоријска и експериментална физика**, положио-ла је испите предвиђене наставним планом и програмом наведене Студијске групе и завршио-ла студије на Физичком факултету 03. oktobra 2014. године, са средњом оценом 9,66 (девет и 66/100) у току студија и постигнутим укупним бројем 240 ЕСПБ (двестачетрдесет ЕСП бодова) и тиме стекао-ла високу стручну спрему и стручни назив

ДИПЛОМИРАНИ ФИЗИЧАР

Уверење се издаје на лични захтев, а служи као доказ о завршеној високој стручној спреми до издавања дипломе.

Уверење је ослобођено плаћања таксе.

Д Е К А Н
ФИЗИЧКОГ ФАКУЛТЕТА

Проф. др Јаблан Дојчиловић

УНИВЕРЗИТЕТ У БЕОГРАДУ ФИЗИЧКИ ФАКУЛТЕТ
UNIVERSITY OF BELGRADE FACULTY OF PHYSICS
Студентски трг 12, 11000 Београд, П.П. 44, Тел: 011-7158-151, Факс: 011-3282-619
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www.ff.bg.ac.rs e-mail: *dekanat@ff.bg.ac.rs*

Број 2472015
Београд, 29.10.2015. године

На основу члана 99. Закона о високом образовању ("Сл. гласник Републике Србије» број 76/05), и члана 9. и 184. Статута Физичког факултета (број 442/1 од 10.10.2006 и дате сагласности Универзитета у Београду број 02 612-1852 од 29.01.2007), у складу са Правилником о садржају и облику образаца јавних исправа које издају високошколске установе ("Сл. гласник Републике Србије» број 21/06, 66/06 и 8/07) издаје се следеће

У В Е Р Е Њ Е

ИВАНИШЕВИЋ (ЗОРАН) ИЛИЈА рођен-а 17. 07. 1991. године у Мостару, Мостар, Република Босна и Херцеговина уписан-а школске 2014/2015. године, завршио-ла је дипломске академске студије – студије другог степена (мастер) на студијском програму Физичког факултета Универзитета у Београду, смер: ТЕОРИЈСКА И ЕКСПЕРИМЕНТАЛНА ФИЗИКА, дана 16. октобра 2015. године, са просечном оценом 9,67 (девет и 67/100) у току студија и постигнутим укупним бројем 60 ЕСПБ (шездесет ЕСП бодова) и тиме стекао-ла високо образовање и академски назив:

МАСТЕР ФИЗИЧАР

Уверење се издаје на лични захтев, а служи као доказ о завршеној стручној спреми до издавања дипломе.

Д Е К А Н

Проф. др Јаблан Дојчиловић

Република Србија
Универзитет у Београду
Физички факултет
Број индекса: 2015/8016
Датум: 24.11.2017.

На основу члана 29. Закона о општем управном поступку и службене евиденције издаје се

УВЕРЕЊЕ О ПОЛОЖЕНИМ ИСПИТИМА

Илија Иванишевић, име једног родитеља Зоран, рођен 17.07.1991.године, Мостар, Република Српска, Босна и Херцеговина, уписан школске 2015/2016. године на докторске академске студије, школске 2017/2018. године уписан на статус финансирање из буџета, студијски програм Физика, током студија положио је испите из следећих предмета:

Р.бр.	Шифра	Назив предмета	Оцена	ЕСПБ	Фонд часова**	Датум
1	ДС15ПЕ4	Квантна теорија градијентних поља	8 (осам)	15	I:(8+0+0)	30.09.2016.
2	ДС15ПЕ8	Суперсиметрије	8 (осам)	15	II:(8+0+0)	01.11.2016.
3	ДС15ФРНД1	Рад на докторату 1. део	П.	30	I:(0+0+12) II:(0+0+12)	
4	ДС15ПЕ9	Теорија струна	10 (десет)	15	III:(8+0+0)	09.10.2017.
5	ДС15ПЕ10	Некомутативна геометрија и примене у физици	10 (десет)	15	IV:(8+0+0)	27.10.2017.
6	ДС15ФРНД2	Рад на докторату 2. део	П.	30	I:(0+0+12) II:(0+0+12)	

* - еквивалентиран/признат испит
** - Фонд часова је у формату (предавања+вежбе+остало)

Општи успех: 9,00 (девет и 00/100) , по годинама студија (8,00, 10,00, /).

Овлашћено лице факултета

ДОКТОРСКЕ СТУДИЈЕ

ПРЕДЛОГ ТЕМЕ ДОКТОРСКЕ ДИСЕРТАЦИЈЕ КОЛЕГИЈУМУ ДОКТОРСКИХ СТУДИЈА

Школска година 20 20 /20 21

Подаци о студенту

Име: Илија

Презиме: Иванишевић

Број индекса: 8016/2015

Научна област дисертације: Квантна поља, честице и гравитација

Подаци о ментору докторске дисертације

Име: Љубица

Презиме: Давидовић

Научна област: Квантна поља, честице и гравитација

Звање: Доктор

Институција: Институт за физику

Предлог теме докторске дисертације

Наслов

Courant algebroids in bosonic string theory
(Courant-ови алгеброиди у бозонској теорији струна)

Уз пријаву теме докторске дисертације Колегијуму докторских студија, потребно је приложити следећа документа:

1. Семинарски рад (дужине до 10 страница)
2. Кратку стручну биографију писану у трећем лицу једнине
3. Фотокопију индекса са докторских студија

Потпис ментора Љубица Давидовић

Датум 30.09.2021

Потпис студента [signature]

Мишљење Колегијума докторских студија

Након образложења теме докторске дисертације Колегијум докторских студија је тему

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