Naučnom veću Instituta za fiziku

Molba za pokretanje postupka za reizbor u zvanje viši naučni saradnik

S obzirom da ispunjavam kriterijume propisane od strane Ministarstva prosvete, nauke i tehnološkog razvoja Republike Srbije za reizbor u zvanje **viši naučni saradnik**, molim Naučno veće Instituta za fiziku da pokrene postupak za moj reizbor u to zvanje.

U prilogu dostavljam:

- mišljenje rukovodica laboratorije,
- kratku biografiju,
- pregled naučne aktivnosti,
- pregled kvalitativnih pokazatelja,
- pregled kvantitativnih pokazatelja,
- spisak radova i njihove kopije,
- dokaz o citiranosti radova,
- rešenje o prethodnom izboru u zvanje.

U Beogradu, 10.04.2021.

dr Marko Vojinović

Mišljenje rukovodioca laboratorije

Dr Marko Vojinović, član Grupe za gravitaciju, čestice i polja, od 2003. godine je zaposlen u Institutu za fiziku, i bavi se proučavanjem klasičnih i kvantnih aspekata gravitacionog polja. Diplomirao je 2002. godine sa prosekom 9.68 i priznanjem studenta generacije Univerziteta u Beogradu. Magistrirao je 2006. godine a doktorirao 2008. godine pod rukovodstvom dr Milovana Vasilića. U zvanje viši naučni saradnik izabran je 28.09.2016. godine odlukom Ministarstva prosvete, nauke i tehnološkog razvoja.

Preporučujem Naučnom veću Instituta za fiziku da odobri **pokretanje postupka za reizbor dr Marka Vojinovića u zvanje višeg naučnog saradnika** iz sledećih razloga:

1. Ispunjenost **kvantitativnih i kvalitativnih kriterijuma**: zbir poena na osnovu objavljenih radova i učešća na konferencijama znatno premašuje uslove za reizbor u zvanje viši naučni saradnik, koji su propisani od strane Ministarstva. Radovi su objavljeni u vrhunskim međunarodnim časopisima u oblasti fizike visokih energija i gravitacije.

2. Pokretanje **nove istraživačke teme** u Grupi za gravitaciju, čestice i polja: dr Vojinović je, nakon postdoktorskog usavršavanja u Lisabonu, pokrenuo istraživački rad u oblasti *spin-foam* i *spin-cube* modela kvantne gravitacije, i tom temom se sada bavi sa troje studenata (dvoje doktoranata i jedan master student). Na toj temi se bazira i njegov predlog za projekt iz programa IDEJE Fonda za nauku, koji je trenutno na evaluaciji.

3. **Međunarodna saradnja**: dr Vojinović je proveo šest godina kao postdok saradnik u Grupi za matematicku fiziku (GFM) na Univerzitetu u Lisabonu, Portugal, i tamo je uspostavio brojne kontakte sa kolegama koji se bave fizičkim i matematičkim aspektima kvantne gravitacije. Po povratku, dr Vojinović je uspostavio međunarodnu saradnju sa Institutom za kvantnu optiku i kvantne informacije (IQOQI) u Beču, Austrija, čija je posledica i bilateralnim projekt sa Austrijom kojim dr Vojinović rukovodi sa srpske strane.

4. **Samostalnost i kvalitet** naučnog rada: dr Vojinović pokazuje visok nivo samostalnosti u radu, što se ogleda kako u broju stručnih seminara koje je održao, radova koje je publikovao sa svojim studentima, tako i u činjenici da sa vrlo velikim uspehom prezentuje svoj istraživački rad na međunarodnim konferencijama. Ovo se prepoznaje i kroz činjenicu da je na nekoliko međunarodnih naučnih skupova držao predavanja po pozivu organizatora. Dr Vojinović objavljuje radove u časopisima sa visokim impakt faktorima, a citiranost radova je, s obzirom na relativno mali broj istraživačkih grupa koje se bave sličnom problematikom, veoma zadovoljavajuća.

5. **Pedagoški rad i popularizacija** fizike: dr Vojinović je tokom školske 2016/2017. godine bio mentor za master rad Tijane Radenković na Fizičkom fakultetu Univerziteta u Beogradu, a počev od školske 2017/2018. godine je i mentor za njenu doktorsku disertaciju, što se vidi iz zajednički objavljena dva naučna rada:

[1] T. Radenković and M. Vojinović, *JHEP* **10**, 222 (2019),

[2] T. Radenković and M. Vojinović, *Symmetry* **12**, 620 (2020),

i dva izvestaja sa međunarodnih skupova štampana u celini:

[3] T. Radenković and M. Vojinović, *SFIN* **XXXIII**, 251 (2020),

[4] T. Radenković and M. Vojinović, *Ann. Univ. Craiova Phys.* **30**, 74 (2020).

Počev od školske 2020/2021. godine, dr Vojinović je mentor i za master rad Mihaila Đorđevića i doktorsku disertaciju Pavla Stipsića, oba na Fizičkom fakultetu.

Dr Vojinović je održao veliki broj naučno-popularnih predavanja pred publikom različitih profila, na poziv i u organizaciji različitih institucija u zemlji i svetu — od stručnih seminara pa sve do intervjua u naucno-popularnim časopisima, TV emisijama i drugim medijima. Osim toga učestvovao je kao spoljni saradnik u radu Istraživačke stanice Petnica, gde je vodio dvoje polaznika kroz realizaciju jedne istraživačke teme, kao i u Matematičkoj gimnaziji, gde vodi jednog đaka u izradi maturskog rada.

Na osnovu svega gore navedenog kao i na osnovu ličnog uvida u kvalitete i posvećenost radu, preporučujem Naučnom veću Instituta za fiziku da **dr Marku Vojinoviću odobri pokretanje postupka za reizbor u zvanje viši naučni saradnik**. Konačno, predlažem sledeće članove komisije za reizbor:

1. dr Branislav Cvetković, naučni savetnik Instituta za fiziku u Beogradu
2. dr Igor Salom, viši naučni saradnik Instituta za fiziku u Beogradu
3. dr Voja Radovanović, redovni profesor Fizičkog fakulteta Univerziteta u Beogradu.

S poštovanjem,

U Beogradu,
06.04.2021.

dr Branislav Cvetković
rukovodilac Grupe za gravitaciju, čestice i polja

Biografija

- **Ime i prezime:** Marko Vojinović
- **Datum i mesto rođenja:** 28.03.1978. Pančevo, Srbija
- **Obrazovanje:**
  - 1997. završio Matematičku Gimnaziju u Beogradu
  - 2002. diplomirao na Fizičkom fakultetu Univerziteta u Beogradu, smer Teorijska i eksperimentalna fizika, sa temom "*Simetrija dualnosti u Born-Infeldovoj elektrodinamici*"
  - 2006. magistrirao na Fizičkom fakultetu Univerziteta u Beogradu, smer Teorijska fizika elementarnih čestica i gravitacije, sa temom "*Kretanje klasične strune u zakrivljenom prostorvremenu*"
  - 2008. doktorirao na Fizičkom fakultetu Univerziteta u Beogradu, smer Teorijska fizika elementarnih čestica i gravitacije, sa temom "*Kretanje ekstendiranih objekata u gravitacionom polju sa torzijom*"
- **Afilijacije:**
  - 10/1997–07/2008: student teorijske fizike na Fizičkom fakultetu Univerziteta u Beogradu
  - 10/2006–03/2009: istraživač u Grupi za gravitaciju, čestice i polja Instituta za fiziku u Beogradu
  - 03/2009–03/2012: postdok istraživač u Grupi za matematičku fiziku Univerziteta u Lisabonu
  - 03/2012–03/2013: naučni saradnik u Grupi za gravitaciju, čestice i polja Instituta za fiziku u Beogradu
  - 03/2013–03/2016: postdok istraživač u Grupi za matematičku fiziku Univerziteta u Lisabonu
  - 03/2016–10/2016: naučni saradnik u Grupi za gravitaciju, čestice i polja Instituta za fiziku u Beogradu
  - nakon 10/2016: viši naučni saradnik u Grupi za gravitaciju, čestice i polja Instituta za fiziku u Beogradu
- **Stečena zvanja:**
  - 2002–2007 *Istraživač pripravnik* u Institutu za fiziku u Beogradu
  - 2007–2009 *Istraživač saradnik* u Institutu za fiziku u Beogradu
  - 2009–2014 *Naučni saradnik* u Institutu za fiziku u Beogradu
  - 2016–2021 *Viši naučni saradnik* u Institutu za fiziku u Beogradu

- **Grantovi:**
  - 04/2008–07/2008: Marie Curie Research Training Network grant (EU FP6, MRTN-CT-2004-005104)
  - 03/2009–03/2012: Portugese postdoctoral fellowship grant (FCT, SFRH-BPD-46376-2008)
  - 03/2013–03/2016: Portugese postdoctoral fellowship grant (FCT, SFRH-BPD-46376-2008)
  - 11/2015: Short Term Scientific Mission grant (STSM, COST Action MP1405)
  - 10/2017–12/2017: Joint Excellence in Science and Humanities grant (JESH, Austrian Academy of Sciences)

- **Angažman na projektima:**
  - Od 01.11.2003. zaposlen na Institutu za Fiziku u Centru za teorijsku fiziku, u Grupi za gravitaciju, čestice i polja
  - Od 01.11.2003. angažovan na projektu 101486 "*Gradijentne teorije gravitacije — simetrije i dinamika*", Ministarstva za nauku i tehnološki razvoj
  - Od 01.01.2006. angažovan na projektu 141036 "*Alternativne teorije gravitacije*", Ministarstva za nauku i zaštitu životne sredine
  - Od 01.01.2012. angažovan na projektu 171031 "*Fizičke implikacije modifikovanog prostor-vremena*", Ministarstva za obrazovanje, nauku i tehnološki razvoj
  - Od 01.03.2008. angažovan na projektu "*Constituents, Fundamental Forces and Symmetries of the Universe*", Marie Curie Research Training Network (EU FP6) and INRNE, Sofia, Bulgaria
  - Od 01.03.2009. angažovan kao post-dok istraživač u Grupi za Matematičku Fiziku Univerziteta u Lisabonu, Portugal, sa učešćem na četiri projekta:
    - projekt "*Algebroids, geometry, quantum groups and applications*", Faculty of Sciences and Technology, University of Coimbra, Portugal
    - projekt "*Strategic Project - UI 208 - 2011-2012*", Group of Mathematical Physics, University of Lisbon, Portugal
    - projekt "*Strategic Project - UI 208 - 2013-2014*", Group of Mathematical Physics, University of Lisbon, Portugal
    - projekt "*Quantum Gravity and Quantum Integrable Models - 2015-2016*" (bilateralni projekt između Portugala i Srbije), Group of Mathematical Physics, University of Lisbon, Portugal
  - Od 01.07.2018. **rukovodi projektom** "*Causality in Quantum Mechanics and Quantum Gravity - 2018-2020*" (bilateralni projekt između Austrije i Srbije), Grupa za gravitaciju, čestice i polja, Institut za fiziku u Beogradu
  - Od 01.01.2021. angažovan na projektu "*Symmetries and Quantization - 2020-2022*" (bilateralni projekt između Portugala i Srbije), Grupa za gravitaciju, čestice i polja, Institut za fiziku u Beogradu
  - Od 01.01.2021. angažovan na projektu "*Symmetries and Quantization 2020*", program DIASPORA Fonda za nauku Republike Srbije

Pregled naučne aktivnosti

dr Marka Vojinovića

Kvantizacija gravitacionog polja je jedan od osnovnih nerešenih problema moderne fundamentalne teorijske fizike. Ljudsko znanje o prirodi se oslanja na dva velika stuba teorijske fizike. Sa jedne strane, Ajnštajnova Opšta teorija relativnosti opisuje osnovne osobine prostora, vremena i gravitacije, od svakodnevnih fenomena na Zemlji pa sve do najvećih kosmoloških skala, ukljucujući i Univerzum kao celinu. Sa druge strane, Standardni Model fizike elementarnih čestica opisuje mikrosvet, od atomskih skala pa sve do veoma malih rastojanja koja se mogu meriti u akceleratorskim eksperimentima (reda 10^{-20} m). Obe ove teorije demonstriraju neprikosnoveno slaganje sa svim eksperimentima koje su je ljudska vrsta ikada izvela, svaka teorija u svom domenu primenljivosti.

Međutim, Opšta teorija relativnosti je klasična teorija, dok Standardni Model predstavlja kvantnu teoriju, što dovodi do njihove međusobne protivrečnosti, već na logičkom nivou aksioma dve teorije. Vodeći se principom da prirodni svet oko nas nije protivrečan samom sebi i da se može opisati konzistentnom teorijom, jedan od glavnih zadataka fundamentalne fizike je da na određeni način modifikuje kako Opštu teoriju relativnosti tako i kvantnu teoriju polja (koja uključuje Standardni Model) sa ciljem da ukloni protivrečnost dvaju teorija, čuvajući pritom sve njihove dobre osobine potvrđene eksperimentima.

Ovaj zadatak je krajnje netrivijalan, i postoji mnogo pristupa problemu kvantizacije Ajnštajnove Opšte teorije relativnosti, među kojima se izdvajaju teorija struna i kvantna gravitacija na petljama kao dva najzastupljenija programa. Takođe, u poslednjih nekoliko godina, pojavila se inicijativa za rešavanje problema kvantne gravitacije od strane istraživača iz oblasti zasnivanja kvantne mehanike i kvantne teorije informacija. Dok su teorija struna i kvantna gravitacija na petljama dominantno geometrijski pristupi konstrukciji teorije kvantne gravitacije, ovaj treći pristup je dominantno orijentisan na informacioni opis fizike, pa samim tim i kvantne gravitacije.

Kandidatova naučna aktivnost je povezana sa sva tri programa, i rezultati kandidatovih istraživanja mogu se grubo podeliti u tri grupe.

1. Istraživanje u okviru teorije struna i klasične gravitacije

Centralni rezultati kandidatovog istraživanja u okviru teorije struna i klasične gravitacije objavljeni su u šest radova:

[1] M. Vasilić, M. Vojinović, "Classical String in Curved Backgrounds", *Phys. Rev. D* **73**, 124013 (2006).

[2] M. Vasilić, M. Vojinović, "Classical Spinning Branes in Curved Backgrounds", *JHEP* 07(2007)028.

[3] M. Vasilić, M. Vojinović, "Zero-size Objects in Riemann-Cartan Spacetime", *JHEP* 08(2008)104.

[4] M. Vasilić, M. Vojinović, "Interaction of particle with the string in pole-dipole approximation", *Fortschr. Phys.* **56**, 542 (2008).

[5] M. Vasilić, M. Vojinović, "Spinning branes in Riemann-Cartan spacetime", *Phys. Rev. D* **78**, 104002 (2008).

[6] M. Vasilić, M. Vojinović, "Test membranes in Riemann-Cartan spacetimes", *Phys. Rev. D* **81**, 024025 (2010).

Radovi [1-6], kao i nekoliko drugih radova (iz raznih štampanih izveštaja sa međunarodnih i domaćih konferencija), predstavljaju program istraživanja koji je započet kroz kandidatov magistarski rad i doktorsku disertaciju. Centralni rezultat radova [1,2,4,5] predstavljaju najopštije realistične jednačine kretanja p-dimenzionalne brane u D-dimenzionalnom prostorvremenu čija geometrija sadrži netrivijalnu krivinu, odnosno krivinu i torziju. Ovo je vrlo važan rezultat, jer daje nov uvid u interpretaciju interakcije materije sa krivinom i torzijom. Rad [6] se nadovezuje na ove rezultate i daje odgovarajuće tumačenje sigma-modela za 1-branu i 2-branu koji je ugrađen u osnove teorije struna. Rad [3] diskutuje kretanje čestice (0-brane) u prostorvremenu sa krivinom i torzijom, sa specijalnim naglaskom na Dirakovu česticu koja ispoljava neke zanimljive osobine kroz interakciju sa torzijom.

Osim toga, rezultati dobijeni ovim programom istraživanja nalaze primenu i u drugim oblastima fizike, kao što je na primer hidrodinamika ne-Njutnovskih fluida.

2. Istraživanje u okviru kvantne gravitacije na petljama

Najznačajniji rezultati kandidatovog dosadašnjeg istraživanja u okviru kvantne gravitacije na petljama objavljeni su u sledećih 11 radova:

[7] A. Miković, M. Vojinović, "Large-spin asymptotics of Euclidean LQG flat-space wavefunctions", *Adv. Theor. Math. Phys.* **15**, 801 (2011).

[8] A. Miković, M. Vojinović, "Effective action and semiclassical limit of spin foam models", *Class. Quant. Grav.* **28**, 225004 (2011).

[9] A. Miković, M. Vojinović, "Poincaré 2-group and quantum gravity", *Class. Quant. Grav.* **29**, 165003 (2012).

[10] A. Miković, M. Vojinović, "A finiteness bound for the EPRL/FK spin foam model", *Class. Quant. Grav.* **30**, 035001 (2013).

[11] M. Vojinović, "Cosine problem in EPRL/FK spinfoam model", *Gen. Relativ. Gravit.* **46**, 1616 (2014).

[12] A. Miković, M. Vojinović, "Solution to the Cosmological Constant Problem in a Regge Quantum Gravity Model", *Europhys. Lett.* **110**, 40008 (2015).

[13] A. Miković, M. A. Oliveira, M. Vojinović, "Hamiltonian analysis of the $BFCG$ theory for the Poincaré 2-group", *Class. Quant. Grav.* **33**, 065007 (2016).

[14] M. Vojinović, "Causal Dynamical Triangulations in the Spincube Model of Quantum Gravity", *Phys. Rev. D* **94**, 024058 (2016).

[15] A. Miković, M. A. Oliveira, M. Vojinović, "Hamiltonian analysis of the $BFCG$ formulation of general relativity", *Class. Quant. Grav.* **36**, 015005 (2019).

[16] T. Radenković, M. Vojinović, "Higher gauge theories based on 3-groups", *JHEP* 10(2019)222.

[17] T. Radenković, M. Vojinović, "Hamiltonian Analysis for the Scalar Electrodynamics as 3BF Theory", *Symmetry* **12**, 620 (2020).

Radovi [7,8,10,11] se bave tzv. *spin-foam* formalizmom za kvantizaciju gravitacije, konkretno otvorenim problemima konačnosti kvantne teorije gravitacije i njenog semiklasičnog limita. U tom smislu ovi radovi su imali veliki doprinos kako razumevanju ova dva problema, tako i metodama za njihovo rešavanje. U radu [10] je dokazano da problem konačnosti teorije može da se reši uspešno odgovarajućom redefinicijom mere u funkcionalnom integralu gravitacionog polja, i da ta redefinicija ne narušava klasičan limes teorije. Ovo je jedan od prvih rezultata te vrste uopšte. Radovi [7,8,11] donose nov uvid u pitanje semiklasičnog limesa kvantne gravitacije, i obezbeđuju efektivan metod za određivanje oblika klasične teorije u zadatoj aproksimaciji. Ovo je takođe prvi rezultat te vrste. Kao dodatni rezultat, pokazano je da u teoriji u kojoj je u kvantnom režimu narušena simetrija difeomorfizama, u klasičnom limesu teorije ova simetrija se ponovo uspostavlja, čime je rešena jedna dugogodišnja zagonetka odnosa diskretizovanog i glatkog prostorvremena.

Radovi [9,12] se bave kategorijskom generalizacijom spin-foam modela na tzv. *spincube* modele kvantne gravitacije. Ovim postupkom se efikasno rešava problem kombinovanja gravitacije sa ostalim interakcijama i fermionskom materijom, što je takođe prvi rezultat te vrste. U spin-foam modelima koji su dosad izučavani u literaturi bilo je efektivno dokazano da fermionsku materiju nije nikako moguće uključiti u formalizam teorije, i ovo je bio veliki problem svih razmatranih spin-foam modela. Kategorijska generalizacija spin-foam modela na spincube modele na neočekivan ali elegantan način prevazilazi ovaj problem, i samim tim predstavlja mnogo bolju osnovu za izgradnju realistične teorije kvantne gravitacije. Istovremeno, teorija sugeriše nov pristup rešavanju fundamentalnog problema kosmološke konstante, a otvara se i mogućnost unifikacije gravitacije sa ostalim interakcijama — mogućnost koja je bila potpuno nedostižna u svim dosadašnjim modelima kvantne gravitacije. Kanonska struktura spincube modela je izučena u radovima [13,15], dok je rad [14] posvećen vezi između spincube modela i jednog tehnički sličnog ali konceptualno različitog pristupa kvantizaciji gravitacionog polja, pod imenom *kauzalne dinamičke triangulacije* (CDT). Uočen je zanimljiv rezultat da se CDT modeli mogu razumeti kao specijalan slučaj spincube modela, što na izvestan način povezuje dva pristupa kvantnoj gravitaciji u jednu zajedničku celinu.

Tokom proučavanja spincube modela uočena je potreba da se izvrši još jedna kategorijska generalizacija, ovaj put prelaskom sa algebarskog pojma grupe (koja odgovara spin-foam modelima) i 2-grupe (koja odgovara spincube modelima) na pojam 3-grupe. Ovo novo uopštenje omogućava da se ne samo gravitaciono, nego i sva ostala polja u prirodi opišu jednom jedinom unificiranom algebarskom strukturom 3-grupe. Odgovarajući model klasične gravitacije kuplovan sa Standardnim Modelom elementarnih čestica baziran na 3-grupi formulisan je u radu [16], i predstavlja pravu polaznu osnovu za kvantizaciju svih polja u prirodi na jedinstven način. Rezultati ovog istraživanja predstavljeni su predavanjem po pozivu na međunarodnom naučnom skupu u Krajovi (Rumunija) u septembru 2020. godine, i štampani u celini:

[18] T. Radenković, M. Vojinović, “Quantum gravity and elementary particles from higher gauge theory”, *Ann. Univ. Craiova Phys.* **30**, 74 (2020).

Nakon formulisanja ovog novog modela, počelo je izučavanje njegovih osobina. Rad [17] predstavlja prvi korak u tom pravcu, i bavi se analizom kanonske strukture specijalnog slučaja modela gravitacije kuplovane sa skalarnom elektrodinamikom.

Rezultati [15,16,17] ostvareni su nakon kandidatovog izbora u prethodno zvanje.

3. Istraživanje u okviru informacione kvantne gravitacije

Najznačajniji dosadašnji rezultati kandidatovog istraživanja u okviru ove informacione kvantne gravitacije, koja je vrlo mlada oblast istraživanja, objavljeni su u sledeća tri rada:

[19] N. Paunković, M. Vojinović, “Gauge protected entanglement between gravity and matter”, *Class. Quant. Grav.* **35**, 185015 (2018).

[20] F. Pipa, N. Paunković, M. Vojinović, “Entanglement-induced deviation from the geodesic motion in quantum gravity”, *Jour. Cosmol. Astropart. Phys.* **09**, 057 (2019).

[21] N. Paunković, M. Vojinović, “Causal orders, quantum circuits and spacetime: distinguishing between definite and superposed causal orders”, *Quantum* **4**, 275 (2020).

Rad [19] se bavi pojavom kvantne upletenosti u kontekstu kvantne gravitacije. Ispostavlja se da grupa difeomorfizama, kao gejdž simetrija gravitacije, nameće ograničenja na moguća kvantna stanja gravitacionog polja i materije, koja praktično eliminišu sva separabilna stanja, ostavljajući isključivo upletena stanja kao moguće početne uslove za bilo kakvu dinamiku. U radu [20] se razmatra kretanje čestice u kvantnoj superpoziciji dve različite konfiguracije gravitacionog polja. Prisustvo drugog gravitacionog polja skreće česticu sa geodezijske putanje u odnosu na prvo gravitaciono polje, dovodeći do narušenja slabog principa ekvivalencije u kvantnoj gravitaciji. Rad [21] proučava tzv. *kvantni prekidač*, kvantni protokol u kome se redosled operacija nad kvantnim sistemom stavlja u superpoziciju. Analizira se prostorvremenski opis kvantnog prekidača u kontekstu kauzalnog poretka događaja. Takođe se uvodi pojam tzv. *gravitacionog prekidača*, koji implementira kvantni prekidač koristeći superpoziciju dve konfiguracije gravitacionog polja. Ispostavlja se da gravitacioni prekidač ispoljava kvalitativno nove osobine, koje ne postoje u slučaju običnog kvantnog prekidača.

Rezultati [19,20,21] ostvareni su nakon kandidatovog izbora u prethodno zvanje.

Pregled kvalitativnih pokazatelja

naučnoistraživačkog rada Marka Vojinovića

1. Kvalitet naučnih rezultata

1.1. Naučni nivo i značaj rezultata, uticaj naučnih radova

Dr Marko Vojinović je u svom dosadašnjem radu objavio preko 30 naučnih publikacija, među kojima 1 rad u kategoriji M21a, 18 u kategoriji M21, 1 u kategoriji M22, 2 u kategoriji M31, 8 u kategoriji M33, kao i dva zbornika radova sa međunarnodnih skupova, kategorija M36.

Od toga, u periodu nakon izbora u prethodno zvanje, dr Vojinović je objavio 1 rad u kategoriji M21a, 4 u kategoriji M21, 1 u kategoriji M22, 1 u kategoriji M31, 2 u kategoriji M33, i 2 u kategoriji M36.

Kao pet najznačajnijih radova kandidata mogu se uzeti:

[1] M. Vasilić and M. Vojinović, "Classical Spinning Branes in Curved Backgrounds", *JHEP* **07** 028 (2007), citiran 13 puta.

[2] A. Miković and M. Vojinović, "Effective action and semiclassical limit of spin foam models", *Class. Quant. Grav.* **28**, 225004 (2011), citiran 7 puta.

[3] M. Vojinović, "Causal Dynamical Triangulations in the Spincube Model of Quantum Gravity", *Phys. Rev. D* **94**, 024058 (2016), citiran 1 put.

[4] T. Radenković and M. Vojinović, "Higher gauge theories based on 3-groups", *JHEP* **10**, 222 (2019), citiran 2 puta.

[5] N. Paunković and M. Vojinović, "Causal orders, quantum circuits and spacetime: distinguishing between definite and superposed causal orders", *Quantum* **4**, 275 (2020), citiran 3 puta.

Prvi rad predstavlja reprezentativan deo programa istraživanja koji je započet kroz kandidatov magistarski rad i doktorsku disertaciju. Centralni rezultat predstavljaju najopštije realistične jednačine kretanja p-dimenzionalne brane u D-dimenzionalnom prostorvremenu čija geometrija sadrži netrivijalnu krivinu. Ovo je vrlo važan rezultat, jer daje nov uvid u interpretaciju interakcije materije sa krivinom, kao i odgovarajuće tumačenje sigma-modela za 1-branu i 2-branu koji je ugrađen u osnove teorije struna. Takođe su detaljno proučene sve simetrije koje imaju dobijene jednačine kretanja. Rezultati dobijeni u ovom radu nalaze primenu čak i u drugim oblastima fizike, kao što je na primer hidrodinamika ne-Njutnovskih fluida.

Drugi rad se bavi spin-foam formalizmom za kvantizaciju gravitacije, konkretno otvorenim problemom semiklasičnog limita spin-foam modela kvantne gravitacije. U tom smislu ovaj rad je imao veliki doprinos kako razumevanju ovog problema, tako i metodama za njegovo rešavanje. Rad uvodi nov efektivan metod za određivanje oblika klasične teorije u zadatoj aproksimaciji, baziran na pojmu efektivnog dejstva u kvantnoj teoriji polja. Ovo je prvi rezultat te vrste. Kao dodatni rezultat, pokazano je da u teoriji u kojoj je u kvantnom režimu narušena simetrija difeomorfizama, u klasičnom limesu teorije ova simetrija se ponovo uspostavlja, čime je rešena jedna dugogodišnja zagonetka odnosa diskretizovanog i glatkog prostorvremena.

Treći rad razmatra neočekivanu vezu između dva bliska ali različita pristupa kvantovanju gravitacionog polja — spincube modela sa jedne strane, i modela kauzalnih dinamičkih triangulacija (CDT) sa druge strane. Ova dva pristupa su nastala nezavisno jedan od drugog, iz različitih premisa o dinamici gravitacionog polja. Zato je sasvim neočekivan i veoma značaja rezultat da se CDT pristup može videti kao specijalan slučaj spincube pristupa kvantnoj gravitaciji. U radu se ta veza eksplicitno demonstrira, i razmatraju njene posledice. Takođe se diskutuju i neki modeli uopštenja CDT pristupa, bazirani na drugim klasama specijalnih slučajeva spincube modela. Značaj ovog rada se dakle najviše ogleda u ujedinjavanju dva različita pristupa kvantnoj gravitaciji u jedinstven matematički formalizam.

Četvrti rad se bavi kategorijskim uopštenjem spin-foam i spincube modela kvantne gravitacije na modele bazirane na pojmu 3-grupe. Ovim postupkom se efikasno rešava problem kombinovanja gravitacije sa ostalim interakcijama i fermionskom materijom. U spin-foam modelima koji su dosad izučavani u literaturi bilo je efektivno dokazano da fermionsku materiju nije nikako moguće uključiti u formalizam teorije, i ovo je bio veliki problem svih razmatranih spin-foam modela. Kategorijska generalizacija spin-foam modela na spincube modele je ukazala na put kojim je moguće prevazići ovaj problem, dok je uopštenje na strukturu 3-grupe realizovalo tu ideju do kraja, definišući mnogo bolju osnovu za izgradnju realistične teorije kvantne gravitacije, uz unifikaciju gravitacije sa ostalim poljima prisutnim u Standardnom Modelu elementarnih čestica.

Peti rad je posvećen pojmu kauzalnosti u tzv. informacionom pristupu kvantnoj gravitaciji. Informaciona kvantna gravitacija je pristup konstrukciji kvantne teorije gravitacije sa tačke gledišta kvantne teorije informacija. U radu se razmatra tzv. *kvantni prekidač*, protokol koji stavlja u superpoziciju dva različita poretka operacija nad kvantnim sistemom. U literaturi je postojala interpretacija koja ulogu kvantnog prekidača posmatra kao superpoziciju dva različita kauzalna poretka između događaja, i ovaj rad detaljno analizira konceptualnu razliku između pojma događaja kao interakcije agenta sa kvantnim sistemom (kvantno-informacioni pristup) i pojma događaja kao prostorvremenske tačke (geometrijski pristup). Osnovni rezultat rada je da ova dva pojma događaja nisu ekvivalentna, i eksplicitno je konstruisana opservabla koja ih razlikuje, na primeru kvantnog prekidača. Za ovu analizu je uveden i nov pojam *gravitacionog prekidača*, protokola nad kvantnim sistemom koji se odvija u superpoziciji dve različite konfiguracije gravitacionog polja. Dokazano je da je samo u slučaju gravitacionog prekidača moguće poistovetiti dva pojma događaja na konzistentan način, i da se samo u tom slučaju može zaista govoriti o pravoj superpoziciji kauzalnih poredaka.

1.2. Pozitivna citiranost naučnih radova

Prema bazi podataka *Web of Science*, radovi kandidata su citirani ukupno 124 puta, od toga 74 puta ne računajući samocitate. Prema istoj bazi, Hiršov indeks kandidata je 6. Relevantni podaci o citiranosti sa internet stranice *Web of Science* baze dati su u prilogu, nakon spiska svih radova.

1.3. Parametri kvaliteta časopisa

Važan element za procenu kvaliteta naučnih rezultata je i kvalitet časopisa u kojima su radovi objavljeni, odnosno njihov *impakt faktor*. Časopisi u kojima je kandidat objavljivao radove i koji imaju impakt faktor pripadaju kategorijama M21a, M21 i M22, i to su sledeći (podvučeni su brojevi koji se odnose na period nakon prethodnog izbora u zvanje):

- *Quantum* — 1 rad, IF=5,381.
- *Journal of High Energy Physics* — 1+2 rada, IF=5,875+5,659+5,659.
- *Journal of Cosmology and Astroparticle Physics* — 1 rad, IF=5,524.
- *Classical and Quantum Gravity* — 2+4 rada, IF=3,487+3,487+3.119+3,562+3,562+3,320.
- *Physical Review D* — 4 rada, IF=4,557+5,050+5,050+4,896.
- *Europhysics Letters* — 1 rad, IF=2,095.
- *General Relativity and Gravitation* — 1 rad, IF=1,902.
- *Advances in Theoretical and Mathematical Physics* — 1 rad, IF=2,034.
- *Fortschritte der Physik* — 1 rad, IF=2,007.
- *Symmetry* — 1 rad, IF=2,645.

Ukupan impakt faktor radova kandidata je 78,691, a u periodu nakon prethodnog izbora u zvanje iznosi 26,399. Časopisi u kojima je kandidat objavljivao radove su po svom ugledu veoma cenjeni u oblastima kojima pripadaju. Među njima se posebno ističu: *Quantum, Journal of High Energy Physics, Physical Review D, Classical and Quantum Gravity* i *Journal of Cosmology and Astroparticle Physics.*

Dodatni bibliometrijski pokazatelji u vezi sa objavljenim radovima kandidata, nakon odluke o prethodnom izboru u zvanje, dati su u donjoj tabeli. Ona sadrži impakt faktore (IF) radova, bodove radova po domaćoj kategorizaciji (M20) naučnoistraživačkih rezultata, kao i impakt faktore normalizovane po impaktu citirajućeg članka (SNIP). U tabeli su date ukupne vrednosti, kao i srednje vrednosti po članku i po broju autora.

	IF	M	SNIP*
Ukupno	26,399	47	5.758
Usrednjeno po članku	4.400	7,833	1.152
Usrednjeno po autoru	11,698	20.833	2.518

*Časopis *Quantum* ima kategoriju M21a i impakt faktor 5,381, ali mu još uvek nije dodeljen SNIP, s obzirom da je počeo da izlazi tek od 2017. godine.

1.4. Stepen samostalnosti i stepen učešća u realizaciji rezultata

U oblasti istraživanja kojom se kandidat bavi uobičajeno je da se autori potpisuju na radove abecednim redom, i ne postoji koncept prvog autora. Ovo je praksa u radovima iz teorijske fizike visokih energija, i primenjena je sistematski u svim kandidatovim radovima. U tom smislu, kandidatov doprinos u objavljenim radovima treba razumeti kao potpuno ravnopravan između svih potpisanih autora.

Takođe, od ukupno 20 objavljenih radova u kategorijama M21a, M21 i M22, kandidat je objavio 2 rada samostalno, 15 radova sa jednim koautorom, i 3 rada sa dva koautora. Od toga, u svim radovima sa dva koautora jedan od koautora je saradnik, a drugi koautor je student (master odnosno doktorskih studija). U periodu nakon prethodnog izbora u zvanje, kandidat je objavio ukupno 6 radova u kategorijama M21a, M21 i M22, pri čemu su 2 rada objavljena sa po jednim saradnikom, 2 rada sa saradnikom i njegovim studentom, i 2 rada sa kandidatovim mlađim saradnikom (student doktorskih studija, T. Radenković).

Budući da su svi radovi objavljeni sa najviše dva koautora, doprinos kandidata izradi svakog od radova je značajan. Kandidat je učestvovao u osmišljavanju, formulaciji i diskusiji problema, analitičkim proračunima i samom pisanju radova. Dodatno, samostalnost kandidata se ogleda kako u dva rada koja je objavio sa svojim mlađim saradnikom (u časopisima *Journal of High Energy Physics* i *Symmetry*), tako i u dva rada koja je kandidat objavio kao jedini autor (u časopisima *Physical Review D* i *General Relativity and Gravitation*).

2. Angažovanost u formiranju naučnih kadrova

U periodu nakon prethodnog izbora u zvanje, kandidat ima tri mlađa saradnika: Tijana Radenković (student doktorskih studija počev od školske 2017/2018. godine), Pavle Stipsić (student doktorskih studija počev od školske 2020/2021. godine), i Mihailo Đorđević (student master studija školske 2020/2021. godine). Kandidatova uloga kao mentora za ove studente tek treba da se ozvaniči[1] odlukom Naučno-nastavnog veća Fizičkog fakulteta Univerziteta u Beogradu, nakon odbrane njihovih tema za doktorske disertacije.

Pritom, do sada je kandidat sa jednim od mlađih saradnika objavio dva rada (M21, M22),

- T. Radenković and M. Vojinović, "Higher gauge theories based on 3-groups", *JHEP* **10**, 222 (2019).
- T. Radenković and M. Vojinović, "Hamiltonian Analysis for the Scalar Electrodynamics as 3BF Theory", *Symmetry* **12**, 620 (2020).

kao i dva saopštenja sa međunarodnih skupova štampana u celini (M31, M33),

- T. Radenković and M. Vojinović, "Quantum gravity and elementary particles from higher gauge theory", *Ann. Univ. Craiova Phys.* **30**, 74 (2020).
- T. Radenković and M. Vojinović, "Construction and examples of higher gauge theories", *SFIN* **XXXIII**, 251 (2020).

Osim toga, Tijana Radenković je pod kandidatovim rukovodstvom odbranila svoju master tezu školske 2016/2017. godine, na Fizičkom fakultetu Univerziteta u Beogradu. Naslovna strana master teze i strana sa zahvalnicom date su u prilogu.

[1]Prema članu 49 važećeg Pravilnika o sticanju istraživačkih i naučnih zvanja ("Službeni glasnik RS" broj 159 od 30.12.2020.), za izbor odnosno reizbor u zvanje viši naučni saradnik formalno mentorstvo za doktorsku disertaciju nije neophodan kriterijum.

U okviru pedagoškog rada, kandidat je bio mentor za istraživački rad dvoje polaznika u Petnici, iz oblasti gravitacionih talasa. Na osnovu tog istraživanja su polaznici objavili rad

- D. Cvijetić, M. Stošić, "Simulacija idealnih detektora gravitacionih talasa", *Petničke Sveske* **78**, 67 (2019).

Prva strana njihovog rada, na kojoj je imenovan i dr Vojinović kao mentor, nalazi se u prilogu.

U periodu pre prethodnog izbora u zvanje, tokom 2013-2015. godine, kandidat je bio komentor doktorske disertacije Migela Angela Oliveire na Univerzitetu u Lisabonu. Disertacija se delom bazira na zajedničkom radu:

- A. Miković, M. A. Oliveira and M. Vojinović, "Hamiltonian analysis of the $BFCG$ theory for the Poincaré 2-group", *Class. Quant. Grav.* **33**, 065007 (2016).

U prilogu se nalaze naslovna strana i strana sa zahvalnicom iz njegove doktorske disertacije, kao i izjava glavnog mentora, dr Aleksandra Mikovića.

