NAUČNOM VEĆU
INSTITUTA ZA FIZIKU U BEOGRADU

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4. Spisak objavljenih radova.
5. Potvrdu o upisanim doktorskim studijama.
6. Diplome završenih osnovnih i master studija.
7. Potvrdu o odbranjenoj temi doktorske disertacije pred Kolegijumom doktorskih studija.
8. Kopije objavljenih radova.

Danijel Obrić

Научном већу Института за физику у Београду

Предмет: *Мишљење руководиоца лабораторије о избору Данијела Обрића у звање истраживач сарадник*

Данијел Обрић , рођен 27.11.1992. године у Бенковцу, је уписао докторске академске студије Физичког факултета Универзитета у Београду у школској 2018/2019. години. Положио је све испите на смеру Квантна поља, честице и гравитација и успешно одбранио предлог теме докторске дисертације под насловом *Т-дуализација бозонске струне и тип II суперструне у присуству координатно зависних позадинских поља* пред Колегијумом докторских студија. Ментор докторске дисертације је др Бојан Николић, виши научни сарадник Института за физику.

Данијел Обрић је од 2018. године запослен у Групи са физику гравитације, честица и поља Института за физику где се бави проблемима везаним за Т-дуализацију (теорија струна). Данијел Обрић је до сада објавио један научни рад категорије М21, као и један научни рад категорије М22. Као што се види из приложеног материјала, он задовољава све предвиђене услове у складу са Правилником о поступку, начину вредновања и квантитативном исказивању научно-истраживачких резултата истраживача Министарства просвете, науке и технолошког развоја Владе Републике Србије, за избор у звање истраживач сарадник, те предлажем Научном већу Института за физику да покрене избор Данијела Обрића у поменуто звање. За чланове комисије предлажем следеће истраживаче:

1. др Бојан Николић, виши научни сарадник, Институт за физику

2. др Љубица Давидовић, виши научни сарадник, Институт за физику

3. проф. др Воја Радовановић, редовни професор Физичког факултета

Београд, 27.12.2021.

др Бранислав Цветковић

Руководилац Групе за физику гравитације, честица и поља

Института за физику

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Radno iskustvo / Zvanja

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Rad na projektima

2019 - 2019:	ON171031 "Fizičke implikacije modifikovanog prostor-vremena"
2022 - :	IDEJE "Quantum Gravity from Higher Gauge Theory"

Konference / Škole

2019:	CERN-SEENET-MTP Balkan School on High Energy and Particle Physics: Theory and Phenomenology

Jezici

Engleski (C1), Srpski (maternji)

Pregled naučne aktivnosti

Danijel Obrić je od aprila 2019. godine zaposlen kao istraživac prirpavnik na Institutu za fiziku u Beogradu u grupi za Gravitaciju, čestice i polja. Bio je aktivan na projektu "Fizičke implikacije modifikovanog prostor-vremena", ON171031, kojim je rukovodila prof. dr Maja Burić. Počev od januara 2022. godine učestvuje na projektu iz programa IDEJE "Quantum Gravity from Higher Gauge Theory" koji je finansiran od strane fonda za nauku Republike Srbije. Do sad ima dva objavljena rada u međunarodnim časopisima SCI liste[1]. Rad [1] je objavljen nakon izbora u zvanje istraživač pripravnik.

Šira naučna oblast kojom se bavi Danijel Obrić je teorijska fizika visokih energija, sa užim fokusom na teoriju struna i njenom povezivanju sa nekomutativnošću i neasocijativnošću koordinata. Teorija struna je značajna zato što predstavlja okvir u kome može da se razmatra problem kvantne gravitacije. Da bi teorija bila matematički konzistentna, kao i da bi opisivala realne fiziķe fenomene, potrenbo je uvesti supersimetriju i proširiti dimenziju prostor-vremena. Ove modifikacije uvode bogatu strukturu u teoriju koja se manifestuje u obliku mreže dualnosti. Sa druge strane, nekomutativnost koordinata je važna jer uvodi prirodnu granicu na energije u kojim teorija važi. Zadatak Danijela Obrića je proučavanje jednog dela strunske mreže dualnosti, naime T-dualnih transformacija, i njenog odnosa sa nekomustatinosti i neasocijativnosti koordinata zatvorene strune. Standardna procedura za postizanje T-dualnosti je Bušerova procedura, procedura je bazirana na lokalizaciji translacione simetrija i radi za slučajeve kad struna propagira u prisustvu konstantnih pozadinskih polja. Proširenjem procedure da obuhvati slučajeve kod kojih imamo i koordinatno zavisna pozadinska polja dobijamo aparat kojim možemo da proučavamo T-dualne teorije gde imamo pojavu nekomutativnosti i neasocijativnosti.

U radovima [1] i [2][2], izvršeno je razmatranje zatvorene bozonske strune u prisustvu koordinatno zavisnog Kalb-Ramondovog polja, gde je pokazano da se u T-dualnoj teoriji javljaju i nekomutativnost i neasocijativnsot koordinata. Dalji rad, planiran je da bude baziran na proučavanju T-dualnosti tip IIB susperstrune u prisustvu koordinatno zavisno Ramond-Ramond polja. Primenom uopštene Bušerove procedure očekujemo da T-dualna teorija poseduje nekomutativne i neasocijativne osobine. Budući da supersimetricne teorije poseduju i fermionske koordinate, takođe je od interesa razmatrati i kakva konfiguracija pozadinskih polja dovodi do fermionske nekomutativnosti.

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[1] pogledati Spisak naučnih radova

[2] *Ibid.*

Spisak objavljenih radova

[1] B. Nikolic and D. Obric, *Directly from H-flux to the family of three nonlocal R-flux theories*, *JHEP* 03 (2019) 136.
Kategorija rada: M21.
IF: 5.875.

[2] B. Nikolic and D. Obric, *Noncommutativity and nonassociativity of closed bosonic string on T-dual toroidal background*, Fortsch. Phys. 66 (2018) 040009.
Kategorija rada: M22.
IF: 3.263.

Република Србија
Универзитет у Београду
Физички факултет
Д.Бр.2018/8013
Датум: 23.12.2021. године

На основу члана 161 Закона о општем управном поступку и службене евиденције издаје се

УВЕРЕЊЕ

Обрић (Милорад) Данијел, бр. индекса 2018/8013, рођен 27.11.1992. године, Бенковац, Република Хрватска, уписан школске 2021/2022. године, у статусу: самофинансирање; тип студија: докторске академске студије; студијски програм: Физика.

Према Статуту факултета студије трају (број година): три.
Рок за завршетак студија: у двоструком трајању студија.

Ово се уверење може употребити за регулисање војне обавезе, издавање визе, права на дечији додатак, породичне пензије, инвалидског додатка, добијања здравствене књижице, легитимације за повлашћену вожњу и стипендије.

Овлашћено лице факултета

Република Србија
Универзитет у Београду
Физички факултет
Д.Бр.2018/8013
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Према Статуту факултета студије трају (број година): три.
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Овлашћено лице факултета

Републикa Србија

Универзитет у Београду
Физички факултет, Београд

УБ

Оснивач: Република Србија
Дозволу за рад број 612-00-02666/2010-04 од 10. децембра 2010.
године је издало Министарство просвете и науке Републике Србије

Диплома

Данијел, Милорад, Обрић

рођен 27. новембра 1992. године у Бенковцу, Република Хрватска, уписан школске 2011/2012. године, а дана 20. септембра 2016. године завршио је основне академске студије, првог степена, на студијском програму Теоријска и експериментална физика, обима 240 (двеста четрдесет) бодова ЕСПБ са просечном оценом 8,16 (осам и 16/100).

На основу тога издаје му се ова диплома о стеченом високом образовању и стручном називу

дипломирани физичар

Број: 6727500
У Београду, 5. јуна 2017. године

Декан
Проф. др Јаблан Дојчиловић

Ректор
Проф. др Владимир Бумбаширевић

00067695

Републикa Србија

УБ

Универзитет у Београду
Физички факултет, Београд

Оснивач: Република Србија
Дозволу за рад број 612-00-02666/2010-04 од 10. децембра 2010. године је издало Министарство просвете и науке Републике Србије

Диплома

Данијел, Милорад, Обрић

рођен 27. новембра 1992. године у Бенковцу, Република Хрватска, уписан школске 2016/2017. године, а дана 27. септембра 2017. године завршио је мастер академске студије, другог степена, на студијском програму Теоријска и експериментална физика, обима 60 (шездесет) бодова ЕСПБ са просечном оценом 10,00 (десет и 0/100).

На основу тога издаје му се ова диплома о стеченом високом образовању и академском називу

мастер физичар

Број: 7295300
У Београду, 25. октобра 2017. године

Декан
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Ректор
Проф. др Владимир Бумбаширевић

00073109

ДОКТОРСКЕ СТУДИЈЕ

ПРЕДЛОГ ТЕМЕ ДОКТОРСКЕ ДИСЕРТАЦИЈЕ КОЛЕГИЈУМУ ДОКТОРСКИХ СТУДИЈА

Школска година 2020/2021

Подаци о студенту

Име	Данијел
Презиме	Обрић
Број индекса	8013/2018

Научна област дисертације: Квантна поља, честице и гравитација

Подаци о ментору докторске дисертације

Име	Бојан
Презиме	Николић
Научна област	Квантна поља, честице и гравитација
Звање	ВИШИ НАУЧНИ САРАДНИК
Институција	ИНСТИТУТ ЗА ФИЗИКУ

Предлог теме докторске дисертације

Наслов

Т-дуализација бозонске струне и тип IIБ суперструне у присуству координатно зависних позадинских поља

Уз пријаву теме докторске дисертације Колегијуму докторских студија, потребно је приложити следећа документа:

1. Семинарски рад (дужине до 10 страница)
2. Кратку стручну биографију писану у трећем лицу једнине
3. Фотокопију индекса са докторских студија

Потпис ментора Бојан Николић

Датум 15.01.2021

Потпис студента [signature]

Мишљење Колегијума докторских студија

Након образложења теме докторске дисертације Колегијум докторских студија је тему

прихватио [x]   није прихватио [ ]

Датум

14.04.2021.

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Directly from H-flux to the family of three nonlocal R-flux theories

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Abstract: In this article we consider T-dualization of the 3D closed bosonic string in the weakly curved background — constant metric and Kalb-Ramond field with one non-zero component, $B_{xy} = Hz$, where field strength H is infinitesimal. We use standard and generalized Buscher T-dualization procedure and make T-dualization starting from coordinate z, via y and finally along x coordinate. All three theories are *nonlocal*, because variable ΔV, defined as line integral, appears as an argument of background fields. After the first T-dualization we obtain commutative and associative theory, while after we T-dualize along y, we get, κ-Minkowski-like, noncommutative and associative theory. At the end of this T-dualization chain we come to the theory which is both noncommutative and nonassociative. The form of the final T-dual action does not depend on the order of T-dualization while noncommutativity and nonassociativity relations could be obtained from those in the $x \to y \to z$ case by replacing $H \to -H$.

Keywords: Bosonic Strings, String Duality

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Contents

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1 Introduction

Noncommutativity of coordinates has come into focus of physics about hundred years ago when the problem with infinite value of physical quantities occurred. The solution was proposed by Heisenberg in the form of noncommutative coordinates. But after developing of renormalization procedure coordinate noncommutativity was forgotten as a tool for cancelling of infinities.

Commuting of coordinates means that there is no minimal possible length in Nature i.e. that we can measure the position of particle with infinite precision. The return of noncommutativity into physics starts with the article of Hartland Snyder [1]. Usually we treat space-time as continuum but Snyder showed that there is Lorentz invariant discrete space-time. Consequently, this means that commutator of coordinates is nonzero, and noncommutativity parameter dictates the scale at which noncommutativity exists.

In the paper [2] existence of noncommutative manifold was shown using propagators in open bosonic string theory with constant metric and constant Kalb-Ramond field. This result is proven in many articles [3–12] after that but using different mathematical methods. Obtained noncommutativity with constant noncommutativity parameter is known in literature as canonical noncommutativity. Consequently, canonical noncommutativity implies that theory is still associative one.

One of the first application of canonical noncommutativity was in Yang-Mills (YM) theories [13–16]. Noncommutative YM theories are constructed and their renormalisability properties are analyzed. It turned out that some processes forbidden in commutative YM are allowed in noncommutative YM theories. Consequently, cross sections for those decays and processes are calculated [17, 18]. Such predictions offer the possibility of indirect check of idea of noncommutativity.

The next type of noncommuatativity which is considered in literature is Lie-algebraic one, which means that commutator of two coordinates is proportional to the coordinate. The κ-Minkowski space-time is an example of this kind of noncommutativity and it is considered in various contexts [19–24]. The κ-Minkowski space is noncommutative but it is easy to check that is associative one. But, in general, if the commutator of the coordinates is proportional to the some linear combination of coordinates, then the space is nonassociative because jacobiator and associator are nonzero. For example, such spaces are closely related to the L_∞ algebra [25].