Takođe, dr Vojinović se bavio i pedagoškim radom. U periodu 01.09.2012.–30.06.2013. godine bio je u svojstvu spoljnjeg saradnika angažovan kao predavač predmeta **fizika** u Matematičkoj Gimnaziji u Beogradu. U toku tog perioda je bio mentor za šest maturskih radova učenika Matematičke Gimnazije, i uručeno mu je priznanje za uspehe njegovih učenika postignute na 51. državnom takmičenju iz fizike za učenike srednjih škola. Priznanje i ugovor o izvođenju nastave se nalaze u prilogu.

3. Normiranje broja koautorskih radova

Kandidat je u periodu od prethodnog izbora u zvanje objavio ukupno 11 publikacija, od toga 1 rad kategorije M21a, 4 rada kategorije M21, 1 rad kategorije M22, 1 rad kategorije M31, 2 rada kategorije M33 i dva zbornika radova kategorije M36. Pritom, sve publikacije su objavljene sa jednim odnosno dva koautora. U skladu sa pravilima, radovi sa ukupno dva odnosno tri autora se ne normiraju, pa je efektivni broj radova u ovom slučaju jednak ukupnom broju radova.

4. Rukovođenje projektima, potprojektima i projektnim zadacima

U periodu nakon prethodnog izbora u zvanje, kandidat rukovodi bilateralnim projektom između Srbije i Austrije, koji sa srpske strane finansira Ministarstvo prosvete, nauke i tehnološkog razvoja. Projekt nosi naziv "Kauzalnost u kvantnoj mehanici i kvantnoj gravitaciji", evidencioni broj 451-03-02141/2017-09/02, i počeo je sa radom 01.07.2018. godine, sa trajanjem od dve godine. Pritom, zbog posledica COVID-19 pandemije, Ministarstvo je donelo odluku (dva puta) da se trajanje projekta produži dok se ponovo ne steknu uslovi za mogućnost putovanja. U tom smislu, projekt još uvek traje, sa trenutno predviđenim završetkom 01.07.2021. godine.

Kao dokaz rukovođenja projektom, u prilogu se nalazi izveštaj poslat Ministarstvu nakon uspešnog završetka prve godine projekta, kao i imejl obaveštenje iz Ministarstva o produžavanju trajanja projekta do 01.07.2021. godine.

Osim rukovođenja, dr Vojinović učestvuje kao član u još dva aktuelna projekta. Prvi je jednogodišnji projekt pod nazivom "Symmetries and Quantization 2020" iz programa DIASPORA Fonda za nauku Republike Srbije, evidencioni broj 6427195, koji je počeo sa radom 01.01.2021. godine. Drugi je dvogodišnji bilateralni projekt između Srbije i Portugala, koji sa srpske strane finansira Ministarstvo prosvete, nauke i tehnološkog razvoja, pod nazivom "Simetrije i kvantizacija", evidencioni broj 337-00-00227/2019-09/57, koji je takođe počeo sa radom 01.01.2021. godine.

U periodu pre prethodnog izbora u zvanje, kandidat je bio rukovodilac za tri zadatka na tri međunarodna projekta, u periodu od 2009. do 2014. godine:

- zadatak "Kvantne grupe i geometrija" na projektu "Algebroids, geometry, quantum groups and applications" Univerziteta u Koimbri, Portugal (2009–2012),
- zadatak "Kvantna gravitacija" na projektu "Strategic Project - UI 208 - 2011-2012" Grupe za Matematičku Fiziku Univerziteta u Lisabonu (2011–2012),
- zadatak "Kvantna gravitacija" na projektu "Strategic Project - UI 208 - 2013-2014" Grupe za Matematičku Fiziku Univerziteta u Lisabonu (2013–2014).

Sva tri projekta finansirala je portugalska nacionalna Fondacija za Nauku i Tehnologiju (FCT). Potvrde o rukovođenju su date u prilogu.

5. Aktivnost u naučnim i naučno-stručnim društvima

Kandidat je recenzent u sledećim među narodnim naučnim časopisima:

- *Classical and Quantum Gravity*,
- *Foundations of Physics*,
- *Symmetry, Integrability and Geometry: Methods and Applications*,
- *Axioms.*

U prilogu se nalaze pisma uredništva svakog od časopisa kandidatu sa pozivima za recenzije.

6. Uticaj naučnih rezultata

Uticajnost naučnih rezultata kandidata navedena je u odeljku 1.2 ovog dokumenta. Pun spisak radova je dat u prilogu, kao i podaci o citiranosti svakog od radova, preuzeti sa internet stranice baze *Web of Science.*

Imajući u vidu da su radovi iz fundamentalne teorijske fizike, ostvaren broj citata (ukupno 124, bez samocitata 74, Hiršov indeks 6) smatra se veoma zadovoljavajućim za oblast istraživanja i teme kojima se kandidat bavi.

7. Doprinos realizaciji radova u naučnim centrima u zemlji i inostranstvu

Kandidat je značajno doprineo svakom radu koji je objavio. Svi radovi objavljeni u periodu nakon prethodnog izbora u zvanje urađeni su sa saradnicima iz inostranstva i sa mlađim kolegama (studentima master i doktorskih studija). Dr Vojinović je imao ključan doprinos u svim publikacijama, bitno je uticao na tok istraživanja tokom izrade radova, učestvovao je u analitičkim proračunima, metodima i tehnikama pristupa problemima, pisanju teksta radova, kao i u komunikaciji sa recenzentima prilikom postupka objavljivanja.

Vezano za dva publikovana zbornika radova sa međunarodnih skupova (kategorija M36), dr Vojinović je učestvovao u selekciji i recenzijama radova obuhvaćenih zbornicima, tehničkoj obradi teksta i pripremi zbornika za publikovanje, kao i pisanju uvoda i ostalih delova zbornika.

8. Međunarodna naučna saradnja

U periodu nakon prethodnog izbora u zvanje, dr Vojinović je tokom dva meseca (u periodu 16.10.–15.12.2017.) boravio u Beču, gostujući u grupi prof. Časlava Bruknera u Institutu za kvantnu optiku i kvantne informacije (IQOQI) Univerziteta u Beču, kao dobitnik JESH granta (Joint Excellence in Science and Humanities) austrijske Akademije nauka. Tokom tog gostovanja, ostvarena je značajna saradnja između Grupe za gravitaciju, čestice i polja Instituta za fiziku u Beogradu i grupe prof. Bruknera u institutu IQOQI. Iako je vreme od dva meseca bilo prekratko za objavljivanje zajedničkih radova, značajan rezultat te saradnje je bilateralni projekt između Srbije i Austrije koji je dobijen 2018. godine i kojim dr Vojinović rukovodi. Zajednički naučnoistraživački rad dvaju grupa je još uvek u toku, i nastaviće se u budućnosti.

U prilogu se nalazi pozivno pismo prof. Antona Cajlingera, predsednika austrijske Akademije nauka, kao i izveštaj o gostovanju i uspešno ostvarenoj saradnji od prof. Časlava Bruknera, direktora IQOQI instituta.

U periodu pre prethodnog izbora u zvanje, kandidat je tokom 01.03.2009.–01.03.2012. i tokom 01.03.2013.–01.01.2016. godine bio angažovan kao post-doktorski istraživač u Grupi za Matematičku Fiziku (GFM) na Univerzitetu u Lisabonu, Portugal. Tokom ukupno šest godina boravka, bio je angažovan na četiri naučno-istraživačka projekta, tri na Univerzitetu u Lisabonu i jedan na Univerzitetu u Koimbri. Kao rezultat te saradnje, objavio je 6 radova u vrhunskim međunarodnim časopisima (M21), u kolaboraciji sa prof. dr Aleksandrom Mikovićem, redovnim profesorom na Lusofona Univerzitetu u Lisabonu i stalnim članom grupe GFM Univerziteta u Lisabonu. Deklaracija o post-doktorskom angažmanu u GFM grupi nalazi se u prilogu.

9. Pokazatelji uspeha u naučnom radu

U periodu nakon prethodnog izbora i zvanje, kandidat je održao tri predavanja po pozivu na međunarodnim naučnim skupovima:

- *QISS 2020 Workshop*, Hong Kong, Kina, 10.01.–19.01.2020. godine,
- *12-th QFND Workshop*, Krajova, Rumunija, 24.–29.09.2020. godine,
- *SAC-19 Conference*, Beograd, Srbija, 13.-17.10.2020. godine.

Pozivna pisma za sva tri skupa se nalaze u prilogu. Pritom, saopštenje sa međunarodnog skupa u Krajovi je štampano u celini (kategorija M31):

- T. Radenković and M. Vojinović, "Quantum gravity and elementary particles from higher gauge theory", *Ann. Univ. Craiova Phys.* **30**, 74 (2020).

Osim toga, dr Vojinović je po pozivu boravio u gostima na Departmanu za filosofiju Univerziteta u Ženevi, u grupi koju predvodi prof. Christian Wüthrich, u periodu 17.-22.10.2016. godine. Tom prilikom je takođe održao predavanje po pozivu, za članove grupe. Pozivno pismo je dato u prilogu.

Kandidat je učestvovao u organizacionim odborima dva međunarodna naučna skupa,

- *9th Meeting on Modern Mathematical Physics*, 18.–23.09.2017, Beograd, Srbija,

  `[http://www.mphys9.ipb.ac.rs/]`

- *10th Meeting on Modern Mathematical Physics*, 09.–14.09.2019, Beograd, Srbija,

  `[http://www.mphys10.ipb.ac.rs/]`

kao i dva domaća skupa:

- *Workshop on Gravity, Holography, Strings and Noncommutative Geometry*, 01.02.2018, Beograd, Srbija,

  `[http://www.gravity.ipb.ac.rs/GHSNG2018/]`

- *Gravity and String Theory: New ideas for unsolved problems III*, 07.–09.09.2018, Zlatibor, Srbija.

  `[http://www.gst2018.ipb.ac.rs/]`

U periodu pre prethodnog izbora u zvanje, dr Vojinović je održao predavanje po pozivu na međunarodnom simpozijumu *V Petrov International Symposium "High Energy Physics, Cosmology and Gravity"*, Kijev, Ukraina, 29.04.–05.05.2012. godine, koje je štampano u celini. Pozivno pismo direktora Bogoljubovljevog Instituta za Teorijsku Fiziku u Kijevu, akademika A. Zagorodny, dato je u prilogu.

Takođe, dr Vojinović je u periodu 2007–2016. godine učestvovao u organizacionim odborima sledećih 5 međunarodnih skupova:

- *5th Mathematical Physics Meeting: Summer School and Conference on Modern Mathematical Physics*, 06.–17.07.2008, Beograd, Srbija.

  `[http://www.mphys5.ipb.ac.rs/]`

- *Gravity: New ideas for unsolved problems*, 12.–14.09.2011, Divčibare, Srbija.

  `[http://www.gravity2011.ipb.ac.rs/]`

- *Quantum Integrable Systems and Geometry*, 03.–07.09.2012, Oljao, Portugal.

  `[http://www.fctec.ualg.pt/qisg/]`

- *7th Mathematical Physics Meeting: Summer School and Conference on Modern Mathematical Physics*, 09.–19.09.2012, Beograd, Srbija.

  [http://www.mphys7.ipb.ac.rs/]

- *8th Mathematical Physics Meeting: Summer School and Conference on Modern Mathematical Physics*, 24.–31.08.2014, Beograd, Srbija.

  [http://www.mphys8.ipb.ac.rs/]

Osim toga, učestvovao je i u organizaciji dva domaća skupa:

- *Gravity: New Ideas for Unsolved Problems II*, 19.–22.09.2013, Divčibare, Srbija.

  [http://www.gravity.ipb.ac.rs/divcibare2013.html]

- *GR100: Centennial of General Relativity*, 23.06.2015, Beograd, Srbija.

  [http://www.gravity.ipb.ac.rs/gr100/]

Pregled kvantitativnih pokazatelja

naučnoistraživačkog rada Marka Vojinovića

Ostvareni rezultati u periodu nakon prethodnog izbora u zvanje:

Kategorija	**M bodova po radu**	**Broj radova**	**Ukupno M bodova**	**Normirani broj M bodova**
M21a	10	1	10	10
M21	8	4	32	32
M22	5	1	5	5
M31	3,5	1	3,5	3,5
M33	1	2	2	2
M36	1,5	2	3	3
Ukupno:		**11**	**55,5**	**55,5**

Poređenje sa minimalnim kvantitativnim uslovima za izbor odnosno reizbor u zvanje viši naučni saradnik:

	Uslov za izbor: viši naučni saradnik	**Uslov za reizbor: viši naučni saradnik**	**Ostvareno**
Ukupno	50	**25**	**55,5**
$M_{10} + M_{20} + M_{31} + M_{32} + M_{33} + M_{41} + M_{42} + M_{90}$	40	**20**	**52,5**
$M_{11} + M_{12} + M_{21} + M_{22} + M_{23}$	30	**15**	**47**

Prema bazi podataka *Web of Science*, radovi kandidata su citirani ukupno 124 puta, odnosno 74 puta ne računajući samocitate. Prema istoj bazi, Hiršov indeks kandidata je 6.

Spisak radova Marka Vojinovića

Radovi objavljeni nakon prethodnog izbora u zvanje

[1] N. Paunković and M. Vojinović, "Causal orders, quantum circuits and spacetime: distinguishing between definite and superposed causal orders", *Quantum* **4**, 275 (2020).

DOI: 10.22331/q-2020-05-28-275

[kategorija: **M21a**; broj heterocitata: **3**; impakt-faktor: **5,381**;

klasifikacija: **Physics, Multidisciplinary 8/85**]

[2] T. Radenković and M. Vojinović, "Higher gauge theories based on 3-groups", *JHEP* **10**, 222 (2019).

DOI: 10.1007/JHEP10(2019)222

[kategorija: **M21**; broj heterocitata: **2**; impakt-faktor: **5,875**;

klasifikacija: **Physics, Particles & Fields 4/29**]

[3] F. Pipa, N. Paunković and M. Vojinović, "Entanglement-induced deviation from the geodesic motion in quantum gravity", *Jour. Cosmol. Astropart. Phys.* **09**, 057 (2019).

DOI: 10.1088/1475-7516/2019/09/057

[kategorija: **M21**; broj heterocitata: **0**; impakt-faktor: **5,524**;

klasifikacija: **Physics, Particles & Fields 5/29**]

[4] A. Miković, M. A. Oliveira and M. Vojinović, "Hamiltonian analysis of the BFCG formulation of general relativity", *Class. Quant. Grav.* **36**, 015005 (2019).

DOI: 10.1088/1361-6382/aaee25

[kategorija: **M21**; broj heterocitata: **1**; impakt-faktor: **3,487**;

klasifikacija: **Physics, Multidisciplinary 13/81**]

[5] N. Paunković and M. Vojinović, "Gauge protected entanglement between gravity and matter", *Class. Quant. Grav.* **35**, 185015 (2018).

DOI: 10.1088/1361-6382/aad7f1

[kategorija: **M21**; broj heterocitata: **0**; impakt-faktor: **3,487**;

klasifikacija: **Physics, Multidisciplinary 13/81**]

[6] T. Radenković and M. Vojinović, "Hamiltonian Analysis for the Scalar Electrodynamics as 3BF Theory", *Symmetry* **12**, 620 (2020).

DOI: 10.3390/sym12040620

[kategorija: **M22**; broj heterocitata: **0**; impakt-faktor: **2,645**;

klasifikacija: **Multidisciplinary Sciences 29/71**]

[7] T. Radenković and M. Vojinović, "Quantum gravity and elementary particles from higher gauge theory", *Ann. Univ. Craiova Phys.* **30**, 74 (2020).

[kategorija: **M31**; broj heterocitata: **0**]

[8] T. Radenković and M. Vojinović, "Construction and examples of higher gauge theories", *SFIN* **XXXIII**, 251 (2020).

[kategorija: **M33**; broj heterocitata: **0**]

[9] A. Miković and M. Vojinović, "Quantum gravity for piecewise flat spacetimes", *SFIN* **XXXI**, 267 (2018).

[kategorija: **M33**; broj heterocitata: **0**]

[10] "Proceedings of the 10th Mathematical Physics Meeting: School and Conference on Modern Mathematical Physics", Ed. B. Dragovich, I. Salom and M. Vojinović, *SFIN* **XXXIII**, 1 – 384 (2020). Published by Institute of Physics, Belgrade, Serbia.

ISBN: 978-86-82441-51-9.

[kategorija: **M36**]

[11] "Proceedings of the 9th Mathematical Physics Meeting: School and Conference on Modern Mathematical Physics", Ed. B. Dragovich, I. Salom and M. Vojinović, *SFIN* **XXXI**, 1 – 324 (2018). Published by Institute of Physics, Belgrade, Serbia.

ISBN: 978-86-82441-48-9.

[kategorija: **M36**]

Radovi objavljeni pre prethodnog izbora u zvanje

[12] M. Vojinović, "Causal Dynamical Triangulations in the Spincube Model of Quantum Gravity", *Phys. Rev. D* **94**, 024058 (2016).

DOI: 10.1103/PhysRevD.94.024058

[kategorija: **M21**; broj heterocitata: **1**; impakt-faktor: **4,557**;

klasifikacija: **Physics, Particles & Fields 8/29**]

[13] A. Miković, M. A. Oliveira and M. Vojinović, "Hamiltonian analysis of the BFCG theory for the Poincaré 2-group", *Class. Quant. Grav.* **33**, 065007 (2016).

DOI: 10.1088/0264-9381/33/6/065007

[kategorija: **M21**; broj heterocitata: **2**; impakt-faktor: **3.119**;

klasifikacija: **Physics, Particles & Fields 8/29**]

[[impakt-faktor: 3.119]]

[14] A. Miković and M. Vojinović, "Solution to the Cosmological Constant Problem in a Regge Quantum Gravity Model", *Europhys. Lett.* **110**, 40008 (2015).

DOI: 10.1209/0295-5075/110/40008

[kategorija: **M21**; broj heterocitata: **4**; impakt-faktor: **2,095**;

klasifikacija: **Physics, Multidisciplinary 20/78**]

[15] M. Vojinović, "Cosine problem in EPRL/FK spinfoam model", *Gen. Relativ. Gravit.* **46**, 1616 (2014).

DOI: 10.1007/s10714-013-1616-8

[kategorija: **M21**; broj heterocitata: **1**; impakt-faktor: **1,902**;

klasifikacija: **Physics, Multidisciplinary 20/83**]

[16] A. Miković and M. Vojinović, "A finiteness bound for the EPRL/FK spin foam model", *Class. Quant. Grav.* **30**, 035001 (2013).

DOI: 10.1088/0264-9381/30/3/035001

[kategorija: **M21**; broj heterocitata: **3**; impakt-faktor: **3,562**;

klasifikacija: **Physics, Multidisciplinary 11/83**]

[[impakt-faktor: 3.320, 3 citata]]

[17] A. Miković and M. Vojinović, "Poincaré 2-group and quantum gravity", *Class. Quant. Grav.* **29**, 165003 (2012).

DOI: 10.1088/0264-9381/29/16/165003

[kategorija: **M21**; broj heterocitata: **4**; impakt-faktor: **3,562**;

klasifikacija: **Physics, Multidisciplinary 11/83**]

[18] A. Miković and M. Vojinović, "Effective action and semiclassical limit of spin foam models", *Class. Quant. Grav.* **28**, 225004 (2011).

DOI: 10.1088/0264-9381/28/22/225004

[kategorija: **M21**; broj heterocitata: **7**; impakt-faktor: **3,320**;

klasifikacija: **Physics, Multidisciplinary 10/84**]

[19] A. Miković and M. Vojinović, “Large-spin asymptotics of Euclidean LQG flat-space wavefunctions”, *Adv. Theor. Math. Phys.* **15**, 801 (2011).

DOI: 10.4310/ATMP.2011.v15.n3.a5

[kategorija: **M21**; broj heterocitata: **0**; impakt-faktor: **2,034**;

klasifikacija: **Physics, Mathematical 10/47**]

[20] M. Vasilić and M. Vojinović, “Test membranes in Riemann-Cartan spacetimes”, *Phys. Rev. D* **81**, 024025 (2010).

DOI: 10.1103/PhysRevD.81.024025

[kategorija: **M21**; broj heterocitata: **0**; impakt-faktor: **5,050**;

klasifikacija: **Physics, Particles & Fields 5/25**]

[21] M. Vasilić and M. Vojinović, “Spinning branes in Riemann-Cartan spacetime”, *Phys. Rev. D* **78**, 104002 (2008).

DOI: 10.1103/PhysRevD.78.104002

[kategorija: **M21**; broj heterocitata: **4**; impakt-faktor: **5,050**;

klasifikacija: **Physics, Particles & Fields 5/25**]

[22] M. Vasilić and M. Vojinović, “Zero-size Objects in Riemann-Cartan Spacetime”, *JHEP* 08(2008)104.

DOI: 10.1088/1126-6708/2008/08/104

[kategorija: **M21**; broj heterocitata: **2**; impakt-faktor: **5,659**;

klasifikacija: **Physics, Particles & Fields 3/24**]

[23] M. Vasilić and M. Vojinović, “Interaction of particle with the string in pole-dipole approximation”, *Fortschr. Phys.* **56**, 542-546 (2008).

DOI: 10.1002/prop.200710532

[kategorija: **M21**; broj heterocitata: **0**; impakt-faktor: **2,007**;

klasifikacija: **Physics, Multidisciplinary 16/68**]

[24] M. Vasilić and M. Vojinović, “Classical Spinning Branes in Curved Backgrounds”, *JHEP* 07(2007)028.

DOI: 10.1088/1126-6708/2007/07/028

[kategorija: **M21**; broj heterocitata: **13**; impakt-faktor: **5,659**;

klasifikacija: **Physics, Particles & Fields 3/24**]

[25] M. Vasilić and M. Vojinović, "Classical String in Curved Backgrounds", *Phys. Rev. D* **73**, 124013 (2006).

DOI: 10.1103/PhysRevD.73.124013

[kategorija: **M21**; broj heterocitata: **4**; impakt-faktor: **4,896**;

klasifikacija: **Physics, Particles & Fields 5/21**]

[26] M. Vojinović, "Category Theory in Spincube Model of Quantum Gravity", *Proceedings of the Vth Petrov international Symposium "High Energy Physics, Cosmology and Gravity"*, 29. April — 5. May 2012, BITP, Kyiv, Ukraine, ed. Stepan S. Moskaliuk (TIMPANI), p. 287–299 (2012).

[kategorija: **M31**; broj heterocitata: **0**]

[27] N. Paunković and M. Vojinović, "Gravity-Matter entanglement in Regge quantum gravity", *Jour. Phys. Conf. Ser.* **701**, 012035 (2016).

DOI: 10.1088/1742-6596/701/1/012035

[kategorija: **M33**; broj heterocitata: **0**]

[28] A. Miković and M. Vojinović, "Categorical generalization of spinfoam models", *Jour. Phys. Conf. Ser.* **532**, 012020 (2014).

DOI: 10.1088/1742-6596/532/1/012020

[kategorija: **M33**; broj heterocitata: **0**]

[29] M. Vojinović, "Spincube model of quantum gravity", *SFIN* **XXVI**, 361 (2013).

[kategorija: **M33**; broj heterocitata: **0**]

[30] A. Miković and M. Vojinović, "Effective action for EPRL/FK spin foam models", *Jour. Phys. Conf. Ser.* **360**, 012049 (2012).

DOI: 10.1088/1742-6596/360/1/012049

[kategorija: **M33**; broj heterocitata: **5**]

[31] M. Vasilić and M. Vojinović, "Single-pole interaction of the particle with the string", *SIGMA* **4**, 019 (2008).

DOI: 10.3842/SIGMA.2008.019

[kategorija: **M33**; broj heterocitata: **0**]

[32] M. Vasilić and M. Vojinović, "Classical String in Curved Backgrounds", *SFIN* **XX**, 403 (2007).

[kategorija: **M33**; broj heterocitata: **0**]

[33] M. Vasilić and M. Vojinović, "Interaction of Particle With the String", *BW2007 III Southeastern European Workshop "Challenges Beyond the Standard Model"*, Kladovo, Serbia, 2–9. September 2007, editors: G. Djordjevic, M. Haack, Lj. Nesic (BW2007 Book of Abstracts, Faculty of Sciences and Mathematics, Section of Serbian Physical Society, Niš 2007), p. 40.

[kategorija: **M34**; broj heterocitata: **0**]

[34] M. Vasilić and M. Vojinović, "Spin-Orbit Interaction of the Particle with the String", *J. Res. Phys.* **31**, 94 (2007).

[kategorija: **M63**; broj heterocitata: **0**]

[35] M. Vojinović, "Self–Dual Lagrangians for Massless Vector Field", *Zbornik radova sa 11. kongresa fizičara Srbije i Crne Gore*, Petrovac na moru, Srbija i Crna Gora, 3–5. jun 2004. godine, urednici: N. Konjević, B. Vujičić i P. Miranović (2004) 5-161 — 5-164.

[kategorija: **M63**; broj heterocitata: **0**]

[36] M. Vojinović, "Kretanje ekstendiranih objekata u gravitacionom polju sa torzijom", Fizički fakultet Univerziteta u Beogradu, 2008, mentor: prof. dr Milovan Vasilić.

[kategorija: **M70**; broj heterocitata: **0**]

Web of Science

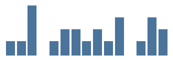
Citation report for 24 results from Web of Science Core Collection between 2006 and 2021 Go

You searched for: **Article Group for:** Vojinovic, M. ...More

This report reflects citations to source items indexed within Web of Science Core Collection. Perform a Cited Reference Search to include citations to items not indexed within Web of Science Core Collection.

Export Data: Save to Excel File

Total Publications
24 Analyze



h-index
6

Average citations per item
5.17

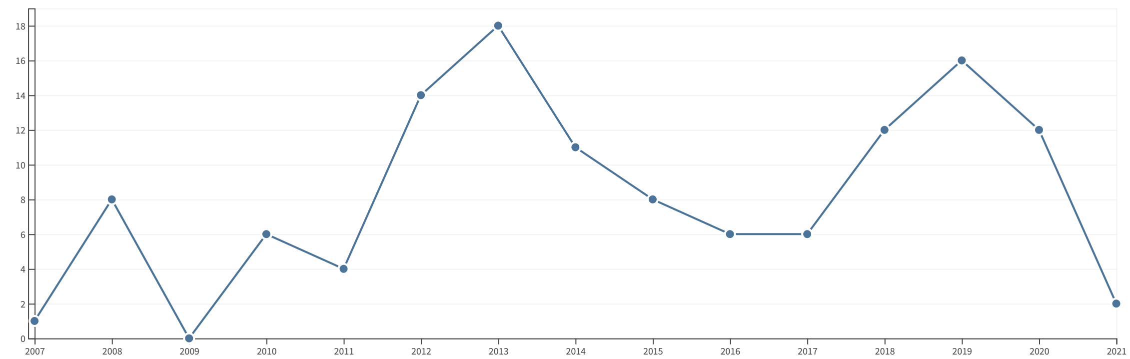
Sum of Times Cited
124

Without self citations
74

Citing articles
69 Analyze

Without self citations
49 Analyze

Sum of Times Cited per Year



Sort by: Times Cited **Date** More

1 of 3

How are these totals calculated?

Use the checkboxes to remove individual items from this Citation Report

or restrict to items published between 2006 and 2021 Go

	2017	2018	2019	2020	2021	Total	Average Citations per Year
	6	12	16	12	2	124	8.27
1. **Causal orders, quantum circuits and spacetime: distinguishing between definite and superposed causal orders** By: Paunkovic, Nikola; Vojinovic, Marko QUANTUM Volume: 4 Published: MAY 23 2020	0	0	0	0	0	0	0.00
2. **Hamiltonian Analysis for the Scalar Electrodynamics as 3BF Theory** By: Radenkovic, Tijana; Vojinovic, Marko SYMMETRY-BASEL Volume: 12 Issue: 4 Article Number: 620 Published: APR 2020	0	0	0	0	0	0	0.00
3. **Higher gauge theories based on 3-groups** By: Radenkovic, T.; Vojinovic, M. JOURNAL OF HIGH ENERGY PHYSICS Issue: 10 Article Number: 222 Published: OCT 22 2019	0	0	0	2	1	3	1.00
4. **Entanglement-induced deviation from the geodesic motion in quantum gravity** By: Pipa, Francisco; Paunkovic, Nikola; Vojinovic, Marko JOURNAL OF COSMOLOGY AND ASTROPARTICLE PHYSICS Issue: 9 Article Number: 057 Published: SEP 2019	0	0	0	0	0	0	0.00
5. **Hamiltonian analysis of the BFCG formulation of general relativity** By: Mikovic, A.; Oliveira, M. A.; Vojinovic, M. CLASSICAL AND QUANTUM GRAVITY Volume: 36 Issue: 1 Article Number: 015005 Published: JAN 10 2019	0	0	1	2	0	3	1.00

6. **Gauge protected entanglement between gravity and matter** By: Paunkovic, N.; Vojinovic, M. CLASSICAL AND QUANTUM GRAVITY Volume: 35 Issue: 18 Article Number: 185015 Published: SEP 20 2018	0	0	1	0	0	1	0.25
7. **Causal dynamical triangulations in the spincube model of quantum gravity** By: Vojinovic, Marko PHYSICAL REVIEW D Volume: 94 Issue: 2 Article Number: 024058 Published: JUL 28 2016	0	1	1	0	0	2	0.33
8. **Hamiltonian analysis of the BFCG theory for the Poincare 2-group** By: Mikovic, A.; Oliveira, M. A.; Vojinovic, M. CLASSICAL AND QUANTUM GRAVITY Volume: 33 Issue: 6 Article Number: 065007 Published: MAR 17 2016	0	1	1	2	0	4	0.67
9. **Gravity-matter entanglement in Regge quantum gravity** By: Paunkovic, Nikola; Vojinovic, Marko Conference: International Symposium on Emergent Quantum Mechanics (EmQM) Location: Vienna Univ Technol, Vienna, AUSTRIA Date: OCT 23-25, 2015 EMQM15: EMERGENT QUANTUM MECHANICS 2015 Book Series: Journal of Physics Conference Series Volume: 701 Article Number: 012035 Published: 2016	0	1	0	0	0	1	0.17
10. **Solution to the cosmological-constant problem in a Regge quantum gravity model** By: Mikovic, A.; Vojinovic, M. EPL Volume: 110 Issue: 4 Article Number: 40008 Published: MAY 2015	0	1	1	2	1	6	0.86

Select Page Save to Excel File

Sort by: Times Cited Date More

1 of 3

24 records matched your query of the 37,652,459 in the data limits you selected.

Web of Science

Citation report for 24 results from Web of Science Core Collection between 2006 and 2021 Go

You searched for: **Article Group for:** Vojinovic, M. ...More

This report reflects citations to source items indexed within Web of Science Core Collection. Perform a Cited Reference Search to include citations to items not indexed within Web of Science Core Collection.

Sort by: Times Cited Date More 2 of 3

How are these totals calculated?

Use the checkboxes to remove individual items from this Citation Report

or restrict to items published between 2006 and 2021 Go

	2017	2018	2019	2020	2021	Total	Average Citations per Year
	6	12	16	12	2	124	8.27
11. **Categorical generalization of spinfoam models** By: Mikovic, A.; Vojinovic, M. Conference: Conference on 3Quantum - Algebra Geometry Information (QQQ) Location: Tallinn Univ Technol, Tallinn, ESTONIA Date: JUL 10-13, 2012 Sponsor(s): ITGP N W ESF; AGMP 3QUANTUM: ALGEBRA GEOMETRY INFORMATION (QQQ CONFERENCE 2012) Book Series: Journal of Physics Conference Series Volume: 532 Article Number: 012020 Published: 2014	0	0	0	0	0	0	0.00
12. **Cosine problem in EPRL/FK spinfoam model** By: Vojinovic, Marko GENERAL RELATIVITY AND GRAVITATION Volume: 46 Issue: 1 Article Number: 1616 Published: JAN 2014	0	1	0	0	0	1	0.13
13. **A finiteness bound for the EPRL/FK spin foam model** By: Mikovic, Aleksandar; Vojinovic, Marko CLASSICAL AND QUANTUM GRAVITY Volume: 30 Issue: 3 Article Number: 035001 Published: FEB 7 2013	1	2	0	0	0	6	0.67
14. **Poincare 2-group and quantum gravity** By: Mikovic, A.; Vojinovic, M. CLASSICAL AND QUANTUM GRAVITY Volume: 29 Issue: 16 Article Number: 165003 Published: AUG 21 2012	1	2	2	3	0	17	1.70
15. **Effective action for EPRL/FK spin foam models** By: Mikovic, Aleksandar; Vojinovic, Marko Conference: International Conference on Non-Perturbative/Background Independent Quantum Gravity (LOOPS) Location: Consejo Superior Investigaciones Cientificas (CSIC), Inst Estructura Mater, Madrid, SPAIN Date: MAY 23-28, 2011 Sponsor(s): Spanish Minist Sci & Innovat; Spanish Res Council; Consejo Superior Investigaciones Cientificas (CSIC); BBVA Fdn; CONSOLIDER-CPAN Project; Spanish Soc Gravitat & Relativ (SEGRE); Univ Carlos III Madrid (UC3M); European Sci Fdn LOOPS 11: NON-PERTURBATIVE / BACKGROUND INDEPENDENT QUANTUM GRAVITY Book Series: Journal of Physics Conference Series Volume: 360 Article Number: 012049 Published: 2012	1	0	0	0	0	9	0.90
16. **Effective action and semi-classical limit of spin-foam models** By: Mikovic, A.; Vojinovic, M. CLASSICAL AND QUANTUM GRAVITY Volume: 28 Issue: 22 Article Number: 225004 Published: NOV 21 2011	1	0	1	0	0	17	1.55
17. **Large-spin asymptotics of Euclidean LQG flat-space wavefunctions** By: Mikovic, Aleksandar; Vojinovic, Marko ADVANCES IN THEORETICAL AND MATHEMATICAL PHYSICS Volume: 15 Issue: 3 Pages: 801-847 Published: JUN 2011	0	0	0	0	0	1	0.09
18. **Test membranes in Riemann-Cartan spacetimes** By: Vasilic, Milovan; Vojinovic, Marko PHYSICAL REVIEW D Volume: 81 Issue: 2 Article Number: 024025 Published: JAN 15 2010	0	0	1	0	0	2	0.17
19. **Spinning branes in Riemann-Cartan spacetime** By: Vasilic, Milovan; Vojinovic, Marko PHYSICAL REVIEW D Volume: 78 Issue: 10 Article Number: 104002 Published: NOV 2008	0	0	1	1	0	8	0.57
20. **Zero-size objects in Riemann-Cartan spacetime** By: Vasilic, Milovan; Vojinovic, Marko JOURNAL OF HIGH ENERGY PHYSICS Issue: 8 Article Number: 104 Published: AUG 2008	1	0	1	0	0	4	0.29

Select Page Save to Excel File

Sort by: Times Cited Date More 2 of 3

24 records matched your query of the 37,652,459 in the data limits you selected.

Citation report for 24 results from Web of Science Core Collection between 2006 and 2021 Go

You searched for: **Article Group for:** Vojinovic, M. ...More

This report reflects citations to source items indexed within Web of Science Core Collection. Perform a Cited Reference Search to include citations to items not indexed within Web of Science Core Collection.

Sort by: Times Cited Date More 3 of 3

How are these totals calculated?

Use the checkboxes to remove individual items from this Citation Report

or restrict to items published between 2006 and 2021 Go

	2017	2018	2019	2020	2021	Total	Average Citations per Year
	6	12	16	12	2	124	8.27
21. **Interaction of the particle with the string in pole-dipole approximation** By: Vasilic, Milovan; Vojinovic, Marko Conference: 3rd Southeastern European Workshop Location: Kladovo, SERBIA Date: SEP 02-09, 2007 Sponsor(s): Univ Nis, Dept Phys; Max Planck Inst Phys; Ludwig Maximilians Univ FORTSCHRITTE DER PHYSIK-PROGRESS OF PHYSICS Volume: 56 Issue: 4-5 Pages: 542-546 Published: APR-MAY 2008	0	0	1	0	0	1	0.07
22. **Single-Pole Interaction of the Particle with the String** By: Vasilic, Milovan; Vojinovic, Marko SYMMETRY INTEGRABILITY AND GEOMETRY-METHODS AND APPLICATIONS Volume: 4 Article Number: 019 Published: 2008	0	0	1	0	0	1	0.07
23. **Classical spinning branes in curved backgrounds** By: Vasilic, Milovan; Vojinovic, Marko JOURNAL OF HIGH ENERGY PHYSICS Issue: 7 Article Number: 028 Published: JUL 2007	1	2	1	0	0	22	1.47
24. **Classical string in curved backgrounds** By: Vasilic, Milovan; Vojinovic, Marko PHYSICAL REVIEW D Volume: 73 Issue: 12 Article Number: 124013 Published: JUN 2006	0	1	2	0	0	15	0.94

Select Page Save to Excel File

Sort by: Times Cited Date More 3 of 3

24 records matched your query of the 37,652,459 in the data limits you selected.

Република Србија
МИНИСТАРСТВО ПРОСВЕТЕ, НАУКЕ И ТЕХНОЛОШКОГ РАЗВОЈА
Комисија за стицање научних звања

Број:660-01-00001/83
28.09.2016. године
Б е о г р а д

ИНСТИТУТ ЗА ФИЗИКУ

ПРИМЉЕНО: 02-11-2016			
Рад.јед.	број	Арх.шифра	Прилог
0801	1837/1		

На основу члана 22. става 2. члана 70. став 6. Закона о научноистраживачкој делатности ("Службени гласник Републике Србије", број 110/05 и 50/06 – исправка и 18/10), члана 50. став 1. Закона о изменама и допунама Закона о научноистраживачкој делатности ("Службени гласник Републике Србије", број 112/15) члана 2. става 1. и 2. тачке 1 – 4.(прилози) и члана 38. Правилника о поступку и начину вредновања и квантитативном исказивању научноистраживачких резултата истраживача ("Службени гласник Републике Србије", број 38/08) и захтева који је поднео

Институт за физику у Београду

Комисија за стицање научних звања на седници одржаној 28.09.2016. године, донела је

ОДЛУКУ
О СТИЦАЊУ НАУЧНОГ ЗВАЊА

Др Марко Којиновић

стиче научно звање
Виши научни сарадник

у области природно-математичких наука - физика

О Б Р А З Л О Ж Е Њ Е

Институт за физику у Београду

утврдио је предлог број 202/1 од 09.02.2016. године на седници Научног већа Института и поднео захтев Комисији за стицање научних звања број 237/1 од 17.02.2016. године за доношење одлуке о испуњености услова за стицање научног звања ***Виши научни сарадник***.

Комисија за стицање научних звања је по претходно прибављеном позитивном мишљењу Матичног научног одбора за физику на седници одржаној 28.09.2016. године разматрала захтев и утврдила да именовани испуњава услове из члана 70. став 6. Закона о научноистраживачкој делатности ("Службени гласник Републике Србије", број 110/05 и 50/06 – исправка и 18/10), члана 2. става 1. и 2. тачке 1 – 4.(прилози) и члана 38. Правилника о поступку и начину вредновања и квантитативном исказивању научноистраживачких резултата истраживача ("Службени гласник Републике Србије", број 38/08) за стицање научног звања ***Виши научни сарадник***, па је одлучила као у изреци ове одлуке.

Доношењем ове одлуке именовани стиче сва права која му на основу ње по закону припадају.

Одлуку доставити подносиоцу захтева, именованом и архиви Министарства просвете, науке и технолошког развоја у Београду.

ПРЕДСЕДНИК КОМИСИЈЕ
Др Станислава Стошић-Грујичић,
научни саветник
С. Стошић-Грујичић

РЕПУБЛИКА СРБИЈА МИНИСТАРСТВО ПРОСВЕТЕ, НАУКЕ И ТЕХНОЛОШКОГ РАЗВОЈА БЕОГРАД

МИНИСТАР
Младен Шарчевић

Univerzitet u Beogradu

Master teza

Kvantna gravitacija na deo-po-deo ravnim mnogostrukostima

student
Tijana Radenković

mentor
dr Marko Vojinović

27.9.2017.

Zahvalnica

Želim da se zahvalim mentoru Marku Vojinoviću na posvećenom vremenu i beskonačno strpljenja koje je pokazao dok me je uvodio u ovu divnu teoriju i pomagao da dovršim tekst teze. Mogu bez preterivanja da kažem da su naši razgovori ponedeljkom probudili u meni potpuno zaboravljeni entuzijazam i učinili da se jako radujem narednim godinama.

Zahvaljujem se kolegama Mariji Tomašević i Banetu Avramovu na moralnoj podršci koju su mi pružali u danima koje smo proveli skučeni u istoj sobi ne razgovarajući međusobno i pišući svoje teze i prijatelju Marku Kuzmanoviću koji je uvek bio tu da me podseti šta mi je bitno u periodima kada ja nisam bila u stanju da se setim.

Takođe, ne smem da zaboravim prijatelje: Tijanu Aćimović, Jelenu Matković, Tamaru Hadži-Đorđević, Ivana Babića, Gvida D'Amika, Vladimira Sivčevića, Aleksandra Samardžiju, Stefana Stojkua i Đorđa Rakića i njihovu neizmernu podršku.

Na kraju, najviše se zahvaljujem svojoj porodici na tome što me tolerišu ovoliko dugo.

Dušan Cvijetić i Maja Stošić

Simulacija idealnih detektora gravitacionih talasa

Nakon prvog direktnog detektovanja gravitacionih talasa 2015. godine, istraživanja gravitacionih fenomena stekla su pažnju šire javnosti. Proučavanje polarizacije gravitacionih talasa može se iskoristiti za proveravanje valjanosti različitih metričkih teorija gravitacije, od kojih je najvažnija Ajnštajnova Opšta teorija relativnosti. U radu je uveden analitički model za opisivanje ravnog monohromatskog gravitacionog talasa male amplitude pomoću promenljive Rimanove metrike prostora, prvo za talas koji se prostire duž koordinatne ose, a zatim je model uopšten za talas bilo kakvog pravca prostiranja. Potom je ispitivana interakcija između detektora i talasa. Dati su dvodimenzionalni i trodimenzionalni modeli idealnog detektora. Prvo je razmatran slučaj ravanskog detektora u obliku kružnice, koji je zatim uopšten do trodimenzionalnog sfernog idealnog detektora. Određena je funkcija odziva detektora na gravitacioni talas date polarizacije i pravca prostiranja. Dalje proučavanje analitičkih rezultata izvršeno je korišćenjem dve simulacije. Prva prati ponašanje idealnog sfernog detektora koji intereaguje sa gravitacionim talasom. Druga prati ponašanje sistema idealnih detektora raspoređenih po Zemlji analogno stvarnim detektorima. Predložene su mogućnosti za dalji razvoj i implementaciju modela u određivanju preciznosti merenja sistema detektora.

Uvod

Teorije gravitacije i sam koncept gravitacije su se značajno menjali kroz vreme. Dugo je smatrano da je gravitacija sila koja deluje između tela sa masom, a značaju prekretnicu uveo je Ajnštajn 1915. godine u svojoj Opštoj teoriji relativnosti (OTR) predstavljajući gravitaciju kao posledicu zakrivljenosti prostorvremena (Misner *et al.* 1973; Vojinović 2019). Pošto se informacija o zakrivljenosti prostorvremena prenosi konačnom brzinom, a zavisi od rasporeda mase u prostoru, prilikom pomeranja tela sa masom nastaju gravitacioni talasi kao posledica konačne brzine prostiranja te informacije (Vojinović 2000). Oni su vrlo interesantni za posmatranje zbog mogućnosti da se koriste za testiranje raznih teorija gravitacije. U tom testiranju značajnu ulogu ima polarizacija talasa, kojom ćemo se mi baviti.

Za predstavljanje četvorodimenzionalnog prostorvremena u kom ćemo modelovati gravitacioni talas koristićemo geometriju Rimanovih prostora (Misner *et al*. 1973; Vojinović 2010*a*, 2010*b*). U analizu ćemo uzimati infinitezimalno bliske tačke koje će imati isti metrički tenzor. Gravitacioni talas ćemo modelovati kao perturbaciju metričkog tenzora i razmtraćemo ravan, monohromatski talas male amplitude sa tri parametra: talasni četvorovektor, početna faza i tenzor polarizacije.

Najpre ćemo modelovati idelani dvodimenzionalni detektor. Model ćemo potom uopštiti prelaskom na trodimenzionalni detektor i na taj način ćemo ostvariti mogućnost da detektor registruje talase koji dolaze duž svih pravaca, a ne samo duž z-ose, kao što je bio slučaj kod 2D

Dušan Cvijetić (2000), Pančevo, učenik 4. razreda Gimnazije „Uroš Predić" u Pančevu

Maja Stošić (2000), Čačak, učenica 4. razreda Gimnazije u Čačku

MENTOR: dr Marko Vojinović, Institut za fiziku Univerziteta u Beogradu

The $BFCG$ theory and canonical quantization of gravity. Draft

Miguel A. Oliveira
Grupo de Fisica Matemática da Universidade de Lisboa
Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal

Thesis submitted for a doctoral degree in Physics at University of Lisbon, March 2015.

Acknowledgements

I would like to thank Doctor Marko Vojinović for help and useful discussions in the context of my PhD, and especially regarding the canonical formalism.

Declaração

Declaro que o Doutor Marko Vojinović é o co-orientador da tese de doutoramento do Miguel Ângelo Oliveira desde Março de 2013 até à conclusão da tese, prevista para Março de 2016.

Prof. Doutor Aleksandar Miković
Grupo de Física Matemática
Universidade de Lisboa

Lisboa, 2 de Novembro de 2015

Deklaracija

Izjavljujem da je dr Marko Vojinović komentor za doktorsku disertaciju doktoranta Miguel Angelo Oliveira počev od marta 2013. pa do završetka teze, očekivanog u martu 2016.

Prof. dr Aleksandar Miković
Grupa za Matematicku Fiziku
Univerzitet u Lisabonu

Lisabon, 2. novembar 2015.

МИНИСТАРСТВО ПРОСВЕТЕ, НАУКЕ И ТЕХНОЛОШКОГ РАЗВОЈА
РЕПУБЛИКЕ СРБИЈЕ
И
ДРУШТВО ФИЗИЧАРА СРБИЈЕ

додељују

ПРИЗНАЊЕ

Наставнику-ци **Марку Војиновићу**

школа **Математичка гимназија**

место **Београд**

на 51. државном такмичењу из физике за ученике средњих школа

одржаном од 13.04. до 14.04.2013. године у Математичкој гимназији

у Београду.

првих награда -

других награда -

трећих награда 1

У Београду, 14.04.2013. године.

Председник Комисије за такмичење ученика средњих школа ДФС

др Александар Крмпот

Председник ДФС

доц. др Иван Дојчиновић

УГОВОР О ИЗВОЂЕЊУ НАСТАВЕ

Закључен између:

1. Математичке гимназије из Београда, Краљице Наталије 37, коју заступа директор мр СрЂан Огњановић и

2. Марко Војновић, професора физике.

Члан 1.

Уговорене стране сагласно констатују да извођач наставе за наставника испуњава све услове да обавља послове и радне задатке професора физике у Математичкој гимназији.

Члан 2.

На основу овог уговора, извођач наставе се обавезује да за потребе Школе изводи наставу из физике, са 12 часа недељно.

Члан 3.

Уговор се закључује за период од 01.09.2012. године до 30.06.2013. године.

Члан 4.

За извођење наставе из члана 1. овог уговора извођачу наставе припада накнада у износу од 20.025,00 бруто динара, на коју накнаду се обрачунавају порези и доприноси у складу са законом.
Обрачун и исплата уговорене накнаде вршиће се када и обрачун и исплата плата запосленима у Школи, на жиро-рачун бр. 275-0000320071855-87.

Члан 5.

Ради праћења извођења наставе од стране извођача наставе, директор Математичке гимназије може да формира комисију од три члана из реда наставника (од којих најмање један мора бити наставник из предмета за чије извођење се закључује овај уговор), која је дужна да о стручном раду извођача наставе да своје мишљење, посебно уколико постоје разлози за отказ из члана 7. став 2. овог уговора.

Члан 6.

Овај уговор престаје:

1. истеком рока на који је закључен
2. отказом од стране било које уговорне стране
3. смрћу извођача наставе или наступањем околности које га из здравствених и других разлога чине неспособним за извођење наставе

Члан 7.

Извођач наставе може отказати овај уговор пре истека рока на који је закључен, са отказним роком од 15 дана.

Школа може отказати овај уговор у следећим случајевима:

1. уколико се на основу извештаја надлежног органа школе покаже да извођач наставе нестручно обавља послове за које је ангажован по овом уговору.
2. уколико извођач наставе несавесно обавља уговорене послове (кашњење на час, несавестан рад са ученицима, недовољна припремљеност за рад, долазак у пијаном стању, неприсуствовање седницама стручних органа школе и сл.).

У случају отказа уговора од стране Школе, извођач наставе је у обавези да, на захтев Школе, остане у настави до ангажовања другог лица које ће обављати исте послове, а најкасније до истека времена на које је уговор закључен.

Члан 8.

Школа се обавезује да извођачу наставе након истека рока на који је уговор закључен исплати све доспеле накнаде, у року од 30 дана од дана истека времена на које је уговор закључен.

Члан 9.

Овај уговор је закључен у 4 (четири) истоветна примерка, од којих по 2 (два) за уговорне стране.

Извођач наставе — директор

Марко Војновић — *мр Срђан Огњановић*

ЈМБГ 2803978860020

УНИВЕРЗИТЕТ У БЕОГРАДУ
ИНСТИТУТ ЗА ФИЗИКУ | БЕОГРАД
ИНСТИТУТ ОД НАЦИОНАЛНОГ ЗНАЧАЈА ЗА РЕПУБЛИКУ СРБИЈУ
Прегревица 118, 11080 Земун - Београд, Република Србија
Телефон: +381 11 3713000, Факс: +381 11 3162190, www.ipb.ac.rs
ПИБ: 100105980, Матични број: 07018029, Текући рачун: 205-66984-23

Република Србија
Министарство просвете, науке и технолошког развоја

Извештај о реализацији билатераланог пројекта за период од 01.07.2018. до 01.07.2019. године

Билатерални програм са: *Навести државу*	**Аустријом**
Пројектни циклус:	**2018-2019**
Назив пројекта: *На српском језику*	**Каузалност у квантној механици и квантној гравитацији**
Евиденциони бр. Пројекта:	**451-03-02141/2017-09/02**
Руководилац српског пројектног тима:	**др Марко Војиновић**
Научноистраживачка организација у РС:	**Институт за физику у Београду**

Место, датум: **Београд, 15.07.2019.**

Марко Војиновић
Руководилац пројекта

Директор/Декан

Извештај је сачињен у два дела:
- први део представља финансијски извештај са потпуним и ажурним информацијама о наменском трошењу средстава на реализацији активности у билатералном пројекту;
- други део обухвата техничко-технолошки извештај о: реализованим активностима у складу са предлогом пројекта, предметом, садржајем, циљем и планом реализације, као и преглед остварених резултата.

Subject **RE: BIlateralni projekat sa Austrijom**
From Snezana Omic <snezana.omic@mpn.gov.rs>
To 'Marko Vojinovic' <vmarko@ipb.ac.rs>
Date 2020-12-24 12:34

Поштовани господине Војиновићу,

Обавештавамо вас да је Министарство просвете, науке и технолошког развоја, имајући у виду немогућност да се нерализоване активности из Програма билатералне научне и технолошке сарадње са Аустријом, реализују до краја 2020., одлучило да омогући продужење реализације билатералних пројеката са Аустријом до краја јуна 2021. године у већ опредељеним износима по пројекту и пројектној години.

Српска страна ће по устаљеној процедури, финансирати нереализоване активности у оквиру максимално одобреног износа по пројектној години од 2000 евра у динарској противвредности, по примљеном Захтеву за финансирање (који можете преузети са нашег сајта) са пратећим прилозима.

Молимо да извештај о реализацији пројекта доставите до 15. јула 2021. године.

Срдачно,
Снежана Омић
Сектор за међународну сарадњу и ЕУ интеграције
Министарство просвете, науке и технолошког развоја
Немањина 22-26
11 000 Београд
Тел: +11 3616 589

-----Original Message-----
From: Marko Vojinovic [mailto:vmarko@ipb.ac.rs]
Sent: Tuesday, 7 July 2020 7:51 PM
To: Snezana Omic <snezana.omic@mpn.gov.rs>
Subject: RE: BIlateralni projekat sa Austrijom

Postovana gospodjo Omic,

Hvala najlepse za Zahtev!

Sve najbolje, :-)
Marko

Dr. Marko Vojinovic
Group for Gravitation, Particles and Fields Institute of Physics University of Belgrade
====================== home page: www.markovojinovic.com
e-mail: vmarko@ipb.ac.rs

On Mon, 6 Jul 2020, Snezana Omic wrote:

> Poštovani gospodine Vojinoviću,
>
> U prilogu dostavljam Zahtev za finansiranje.
>
> Srdačno,
> Snežana Omić

Declaração

Declaro que o Doutor Marko Vojinović foi o investigador principal na tarefa "Grupos Quânticos e Geometria" do projecto "Algebróides, Geometria, Grupos Quânticos e Aplicações" (PTDC/MAT/099880/2008) da Fundação para a Ciência e a Tecnologia no período de Outubro de 2009 até Outubro de 2012.

Prof. Doutor Aleksandar Miković
Grupo de Física Matemática
Universidade de Lisboa

Lisboa, 2 de Novembro de 2015

Deklaracija

Izjavljujem da je dr Marko Vojinović bio vodeći istraživač za zadatak "Kvantne Grupe i Geometrija" projekta "Algebroidi, Geometrija, Kvantne Grupe i Primene" (PDTC/MAT/099880/2008) Fondacije za Nauku i Tehnologiju u periodu od oktobra 2009 do oktobra 2012.

Prof. dr Aleksandar Miković
Grupa za Matematicku Fiziku
Univerzitet u Lisabonu

Lisabon, 2. novembar 2015.

Declaração

Pela presente se declara que o Doutor Marko Vojinovic, investigador de pós-doutoramento no Grupo de Física-Matemática da Universidade de Lisboa (GFMUL), desempenhou um papel de liderança no programa sobre Gravidade Quântica desenvolvido neste centro de investigação durante dois projectos sucessivos da Fundação para a Ciência e Tecnologia (FCT), com referências, nomeadamente,

Pest-OE/MAT/UI0208/2013 e Pest-OE/MAT/UI0208/2011

Em particular, o Dr. Vojinovic esteve na origem de métodos inovadores para a abordagem de integrais de caminho para modelos "spin-cube", bem como de generalizações de modelos de "spin-foam" da gravidade quântica e algumas teorias de campo topológicas.

No que respeita ao mais antigo destes projectos (2011), Marko Vojinovic publicou alguns dos seus resultados em *Gen. Relativ. Gravit. 46, 1616 (2014)*. Quanto ao projecto de 2013 poderá consultar-se *Mikovic, Vojinovic, Europhys. Letters 110, 40008 (2015)*. Outras duas publicações estão ainda em processo de revisão.

O Dr. Marko Vojinovic demonstrou, em consequência, independência científica e capacidades de liderança na direcção dos programas de investigação.

Lisboa, 3/10/2015

GRUPO DE FÍSICA-MATEMÁTICA UNIVERSIDADE DE LISBOA

Jean-Claude Zambrini
(Director of GFMUL)

Grupa za Matematičku Fiziku

Deklaracija

Ovime se izjavljuje da je dr Marko Vojinović, postdok istraživač u Grupi za Matematičku Fiziku Univerziteta u Lisabonu (GFMUL) imao vodeću ulogu za program o Kvantnoj Gravitaciji razvijan u ovom istraživačkom centru tokom dva sukcesivna projekta Fondacije za Nauku i Tehnologiju (FCT), konkretno sa referencama

Pest-OE/MAT/UI0208/2013 i Pest-OE/MAT/UI0208/2011

Konkretno, dr Vojinović je doveo do inovativnih metoda za path-integral pristup u "spin-cube" modelu, kao i do uopštavanja "spin-foam" modela kvantne gravitacije i nekih topoloških teorija polja.

Vezano za stariji projekt (2011), Marko Vojinović je publikovao neke od svojih rezultata u Gen. Relativ. Gravit. 46, 1616 (2014). Vezano za projekt iz 2013 možete konsultovati Mikovic, Vojinovic, Europhys. Letters 110, 40008 (2015). Jodve publikacije su pod recenzijom.

Kao posledica, sledi da je dr Marko Vojinović demonstrirao naučnu nezavisnost i sposobnost za rukovođenje i upravljanje istraživačkim projektima.

Lisabon, 03.10.2015.

Jean-Claude Zambrini

(Direktor GFMUL)

Subject A request to referee for Classical and Quantum Gravity - CQG-[redacted]

From Classical and Quantum Gravity <onbehalfof+cqg+iop.org@manuscriptcentral.com>

Sender <onbehalfof+cqg+iop.org@manuscriptcentral.com>

To <vmarko@ipb.ac.rs>

Reply-To <cqg@iop.org>

Date 2016-09-19 17:46

- [redacted]

Dear Dr Vojinovic,

Re: [redacted]

Article reference: CQG-[redacted]

This Note has been submitted to Classical and Quantum Gravity for consideration and you have been suggested as a possible expert who could referee the article. We would be very grateful if you could help us.

The abstract appears at the end of this letter, along with the names of the author(s). Please let us know as soon as possible if you will be able to accept our invitation to referee. You may follow the appropriate link at the bottom of the page or e-mail us with your reply. Once you accept our invitation to referee this manuscript, you will be notified via e-mail about how to access it through our online referee system. You will then have access to the manuscript and referee instructions in your Referee Centre.

We require your comments and recommendation by 03-Oct-2016. However, if you need more time, please contact the journal at cqg@iop.org giving a provisional date you hope to report by. If you are unable to report on this occasion we would be grateful if you could provide the names and e-mail addresses of possible alternative referees when prompted.

Thank you for considering this manuscript. We look forward to hearing from you soon.

Yours sincerely

Emily Tapp

On behalf of the CQG peer review team

Peer review operations
Jennifer Sanders - Editor
Emily Tapp and David Jones - Associate Editors
James Dimond - Editorial Assistant

cqg@iop.org

Adam Day - Publisher

http://iopscience.iop.org/cqg

To respond automatically, click below:

Agreed: [redacted]

Declined - Conflict of Interest: [redacted]

Subject **Invitation to review for Classical and Quantum Gravity - CQG-**[redacted]
From Classical and Quantum Gravity <onbehalfof@manuscriptcentral.com>
To <vmarko@ipb.ac.rs>
Reply-To <cqg@ioppublishing.org>
Date 2021-01-28 15:57

Dear Dr Vojinović,

Re: [redacted]
Article reference: CQG-[redacted]

This Paper has been submitted to Classical and Quantum Gravity for consideration and we have identified you as a possible expert who could review the manuscript. We would be very grateful if you could offer your opinion on the manuscript and whether it is suitable for publication. You can review the full abstract and further article information at the end of this email before making your choice.

Our expert reviewers greatly contribute to the high standards of the Journal, and we thank you for your present and/or future participation.

To respond automatically, click below:

*** PLEASE NOTE: This is a two-step process. After clicking on the link, you will be directed to a webpage to confirm. ***

Agreed: [redacted]

Declined - Conflict of Interest: [redacted]

Declined - Out of Field: [redacted]

Declined - Unavailable: [redacted]

Declined - No Reason: [redacted]

If you accept our invitation to review this manuscript, you will be notified via e-mail with instructions on how to access and review the manuscript in your Reviewer Centre. We would appreciate your comments and recommendation by 11-Feb-2021.

If you need more time, please reply to this email, providing a date you can report by. If you are unable to report on this occasion, we would be grateful if you could provide the names and e-mail addresses of possible alternative reviewers when prompted.

You can gain credit for this review and connect with ORCID through our connection with Publons.

Yours sincerely

On behalf of:
Classical and Quantum Gravity

Subject **Invitation to review for Classical and Quantum Gravity - CQG-[redacted]**
From Classical and Quantum Gravity <onbehalfof@manuscriptcentral.com>
To <vmarko@ipb.ac.rs>
Reply-To <cqg@ioppublishing.org>
Date 2021-03-15 17:57

Dear Dr Vojinović,

Re: [redacted]
Article reference: CQG-[redacted]

This Letter has been submitted to Classical and Quantum Gravity for consideration and we have identified you as a possible expert who could review the manuscript. We would be very grateful if you could offer your opinion on the manuscript and whether it is suitable for publication. You can review the full abstract and further article information at the end of this email before making your choice.

Our expert reviewers greatly contribute to the high standards of the Journal, and we thank you for your present and/or future participation.

To respond automatically, click below:

*** PLEASE NOTE: This is a two-step process. After clicking on the link, you will be directed to a webpage to confirm. ***

Agreed: [redacted]

Declined - Conflict of Interest: [redacted]

Declined - Out of Field: [redacted]

Declined - Unavailable: [redacted]

Declined - No Reason: [redacted]

If you accept our invitation to review this manuscript, you will be notified via e-mail with instructions on how to access and review the manuscript in your Reviewer Centre. We would appreciate your comments and recommendation by 22-Mar-2021.

If you need more time, please reply to this email, providing a date you can report by. If you are unable to report on this occasion, we would be grateful if you could provide the names and e-mail addresses of possible alternative reviewers when prompted.

You can gain credit for this review and connect with ORCID through our connection with Publons.

Yours sincerely

On behalf of:
Classical and Quantum Gravity

Subject **Manuscript FOOP[redacted] for review**
From Foundations of Physics <jenna.cataluna@springer.com>
Sender <em.foop.0.[redacted]@editorialmanager.com>
To <vmarko@phy.bg.ac.yu>
Date 2008-08-05 16:40

webmail

Dear Dr. Vojinovic,

In view of your expertise I would be very grateful if you could review the following manuscript which has been submitted to Foundations of Physics.

Manuscript Number: FOOP[redacted]

Title: [redacted]

Abstract: [redacted]

In case you accept to review this submission please click on this link:

[redacted]

If you do not have time to do this, or do not feel qualified, please click on this link:

[redacted]

Kindly suggest names of other possible referees in case you are not able to review this paper.

You may also click here to view the PDF of the submission [redacted]

We hope you are willing to review the manuscript. If so, would you be so kind as to return your review to us within 33 days of agreeing to review? Thank you.

It is a policy of this journal that the reviewer's identity will not be disclosed to the author, unless the reviewer explicitly requests otherwise.

Our primary concerns are:

- Is the paper technically correct and accurate?
- Does it contain novel results and do you consider it of interest to readers of Foundations of Physics?
- Does it give proper references to earlier work?
- Are there parts that could be considered superfluous or misleading? Is its length appropriate?
- Is the language sufficiently clear and proper?
- Can you propose explicit corrections or further improvements?
- Do you have any other recommendations or suggestions concerning this manuscript?

Subject	**Manuscript FOOP-[redacted] for review**
From	Foundations of Physics (FOOP) <em@editorialmanager.com>
Sender	<em.foop.0.[redacted]@editorialmanager.com>
To	Marko Vojinovic <vmarko@ipb.ac.rs>
Reply-To	Foundations of Physics (FOOP) <roseline.periyanayagam@springernature.com>
Date	2020-09-18 08:52

Dear Dr Vojinovic,

In view of your expertise I would be very grateful if you could review the following manuscript which has been submitted to Foundations of Physics.

Manuscript Number: FOOP-[redacted]

Title: [redacted]

Abstract: [redacted]

In case you accept to review this submission please click on this link:

[redacted]

If you do not have time to do this, or do not feel qualified, please click on this link:

[redacted]

Kindly suggest names of other possible referees in case you are not able to review this paper.

You may also click here to view the PDF of the submission
[redacted]

We hope you are willing to review the manuscript. If so, would you be so kind as to return your review to us within 21 days of agreeing to review? Thank you.

It is a policy of this journal that the reviewer's identity will not be disclosed to the author, unless the reviewer explicitly requests otherwise.

Foundations of Physics' Editor-in-Chief, Professor Carlo Rovelli
, express the guidelines for reviewers as follows:

Subject **Request to referee SIGMA-[redacted]**
From SIGMA <editor@sigma-journal.com>
To Marko Vojinovic <vmarko@cii.fc.ul.pt>
Date 2011-08-30 08:48
Priority Normal

webmail

- [redacted]

Ref: SIGMA-[redacted]
TITLE: [redacted]
AUTHOR(S): [redacted]

Dear Professor Vojinovic,

The above paper (see attached PDF) has been submitted for consideration by SIGMA (Symmetry, Integrability and Geometry: Methods and Applications)
http://www.emis.de/journals/SIGMA/
and as an expert in the field we would be very grateful if you could referee it for us.

There could be several options of the referee's recommendation - publication, publication after minor recommended revisions, revision and repeated refereeing and rejection.
The detailed evaluation criteria are outlined on the journal's web site.

We will be grateful if you are able to send your report within ONE MONTH after receiving this letter.

If you believe that for any reason you are unable to referee the paper within this period, please ask for more time,
or, otherwise, we would be grateful if you recommend another referee(s).

We would like to ask you to acknowledge receiving this request, and inform us whether you are able to referee the paper.

Sincerely yours,

Mrs. Vira Pobyzh
Executive Assistant
SIGMA
http://www.emis.de/journals/SIGMA/

Subject [Axioms] Manuscript ID: axioms-[redacted] - Review Request
From Axioms Editorial Office <axioms@mdpi.com>
Sender <agnes.zhao@mdpi.com>
To Marko Vojinovic <vmarko@ipb.ac.rs>
Cc Axioms Editorial Office <axioms@mdpi.com>, Agnes Zhao <agnes.zhao@mdpi.com>
Reply-To Agnes Zhao <agnes.zhao@mdpi.com>
Date 2020-09-23 08:01

IPB webmail

Dear Dr. Vojinovic,

We have received the following manuscript to be considered for publication in Axioms (https://www.mdpi.com/journal/axioms/) and kindly invite you to provide a review to evaluate its suitability for publication:

Type of manuscript: Article
Title: [redacted]

The abstract is available at the end of this message. Please click on the link below to access the manuscript and review report form, and inform us whether or not you will be able to provide a review.

[redacted]

If you accept this invitation we would appreciate receiving your comments within 10 days. Please let us know if you will need more time. Advice for completing your review can be found at: https://www.mdpi.com/reviewers. We would like to stress that we rely on the critical reviews of external experts to maintain the quality of Axioms. Along with the authors, we would greatly value your contribution to the peer-review process.

If you are not able to review this manuscript, we kindly ask you to decline by clicking on the above link so that we can continue processing this submission. We would also appreciate any suggestions for alternative expert reviewers.

MDPI partners with Publons (https://publons.com/in/mdpi/) to provide recognition for reviewers. Your credit will appear on Publons after a final decision on the paper and once you have claimed your review on the Publons website. For well-prepared review reports submitted in a timely manner, we also provide APC discount vouchers for publication in any MDPI journal.

Please note that this peer-review request and the contents of the manuscript are highly confidential. You must not distribute the manuscript in part or whole to a third party, including other members of your research group, without explicit permission from the editorial office. You must also disclose if you have a conflict of interest with the content of the manuscript or the authors. We discourage reviewers from recommending citation of their own work when not clearly necessary to improve the quality of the manuscript under review. Please state in your comments to the editor if you recommend citation of your own work and the reason for this recommendation. Please also look at the reference list of the manuscript and check if there are inappropriate self-citations.

Axioms is one of MDPI's open access journals. Our aim is to process manuscripts quickly and publish them shortly after peer-review. Some high quality manuscripts have been peer-reviewed and published in less than four weeks from submission.

ÖAW AUSTRIAN ACADEMY OF SCIENCES

ANTON ZEILINGER
PRESIDENT

To
Ass. Prof. Dr. Marko Vojinovic
University of Belgrade
Group for Gravitation, Particles and Fields
vmarko@ipb.ac.rs

Vienna, April 20, 2017
AZ/sb

JESH Application [Joint Excellence in Science and Humanities]

Dear Professor Vojinovic,

Thank you very much for your interest in the JESH programme and for sending a most intriguing proposal. I am very pleased to be able to inform you that you have been awarded JESH funding by the Austrian Academy of Sciences for a period of **2 months**, subject to compliance with the requirements set out in the attached information sheet.

I congratulate you most warmly, and wish you a successful research stay in Austria.

Sincerely yours,

DR. IGNAZ SEIPEL-PLATZ 2 | 1010 VIENNA, AUSTRIA | T: +43 1 51581-1202 | F: +43 1 51581-1209
PRAESIDENT@OEAW.AC.AT

February 15, 2018

Prof. Dr. Caslav Brukner | caslav.brukner@univie.ac.at
Tel +43 1 4277 2582

S T A T E M E N T

about the research visit of Dr. Marko Vojinović to IQOQI,
supported by the JESH programme of Austrian Academy of Sciences (ÖAW)

Dr. Marko Vojinović has been a guest visitor to our group at IQOQI from 15. October to 16. December 2017. During this time, he gave a seminar titled "Causal ordering and quantum gravity", and has actively participated in many discussions related to this topic. In addition, he gave valuable contributions at our group meetings and seminars, proposing an interesting research topic for joint collaboration.

As a result, after his visit we have jointly applied for a bilateral project between ÖAD and Serbian Ministry of Science for the period 2018-2019. The project proposal is currently under review, and it will intensify the collaboration between the two groups in the research of causality in quantum mechanics and quantum gravity.

Overall, in my opinion the visit of Dr. Vojinovic has been very successful, and our group is looking forward to collaborate with him even more in the future.

Sincerely,

Yours sincerely,

Prof. Dr. Časlav Brukner
Institute Director

UNIVERSIDADE DE LISBOA

GRUPO DE FÍSICA - MATEMÁTICA

DECLARAÇÃO

Declaro para os devidos efeitos que o **Doutor Marko Vojinovik** é investigador do **Grupo de Física-Matemática da Universidade de Lisboa**, onde realiza trabalhos de investigação no âmbito de uma bolsa de Pós-doutoramento da FCT, com início em Maio de 2009, pelo período de 3 anos.

Lisboa, 27 de Julho de 2009

GRUPO DE FÍSICA MATEMÁTICA UNIVERSIDADE DE LISBOA

Professor Jean-Claude Zambrini
(Coordenador Científico do GFMUL)

THE UNIVERSITY OF HONG KONG
DEPARTMENT OF
COMPUTER SCIENCE

Pokfulam Road, Hong Kong
Tel : (+852) 2859 2180
Fax : (+852) 2559 8447
Email : dept@cs.hku.hk
Web site : www.cs.hku.hk

香 港 大 學 計 算 機 科 學 系

By e-mail December 11, 2019

Prof. Marko Vojinovic
Institute of Physics,
University of Belgrade,
Belgrade, Serbia
vmarko@ipb.ac.rs

Dear Prof. Vojinovic,

Invitation Letter

We would like to invite you to visit our research group for the period January 10 to 19 in the year 2020, in order to attend the workshop QISS 2020 and give a scientific talk. The workshop is organised by the group of Professor Giulio Chiribella and our Department at The University of Hong Kong.

As to travel and medical insurance, please kindly make your own arrangements as necessary. Please also be reminded to follow the immigration entry requirements as applicable.

I look forward to meeting you in Hong Kong soon and having fruitful discussions and collaborations.

Yours sincerely,

Dr Christodoulou Marios

THE JOINT MEETING ON
QUANTUM FIELDS AND NONLINEAR DYNAMICS
24-29 September 2020,Craiova, Romania

- ***The 12-th Workshop "Quantum Fields and Nonlinear Dynamics" (QFND)***
- ***The SEENET PhD School on Computational Methods in Theoretical Physics***

.

To:
Dr. Marko Vojinovic
Group for Gravitation, Particles and Fields
Institute of Physics, University of Belgrade

INVITATION LETTER

It is our pleasure of inviting you to participate in the ***12-th Workshop "Quantum Fields and Nonlinear Dynamics" (QFND)*** that will be held online, from September 27 to September 29, 2020, in the organization of the University of Craiova, Romania.

We will be happy if you will accept to include in the Workshop's Program a scientific presentation, paper that can be considered for publication in the Proceedings, a special issue of "Physics AUC", a journal edited by the University of Craiova and indexed by SCOPUS.

We are waiting for your reply, hoping you will be able to adjust the meeting to your schedule.

Coordinator of the Organizing Committee,
Prof. Univ. dr. Radu Constantinescu

Craiova, September 2, 2020

Serbian Astronomical Conference
Belgrade, October 13-17, 2020

To: Dr. Marko Vojinović
Institution: Institute of Physics
Address: Pregrevica 118, 11080 Belgrade, Serbia
Email: vmarko@ipb.ac.rs

Dear Dr. Vojinović,

On behalf of the Scientific Organizing Committee we are pleased to invite you to participate and to present your work *Cosmological constant problem in discretized quantum gravity* as an Invited Lecture at **19th Serbian Astronomical Conference (19th SAC)**, to be held in Belgrade, Serbia, October 13-17, 2020 (http://astro.math.rs/kas19/).

We will be pleased if you accept our invitation, and we are looking forward to seeing you at **19th SAC**.

Yours Sincerely,

Prof. dr Anđelka Kovačević
Co-chair of the SOC
Department of astronomy
Faculty of Mathematics
University of Belgrade

UNIVERSITÉ DE GENÈVE
FACULTÉ DES LETTRES
Département de philosophie

Christian Wüthrich
Professeur associé
Ligne directe: 022 379 70 53
christian.wuthrich@unige.ch

Dr. Marko Vojinovic
Group for Gravitation, Particles and Fields
Institute of Physics
University of Belgrade

Geneva, 10 May 2016

Letter on invitation for Dr Marko Vojinovic

Dear Dr Vojinovic,

I am happy to inform you that your application for a visiting fellowship to the University of Geneva to visit the Geneva Center of the Templeton-funded project 'Space and Time after Quantum Gravity', which is a joint project with Professor Nick Huggett Huggett at the University of Illinois at Chicago, has been accepted.

Given our budget we would cover your expenses for travel and accommodation up to 500 Swiss francs. If this doesn't suffice to cover your expenses, we can revisit the issue, but we cannot guarantee that we can offer more. Also, we will provide office space. We are working on a list of potential accommodations, but you can easily check online to see what kinds of things are available. Even though Geneva is expensive, there are a number of hotels (and hostels) available that are reasonably priced.

As for the date, the week of 17-22 October 2016 would suit us best.

You will be welcome to give a talk during your visit. We plan to skype this talk to our partners at Chicago and will record it for publication on our YouTube channel. Ideally, your talk would happen during a time slot when this is possible.

We can work out the other activities during your visit later, but we hope that you will engage with the group here.

We very much hope that you can accept our invitation.

Please let me know if you need any further information.

Yours sincerely,

Christian Wüthrich

Christian Wüthrich

НАЦІОНАЛЬНА АКАДЕМІЯ НАУК УКРАЇНИ

ІНСТИТУТ ТЕОРЕТИЧНОЇ ФІЗИКИ
ім. М.М. БОГОЛЮБОВА

NATIONAL ACADEMY OF SCIENCES OF UKRAINE

BOGOLYUBOV INSTITUTE FOR
THEORETICAL PHYSICS

вул. Метрологічна, 14-б,
Київ, 03680, Україна

Tel: 38 (044) 526-5362
E-mail: itp@bitp.kiev.ua

Fax: 38 (044) 526-5998
www.bitp.kiev.ua

14-b Metrologichna St.
Kyiv, 03680, Ukraine

05.04.2012 № 69-138/16

Invitation to the V Petrov Symposium

Prof. V. VOJINOVIC
Belgrade Institute of Physics
84/32, Pregrevica 118, Zemun
Belgrade, 11000
Serbia
email: vmarko@ipb.ac.rs

Kiev, April 6, 2012

Dear Prof. VOJINOVIC,

on behalf of the Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine (BITP) I have a pleasure to invite you to participate in the V Petrov International Symposium "High Energy Physics, Cosmology and Gravity" to be held in Kyiv, Ukraine from April 29 to May 05, 2012 (http://quark.itp.tuwien.ac.at/~diefaust/Symposium/2012/) and give a plenary talk.

Looking forward to seeing you in Kiev.

Yours sincerely,

НАЦІОНАЛЬНА АКАДЕМІЯ НАУК УКРАЇНИ
ІНСТИТУТ ТЕОРЕТИЧНОЇ ФІЗИКИ ім. М.М. БОГОЛЮБОВА
Ідентифікаційний код 05417124
м. КИЇВ

A. Zagorodny
Academician
Director of the BITP

Causal orders, quantum circuits and spacetime: distinguishing between definite and superposed causal orders

Nikola Paunković[1] and Marko Vojinović[2]

[1]Instituto de Telecomunicações and Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Avenida Rovisco Pais 1049-001, Lisboa, Portugal

[2]Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia

We study the notion of causal orders for the cases of (classical and quantum) circuits and spacetime events. We show that every circuit can be immersed into a classical spacetime, preserving the compatibility between the two causal structures. Using the process matrix formalism, we analyse the realisations of the quantum switch using 4 and 3 spacetime events in classical spacetimes with fixed causal orders, and the realisation of a gravitational switch with only 2 spacetime events that features superpositions of different gravitational field configurations and their respective causal orders. We show that the current quantum switch experimental implementations do not feature superpositions of causal orders between spacetime events, and that these superpositions can only occur in the context of superposed gravitational fields. We also discuss a recently introduced operational notion of an event, which does allow for superpositions of respective causal orders in flat spacetime quantum switch implementations. We construct two observables that can distinguish between the quantum switch realisations in classical spacetimes, and gravitational switch implementations in superposed spacetimes. Finally, we discuss our results in the light of the modern relational approach to physics.