The mathematical framework for T-dualization is standard Buscher procedure [26, 27]. It consists of the localization of the shift symmetry and adding a term with Lagrange multiplier in order to make gauge fields unphysical degrees of freedom. Also there is an improvement of standard Buscher procedure developed and applied in refs. [28–31], generalized Buscher procedure. In the application of the generalized procedure of T-dualization there is one additional step with respect to the standard one. We introduce invariant coordinate in order to localize shift symmetry in the coordinate dependent backgrounds.

The first articles addressing the subject of coordinate dependent backgrounds appear in the last ten years [32–43]. A 3-torus with constant metric and Kalb-Ramond field with just one nonzero component, $B_{xy} = Hz$, was considered within standard Buscher procedure [33]. Authors made two successive T-dualzation along isometry directions x and y, and, using nontrivial winding conditions, obtained noncommutativity with parameter proportional to field strength H and winding number N_3.

Using generalized T-duality procedure [30, 44] we obtained coordinate dependent noncommutativity and, consequently, nonassociativity. Also it is shown that final theory is nonlocal. In ref. [30] the bosonic string is considered in the weakly curved background — constant metric and linearly coordinate dependent Kalb-Ramond field with infinitesimal field strength, while in [44] we consider the same model as in [33], but T-dualizing along all three directions and imposing trivial winding conditions. Obtained nonlocality comes from the coordinate dependent background, or more precisely, from invariant coordinates. At the end of T-dualization procedure background fields depend on ΔV, defined as line integral. Nonlocality has been become very important issue in the quantum mechanical considerations [45].

In this article we will deal with closed bosonic string propagating in the constant metric and linear dependent Kalb-Ramond field with $B_{xy} = Hz$, the same background as in [33, 44]. But our goal here is to examine the influence of order of T-dualizations. In ref. [44] we T-dualize first along isometry directions, first along x and then along y, and at

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the end, along direction z. The first T-dualization produces configuration known as twisted torus which is commutative, and it is globally and locally well defined. After second T-dualization we obtained nongeometric theory with Q flux which is still locally well defined and it is commutative. The final T-dualization along z direction produces nonlocal theory which is noncommutative and nonassociative one. This line of T-dualizations we will call xyz one.

But what it will happen, if we change the order of T-dualizations, regrading (non)locality issue as well as (non)commutativity and (non)associativity? It is quite obvious that nothing will be changed if we T-dualize along line yxz, because the first two directions, which are T-dualized, are isometry ones. Some nontrivial issues could be expected if we T-dualize first along z direction. In this article we will present T-dualization of the model from [33, 44] along the T-dualization line zyx. After every step of T-dualization we will rewrite the T-dual transformation law in canonical form using the expressions for canonical momenta of the initial theory. Also we will check whether the obtained theory is commutative or not and, consequently, we will see whether it is associative or not.

The fact which is quite sure is that all three theories which we will obtain from the T-dualization line zyx are *nonlocal*. The explanation comes from the fact that background field $B_{\mu\nu}$ is z dependent and according to the generalized T-dualization procedure, after T-dualization along z, we obtain quantity ΔV which is defined as line integral. Consequently, the theory is nonlocal. But because y and x T-dualizations do not affect ΔV, all three theories obtained in zyx T-dualization line are nonlocal. That is a difference with respect to the xyz T-dualization line considered in [44].

The interesting thing is that transformation laws can be obtained from the corresponding ones in [44] by replacing $H \to -H$, but because in this article we T-dualize in the opposite direction, that produces theories of the different commutative and associative features with respect to [44]. After first T-dualization we get commutative and associative theory which is the same as in xyz case from [44]. But the second T-dualization here produces *noncommutative* and associative theory of κ-Minkowski type. That is different with respect to the xyz case, where second theory in the line is both commutative and associative. At the end we obtain the same theory as in [44] which is nonassociative and noncommutative. The noncommutativity and nonassociativity parameters have one additional "$-$" sign comparing with the corresponding ones in [44]. In this article as well as in [44], we impose trivial winding conditions which means $x^\mu(\sigma + 2\pi) = x^\mu(\sigma) + 2\pi N^\mu$, where N^μ is a winding number.

At the end we comment some quantum aspects of the problem and add two appendices. The first one contains conventions regarding light-cone coordinates, while the second one is related to the mathematical details concerning derivation of two kinds of Poisson brackets appearing in the article.

2 Bosonic string action and choice of background fields

In this section we will introduce the action for bosonic string propagating in 3D space with constant metric and Kalb-Ramond field which single component is different from zero,

$B_{xy} = Hz$. This model is well known in literature as torus with H-flux. Since we are working with the same model as in [33, 44], for completeness we will repeat most of the steps from introductory part in the [44].

The closed bosonic string which propagates in the presence of the space-time metric $G_{\mu\nu}(x)$, Kalb-Ramond field $B_{\mu\nu}(x)$, and dilaton field $\Phi(x)$ is described by action [46–48]

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\} , \tag{2.1}$$

where world-sheet surface Σ is parameterized by $\xi^\alpha = (\tau\,,\sigma)$ $[(\alpha = 0\,,1),\ \sigma \in (0\,,\pi)]$, while x^μ $(\mu = 0, 1, 2, \ldots, D-1)$ are space-time coordinates. Intrinsic world sheet metric is denoted by $g_{\alpha\beta}$, and the corresponding scalar curvature with $R^{(2)}$.

Conformal symmetry on the quantum level is not preserved for any choice of background fields. If we want to keep conformal symmetry on the quantum level, background fields must obey the space-time field equations [49]

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2D_\mu a_\nu = 0\,, \tag{2.2}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2a_\rho B^\rho{}_{\mu\nu} = 0\,, \tag{2.3}$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - R - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = c\,, \tag{2.4}$$

where c is an arbitrary constant. From

$$D^\nu \beta^G_{\nu\mu} + \partial_\mu \beta^\Phi = 0\,, \tag{2.5}$$

it follows that third beta function, β^Φ, is equal to an arbitrary constant. Here $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to the space-time metric $G_{\mu\nu}$. Field strength for Kalb-Ramond field $B_{\mu\nu}$ and dilaton gradient are defined as

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\,, \quad a_\mu = \partial_\mu \Phi\,. \tag{2.6}$$

One of the solutions of these equations which is important for us here is the solution where some background fields are coordinate dependent. Let us choose Kalb-Ramond field to be linearly coordinate dependent and dilaton field to be constant. The equation (2.2) turns into

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0\,. \tag{2.7}$$

If we assume that field strength is infinitesimal, then we take $G_{\mu\nu}$ to be constant in approximation linear in $B_{\mu\nu\rho}$. Consequently, the third equation (2.4) is of the form

$$2\pi\kappa \frac{D-26}{6} = c\,. \tag{2.8}$$

The constant c is arbitrary, and fixing its value at $c = -\frac{23\pi\kappa}{3}$, we obtain $D = 3$, dimension of the space in which we will work further.

The choice of background fields in the case we will consider is

$$G_{\mu\nu} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.9}$$

where $R_\mu(\mu = 1, 2, 3)$ are radii of the compact dimensions. In terms of radii, the imposed condition that H is infinitesimal, can be rewritten as

$$\left(\frac{H}{R_1 R_2 R_3}\right)^2 = 0\,. \tag{2.10}$$

Physically, infinitesimality of H means that we work with sufficiently large torus (diluted flux approximation). If we rescale the coordinates

$$x^\mu \longmapsto x'^\mu = R_\mu x^\mu\,, \tag{2.11}$$

where indices on the right hand-side of equation are not summed, the form of the metric simplifies

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

Taking all assumption into consideration, the action is of the form

$$\begin{aligned} S &= \kappa \int_\Sigma d^2\xi \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \\ &= \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\left(\partial_+ x \partial_- x + \partial_+ y \partial_- y + \partial_+ z \partial_- z\right) + \partial_+ x H z \partial_- y - \partial_+ y H z \partial_- x\right], \end{aligned} \tag{2.13}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ is world-sheet derivative with respect to the light-cone coordinates $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$ and

$$x^\mu = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2.14}$$

T-dualization of dilaton is done within quantum formalism and here it will not be presented.

3 Family of three R flux non-local theories

In this section we will perform T-dualization of closed bosonic string equipped by H-flux torus background fields, one direction at time. T-dualization procedure will go along zyx line. We will show that all three theories are nonlocal with R-flux. Also we will find expressions connecting initial and T-dual variables, so called T-dual transformation laws. Using transformation laws in canonical form, we will check after every step whether obtained theory is (non)commutative and/or (non)associative.

3.1 T-dualization along z direction — shortcut to R-flux

Unlike the cases considered in [33, 44], where T-dualization drives along xyz line, let us do that in opposite direction and perform generalized T-dualization [28] of action (2.13) along z direction.

3.1.1 T-dualization procedure

It looks like that this direction is not isometry one. But we can show that it can be treated like isometry direction. Let us consider the global transformation

$$\delta x^{\mu}=\lambda^{\mu}\,, \tag{3.1}$$

and vary the action with respect to this transformation

$$\delta S=\frac{\kappa}{3}B_{\mu\nu\rho}\lambda^{\rho}\int_{\Sigma}d^2\xi\partial_+x^{\mu}\partial_-x^{\nu}=\frac{2k}{3}B_{\mu\nu\rho}\lambda^{\rho}\epsilon^{\alpha\beta}\int_{\Sigma}d^2\xi[\partial_{\alpha}(x^{\mu}\partial_{\beta}x^{\nu})-x^{\mu}(\partial_{\alpha}\partial_{\beta}x^{\nu})]\,. \tag{3.2}$$

The second term vanishes as a consequence of contraction of antisymmetric ($\epsilon^{\alpha\beta}$) and symmetric ($\partial_{\alpha}\partial_{\beta}$) tensors, while the first one, surface term, survives, and it is, in general, different from zero. But, the expression δS is an topological invariant, so it vanishes if the map from the world-sheet to D-dimensional space-time is topologically trivial. Essentially, infinitesimal field strength H does not affect the vanishing of the surface term.

There is one more explanation of vanishing of this surface term. It is more technical and adjusted to the approximation we used in this article which essence is the explanation in paragraph above. Because we work in the approximation up to the linear terms in H, x^{μ} satisfies equation of motion for constant $G_{\mu\nu}$ and $B_{\mu\nu}$, $\partial_+\partial_-x^{\mu}=0$, which solution is well known in literature. If the winding number is equal to zero, it holds $x^{\mu}(2\pi+\sigma)=x^{\mu}(\sigma)$, and since the configuration in the initial τ_i and final moment τ_f is fixed, the surface term vanishes.

So, in the weakly curved background case (H-flux torus background is such like that), z direction is an isometry one. Localization of the shift symmetry of the action (2.13) along z starts with introducing the covariant derivative

$$\partial_{\pm}z\longrightarrow D_{\pm}z=\partial_{\pm}z+v_{\pm}\,, \tag{3.3}$$

where $v_{\pm}$ is a gauge field. In order to make gauge fields unphysical ones, we introduce term with Lagrange multiplier

$$S_{add}=\frac{\kappa}{2}\int_{\Sigma}d^2\xi y_3(\partial_+v_--\partial_-v_+)\,. \tag{3.4}$$

These two steps are the part of the standard Buscher procedure. Because of coordinate dependent background field $B_{\mu\nu}$, generalized T-dualization procedure has an additional step, introducing of an invariant coordinate

$$z^{inv}=\int_P d\xi^{\alpha}D_{\alpha}z=\int_P d\xi^+D_+z+\int_P d\xi^-D_-z=z(\xi)-z(\xi_0)+\Delta V\,, \tag{3.5}$$

where

$$\Delta V=\int_P d\xi^\alpha v_\alpha=\int_P (d\xi^+ v_+ + d\xi^- v_-)\,. \tag{3.6}$$

The form of the action is now

$$\bar S=\kappa\int_\Sigma d^2\xi\left[Hz^{inv}(\partial_+x\partial_-y-\partial_+y\partial_-x)+\frac{1}{2}(\partial_+x\partial_-x+\partial_+y\partial_-y+D_+zD_-z)\right.$$
$$\left.+\,\frac{1}{2}\,y_3(\partial_+v_- - \partial_-v_+)\right]\,. \tag{3.7}$$

Fixing the gauge, $z(\xi)=z(\xi_0)$, we get gauged fixed action in the form

$$S_{fix}=\kappa\int_\Sigma d^2\xi\left[H\Delta V(\partial_+x\partial_-y-\partial_+y\partial_-x)+\frac{1}{2}(\partial_+x\partial_-x+\partial_+y\partial_-y+v_+v_-)\right.$$
$$\left.+\,\frac{1}{2}y_3(\partial_+v_- - \partial_-v_+)\right]\,. \tag{3.8}$$

The equation of motion for Lagrange multiplier y_3 obtained from above action (3.8) produces

$$\partial_+v_- - \partial_-v_+ = 0 \Longrightarrow v_\pm=\partial_\pm z\,, \tag{3.9}$$

which drives us back to the initial action (2.13). On the other side, if we found equations of motion for gauge fields $v_\pm$, we get

$$v_\pm=\pm\partial_\pm y_3-2\beta^\mp\,, \tag{3.10}$$

where $\beta^\pm$ functions are defined as

$$\beta^\pm=\mp\frac{1}{2}H(x\partial_\mp y-y\partial_\mp x)\,. \tag{3.11}$$

The $\beta^\pm$ functions stem from the variation of the term containing ΔV. The derivation of beta functions $\beta^\pm$ is based on the relation $\partial_\pm\Delta V=v_\pm$. In the derivation of the beta functions there is one nontrivial technical point and that is vanishing of the surface term after one partial integration. That surface term is of the same form as in eq. (3.2), so the same reasons for surface term vanishing hold here. Mathematical details regarding derivation of $\beta^\pm$ functions can be found in refs. [28–31, 44].

Inserting the relations (3.10) into the gauge fixed action, keeping linear terms in H, we obtain the T-dual action

$${}_zS=\kappa\int_\Sigma d^2\xi\partial_+{}_zX^\mu{}_z\Pi_{+\mu\nu}\partial_-{}_zX^\nu\,, \tag{3.12}$$

where

$${}_zX^\mu=\begin{pmatrix}x\\ y\\ y_3\end{pmatrix},\qquad {}_z\Pi_{+\mu\nu}={}_zB_{\mu\nu}+\frac{1}{2}{}_zG_{\mu\nu}\,, \tag{3.13}$$

$${}_zB_{\mu\nu}=\begin{pmatrix}0 & H\Delta V & 0\\ -H\Delta V & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad {}_zG_{\mu\nu}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}. \tag{3.14}$$

Let us note that presence of ΔV, defined as line integral, represents the source of nonlocality of the T-dual theory.

3.1.2 T-dual transformation law

Combining the equations of motion for Lagrange multiplier (3.9) and for gauge fields (3.10), we obtain T-dual transformation laws

$$\partial_\pm z \cong \pm\partial_\pm y_3 \mp H(x\partial_\pm y - y\partial_\pm x)\,, \tag{3.15}$$

where $\cong$ is used here to mark T-dual relation. Momentum of the initial theory (2.13) canonically conjugated to the coordinate z is of the form

$$\pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \kappa \dot{z}\,, \tag{3.16}$$

where $\mathcal{L}$ is a Lagrangian density defined as $S = \int_\Sigma d^2\xi \mathcal{L}$. Calculating $\dot{z}$ using T-dual transformation law (3.15), we get the T-dual transformation law in canonical form

$$y_3' \cong \frac{1}{\kappa}\pi_z + H(xy' - yx')\,, \tag{3.17}$$

which is of the same form as in the xyz case.

In all further expressions we will keep the symbol ΔV, but we must have in mind that we used equations of motion for Lagrange multipliers (3.9) at the end of T-dulization procedure along z coordinate, so, having in mind (3.6) and (3.15), we get

$$\Delta V = \Delta z \cong \int d\xi^+ \partial_+ y_3 - \int d\xi^- \partial_- y_3 \equiv \tilde{y}_3\,. \tag{3.18}$$

The variable ΔV is multiplied by infinitesimal field strength H, so, in the above expression we used $\partial_\pm z \cong \pm\partial_\pm y_3$, as a consequence of diluted flux approximation.

3.1.3 (Non)commutativity and (non)associativity

The initial theory is geometric one and its variables satisfy the standard Poisson algebra

$$\{x^\mu(\sigma), x^\nu(\bar\sigma)\} = \{\pi_\mu(\sigma), \pi_\nu(\bar\sigma)\} = 0\,, \quad \{x^\mu, \pi_\nu(\bar\sigma)\} = \delta^\mu{}_\nu \delta(\sigma - \bar\sigma)\,, \tag{3.19}$$

where x^μ are the coordinates of the initial theory, while π_μ are their canonically conjugated momenta. Using expression (3.17) and standard Poisson algebra (3.19), we obtain that coordinates of the theory obtained after one T-dualization, ${}_zX^\mu$, are commutative. Consequently, Jacobiator is equal to zero, which means that theory is associative.

Summarizing this first step of T-dualization, obtained theory is *commutative and associative nonlocal R-flux theory*. Comparing with the results of the ref. [44] after first T-dualization, qualitatively we obtain the same result, but with the essential difference that here obtained theory is nonlocal R-flux theory unlike that in [44] which is geometrical one, locally and globally well defined.

3.2 Step 2 — T-dualization along y direction

Our starting point is the action given in eq. (3.12). The background fields are independent of y, so, we apply standard Buscher procedure. This means that, unlike the previous case, we perform just first two steps in T-dualization procedure and skip the third one — introducing of invariant coordinate. The T-dualization procedure is already presented, so, we will skip explaining procedure steps further.

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3.2.1 T-dualization procedure

The gauge fixed action is of the form

$$S_{fix} = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2} \left(\partial_+ x \partial_- x + v_+ v_- + \partial_+ y_3 \partial_- y_3 \right) + H\Delta V (v_- \partial_+ x - v_+ \partial_- x) \right] + \frac{\kappa}{2} \int_\Sigma d^2\xi y_2 (\partial_+ v_- - \partial_- v_+) \,. \tag{3.20}$$

Varying with respect to the Lagrange multiplier y_2 we get

$$v_\pm = \partial_\pm y \,, \tag{3.21}$$

while the equations of motion for gauge fields are

$$v_\pm = \pm \partial_\pm y_2 \mp 2H\Delta V \partial_\pm x \,. \tag{3.22}$$

Inserting the expression for gauge fields (3.22) into gauge fixed action (3.20), we obtain the T-dual action

$${}_{zy}S = \kappa \int_\Sigma d^2\xi \; \partial_+ \, {}_{zy}X^\mu \; {}_{zy}\Pi_{+\mu\nu} \; \partial_- \, {}_{zy}X^\nu \,, \tag{3.23}$$

where

$${}_{zy}X^\mu = \begin{pmatrix} x \\ y_2 \\ y_3 \end{pmatrix} , \qquad {}_{zy}\Pi_{+\mu\nu} = {}_{zy}B_{\mu\nu} + \frac{1}{2} \, {}_{zy}G_{\mu\nu} \,, \tag{3.24}$$

$${}_{zy}B_{\mu\nu} = 0 \,, \qquad {}_{zy}G_{\mu\nu} = \begin{pmatrix} 1 & -2H\Delta V & 0 \\ -2H\Delta V & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.25}$$

Let us note that after two T-dualizations in the xyz case in [44] we also obtained that T-dual Kalb-Ramond field is zero.

3.2.2 T-dual transformation law

Combining equations of motion (3.21) and (3.22) we get the corresponding transformation law

$$\partial_\pm y \cong \pm \partial_\pm y_2 \mp 2H\Delta V \partial_\pm x \,. \tag{3.26}$$

Let us now prescribe the transformation law in canonical form. The momentum canonically conjugated to the initial coordinate y is obtained by variation of the initial action (2.13) with respect to the $\dot{y}$ and it is of the form

$$\pi_y = \kappa (\dot{y} + 2Hzx') \,, \tag{3.27}$$

while from transformation law (3.26) we have

$$\dot{y} \cong y_2' - 2H\Delta V x' \,. \tag{3.28}$$

Combining last two equations and using the fact that, in the approximation linear in H, ΔV and z are T-dual to each other, we get

$$y_2' \cong \frac{1}{\kappa} \pi_y \,. \tag{3.29}$$

As we see the transformation law is the same as in the xyz case.

3.2.3 (Non)commutativity and (non)associativity

In this paragraph we will calculate Poisson brackets of the coordinates ${}_{zy}X^\mu$ using transformation laws in canonical form given by eqs. (3.17) and (3.29).

With the help of the standard Poisson algebra (3.19) and instructions from appendix B, it is easy to see that

$$\{x(\sigma), x(\bar\sigma)\} = \{y_2(\sigma), y_2(\bar\sigma)\} = \{y_3(\sigma), y_3(\bar\sigma)\} = \{x(\sigma), y_2(\bar\sigma)\} = \{x(\sigma), y_3(\bar\sigma)\} = 0\,. \tag{3.30}$$

The only non-zero Poisson bracket is

$$\{y_2'(\sigma), y_3'(\bar\sigma)\} \cong \frac{H}{\kappa}\left[2x'(\sigma)\delta(\sigma-\bar\sigma) + x(\sigma)\delta'(\sigma-\bar\sigma)\right]\,, \tag{3.31}$$

where $\delta' \equiv \partial_\sigma \delta(\sigma-\bar\sigma)$. This result is obtained by straightforward calculation using T-dual transformation laws, (3.17) and (3.29), and standard Poisson algebra (3.19). The relation (3.31) is of the form (B.1), where $A'(\sigma) = y_2'(\sigma)$, $B'(\bar\sigma) = y_3'(\bar\sigma)$, $U'(\sigma) = \frac{H}{\kappa}2x'(\sigma)$ and $V(\sigma) = \frac{H}{\kappa}x(\sigma)$. With these substitutions in mind, we have that final expression is of the form (B.8)

$$\{y_2(\sigma), y_3(\bar\sigma)\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar\sigma)\right]\theta(\sigma-\bar\sigma)\,. \tag{3.32}$$

For $\sigma \to \sigma + 2\pi$ and $\bar\sigma \to \sigma$ we have

$$\{y_2(\sigma+2\pi), y_3(\sigma)\} \cong -\frac{H}{\kappa}\left[x(\sigma) + 4\pi N_x\right]\,, \tag{3.33}$$

because $\theta(2\pi) = 1$ (B.6), while N_x is winding number for x coordinate

$$x(\sigma+2\pi) - x(\sigma) = 2\pi N_x\,. \tag{3.34}$$

As we can see the noncommutativity relation (3.32) is of κ-Minkowski type. It is straightforward to see that

$$\{x(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\} + \{y_2(\sigma_2), \{y_3(\sigma_3), x(\sigma_1)\}\} + \{y_3(\sigma_3), \{x(\sigma_1), y_2(\sigma_2)\}\} \cong 0\,. \tag{3.35}$$

Because the Jacobiator is zero, we conclude that this R-flux theory is **noncommutative** and **associative** one.

3.3 Step 3 — T-dualization along x direction

In this subsection we will finish T-dualization procedure not repeating the mathematical details, but giving just the important equations and results.

The gauge fixed action is given by the following equation

$$S_{fix} = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}\left(v_+v_- + \partial_+y_2\partial_-y_2 + \partial_+y_3\partial_-y_3\right) - H\Delta V\left(v_+\partial_-y_2 + \partial_+y_2\, v_-\right)\right]$$
$$+ \frac{\kappa}{2}\int_\Sigma d^2\xi y_1(\partial_+v_- - \partial_-v_+)\,. \tag{3.36}$$

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The equations of motion for Lagrange multiplier produces

$$v_\pm = \partial_\pm x\,, \tag{3.37}$$

while the equations of motion for gauge fields $v_\pm$ give

$$v_\pm = \pm\partial_\pm y_1 + 2H\Delta V\ \partial_\pm y_2\,. \tag{3.38}$$

Inserting expressions for $v_\pm$ into gauge fixed action we get the T-dual action

$$_{zyx}S = \kappa \int_\Sigma d^2\xi \partial_+\ _{zyx}X^\mu\ _{zyx}\Pi_{+\mu\nu}\ _{zyx}\ X^\nu\,, \tag{3.39}$$

where

$$_{zyx}X^\mu = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad _{zyx}\Pi_{+\mu\nu} = {}_{zyx}B_{\mu\nu} + \frac{1}{2}\ _{zyx}G_{\mu\nu} \tag{3.40}$$

$$_{zyx}B_{\mu\nu} = \begin{pmatrix} 0 & -H\Delta V & 0 \\ H\Delta V & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad _{zyx}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.41}$$

Combining the equations of motion (3.37) and (3.38) we obtain the T-dual transformation law

$$\partial_\pm x \cong \pm\partial_\pm y_1 + 2H\Delta V \partial_\pm y_2\,. \tag{3.42}$$