1 Introduction

The notion of causality is one of the most prominent in science, and also in philosophy of Nature. Its treatment separates Aristotelian from the modern physics, and its *formal* meaning within the latter is likely to have played a significant role, over the past centuries since Galileo, in forming our current *everyday* understanding of the notion of causality. While in Newtonian physics the cause-effect relations were encompassed by a rather simple linear and absolute time, Einstein's analysis of causal relations was pivotal in the formulation of the theory of relativity. But it was quantum mechanics (QM) that, through the EPR argument [1], further formalised by Bell [2], showed how quantum nonlocality, rooted in the superposition principle of QM, revolutionised our everyday notion of causality. Finally, strong theoretical evidence that, when combining the two fundamental theories of the modern physics, one is to expect explicit dynamical nonlocal effects in quantum gravity (QG), shows that our basic understanding of causality and causal orders might be crucial in the development of new physics.

Recently, causal orders were, mainly within the quantum information community, discussed in the context of controlled operations. In particular, it was argued that the quantum switch, a specific controlled operation introduced in [3], exhibits superpositions of causal orders, not only in the context of quantised gravity, where genuine superpositions of different states of gravity are present, but also in the experimental realisations performed in classical spacetimes with fixed causal structure [4, 5, 6]. Note that the notion of causal order discussed in these papers is *different* from the causal order of the underlying spacetime structure. We discuss in detail the relation between the two.

In this paper, we analyse the notion of causal orders in the context of classical and quantum circuits, and relate it to the spacetime causal structures. We prove that each circuit can be

Nikola Paunković: npaunkov@math.tecnico.ulisboa.pt
Marko Vojinović: vmarko@ipb.ac.rs

Accepted in Quantum 2020-05-12, click title to verify.

realised in a classical spacetime, preserving the fixed causal relations of the former, with respect to the causal relations between spacetime events of the latter (see the next section for the details of the theorem). Further, we analyse possible realisations of the quantum switch, showing that those performed in everyday labs do not feature superpositions of causal orders between spacetime events (consistent with our theorem), but rather standard non-relativistic quantum mechanical (coherent) superpositions of different evolutions of a system. On the other hand, we argue that genuine superpositions of different causal orders are indeed to be expected within the QG scenario, where superpositions of different states of the gravitational field, with their corresponding causal orders, are manifestly allowed (Hardy was one of the first to discuss the notion of superpositions of causal orders in the context of QG [7]). In addition, we explicitly construct two distinct observables that can distinguish between the realisations of the quantum switch in classical spacetimes, and implementations of the gravitational switch in superposed spacetimes. This way, we show that the two notions of causal orders, namely one discussed in [4, 5, 6] and the other discussed in this paper, can be experimentally distinguished, in contrast to the opposite claim present in the literature [4]. Finally, we discuss our results in the context of the relational approach to physics.

The layout of the paper is as follows. In Section 2, we introduce the notion of causal order for circuits, and prove the Theorem of the circuit immersion in classical spacetimes. Section 3 is devoted to the analysis of the quantum switch implementations in classical spacetimes that do not feature superpositions of spacetime causal orders, as well as implementations in the context of QG. In Section 4, we compare the quantum switch implementations discussed, and introduce observables that can distinguish between those that feature superpositions of spacetime causal orders, and those that do not. Section 5 is devoted to the discussion of the superpositions of causal orders in the context of the relational approach to physics. Finally, in Section 6, we present and discuss the results, provide some final remarks, and list possible future research directions.

2 Causal orders

We begin by discussing circuits and their realisations in (classical) spacetimes with well defined fixed causal orders. Given a directional acyclic graph $G = (I, E)$, where I is the set of graph nodes, and $E = \{(u, v) \,|\, u, v \in I\}$ is the set of its directed edges (arrows pointing from u to v representing the *wires* of the circuit), a *circuit* $\mathcal{C}$ over the set of operations $\mathcal{G}$ is a pair $\mathcal{C} = (G, g)$, where the mapping $g : I \to \mathcal{G}$ assigns operations to each node. Depending on the type of the operations from $\mathcal{G}$, we will call the circuit *classical* (if the operations are, say, classical logic gates), or *quantum* (if the operations are, say, unitaries, measurements, etc.).

The fact that G is directional and acyclic allows one to define a *partial order* $\prec_I$ over the set I as

$$u \prec_I v \quad \overset{\text{def}}{\iff} \quad \Big(\exists\, n \in \mathbb{N} \;\wedge\; \{u \equiv u_1, u_2, \dots, u_n \equiv v\} \subset I\Big) \quad \Big(\forall\, i \in \{1, 2, \dots, n-1\}\Big)\; (u_i, u_{i+1}) \in E\,, \tag{1}$$

representing the causal relation between the graph nodes. Next, we define the set of *gates of the circuit* $\mathcal{C}$ as $\mathcal{G}_\mathcal{C} = \{g_u \equiv (u, g(u)) \,|\, u \in I\}$. The induced causal order between the circuit gates $\prec_\mathcal{C}$ is by definition given as

$$g_u \prec_\mathcal{C} g_v \quad \overset{\text{def}}{\iff} \quad u \prec_I v\,. \tag{2}$$

Moreover, since there exists a canonical bijection between I and $\mathcal{G}_\mathcal{C}$, the order relations $\prec_I$ and $\prec_\mathcal{C}$ are *isomorphic*.

Finally, we can introduce the set $\mathcal{M}$ of all spacetime events, which is assumed to be a traditional $4D$ manifold. On this spacetime manifold we assume to have a gravitational field, described in a standard way, using a metric tensor $g_{\mu\nu}$. The metric is assumed to be of Minkowski signature, such that the metric-induced light cone structure determines a partial order relation between nearby events, denoted $\prec^g_\mathcal{M}$ (or simply $\prec_\mathcal{M}$ when the choice of the metric is implicit). Note that the causal order over the spacetime events is not an intrinsic property of the spacetime manifold itself, but rather determined by the metric, i.e., the configuration of the gravitational field living on the manifold.

One might pose a question if, given a formal circuit $\mathcal{C}$ with gates $\mathcal{G}_\mathcal{C}$, it is possible to realise it in a lab — if it is possible to "immerse" it into spacetime. More precisely, given an arbitrary spacetime manifold $\mathcal{M}$, our goal is to study if there exists an *order-preserving map* $\mathcal{P} : \mathcal{G}_\mathcal{C} \to \mathcal{M}$, i.e., if the partial order relations satisfy

$$g_u \prec_\mathcal{C} g_v \implies \mathcal{P}(g_u) \prec_\mathcal{M} \mathcal{P}(g_v) , \qquad (3)$$

for every $g_u, g_v \in \mathcal{G}_\mathcal{C}$. To that end, we formulate the following theorem (the proof is given in Appendix A).

Theorem. *Any circuit $\mathcal{C}$ can be immersed into a globally hyperbolic spacetime manifold $\mathcal{M}$, such that its relation of partial order $\prec_\mathcal{C}$ is preserved by the relation of spacetime events $\prec_\mathcal{M}$.*

Regarding the physical interpretation of the Theorem, note that it assigns a spacetime point to each gate in a circuit, as opposed to a point in $3D$ space. Since each spatially localised apparatus may perform the same operation more than once, at different moments in time, it may then correspond to several different gates of the circuit, and thus several different nodes of the graph, instead of just one. In other words, a single piece of experimental equipment *does not* always correspond to a single gate of a circuit.

In addition to the above comment, note that in reality each operation actually takes place in some finite volume of both space and time. However, in theoretical arguments it is convenient to approximate this finite spacetime volume with a single point, ignoring the size and time of activity of the device performing the operation. We adopt this approximation throughout this paper.

Circuits are seen as operations acting upon certain inputs to obtain the corresponding outputs. Usually, the initial/final states (which include instructions, measurement results, etc.) are depicted by the wires. But in our approach, the input state is prepared by the "initial gate" $\mathcal{I}$, while the output state is obtained by the "final gate" $\mathcal{F}$. This way, the circuit $\mathcal{C}$ is seen as an operation $\mathcal{O}_\mathcal{C}$ acting from $\mathcal{I}$ to $\mathcal{F}$.

Note that, given a circuit $\mathcal{C}$, the corresponding overall operation $\mathcal{O}_\mathcal{C}$ (as well as the input and the output gates $\mathcal{I}$ and $\mathcal{F}$) is uniquely defined. The opposite is not the case: given the operation $\mathcal{O}$, one can design different circuits $\mathcal{C}, \mathcal{C}', \dots$ that achieve it. To see this, let us consider the simplest case of the operation which satisfies $\mathcal{O} = \mathcal{O}_2 \circ \mathcal{O}_1$, where $\circ$ represents the composition of operations. This operation can be trivially achieved by the two circuits: (i) $\mathcal{C}$, which consists of three nodes — node i whose gate $\mathcal{I}$ prepares the input state, node o that applies the gate $g_o = \mathcal{O}$, and node f whose gate $\mathcal{F}$ outputs either the quantum state, the classical outcome(s), or the combination of the two; (ii) $\mathcal{C}_{12}$, which consists of four nodes — nodes i and f that perform the same operations as before, and *two* intermediate nodes o_1 and o_2 that perform $g_{o_1} = \mathcal{O}_1$ and $g_{o_2} = \mathcal{O}_2$, respectively. For simplicity, here and elsewhere in the text, by $\mathcal{O}$ we denote both the operation and the gate that implements it. The two situations are depicted in the following diagrams (see Figure 1).

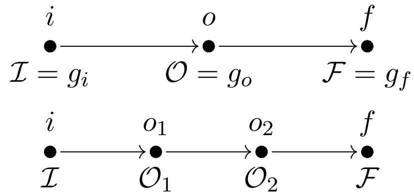


Figure 1: Implementing operation $\mathcal{O}$ with a single gate (upper diagram), and by two consecutive gates $\mathcal{O}_1$ and $\mathcal{O}_2$ (lower diagram).

Finally, in recent literature one can find a notion of an event which is different from the notion of a spacetime point [3, 4, 5, 6, 8, 9, 10, 11]. Namely, one can talk about events as interactions between the quantum system under consideration and the apparatus in the lab. This is motivated by the operational approach to physics, where the interactions between objects are taken as fundamental. Then, one can introduce the relation of partial order, which reflects the causal relationships between such events. Of course, in general, this causal order does not need to coincide with the spacetime causal order. Throughout this paper, if not explicitly stated otherwise, by causal order we mean the order between the spacetime points, which due to our Theorem can also be regarded as the order between the circuit gates. We discuss the difference between the two notions of causal orders in Section 4.

3 Quantum switch

The most prominent feature of quantum systems is that they can be found in *coherent superpositions* of states. This allows for applying the so-called *control operations*. For simplicity, let us assume that operations $\mathcal{O}$ are unitaries, denoted as U. Given a *control* system C in a superposition $|\varphi\rangle_C = a|0\rangle_C + b|1\rangle_C$ (with $\langle 0|1\rangle_C = 0$), the control operation

$$U_{CT} = |0\rangle_C\langle 0| \otimes U_0 + |1\rangle_C\langle 1| \otimes U_1 \qquad (4)$$

transforms the initial product state $|\Psi_i\rangle_{CT} = |\varphi\rangle_C \otimes |\psi_i\rangle_T$ between the control and the *target* systems into the final entangled state $|\Psi_f\rangle_{CT} = a|0\rangle_C \otimes U_0|\psi_i\rangle_T + b|1\rangle_C \otimes U_1|\psi_i\rangle_T$. A simple realisation of such operation by a circuit consisting of three gates is shown below (see Figure 2).

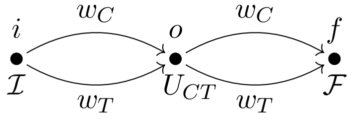


Figure 2: Controlled operation U_{CT}. Applying operation U_b on a system in the wire w_T *controlled* by the state $|b\rangle$ on a system in the wire w_C, with $b = 0, 1$.

Here, the first node and the corresponding gate prepares the initial superposition of the control system, the second implements U_{CT}, and the third is either an identity, a measurement on the two systems, or a combination (say, a measurement of the target qubit, while leaving the control intact). In order to allow for the description of quantum superpositions, we introduce the notion of a vacuum in the analysis of quantum circuits, as is done for example in [12] (for technical details, see Appendix B).

As noted above, given the operation, many different circuits can achieve it. Indeed, in standard optical implementations of the above controlled operation (4), the control qubit is spanned by two spatial modes of a photon, while the target one is its polarisation degree of freedom. The initial superposition state of the control qubit is prepared by a beam splitter, while the two operations U_0 and U_1 are implemented locally in Alice's and Bob's laboratories. Note that, since the control qubit is achieved by the means of two spatial modes of a single photon, while the target qubit is, being the photon's polarisation, "attached to" the control, the target is formally achieved by two degrees of freedom (two wires), one assigned to Alice (T_A), and the other to Bob (T_B). Thus, in such a realization, the control degree of freedom is redundant in the circuit diagram and can be omitted. Nevertheless, since we will later discuss the case of the gravitational quantum switch, in which the gravitational degree of freedom plays the role of the control, here we keep its corresponding wire and gate in the diagram, as presented below (see Figure 3).

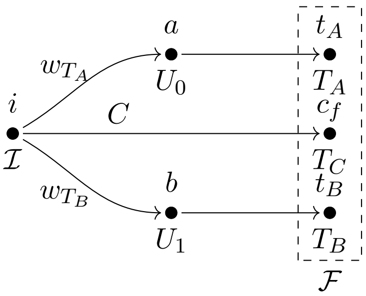


Figure 3: Implementation of the controlled operation using the spatial degree of freedom as a control.

The final gate $\mathcal{F}$ consists of three "elementary gates", represented by the circuit nodes t_A and t_B for the two target wires, and the node c_f for the final control wire.

An important instance of controlled operations is the so-called *quantum switch*, for which the two controlled operations are given by $U_0 = UV$ and $U_1 = VU$, where U and V are two arbitrary unitaries [3]. Having two pairs of equipment, one applying U and the other V, it is straightforward to implement the quantum switch through the circuit similar to the one above, which instead of two gates, one in the node a applying U_0, and another in node b applying U_1, contains four gates placed in the nodes a_U, a_V, b_V and b_U (see Figure 4).

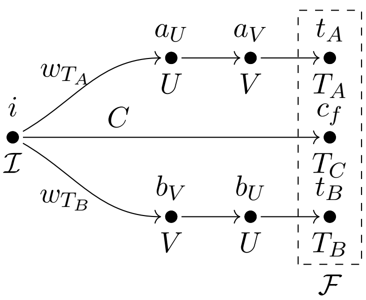


Figure 4: The quantum switch.

The question arises, is it possible to achieve the same using less resources, say, using only two such pieces of equipment, located in two different points (regions) of 3D *space*? Indeed, it is possible to do so, and recently a number of implementations of the quantum switch were performed in flat Minkowski spacetime [4, 5, 6]. Nevertheless, such implementations still correspond to circuits that implement U_0 and U_1 by four, rather than two gates. The difference is that, when immersing it in a flat spacetime, the two pairs of gates are now distinguished only by the temporal, rather than all four spacetime coordinates. Thus, one cannot talk of superpositions of causal orders between spacetime events in such implementations, as flat (indeed, any globally hyperbolic) spacetime has a manifestly fixed causal order. To implement U_0 and U_1 of the quantum switch by a circuit that consists of two gates only (and thus two corresponding spacetime points), one needs a superposition of gravitational fields with different (incompatible) causal orders. In the following two subsections, we analyse in more detail the "4-event" and the "3-event" implementations of the quantum switch, while the "2-event" case is discussed in the last subsection (the numbers 4, 3 and 2 refer to the numbers of spacetime events corresponding to distinct gates used to achieve U_0 and U_1). A detailed mathematical description using the process matrix formalism [8], is presented in the Appendices C, D and E.

Following the previously mentioned distinction between the spacetime event and the operational notion of the event, the 4-event and 3-event quantum switch implementations will have a description within the operational approach that is different from the spacetime description. In particular, in such approach these two implementations of quantum switch would feature only 2 operationally defined events, and thus the superposition of the corresponding causal orders.

3.1 4-event process

The realisations of the quantum switch are performed in table-top experiments in the gravitational field of the Earth, and can be for all practical purposes considered as being performed in flat Minkowski spacetime. In such experiments, Alice performs the unitary U in her localised laboratory, and Bob performs V in his separate localised laboratory, such that both are stationary with respect to each other and the Earth. The operations are applied on a single particle that arrives from the beam splitter, in a superposition of trajectories towards Alice and Bob, and, upon the exchange between the two agents, is finally recombined on the same beam splitter (for simplicity, we chose one beam splitter, but the whole analysis equally holds for two spatially separated beam splitters), and then measured. Below, we present a spacetime diagram of this experimental realisation of the quantum switch, which also represents a circuit of the implementation scheme (see Figure 5).

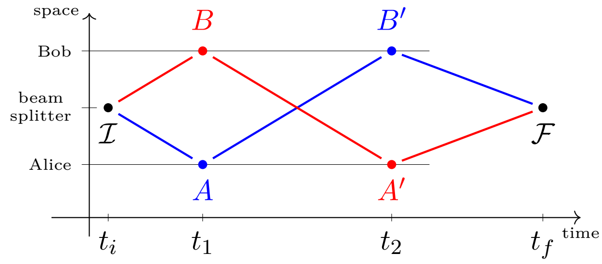


Figure 5: Spacetime diagram, as well as the circuit representation, of the 4-event implementation of the quantum switch.

Black horizontal lines represent world lines for Alice and Bob, as well as the global time coordinate line at the bottom. The black vertical line represents global space coordinate line. Quantum gates are represented by big dots. The composite gate $\mathcal{I}$ consists of the two preparation gates and the initial beam splitter gate, while $\mathcal{F}$ consists of the final beam splitter gate and the target gates that perform the final measurements (for details, see Appendix C). For simplicity, from now on we omit writing the labels of the nodes and keep only the labels of the corresponding circuit gates. The two histories of the particle exchanged between Alice and Bob, representing Alice's and Bob's wires, are full lines coloured in blue and red, respectively.

From the diagram we can see that in the blue history we have the following chain of gates

$$\mathcal{I} \prec_C A \prec_C B' \prec_C \mathcal{F}\,, \tag{5}$$

while for the red history we have

$$\mathcal{I} \prec_C B \prec_C A' \prec_C \mathcal{F}\,. \tag{6}$$

In total, there are four spacetime events involving Alice's and Bob's actions on the particle (gates), namely A, B, A' and B'. Thus, we call the above diagram the "4-event diagram". This setup was already discussed in the literature (see the very end of the Supplementary Notes of [13]).

In order to compare the cases of the quantum and the gravitational switches, it would be interesting to analyse the two examples within recently introduced powerful *process matrix* formalism [8]. To do so, one needs to formulate the formalism involving the vacuum state (see Appendix B for details). The straightforward application of the formalism to the 4-event case is in full accord with the experimental results, as demonstrated in Appendix C.

3.2 3-event process

One can imagine that instead of two, one of the agents implements only one gate. For example, by conveniently choosing the velocity of the particle along its trajectory between Alice and Bob, we can identify Bob's two gates,

$$B \equiv B' \,. \tag{7}$$

We thus arrive to the new spacetime diagram and the associated circuit, called the "3-event diagram" (see Figure 6).

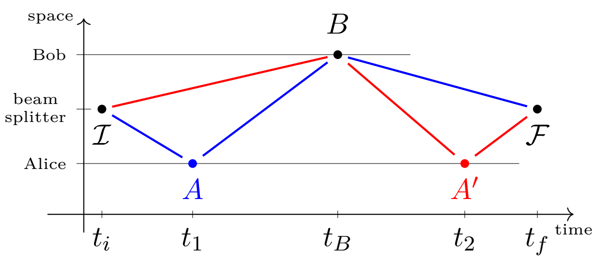


Figure 6: Spacetime diagram, as well as the circuit representation, of the 3-event implementation of the quantum switch.

Now, the obvious question is the following — can we, in addition to (7), impose also that

$$A \equiv A' \,, \tag{8}$$

i.e., also identify Alice's gates into a single spacetime event? In flat Minkowski spacetime, the answer is negative. Namely, by simply looking at the 3-event diagram one can see that the trajectory of the particle between Alice and Bob would require either superluminal speed, or backwards-in-time trajectory in at least one history (note that the diagram assumes that light propagates along the lines that form the 45° angle with the coordinate axes). This is also seen directly from inequalities (5) and (6): identifying both $A \equiv A'$ and $B \equiv B'$ would lead to requiring that *both* $A \prec_{\mathcal{C}} B$ and $B \prec_{\mathcal{C}} A$ are satisfied, i.e., $A \equiv B$. As guaranteed by our Theorem from Section 2, in a curved spacetime it is also impossible to make both identifications (7) and (8), at least if spacetime were globally hyperbolic. Finally, as in the 4-event case, here also the process matrix formalism is consistent with the experimental results, see Appendix D.

3.3 2-event process — gravitational switch

Despite the conclusion of the previous subsection, within the framework of quantum gravity one is allowed to construct superpositions of different gravitational field configurations, leading to superpositions of different causal structures for the spacetime manifold. The assumption of superpositions of different gravitational field configurations is common to all models of QG. Other than that, we will not have any additional assumptions, and thus our approach does not depend on any particular QG model.

In what follows, for the sake of concreteness, we assume the "traditional" approach to the formulation of the QG formalism. Namely, we assume that there exists a smooth $4D$ manifold, called *spacetime*, and denoted as $\mathcal{M}$. Quantum fields, including the gravitational field, live on top of $\mathcal{M}$. The gravitational field is described either via the metric or via some other degrees of freedom (for example, tetrads and spin connection), such that the metric is a function of these. We call this kind of construction "traditional" because it represents a minimal deviation from the mathematical structure of quantum field theory (QFT) in flat Minkowski spacetime, in the sense of preserving the underlying manifold structure. A QG model implementing this approach is, for example, the asymptotic safety framework [14]. Of course, we do not aim to provide a full-fledged model of QG, but rather to only specify the status of the manifold structure within it. As an alternative, in Subsection 5.2, we will discuss the relational framework of QG in which the manifold structure does not exist a priori, but is emergent from relational properties of quantum fields themselves. Finally, note that the discussion of the flat-spacetime cases in the previous sections implicitly assumes the traditional point of view on spacetime manifold. Nevertheless, it has to be compatible with the semiclassical limit of any viable QG model.

As a consequence of the superposition of causal structures in QG, it is possible to achieve a *gravitational switch*, which implements the same quantum switch as described above, with a circuit consisting (in addition to the initial and final gates $\mathcal{I}$ and $\mathcal{F}$) of only two gates: the Alice's gate A that applies U, and Bob's gate B that applies V. Superposing two gravity-matter states, such that in the first the spacetime geometry (described by the metric tensor g_0) establishes the causal structure

$$\mathcal{I} \prec_{\mathcal{M}}^{g_0} A \prec_{\mathcal{M}}^{g_0} B \prec_{\mathcal{M}}^{g_0} \mathcal{F} \,, \tag{9}$$

while in the second (described by the metric g_1) it is

$$\mathcal{I} \prec_{\mathcal{M}}^{g_1} B \prec_{\mathcal{M}}^{g_1} A \prec_{\mathcal{M}}^{g_1} \mathcal{F} \,, \tag{10}$$

the overall circuit applies operations $U_0 = UV$ and $U_1 = VU$, conditioned on the state of gravity. As a side note, it is clear from (9) and (10) that superpositions of the spacetime causal orders can occur only in the framework of quantum gravity.

Such a switch was previously introduced by Zych *et al.* [15], in the context of two spacetimes which are solutions of the Einstein equations. In their proposal, the beam splitter acted *only* on the gravitational degree of freedom (and the accompanied source, the planet), while leaving the rest of the matter, in particular the particle, Alice and Bob, unaffected. Upon the final beam splitter recombination, the matter is left in an *incoherent* mixture of two states proportional to $\{\, U \,,\, V \,\}\, |\Psi\rangle$ and $[\, U \,,\, V \,]\, |\Psi\rangle$. Subsequently, the mass (along with its gravitational degrees of freedom) is being measured in the superposition basis. Upon a post-selection conditioned on the outcome of the measurement, the matter is again in a pure state.

Another way to obtain a genuine superposition of two different causal orders is by using a spatially delocalised beam splitter, that acts on both gravitational and matter fields. This can be depicted by the following 2-event diagram (see Figure 7).

The yellow region in this diagram represents a compact piece of spacetime where the gravitational field is in a superposition of the two distinct states, and plays the role of the control degree of freedom. Along the boundary of that region, both gravitational configurations smoothly join into a single configuration outside. The boundary of the yellow region thus acts as a beam splitter for anything that enters, and again (in the recombining role) for anything that exits. Therefore, all worldlines (namely, of Alice, Bob and the particle) are doubled inside the yellow region. The blue and red colours represent their spacetime trajectories in two different gravitational field backgrounds, respectively.

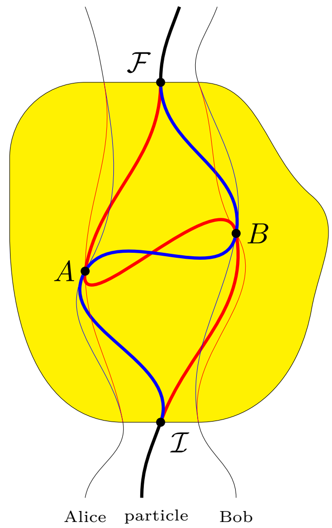


Figure 7: Spacetime diagram of the 2-event implementation of the gravitational switch. Note that formally this is not a circuit diagram, as the control wire, implemented by the state of the gravitational field in the yellow region, is missing.

We model our gravitational switch such that the overall output state is the product between the state of the gravitational field and the state of the particle. The state of the particle is of the form $(\alpha UV + \beta VU)|\Psi\rangle$, obtained without performing final selective measurement. In particular, in order to compare it with the other quantum switch realizations, we choose either $\{\, U \,,\, V \,\}\, |\Psi\rangle$ or $[\, U \,,\, V \,]\, |\Psi\rangle$. In order to achieve this, the gravitational switch should act upon *all* degrees of freedom, both gravitational and matter. Note that our gravitational switch does require certain fine tuning, in the sense that the whole, delocalised beam splitter, that acts nontrivially on the whole joint gravity-matter system, is designed for the particular pair of operations applied by Alice and Bob: only for those operations, the beam splitter will output the product state between gravity and matter. Otherwise, the output will be the entangled gravity-matter state, like in the cases of the optical quantum switch and the gravitational switch introduced by Zych et al. (before the final selective measurement). Still, the process matrix describing the gravitational switch itself is independent of the choice of the gate operations of the agents. See Appendix E for details.

The question whether this kind of diagram is admissible in some theory of quantum gravity is nontrivial, and model dependent, on several grounds. First, it is impossible to construct this diagram by superposing two classical configurations of gravitational field, such that each configuration satisfies Einstein equations. The reason is simple — assuming that the gravitational field is specified outside the yellow region, Einstein equations have a unique solution (up to diffeomorphism symmetry) for the compact yellow region, given such a boundary condition. Therefore, one cannot have two different solutions to superpose inside. The only two options are to either superpose one on-shell and one off-shell configuration of gravity, or two off-shell configurations. This scenario can arguably be considered within the path integral framework for quantum gravity.

Second, the question of the particle trajectory is nontrivial. Namely, given one gravitational configuration in which the particle has the spacetime causal structure (9), corresponding to the blue history, it is not obvious that there can exist another gravitational configuration (with the same boundary conditions at the edge of the yellow region), in which the particle has the spacetime causal structure (10), corresponding to the red history. Even if one admits arbitrary off-shell configurations of gravity, it may turn out that the order of events inside the yellow region must be fixed by the boundary conditions. The only viable way to answer this question is to try and construct an explicit example of two geometries implementing (9) and (10) for the same boundary conditions. Numerical investigations are underway to explore this possibility.

4 Distinguishing 2-, 3-, and 4-event realisations of the quantum switch

In a number of both theoretical proposals [3, 8, 9, 10, 11], as well as experimental realisations [4, 5, 6] of the quantum switch, it is claimed that they feature genuine superpositions of causal orders. The reason for this is the introduction of an alternative, operational notion of the event, which differs from a spacetime point. The motivation for this lies in the claim that the individual spacetime points A and A' (and B and B') do not have an operational meaning. In words of the authors of [4] (see the Discussion section):

> *"The results of the experiment confirm that such [which way] information is not available anywhere and that the interpretation of the experiment in terms of four, causally-ordered events cannot be given any operational meaning. If, on the other hand, one requires events to be defined operationally, in terms of measurable interactions with physical systems [...], then the experiment should be described in terms of only two events — a single use of each of the two gates."*

While it is obvious that the mentioned which-way information is not available in the quantum switch experiment, in what follows we argue that this does not imply that one cannot give an operational meaning to spacetime points, even in the context of the quantum switch in classical geometries.

Below, we first present a critical analysis of the arguments behind introducing the operational notion of event. Then, we show how one can experimentally, at least in principle, distinguish 2-, 3-, and 4-event realisations of the quantum switch.

It is the operational approach to understanding spacetime, applied within the framework of relationalism (see Section 5 for a detailed discussion of the relation between the two frameworks), that is arguably the main argument for introducing the alternative notion of an event. This new notion of an event gives rise to the superposition of respective causal orders in the realisations of the quantum switch even in classical spacetimes. Assuming that the smooth (classical) spacetime is an emergent phenomenon, in the operational approach one considers "closed laboratories" [8] as the primal entities within which one can locally apply standard quantum mechanics, while their connections form the relations from which the spacetime emerges. Indeed, it seems that the process matrix formalism was developed precisely with this idea in mind: to be a mathematical tool in analysing the emergence of the spacetime through the relations between the closed laboratories. We would like to note that, as shown in Appendices C, D and E, the mentioned formalism is also fully applicable within the standard formulation of quantum mechanics in classical Minkowski spacetime.

Given that in the case of coherent superpositions of the two paths (a particle first goes to Alice, then to Bob, and vice versa) it is not possible to know which of the two has actually been taken, one may conclude that one cannot distinguish between spacetime events A and A', and that the two are operationally given by the single action of a *spatially* localised laboratory. However, this point of view is at odds with our understanding of the ordinary double slit experiment. Namely, by exchanging the roles of time and space, and following the above logic, applied to the case of the standard double slit experiment, one could analogously conclude that, since in order to obtain the interference pattern at the screen one must not (and thus cannot) learn which slit the particle went through, the two slits represent one and the same operational "lab", and one operational point (region) in space.

Let us explain our argument in slightly more detail. Consider first the optical quantum switch. Here, a particle passes through Alice's lab, described by the two spacetime points, (x_A, t) and (x_A, t'). Any attempt to distinguish the times t and t' at which the particle passes through Alice's lab would destroy the superposition. Consider now the standard double slit experiment. Here, a particle passes through the two slits, described by the two spacetime points, (x_L, t) and (x_R, t). Any attempt to distinguish the positions of the slits x_L and x_R through which the particle passes would destroy the superposition. Note that by exchanging the roles of space and time, the descriptions of the above two situations are essentially identical.

According to the operational approach, as a consequence of the above, one should describe Alice's actions in the optical quantum switch in terms of only one operational event. Thus, analogously, one should also describe the particle passing through the slits in terms of only one operational event. However, such interpretation of the double slit experiment is, to the best of our knowledge, absent from the literature.

Note also that the 3-event realisation of the quantum switch offers a natural alternative interpretation of this phenomenon, as a well known *time double slit experiment* [16]. Indeed, the two events (gates) A and A' play the role of the two time-like slits, while the event (gate) B separates the two in the same way the closed shutter separates the two time-like slits in the time double slit experiment. This comes as no surprise: quantum superpositions are in general accompanied by the interference effects, and the quantum switch is, as already emphasised in Section 3, just another instance of a superposition of two different states of the standard quantum mechanics in Minkowski spacetime.

The operational interpretation of identifying the events A and A' in the current experimental realisations of the quantum switch indeed seems to be a tempting proposal. Nevertheless, we would like to point out that in fact it does not resolve any open problem. In addition, being similar to Mach's ideas, it too may be at odds with the theory of general relativity (GR), see Subsection 5.1 for a detailed discussion.

4.1 Distinguishing by decohering the particle

In the above quote from [4], the authors claim that in order to directly distinguish points A and A' (as well as B and B'), one *must* destroy the superposition in the apparatus. Conversely, being unable to distinguish those points in any experiment that maintains superposition and realises the quantum switch, one cannot give them operational meaning. Therefore, those spacetime points are redundant in the theory, and each pair should be replaced by a single operational event. In this subsection, we discuss this type of an argument. In the next, we give an explicit example of an observable that does distinguish such spacetime points without obtaining the which way information.

Let us study one concrete way of distinguishing the mentioned pairs of points, which decoheres the particle. For simplicity, we will analyse the 4- and 2-event cases only. To this end, we will introduce a third agent, Alice's and Bob's *Friend*. At each run of the quantum switch experiment, Alice will, independently and at random, decide whether just to apply her operation onto the particle, or in addition to that, send a photon to Friend. The same holds for Bob. In 25% of the cases, both agents just perform their respective operations, thus performing the quantum switch. Next, in the 25% of the cases, both agents decide, in addition to applying their respective operations, to send the photons to Friend, who detects them in his spatially localised lab. The remaining 50% of the cases are essentially the same as the previous ones, so for simplicity we omit their analysis.

First, we present the spacetime diagram of the 4-event quantum switch for the case when the agents decide to send the photons to Friend (see Figure 8).

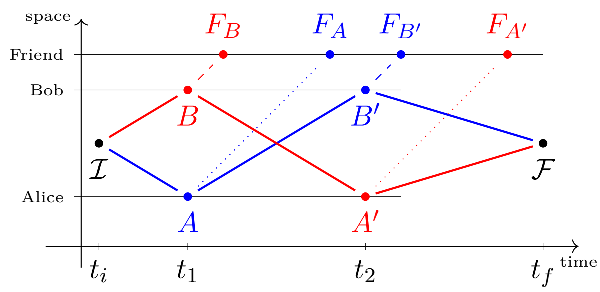


Figure 8: Distinguishing spacetime points by decohering the particle in the 4-event quantum switch. The dotted (dashed) lines represent photons sent by Alice (Bob) to Friend.

The photons coming from Alice are dotted, while the photons coming from Bob are dashed. By knowing the geometry of the whole experiment, Friend would be able to measure, in a generic setup, four *different* times of the photon arrivals: t_A and $t_{A'}$ for spacetime points F_A and $F_{A'}$, and two more for the photons sent by Bob.

On the other hand, in the case of the 2-event gravitational switch realisation, Friend would detect only two times of the photons' arrival. Below, we extend the diagram of the gravitational switch we introduced in Section 3.3, by adding the photons sent to Friend. In order to indicate the fact that the events A and A', etc., are in this setup indeed identified, we write the tilde over the corresponding letters A, B and F (see Figure 9).

Clearly, the two situations are experimentally distinguishable.

Nevertheless, as noted in [4], one might argue that, since the photons sent to Friend in the 4-event case decohere the particle in the switch, this situation does not correspond to the experiment in which the coherence is maintained. Therefore, in the latter, the pair of spacetime events A and

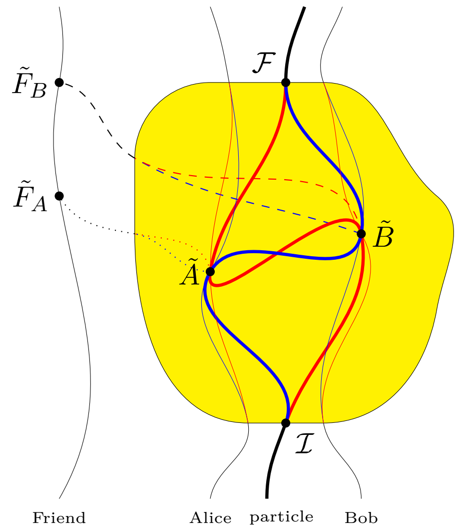


Figure 9: Distinguishing spacetime points by decohering the particle in the 2-event gravitational switch. The dotted (dashed) lines represent photons sent by Alice (Bob) to Friend.

A' still ought to be substituted with a single operational event (and analogously for B and B').

However, even if instead of spacetime points one decides to talk about operational events, such a framework should still be consistent with the experimentally tested theories, GR in particular. According to GR, in flat spacetime (or in any classical configuration of the gravitational field), regardless of whether we decohere the particle or not, both experiments feature four spacetime points, such that A and B (as well as A' and B') can be considered to be simultaneous (see Figure 8). Therefore, the time of execution of both experiments is $\delta t = t_2 - t_1 + C$, where $C \equiv (t_1 - t_i) + (t_f - t_2)$. Note that the time period $t_2 - t_1$ represents the travel time of the particle from one laboratory to the other, and is therefore strictly positive.

From the operational point of view, the decohered version of the experiment also features four operational events, and is thus manifestly consistent with the GR description. Note that a decohered version of the switch still features only two events *per run*: in a classical mixture between "Alice's event before Bob's event" and "Bob's event before Alice's event" each run features just two events, and the duration of the overall experiment in each run is the time between the two events of that run (plus the above constant C).

On the other hand, if the coherence were maintained, the operational point of view features only two operational events, one per laboratory. Then, the total time of execution of the experiment ought to be $\delta\tau = 0 + C$, which is clearly different from the GR prediction. The total time of execution of the quantum switch experiment is a measurable quantity. This means that one can easily determine whether this time is $\delta\tau$ or δt. The former outcome invalidates GR, which would necessitate the formulation of an alternative theory. Note that in this case, a sheer decision to either decohere a particle or not would allow agents to influence the time flow in their labs. Moreover, it raises the question of the time flow in nearby labs *isolated* from the experiment during its execution. The latter outcome poses the problem of the precise formulation of an *operational theory* such that the experiment which features only two operational events lasts precisely the same time as the experiment which features four operational events.

4.2 Distinguishing without decohering the particle

In addition to the above argument, supported by the experimental setup presented in the previous subsection, by erasing the which way information it *is* possible for Friend to distinguish the 4-event and the 2-event realisations even when the "full" quantum switch is executed. For that, one needs to supply Friend with a photon non-demolition measurement. This is in principle possible to construct, although in practice a bit challenging. It thus might be technically easier to use some particles other than photons for sending signals to Friend.

By agreeing in advance of the particular experimental setup, Friend would be able to predict the *distinct* times of arrival of the photons, t_{F_A}, $t_{F_{A'}}$, t_{F_B} and $t_{F_{B'}}$ in the 4-event case, and $t_{\tilde{F}_A}$, $t_{\tilde{F}_B}$ in the 2-event case, thus defining the states of the two photons that arrive to his lab: $|F_A, F_{B'}\rangle$, $|F_{A'}, F_B\rangle$, and $|\tilde{F}_A, \tilde{F}_B\rangle$, respectively. Let us define $\mathcal{H}_{A\prec B'} = \text{span}\{|F_A, F_{B'}\rangle\}$, $\mathcal{H}_{B\prec A'} = \text{span}\{|F_{A'}, F_B\rangle\}$, and $\mathcal{H}_{A\prec B\wedge B\prec A} = \text{span}\{|\tilde{F}_A, \tilde{F}_B\rangle\}$. Then, the relevant Hilbert space of the two photons is

$$\mathcal{H}_{ph} = \mathcal{H}_{A\prec B'} \oplus \mathcal{H}_{B\prec A'} \oplus \mathcal{H}_{A\prec B\wedge B\prec A}\,. \tag{11}$$

Let us define $P_<$, $P_>$ and $P_=$ as orthogonal projectors onto $\mathcal{H}_{A\prec B'}$, $\mathcal{H}_{B\prec A'}$ and $\mathcal{H}_{A\prec B\wedge B\prec A}$, respectively. One can then define a dichotomic photon non-demolition orthogonal observable performed by Friend on the two photons in his laboratory:

$$M = 1\cdot(P_< + P_>) + 0\cdot P_=\,. \tag{12}$$

Provided that the experimental setup is *either* that of the 4-event, or the 2-event type, such measurement would not change the state of the experimental setup (the interferometer, the particle in it, and the photons in the Friend's apparatus), while still leaking the information to Friend (via the measurement outcome) about the type of the quantum switch realisation. Finally, by performing the *quantum erasing* procedure [17, 18], the which way information is lost, and the final state of the particle is restored to a coherent superposition.

Let us examine this more formally. Let the two states of the particle in the quantum switch be $|R\rangle$ and $|B\rangle$, corresponding to the red and the blue trajectory, respectively. After $\mathcal{I}$, the state of the particle in the quantum switch is $\frac{1}{\sqrt{2}}(|R\rangle + |B\rangle)$. As the particle passes through Alice's and Bob's labs, two photons are emitted, which arrive at the Friend's lab. The overall state of the particle and the two photons in the 2-event quantum switch is then

$$\frac{1}{\sqrt{2}}\Big(|R\rangle + |B\rangle\Big)|\tilde{F}_A, \tilde{F}_B\rangle. \tag{13}$$

The particle in the quantum switch is in superposition of the two paths, and it stays so upon measuring M and obtaining the result 0.

On the other hand, the overall state of the particle and the two photons in the 4-event quantum switch is, upon the photons' arrival in the Friend's lab, given by

$$\begin{aligned} &\frac{1}{\sqrt{2}}\Big(|R\rangle|F_{A'}, F_B\rangle + |B\rangle|F_A, F_{B'}\rangle\Big) \qquad (14)\\ &= \frac{1}{2\sqrt{2}}\Big[\Big(|R\rangle + |B\rangle\Big)\Big(|F_{A'}, F_B\rangle + |F_A, F_{B'}\rangle\Big)\\ &\quad + \Big(|R\rangle - |B\rangle\Big)\Big(|F_{A'}, F_B\rangle - |F_A, F_{B'}\rangle\Big)\Big]. \end{aligned}$$

The particle is now decohered by the two photons, and it remains so upon measuring M and obtaining 1 as the result. Therefore, to erase the which way information, Friend has to perform an additional measurement in the basis

$$|\pm\rangle = \frac{1}{\sqrt{2}}\Big(|F_{A'}, F_B\rangle \pm |F_A, F_{B'}\rangle\Big), \qquad (15)$$

thus collapsing the state of the particle in one of the two pure states

$$\frac{1}{\sqrt{2}}\Big(|R\rangle \pm |B\rangle\Big). \qquad (16)$$

Knowing the outcome of the measurement of M, Friend can post-select the output of the particle coming out of the quantum switch. Alternatively, in the case of obtaining the $|-\rangle$ result, Friend can change the relative phase between the two of the particle's superposed states.

4.3 Other types of gravitational switches

It is important to note that the framework of QG also allows for the construction of 3- and 4-event switches, in addition to the 2-event one. This is straightforward to see, for example by immersing the above 3- or 4-event spacetime diagram into a superposition of different geometries.

Moreover, all of these gravitational switches may give different outcomes when measuring the observable M, given by (12), followed by the quantum erasing procedure (15). The criteria to necessarily obtain the outcome 0 are: (i) that the photons in red and blue histories meet at the boundary of the yellow region, and (ii) from that point on they recombine into a single photon history. Depending on the details of their construction, all gravitational switches either may or may not satisfy the criteria (i) and (ii). On the other hand, no quantum switch realisations in classical spacetimes with definite causal order could ever yield result 0. Finally, we note that even though some of 2-event gravitational switches may give the outcome 1 when measuring M, it does not necessarily mean that there exist no other observable that could distinguish them from the 4-event quantum switches in a classical geometry. This is a matter for further research.

Detailed graphical visualisations of various gravitational switches are presented in the Appendix F.

5 Relational approach to physics

In the light of the operational framework, which suggests the substitution of the spacetime events A and A' with a single operational event (and analogously for B and B'), it is important to comment on one different but related approach to understanding spacetime, called relationalism. Note that by this promotion of operational events as fundamental entities that ought to replace and play the role of the spacetime events, effectively means the identification of A with A', and B with B'. In this section, we first present a historical review of the relational approach to physics. Then, we discuss the operational framework within the context of the modern approach to relationalism.

5.1 Mach principle and the history of relationalism

The idea of relationalism is an old one, it traces back at least as far as Decartes, and is very important in human thought, in particular in the history of physics. It was brought back to science by Mach in the second half of the XIX century (for an overview and history of the Mach principle and the relational approach to space, from its origins in ancient Greece, see for example [19] and the references therein). Based on the Leibniz ideas of a relational world, Mach formulated his famous Mach principle, an intuitively reasonable approach in analysing physics, and space(time) relations in particular. One of the main characteristics of the Mach principle is that (see [20], page 17):

> *"Space as such plays no role in physics; it is merely an abstraction from the totality of spatial relations between material objects."*

The same formulation can be found in [21], slightly re-phrased as *"Mach7: If you take away all matter, there is no more space."* It is interesting to note that the authors attribute this formulation to A. S. Eddington [22], page 164.

As discussed at the beginning of Section 4, in the operational approach one attributes the ultimate existence to the "closed laboratories" only, while their mutual relations, epitomised by the process matrix, are then giving rise to higher level emergent entities. This clearly shows striking similarities between the Mach's and the operational approaches to space(time).

Mach's ideas were crucial for Einstein in formulating the theory of relativity. And while many of Mach's predictions were indeed realised in the new theory, some of them were not. Mach's idea that the matter is the basic entity, and that by abstracting the relations between the objects the space emerges, led him to the following statement: if the matter in the universe were finite and had 3D rotational symmetry, it would be impossible to determine its angular momentum (indeed, even talking about it would have no meaning). This is a plausible idea. Nevertheless, it does not hold in general relativity (GR), where one can find two solutions of the Einstein equations for the isolated black hole (the stationary Schwarzschild solution and the rotating Kerr solution [23]). Moreover, while according to the Mach principle the matter completely determines the space, this is not the case in GR: not only that there exists a solution for the gravitational field in the absence of matter (when the stress-energy tensor T is identically zero), but the solution is not unique, as it depends on the boundary conditions as well (i.e., flat Minkowski spacetime is not the only solution — gravitational waves being a possible alternative [23]). This also holds for the general $T \neq 0$ case, as there too boundary conditions play an important role. Thus, matter does not fully determine the inertia, as should according to Mach principle, which states that the inertia of a massive body is given solely in terms of its relations with the other massive bodies.

Motivated by giving the ultimate reality to material objects only (closed laboratories in the case of the operational approach), Mach formulated the above list of claims. Nevertheless, they were later shown not to hold in GR. Provided the similarities between the Mach ideas and the operational approach, the latter might face similar problems as well. We thus believe that introducing the operationalist notion of an event should be accompanied by more elaborate proposals of new physical hypotheses and theories. We hope that our discussion may serve as a small step towards achieving this goal.

5.2 Modern approach to relationalism

In contrast to the historical approach to relationalism and Mach's ideas, that sounded plausible at the time but ultimately failed with the development of GR, the more elaborate modern approach to relationalism is epitomised in the words of Carlo Rovelli (see Section 2.3 of [24]):

> *"The world is made up of fields. Physically, these do not live on spacetime. They live, so to say, on one another. No more fields on spacetime, just fields on fields."*

In particular, the modern relational approach to spacetime defines a particular spacetime point by the physical processes that are "happening at that point". More technically, given an ordered set of classical fields $\phi \equiv (\phi_1, \dots, \phi_n)$ used to describe physics in a given classical theoretical framework, one traditionally starts from some spacetime point $\tilde{x}$ and evaluates the fields at that point, $\tilde{\phi}_i = \phi_i(\tilde{x})$, obtaining an n-tuple of numbers $(\tilde{\phi}_1, \dots, \tilde{\phi}_n)$. The idea of relationalism does the opposite — one starts from n-tuples of field values, and then defines a spacetime point using an n-tuple, $\tilde{x} \equiv (\tilde{\phi}_1, \dots, \tilde{\phi}_n)$, so that the same equation $\tilde{\phi}_i = \phi_i(\tilde{x})$ holds. The question of how to operationally relate values of different fields, and assign and distribute them into n-tuples, is a matter of a separate study [25]. In this work, we assume that this problem is already solved. Moreover, note that fields ϕ need not be observable, due to potential gauge symmetries (for example, the electromagnetic potential A_μ and the metric $g_{\mu\nu}$). To that end, we introduce an ordered set of gauge invariant functions $\mathcal{O}(\phi) \equiv (\mathcal{O}_1(\phi), \dots, \mathcal{O}_m(\phi))$, where $m \geq n$ (for example, the electromagnetic field strength $F_{\mu\nu}$ and the curvature $R^\lambda{}_{\mu\nu\rho}$), and *define a spacetime point* as an m-tuple of their values $\tilde{\mathcal{O}}$.

Unless the physical system features some global symmetry, each m-tuple $\tilde{\mathcal{O}}$ defines a unique point in spacetime. Note that, in the context of GR, the absence of global symmetries is actually the generic case. Thus, the essential feature of this definition is that it does not make sense to say that the same m-tuple of field strengths can occur in two different spacetime points, since "both" spacetime points in question are defined in terms of the one and the same m-tuple, and therefore represent a *single* point.

Moving from classical to the quantum framework, where no system has predetermined physical properties independent of observation, one needs to talk about observables. Given an ordered set of quantum fields $\phi \equiv (\phi_1, \dots, \phi_n)$, one constructs one *specific* complete set of compatible observables $\mathcal{O} \equiv (\mathcal{O}_1(\phi), \dots . \mathcal{O}_m(\phi))$, where

- *compatible* means that all observables mutually commute, $[\mathcal{O}_i, \mathcal{O}_j] = 0$ for every i and j, while

- *complete* means that the eigenspaces common for all these observables are nondegenerate, i.e., they are one-dimensional subspaces of the total Hilbert space.

Here, by "specific" we mean the set of observables which depend *only* on fields ϕ, but *not* on their conjugated momenta. This fixes the coordinate representation, such that each common eigenvector corresponds to one classical configuration of fields. The outcomes of the measurements of these observables can then be grouped into m-tuples and used to *define individual spacetime points*, as in the classical case above, thus giving rise to an emergent classical spacetime. On the other hand, if the state is not an eigenvector of $\mathcal{O}$, one cannot speak of a single classical configuration of fields, and thus the notion of emergent spacetime and its points ceases to make sense globally, according to the relational approach. At most, one could speak of a superposition of classical configurations and corresponding emergent spacetimes, but without any natural way to relate spacetime points across different branches in the superposition. Nevertheless, this does not mean that establishing such a relation is impossible for certain subregions of spacetime. Indeed, the whole non-yellow "outside" part of the gravitational switch picture from Subsection 3.3 represents a subregion with a locally classical configuration and thus well defined spacetime points.

In order to better appreciate the relational definition of spacetime points given above, it is instructive to look at the realisation of spacetime in the context of a relational quantum gravity model, such as a spinfoam model in the Loop Quantum Gravity (LQG) framework [24, 26]. There, the spacetime is "built" out of the spin foam — a lattice-like structure with vertices, edges and faces, each labeled by the eigenvalues of particular field operators that "live" on these structures, depicted as follows (see Figure 10):

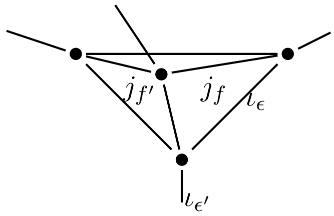


Figure 10: A piece of a spin foam diagram. The field j labels the faces $f, f', \dots$, while the field ι labels the edges $\epsilon, \epsilon', \dots$, of the diagram.

For example, the *area operator*, which is a function of the gravitational field, has eigenvalues determined by a half-integer label $j \in \mathbb{N}/2$, and each face of the spin foam carries one such label, specifying the area of the surface dual to that face. In particular, the spectrum of the area operator is given as

$$A(j) = 8\pi\gamma l_p^2 \sum_f \sqrt{j_f(j_f+1)}\,, \qquad (17)$$

where l_p is the Planck length, γ is the Barbero-Immirzi parameter, while the sum goes over all faces f of the spin foam that intersect the surface whose area we are interested in, see [24, 26] for details. All other physical observables similarly provide appropriate labels for each vertex, edge and face of the spin foam. Since edges and faces meet at vertices, a given vertex carries labels of all observables of all edges and faces that are connected to that vertex. These observables form the complete set of compatible observables $\mathcal{O}$, and their eigenvalues label each vertex, determining the identity of that vertex. In other words, each labeled vertex of a spin foam defines a "spacetime point", and if two vertices have completely identical properties in the sense of their labels and their connectedness to neighbouring objects, they actually represent the one and the same vertex.

At first sight, it is tempting to apply the ideas of relational spacetime to the case of the quantum switch, as follows. At the spacetime event A, Alice interacts with the particle as it enters and exits her lab, while at the spacetime event A' Alice also interacts (in exactly the same way) with the same particle. The idea of relational spacetime then might suggest that one should *define* the spacetime events A and A' by the physical event of interaction between Alice and the particle. Since this interaction is the same in both cases, one ought to identify the two points, $A \equiv A'$, and claim that both of these correspond to the same spacetime event, defined by the interaction between Alice and the particle. The same argument applies to Bob, and events B and B'.

Unfortunately, this argument is not fully in line with relationalism. The reason lies in the fact that the interaction between Alice and the particle (and also between Bob and the particle) does not meet the criteria given in the above relational definition of a spacetime point. Namely, neither Alice, nor Bob, performs a measurement of a *complete set of compatible observables* $\mathcal{O}$. The mentioned interaction with the particle is merely a subset of this. In particular, the interaction of Alice with the particle does not uniquely fix the state of, say, the gravitational field, or the electromagnetic field, or the Higgs field, etc. Therefore, it may happen that the measurement outcomes of the whole set of observables $\mathcal{O}$ at spacetime events A and A' are still mutually distinct, thereby defining the events A and A' as two distinguishable spacetime points. In order to be certain that A and A' are really the same spacetime event, Alice would need to measure the complete set of observables $\mathcal{O}$, and convince herself that the results of all those measurements at A and at A' are identical. The mere interaction with the particle is not enough to achieve this, and the experimental setups such as [4, 5, 6] obviously fall short of accounting for the state of all other possible physical fields that Alice and Bob can interact with, in addition to the interaction with the particle.

We see that, when applied to the case of the quantum switch in classical gravitational field, the relational framework is at odds with the operational approach — the former distinguishes A and A' while the latter regards them as identical. This is because the matter fields of the particle are in a superposition of two classical configurations. Similarly, in the case of the 2-event gravitational switch introduced in Subsection 3.3, the overall state of gravity and matter is a superposition of two distinct classical configurations. Therefore, within the relational framework, it is not possible to talk about a single emergent spacetime, nor to compare the points that belong to different branches. This is different from the operational approach, which aims to identify points from different branches. It is also different from the traditional approach, since the latter postulates a unique classical spacetime manifold.

Note that, if understood as an *interpretation*, relational framework ought to have all experimental predictions the same as those from the traditional approach. Thus, the observable constructed in Subsection 4.2 should distinguish the quantum from the gravitational switch, in the same way as in the traditional approach. On the other hand, potential new physics formulated based on the relational framework might, or might not, feature different experimental predictions.

It is important to emphasise that, as discussed in Subsection 4.3, various realisations of the quantum switch are possible by superposing different causal orders in the framework of QG. In particular, regarding the 2-event realisations, one can consider the following diagram (see Figure 11):

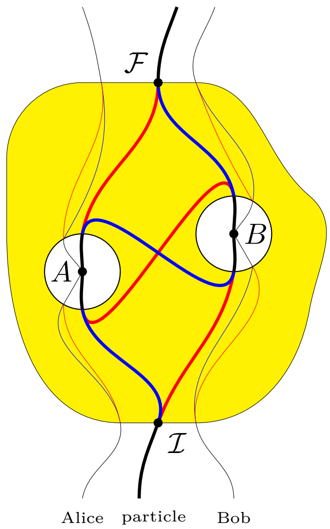


Figure 11: Spacetime diagram of a version of a 2-event gravitational switch, in which Alice and Bob perform their respective operations in the regions of spacetime with a single gravitational configuration.

This diagram features two *classical* spacetime subregions surrounding Alice's and Bob's laboratories. As such, Alice and Bob can measure the complete set of compatible observables within their laboratories, without obtaining which-way information and destroying the superposition. Therefore, even from the relational point of view, this represents an implementation of a 2-event gravitational switch. Note that in this case Alice and Bob do not even need Friend in order to verify the 2-event nature of their gravitational switch.

It is interesting to observe that this realisation of the quantum switch implements the operational idea of a 2-event quantum switch, in terms of closed laboratories. However, to achieve such an implementation, it is necessary to have a genuine superposition of metric-induced spacetime causal orders in the yellow region of spacetime, which does not feature in experimental realisations [4, 5, 6].

6 Conclusions

In this paper, we analysed the notion of causal orders both in classical and quantum worlds, with the emphasis on the latter. We defined the notion of the causal order for the case of (classical and quantum) circuits, in terms of partial ordering between the nodes of the circuit's underlying graph that defines the cause-effect structure. We discussed the possibility of implementing an abstract circuit in the real world, showing that it is always possible to do so for the case of a globally hyperbolic (classical) spacetime, in which the circuit's causal order is preserved by the metric-induced relation between the spacetime events.

The superposition principle of quantum mechanics offers the possibility of controlled operations, in particular the quantum switch, whose experimental realisations have been claimed to present genuine superpositions of causal orders. Within the process matrix formalism, we have analysed the 4- and 3-event realisations of the quantum switch in classical spacetimes with fixed spacetime causal orders, and the 2-event realisation of a gravitational switch that features superpositions of different gravitational field configurations and their respective spacetime causal orders. To that end, we have extended the process matrix formalism, by introducing the notion of a vacuum state. Our analysis shows that the process matrix formalism can explain the quantum switch realisations within the standard physics, and is thus consistent with it.

Thus, as a consequence of our Theorem, and the analysis of the quantum switch implementations, we argued that, in contrast to the gravitational switch, the current experimental implementations do not feature superpositions of spacetime causal orders, and that they are variants of the time double slit experiment. Moreover, by explicitly constructing two different observables, presented in Sections 4.1 and 4.2, respectively, we showed that it is possible to experimentally distinguish between different realisations of the quantum switch.

Finally, in Section 5, we analysed the relation among the traditional QFT approach to QG (used throughout this paper), the operational point of view, and the relational framework of QG. On the example of the quantum switch, we showed that the operational viewpoint, while consistent with the approach advocated by Mach, is nevertheless at odds with the modern relational framework. On the other hand, the traditional QFT approach and the relational framework may or may not be compatible, depending on the concrete realisation of the quantum switch. In particular, for the specific realisation of the gravitational switch given in Subsection 5.2, the two frameworks are compatible in the prediction that Alice and Bob can locally (without the help of Friend) verify that the switch is implemented on 2 events.

In a recent work [27], the authors report on a violation of the causal inequality [8] in flat Minkowski spacetime with a definite causal order. To achieve it, they consider laboratories that are localised in space only, while delocalised in time. Therefore, their alternative notion of a "closed laboratory", and that considered in [8], do not coincide, this way manifestly violating the conditions necessary for the causal inequality to hold. For the same reason, the scenario considered in [27] falls out of the scope of the current work as well. Additionally, in another recent work [28], the author discusses the quantum switch in terms of the time-delocalised quantum subsystems and operations, and generalises it to more complex quantum circuits and processes. The results of these two papers deserve further analysis and remain to be a subject of future research.

Exploring possible generalisations of our Theorem, as suggested at the end of Appendix A, presents a straightforward future line of research. Also, one could further analyse the process matrix formalism, in particular by exploring the situations in which the operational approach interpretation fails to describe the known processes. Or, to search for the opposite — the instances of physical processes that cannot be explained by the process matrix formalism, when applied within the standard physics. In order to show that the process matrix formalism is perfectly suitable for describing the quantum switch implementations within the standard physics, we formulated its version that features the vacuum state. One can thus further study possible generalisations of this formalism and its applications to the cases that go beyond simple non-relativistic mechanics. Finally, motivated by our analysis and discussion from Subsections 4.1 and 5.1, one can try to formulate alternative theories that would be consistent with the experimentally tested known physics (GR in particular), while at the same time substituting the spacetime events A and A', from the quantum switch realisations in classical spacetimes, with a single operational event (and analogously for B and B').

Acknowledgments

The authors would like to thank Časlav Brukner, Esteban Castro-Ruiz and Flaminia Giacomini for helpful comments and discussions.

NP acknowledges the support of SQIG — Security and Quantum Information Group, the Instituto de Telecomunicações (IT) Research Unit, ref. UIDB/50008/2020, funded by Fundação para a Ciência e Tecnologia (FCT), and the FCT projects QuantumMining POCI-01-0145-FEDER-031826 and Predict PTDC/CCI-CIF/29877/2017, supported by the European Regional Development Fund (FEDER), through the Competitiveness and Internationalisation Operational Programme (COMPETE 2020), and by the Regional Operational Program of Lisbon, as well as the FCT Estímulo ao Emprego Científico grant no. CEECIND/04594/2017/CP1393/CT0006. The bilateral scientific cooperation between Portugal and Serbia through the project "Noise and measurement errors in multi-party quantum security protocols", no. 451-03-01765/2014-09/04 supported by the FCT, Portugal, and the Ministry of Education, Science and Technological Development of the Republic of Serbia is also acknowledged.

MV was supported by the project ON 171031 of the Ministry of Education, Science and Technological Development of the Republic of Serbia, and the bilateral scientific cooperation between Austria and Serbia through the project "Causality in Quantum Mechanics and Quantum Gravity — 2018-2019", no. 451-03-02141/2017-09/02 supported by the Austrian Academy of Sciences (ÖAW), Austria, and the Ministry of Education, Science and Technological Development of the Republic of Serbia.

References

[1] A. Einstein, B. Podolsky and N. Rosen, *Physical Review* **47**, 777 (1935).

[2] J. S. Bell, *Physics Physique Fizika* **1**, 195 (1964).

[3] G. Chiribella, G. M. D'Ariano, P. Perinotti and B. Valiron, *Physical Review A* **88**, 022318 (2013).

[4] L. M. Procopio, A. Moqanaki, M. Araújo, F. Costa, I. A. Calafell, E. G. Dowd, D. R. Hamel, L. A. Rozema, Č. Brukner and P. Walther, *Nature Communications* **6**, 7913 (2015).

[5] G. Rubino, L. A. Rozema, A. Feix, M. Araújo, J. M. Zeuner, L. M. Procopio, Č. Brukner and P. Walther, *Science Advances* **3**, e1602589 (2017).

[6] G. Rubino, L. A. Rozema, F. Massa, M. Araújo, M. Zych, Č. Brukner and P. Walther, `arXiv:1712.06884` .

[7] L. Hardy, *Journal of Physics A: Mathematical and Theoretical* **40**, 3081 (2007).

[8] O. Oreshkov, F. Costa and Č. Brukner, *Nature Communications* **3**, 1092 (2012).

[9] M. Araújo, C. Branciard, F. Costa, A. Feix, C. Giarmatzi and Č. Brukner, *New Journal of Physics* **17**, 102001 (2015).

[10] J. Bavaresco, M. Araújo, Č. Brukner and M. T. Quintino, *Quantum* **3** 176 (2019).

[11] P. Allard Guérin, G. Rubino and Č. Brukner, *Phys. Rev. A* **99** 062317 (2019).

[12] C. Portmann, C. Matt, U. Maurer, R. Renner and B. Tackmann, *IEEE Transactions on Information Theory* **63**, 3277 (2017).

[13] J.-P. W. MacLean, K. Ried, R. W. Spekkens and K. J. Resch, *Nature Communications* **8**, 15149 (2017).

[14] M. Niedermaier and M. Reuter, *Living Rev. Rel.* **9**, 5 (2006).

[15] M. Zych, F. Costa, I. Pikovski and Č. Brukner, *Nature Communications* **10**, 3772 (2019).

[16] M. Bauer, *American Journal of Physics* **82**, 1087 (2014).

[17] M. O. Scully and K. Drühl, *Physical Review A* **25**, 2208 (1982).

[18] M. O. Scully, B.-G. Englert and H. Walther, *Nature* **351**, 111 (1991).

[19] H. Lichtenegger and B. Mashhoon, in *The Measurement of Gravitomagnetism: A Challenging Enterprise*, edited by L. Iorio, 13, Nova Science Pub Inc, New York (2007).

[20] M. Blagojević, *Gravitation and Gauge Symmetries*, Institute of Physics Publishing, Bristol (2002).

[21] H. Bondi and J. Samuel, *Physics Letters A* **228**, 121 (1997).

[22] A. S. Eddington, *Space time and gravitation*, Cambridge University Press, Cambridge (1921).

[23] C. Misner, K. Thorne and J. Wheeler, *Gravitation*, W. H. Freeman, San Francisco (1973).

[24] C. Rovelli, *Quantum Gravity*, Cambridge University Press, Cambridge (2004).

[25] J. Janjić, N. Paunković and M. Vojinović, *in preparation*.

[26] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory*, Cambridge University Press, Cambridge (2014).

[27] C. T. Marco Ho, F. Costa, C. Giarmatzi and T. C. Ralph, `arXiv:1804.05498` .

[28] O. Oreshkov, *Quantum* **3**, 206 (2019).

[29] R. A. Bertlmann and P. Krammer, *Journal of Physics A: Mathematical and Theoretical* **41**, 235303 (2008).

A Proof of the Theorem

Here we give an explicit constructive proof of the Theorem from the main text.

Given the graph G, we begin the proof by partitioning its set of nodes I into disjoint subsets, in the following way. Since the graph is finite, we introduce the subset $M_1 \subset I$ which consists of all minimal nodes of the graph G:

$$M_1 = \{u \in I \,|\, (\neg \exists v \in I)\, v \prec_I u\} \,. \tag{18}$$

Since all nodes in M_1 are minimal, there is no order relation $\prec_I$ between any two of them. Therefore, we can intuitively understand them as "simultaneous". As a next step, we remove these nodes and the corresponding edges from G, reducing it to a subgraph $G_2 = (I_2, E_2)$, where

$$I_2 = I \backslash N_1 \,, \qquad E_2 = \{(u,v) \,|\, u, v \in I_2 \,,\, (u,v) \in E\} \,. \tag{19}$$

Then we repeat the construction for the graph G_2, obtaining the new minimal set M_2, and the next subgraph G_3, in an analogous way. Since the graph G is finite, after a certain finite number of steps we will exhaust all nodes in I, ending up with a partition of "simultaneous" subsets $M_1, \dots, M_m$ ($m \in \mathbb{N}$), such that

$$(\forall i \neq j) \quad M_i \cap M_j = \emptyset \,, \qquad \bigcup_{i=1}^{m} M_i = I \,. \tag{20}$$

Once we have partitioned the set of nodes I into subsets, we turn to the construction of the immersing map $P : I \to \mathcal{M}$, in the following way. Since spacetime is globally hyperbolic, we can write $\mathcal{M} = \Sigma \times \mathbb{R}$, where Σ is a spatial 3-dimensional hypersurface, and $\mathbb{R}$ is timelike. Without loss of generality, one can then introduce a foliation of spacetime into a family of such hypersurfaces, denoted Σ_t and labeled by a parameter $t \in \mathbb{R}$. Start from some initial parameter t_1, and choose a compact subset $S_{t_1} \subset \Sigma_{t_1}$. Denoting the number of elements in the partition M_i as $\|M_i\|$, we pick in an arbitrary way the set of $\|M_1\|$ points $\vec{x}_k \in S_{t_1}$ (here, $k = 1, \dots, \|M_1\|$), and define the map P to assign a node from M_1 to each point $\vec{x}_k$ in a one-to-one fashion:

$$P(u_k) = (t_1, \vec{x}_k) \in \mathcal{M} \,, \qquad k = 1, \dots, \|M_1\| \,. \tag{21}$$

Once this assignment has been defined, construct a future-pointing light cone from each spacetime point $(t_1, \vec{x}_k)$. Then we find a new hypersurface, Σ_{t_2}, which contains a common intersection with all constructed light cones, and denote this intersection $S_{t_2} \subset \Sigma_{t_2}$. In this way, by construction, all points $(t_1, \vec{x}_k)$ are in the past of all points in S_{t_2},

$$(t_1, \vec{x}_k) \prec_{\mathcal{M}} S_{t_2} \,, \qquad k = 1, \dots, \|M_1\| \,. \tag{22}$$

Now extend the definition of P such that it assigns the nodes from the next partition, M_2, to a randomly chosen set of points in S_{t_2} in a similar way as before, then construct a set of light cones from them, and repeat the construction for all partitions M_i. Constructed in this way, the map P ensures that for every pair of nodes $u, v \in I$, we have

$$u \prec_I v \quad \Longrightarrow \quad P(u) \prec_{\mathcal{M}} P(v) \,, \qquad \forall u, v \in I \,. \tag{23}$$

Once we have constructed the map $P : I \to \mathcal{M}$ satisfying (23), using the definition (2), it induces the map $\mathcal{P} : \mathcal{G}_{\mathcal{C}} \to \mathcal{M}$, which satisfies the required statement (3).

This completes the proof. □

Note that, while the causal order $\prec_{\mathcal{M}}$ indeed preserves the causal order $\prec_{\mathcal{C}}$, it is "stronger" in the sense that it may introduce additional relations between the images of nodes, which do not hold in the graph itself. Indeed, the construction of the map P in the above proof is such that *each* image of a node from some given partition M_i is in the causal past of *all* images from the previous partition M_{i-1}, which is not necessarily the case for the nodes themselves. One might study if the causal orders over the set of nodes and over the set of its images can be equivalent, i.e., if the opposite implication from equation (23) also holds (in this case the immersion P is called an *embedding* of G into $\mathcal{M}$). Whether such an embedding exists for all hyperbolic spacetimes, or at least for some, is an open question.

Next, one could also discuss the generalisation of the above theorem to the case of countably infinite graphs G. However, for our purposes, the existence of a partially ordered map $\mathcal{P}$ over the set of finite graphs will suffice.

Regarding the proof itself, one can formulate an alternative (and simpler) approach to the proof of the theorem. Namely, one can first prove that every circuit can be immersed into the flat Minkowski spacetime. Then, knowing that the sufficiently small neighbourhood of every spacetime point in an arbitrary manifold $\mathcal{M}$ can be well approximated with its tangent space, one can always immerse the whole circuit into this small neighbourhood. However, this implies that the geometric size of the circuit can be considered negligible compared to the curvature scale of the manifold, which may render such implementation practically unfeasible. Moreover, this alternative approach does not cover the cases where one actually wants the scale of the circuit to be comparable to the curvature scale. Specifically, if one wishes to employ the circuit to study gravitational phenomena, its gates must be distributed across spacetime precisely in a way that is sensitive to curvature. Therefore, the construction of the map P used in the proof of the theorem is more general than the construction in this alternative approach.

Finally, given the construction in the proof, the gates of the set of minimal nodes M_1 define the initial gate $\mathcal{I}$, the set of maximal nodes M_m define the final gate $\mathcal{F}$, while the the gates of the remaining intermediary sets of nodes $M_2, \ldots M_{m-1}$ define the operation $\mathcal{O}_\mathcal{C}$. This is illustrated in the diagram below (see Figure 12).

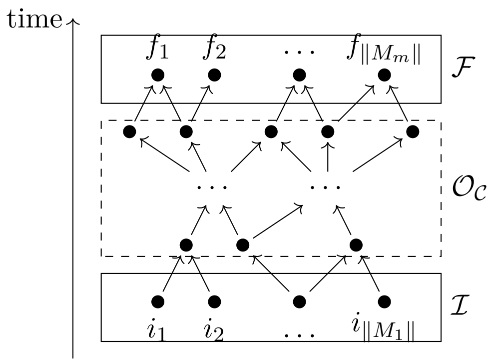


Figure 12: The spacetime diagram of the circuit $\mathcal{C}$, with the initial gate $\mathcal{I}$, the operation gate $\mathcal{O}_\mathcal{C}$, and the final gate $\mathcal{F}$.

B Qutrit states, operators and bases

The notion of a qubit can be generalised from a 2-dimensional Hilbert space to a d-dimensional Hilbert space. The generalised object is called "qudit" in d dimensions [29]. Since we are interested in describing ordinary 2-dimensional qubits with an additional vacuum state, it is natural to consider qudits in $d = 3$, called "qutrits". We introduce the following notation for the basis states of a qutrit in $\mathcal{H}_3 = \mathbb{C}^3$:

$$|0\rangle \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad |1\rangle \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad |v\rangle \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{24}$$

The states $|0\rangle$ and $|1\rangle$ will be understood as the usual computational basis for a 2-dimensional qubit, while the state $|v\rangle$ will represent the vacuum, i.e., the "absence of a qubit". In cases when we take sums over the basis vectors, we will assume that the vacuum state carries the index 2, i.e., $|v\rangle \equiv |2\rangle$, so that we can write

$$\sum_{i=0}^{2} |i\rangle = |0\rangle + |1\rangle + |v\rangle\,, \qquad \text{and} \qquad \sum_{i=0}^{1} |i\rangle = |0\rangle + |1\rangle\,. \tag{25}$$

Using this notation, we write the unnormalised maximally correlated states for the qutrit and the qubit as

$$|\mathbb{1}\rangle\!\rangle = \sum_{i=0}^{2} |i\rangle|i\rangle = |0\rangle|0\rangle + |1\rangle|1\rangle + |v\rangle|v\rangle \in \mathcal{H}_3 \otimes \mathcal{H}_3\,, \qquad |1\rangle\!\rangle = \sum_{i=0}^{1} |i\rangle|i\rangle = |0\rangle|0\rangle + |1\rangle|1\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_2\,, \tag{26}$$

so that

$$|\mathbb{1}\rangle\!\rangle = |1\rangle\!\rangle + |v\rangle|v\rangle\,. \tag{27}$$

One can also introduce the standard Hilbert-Schmidt basis in the space $\mathcal{L}(\mathcal{H}_3)$ of linear operators on $\mathcal{H}_3$. This basis consists of 9 matrices 3×3, labeled as $\lambda_0, \dots, \lambda_8$, as follows:

- the three symmetric matrices

$$\lambda_1 = \sqrt{\frac{3}{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \lambda_2 = \sqrt{\frac{3}{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \lambda_3 = \sqrt{\frac{3}{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{28}$$

- the three antisymmetric matrices

$$\lambda_4 = \sqrt{\frac{3}{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \lambda_5 = \sqrt{\frac{3}{2}} \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \qquad \lambda_6 = \sqrt{\frac{3}{2}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \tag{29}$$

- and the three diagonal matrices

$$\lambda_7 = \sqrt{\frac{3}{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad \lambda_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{30}$$

The matrix λ_0 is the unit matrix, while $\lambda_1, \dots, \lambda_8$ are self-adjoint, traceless, and orthogonal with respect to the standard scalar product:

$$\lambda_i^\dagger = \lambda_i\,, \qquad \mathrm{Tr}\,\lambda_i = 0\,, \qquad \mathrm{Tr}\,\lambda_i^\dagger \lambda_j = 3\delta_{ij}\,, \qquad i = 1, \dots, 8\,. \tag{31}$$

They represent the generators of the $SU(3)$ group, and are known as the Gell-Mann matrices (up to a normalisation factor $\sqrt{3/2}$).