It directly follows that

$$\dot{x} \cong y_1' + 2H\Delta V \dot{y}_2\,. \tag{3.43}$$

From the initial action (2.13) it is obvious that momentum canonically conjugated to x is of the form

$$\pi_x = \kappa\dot{x} - 2\kappa H z y'\,. \tag{3.44}$$

The T-dual transformation law for y (3.26), in the approximation linear in H, produces that $y' \cong \dot{y}_2$. Taking into account the relation (3.43), we get the canonical form of the T-dual transformation law

$$y_1' \cong \frac{1}{\kappa}\pi_x\,. \tag{3.45}$$

As we see the full set of T-dual transformation laws, (3.17), (3.29) and (3.45), are the same as in the case where T-dualization was along xyz line [44] up to $H \to -H$. The full T-dualized theory is of the same form as in [44] with the expressions for **noncommutativity**

$$\{y_1(\sigma), y_3(\bar\sigma)\} \cong \frac{H}{\kappa}\left[2y(\sigma) - y(\bar\sigma)\right]\theta(\sigma - \bar\sigma)\,, \tag{3.46}$$

$$\{y_2(\sigma), y_3(\bar\sigma)\} \cong -\frac{H}{\kappa}\left[2x(\sigma) - x(\bar\sigma)\right]\theta(\sigma - \bar\sigma)\,, \tag{3.47}$$

and **nonassociativity**

$$\begin{aligned}&\{y_1(\sigma_1), y_2(\sigma_2), y_3(\sigma_3)\} \equiv \\ &\{y_1(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\} + \{y_2(\sigma_2), \{y_3(\sigma_3), y_1(\sigma_1)\}\} + \{y_3(\sigma_3), \{y_1(\sigma_1), y_2(\sigma_2)\}\} \cong \\ &\frac{2H}{\kappa^2}\left[\theta(\sigma_1-\sigma_2)\theta(\sigma_2-\sigma_3) + \theta(\sigma_2-\sigma_1)\theta(\sigma_1-\sigma_3) + \theta(\sigma_1-\sigma_3)\theta(\sigma_3-\sigma_2)\right],\end{aligned} \tag{3.48}$$

which can be obtained from the corresponding ones in xyz case [44] by replacing $H \to -H$.

4 Quantum aspects of T-dualization in the weakly curved background

In proving isometry and computing the $\beta^{\pm}$ functions we assumed the trivial topology and the surface term occurring there vanishes. Now we want to discuss some quantum aspects of the considered problems in nontrivial topologies. We will consider the action for bosonic string in the weakly curved background — constant metric and Kalb-Ramond field depending on all coordinates and with infinitesimal field strength. Torus with infinitesimal H-flux is special case of this model.

On th classical level there are a few problems in the theory. In order to perform the generalized T-dualization procedure the invariant coordinate x^{μ}_{inv} is introduced. But it is multivalued and the proof of equivalence of gauged and initial theories needs the part considering global characteristics. Moreover, in the quantum theory at higher genus, the holonomies of the world-sheet gauge fields complicate the situation a little bit. Fortunately, these problems can be resolved in Abelian case in the quantum theory [50–52].

First, we make Wick rotation $\tau \to -i\tau$, which makes the term which contains metric tensor $G_{\mu\nu}$ gets multiplier i, while the terms which contain Kalb-Ramond field $B_{\mu\nu}$ and Lagrange multiplier y_μ stay unchanged. Then the partition function is of the form

$$Z = \sum_{g=0}^{\infty} \int \mathcal{D}y\mathcal{D}v\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\, {}^\star v + i\kappa \int_\Sigma v B[V] v + \frac{i\kappa}{2}\int_\Sigma v dy}\,. \tag{4.1}$$

We use differential forms and omit the space-time indices to simplify writing of equations. The Hodge duality operator is denoted by star. The index g denotes the genus of manifold.

The first step in the calculation process is separation the one form dy into the exact part dy_e (y_e is single valued) and the harmonic part y_h ($dy_h = 0 = d^\dagger y_h$)

$$dy = dy_e + y_h. \tag{4.2}$$

For the closed forms the co-exact term $d^\dagger y_{co}$ in the Hodge decomposition is missing.

The path integral (4.1) goes over all degrees of freedom including local degrees of freedom as well as the sum over different topologies. Consequently, according to the (4.2), we substitute $\mathcal{D}y$ with the path integral over y_e and the sum over all possible topologically nontrivial states contained in y_h (marked by H_y)

$$\mathcal{D}y \to \mathcal{D}y_e \sum_{H_y}. \tag{4.3}$$

The integration over y_e induces vanishing of the field strength

$$Z = \int \mathcal{D}v\, \delta(dv)\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G\, {}^\star v + i\kappa \int_\Sigma v B[V] v} \sum_{H_y} e^{\frac{i\kappa}{2}\int_\Sigma v y_h}. \tag{4.4}$$

The 1-form v can be expressed as sum of exact, co-exact and the harmonic parts

$$v = dx + d^\dagger v_{ce} + v_h, \tag{4.5}$$

which means that

$$\mathcal{D}v \to \mathcal{D}x\mathcal{D}d^\dagger v_{ce}\, dH_v. \tag{4.6}$$

The functional integration over harmonic part v_h drives to the ordinary integration over topologically nontrivial periods (marked by symbol H_v). After integration over $d^\dagger v_{ce}$ we get

$$Z = \int \mathcal{D}x dH_v\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G^\star v + i\kappa \int_\Sigma vB[V]v} \sum_{H_y} e^{\frac{i\kappa}{2}\int_\Sigma v y_h}. \tag{4.7}$$

The last term in the exponent is responsible for nontrivial holonomies. Eliminating v_{ce} part, the 1-form v becomes closed and the Riemann bilinear relation becomes usable

$$\int_\Sigma v y_h = \sum_{i=1}^{g}\Big[\oint_{a_i} v \oint_{b_i} y_h - \oint_{a_i} y_h \oint_{b_i} v\Big]. \tag{4.8}$$

The symbols $a_i, b_i\, (i = 1, 2, \ldots, g)$ represent the canonical homology basis for the world-sheet. Because of the periodicity of the Lagrange multiplier y, its periods are just the winding numbers around cycles a_i and b_i

$$N_{a_i} = \oint_{a_i} y_h, \quad N_{b_i} = \oint_{b_i} y_h\,. \tag{4.9}$$

Denoting the periods with

$$A_i = \oint_{a_i} v, \quad B_i = \oint_{b_i} v\,, \tag{4.10}$$

we get

$$\int_\Sigma v y_h = \sum_{i=1}^{g}(N_{b_i}A_i - N_{a_i}B_i). \tag{4.11}$$

Now the partition function (4.7) gets the form

$$Z = \int \mathcal{D}x\, dA_i dB_i e^{-\frac{\kappa}{2}\int_\Sigma v\, G^\star v + i\kappa\int_\Sigma vB[V]v} \sum_{N_{a_i}, N_{b_i}\in Z} e^{\frac{i\kappa}{2}\sum_{i=1}^{g}(N_{b_i}A_i - N_{a_i}B_i)}. \tag{4.12}$$

The periodic delta function is defined as $\delta(x) = \frac{1}{2\pi}\sum_{n\in Z} e^{inx}$, which produces

$$Z = \int \mathcal{D}x\, dA_i dB_i \delta\left(\frac{\kappa}{2}A_i\right)\delta\left(\frac{\kappa}{2}B_i\right)\, e^{-\frac{\kappa}{2}\int_\Sigma v\, G^\star v + i\kappa\int_\Sigma vB[V]v}\,. \tag{4.13}$$

It is useful to examine the path dependence of the variable V^μ, which form is now

$$V^\mu(\xi) = x^\mu(\xi) - x^\mu(\xi_0) + \int_P v_h^\mu. \tag{4.14}$$

Let us consider two paths, P_1 and P_2, with the same initial ξ_0^α and the final points ξ^α. Now we will subtract from the value of V^μ along P_1 the value along path P_2 and obtain the integral over closed curve $P_1P_2^{-1}$ of the harmonic form

$$V^\mu[P_1] - V^\mu[P_2] = \oint_{P_1P_2^{-1}} v_h^\mu. \tag{4.15}$$

Establishing the homology between the closed curve $P_1P_2^{-1}$ and curve $\sum_i\left[n_ia_i+m_ib_i\right]$, $(n_i,m_i\in Z)$ we get finally

$$V^\mu[P_1]=V^\mu[P_2]+\sum_i(n_iA_i^\mu+m_iB_i^\mu). \tag{4.16}$$

The variable $V^\mu(\xi)$ in classical theory is path dependent if holonomies are nontrivial.

Integrating eq. (4.13) over A_i and B_i implies that periods A_i and B_i are zero. Consequently

$$v=dx. \tag{4.17}$$

The variable V^μ becomes single valued, and the initial theory is restored

$$Z=\int\mathcal{D}xe^{-\frac{\kappa}{2}\int_\Sigma dx\,G\,{}^\star dx+i\kappa\int_\Sigma dxB[x]dx}=\int\mathcal{D}xe^{-\kappa\int_\Sigma d^2\xi\partial x\Pi_+[x]\bar\partial x}. \tag{4.18}$$

Consequently, starting with partition function of the gauged fixed action of bosonic string in the weakly curved background, within path integral formalism and in the presence of nontrivial topologies, we came to the partition function of the initial theory. That means that introducing coordinate dependent Kalb-Ramond field is consistent with path integral quantization process.

5 Conclusion

In this article we studied the 3D closed bosonic string propagating in the geometry known as torus with H-flux — constant metric and Kalb-Ramond field with just one nonzero component, $B_{xy}=-B_{yx}=Hz$. The choice of background fields is consistent with the consistency conditions if we work in the diluted flux approximation which assumes that in all calculations we keep just the constant terms and those linear in the infinitesimal field strength H. Our goal was to study the T-dualization line which goes in the opposite direction from the standard one. First, we T-dualize z direction, then y and at the end along x direction — so-called zyx T-dualization line. We analyzed in every step the (non)commutativity and (non)associativity of the obtained theory and made comparisons with the case of xyz T-dualization line considered in [33, 44].

The common fact for all three theories obtained in the process of T-dualization step by step is that all three ones are nonlocal R-flux theories. The nonlocality comes as a result of the first step in T-dualization procedure, T-dualization along z direction. Generalized T-dualization procedure has one additional step with respect to the standard Buscher procedure and that is introduction of invariant coordinate. In the process of T-dualization invariant coordinate turns into variable ΔV which is defined as line integral. Consequently, this means that obtained theory is nonlocal. Further T-dualizations does not affect ΔV and, all three theories are nonlocal ones. As we know, in the case of xyz T-dualization line [44], we obtained three different theories in geometrical sense — twisted torus, Q-flux theory (which is local) and nonlocal R-flux theory.

The dualization along z direction produces nonlocal R-flux theory unlike the xyz case [33, 44] where the theory obtained after first T-dualization is locally and globally well defined. Because initial theory is geometrical one, its variables satisfy standard Pois-

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son algebra (3.19). Using (3.19) and T-dual transformation law written in the canonical form (3.17), we showed that theory obtained after T-dualization along z coordinate (using generalized T-dualization procedure) is *commutative* and, consequently, *associative* one as in [44].

The second step in T-dualization is T-dualization along y direction. Using standard Buscher procedure, we obtained the form of the T-dual theory and the corresponding T-dual transformation law, which is rewritten in the canonical form (3.29) in terms of the coordinates and momenta of the initial theory. Using standard Poisson algebra (3.19) and T-dual transformation laws in canonical form, (3.17) and (3.29), we easily proved that theory after two T-dualizations is *noncommutative*, but it is still *associative* one. In this article we used trivial winding condition (3.34) and showed that T-dual coordinates $y_2(\sigma)$ and $y_3(\bar{\sigma})$ are commutative for equal arguments, $\sigma = \bar{\sigma}$, but they are noncommutative if $\sigma - \bar{\sigma} = 2\pi$. The result is qualitatively similar to the result of [33], where after two T-dualizations the obtained theory is noncommutative one. But, the difference is in the winding condition which is nontrivial in [33], mixing different coordinates. The different winding condition induces the noncommutativity for $\sigma = \bar{\sigma}$ (for more details see [33]). On the other hand in the analysis presented in [44] (xyz T-dualization line) the theory obtained after two T-dualizations is commutative under trivial winding condition.

The final step in T-dualization procedure is T-dualization along x direction. The theory after full T-dualization is the same as in xyz case [44] with the noncommutativity and nonassociativity parameters which can be obtained from those in xyz case [44] adding "$-$" sign. This is a consequence of the fact that the full set of T-dual transformation laws is the same as in [44] up to the replacing $H \to -H$. This difference up to the "$-$" sign stems from the initial actions. In this article we start from (2.13), while in [44] the starting action for z T-dualization is Q-flux action, formally the same as (2.13) up to the replacing $H \to -H$.