If we denote $\mathcal{H}_v$ as the 1-dimensional vacuum-spanned subspace of $\mathcal{H}_3$, one can see that $\mathcal{L}(\mathcal{H}_2) \oplus \mathcal{L}(\mathcal{H}_v) \subset \mathcal{L}(\mathcal{H}_3)$. In particular, if we denote the standard Pauli matrices as $\sigma_x, \sigma_y, \sigma_z$ and the unit 2×2 matrix as I_2, they form the basis in $\mathcal{L}(\mathcal{H}_2)$, and the qubit basis can thus be constructed as

$$\sqrt{\frac{2}{3}}\lambda_1 = \left[\begin{array}{c|c} \sigma_x & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 0 \end{array}\right], \qquad \sqrt{\frac{2}{3}}\lambda_4 = \left[\begin{array}{c|c} \sigma_y & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 0 \end{array}\right], \qquad \sqrt{\frac{2}{3}}\lambda_7 = \left[\begin{array}{c|c} \sigma_z & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 0 \end{array}\right], \tag{32}$$

along with

$$\frac{2}{3}\lambda_0 + \frac{\sqrt{2}}{3}\lambda_8 = \left[\begin{array}{c|c} I_2 & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 0 \end{array}\right]. \tag{33}$$

Also, the vacuum space $\mathcal{L}(\mathcal{H}_v)$ is one-dimensional, and the basis is

$$\frac{1}{3}\lambda_0 - \frac{\sqrt{2}}{3}\lambda_8 = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]. \tag{34}$$

C Process matrix evaluation

Let us give an explicit step by step evaluation of the probability distribution for the 4-event process discussed in the text, using the process matrix formalism. The complete spacetime diagram of the process is given as (see Figure 13):

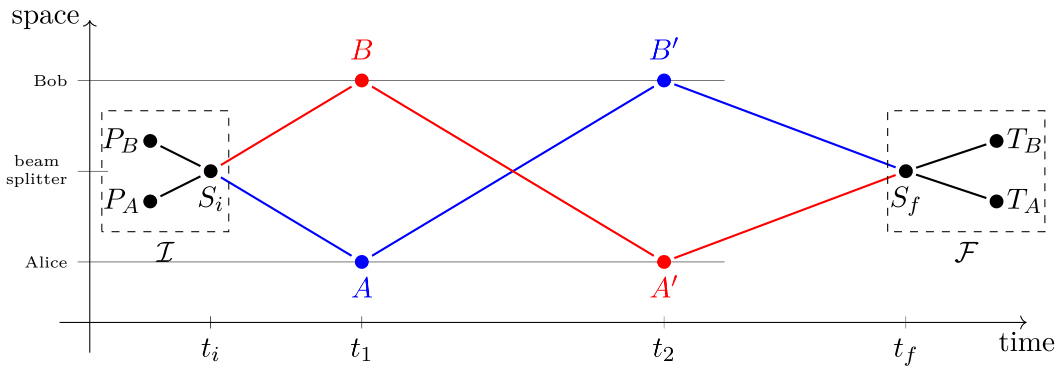


Figure 13: Spacetime diagram of the 4-event implementation of the quantum switch. The internal structures of the composite gates $\mathcal{I}$ and $\mathcal{F}$ are explicitly depicted.

The composite event $\mathcal{I}$ consists of the two preparation events P_A, P_B, and the initial beam splitting event S_i, while $\mathcal{F}$ consists of the recombination event S_f and the measurement events T_A and T_B.

The corresponding circuit diagram is obtained from the above one by promoting each event of interaction to a gate, and the propagation of each particle to a channel. This leads to the following circuit diagram (see Figure 14):

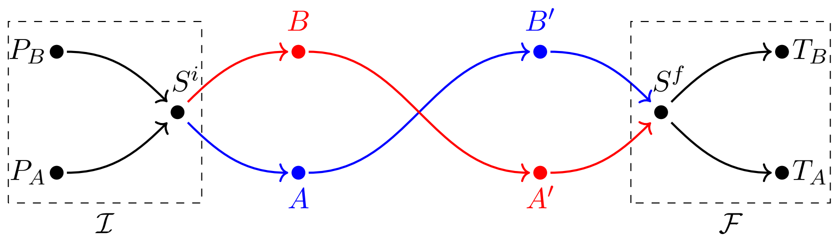


Figure 14: Circuit diagram of the 4-event implementation of the quantum switch. The internal structures of the composite gates $\mathcal{I}$ and $\mathcal{F}$ are explicitly depicted.

Its structure is in one-to-one correspondence with the spacetime diagram for the 4-event process, where the preparation and measurement spacetime events $\mathcal{I}$ and $\mathcal{F}$ have been split into three sub-gates each, for clarity.

The operations on each of the gates are given as follows. The preparation gate P_A maps from the input Hilbert space P_{A_I} to the output Hilbert space P_{A_O}, and analogously for gate P_B. The input spaces are trivial, $\dim P_{A_I} = \dim P_{B_I} = 1$, while each output space is spanned by vectors $|0\rangle$, $|1\rangle$ and $|v\rangle$. Here, $|0\rangle$ and $|1\rangle$ represent the two orthogonal qubit states (say, vertical and horizontal polarisations along certain axis in 3D space), while $|v\rangle$ is the vacuum state, representing the absence of particles in the corresponding arm of the interferometer. The operations performed at these gates, $P_A = |\Psi\rangle$ and $P_B = |v\rangle$, specify the initial conditions for the rest of the circuit diagram, and are described by the Choi-Jamiołkowski (CJ) states as

$$|P_A^*\rangle\rangle^{P_{A_I}P_{A_O}} = |\Psi^*\rangle^{P_{A_O}} , \qquad |P_B^*\rangle\rangle^{P_{B_I}P_{B_O}} = |v\rangle^{P_{B_O}} . \tag{35}$$

Here, $*$ denotes the complex conjugation.

Analogously, the target gates T_A and T_B facilitate the final measurement outcomes of the circuit diagram. The input spaces T_{A_I} and T_{B_I} are three-dimensional, spanned over the two qubit states and the vacuum, while the output spaces are one-dimensional. The operations performed at these gates, $T_\alpha = \langle\alpha|$ and $T_\beta = \langle\beta|$, read out the measurement results $\alpha, \beta \in \{0, 1, v\}$. The corresponding CJ states are given as

$$|T^*_\alpha\rangle\!\rangle^{T_{A_I}T_{A_O}} = |\alpha\rangle^{T_{A_I}}\,, \qquad |T^*_\beta\rangle\!\rangle^{T_{B_I}T_{B_O}} = |\beta\rangle^{T_{B_I}}\,. \tag{36}$$

The gates A, A', B and B' perform the unitaries U and V. The input and output spaces A_I and A_O of the Alice's gate A are both spanned by vectors $|0\rangle$, $|1\rangle$ and $|v\rangle$, and analogously for the input and output spaces of the remaining three gates. Assuming that in her (spatially) local laboratory Alice performs the unitary U on the particle's internal degree of freedom, the induced operation between the three-dimensional spaces A_I and A_O that include the vacuum states is given by

$$\tilde{U}^{A_O A_I} = U^{A_O A_I} P_{01}^{A_I A_I} + I^{A_O A_I} P_v^{A_I A_I}\,, \tag{37}$$

where $P_{01}^{A_I A_I} = |0\rangle^{A_I}\langle 0|^{A_I} + |1\rangle^{A_I}\langle 1|^{A_I}$, $P_v^{A_I A_I} = |v\rangle^{A_I}\langle v|^{A_I}$, and $I^{A_O A_I}$ represents the identity map between the Hilbert spaces A_O and A_I. The analogous construction also holds for the gate A', so the respective CJ states for the gates A and A' are then given by:

$$\begin{aligned} |\tilde{U}^*\rangle\!\rangle^{A_I A_O} &= \left[I^{A_I A_I} \otimes (\tilde{U}^*)^{A_O A_I}\right] |\mathbb{1}\rangle\!\rangle^{A_I A_I}\,, \\ |\tilde{U}^*\rangle\!\rangle^{A'_I A'_O} &= \left[I^{A'_I A'_I} \otimes (\tilde{U}^*)^{A'_O A'_I}\right] |\mathbb{1}\rangle\!\rangle^{A'_I A'_I}\,. \end{aligned} \tag{38}$$

Here, the "transport vector" is given by (for details of the process matrix formalism for the case of three-dimensional spaces — qutrits, see Appendix B):

$$|\mathbb{1}\rangle\!\rangle = |0\rangle|0\rangle + |1\rangle|1\rangle + |v\rangle|v\rangle\,. \tag{39}$$

Bob performs V in his (spatially) local laboratory, and therefore the CJ states for the gates B and B' are given as:

$$\begin{aligned} |\tilde{V}^*\rangle\!\rangle^{B_I B_O} &= \left[I^{B_I B_I} \otimes (\tilde{V}^*)^{B_O B_I}\right] |\mathbb{1}\rangle\!\rangle^{B_I B_I}\,, \\ |\tilde{V}^*\rangle\!\rangle^{B'_I B'_O} &= \left[I^{B'_I B'_I} \otimes (\tilde{V}^*)^{B'_O B'_I}\right] |\mathbb{1}\rangle\!\rangle^{B'_I B'_I}\,. \end{aligned} \tag{40}$$

The gates S^i and S^f act as beam splitters, i.e., they both perform the same Hadamard operation H, given as follows. The beam splitter input and output spaces consist of the Alice's and Bob's factor spaces. For the case of the Alice's input space, we have $S_{A_I} = \text{span}\{|0\rangle^{S_{A_I}}, |1\rangle^{S_{A_I}}, |v\rangle^{S_{A_I}}\}$, and analogously for the output space, as well as for Bob's factor spaces. The overall input and output beam splitter spaces are therefore defined as $S_I = S_{(AB)_I} = S_{A_I} \otimes S_{B_I}$ and $S_O = S_{(AB)_O} = S_{A_O} \otimes S_{B_O}$. Finally, the unitary matrix associated to gate S representing the action of the balanced Hadamard beam splitter is given by:

$$\begin{aligned} H^{S_O S_I} = \; & \frac{1}{\sqrt{2}}\left(|0\rangle^{S_{A_O}}|v\rangle^{S_{B_O}} + |v\rangle^{S_{A_O}}|0\rangle^{S_{B_O}}\right)\langle 0|^{S_{A_I}}\langle v|^{S_{B_I}} \\ & + \frac{1}{\sqrt{2}}\left(|1\rangle^{S_{A_O}}|v\rangle^{S_{B_O}} + |v\rangle^{S_{A_O}}|1\rangle^{S_{B_O}}\right)\langle 1|^{S_{A_I}}\langle v|^{S_{B_I}} \\ & + \frac{1}{\sqrt{2}}\left(|0\rangle^{S_{A_O}}|v\rangle^{S_{B_O}} - |v\rangle^{S_{A_O}}|0\rangle^{S_{B_O}}\right)\langle v|^{S_{A_I}}\langle 0|^{S_{B_I}} \\ & + \frac{1}{\sqrt{2}}\left(|1\rangle^{S_{A_O}}|v\rangle^{S_{B_O}} - |v\rangle^{S_{A_O}}|1\rangle^{S_{B_O}}\right)\langle v|^{S_{A_I}}\langle 1|^{S_{B_I}}\,. \end{aligned} \tag{41}$$

The beam splitter acts such that the system coming from the Alice's side comes into an equal superposition of the two output spatial modes coming to Alice and Bob, with zero relative phase, while the system coming from the Bob's side (blue line) comes into an equal superposition of the two output spatial modes with relative phase π. Thus, in the output space the correlation between the Alice's and Bob's vacuum state is the opposite as in the input case. The corresponding CJ state is then

$$|H^*\rangle\rangle^{S_I S_O} = \left[I^{S_I S_I} \otimes (H^*)^{S_O S_I} \right] |\mathbb{1}\rangle\rangle^{S_I S_I} , \tag{42}$$

where the transport vector $|\mathbb{1}\rangle\rangle$ for the beam splitter, when projected to a single-particle subspace, is given by

$$|\mathbb{1}\rangle\rangle = |0v\rangle|0v\rangle + |1v\rangle|1v\rangle + |v0\rangle|v0\rangle + |v1\rangle|v1\rangle . \tag{43}$$

Note that the full transport vector contains nine terms instead of the above four, but for the purpose of this paper, we do not need those five additional terms.

The process vector encodes the wires between the gates, and it is being constructed by taking the tensor product over appropriate transport vectors $|\mathbb{1}\rangle\rangle$ for Alice's and Bob's qutrits, see equations (26) and (27), such that each transport vector corresponds to one wire in the circuit diagram, connecting the output of the source gate to the input of the target gate. The process vector is thus given as:

$$\begin{aligned} |W_{\text{4-event}}\rangle\rangle \; = \; & \underbrace{|\mathbb{1}\rangle\rangle^{P_{A_O} S^i_{A_I}} |\mathbb{1}\rangle\rangle^{P_{B_O} S^i_{B_I}}}_{\text{initial}} \underbrace{|\mathbb{1}\rangle\rangle^{S^i_{A_O} A_I} |\mathbb{1}\rangle\rangle^{A_O B'_I} |\mathbb{1}\rangle\rangle^{B'_O S^f_{B_I}}}_{\text{blue}} \\ & \underbrace{|\mathbb{1}\rangle\rangle^{S^i_{B_O} B_I} |\mathbb{1}\rangle\rangle^{B_O A'_I} |\mathbb{1}\rangle\rangle^{A'_O S^f_{A_I}}}_{\text{red}} \underbrace{|\mathbb{1}\rangle\rangle^{S^f_{A_O} T_{A_I}} |\mathbb{1}\rangle\rangle^{S^f_{B_O} T_{B_I}}}_{\text{final}} . \end{aligned} \tag{44}$$

One can now evaluate the probability distribution

$$p(\alpha, \beta) = \left\| \mathcal{M}(\alpha, \beta) \right\|^2 , \tag{45}$$

where the probability amplitude $\mathcal{M}(\alpha, \beta)$ is constructed by acting with the tensor product of all gate operations (35), (42), (38), (40), (42) and (36), on the process vector (44). Since each of the gate operations acts in its own part of the total Hilbert space, the order of application of these operations is immaterial, and we are free to choose the most convenient one.

To see what happens when the operations (35) of the preparation gates act on the process vector, let us evaluate the action of $|P^*_A\rangle\rangle^{P_{A_I} P_{A_O}}$ on $|\mathbb{1}\rangle\rangle^{P_{A_O} S^i_{A_I}}$:

$$\langle\langle P^*_A|^{P_{A_I} P_{A_O}} |\mathbb{1}\rangle\rangle^{P_{A_O} S^i_{A_I}} = \langle \Psi^* |^{P_{A_O}} \sum_{k=0}^{2} |k\rangle^{P_{A_O}} |k\rangle^{S^i_{A_I}} = \sum_{k=0}^{2} \Big(\langle \Psi | k \rangle \Big)^* |k\rangle^{S^i_{A_I}} = |\Psi\rangle^{S^i_{A_I}} . \tag{46}$$

An analogous calculation can be performed for $|P^*_B\rangle\rangle^{P_{B_I} P_{B_O}}$, so the action of both preparation operations (35) on the process vector (44) evaluates to:

$$\begin{aligned} & \Big(\langle\langle P^*_A|^{P_{A_I} P_{A_O}} \otimes \langle\langle P^*_B|^{P_{B_I} P_{B_O}} \Big) |W_{\text{4-event}}\rangle\rangle = \\ & |\Psi\rangle^{S^i_{A_I}} |v\rangle^{S^i_{B_I}} \underbrace{|\mathbb{1}\rangle\rangle^{S^i_{A_O} A_I} |\mathbb{1}\rangle\rangle^{A_O B'_I} |\mathbb{1}\rangle\rangle^{B'_O S^f_{B_I}}}_{\text{blue}} \\ & \underbrace{|\mathbb{1}\rangle\rangle^{S^i_{B_O} B_I} |\mathbb{1}\rangle\rangle^{B_O A'_I} |\mathbb{1}\rangle\rangle^{A'_O S^f_{A_I}}}_{\text{red}} \underbrace{|\mathbb{1}\rangle\rangle^{S^f_{A_O} T_{A_I}} |\mathbb{1}\rangle\rangle^{S^f_{B_O} T_{B_I}}}_{\text{final}} . \end{aligned} \tag{47}$$

Next one acts with the initial Hadamard operation (42) on this process vector, transforming it into

$$\begin{aligned} & \Big(\langle\langle P^*_A|^{P_{A_I} P_{A_O}} \otimes \langle\langle P^*_B|^{P_{B_I} P_{B_O}} \otimes \langle\langle S^*|^{S^i_{(AB)_I} S^i_{(AB)_O}} \Big) |W_{\text{4-event}}\rangle\rangle = \\ & \frac{1}{\sqrt{2}} \Big(|\Psi\rangle^{A_I} |v\rangle^{B_I} + |v\rangle^{A_I} |\Psi\rangle^{B_I} \Big) \underbrace{|\mathbb{1}\rangle\rangle^{A_O B'_I} |\mathbb{1}\rangle\rangle^{B'_O S^f_{B_I}}}_{\text{blue}} \\ & \underbrace{|\mathbb{1}\rangle\rangle^{B_O A'_I} |\mathbb{1}\rangle\rangle^{A'_O S^f_{A_I}}}_{\text{red}} \underbrace{|\mathbb{1}\rangle\rangle^{S^f_{A_O} T_{A_I}} |\mathbb{1}\rangle\rangle^{S^f_{B_O} T_{B_I}}}_{\text{final}} \equiv |W_{QS_4}\rangle\rangle . \end{aligned} \tag{48}$$

The resulting process vector is the outcome of the action of the composite gate $\mathcal{I}$ on (44), before the actions of Alice and Bob (note that often in the literature this is taken as the initial process vector in the analysis):

$$|W_{QS_4}\rangle\rangle = \frac{1}{\sqrt{2}} \left(|\Psi\rangle^{A_I} |v\rangle^{B_I} + |v\rangle^{A_I} |\Psi\rangle^{B_I} \right) |\mathbb{1}\rangle\rangle^{A_O B'_I} |\mathbb{1}\rangle\rangle^{B_O A'_I} |\mathbb{1}\rangle\rangle^{(A'_O B'_O) S_{(AB)_I}} |\mathbb{1}\rangle\rangle^{S_{(AB)_O} T_{(AB)_I}} . \quad (49)$$

Continuing the computation, the action of the remaining gate operations (38), (40), (36) and (42) on the process vector (49) gives us the probability amplitude,

$$\mathcal{M}(\alpha, \beta) \equiv$$

$$\left(\langle\langle \tilde{U}^* |^{A_I A_O} \otimes \langle\langle \tilde{U}^* |^{A'_I A'_O} \otimes \langle\langle \tilde{V}^* |^{B_I B_O} \otimes \langle\langle \tilde{V}^* |^{B'_I B'_O} \otimes \langle\langle H^* |^{S_I S_O} \otimes \langle\langle T^*_\alpha |^{T_{A_I} T_{A_O}} \otimes \langle\langle T^*_\beta |^{T_{B_I} T_{B_O}} \right) |W_{QS_4}\rangle\rangle . \quad (50)$$

Let us now calculate the action of $\langle\langle \tilde{U}^* |^{A_I A_O}$ on (49):

$$\langle\langle \tilde{U}^* |^{A_I A_O} |W_{QS_4}\rangle\rangle = \langle\langle \mathbb{1} |^{A_I A_I} \left[I^{A_I A_I} \otimes (\tilde{U}^T)^{A_O A_I} \right] |W_{QS_4}\rangle\rangle . \quad (51)$$

Looking at the structure of the process vector, one sees that the resulting new process vector will have the form

$$\langle\langle \tilde{U}^* |^{A_I A_O} |W_{QS_4}\rangle\rangle = \frac{1}{\sqrt{2}} \left(|X\rangle^{B'_I} |v\rangle^{B_I} + |Y\rangle^{B'_I} |\Psi\rangle^{B_I} \right) |\mathbb{1}\rangle\rangle^{B_O A'_I} |\mathbb{1}\rangle\rangle^{(A'_O B'_O) S_{(AB)_I}} |\mathbb{1}\rangle\rangle^{S_{(AB)_O} T_{(AB)_I}} , \quad (52)$$

where $|X\rangle^{B'_I}$ and $|Y\rangle^{B'_I}$ are shorthands for the expressions

$$|X\rangle^{B'_I} \equiv \langle\langle \mathbb{1} |^{A_I A_I} \left[I^{A_I A_I} \otimes (\tilde{U}^T)^{A_O A_I} \right] |\Psi\rangle^{A_I} |\mathbb{1}\rangle\rangle^{A_O B'_I} \quad (53)$$

and

$$|Y\rangle^{B'_I} \equiv \langle\langle \mathbb{1} |^{A_I A_I} \left[I^{A_I A_I} \otimes (\tilde{U}^T)^{A_O A_I} \right] |v\rangle^{A_I} |\mathbb{1}\rangle\rangle^{A_O B'_I} , \quad (54)$$

which need to be evaluated. The explicit computation of the first expression goes as follows:

$$\begin{aligned} |X\rangle^{B'_I} &= \langle\langle \mathbb{1} |^{A_I A_I} \left[I^{A_I A_I} \otimes (\tilde{U}^T)^{A_O A_I} \right] |\Psi\rangle^{A_I} |\mathbb{1}\rangle\rangle^{A_O B'_I} \\ &= \sum_{k=0}^{2} \langle k|^{A_I} \langle k|^{A_I} \left[I^{A_I A_I} \otimes (\tilde{U}^T)^{A_O A_I} \right] |\Psi\rangle^{A_I} \sum_{m=0}^{2} |m\rangle^{A_O} |m\rangle^{B'_I} \\ &= \sum_{m=0}^{2} \left[\sum_{k=0}^{2} \left(\langle k|^{A_I} I^{A_I A_I} |\Psi\rangle^{A_I} \right) \left(\langle k|^{A_I} (\tilde{U}^T)^{A_O A_I} |m\rangle^{A_O} \right) \right] |m\rangle^{B'_I} \\ &= \sum_{m=0}^{2} \left[\sum_{k=0}^{2} \langle k|\Psi\rangle^{A_I} \; \langle m|^{A_O} \tilde{U}^{A_O A_I} |k\rangle^{A_I} \right] |m\rangle^{B'_I} \\ &= \sum_{m=0}^{2} \left[\langle m|^{A_O} \tilde{U}^{A_O A_I} |\Psi\rangle^{A_I} \right] |m\rangle^{B'_I} . \end{aligned} \quad (55)$$

Using (37), the coefficient in the brackets can be evaluated as

$$\langle m|^{A_O} \tilde{U}^{A_O A_I} |\Psi\rangle^{A_I} = \langle m|^{A_O} \left(U^{A_O A_I} P_{01}^{A_I A_I} + I^{A_O A_I} P_v^{A_I A_I} \right) |\Psi\rangle^{A_I} = \langle m|U|\Psi\rangle , \quad (56)$$

since $P_{01}^{A_I A_I} |\Psi\rangle^{A_I} = |\Psi\rangle^{A_I}$ and $P_v^{A_I A_I} |\Psi\rangle^{A_I} = 0$. Thus, we have

$$|X\rangle^{B'_I} = \sum_{m=0}^{2} \langle m|U|\Psi\rangle \; |m\rangle^{B'_I} = U|\Psi\rangle^{B'_I} \equiv |U\Psi\rangle^{B'_I} . \quad (57)$$

The computation of $|Y\rangle^{B'_I}$ proceeds in an analogous way to (55), and the result is

$$|Y\rangle^{B'_I} = \sum_{m=0}^{2} \left[\langle m|^{A_O} \tilde{U}^{A_O A_I} |v\rangle^{A_I} \right] |m\rangle^{B'_I} . \tag{58}$$

Again, using (37), the coefficient in the brackets can be evaluated as

$$\langle m|^{A_O} \tilde{U}^{A_O A_I} |v\rangle^{A_I} = \langle m|^{A_O} \left(U^{A_O A_I} P_{01}^{A_I A_I} + I^{A_O A_I} P_v^{A_I A_I} \right) |v\rangle^{A_I} = \langle m|v\rangle = \delta_{mv} , \tag{59}$$

since $P_{01}^{A_I A_I} |v\rangle^{A_I} = 0$ and $P_v^{A_I A_I} |v\rangle^{A_I} = |v\rangle^{A_I}$. Thus, we have

$$|Y\rangle^{B'_I} = \sum_{m=0}^{2} \delta_{mv} \, |m\rangle^{B'_I} = |v\rangle^{B'_I} . \tag{60}$$

Finally, substituting (57) and (60) back into (52), we obtain:

$$\langle\!\langle \tilde{U}^* |^{A_I A_O} | W_{QS_4} \rangle\!\rangle = \frac{1}{\sqrt{2}} \left(|U\Psi\rangle^{B'_I} |v\rangle^{B_I} + |v\rangle^{B'_I} |\Psi\rangle^{B_I} \right) |\mathbb{1}\rangle\!\rangle^{B_O A'_I} |\mathbb{1}\rangle\!\rangle^{(A'_O B'_O) S_{(AB)_I}} |\mathbb{1}\rangle\!\rangle^{S_{(AB)_O} T_{(AB)_I}} . \tag{61}$$

One should note, comparing (61) with (49), that the action of the gate A operation onto the process vector effectively performs the following transformation,

$$|\Psi\rangle^{A_I} \to |U\Psi\rangle^{A_O} \to |U\Psi\rangle^{B'_I} , \qquad |v\rangle^{A_I} \to |v\rangle^{A_O} \to |v\rangle^{B'_I} , \tag{62}$$

where the transport vector $|\mathbb{1}\rangle\!\rangle^{A_O B'_I}$ has been utilised for "transporting" the state from the output A_O of gate A to the input B'_I of the gate B', in line with the spacetime diagram. This scheme repeats itself with the action of all remaining gate operations on (61). In particular, the subsequent action of the gate B operation gives:

$$\begin{aligned} &\left(\langle\!\langle \tilde{V}^* |^{B_I B_O} \otimes \langle\!\langle \tilde{U}^* |^{A_I A_O} \right) | W_{QS_4} \rangle\!\rangle = \\ &\qquad \frac{1}{\sqrt{2}} \left(|U\Psi\rangle^{B'_I} |v\rangle^{A'_I} + |v\rangle^{B'_I} |V\Psi\rangle^{A'_I} \right) |\mathbb{1}\rangle\!\rangle^{(A'_O B'_O) S_{(AB)_I}} |\mathbb{1}\rangle\!\rangle^{S_{(AB)_O} T_{(AB)_I}} , \end{aligned} \tag{63}$$

which can also be verified with an explicit calculation similar to the above. Continuing on, the operations at the gates A' and B' give:

$$\begin{aligned} &\left(\langle\!\langle \tilde{V}^* |^{B'_I B'_O} \otimes \langle\!\langle \tilde{U}^* |^{A'_I A'_O} \otimes \langle\!\langle \tilde{V}^* |^{B_I B_O} \otimes \langle\!\langle \tilde{U}^* |^{A_I A_O} \right) | W_{QS_4} \rangle\!\rangle = \\ &\qquad \frac{1}{\sqrt{2}} \left(|VU\Psi\rangle^{S_{B_I}} |v\rangle^{S_{A_I}} + |v\rangle^{S_{B_I}} |UV\Psi\rangle^{S_{A_I}} \right) |\mathbb{1}\rangle\!\rangle^{S_{(AB)_O} T_{(AB)_I}} . \end{aligned} \tag{64}$$

Next, the action of the beam splitter at the gate S_f gives

$$\begin{aligned} &\left(\langle\!\langle \tilde{H}^* |^{S_I S_O} \otimes \langle\!\langle \tilde{V}^* |^{B'_I B'_O} \otimes \langle\!\langle \tilde{U}^* |^{A'_I A'_O} \otimes \langle\!\langle \tilde{V}^* |^{B_I B_O} \otimes \langle\!\langle \tilde{U}^* |^{A_I A_O} \right) | W_{QS_4} \rangle\!\rangle = \\ &\qquad \frac{1}{2} \sum_{i=0}^{1} \left(\langle i| \{ U , V \} |\Psi\rangle \, |i\rangle^{T_{A_I}} |v\rangle^{T_{B_I}} + \langle i| [U , V] |\Psi\rangle \, |v\rangle^{T_{A_I}} |i\rangle^{T_{B_I}} \right) . \end{aligned} \tag{65}$$

Finally, the action of the operations of the target gates T_A and T_B gives us the probability amplitude as a function of the measurement outcomes α and β,

$$\begin{aligned} &\mathcal{M}(\alpha, \beta) \equiv \\ &\left(\langle\!\langle \tilde{U}^* |^{A_I A_O} \otimes \langle\!\langle \tilde{U}^* |^{A'_I A'_O} \otimes \langle\!\langle \tilde{V}^* |^{B_I B_O} \otimes \langle\!\langle \tilde{V}^* |^{B'_I B'_O} \otimes \langle\!\langle H^* |^{S_I S_O} \otimes \langle\!\langle T_\alpha^* |^{T_{A_I} T_{A_O}} \otimes \langle\!\langle T_\beta^* |^{T_{B_I} T_{B_O}} \right) | W_{QS_4} \rangle\!\rangle \end{aligned}$$

$$= \frac{1}{2}\Big[\delta_{\beta v}\langle\alpha|\,\{\,U\,,\,V\,\}\,|\Psi\rangle + \delta_{\alpha v}\langle\beta|\,[\,U\,,\,V\,]\,|\Psi\rangle\Big]\,. \tag{66}$$

At this point we can employ (45) and calculate the probability distribution,

$$p(\alpha,\beta) = \frac{1}{4}\Big[\delta_{\beta v}\left|\langle\alpha|\,\{\,U\,,\,V\,\}\,|\Psi\rangle\right|^2 + \delta_{\alpha v}\left|\langle\beta|\,[\,U\,,\,V\,]\,|\Psi\rangle\right|^2\Big]\,, \tag{67}$$

where we have used the fact that the vacuum state $|v\rangle$ is orthogonal to both $\{\,U\,,\,V\,\}\,|\Psi\rangle$ and $[\,U\,,\,V\,]\,|\Psi\rangle$. In particular, we see that for $i \in \{0,1\}$ we have

$$p(i,v) = \frac{1}{4}\left|\langle i|\,\{\,U\,,\,V\,\}\,|\Psi\rangle\right|^2\,, \qquad p(v,i) = \frac{1}{4}\left|\langle i|\,[\,U\,,\,V\,]\,|\Psi\rangle\right|^2\,, \tag{68}$$

while all other choices of α and β give $p(\alpha,\beta) = 0$. The total probability that Alice will detect a particle is given by the marginal

$$p_A = \sum_{i=1}^{2} p(i,v) = \frac{1}{2}\left(1 + \mathrm{Re}\,\langle\Psi|U^\dagger V^\dagger U V|\Psi\rangle\right)\,, \tag{69}$$

while the corresponding total probability that Bob will detect a particle is

$$p_B = \sum_{i=1}^{2} p(v,i) = \frac{1}{2}\left(1 - \mathrm{Re}\,\langle\Psi|U^\dagger V^\dagger U V|\Psi\rangle\right)\,. \tag{70}$$

As a final point, we can verify that the probability distribution is correctly normalised. Using the fact that the only nonzero values for the probability are given in (68), we have

$$p_{\text{total}} = \sum_{\alpha=0}^{2}\sum_{\beta=0}^{2} p(\alpha,\beta) = \underbrace{p(0,v) + p(1,v)}_{p_A} + \underbrace{p(v,0) + p(v,1)}_{p_B} = 1\,, \tag{71}$$

as expected.

Instead of recombining the particle on the second beam splitter, one can consider the case in which the final gate $\mathcal{F}$ consists only of local measurements performed onto a particle in the Alice's and Bob's paths. In this case, the final gate is equivalent to two target gates T_A and T_B. Given that all the processes considered are pure, the corresponding process vector for the 4-event quantum switch implementation is given as (in order to compare the current with the previous works, we present the process that starts after $\mathcal{I}$, as was done in, say, [9]):

$$|W_{QS_4}\rangle\!\rangle = \frac{1}{\sqrt{2}}\left(|\Psi\rangle^{A_I}|v\rangle^{B_I} + |v\rangle^{A_I}|\Psi\rangle^{B_I}\right)|\mathbb{1}\rangle\!\rangle^{A_O B'_I}|\mathbb{1}\rangle\!\rangle^{B_O A'_I}|\mathbb{1}\rangle\!\rangle^{A'_O T_{A_I}}|\mathbb{1}\rangle\!\rangle^{B'_O T_{B_I}}. \tag{72}$$

D 3-event process vector

The detailed spacetime diagram for the 3-event quantum switch is given below (see Figure 15).

The process vector for this case is obtained from (44) by identifying the spacetime positions of the gates B and B', yet keeping the corresponding Hilbert spaces in the mathematical description (i.e., keeping the dimensionality of the problem). Thus, the corresponding circuit is *identical* to the 4-event circuit, and the process vector has the *identical* mathematical form as in the case of four gates. In order to emphasise the physical difference between the two cases, instead of $B_{I/O}$ and $B'_{I/O}$, we write B_{I_1/O_1} and B_{I_2/O_2}, respectively:

$$\begin{aligned} |W_{\text{3-event}}\rangle\!\rangle \;=\; & \underbrace{|\mathbb{1}\rangle\!\rangle^{P_{A_O} S^i_{A_I}}|\mathbb{1}\rangle\!\rangle^{P_{B_O} S^i_{B_I}}}_{\text{initial}}\underbrace{|\mathbb{1}\rangle\!\rangle^{S^i_{A_O} A_I}|\mathbb{1}\rangle\!\rangle^{A_O B_{I_2}}|\mathbb{1}\rangle\!\rangle^{B_{O_2} S^f_{B_I}}}_{\text{blue}} \\ & \underbrace{|\mathbb{1}\rangle\!\rangle^{S^i_{B_O} B_{I_1}}|\mathbb{1}\rangle\!\rangle^{B_{O_1} A'_I}|\mathbb{1}\rangle\!\rangle^{A'_O S^f_{A_I}}}_{\text{red}}\underbrace{|\mathbb{1}\rangle\!\rangle^{S^f_{A_O} T_{A_I}}|\mathbb{1}\rangle\!\rangle^{S^f_{B_O} T_{B_I}}}_{\text{final}}\,. \end{aligned} \tag{73}$$

The final probability distribution is identical to the one for the 4-event process, given by (67).

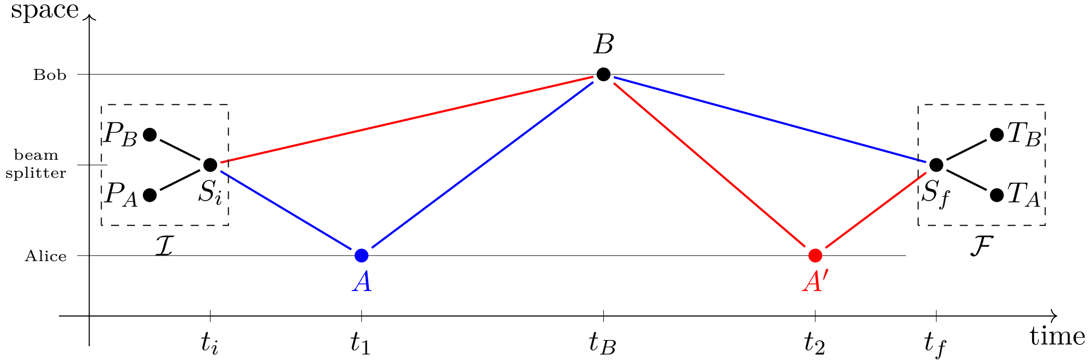


Figure 15: Spacetime diagram of the 3-event implementation of the quantum switch. The internal structures of the composite gates $\mathcal{I}$ and $\mathcal{F}$ are explicitly depicted.

E 2-event

In this Appendix we present process vectors for the two gravitational switches discussed in the main text. First, the process vector of the gravitational switch without recombination [15], is given by (since the "control" is now played by gravity, it is thus denoted as G, instead of C):

$$|W_{QS_2}\rangle\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^G|\Psi\rangle^{A_I}|\mathbb{1}\rangle\rangle^{A_OB_I}|\mathbb{1}\rangle\rangle^{B_OT_{B_I}} + |1\rangle^G|\Psi\rangle^{B_I}|\mathbb{1}\rangle\rangle^{B_OA_I}|\mathbb{1}\rangle\rangle^{A_OT_{A_I}}\right)|\mathbb{1}\rangle\rangle^{GT_{G_I}}. \tag{74}$$

It is then straightforward to obtain the process vector for the case of recombining only the gravity on the final beam splitter S_f (a part of a bigger final gate $\mathcal{F}$), obtaining the analogue of (49), while the particle is not being recombined. Note that in this case the introduction of the vacuum state is not necessary, as in each branch of superposition all gates are acting upon a particle.

In contrast to the above case, the process vector describing the gravitational 2-event quantum switch with the recombination of both gravity and the particle is given as follows:

$$\begin{aligned}|W^{(r)}_{QS_2}\rangle\rangle = \frac{1}{\sqrt{2}}&\left(|0\rangle^G|\Psi\rangle^{A_I}|\mathbb{1}\rangle\rangle^{A_OB_I}|\mathbb{1}\rangle\rangle^{B_OS_{P_I}} + |1\rangle^G|\Psi\rangle^{B_I}|\mathbb{1}\rangle\rangle^{B_OA_I}|\mathbb{1}\rangle\rangle^{A_OS_{P_I}}\right)\\ &\otimes|\mathbb{1}\rangle\rangle^{GS_{G_I}}|\mathbb{1}\rangle\rangle^{(S_{G_O}S_{P_O})(T_{G_I}T_{P_I})}.\end{aligned} \tag{75}$$

Here, P stands for "the particle" (whose corresponding input space S_{P_I} is isomorphic to the tensor product of Alice's and Bob's output spaces, $S_{P_I} \simeq A_O \otimes B_O$), and S stands for a "beam splitter" (whose corresponding input space is $S_I = S_{G_I} \otimes S_{P_I}$, and analogously for the output space).

While defining the spaces $S_{G_{I/O}}$, $S_{P_{I/O}}$, T_{G_I}, T_{P_I}, and the vector $|\mathbb{1}\rangle\rangle^{(S_{G_O}S_{P_O})(T_{G_I}T_{P_O})}$ is straightforward, it is not so for the "final" recombination vector $|U^*_{BS}\rangle\rangle^{(S_{G_I}S_{P_I})(S_{G_O}S_{P_O})}$. Namely, note that in our gravitational switch *all* degrees of freedom, both gravitational and matter, are recombined by U_{BS} such that, by acting on the beam splitter input entangled state,

$$|\Psi_i\rangle^{S_{G_I}S_{P_I}} = \frac{1}{\sqrt{2}}\left(|0\rangle^{S_{G_I}} \otimes \left[UV|\Psi\rangle^{S_{P_I}}\right] + |1\rangle^{S_{G_I}} \otimes \left[VU|\Psi\rangle^{S_{P_I}}\right]\right), \tag{76}$$

the overall output state is a product one, of the form

$$|\Psi_o\rangle^{S_{G_O}S_{P_O}} = |\Gamma\rangle^{S_{G_O}} \otimes (\alpha UV + \beta VU)\,|\Psi\rangle^{S_{P_O}}, \tag{77}$$

where $|\Gamma\rangle^{S_{G_O}}$ is some (not necessarily classical) state of gravity. The above evolution is, at least in principle, allowed by the quantum laws, which is all that we need to know regarding the action of U_{BS} at this point. Its action on the rest of the overall Hilbert space is, for the purpose of our argument, irrelevant, and can thus be chosen arbitrarily.

Finally, we would like to note that the same type of the 4-, or 3-event quantum switches in classical spacetimes can also be defined, resulting in the same type of the output state as (77), with the gravity degree of freedom being replaced by any additional matter degree of freedom that plays the role of the control.