Finishing the discussion of the results obtained in this paper it is interesting to make comparison with some similar efforts. We studied the abelian isometries using both standard and generalized T-duality procedure, while in the paper [53] nonabelian isometries using standard Buscher procedure are considered. The authors of [53] showed that spaces with isometry maps to the nonisometry spaces, while in this paper there is isometry in every T- dualization step. One of their conclusions that T-dual transformations are more than continuous isometry can be added to the concluding remarks of this paper. In the ref. [43] generalized T-duality and nongeometric background are considered, but using low energy effective action, unlike here, where we used sigma model action. The paper [54] deals with T-dualizations along nonisometry directions like in [31], using extension of gauge symmetry, while the authors of [31] use the generalized T-dualization procedure introducing invariant coordinates (in [54] they call them "covariant" coordinates). In this paper we use this generalized T-dualization procedure but all directions considered here are isometry ones. It is useful to mention that in the paper [55] bosonic string in the presence of the weakly curved backgrounds is considered using double space formalism as well as the influence of the order of T-dualizations. The double space formalism gives the result which is in accordance with the result of the current paper.

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Consequently, we conclude that in the case of the full T-dualization the form of the T-dual theory do not depend on the order of T-dualization, while parameters of noncommutativity and nonassociativity change sign.

A Light-cone coordinates

In the paper we often use light-cone coordinates defined as

$$\xi^{\pm} = \frac{1}{2}(\tau \pm \sigma)\,. \tag{A.1}$$

The corresponding partial derivatives are

$$\partial_{\pm} \equiv \frac{\partial}{\partial \xi^{\pm}} = \partial_{\tau} \pm \partial_{\sigma}\,. \tag{A.2}$$

Two dimensional Levi-Civita $\varepsilon^{\alpha\beta}$ is chosen in (τ,σ) basis as $\varepsilon^{\tau\sigma} = -1$. Consequently, in the light-cone basis the form of tensor is

$$\varepsilon^{\alpha\beta}_{lc} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}\,. \tag{A.3}$$

The flat world-sheet metric is of the form in (τ,σ) and light-cone basis, respectively

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad \eta^{lc}_{\alpha\beta} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\,. \tag{A.4}$$

Let us stress that in whole article we use standard notation for τ and σ derivatives — $\dot{A} \equiv \partial_\tau A$ and $A' \equiv \partial_\sigma A$, where A is an arbitrary variable.

B Two types of Poisson brackets used in the paper

In this paper, we have seen that T-dual transformation laws connect derivatives of T-dual coordinates with coordinates and momenta of initial theory. While initial theory satisfies standard Poisson brackets, in order to find Poisson brackets for T-dual theory, we first need to find Poisson brackets between σ derivatives of T-dual coordinates. This type of Poisson bracket will, in general case, be some function of initial coordinates, Dirac delta functions and their derivatives with respect to σ. Having this in mind, general case for our Poisson brackets will have following form

$$\{A'(\sigma), B'(\bar{\sigma})\} = U'(\sigma)\delta(\sigma-\bar{\sigma}) + V(\sigma)\delta'(\sigma-\bar{\sigma})\,, \tag{B.1}$$

where $\delta'(\sigma-\bar{\sigma}) \equiv \partial_\sigma\delta(\sigma-\bar{\sigma})$. For terms $A'(\sigma)$, $U'(\sigma)$ and $B'(\bar{\sigma})$, symbol $'$ stands for partial derivative with respect to σ and $\bar{\sigma}$, respectively. If we want to calculate the Poisson bracket

$$\{A(\sigma), B(\bar{\sigma})\}\,,$$

first we have to calculate the following one

$$\{\Delta A(\sigma,\sigma_0), \Delta B(\bar{\sigma},\bar{\sigma}_0)\}\,, \tag{B.2}$$

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where

$$\Delta A(\sigma,\sigma_0)=\int_{\sigma_0}^{\sigma}dxA'(x)=A(\sigma)-A(\sigma_0)\,,\quad \Delta B(\bar\sigma,\bar\sigma_0)=\int_{\bar\sigma_0}^{\bar\sigma}dxB'(x)=B(\bar\sigma)-B(\bar\sigma_0)\,. \tag{B.3}$$

Substituting the expressions (B.3) into (B.2), we have

$$\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=\int_{\sigma_0}^{\sigma}dx\int_{\bar\sigma_0}^{\bar\sigma}dy\;\left[U'(x)\delta(x-y)+V(x)\delta'(x-y)\right]\,. \tag{B.4}$$

After integration over y we get

$$\begin{aligned}&\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=\\ &=\int_{\sigma_0}^{\sigma}dx\{U'(x)\left[\theta(x-\bar\sigma_0)-\theta(x-\bar\sigma)\right]+V(x)\left[\delta(x-\bar\sigma_0)-\delta(x-\bar\sigma)\right]\},\end{aligned} \tag{B.5}$$

where $\theta(x)$ is defined as

$$\theta(x)=\int_0^x d\eta\delta(\eta)=\frac{1}{2\pi}\left[x+2\sum_{n\geq1}\frac{1}{n}\sin(nx)\right]=\begin{cases}0 & \text{if } x=0\\ 1/2 & \text{if } 0<x<2\pi\,,\\ 1 & \text{if } x=2\pi\end{cases} \tag{B.6}$$

where $\delta(x)=\frac{1}{2\pi}\sum_{n\in Z}e^{inx}$. Finally, integrating over x, we obtain

$$\begin{aligned}\{\Delta A(\sigma,\sigma_0),\Delta B(\bar\sigma,\bar\sigma_0)\}=&\,U(\sigma)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma-\bar\sigma)]-U(\sigma_0)[\theta(\sigma_0-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma)]\\ &-U(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0)]+U(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)]\\ &+V(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0]-V(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)].\end{aligned} \tag{B.7}$$

From the last expression, using (B.3), we extract the searched Poisson bracket

$$\{A(\sigma),B(\bar\sigma)\}=-[U(\sigma)-U(\bar\sigma)+V(\bar\sigma)]\theta(\sigma-\bar\sigma)\,. \tag{B.8}$$

In order to calculate Jacobiator we have to find Poisson brackets of type $\{y(\sigma),x(\bar\sigma)\}$, where $y(\sigma)$ is coordinate T-dual to initial one $x(\sigma)$. Having this in mind, we start with the following Poisson bracket

$$\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}=\left\{\int_{\sigma_0}^{\sigma}d\eta y'(\eta),x(\bar\sigma)\right\}\,, \tag{B.9}$$

and using T-dual transformation law in canonical form

$$\pi\cong\kappa y'\,, \tag{B.10}$$

we get

$$\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}\cong\frac{1}{\kappa}\left\{\int_{\sigma_0}^{\sigma}d\eta\pi(\eta),x(\bar\sigma)\right\}\,, \tag{B.11}$$

where $\pi(\sigma)$ is momentum canonically conjugated to the coordinate $x(\sigma)$. Initial theory is geometric one which variables satisfy standard Poisson algebra, so, the final result is of the form

$$\{\Delta y(\sigma,\sigma_0),x(\bar\sigma)\}\cong-\frac{1}{\kappa}\left[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)\right]\quad\Longrightarrow\{y(\sigma),x(\bar\sigma)\}\cong-\frac{1}{\kappa}\theta(\sigma-\bar\sigma)\,. \tag{B.12}$$

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ORIGINAL PAPER

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Noncommutativity and Nonassociativity of Closed Bosonic String on T-dual Toroidal Backgrounds

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In this article we consider closed bosonic string in the presence of constant metric and Kalb-Ramond field with one non-zero component, $B_{xy} = Hz$, where field strength H is infinitesimal. Using Buscher T-duality procedure we dualize along x and y directions and using generalized T-duality procedure along z direction imposing trivial winding conditions. After first two T-dualizations we obtain Q flux theory which is just locally well defined, while after all three T-dualizations we obtain nonlocal R flux theory. Origin of non-locality is variable ΔV defined as line integral, which appears as an argument of the background fields. Rewriting T-dual transformation laws in the canonical form and using standard Poisson algebra, we obtained that Q flux theory is commutative one and the R flux theory is noncommutative and nonassociative one. Consequently, there is a correlation between non-locality and closed string noncommutativity and nonassociativity.

1. Introduction

Coordinate noncommutativity means that there exists minimal possible length, which imposes natural UV cutoff. Idea of coordinate noncommutativity is very old. Heisenberg suggested coordinate noncommutativity to solve the problem of the occurrence of infinite quantities before renormalization procedure was developed and accepted. The first scientific paper considering this subject appeared 1947[1] where construction of discrete Lorentz invariant space-time is presented. Later in the period of 1980s A. Connes developed noncommutative geometry as a generalization of the standard commutative geometry.[2]

Noncommutativity became again interesting for particle physicists when the paper[3] appeared. In this article it is shown using propagators that open string endpoints in the presence of the constant metric and Kalb-Ramond field become noncommutative. D-brane on which the string endpoints are forced to move becomes noncommutative manifold. After this article many articles[4] appeared addressing the same subject but using different approaches - Fourier expansion, canonical methods, solving of boundary conditions etc.

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In the last two articles of[4] the method of solving of boundary conditions is presented. The basic idea is that open string boundary condition is treated as canonical constraint. Investigating the consistency of the canonical constraint we obtained the σ dependent form of the boundary condition. Further, we can proceed twofold: to introduce Dirac brackets or solve the constraint. Solving the constraint, we obtained the initial coordinate as a linear combination of the effective coordinate and momenta. Consequently, initial coordinates are noncommutative and the main contribution to noncomutativity parameter comes from Kalb-Ramond field as it was expected.

Following the result of the article[5] it can be proven that gauge fields "live" at the open string endpoints. Consequently, many interesting papers concerning non-commutative Yang-Mills theories and their renormalisability appeared.[6] In the papers[7] cross sections for some decays, allowed in noncommutative Yang-Mills theories and forbidden in commutative ones, are calculated, which offers a possibility of the experimental check of the noncommutativity idea and further, indirectly, idea of strings.

It is obvious that closed bosonic string in the presence of constant background fields remains commutative. There are no boundaries and, consequently, boundary conditions constraining string dynamics. In the case of open string we obtained initial coordinate in the form of linear combination of effective coordinates and momenta using boundary condition. That is achieved in the closed string case[8] using T-duality procedure and coordinate dependent background.

T-duality as a fundamental feature of string theory,[9–15] unexperienced by point particle, makes that there is no physical difference between string theory compactified on a circle of radius R and circle of radius $1/R$. Buscher T-dualization procedure[10] represents a mathematical frame in which T-dualization is realized. If the background fields do not depend on some coordinates then those coordinates are isometry directions. Consequently, that symmetry can be localized replacing ordinary world-sheet derivatives $\partial_\pm$ by covariant ones $D_\pm x^\mu = \partial_\pm x^\mu + v^\mu_\pm$, where $v^\mu_\pm$ are gauge fields. In order to make T-dual theory has the same number of degrees of freedom, the new term with Lagrange multipliers is added to the action which forces the gauge fields to be unphysical degrees of freedom. Because of the shift symmetry, using gauge freedom we fix initial coordinates. Variation of this gauge fixed action with respect to the Lagrange multipliers produces initial action and with respect to the gauge fields produces T-dual action.

Standard Buscher T-dualization was applied in closed string case in the papers.[8,16–19] In Ref. [16] authors consider 3-torus in the presence of constant metric and Kalb-Ramond field with one nonzero component $B_{xy} = Hz$, where field strength H is infinitesimal. They systematically apply Buscher procedure and, after two T-dualizations along isometry directions, obtain theory with Q flux which is noncommutative. In the calculations they used nontrivial boundary conditions (winding conditions). The result is that T-dual closed string coordinates are noncommutative for the same values of parameters $\sigma = \bar{\sigma}$ with noncommutativity parameter proportional to field strength H and N_3, winding number for z coordinate.

But, except this standard Buscher procedure, there is a generalized Buscher procedure dealing with background fields depending on all coordinates. The generalized procedure was applied to the case of bosonic string moving in the weakly curved background[20–22] and in the case where metric is quadratic in coordinates and Kalb-Ramond field is linear function of coordinates.[23] The generalized procedure enables us to make T-dualization in mentioned cases along arbitrary subset of coordinates.

Double space is one picturesque framework for representation of T-duality. Double space is introduced two to three decades ago.[24–28] It is spanned by double coordinates $Z^M = (x^\mu, y_\mu)$ $(\mu = 0, 1, 2, \dots, D-1)$, where x^μ are the coordinates of the initial theory and y_μ are T-dual coordinates. In this space T-dualization is represented as $O(d, d)$ transformation.[29–33] Permutation of the appropriate subsets of the initial and T-dual coordinates is interpreted as partial T-dualization[34,35] expanding Duff's idea.[24] The newly invented intrinsic noncommutativity[36] is related to double space. Intrinsic noncommutativity exists in the constant background case because it is considered within double space framework.