F Various implementations of the gravitational switch

In this Appendix we present a few representative additional constructions of the gravitational quantum switch. First, we start with a 2-event switch for which the requirement (i) from Subsection 4.3 is not satisfied. It is obvious from the diagram on the left that each of the photon's superposed trajectories fail to meet at the boundary region, violating requirement (i), see the left diagram of Figure 16. Next, we proceed with the 2-event implementation for which requirement (i) is satisfied, but the requirement (ii) is not, since the superposed trajectories fail to recombine. This is depicted on the right diagram of Figure 16.

Finally, we present a 4-event implementation for which *both* requirements (i) and (ii) are satisfied (see Figure 17).

Of course, other variations are possible as well.

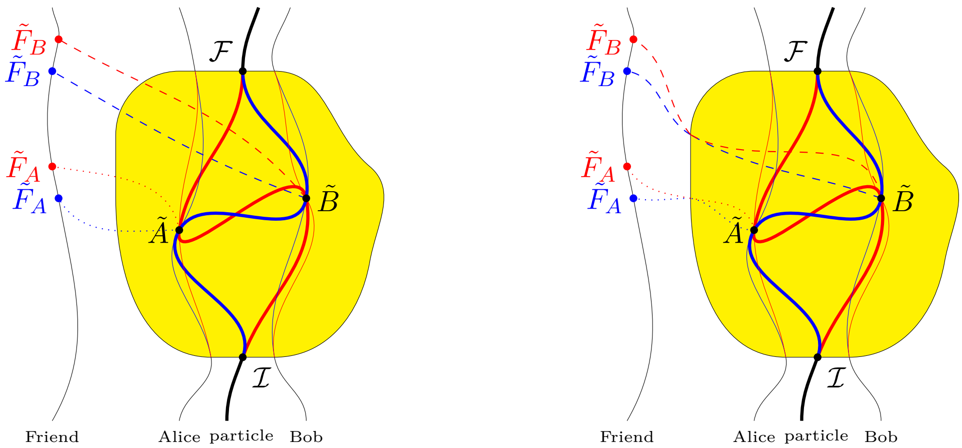


Figure 16: The spacetime diagrams of a 2-event gravitational switch implementations, with Friend's measurements, which fail to distinguish them from the optical implementation of the quantum switch.

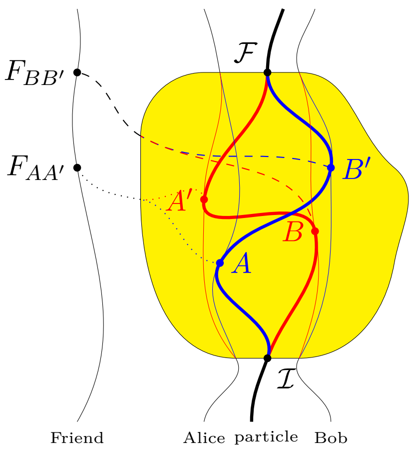


Figure 17: The spacetime diagram of a 4-event gravitational switch implementation, with Friend's measurement, which fails to distinguish it from the 2-event implementation of the gravitational switch.

Published for SISSA by Springer
Received: June 12, 2019
Accepted: October 6, 2019
Published: October 22, 2019

Higher gauge theories based on 3-groups

T. Radenković[1] and M. Vojinović

*Institute of Physics, University of Belgrade,
Pregrevica 118, 11080 Belgrade, Serbia*

E-mail: rtijana@ipb.ac.rs, vmarko@ipb.ac.rs

Abstract: We study the categorical generalizations of a BF theory to $2BF$ and $3BF$ theories, corresponding to 2-groups and 3-groups, in the framework of higher gauge theory. In particular, we construct the constrained $3BF$ actions describing the correct dynamics of Yang-Mills, Klein-Gordon, Dirac, Weyl, and Majorana fields coupled to Einstein-Cartan gravity. The action is naturally split into a topological sector and a sector with simplicity constraints, adapted to the spinfoam quantization programme. In addition, the structure of the 3-group gives rise to a novel gauge group which specifies the spectrum of matter fields present in the theory, just like the ordinary gauge group specifies the spectrum of gauge bosons in the Yang-Mills theory. This allows us to rewrite the whole Standard Model coupled to gravity as a constrained $3BF$ action, facilitating the nonperturbative quantization of both gravity and matter fields. Moreover, the presence and the properties of this new gauge group open up a possibility of a nontrivial unification of all fields and a possible explanation of fermion families and all other structure in the matter spectrum of the theory.

Keywords: Models of Quantum Gravity, Topological Field Theories, Gauge Symmetry, Beyond Standard Model

ArXiv ePrint: 1904.07566

[1]Corresponding author.

https://doi.org/10.1007/JHEP10(2019)222

JHEP10(2019)222

Contents

1 Introduction

The quantization of the gravitational field is one of the most prominent open problems in modern theoretical physics. Within the Loop Quantum Gravity framework, one can study the nonperturbative quantization of gravity, both canonically and covariantly, see [1–3] for an overview and a comprehensive introduction. The covariant approach focuses on the definition of the path integral for the gravitational field,

$$Z = \int \mathcal{D}g \; e^{iS[g]} \,, \tag{1.1}$$

by considering a triangulation of a spacetime manifold, and defining the path integral as a discrete state sum of the gravitational field configurations living on the simplices in the triangulation. This quantization technique is known as the *spinfoam* quantization method, and roughly goes along the following lines:

1. first, one writes the classical action $S[g]$ as a topological BF action plus a simplicity constraint,

2. then one uses the algebraic structure (a Lie group) underlying the topological sector of the action to define a triangulation-independent state sum Z,

3. and finally, one imposes the simplicity constraints on the state sum, promoting it into a path integral for a physical theory.

This quantization prescription has been implemented for various choices of the action, the Lie group, and the spacetime dimension. For example, in 3 dimensions, the prototype spinfoam model is known as the Ponzano-Regge model [4]. In 4 dimensions there are multiple models, such as the Barrett-Crane model [5, 6], the Ooguri model [7], and the most sophisticated EPRL/FK model [8, 9]. All these models aim to define a viable theory of quantum gravity, with variable success. However, virtually all of them are focused on pure gravity, without matter fields. The attempts to include matter fields have had limited success [10], mainly because the mass terms could not be expressed in the theory due to the absence of the tetrad fields from the BF sector of the theory.

In order to resolve this issue, a new approach has been developed, using the categorical generalization of the BF action, within the framework of *higher gauge theory* (see [11] for a review). In particular, one uses the idea of a categorical ladder to promote the BF action, which is based on some Lie group, into a $2BF$ action, which is based on the so-called 2-group structure. If chosen in a suitable way, the 2-group structure should hopefully introduce the tetrad fields into the action. This approach has been successfully implemented [12], rewriting the action for general relativity as a constrained $2BF$ action, such that the tetrad fields are present in the topological sector. This result opened up a possibility to couple all matter fields to gravity in a straightforward way. Nevertheless, the matter fields could not be naturally expressed using the underlying algebraic structure of a 2-group, rendering the spinfoam quantization method only half-implementable, since the matter sector of the classical action could not be expressed as a topological term plus a simplicity constraint, which means that the steps 2 and 3 above could not be performed for the matter sector of the action.

We address this problem in this paper. As we will show, it turns out that it is necessary to perform one more step in the categorical ladder, generalizing the underlying algebraic structure from a 2-group to a 3-group. This generalization then naturally gives rise to the so-called $3BF$ action, which proves to be suitable for a unified description of both gravity and matter fields. The steps of the categorical ladder can be conveniently summarized in the following table:

categorical structure	algebraic structure	linear structure	topological action	degrees of freedom
Lie group	Lie group	Lie algebra	BF theory	gauge fields
Lie 2-group	Lie crossed module	differential Lie crossed module	$2BF$ theory	tetrad fields
Lie 3-group	Lie 2-crossed module	differential Lie 2-crossed module	$3BF$ theory	scalar and fermion fields

Once the suitable gauge 3-group has been specified and the corresponding $3BF$ action constructed, the most important thing that remains, in order to complete the step 1 of the spinfoam quantization programme, is to impose appropriate simplicity constraints onto the degrees of freedom present in the $3BF$ action, so that we obtain the desired classical dynamics of the gravitational and matter fields. Then one can proceed with steps 2 and 3 of the spinfoam quantization, hopefully ending up with a viable model of quantum gravity and matter.

In this paper, we restrict our attention to the first of the above steps: we will construct a constrained $3BF$ action for the cases of Klein-Gordon, Dirac, Weyl and Majorana fields, as well as Yang-Mills and Proca vector fields, all coupled to the Einstein-Cartan gravity in the standard way. This construction will lead us to an unexpected novel result. As we shall see, the scalar and fermion fields will be *naturally associated to a new gauge group*, generalizing the notion of a gauge group in the Yang-Mills theory, which describes vector bosons. This new group opens up a possibility to use it as an algebraic way of classifying matter fields, describing the structures such as quark and lepton families, and so on. The insight into the existence of this new gauge group is the consequence of the categorical ladder and is one of the main results of the paper. However, given the complexity of the algebraic properties of 3-groups, we will restrict ourselves only to the reconstruction of the already known theories, such as the Standard Model (SM), in the new framework. In this sense, any potential explanation of the spectrum of matter fields in the SM will be left for future work.

The layout of the paper is as follows. In subsection 2.1 we will give a short overview of the constrained BF actions, including the well-known example of the Plebanski action for general relativity, and a completely new example of the Yang-Mills theory rewritten as a constrained BF model. In the subsection 2.2 we also introduce the formalism of the constrained $2BF$ actions, reviewing the example of general relativity as a constrained $2BF$ action, first introduced in [12]. In addition, we will demonstrate how to couple gravity in a natural way within the formalism of 2-groups. Section 3 contains the main results of the paper and is split into 4 subsections. The subsection 3.1 introduces the formalism of 3-groups, and the definition and properties of a $3BF$ action, including the three types of gauge transformations. The subsection 3.2 focuses on the construction of a constrained $3BF$ action which describes a single real scalar field coupled to gravity. It provides the most elementary example of the insight that matter fields correspond to a gauge group. Encouraged by these results, in the subsection 3.3 we construct the constrained $3BF$ action for the Dirac field coupled to gravity and specify its gauge group. Finally, the subsection 3.4 deals with the construction of the constrained $3BF$ action for the Weyl and Majorana fields coupled to gravity, thereby covering all types of fields potentially relevant for the Standard Model and beyond. After the construction of all building blocks, in section 4 we apply the results of sections 2 and 3 to construct the constrained $3BF$ action corresponding to the full Standard Model coupled to Einstein-Cartan gravity. Finally, section 5 is devoted to the discussion of the results and the possible future lines of research. The appendices contain some mathematical reminders and technical details.

The notation and conventions are as follows. The local Lorentz indices are denoted by the Latin letters $a, b, c, \dots$, take values $0, 1, 2, 3$, and are raised and lowered using the

JHEP10(2019)222

Minkowski metric η_{ab} with signature $(-,+,+,+)$. Spacetime indices are denoted by the Greek letters $\mu,\nu,\dots$, and are raised and lowered by the spacetime metric $g_{\mu\nu}=\eta_{ab}e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ are the tetrad fields. The inverse tetrad is denoted as $e^\mu{}_a$. All other indices that appear in the paper are dependent on the context, and their usage is explicitly defined in the text where they appear. A lot of additional notation is defined in appendix A. We work in the natural system of units where $c=\hbar=1$, and $G=l_p^2$, where l_p is the Planck length.

2 BF and $2BF$ models, ordinary gauge fields and gravity

Let us begin by giving a short review of BF and $2BF$ theories in general. For additional information on these topics, see for example [11, 13–18].

2.1 BF theory

Given a Lie group G and its corresponding Lie algebra $\mathfrak{g}$, one can introduce the so-called BF action as

$$S_{BF}=\int_{\mathcal{M}_4}\langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}}\,. \tag{2.1}$$

Here, $\mathcal{F}\equiv \mathrm{d}\alpha+\alpha\wedge\alpha$ is the curvature 2-form for the algebra-valued connection 1-form $\alpha\in\mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$ on some 4-dimensional spacetime manifold $\mathcal{M}_4$. In addition, $B\in\mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{g})$ is a Lagrange multiplier 2-form, while $\langle\_\,,\_\rangle_{\mathfrak{g}}$ denotes the G-invariant bilinear symmetric nondegenerate form.

From the structure of (2.1), one can see that the action is diffeomorphism invariant, and it is usually understood to be gauge invariant with respect to G. In addition to these properties, the BF action is topological, in the following sense. Varying the action (2.1) with respect to B^β and α^β, where the index β counts the generators of $\mathfrak{g}$ (see appendix A for notation and conventions), one obtains the equations of motion of the theory,

$$\mathcal{F}=0\,,\qquad \nabla B\equiv \mathrm{d}B+\alpha\wedge B=0\,. \tag{2.2}$$

From the first equation of motion, one immediately sees that α is a flat connection, which then together with the second equation of motion implies that B is constant. Therefore, there are no local propagating degrees of freedom in the theory, and one then says that the theory is topological.

Usually, in physics one is interested in theories which are nontopological, i.e., which have local propagating degrees of freedom. In order to transform the BF action into such a theory, one adds an additional term to the action, commonly called the *simplicity constraint*. A very nice example is the Yang-Mills theory for the $\mathrm{SU}(N)$ group, which can be rewritten as a constrained BF theory in the following way:

$$S=\int B_I\wedge F^I+\lambda^I\wedge\left(B_I-\frac{12}{g}M_{abI}\delta^a\wedge\delta^b\right)+\zeta^{abI}\left(M_{abI}\varepsilon_{cdef}\delta^c\wedge\delta^d\wedge\delta^e\wedge\delta^f-g_{IJ}F^J\wedge\delta_a\wedge\delta_b\right). \tag{2.3}$$

Here $F\equiv\mathrm{d}A+A\wedge A$ is again the curvature 2-form for the connection $A\in\mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{su}(N))$, and $B\in\mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{su}(N))$ is the Lagrange multiplier 2-form. The Killing form $g_{IJ}\equiv$

JHEP10(2019)222

$\langle\tau_I,\tau_J\rangle_{\mathfrak{su}(N)} \propto f_{IK}{}^L f_{JL}{}^K$ is used to raise and lower the indices $I, J, \dots$ which count the generators of $\mathrm{SU}(N)$, where $f_{IJ}{}^K$ are the structure constants for the $\mathfrak{su}(N)$ algebra. In addition to the topological $B\wedge F$ term, we also have two simplicity constraint terms, featuring the Lagrange multiplier 2-form λ^I and the Lagrange multiplier 0-form ζ^{abI}. The 0-form M_{abI} is also a Lagrange multiplier, while g is the coupling constant for the Yang-Mills theory.

Finally, δ^a is a nondynamical 1-form, such that there exists a global coordinate frame in which its components are equal to the Kronecker symbol $\delta^a{}_\mu$ (hence the notation δ^a). The 1-form δ^a plays the role of a background field, and defines the global spacetime metric, via the equation

$$\eta_{\mu\nu} = \eta_{ab}\delta^a{}_\mu\delta^b{}_\nu\,, \tag{2.4}$$

where $\eta_{ab} \equiv \mathrm{diag}(-1,+1,+1,+1)$ is the Minkowski metric. Since the coordinate system is global, the spacetime manifold $\mathcal{M}_4$ is understood to be flat. The indices $a, b, \dots$ are local Lorentz indices, taking values $0,\dots,3$. Note that the field δ^a has all the properties of the tetrad 1-form e^a in the flat Minkowski spacetime. Also note that the action (2.3) is manifestly diffeomorphism invariant and gauge invariant with respect to $\mathrm{SU}(N)$, but not background independent, due to the presence of δ^a.

The equations of motion are obtained by varying the action (2.3) with respect to the variables ζ^{abI}, M_{abI}, A^I, B_I, and λ^I, respectively (note that we do not take the variation of the action with respect to the background field δ^a):

$$M_{abI}\varepsilon_{cdef}\delta^c\wedge\delta^d\wedge\delta^e\wedge\delta^f - F_I\wedge\delta_a\wedge\delta_b = 0\,, \tag{2.5}$$

$$-\frac{12}{g}\lambda^I\wedge\delta^a\wedge\delta^b + \zeta^{abI}\varepsilon_{cdef}\delta^c\wedge\delta^d\wedge\delta^e\wedge\delta^f = 0\,, \tag{2.6}$$

$$-\mathrm{d}B_I + f_{JI}{}^K B_K\wedge A^J + \mathrm{d}(\zeta^{ab}{}_I\delta_a\wedge\delta_b) - f_{JI}{}^K\zeta^{ab}{}_K\delta_a\wedge\delta_b\wedge A^J = 0\,, \tag{2.7}$$

$$F_I + \lambda_I = 0\,, \tag{2.8}$$

$$B_I - \frac{12}{g}M_{abI}\delta^a\wedge\delta^b = 0\,, \tag{2.9}$$

From the algebraic equations (2.5), (2.6), (2.8) and (2.9) one obtains the multipliers as functions of the dynamical field A^I:

$$M_{abI} = \frac{1}{48}\varepsilon_{abcd}F_I{}^{cd}\,,\quad \zeta^{abI} = \frac{1}{4g}\varepsilon^{abcd}F^I{}_{cd}\,,\quad \lambda_{Iab} = F_{Iab}\,,\quad B_{Iab} = \frac{1}{2g}\varepsilon_{abcd}F_I{}^{cd}\,. \tag{2.10}$$

Here we used the notation $F_{Iab} = F_{I\mu\nu}\delta_a{}^\mu\delta_b{}^\nu$, where we used the fact that $\delta^a{}_\mu$ is invertible, and similarly for other variables. Using these equations and the differential equation (2.7) one obtains the equation of motion for gauge field A^I,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0\,. \tag{2.11}$$

This is precisely the classical equation of motion for the free Yang-Mills theory. Note that in addition to the Yang-Mills theory, one can easily extend the action (2.3) in order to describe the massive vector field and obtain the Proca equation of motion. This is done by adding a mass term

$$-\frac{1}{4!}m^2 A_{I\mu}A^I{}_\nu\eta^{\mu\nu}\varepsilon_{abcd}\delta^a\wedge\delta^b\wedge\delta^c\wedge\delta^d \tag{2.12}$$

to the action (2.3). Of course, this term explicitly breaks the SU(N) gauge symmetry of the action.

Another example of the constrained BF theory is the Plebanski action for general relativity [15], see also [13] for a recent review. Starting from a gauge group SO(3, 1), one constructs a constrained BF action as

$$S = \int_{\mathcal{M}_4} B_{ab} \wedge R^{ab} + \phi_{abcd} B^{ab} \wedge B^{cd} \,. \tag{2.13}$$

Here R^{ab} is the curvature 2-form for the spin connection ω^{ab}, B_{ab} is the usual Lagrange multiplier 2-form, while ϕ_{abcd} is the Lagrange multiplier 0-form corresponding to the simplicity constraint term $B^{ab} \wedge B^{cd}$. It can be shown that the variation of this action with respect to B_{ab}, ω^{ab} and ϕ_{abcd} gives rise to equations of motion which are equivalent to vacuum general relativity. However, the tetrad fields appear in the model as a solution to the simplicity constraint equation of motion $B^{ab} \wedge B^{cd} = 0$. Thus, being intrinsically on-shell objects, they are not present in the action and cannot be quantized. This renders the Plebanski model unsuitable for coupling of matter fields to gravity [10, 12, 19]. Nevertheless, as a model for pure gravity, the Plebanski model has been successfully quantized in the context of spinfoam models, see [1, 2, 8, 9] for details and references.

2.2 2BF theory

In order to circumvent the issue of coupling of matter fields, a recent promising approach has been developed [12, 19–23] in the context of higher category theory [11]. In particular, one employs the higher category theory construction to generalize the BF action to the so-called 2BF action, by passing from the notion of a gauge group to the notion of a gauge 2-group. In order to introduce it, let us first give a short review of the 2-group formalism.

In the framework of category theory, the group as an algebraic structure can be understood as a specific type of category, namely a category with only one object and invertible morphisms [11]. The notion of a category can be generalized to the so-called *higher categories*, which have not only objects and morphisms, but also 2-morphisms (morphisms between morphisms), and so on. This process of generalization is called the *categorical ladder*. Similarly to the notion of a group, one can introduce a 2-group as a 2-category consisting of only one object, where all the morphisms and 2-morphisms are invertible. It has been shown that every strict 2-group is equivalent to a crossed module $(H \stackrel{\partial}{\to} G \,, \rhd)$, see appendix A for definition. Here G and H are groups, δ is a homomorphism from H to G, while $\rhd : G \times H \to H$ is an action of G on H.

An important example of this structure is a vector space V equipped with an isometry group O. Namely, V can be regarded as an Abelian Lie group with addition as a group operation, so that a representation of O on V is an action $\rhd$ of O on the group V, giving rise to the crossed module $(V \stackrel{\partial}{\to} O \,, \rhd)$, where the homomorphism ∂ is chosen to be trivial, i.e., it maps every element of V into a unit of O. We will make use of this example below to introduce the Poincaré 2-group.

Similarly to the case of an ordinary Lie group G which has a naturally associated notion of a connection α, giving rise to a BF theory, the 2-group structure has a naturally associated notion of a 2-connection $(\alpha\,,\beta)$, described by the usual $\mathfrak{g}$-valued 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$ and an $\mathfrak{h}$-valued 2-form $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$, where $\mathfrak{h}$ is a Lie algebra of the Lie group H. The 2-connection gives rise to the so-called *fake 2-curvature* $(\mathcal{F},\mathcal{G})$, given as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,, \qquad \mathcal{G} = d\beta + \alpha\wedge^{\triangleright}\beta\,. \tag{2.14}$$

Here $\alpha\wedge^{\triangleright}\beta$ means that α and β are multiplied as forms using $\wedge$, and simultaneously multiplied as algebra elements using $\triangleright$, see appendix A. The curvature pair $(\mathcal{F},\mathcal{G})$ is called fake because of the presence of the $\partial\beta$ term in the definition of $\mathcal{F}$, see [11] for details.

Using these variables, one can introduce a new action as a generalization of the BF action, such that it is gauge invariant with respect to both G and H groups. It is called the $2BF$ action and is defined in the following way [16, 17]:

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}} + \langle C\wedge\mathcal{G}\rangle_{\mathfrak{h}}\,, \tag{2.15}$$

where the 2-form $B\in\mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{g})$ and the 1-form $C\in\mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{h})$ are Lagrange multipliers. Also, $\langle\_\,,\_\rangle_{\mathfrak{g}}$ and $\langle\_\,,\_\rangle_{\mathfrak{h}}$ denote the G-invariant bilinear symmetric nondegenerate forms for the algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. As a consequence of the axiomatic structure of a crossed module (see appendix A), the bilinear form $\langle\_\,,\_\rangle_{\mathfrak{h}}$ is H-invariant as well. See [16, 17] for review and references.

Similarly to the BF action, the $2BF$ action is also topological, which can be seen from equations of motion. Varying with respect to B and C one obtains

$$\mathcal{F} = 0\,, \qquad \mathcal{G} = 0\,, \tag{2.16}$$

while varying with respect to α and β one obtains the equations for the multipliers,

$$\mathrm{d}B_\alpha - g_{\alpha\beta}{}^{\gamma}B_\gamma\wedge\alpha^\beta - \triangleright_{\alpha a}{}^{b}C_b\wedge\beta^a = 0\,, \tag{2.17}$$

$$\mathrm{d}C_a - \partial_a{}^{\alpha}B_\alpha + \triangleright_{\alpha a}{}^{b}C_b\wedge\alpha^\alpha = 0\,. \tag{2.18}$$

One can either show that these equations have only trivial solutions, or one can use the Hamiltonian analysis to show that there are no local propagating degrees of freedom (see for example [21, 22]), demostrating the topological nature of the theory.

An example of a 2-group relevant for physics is the Poincaré 2-group, which is constructed using the aforementioned example of a vector space equipped with an isometry group. One constructs a crossed module by choosing

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \tag{2.19}$$

while $\triangleright$ is a natural action of $\mathrm{SO}(3,1)$ on $\mathbb{R}^4$, and the map ∂ is trivial. The 2-connection (α,β) is given by the algebra-valued differential forms

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \tag{2.20}$$

where ω^{ab} is the spin connection, while M_{ab} and P_a are the generators of groups $\mathrm{SO}(3,1)$ and $\mathbb{R}^4$, respectively. The corresponding 2-curvature in this case is given by

$$\mathcal{F} = (\mathrm{d}\omega^{ab} + \omega^a{}_c\wedge\omega^{cb})M_{ab} \equiv R^{ab}M_{ab}\,, \quad \mathcal{G} = (\mathrm{d}\beta^a + \omega^a{}_b\wedge\beta^b)P_a \equiv \nabla\beta^a P_a \equiv G^a P_a\,, \tag{2.21}$$

JHEP10(2019)222

where we have evaluated $\wedge^{\triangleright}$ using the equation $M_{ab} \triangleright P_c = \eta_{[bc} P_{a]}$. Note that, since ∂ is trivial, the fake curvature is the same as ordinary curvature. Using the bilinear forms

$$\langle M_{ab}, M_{cd} \rangle_{\mathfrak{g}} = \eta_{a[c}\eta_{bd]}\,, \qquad \langle P_a, P_b \rangle_{\mathfrak{h}} = \eta_{ab}\,, \tag{2.22}$$

one can show that 1-forms C^a transform in the same way as the tetrad 1-forms e^a under the Lorentz transformations and diffeomorphisms, so the fields C^a can be identified with the tetrads. Then one can rewrite the $2BF$ action (2.15) for the Poincaré 2-group as

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a\,. \tag{2.23}$$

In order to obtain general relativity, the topological action (2.23) can be modified by adding a convenient simplicity constraint, like it is done in the BF case:

$$S = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right). \tag{2.24}$$

Here λ_{ab} is a Lagrange multiplier 2-form associated to the simplicity constraint term, and l_p is the Planck length. Varying the action (2.24) with respect to B_{ab}, e_a, ω_{ab}, β_a and λ_{ab}, one obtains the following equations of motion:

$$R_{ab} - \lambda_{ab} = 0\,, \tag{2.25}$$

$$\nabla \beta_a + \frac{1}{8\pi l_p^2} \varepsilon_{abcd} \lambda^{bc} \wedge e^d = 0\,, \tag{2.26}$$

$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} = 0\,, \tag{2.27}$$

$$\nabla e_a = 0\,, \tag{2.28}$$

$$B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d = 0\,. \tag{2.29}$$

The only dynamical fields are the tetrads e^a, while all other fields can be algebraically determined, as follows. From the equations (2.28) and (2.29) we obtain that $\nabla B^{ab} = 0$, from which it follows, using the equation (2.27), that $e_{[a} \wedge \beta_{b]} = 0$. Assuming that the tetrads are nondegenerate, $e \equiv \det(e^a{}_\mu) \neq 0$, it can be shown that this is equivalent to the condition $\beta^a = 0$ (for the proof see appendix in [12]). Therefore, from the equations (2.25), (2.27), (2.28) and (2.29) we obtain

$$\lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu}\,, \quad \beta^a{}_{\mu\nu} = 0\,, \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu\,, \quad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,. \tag{2.30}$$

Here the Ricci rotation coefficients are defined as

$$\triangle^{ab}{}_\mu \equiv \frac{1}{2}(c^{abc} - c^{cab} + c^{bca}) e_{c\mu}\,, \tag{2.31}$$

where

$$c^{abc} = e^\mu{}_b e^\nu{}_c \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu \right)\,. \tag{2.32}$$

JHEP10(2019)222

Finally, the remaining equation (2.26) reduces to

$$\varepsilon_{abcd} R^{bc} \wedge e^d = 0\,, \tag{2.33}$$

which is nothing but the vacuum Einstein field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$. Therefore, the action (2.24) is classically equivalent to general relativity.

The main advantage of the action (2.24) over the Plebanski model and similar approaches lies in the fact that the tetrad fields are explicitly present in the topological sector of the theory. This allows one to couple matter fields in a straightforward way, as demonstrated in [12]. However, one can do even better, and couple gauge fields to gravity within a unified framework of 2-group formalism.

Let us demonstrate this on the example of the SU(N) Yang-Mills theory. Begin by modifying the Poincaré 2-group structure to include the SU(N) gauge group, as follows. We choose the two Lie groups as

$$G = \mathrm{SO}(3,1) \times \mathrm{SU}(N)\,, \qquad H = \mathbb{R}^4\,, \tag{2.34}$$

and we define the action $\triangleright$ of the group G in the following way. As in the case of the Poincaré 2-group, it acts on itself via conjugation. Next, it acts on H such that the SO(3,1) subgroup acts on $\mathbb{R}^4$ via the vector representation, while the action of SU(N) subgroup is trivial. The map ∂ also remains trivial, as before. The 2-connection (α, β) now obtains the form which reflects the structure of the group G,

$$\alpha = \omega^{ab} M_{ab} + A^I \tau_I\,, \qquad \beta = \beta^a P_a\,, \tag{2.35}$$

where A^I is the gauge connection 1-form, while τ_I are the SU(N) generators. The curvature for α is thus

$$\mathcal{F} = R^{ab} M_{ab} + F^I \tau_I\,, \qquad F^I \equiv \mathrm{d}A^I + f_{JK}{}^I A^J \wedge A^K\,. \tag{2.36}$$

The curvature for β remains the same as before, since the action $\triangleright$ of SU(N) on $\mathbb{R}^4$ is trivial, i.e., $\tau_I \triangleright P_a = 0$. Finally, the product structure of the group G implies that its Killing form $\langle \_ , \_ \rangle_{\mathfrak{g}}$ reduces to the Killing forms for the SO(3,1) and SU(N), along with the identity $\langle M_{ab}, \tau_I \rangle_{\mathfrak{g}} = 0$.

Given a crossed module defined in this way, its corresponding topological $2BF$ action (2.15) becomes

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a\,, \tag{2.37}$$

where $B^I \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{su}(N))$ is the new Lagrange multiplier. In order to transform this topological action into action with nontrivial dynamics, we again introduce the appropriate simplicity constraints. The constraint giving rise to gravity is the same as in (2.24), while the constraint for the gauge fields is given as in the action (2.3) with the substitution $\delta^a \to e^a$:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda^I \wedge \left(B_I - \frac{12}{g} M_{abI} e^a \wedge e^b \right) + \zeta^{abI} \left(M_{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - g_{IJ} F^J \wedge e_a \wedge e_b \right). \end{aligned} \tag{2.38}$$

JHEP10(2019)222

It is crucial to note that the action (2.38) is a combination of the pure gravity action (2.24) and the Yang-Mills action (2.3), such that the nondynamical background field δ^a from (2.3) gets promoted to a dynamical field e^a. The relationship between these fields has already been hinted at in the equation (2.4), which describes the connection between δ^a and the flat spacetime metric $\eta_{\mu\nu}$. Once promoted to e^a, this field becomes dynamical, while the equation (2.4) becomes the usual relation between the tetrad and the metric,

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \,, \tag{2.39}$$

further confirming that the Lagrange multiplier C^a should be identified with the tetrad. Moreover, the total action (2.38) now becomes background independent, as expected in general relativity. All this is a consequence of the fact that the tetrad field is explicitly present in the topological sector of the action (2.24), establishing an improvement over the Plebanski model.

By varying the action (2.38) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, ζ^{abI}, M_{abI}, B_I, λ^I, A^I, and e^a, we obtain the following equations of motion, respectively:

$$R^{ab} - \lambda^{ab} = 0 \,, \tag{2.40}$$

$$\nabla B^{ab} - e^{[a} \wedge \beta^{b]} = 0 \,, \tag{2.41}$$

$$\nabla e^a = 0 \,, \tag{2.42}$$

$$B_{ab} - \frac{1}{16\pi l_p^2} \varepsilon_{abcd} e^c \wedge e^d = 0 \,, \tag{2.43}$$

$$M_{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F_I \wedge e_a \wedge e_b = 0 \,, \tag{2.44}$$

$$-\frac{12}{g} \lambda^I \wedge e^a \wedge e^b + \zeta^{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f = 0 \,, \tag{2.45}$$

$$F_I + \lambda_I = 0 \,, \tag{2.46}$$

$$B_I - \frac{12}{g} M_{abI} e^a \wedge e^b = 0 \,, \tag{2.47}$$

$$-\mathrm{d}B_I + B_K \wedge g_{JI}{}^K A^J + \mathrm{d}(\zeta_I^{ab} e_a \wedge e_b) - \zeta_K^{ab} e_a \wedge e_b \wedge g_{JI}{}^K A^J = 0 \,, \tag{2.48}$$

$$\begin{aligned} &\nabla \beta_a + \frac{1}{8\pi l_p^2} \varepsilon_{abcd} \lambda^{bc} \wedge e^d - \frac{24}{g} M_{abI} \lambda^I \wedge e^b \\ &+ 4\zeta^{ef\,I} M_{ef\,I} \varepsilon_{abcd} e^b \wedge e^c \wedge e^d - 2\zeta_{ab}{}^I F_I \wedge e^b = 0 \,. \end{aligned} \tag{2.49}$$

In the above system of equations, we have two dynamical equations for e^a and A^I, while all other variables are algebraically determined from these. In particular, from equations (2.40)–(2.47), we have:

$$\lambda_{ab\mu\nu} = R_{ab\mu\nu} \,, \quad \beta_{a\mu\nu} = 0 \,, \quad \omega_{ab\mu} = \triangle_{ab\mu} \,, \quad \lambda_{abI} = F_{abI} \,, \quad B_{\mu\nu I} = -\frac{e}{2g} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}{}_I \,, \tag{2.50}$$

$$B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu \,, \quad M_{abI} = -\frac{1}{4eg} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma \,, \quad \zeta^{abI} = \frac{1}{4eg} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma \,.$$

Then, substituting all these into (2.48) and (2.49) we obtain the differential equation of motion for A^I,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + \Gamma^\rho{}_{\lambda\rho} F^{I\lambda\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0 \,, \tag{2.51}$$

where $\Gamma^{\lambda}{}_{\mu\nu}$ is the standard Levi-Civita connection, and a differential equation of motion for e^a,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \qquad T^{\mu\nu} \equiv -\frac{1}{4g}\left(F_{\rho\sigma}{}^{I}F^{\rho\sigma}{}_{I}g^{\mu\nu} + 4F^{\mu\rho}{}_{I}F_{\rho}{}^{\nu I}\right)\,. \tag{2.52}$$

The system of equations (2.50)–(2.52) is equivalent to the system (2.40)–(2.49). Note that we have again obtained that $\beta^a = 0$, as in the pure gravity case.

In this way, we see that both gravity and gauge fields can be represented within a unified framework of higher gauge theory based on a 2-group structure.

3 3*BF* models, scalar and fermion matter fields

While the structure of a 2-group can successfully accommodate both gravitational and gauge fields, unfortunately it cannot include other matter fields, such as scalars or fermions. In order to construct a unified description of all matter fields within the framework of higher gauge theory, we are led to make a further generalization, passing from the notion of a 2-group to the notion of a 3-group. As it turns out, the 3-group structure is a perfect fit for the description of all fields that are present in the Standard Model, coupled to gravity. Moreover, this structure gives rise to a new gauge group, which corresponds to the choice of the scalar and fermion fields present in the theory. This is a novel and unexpected result, which has the potential to open up a new avenue of research with the aim of explaining the structure of the matter sector of the Standard Model and beyond.

In order to demonstrate this in more detail, we first need to introduce the notion of a 3-group, which we will afterward use to construct constrained 3*BF* actions describing scalar and fermion fields on an equal footing with gravity and gauge fields.

3.1 3-groups and topological 3*BF* action

Similarly to the concepts of a group and a 2-group, one can introduce the notion of a 3-group in the framework of higher category theory, as a 3-category with only one object where all the morphisms, 2-morphisms and 3-morphisms are invertible. It has been proved that a strict 3-group is equivalent to a 2-*crossed module* [24], in the same way as a 2-group is equivalent to a crossed module.

A Lie 2-crossed module, denoted as $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$, is a algebraic structure specified by three Lie groups G, H and L, together with the homomorphisms δ and ∂, an action $\triangleright$ of the group G on all three groups, and a G-equivariant map

$$\{\_\,,\_\} : H \times H \to L\,.$$

called the Peiffer lifting. See appendix A for more details.

In complete analogy to the construction of BF and $2BF$ topological actions, one can define a gauge invariant topological $3BF$ action for the manifold $\mathcal{M}_4$ and 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$. Given $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ as Lie algebras corresponding to the groups G, H and L, one can introduce a 3-connection (α, β, γ) given by the algebra-valued differential forms $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$ and $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,,\mathfrak{l})$. The corresponding fake 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is then defined as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha\wedge^{\triangleright}\beta - \delta\gamma\,, \qquad \mathcal{H} = \mathrm{d}\gamma + \alpha\wedge^{\triangleright}\gamma + \{\beta\wedge\beta\}\,. \tag{3.1}$$

see [24, 25] for details. Then, a $3BF$ action is defined as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}} + \langle C\wedge\mathcal{G}\rangle_{\mathfrak{h}} + \langle D\wedge\mathcal{H}\rangle_{\mathfrak{l}}\,, \tag{3.2}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4,\mathfrak{g})$, $C \in \mathcal{A}^1(\mathcal{M}_4,\mathfrak{h})$ and $D \in \mathcal{A}^0(\mathcal{M}_4,\mathfrak{l})$ are Lagrange multipliers. The forms $\langle\_,\_\rangle_{\mathfrak{g}}$, $\langle\_,\_\rangle_{\mathfrak{h}}$ and $\langle\_,\_\rangle_{\mathfrak{l}}$ are G-invariant bilinear symmetric nondegenerate forms on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, respectively. Under certain conditions, the forms $\langle\_,\_\rangle_{\mathfrak{h}}$ and $\langle\_,\_\rangle_{\mathfrak{l}}$ are also H-invariant and L-invariant, see appendix B for details.