In this article we will deal with closed bosonic string propagating in the constant metric and linear dependent Kalb-Ramond field with $B_{xy} = Hz$, the same background as in [16]. This configuration is known in literature as torus with H-flux. As in the Ref. [16] we will use approximation of diluted flux, which means that in all calculations we keep constant and linear terms in infinitesimal field strength H. Transformation laws, relations which connect initial and T-dual variables, we will write in canonical form expressing initial momenta in terms of the T-dual coordinates. Unlike Ref. [16], except T-dualization along two isometry directions, we will make one step more and T-dualize along z coordinate using generalized T-dualization procedure. During dualization procedure we will use trivial boundary (winding) conditions.

Transformation laws in canonical form enable us to express sigma derivative of the T-dual coordinate as a linear combination of the initial momenta and coordinates. Because initial theory is geometrical locally and globally, its coordinates and canonically conjugated momenta satisfy standard Poisson algebra. This fact means that we can calculate the Poisson brackets of the T-dual coordinates using technical instruction given in subsection 4.1.

After T-dualizations along isometry directions (along x and y) we obtain the same background as in Ref. [16] but, obtained Q flux theory, which is still locally well defined, is commutative. This is a consequence of the imposed trivial winding conditions. Having in mind the generalized T-duality procedure,[20,21,23] T-dualization along z coordinate produces R flux nonlocal theory because it depends on the variable ΔV which is defined as line integral. Calculating Poisson brackets of the T-dual coordinates we obtain two nonzero Poisson brackets and show that there is a correlation between non-locality and closed string noncommutativity.

The form of noncommutativity is such that it exists when arguments of the coordinates are different, $\sigma \neq \bar{\sigma}$. That is another difference with respect to the result of Ref. [16] but there is no contradiction because the origins of noncommutativity are different. In this article non-locality is related with noncommutativity of R flux theory under trivial winding conditions while in Ref. [16] it is about noncommutativity of Q flux theory under nontrivial winding conditions.

From the noncommutativity relations it follows that Jacobi identity is broken i.e. nonassociativity occurs. Nonassociativity parameter, R flux, is proportional to the field strength H. Using generalized T-duality[20,21,23] we obtain the concrete form of nonassociativity from string dynamics. Similar as noncommutativity, discovery of nonassociativity pushes the scientists to explore the effects of nonassociativity in the field of renormalisability of ϕ^4 theory[37] as well as formulation of nonassociative gravity.[38]

At the end we add an appendix containing some conventions used in the paper.

2. Bosonic String Action and Choice of Background Fields

The action of the closed bosonic string in the presence of the space-time metric $G_{\mu\nu}(x)$, Kalb-Ramond antisymmetric field $B_{\mu\nu}(x)$, and dilaton scalar field $\Phi(x)$ is given by the following expression[9]

$$S = \kappa \int_\Sigma d^2\xi \sqrt{-g} \times \left\{ \left[\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x) \right] \partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x) R^{(2)} \right\}, \tag{2.1}$$

where Σ is the world-sheet surface parameterized by $\xi^\alpha = (\tau, \sigma)$ $[(\alpha = 0, 1),\ \sigma \in (0, \pi)]$, while the D-dimensional space-time is spanned by the coordinates x^μ $(\mu = 0, 1, 2, \dots, D-1)$. We denote intrinsic world sheet metric with $g_{\alpha\beta}$, and the corresponding scalar curvature with $R^{(2)}$.

In order to keep conformal symmetry on the quantum level background fields must obey space-time field equations[39]

$$\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2 D_\mu a_\nu = 0, \tag{2.2}$$

$$\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2 a_\rho B^\rho{}_{\mu\nu} = 0, \tag{2.3}$$

$$\beta^\Phi \equiv 2\pi\kappa \frac{D-26}{6} - R - \frac{1}{24} B_{\mu\rho\sigma} B^{\mu\rho\sigma} - D_\mu a^\mu + 4a^2 = c, \tag{2.4}$$

where c is an arbitrary constant. The function β^Φ could be a constant because of the relation

$$D^\nu \beta^G_{\nu\mu} + \partial_\mu \beta^\Phi = 0. \tag{2.5}$$

Further, $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to the space-time metric $G_{\mu\nu}$, while

$$B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \quad a_\mu = \partial_\mu \Phi, \tag{2.6}$$

are field strength for Kalb-Ramond field $B_{\mu\nu}$ and dilaton gradient, respectively. Trivial solution of these equations is that all three background fields are constant. This case was pretty exploited in the analysis of the open string noncommutativity.

The less trivial case would be a case where some background fields are coordinate dependent. If we choose Kalb-Ramond field to be linearly coordinate dependent and dilaton field to be constant then the first equation (2.2) becomes

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} = 0. \tag{2.7}$$

The field strength $B_{\mu\nu\rho}$ is constant and, if we assume that it is infinitesimal, then we can take $G_{\mu\nu}$ to be constant in approximation linear in $B_{\mu\nu\rho}$. Consequently, all three space-time field equations are satisfied. Especially, the third one is of the form

$$2\pi\kappa \frac{D-26}{6} = c, \tag{2.8}$$

which enables us to work in arbitrary number of space-time dimensions.

In this article we will work in $D = 3$ dimensions with the following choice of background fields

$$G_{\mu\nu} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & Hz & 0 \\ -Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.9}$$

where $R_\mu (\mu = 1, 2, 3)$ are radii of the compact dimensions. This choice of background fields is known in geometry as torus with flux (field strength) H.[16] Our choice of infinitesimal H can be understood in terms of the radii as that

$$\left(\frac{H}{R_1 R_2 R_3}\right)^2 = 0. \tag{2.10}$$

This approximation is known in literature as the approximation of diluted flux. Physically, this means that we work with the torus which is sufficiently large. Consequently, we can rescale the coordinates

$$x^\mu \longmapsto \frac{x^\mu}{R_\mu}, \tag{2.11}$$

which simplifies the form of the metric

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

The final form of the closed bosonic string action is

$$\begin{aligned} S &= \kappa \int_\Sigma d^2\xi\, \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu \\ &= \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}(\partial_+ x \partial_- x + \partial_+ y \partial_- y + \partial_+ z \partial_- z) \right. \\ &\quad \left. + \partial_+ x Hz \partial_- y - \partial_+ y Hz \partial_- x\right], \end{aligned} \tag{2.13}$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$ is world-sheet derivative with respect to the light-cone coordinates $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}$ and

$$x^\mu = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{2.14}$$

Let us note that we do not write dilaton term because its T-dualization is performed separately within quantum formalism and here will be skipped.

3. T-dualization of the Bosonic Closed String Action

In this section we will perform T-dualization along three directions, one direction at time. Our goal is to find the relations connecting initial variables with T-dual ones called transformation laws. Using transformation laws we will find noncommutativity and nonassociativity relations.

3.1. T-dualization Along x Direction – from Torus with H Flux to the Twisted Torus

Let us perform standard Buscher T-dualization[10] of action (2.13) along x direction. Note that x direction is an isometry direction which means that action has a global shift symmetry, $x \longrightarrow x + a$. In order to perform Buscher procedure, we have to localize this symmetry introducing covariant world-sheet derivatives instead of the ordinary ones

$$\partial_\pm x \longrightarrow D_\pm x = \partial_\pm x + v_\pm, \tag{3.1}$$

where $v_\pm$ are gauge fields which transform as $\delta v_\pm = -\partial_\pm a$. Because T-dual action must have the same number of degrees of freedom as initial one, we have to make these fields $v_\pm$ be unphysical degrees of freedom. This is accomplished by adding following term to the action

$$S_{add} = \frac{\kappa}{2} \int_\Sigma d^2\xi\, y_1 (\partial_+ v_- - \partial_- v_+), \tag{3.2}$$

where y_1 is a Lagrange multiplier. After gauge fixing, $x = const.$, the action gets the form

$$\begin{aligned} S_{fix} = \kappa \int d^2\xi & \left[\frac{1}{2}(v_+ v_- + \partial_+ y \partial_- y + \partial_+ z \partial_- z) + v_+ Hz \partial_- y \right. \\ & \left. - \partial_+ y Hz v_- + \frac{1}{2} y_1 (\partial_+ v_- - \partial_- v_+)\right]. \end{aligned} \tag{3.3}$$

From the equations of motion for y_1 we obtain that field strength for the gauge field $v_\pm$ is equal to zero

$$F_{+-} = \partial_+ v_- - \partial_- v_+ = 0, \tag{3.4}$$

which gives us the solution for gauge field

$$v_\pm = \partial_\pm x. \tag{3.5}$$

Inserting this solution for gauge field into gauge fixed action (3.3) we obtain initial action given by Eq. (2.13). Equations of motion for $v_\pm$ will lead to the T-dual action. Varying the gauge fixed action (3.3) with respect to the gauge field v_+ we get

$$v_- = -\partial_- y_1 - 2Hz\partial_- y, \tag{3.6}$$

while on the equation of motion for v_- it holds

$$v_+ = \partial_+ y_1 + 2Hz\partial_+ y. \tag{3.7}$$

Inserting relations (3.6) and (3.7) into expression for gauge fixed action (3.3), keeping terms linear in H, we obtain the T-dual action

$${}_xS = \kappa \int_\Sigma d^2\xi\, \partial_+ ({}_xX)^\mu\, {}_x\Pi_{+\mu\nu} \partial_- ({}_xX)^\nu, \tag{3.8}$$

where subscript $_x$ denotes quantity obtained after T-dualization along x direction and

$${}_xX^\mu = \begin{pmatrix} y_1 \\ y \\ z \end{pmatrix}. \tag{3.9}$$

Further we have the T-dual background fields

$${}_x\Pi_{+\mu\nu} = {}_xB_{\mu\nu} + \frac{1}{2}{}_xG_{\mu\nu}, \quad {}_xB_{\mu\nu} = 0,$$
$${}_xG_{\mu\nu} = \begin{pmatrix} 1 & 2Hz & 0 \\ 2Hz & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.10}$$

Obtained background fields (3.10) define that what is known in literature as *twisted torus geometry*. String theory after one T-dualization is geometrically well defined globally and locally or, simply, theory is geometrical (flux H takes the role of connection).

Combining the solutions of equations of motion for Lagrange multiplier (3.5) and for gauge fields, (3.6) and (3.7), we get the transformation laws connecting initial, x^μ, and T-dual, ${}_xX^\mu$, coordinates

$$\partial_\pm x \cong \pm\partial_\pm y_1 \pm 2Hz\partial_\pm y, \tag{3.11}$$

where $\cong$ denotes T-duality relation. The momentum π_x is canonically conjugated to the initial coordinate x. Using the initial action (2.13) we get

$$\pi_x = \frac{\delta S}{\delta \dot{x}} = \kappa(\dot{x} - 2Hzy'), \tag{3.12}$$

where $\dot{A} \equiv \partial_\tau A$ and $A' \equiv \partial_\sigma A$. From transformation law (3.11) it is straightforward to obtain

$$\dot{x} \cong y_1' + 2Hzy', \tag{3.13}$$

which, inserted in the expression for momentum π_x, gives transformation law in canonical form

$$\pi_x \cong \kappa y_1'. \tag{3.14}$$

3.2. From Twisted Torus to Non-geometrical Q Flux

In this subsection we will continue the T-dualization of action (3.8) along y direction. After x and y T-dualization we obtain the structure which has local geometrical interpretation but global omissions. Such structure is known in literature as non-geometry.

We repeat the procedure from the previous subsection and form the gauge fixed action

$$S_{fix} = \kappa \int_\Sigma d^2\xi \left[\frac{1}{2}(\partial_+ y_1 \partial_- y_1 + v_+ v_- + \partial_+ z \partial_- z) + \partial_+ y_1 Hz v_- \right.$$
$$\left. + v_+ Hz \partial_- y_1 + \frac{1}{2} y_2 (\partial_+ v_- - \partial_- v_+)\right]. \tag{3.15}$$

From the equation of motion for Lagrange multiplier y_2

$$\partial_+ v_- - \partial_- v_+ = 0 \longrightarrow v_\pm = \partial_\pm y, \tag{3.16}$$

gauge fixed action becomes initial one (3.8). Varying the gauge fixed action (3.15) with respect to the gauge fields we get

$$v_\pm = \pm\partial_\pm y_2 - 2Hz\partial_\pm y_1. \tag{3.17}$$

Inserting these expressions for gauge fields into gauge fixed action, keeping the terms linear in H, gauge fixed action is driven into T-dual action

$${}_{xy}S = \kappa \int d^2\xi\, \partial_+ ({}_{xy}X)^\mu\, {}_{xy}\Pi_{+\mu\nu} \partial_- ({}_{xy}X)^\nu, \tag{3.18}$$

where

$$({}_{xy}X)^\mu = \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix},$$

$${}_{xy}\Pi_{+\mu\nu} = {}_{xy}B_{\mu\nu} + \frac{1}{2}{}_{xy}G_{\mu\nu} = \begin{pmatrix} \frac{1}{2} & -Hz & 0 \\ Hz & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{3.19}$$

Explicit expressions for background fields are

$${}_{xy}B_{\mu\nu} = \begin{pmatrix} 0 & -Hz & 0 \\ Hz & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -B_{\mu\nu}, \quad {}_{xy}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.20}$$

Let us note that background fields obtained after two T-dualizations are similar to the geometric background of torus with H flux, but they should be considered only locally. Their global properties are non-trivial and because of that the term "non-geometry" is introduced.

Combining the equations of motion for Lagrange multiplier y_2 and for gauge fields $v_\pm$, we obtain T-dual transformation laws

$$\partial_\pm y \cong \pm\partial_\pm y_2 - 2Hz\partial_\pm y_1. \tag{3.21}$$

The y component of the initial canonical momentum π_y is a variation of the initial action with respect to the $\dot{y}$

$$\pi_y = \frac{\delta S}{\delta \dot{y}} = \kappa(\dot{y} + 2Hzx'). \tag{3.22}$$

Using T-dual transformation laws (3.21) we easily get

$$\dot{y} \cong y_2' - 2Hz\dot{y}_1, \tag{3.23}$$

while from the transformation law (3.11), at zeroth order in H, it holds $x' \cong \dot{y}_1$. Inserting last two expression into π_y we obtain transformation law in canonical form

$$\pi_y \cong \kappa y_2'. \tag{3.24}$$

After two T-dualizations along isometry directions, in the approximation of the diluted flux (keeping just terms linear in H), according to the canonical forms of the transformation laws (3.14) and (3.24), we see that T-dual coordinates y_1 and y_2 are still commutative. This is a consequence of the simple fact that variables of the initial theory, which is geometrical one, satisfy standard Poisson algebra

$$\left\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\right\} = \delta^\mu{}_\nu\delta(\sigma-\bar{\sigma}), \quad \{x^\mu, x^\nu\} = \left\{\pi_\mu, \pi_\nu\right\} = 0, \tag{3.25}$$

where

$$\pi_\mu = \begin{pmatrix} \pi_x \\ \pi_y \\ \pi_z \end{pmatrix}. \tag{3.26}$$

3.3. From Q to R Flux – T-dualization Along z Coordinate

In this subsection we will finalize the process of T-dualization dualizing along remaining z direction. For this purpose we will use generalized T-dualization procedure.[20,21,23] The result is a theory which is not well defined even locally and is known in literature as theory with R-flux.

We start with the action obtained after T-dualizations along x and y directions (3.18). The Kalb-Ramond field (3.20) depends on z and it seems that it is not possible to perform T-dualization. Let us assume that Kalb-Ramond field linearly depends on all coordinates, $B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho$ and check if some global transformation can be treated as isometry one. We start with global shift transformation

$$\delta x^\mu = \lambda^\mu, \tag{3.27}$$

and make a variation of action

$$\begin{aligned}\delta S &= \frac{\kappa}{3}B_{\mu\nu\rho}\lambda^\rho\int_\Sigma d^2\xi\,\partial_+x^\mu\partial_-x^\nu \\ &= \frac{2k}{3}B_{\mu\nu\rho}\lambda^\rho\epsilon^{\alpha\beta}\int_\Sigma d^2\xi[\partial_\alpha(x^\mu\partial_\beta x^\nu) - x^\mu(\partial_\alpha\partial_\beta x^\nu)].\end{aligned} \tag{3.28}$$

The second term vanishes explicitly, while the first term is surface one. Consequently, in the case of constant metric and linearly dependent Kalb-Ramond field, global shift transformation is an isometry transformation. This means that we can make T-dualization along z coordinate using generalized T-dualization procedure.

The generalized T-dualization procedure is presented in detail in Ref. [20]. In order to localize shift symmetry of the action (3.18) along z direction we introduce covariant derivative

$$\partial_\pm z \longrightarrow D_\pm z = \partial_\pm z + v_\pm, \tag{3.29}$$

which is a part of the standard Buscher procedure. The novelty is introduction of the invariant coordinate as line integral

$$\begin{aligned}z^{inv} &= \int_P d\xi^\alpha D_\alpha z \\ &= \int_P d\xi^+ D_+z + \int_P d\xi^- D_-z = z(\xi) - z(\xi_0) + \Delta V,\end{aligned} \tag{3.30}$$

where

$$\Delta V = \int_P d\xi^\alpha v_\alpha = \int_P (d\xi^+ v_+ + d\xi^- v_-). \tag{3.31}$$

Here ξ and ξ_0 are the current and initial point of the world-sheet line P. At the end, as in the standard Buscher procedure, in order to make $v_\pm$ to be unphysical degrees of freedom we add to the action term with Lagrange multiplier

$$S_{add} = \frac{\kappa}{2}\int_\Sigma d^2\xi\ y_3(\partial_+v_- - \partial_+v_-). \tag{3.32}$$

The final form of the action is

$$\begin{aligned}\bar{S} = \kappa\int_\Sigma d^2\xi\Big[&-Hz^{inv}(\partial_+y_1\partial_-y_2 - \partial_+y_2\partial_-y_1) \\ &+ \frac{1}{2}(\partial_+y_1\partial_-y_1 + \partial_+y_2\partial_-y_2 + D_+zD_-z) \\ &+ \frac{1}{2}\ y_3(\partial_+v_- - \partial_-v_+)\Big].\end{aligned} \tag{3.33}$$

Because of existing shift symmetry we fix the gauge, $z(\xi) = z(\xi_0)$, and then the gauge fixed action takes the form

$$S_{fix} = \kappa \int_\Sigma d^2\xi \Big[-H\Delta V(\partial_+ y_1 \partial_- y_2 - \partial_+ y_2 \partial_- y_1) + \frac{1}{2}(\partial_+ y_1 \partial_- y_1 + \partial_+ y_2 \partial_- y_2 + v_+ v_-) + \frac{1}{2} y_3 (\partial_+ v_- - \partial_- v_+) \Big]. \tag{3.34}$$

From the equation of motion for Lagrange multiplier y_3 we obtain

$$\partial_+ v_- - \partial_- v_+ = 0 \Longrightarrow v_\pm = \partial_\pm z, \quad \Delta V = \Delta z, \tag{3.35}$$

which drives back the gauge fixed action to the initial action (3.18). Varying the gauge fixed action (3.34) with respect to the gauge fields $v_\pm$ we get the following equations of motion

$$v_\pm = \pm \partial_\pm y_3 - 2\beta^\mp, \tag{3.36}$$

where $\beta^\pm$ functions are defined as

$$\beta^\pm = \pm \frac{1}{2} H(y_1 \partial_\mp y_2 - y_2 \partial_\mp y_1). \tag{3.37}$$

The $\beta^\pm$ functions are obtained as a result of the variation of the term containing ΔV

$$\delta_v \left(-2\kappa \int d^2\xi \varepsilon^{\alpha\beta} H \partial_\alpha y_1 \partial_\beta y_2 \Delta V \right) = \kappa \int d^2\xi \left(\beta^+ \delta v_+ + \beta^- \delta v_- \right), \tag{3.38}$$

using partial integration and the fact that $\partial_\pm V = v_\pm$. Inserting the relations (3.36) into the gauge fixed action, keeping linear terms in H, we obtain the T-dual action

$$_{xyz}S = \kappa \int_\Sigma d^2\xi \partial_{+\,xyz} X^\mu {}_{xyz}\Pi_{+\mu\nu} \partial_{-\,xyz} X^\nu, \tag{3.39}$$

where

$$_{xyz}X^\mu = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad {}_{xyz}\Pi_{+\mu\nu} = {}_{xyz}B_{\mu\nu} + \frac{1}{2} {}_{xyz}G_{\mu\nu}, \tag{3.40}$$

$$_{xyz}B_{\mu\nu} = \begin{pmatrix} 0 & -H\Delta\tilde{y}_3 & 0 \\ H\Delta\tilde{y}_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad {}_{xyz}G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.41}$$

Here we introduced double coordinate $\tilde{y}_3$ defined as

$$\partial_\pm y_3 \equiv \pm \partial_\pm \tilde{y}_3. \tag{3.42}$$

Let us note that ΔV stands beside field strength H, which implicates that, according to the diluted flux approximation, we calculate ΔV in the zeroth order in H

$$\Delta V = \int d\xi^+ \partial_+ y_3 - \int d\xi^- \partial_- y_3. \tag{3.43}$$

Having this into account it is clear why we defined double coordinate $\tilde{y}_3$ as in Eq. (3.42). Also it is useful to note that presence of ΔV, which is defined as line integral, represents the source of non-locality of the T-dual theory. the result of the three T-dualization is a theory with R flux as it is known in the literature.

Combining the equations of motion for Lagrange multiplier (3.35), $v_\pm = \partial_\pm z$, and equations of motion for gauge fields (3.36), we obtain the T-dual transformation law

$$\partial_\pm z \cong \pm \partial_\pm y_3 - 2\beta^\mp. \tag{3.44}$$

Adding transformation laws for $\partial_\pm z$ and $\partial_- z$ we get the transformation law for $\dot{z}$

$$\dot{z} \cong y_3' + H(y_1 y_2' - y_2 y_1'), \tag{3.45}$$

which enables us to write down the transformation law in the canonical form

$$y_3' \cong \frac{1}{\kappa}\pi_z - H(xy' - yx'). \tag{3.46}$$

Here we used the expression for the canonical momentum of the initial theory (2.13)

$$\pi_z = \frac{\delta S}{\delta \dot{z}} = \kappa \dot{z}. \tag{3.47}$$

4. Noncommutativity and Nonassociativity Using T-duality

In the open string case noncommutativity comes from the boundary conditions which makes that coordinates x^μ depend both on the effective coordinates and on the effective momenta.[4] Effective coordinates and momenta do not commute and, consequently, coordinates x^μ do not commute. In the closed bosonic string case the logic is the same but the execution is different. Using T-duality we obtained transformation laws, (3.11), (3.21) and (3.44), which relate T-dual coordinates with the initial coordinates and their canonically conjugated momenta. In this section we will use these relations to get noncommutativity and nonassociativity relations.

4.1. Noncommutativity Relations

Let us start with the Poisson bracket of the σ derivatives of two arbitrary coordinates in the form

$$\{A'(\sigma), B'(\bar{\sigma})\} = U'(\sigma)\delta(\sigma - \bar{\sigma}) + V(\sigma)\delta'(\sigma - \bar{\sigma}), \tag{4.1}$$

where $\delta'(\sigma-\bar\sigma)\equiv\partial_\sigma\delta(\sigma-\bar\sigma)$. In order to find the form of the Poisson bracket

$$\{A(\sigma),\,B(\bar\sigma)\},$$

we have to find the form of the Poisson bracket

$$\{\Delta A(\sigma,\sigma_0),\,\Delta B(\bar\sigma,\bar\sigma_0)\},$$

where

$$\Delta A(\sigma,\sigma_0)=\int_{\sigma_0}^{\sigma}dx A'(x)=A(\sigma)-A(\sigma_0),$$
$$\Delta B(\bar\sigma,\bar\sigma_0)=\int_{\bar\sigma_0}^{\bar\sigma}dx B'(x)=B(\bar\sigma)-B(\bar\sigma_0). \tag{4.2}$$

Now we have

$$\{\Delta A(\sigma,\sigma_0),\,\Delta B(\bar\sigma,\bar\sigma_0)\}
=\int_{\sigma_0}^{\sigma}dx\int_{\bar\sigma_0}^{\bar\sigma}dy\,\left[U'(x)\delta(x-y)+V(x)\delta'(x-y)\right]. \tag{4.3}$$

After integration over y we get

$$\{\Delta A(\sigma,\sigma_0),\,\Delta B(\bar\sigma,\bar\sigma_0)\}=\int_{\sigma_0}^{\sigma}dx\,\Big\{U'(x)\left[\theta(x-\bar\sigma_0)-\theta(x-\bar\sigma)\right] + V(x)\left[\delta(x-\bar\sigma_0)-\delta(x-\bar\sigma)\right]\Big\}, \tag{4.4}$$

where function $\theta(x)$ is defined as

$$\theta(x)=\int_0^x d\eta\delta(\eta)=\frac{1}{2\pi}\left[x+2\sum_{n\geq1}\frac{1}{n}\sin(nx)\right]
=\begin{cases}0 & \text{if } x=0\\ 1/2 & \text{if } 0<x<2\pi.\\ 1 & \text{if } x=2\pi\end{cases} \tag{4.5}$$

Integrating over x using partial integration finally we obtain

$$\begin{aligned}\{\Delta A(\sigma,\sigma_0),\,\Delta B(\bar\sigma,\bar\sigma_0)\}&=U(\sigma)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma-\bar\sigma)]\\
&-U(\sigma_0)[\theta(\sigma_0-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma)]-U(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0)]\\
&+U(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)]+V(\bar\sigma_0)[\theta(\sigma-\bar\sigma_0)-\theta(\sigma_0-\bar\sigma_0]\\
&-V(\bar\sigma)[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)].\end{aligned} \tag{4.6}$$

From the last expression, using the right-hand sides of the expressions in Eq. (4.2), we extract the desired Poisson bracket

$$\{A(\sigma),\,B(\bar\sigma)\}=-[U(\sigma)-U(\bar\sigma)+V(\bar\sigma)]\theta(\sigma-\bar\sigma). \tag{4.7}$$

Let us rewrite the canonical forms of the transformation laws, (3.14), (3.24) and (3.46), in the following way

$$y_1'\cong\frac{1}{\kappa}\pi_x,\quad y_2'\cong\frac{1}{\kappa}\pi_y,\quad y_3'\cong\frac{1}{\kappa}\pi_z-H(xy'-yx'). \tag{4.8}$$

In order to find the Poisson brackets between T-dual coordinates y_μ we will use the algebra of the coordinates and momenta of the initial theory (3.25). It is obvious that only nontrivial Poisson brackets will be $\{y_1(\sigma),\,y_3(\bar\sigma)\}$ and $\{y_2(\sigma),\,y_3(\bar\sigma)\}$.

Let us first write the corresponding Poisson brackets of the sigma derivatives of T-dual coordinates y_μ using (4.8)

$$\{y_1'(\sigma),\,y_3'(\bar\sigma)\}\cong\frac{2}{\kappa}Hy'(\sigma)\delta(\sigma-\bar\sigma)+\frac{1}{\kappa}Hy(\sigma)\delta'(\sigma-\bar\sigma), \tag{4.9}$$

$$\{y_2'(\sigma),\,y_3'(\bar\sigma)\}\cong-\frac{2}{\kappa}Hx'(\sigma)\delta(\sigma-\bar\sigma)-\frac{1}{\kappa}Hx(\sigma)\delta'(\sigma-\bar\sigma), \tag{4.10}$$

while all other Poisson brackets are zero. We see that these Poisson brackets are of the form (4.1), so, we can apply the result (4.7). Consequently, we get

$$\{y_1(\sigma),\,y_3(\bar\sigma)\}\cong-\frac{H}{\kappa}\left[2y(\sigma)-y(\bar\sigma)\right]\theta(\sigma-\bar\sigma), \tag{4.11}$$

$$\{y_2(\sigma),\,y_3(\bar\sigma)\}\cong\frac{H}{\kappa}\left[2x(\sigma)-x(\bar\sigma)\right]\theta(\sigma-\bar\sigma), \tag{4.12}$$

where function $\theta(x)$ is defined in (4.5). Let us note that these two Poisson brackets are zero when $\sigma=\bar\sigma$ and/or field strength H is equal to zero. But if we take that $\sigma-\bar\sigma=2\pi$ then we have $\theta(2\pi)=1$ and it follows

$$\{y_1(\sigma+2\pi),\,y_3(\sigma)\}\cong-\frac{H}{\kappa}\left[4\pi N_y+y(\sigma)\right], \tag{4.13}$$

$$\{y_2(\sigma+2\pi),\,y_3(\sigma)\}\cong\frac{H}{\kappa}\left[4\pi N_x+x(\sigma)\right], \tag{4.14}$$

where N_x and N_y are winding numbers defined as

$$x(\sigma+2\pi)-x(\sigma)=2\pi N_x,\quad y(\sigma+2\pi)-y(\sigma)=2\pi N_y. \tag{4.15}$$

From these relations we can see that if we choose such σ for which $x(\sigma)=0$ and $y(\sigma)=0$ then noncommutativity relations are proportional to winding numbers. On the other side, for winding numbers which are equal to zero there is still noncommutativity between T-dual coordinates.

4.2. Nonassociativity

In order to calculate Jacobi identity of the T-dual coordinates we first have to find Poisson brackets $\{y_1(\sigma),\,x(\bar\sigma)\}$ as well as $\{y_2(\sigma),\,y(\bar\sigma)\}$. We start with

$$\{\Delta y_1(\sigma,\sigma_0),\,x(\bar\sigma)\}=\left\{\int_{\sigma_0}^{\sigma}d\eta y_1'(\eta),\,x(\bar\sigma)\right\}, \tag{4.16}$$

and then use the T-dual transformation for x-direction in canonical form

$$\pi_x\cong\kappa y_1'. \tag{4.17}$$

From these two equations it follows

$$\{\Delta y_1(\sigma,\sigma_0), x(\bar\sigma)\} \cong \frac{1}{\kappa}\left\{\int_{\sigma_0}^{\sigma} d\eta \pi_x(\eta), x(\bar\sigma)\right\}, \tag{4.18}$$

which, using the standard Poisson algebra, produces

$$\{\Delta y_1(\sigma,\sigma_0), x(\bar\sigma)\} \cong -\frac{1}{\kappa}\left[\theta(\sigma-\bar\sigma)-\theta(\sigma_0-\bar\sigma)\right]$$

$$\Longrightarrow \{y_1(\sigma), x(\bar\sigma)\} \cong -\frac{1}{\kappa}\theta(\sigma-\bar\sigma). \tag{4.19}$$

The relation $\{y_2(\sigma), y(\bar\sigma)\}$ can be obtained in the same way. Because the transformation law for y-direction is of the same form as for x-direction, the Poisson bracket is of the same form

$$\{y_2(\sigma), y(\bar\sigma)\} \cong -\frac{1}{\kappa}\theta(\sigma-\bar\sigma). \tag{4.20}$$

Now we can calculate Jacobi identity using noncommutativity relations (4.11) and (4.12) and above two Poisson brackets

$$\begin{aligned}
&\{y_1(\sigma_1), y_2(\sigma_2), y_3(\sigma_3)\} \equiv \{y_1(\sigma_1), \{y_2(\sigma_2), y_3(\sigma_3)\}\}\\
&\quad +\{y_2(\sigma_2), \{y_3(\sigma_3), y_1(\sigma_1)\}\} + \{y_3(\sigma_3), \{y_1(\sigma_1), y_2(\sigma_2)\}\}\\
&\cong -\frac{2H}{\kappa^2}\big[\theta(\sigma_1-\sigma_2)\theta(\sigma_2-\sigma_3)\\
&\quad +\theta(\sigma_2-\sigma_1)\theta(\sigma_1-\sigma_3)+\theta(\sigma_1-\sigma_3)\theta(\sigma_3-\sigma_2)\big].
\end{aligned} \tag{4.21}$$

Jacobi identity is nonzero which means that theory with R-flux is nonassociative. For $\sigma_2=\sigma_3=\sigma$ and $\sigma_1=\sigma+2\pi$ we get

$$\{y_1(\sigma+2\pi), y_2(\sigma), y_3(\sigma)\} \cong \frac{2H}{\kappa^2}. \tag{4.22}$$

From the last two equations, general form of Jacobi identity and Jacobi identity for special choice of σ's, we see that presence of the coordinate dependent Kalb-Ramond field is a source of noncommutativity and nonassociativity.

5. Conclusion

In this article we have considered the closed bosonic string propagating in the three-dimensional constant metric and Kalb-Ramond field with just one nonzero component $B_{xy}=Hz$. This choice of background is in accordance with consistency conditions in the sense that all calculations were made in approximation linear in Kalb-Ramond field strength H. Geometrically, this settings corresponds to the torus with H flux. Then we performed standard Buscher T-dualization procedure along isometry directions, first along x and then along y direction. At the end we performed generalized T-dualization procedure along z direction and obtained nonlocal theory with R flux. Using the relations between initial and T-dual variables, called T-dual transformation laws, in canonical form we find the noncommutativity and nonassociativity relations between T-dual coordinates.

After T-dualization along x direction we obtained theory embedded in geometry known in literature as twisted torus geometry. The relation between initial and T-dual variables is trivial, $\pi_x \cong \kappa y_1'$, where π_x is x component of the canonical momentum of the initial theory and y_1 is coordinate T-dual to x. Consequently, flux H takes a role of connection, obtained theory is globally and locally well defined and commutative, because the coordinates and their canonically conjugated momenta satisfy the standard Poisson algebra (3.25).

The second T-dualization, along y direction, produces nongeometrical theory, in literature known as Q flux theory. The metric is the same as initial one and Kalb-Ramond field have the same form as initial up to minus sign. But, this theory has just local geometrical interpretation. We obtained that, in approximation linear in H, the transformation law in canonical form is again trivial, $\pi_y \cong \kappa y_2'$, where π_y is y component od the canonical momentum of the initial theory and y_2 is coordinate T-dual to y. As a consequence of the standard Poisson algebra (3.25), we conclude that Q flux theory is still commutative. This result seems to be opposite from the result of the reference [16] where in detailed calculation it is shown that Q flux theory is noncommutative. The difference is in the so called boundary condition i.e. winding condition. In the Ref. [16] they imposed nontrivial winding condition which mixes the coordinates and their T-dual partners (condition given in Eq. (C.18) of Ref. [16]) and the result is noncommutativity. In this article the trivial winding condition is imposed on x and y coordinates. The consequence is that Q flux theory is commutative. But as it is written in Ref. [16] on page 42, "a priori other reasonings could as well be pursued".

T-dualizing along coordinate z using the machinery of the generalized T-dualization procedure[20,21,23] we obtain the nonlocal theory (theory with R flux) and nontrivial transformation law in canonical form. Non-locality stems from the fact that background fields are expressed in terms of the variable ΔV which is defined as line integral. On the other side, dependence of the Kalb-Ramond field on z coordinate produces the $\beta^{\pm}(x,y)$ functions and nontrivial transformation law for π_z. Consequently, coordinate dependent background gives non-locality and, further, nonzero Poisson brackets of the T-dual coordinates. We can claim that there is a correlation between non-locality (R-flux theory) and closed string noncommutativity and nonassociativity. In addition, nonzero Poisson bracket implies nonzero Jacobi identity which is a signal of nonassociativity.

From the expressions (4.11), (4.12) and (4.21) it follows that parameters of noncommutativity and nonassociativity are proportional to the field strength H. That means that closed string noncommuatativity and nonassociativity are consequence of the fact that Kalb-Ramond field is coordinate dependent, $B_{xy}=Hz$, where H is an infinitesimal parameter according to the approximation of diluted flux. Using T-duality and trivial winding conditions we obtained noncommutativity relations. The noncommutativity relations are zero if $\sigma=\bar\sigma$ because in noncommuatativity relations function $\theta(\sigma-\bar\sigma)$ is present, which is zero if its argument is zero. This is also at the first glance opposite to the result of Ref. [16], but, having in mind that origin of noncommutativity is not same, this difference is not surprising. If we made a round in sigma choosing $\sigma\to\sigma+2\pi$ and $\bar\sigma\to\sigma$, because of $\theta(2\pi)=1$, we obtained nonzero Poisson brackets. From the relations (4.13) and (4.14) we see that noncommutativity exists even in the case when winding numbers are zero, noncommutativity relations still stand unlike the result in [16]. Consequently, we can speak about some essential noncommutativity originating from non-locality.

We showed that in *ordinary* space coordinate dependent background is a sufficient condition for closed string noncommutativity. Some papers[36] show that noncommutativity is possible even in the constant background case. But that could be realized using the *double space formalism*. At the zeroth order the explanation follows from the fact that transformation law in canonical form is of the form $\pi_\mu \cong \kappa y'_\mu$, where y_μ is T-dual coordinate. Forming double space spanned by $Z^M = (x^\mu, y_\mu)$, we obtained noncommuative (double) space. In literature this kind of noncommutativity is called intrinsic one.

Appendix: Light-Cone Coordinates

In the paper we often use light-cone coordinates defined as

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma). \tag{A.1}$$

The corresponding partial derivatives are

$$\partial_\pm \equiv \frac{\partial}{\partial \xi^\pm} = \partial_\tau \pm \partial_\sigma. \tag{A.2}$$

Two dimensional Levi-Civita $\varepsilon^{\alpha\beta}$ is chosen in (τ, σ) basis as $\varepsilon^{\tau\sigma} = -1$. Consequently, in the light-cone basis the form of tensor is

$$\varepsilon_{lc} = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{A.3}$$

The flat world-sheet metric is of the form in (τ, σ) and light-cone basis, respectively

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta_{lc} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}. \tag{A.4}$$

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Conflict of Interest

The authors have declared no conflict of interest.

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