One can see that varying the action with respect to the variables B, C and D, one obtains the equations of motion

$$\mathcal{F} = 0\,, \qquad \mathcal{G} = 0\,, \qquad \mathcal{H} = 0\,, \tag{3.3}$$

while varying with respect to α, β, γ one obtains

$$\mathrm{d}B_\alpha - g_{\alpha\beta}{}^{\gamma}B_\gamma\wedge\alpha^\beta - \triangleright_{\alpha a}{}^{b}C_b\wedge\beta^a + \triangleright_{\alpha B}{}^{A}D_A\wedge\gamma^B = 0\,, \tag{3.4}$$

$$\mathrm{d}C_a - \partial_a{}^{\alpha}B_\alpha + \triangleright_{\alpha a}{}^{b}C_b\wedge\alpha^\alpha + 2X_{\{ab\}}{}^{A}D_A\wedge\beta^b = 0\,, \tag{3.5}$$

$$\mathrm{d}D_A - \triangleright_{\alpha A}{}^{B}D_B\wedge\alpha^\alpha + \delta_A{}^{a}C_a = 0\,. \tag{3.6}$$

Regarding the gauge transformations, the $3BF$ action is invariant with respect to three different types of transformations, generated by the groups G, H and L, respectively. Under the G-gauge transformations, the 3-connection transforms as

$$\alpha' = g^{-1}\alpha g + g^{-1}\mathrm{d}g\,, \qquad \beta' = g^{-1}\triangleright\beta\,, \qquad \gamma' = g^{-1}\triangleright\gamma\,, \tag{3.7}$$

where $g : \mathcal{M}_4 \to G$ is an element of the G-principal bundle over $\mathcal{M}_4$. Next, under the H-gauge transformations, generated by $\eta \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{h})$, the 3-connection transforms as

$$\alpha' = \alpha + \partial\eta\,, \qquad \beta' = \beta + \mathrm{d}\eta + \alpha'\wedge^{\triangleright}\eta - \eta\wedge\eta\,, \qquad \gamma' = \gamma - \{\beta'\wedge\eta\} - \{\eta\wedge\beta\}\,. \tag{3.8}$$

Finally, under the L-gauge transformations, generated by $\theta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{l})$, the 3-connection transforms as

$$\alpha' = \alpha\,, \qquad \beta' = \beta - \delta\theta\,, \qquad \gamma' = \gamma - \mathrm{d}\theta - \alpha\wedge\theta\,. \tag{3.9}$$

As a consequence of the definition (3.1) and the above transformation rules, the curvatures transform under the G-gauge transformations as

$$\mathcal{F} \to g^{-1}\mathcal{F}g\,, \qquad \mathcal{G} \to g^{-1}\triangleright\mathcal{G}\,, \qquad \mathcal{H} \to g^{-1}\triangleright\mathcal{H}\,, \tag{3.10}$$

under the H-gauge transformations as

$$\mathcal{F} \to \mathcal{F}\,, \qquad \mathcal{G} \to \mathcal{G} + \mathcal{F}\wedge^{\triangleright}\eta\,, \qquad \mathcal{H} \to \mathcal{H} - \{\mathcal{G}'\wedge\eta\} + \{\eta\wedge\mathcal{G}\}\,, \tag{3.11}$$

JHEP10(2019)222

and under the L-gauge transformations as

$$\mathcal{F} \to \mathcal{F}\,, \qquad \mathcal{G} \to \mathcal{G}\,, \qquad \mathcal{H} \to \mathcal{H} - \mathcal{F} \wedge^{\rhd} \theta\,. \tag{3.12}$$

For more details, the reader is referred to [25].

In order to make the action (3.2) gauge invariant with respect to the transformations (3.7), (3.8) and (3.9), the Lagrange multipliers B, C and D must transform under the G-gauge transformations as

$$B \to g^{-1} B g\,, \qquad C \to g^{-1} \rhd C\,, \qquad D \to g^{-1} \rhd D\,, \tag{3.13}$$

under the H-gauge transformations as

$$B \to B + C' \wedge^{\mathcal{T}} \eta - \eta \wedge^{\mathcal{D}} \eta \wedge^{\mathcal{D}} D\,, \qquad C \to C + D \wedge^{\mathcal{X}_1} \eta + D \wedge^{\mathcal{X}_2} \eta\,, \qquad D \to D\,, \tag{3.14}$$

while under the L-gauge transformations they transform as

$$B \to B - D \wedge^{\mathcal{S}} \theta\,, \qquad C \to C\,, \qquad D \to D\,. \tag{3.15}$$

See appendix B for details, for the definition of the maps $\mathcal{T}$, $\mathcal{D}$, $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{S}$, and for the notation of the $\wedge^{\mathcal{T}}$, $\wedge^{\mathcal{D}}$, $\wedge^{\mathcal{X}_1}$, $\wedge^{\mathcal{X}_2}$, and $\wedge^{\mathcal{S}}$ products.

3.2 Constrained 3BF action for a real Klein-Gordon field

Once the topological 3BF action is specified, we can proceed with the construction of the constrained 3BF action, describing a realistic case of a scalar field coupled to gravity. In order to perform this construction, we have to define a specific 2-crossed module which gives rise to the topological sector of the action, and then we have to impose convenient simplicity constraints.

We begin by defining a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \rhd\,, \{\_\,,\_\})$, as follows. The groups are given as

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}\,. \tag{3.16}$$

The group G acts on itself via conjugation, on H via the vector representation, and on L via the trivial representation. This specifies the definition of the action $\rhd$. The map ∂ is chosen to be trivial, as before. The map δ is also trivial, that is, every element of L is mapped to the identity element of H. Finally, the Peiffer lifting is trivial as well, mapping every ordered pair of elements in H to an identity element in L. This specifies one concrete 2-crossed module.

Given this choice of a 2-crossed module, the 3-connection $(\alpha\,, \beta\,, \gamma)$ takes the form

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma \mathbb{I}\,, \tag{3.17}$$

where $\mathbb{I}$ is the sole generator of the Lie group $\mathbb{R}$. From (3.1), the fake 3-curvature $(\mathcal{F}\,, \mathcal{G}\,, \mathcal{H})$ reduces to the ordinary 3-curvature,

$$\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla \beta^a P_a\,, \qquad \mathcal{H} = \mathrm{d}\gamma\,, \tag{3.18}$$

JHEP10(2019)222

where we used the fact that G acts trivially on L, that is, $M_{ab} \rhd \mathbb{I} = 0$. The topological $3BF$ action (3.2) now becomes

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \phi\, \mathrm{d}\gamma\,, \tag{3.19}$$

where the bilinear form for L is $\langle \mathbb{I}, \mathbb{I} \rangle_{\mathfrak{l}} = 1$.

It is important to note that the Lagrange multiplier D in (3.2) is a 0-form and transforms trivially with respect to G, H and L gauge transformations for our choice of the 2-crossed module, as can be seen from (3.13), (3.14) and (3.15). Thus, D has all the *hallmark properties of a real scalar field*, allowing us to make identification between them, and conveniently relabel D into ϕ in (3.19). This is a crucial property of the 3-group structure in a 4-dimensional spacetime and is one of the main results of the paper. It follows the line of reasoning used in recognizing the Lagrange multiplier C^a in the $2BF$ action for the Poincaré 2-group as a tetrad field e^a. It is also important to stress that the choice of the third gauge group, L, dictates the number and the structure of the matter fields present in the action. In this case, $L = \mathbb{R}$ implies that we have only one real scalar field, corresponding to a single generator $\mathbb{I}$ of $\mathbb{R}$. The trivial nature of the action $\rhd$ of $\mathrm{SO}(3,1)$ on $\mathbb{R}$ also implies that ϕ transforms as a scalar field. Finally, the scalar field appears as a degree of freedom in the topological sector of the action, making the quantization procedure feasible.

As in the case of BF and $2BF$ theories, in order to obtain nontrivial dynamics, we need to impose convenient simplicity constraints on the variables in the action (3.19). Since we are interested in obtaining the scalar field ϕ of mass m coupled to gravity in the standard way, we choose the action in the form:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \phi\, \mathrm{d}\gamma \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda \wedge \left(\gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c \right) + \Lambda^{ab} \wedge \left(H_{abc} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b \right) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi^2 \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d\,. \end{aligned} \tag{3.20}$$

Note that the first row is the topological sector (3.19), the second row is the familiar simplicity constraint for gravity from the action (2.24), the third row contains the new simplicity constraints corresponding to the Lagrange multiplier 1-forms λ and Λ^{ab} and featuring the Lagrange multiplier 0-form H_{abc}, while the fourth row is the mass term for the scalar field.

Varying the total action (3.20) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, Λ_{ab}, γ, λ, H_{abc}, ϕ and e^a one obtains the equations of motion:

$$R^{ab} - \lambda^{ab} = 0\,, \tag{3.21}$$

$$\nabla B^{ab} - e^{[a} \wedge \beta^{b]} = 0\,, \tag{3.22}$$

$$\nabla e^a = 0\,, \tag{3.23}$$

JHEP10(2019)222

$$B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd} e^c \wedge e^d = 0\,, \tag{3.24}$$

$$H_{abc}\varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b = 0\,, \tag{3.25}$$

$$\mathrm{d}\phi - \lambda = 0\,, \tag{3.26}$$

$$\gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c = 0\,, \tag{3.27}$$

$$-\frac{1}{2}\lambda \wedge e^a \wedge e^b \wedge e^c + \varepsilon^{cdef}\Lambda^{ab} \wedge e_d \wedge e_e \wedge e_f = 0\,, \tag{3.28}$$

$$\mathrm{d}\gamma - \mathrm{d}(\Lambda^{ab} \wedge e_a \wedge e_b) - \frac{1}{4!} m^2 \phi \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d = 0\,, \tag{3.29}$$

$$\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{3}{2} H_{abc}\lambda \wedge e^b \wedge e^c + 3H^{def}\varepsilon_{abcd}\Lambda_{ef} \wedge e^b \wedge e^c$$
$$-2\Lambda_{ab} \wedge \mathrm{d}\phi \wedge e^b - 2\frac{1}{4!} m^2 \phi \varepsilon_{abcd} e^b \wedge e^c \wedge e^d = 0\,. \tag{3.30}$$

The dynamical degrees of freedom are e^a and ϕ, while the remaining variables are algebraically determined in terms of them. Specifically, the equations (3.21)–(3.28) give

$$\begin{aligned} &\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, && {\omega^{ab}}_\mu = {\triangle^{ab}}_\mu\,, \quad \gamma_{\mu\nu\rho} = -\frac{e}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\sigma\phi\,, \\ &{\Lambda^{ab}}_\mu = \frac{1}{12e} g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\partial_\nu\phi {e^a}_\rho {e^b}_\sigma\,, && {\beta^a}_{\mu\nu} = 0\,, \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd} {e^c}_\mu {e^d}_\nu\,, \\ &H^{abc} = \frac{1}{6e}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\phi {e^a}_\nu {e^b}_\rho {e^c}_\sigma\,, && \lambda_\mu = \partial_\mu\phi\,. \end{aligned} \tag{3.31}$$

Note that from the equations (3.22), (3.23) and (3.24) it follows that $\beta^a = 0$, as in the pure gravity case. The equation of motion (3.29) reduces to the covariant Klein-Gordon equation for the scalar field,

$$\left(\nabla_\mu\nabla^\mu - m^2\right)\phi = 0\,. \tag{3.32}$$

Finally, the equation of motion (3.30) for e^a becomes:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2\, T^{\mu\nu}\,, \qquad T^{\mu\nu} \equiv \partial^\mu\phi\partial^\nu\phi - \frac{1}{2} g^{\mu\nu}\left(\partial_\rho\phi\partial^\rho\phi + m^2\phi^2\right)\,. \tag{3.33}$$

The system of equations (3.21)–(3.30) is equivalent to the system of equations (3.31)–(3.33). Note that in addition to the correct covariant form of the Klein-Gordon equation, we have also obtained the correct form of the stress-energy tensor for the scalar field.

3.3 Constrained 3BF action for the Dirac field

Now we pass to the more complicated case of the Dirac field. We first define a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\})$ as follows. The groups are:

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^8(\mathbb{G})\,, \tag{3.34}$$

where $\mathbb{G}$ is the algebra of complex Grassmann numbers. The maps ∂, δ and the Peiffer lifting are trivial. The action of the group G on itself is given via conjugation, on H via vector representation, and on L via spinor representation, as follows. Denoting the

JHEP10(2019)222

8 generators of the Lie group $\mathbb{R}^8(\mathbb{G})$ as P_α and P^α, where the index α takes the values $1,\dots,4$, the action of G on L is thus given explicitly as

$$M_{ab} \triangleright P_\alpha = \frac{1}{2}(\sigma_{ab})^\beta{}_\alpha P_\beta \,, \qquad M_{ab} \triangleright P^\alpha = -\frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta \,, \tag{3.35}$$

where $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, and γ_a are the usual Dirac matrices, satisfying the anticommutation rule $\{\gamma_a \,, \gamma_b\} = -2\eta_{ab}$.

As in the case of the scalar field, the choice of the group L dictates the matter content of the theory, while the action $\triangleright$ of G on L specifies its transformation properties. To see this explicitly, let us construct the corresponding $3BF$ action. The 3-connection $(\alpha\,,\beta\,,\gamma)$ now takes the form

$$\alpha = \omega^{ab} M_{ab} \,, \qquad \beta = \beta^a P_a \,, \qquad \gamma = \gamma^\alpha P_\alpha + \bar\gamma_\alpha P^\alpha \,, \tag{3.36}$$

while the 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$, defined in (3.1), is given as

$$\mathcal{F} = R^{ab} M_{ab} \,, \mathcal{G} = \nabla\beta^a P_a \,, \tag{3.37}$$
$$\mathcal{H} = \left(\mathrm{d}\gamma^\alpha + \frac{1}{2}\omega^{ab}(\sigma_{ab})^\alpha{}_\beta \gamma^\beta \right) P_\alpha + \left(\mathrm{d}\bar\gamma_\alpha - \frac{1}{2}\omega^{ab}\bar\gamma_\beta(\sigma_{ab})^\beta{}_\alpha \right) P^\alpha \equiv (\overset{\rightarrow}{\nabla}\gamma)^\alpha P_\alpha + (\bar\gamma\overset{\leftarrow}{\nabla})_\alpha P^\alpha \,,$$

where we have used (3.35). The bilinear form $\langle \_ , \_ \rangle_{\mathfrak{l}}$ is defined as

$$\langle P_\alpha, P_\beta \rangle_{\mathfrak{l}} = 0 \,, \qquad \langle P^\alpha, P^\beta \rangle_{\mathfrak{l}} = 0 \,, \qquad \langle P_\alpha, P^\beta \rangle_{\mathfrak{l}} = -\delta_\alpha^\beta \,, \qquad \langle P^\alpha, P_\beta \rangle_{\mathfrak{l}} = \delta^\alpha_\beta \,. \tag{3.38}$$

Note that, for general $A, B \in \mathfrak{l}$, we can write

$$\langle A, B \rangle_{\mathfrak{l}} = A^I B^J g_{IJ} \,, \qquad \langle B, A \rangle_{\mathfrak{l}} = B^J A^I g_{JI} \,. \tag{3.39}$$

Since we require the bilinear form to be symmetric, the two expressions must be equal. However, since the coefficients in $\mathfrak{l}$ are Grassmann numbers, we have $A^I B^J = -B^J A^I$, so it follows that $g_{IJ} = -g_{JI}$. Hence the antisymmetry of (3.38).

Now we use the properties of the group L and the action $\triangleright$ of G on L to recognize the physical nature of the Lagrange multiplier D in (3.2). Indeed, the choice of the group L dictates that D contains 8 independent complex Grassmannian matter fields as its components. Moreover, due to the fact that D is a 0-form and that it transforms according to the spinorial representation of SO(3, 1), we can identify its components with the Dirac bispinor fields, and write

$$D = \psi^\alpha P_\alpha + \bar\psi_\alpha P^\alpha \,, \tag{3.40}$$

where it is assumed that ψ and $\bar\psi$ are independent fields, as usual. This is again an illustration of the fact that information about the structure of the matter sector in the theory is specified by the choice of the group L in the 2-crossed module, and another main result of the paper.

Given all of the above, now we can finally write the $3BF$ action (3.2) corresponding to this choice of the 2-crossed module as

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + (\bar\gamma\overset{\leftarrow}{\nabla})_\alpha \psi^\alpha + \bar\psi_\alpha (\overset{\rightarrow}{\nabla}\gamma)^\alpha \,. \tag{3.41}$$

JHEP10(2019)222

In order to promote this action into a full theory of gravity coupled to Dirac fermions, we add the convenient constraint terms to the action, as follows:

$$
\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + (\bar{\gamma} \overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla} \gamma)^\alpha \\
& - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\
& - \lambda^\alpha \wedge \left(\bar{\gamma}_\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar{\psi} \gamma^d)_\alpha \right) + \bar{\lambda}_\alpha \wedge \left(\gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^\alpha \right) \\
& - \frac{1}{12} m \, \bar{\psi} \psi \, \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d + 2\pi i l_p^2 \, \bar{\psi} \gamma_5 \gamma^a \psi \, \varepsilon_{abcd} e^b \wedge e^c \wedge \beta^d \,. \qquad (3.42)
\end{aligned}
$$

Here the first row is the topological sector, the second row is the gravitational simplicity constraint term from (2.24), while the third row contains the new simplicity constraints for the Dirac field corresponding to the Lagrange multiplier 1-forms λ^α and $\bar{\lambda}_\alpha$. The fourth row contains the mass term for the Dirac field, and a term which ensures the correct coupling between the torsion and the spin of the Dirac field, as specified by the Einstein-Cartan theory. Namely, we want to ensure that the torsion has the form

$$
T_a \equiv \nabla e_a = 2\pi l_p^2 s_a \,, \qquad (3.43)
$$

where

$$
s_a = i \varepsilon_{abcd} e^b \wedge e^c \bar{\psi} \gamma_5 \gamma^d \psi \qquad (3.44)
$$

is the spin 2-form. Of course, other couplings should also be straightforward to implement, but we choose this particular coupling because we are interested in reproducing the standard Einstein-Cartan gravity coupled to the Dirac field.

Varying the action (3.42) with respect to B_{ab}, λ^{ab}, $\bar{\gamma}_\alpha$, γ^α, λ^α, $\bar{\lambda}_\alpha$, $\bar{\psi}_\alpha$, ψ^α, e^a, β^a and ω^{ab} one obtains the equations of motion:

$$
R^{ab} - \lambda^{ab} = 0 \,, \qquad (3.45)
$$

$$
B_{ab} - \frac{1}{16\pi l_p^2} \varepsilon_{abcd} e^c \wedge e^d = 0 \,, \qquad (3.46)
$$

$$
(\overrightarrow{\nabla} \psi)^\alpha - \lambda^\alpha = 0 \,, \qquad (3.47)
$$

$$
(\bar{\psi} \overleftarrow{\nabla})_\alpha - \bar{\lambda}_\alpha = 0 \,, \qquad (3.48)
$$

$$
\bar{\gamma}_\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar{\psi} \gamma^d)_\alpha = 0 \,, \qquad (3.49)
$$

$$
\gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^\alpha = 0 \,, \qquad (3.50)
$$

$$
\begin{aligned}
\mathrm{d}\gamma^\alpha + \omega^\alpha{}_\beta \wedge \gamma^\beta + \frac{i}{6} \lambda^\beta \wedge \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \gamma^{d\alpha}{}_\beta + \frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \psi^\alpha \\
+ i 2\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\gamma_5 \gamma^d \psi)^\alpha = 0 \,, \qquad (3.51)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\bar{\gamma}_\alpha - \bar{\gamma}_\beta \wedge \omega^\beta{}_\alpha + \frac{i}{6} \bar{\lambda}_\beta \wedge \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \gamma^{d\beta}{}_\alpha - \frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \bar{\psi}_\alpha \\
- i 2\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\bar{\psi} \gamma_5 \gamma^d)_\alpha = 0 \,, \qquad (3.52)
\end{aligned}
$$

JHEP10(2019)222

$$\nabla\beta_a + 2\varepsilon_{abcd}\lambda^{bc}\wedge e^d - \frac{i}{2}\varepsilon_{abcd}\lambda^\alpha\wedge e^b\wedge e^c(\bar\psi\gamma^d)_\alpha + \frac{i}{2}\varepsilon_{abcd}\bar\lambda_\alpha\wedge e^b\wedge e^c(\gamma^d\psi)^\alpha$$
$$-\frac{1}{3}\varepsilon_{abcd}e^b\wedge e^c\wedge e^d m\bar\psi\psi - 4\pi l_p^2 i\varepsilon_{abcd}e^b\wedge\beta^c\bar\psi\gamma_5\gamma^d\psi = 0\,, \quad (3.53)$$
$$\nabla e_a - i2\pi l_p^2\varepsilon_{abcd}e^b\wedge e^c\bar\psi\gamma_5\gamma^d\psi = 0\,, \quad (3.54)$$
$$\nabla B_{ab} - e_{[a}\wedge\beta_{b]} + \bar\gamma\frac{1}{8}[\gamma_a,\gamma_b]\psi + \bar\psi\frac{1}{8}[\gamma_a,\gamma_b]\gamma = 0\,. \quad (3.55)$$

The dynamical degrees of freedom are e^a, ψ^α and $\bar\psi_\alpha$, while the remaining variables are determined in terms of the dynamical variables, and are given as:

$$\begin{aligned} &B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, && \lambda^\alpha{}_\mu = (\vec\nabla_\mu\psi)^\alpha\,, && \bar\lambda_{\alpha\mu} = (\bar\psi\overleftarrow\nabla_\mu)_\alpha\,, \\ &\bar\gamma_{\alpha\mu\nu\rho} = i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\bar\psi\gamma^d)_\alpha\,, && \gamma^\alpha{}_{\mu\nu\rho} = -i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\gamma^d\psi)^\alpha\,, && \\ &\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, && \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu + K^{ab}{}_\mu\,. && \end{aligned} \quad (3.56)$$

Here $K^{ab}{}_\mu$ is the contorsion tensor, constructed in the standard way from the torsion tensor, whereas from (3.54) we have

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a\,, \quad (3.57)$$

which is precisely the desired equation (3.43). Further, from the equation (3.46) one obtains

$$\nabla B^{ab} = -\frac{1}{8\pi l_p^2}\varepsilon^{abcd}\left(e_c\wedge\nabla e_d\right)\,. \quad (3.58)$$

Substituting this expression in the equation (3.55) it follows that

$$2\varepsilon_{abcd}e^c\wedge\left(-\frac{1}{16\pi l_p^2}\nabla e^d + \frac{1}{8}s^d\right) - e_{[a}\wedge\beta_{b]} = 0\,. \quad (3.59)$$

The expression in the parentheses is equal to zero, according to the equation (3.54). From the remaining term $e_{[a}\wedge\beta_{b]} = 0$ it again follows that

$$\beta = 0\,. \quad (3.60)$$

Using this result, the equation of motion (3.51) for fermions becomes

$$\frac{i}{6}\varepsilon_{abcd}e^a\wedge e^b\wedge\left(2e^c\wedge\gamma^d\vec\nabla + \frac{im}{2}e^c\wedge e^d - 3(\nabla e^c)\gamma^d\right)\psi = 0\,. \quad (3.61)$$

Using equation (3.54), the last term in the parentheses vanishes, and the equation reduces to the covariant Dirac equation,

$$(i\gamma^a e^\mu{}_a\vec\nabla_\mu - m)\psi = 0\,, \quad (3.62)$$

where $e^\mu{}_a$ is the inverse tetrad. Similarly, the equation (3.52) gives the conjugated Dirac equation:

$$\bar\psi(i\overleftarrow\nabla_\mu e^\mu{}_a\gamma^a + m) = 0\,. \quad (3.63)$$

Finally, the equation of motion (3.53) for tetrad field reduces to

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \qquad T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\gamma^{\nu}\overset{\leftrightarrow}{\nabla}{}^{a}e^{\mu}{}_{a}\psi - \frac{1}{2}g^{\mu\nu}\bar{\psi}\Big(i\gamma^{a}\overset{\leftrightarrow}{\nabla}_{\rho}e^{\rho}{}_{a} - 2m\Big)\psi\,, \quad (3.64)$$

Here, we used the notation $\overset{\leftrightarrow}{\nabla} = \overset{\rightarrow}{\nabla} - \overset{\leftarrow}{\nabla}$. The system of equations (3.45)–(3.55) is equivalent to the system of equations (3.56), (3.60), (3.62)–(3.64). As we expected, the equations of motion (3.57), (3.62), (3.63) and (3.64) are precisely the equations of motion of the Einstein-Cartan theory coupled to a Dirac field.

3.4 Constrained 3BF action for the Weyl and Majorana fields

A general solution of the Dirac equation is not an irreducible representation of the Lorentz group, and one can rewrite Dirac fermions as left-chiral and right-chiral fermion fields that both retain their chirality under Lorentz transformations, implying their irreducibility. Hence, it is useful to rewrite the action for left and right Weyl spinors as a constrained 3BF action. For simplicity, we will discuss only left-chiral spinor field, while the right-chiral field can be treated analogously. Both Weyl and Majorana fermions can be treated in the same way, the only difference being the presence of an additional mass term in the Majorana action.

We being by defining a 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$, as follows. The groups are:

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^4(\mathbb{G})\,. \quad (3.65)$$

The maps ∂, δ and the Peiffer lifting are trivial. The action $\triangleright$ of the group G on G, H and L is given in the same way as for the Dirac case, whereas the spinorial representation reduces to

$$M_{ab} \triangleright P^{\alpha} = \frac{1}{2}(\sigma_{ab})^{\alpha}{}_{\beta}P^{\beta}\,, \qquad M_{ab} \triangleright P_{\dot\alpha} = \frac{1}{2}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha}P_{\dot\beta}\,, \quad (3.66)$$

where $\sigma^{ab} = -\bar\sigma^{ab} = \frac{1}{4}(\sigma^a\bar\sigma^b - \sigma^b\bar\sigma^a)$, for $\sigma^a = (1,\vec\sigma)$ and $\bar\sigma^a = (1,-\vec\sigma)$, in which $\vec\sigma$ denotes the set of three Pauli matrices. The four generators of the group L are denoted as P^{α} and $P_{\dot\alpha}$, where the Weyl indices $\alpha, \dot\alpha$ take values $1,2$.

The 3-connection $(\alpha\,,\beta\,,\gamma)$ now takes the form corresponding to this choice of Lie groups,

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma_{\alpha}P^{\alpha} + \bar\gamma^{\dot\alpha}P_{\dot\alpha}\,, \quad (3.67)$$

while the fake 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ defined in (3.1) is

$$\mathcal{F} = R^{ab}M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,, \quad (3.68)$$
$$\mathcal{H} = \Big(\mathrm{d}\gamma_{\alpha} + \frac{1}{2}\omega^{ab}(\sigma^{ab})^{\beta}{}_{\alpha}\gamma_{\beta}\Big)P^{\alpha} + \Big(\mathrm{d}\bar\gamma^{\dot\alpha} + \frac{1}{2}\omega_{ab}(\bar\sigma^{ab})^{\dot\alpha}{}_{\dot\beta}\bar\gamma^{\dot\beta}\Big)P_{\dot\alpha} \equiv (\overset{\rightarrow}{\nabla}\gamma)_{\alpha}P^{\alpha} + (\bar\gamma\overset{\leftarrow}{\nabla})^{\dot\alpha}P_{\dot\alpha}\,.$$

Introducing the spinor fields ψ_{α} and $\bar\psi^{\dot\alpha}$ via the Lagrange multiplier D as

$$D = \psi_{\alpha}P^{\alpha} + \bar\psi^{\dot\alpha}P_{\dot\alpha}\,, \quad (3.69)$$

and using the bilinear form $\langle\_,\_\rangle_{\mathfrak{l}}$ for the group L,

$$\langle P^{\alpha}, P^{\beta}\rangle_{\mathfrak{l}} = \varepsilon^{\alpha\beta}\,, \qquad \langle P_{\dot\alpha}, P_{\dot\beta}\rangle_{\mathfrak{l}} = \varepsilon_{\dot\alpha\dot\beta}\,, \qquad \langle P^{\alpha}, P_{\dot\beta}\rangle_{\mathfrak{l}} = 0\,, \qquad \langle P_{\dot\alpha}, P^{\beta}\rangle_{\mathfrak{l}} = 0\,, \quad (3.70)$$

where $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$ are the usual two-dimensional antisymmetric Levi-Civita symbols, the topological $3BF$ action (3.2) for spinors coupled to gravity becomes

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \psi^\alpha \wedge (\overset{\rightarrow}{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}} \wedge (\bar{\gamma}\overset{\leftarrow}{\nabla})^{\dot{\alpha}} . \tag{3.71}$$

In order to obtain the suitable equations of motion for the Weyl spinors, we again introduce appropriate simplicity constraints, so that the action becomes:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \psi^\alpha \wedge (\overset{\rightarrow}{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}} \wedge (\bar{\gamma}\overset{\leftarrow}{\nabla})^{\dot{\alpha}} \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & - \lambda^\alpha \wedge \left(\gamma_\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} \right) - \bar{\lambda}_{\dot{\alpha}} \wedge \left(\bar{\gamma}^{\dot{\alpha}} + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta \right) \\ & - 4\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\bar{\psi}_{\dot{\alpha}} \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta) . \end{aligned} \tag{3.72}$$

The new simplicity constraints are in the third row, featuring the Lagrange multiplier 1-forms λ_α and $\bar{\lambda}^{\dot{\alpha}}$. Also, using the coupling between the Dirac field and torsion from Einstein-Cartan theory as a model, the term in the fourth row is chosen to ensure that the coupling between the Weyl spin tensor

$$s_a \equiv i\varepsilon_{abcd} e^b \wedge e^c \ \psi^\alpha \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} , \tag{3.73}$$

and torsion is given as:

$$T_a = 4\pi l_p^2 s_a . \tag{3.74}$$

The case of the Majorana field is introduced in exactly the same way, albeit with an additional mass term in the action, of the form:

$$-\frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d (\psi^\alpha \psi_\alpha + \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}) . \tag{3.75}$$

Varying the action (3.72) with respect to the variables B_{ab}, λ^{ab}, γ_α, $\bar{\gamma}^{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, e^a, β^a and ω^{ab} one again obtains the complete set of equations of motion, displayed in the appendix C. The only dynamical degrees of freedom are ψ_α, $\bar{\psi}^{\dot{\alpha}}$ and e^a, while the remaining variables are algebraically determined in terms of these as:

$$\begin{gathered} \lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu} , \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu , \quad \lambda_{\alpha\mu} = \nabla_\mu \psi_\alpha , \quad \bar{\lambda}^{\dot{\alpha}}{}_\mu = \nabla_\mu \bar{\psi}^{\dot{\alpha}} , \\ \gamma_{\alpha\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} , \quad \bar{\gamma}^{\dot{\alpha}}{}_{\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta , \quad \omega_{ab\mu} = \triangle_{ab\mu} + K_{ab\mu} . \end{gathered} \tag{3.76}$$

In addition, one also maintains the result $\beta = 0$ as before. Finally, the equations of motion for the dynamical fields are

$$\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta = 0 , \qquad \sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} = 0 , \tag{3.77}$$

and

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2 \, T^{\mu\nu} , \tag{3.78}$$

where

$$T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} - g^{\mu\nu}\frac{1}{2}\left(i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi}\right). \tag{3.79}$$

Here we have suppressed the spinor indices. In the case of the Majorana field, the equations of motion (3.76) remain the same, while the equations of motion for ψ_α and $\bar{\psi}^{\dot{\alpha}}$ take the form

$$i\sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} - m\psi_\alpha = 0\,, \qquad i\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta - m\bar{\psi}^{\dot{\alpha}} = 0\,, \tag{3.80}$$

whereas the stress-energy tensor takes the form

$$\begin{aligned} T^{\mu\nu} \equiv\; & \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} \\ & - g^{\mu\nu}\frac{1}{2}\left[i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi} - \frac{1}{2}m\left(\psi\psi + \bar{\psi}\bar{\psi}\right)\right]. \end{aligned} \tag{3.81}$$

4 The Standard Model

The Standard Model 3-group can be defined as:

$$G = \mathrm{SO}(3,1)\times\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\,, \quad H = \mathbb{R}^4\,, \quad L = \mathbb{R}^4(\mathbb{C})\times\mathbb{R}^{64}(\mathbb{G})\times\mathbb{R}^{64}(\mathbb{G})\times\mathbb{R}^{64}(\mathbb{G})\,, \tag{4.1}$$

where $\mathbb{C}$ denotes the field of complex numbers. The motivation for this choice of the group L is given in the table below.

1. lepton generation	red color 1. quark generation	green color 1. quark generation	blue color 1. quark generation
$\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L$	$\begin{pmatrix}u_r \\ d_r\end{pmatrix}_L$	$\begin{pmatrix}u_g \\ d_g\end{pmatrix}_L$	$\begin{pmatrix}u_b \\ d_b\end{pmatrix}_L$
$(\nu_e)_R$	$(u_r)_R$	$(u_g)_R$	$(u_b)_R$
$(e^-)_R$	$(d_r)_R$	$(d_g)_R$	$(d_b)_R$

We see that in order to introduce one generation of matter one needs to provide 16 spinors, or equivalently the group L has to be chosen as $L = \mathbb{R}^{64}(\mathbb{G})$. As there are three generations of matter, the part of the group L that corresponds to the fermion fields in the theory is chosen to be $L = \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G})$. To define the Higgs sector one needs two complex scalar fields $\begin{pmatrix}\phi^+ \\ \phi_0\end{pmatrix}$, or equivalently the scalar sector of the group L is given as $L = \mathbb{R}^4(\mathbb{C})$.

The maps ∂, δ and the Peiffer lifting are trivial. The action of the group G on itself is given via conjugation. The action of the $\mathrm{SO}(3,1)$ subgroup of G on H is via vector representation and the action of $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ subgroup on H is via trivial representation. The action of the $\mathrm{SO}(3,1)$ on L is via trivial representation for the generators corresponding to the scalar fields, i.e. the $\mathbb{R}^4(\mathbb{C})$ subgroup of L, and via spinor representation for the every quadruple of generators corresponding to the fermion fields, given as in the section 3. The information how spinors transform under the SU(3) $\times$ SU(2) $\times$ U(1) group is encoded in the action of that subgroup of G on L, as specified in the table above. For simplicity, in the following, only one family of the lepton sector and only electroweak part of the gauge sector of the Standard model is considered.

The groups are chosen as:

$$G = \mathrm{SO}(3,1)\times \mathrm{SU}(2)\times \mathrm{U}(1)\,, \quad H=\mathbb{R}^4\,, \quad L^{\text{leptons}} = \mathbb{R}^{16}(\mathbb{G})\times \mathbb{R}^4(\mathbb{C})\,. \tag{4.2}$$

The 3-connection then takes the form

$$\begin{aligned} &\alpha = \omega^{ab}M_{ab} + W^I T_I + AY\,, \qquad\qquad \beta=\beta^a P_a\,, \\ &\gamma = \gamma_\alpha{}^{\tilde L} P^\alpha{}_{\tilde L} + \gamma^{\dot\alpha}{}_{\tilde L} P_{\dot\alpha}{}^{\tilde L} + \gamma_\alpha{}^{\tilde R} P^\alpha{}_{\tilde R} + \gamma^{\dot\alpha}{}_{\tilde R} P_{\dot\alpha}{}^{\tilde R} + \gamma^{\tilde a} P_{\tilde a}\,. \end{aligned} \tag{4.3}$$

Here the indices $I,J,\dots$ take the values $1,2,3$ and counts the Pauli matrices, generators of the group SU(2), the indices $\tilde L, \tilde L',\dots$ take the values $1,2$ and count the components of left doublet, $\tilde R$ denotes the right singlet $(e^-)_R$ and right singlet $(\nu_e)_R$, and indices $\tilde a, \tilde b,\dots$ take values $1,2$ and count the components of the scalar doublet. It is also useful to define $\tilde i = (\tilde L, \tilde R)$ which takes values $1,\dots,4$.

The action of the group G on L is defined as:

$$\begin{aligned} &M_{ab} \rhd P^\alpha{}_i = \frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta{}_i\,, \qquad M_{ab}\rhd P_{\dot\alpha i} = \frac12 (\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha} P_{\dot\beta i}\,, \qquad M_{ab}\rhd P_{\tilde a}=0\,, \\ &T_I \rhd P^\alpha{}_{\tilde L} = \frac12 (\sigma_I)^{\tilde L'}{}_{\tilde L} P^\alpha{}_{\tilde L'}\,, \qquad T_I\rhd P_{\dot\alpha \tilde L} = \frac12(\sigma_I)^{\tilde L'}{}_{\tilde L} P_{\dot\alpha\tilde L'}\,, \\ &T_I\rhd P^\alpha{}_{\tilde R}=0\,, \qquad T_I\rhd P_{\dot\alpha\tilde R}=0\,, \qquad T_I \rhd P_{\tilde a} = \frac12(\sigma_I)^{\tilde b}{}_{\tilde a} P_{\tilde b}\,, \\ &Y\rhd P^\alpha{}_{\tilde L} = -P^\alpha{}_{\tilde L}\,, \quad Y\rhd P^\alpha{}_{e_R} = -2P^\alpha{}_{e_R}\,, \quad Y\rhd P^\alpha{}_{\nu_R} = -2P^\alpha{}_{\nu_R}\,, \quad Y\rhd P_{\tilde a} = P_{\tilde a}\,, \\ &Y\rhd P_{\dot\alpha\tilde L} = -P_{\dot\alpha \tilde L}\,, \qquad Y\rhd P_{\dot\alpha e_R} = -2P_{\dot\alpha e_R}\,, \qquad Y\rhd P_{\dot\alpha\nu_R} = -2P_{\dot\alpha\nu_R}\,. \end{aligned} \tag{4.4}$$

The 3-curvatures are given as:

$$\begin{aligned} &\mathcal{F} = R^{ab}M_{ab} + F^I T_I + FY\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,, \\ &\mathcal{H} = (\overrightarrow{\nabla}\gamma^{\tilde L})_\alpha P^\alpha{}_{\tilde L} + (\bar\gamma_{\tilde L}\overleftarrow{\nabla})^{\dot\alpha} P_{\dot\alpha}{}^{\tilde L} + (\overrightarrow{\nabla}\gamma^{\tilde R})_\alpha P^\alpha{}_{\tilde R} + (\bar\gamma_{\tilde R}\overleftarrow{\nabla})^{\dot\alpha} P_{\dot\alpha}{}^{\tilde R} + \mathrm{d}\gamma^{\tilde a} P_{\tilde a}\,. \end{aligned} \tag{4.5}$$

The topological $3BF$ action is defined as:

$$S = \int B_{ab}R^{ab} + B_I F^I + BF + e_a\nabla\beta^a + \psi^\alpha{}_{\tilde i}(\overrightarrow{\nabla}\gamma^{\tilde i})_\alpha + \bar\psi_{\dot\alpha}{}^{\tilde i}(\bar\gamma_{\tilde i}\overleftarrow{\nabla})^{\dot\alpha} + \phi^{\tilde a}\mathrm{d}\gamma_{\tilde a}\,. \tag{4.6}$$

At this point, it is useful to simplify the notation and denote all indices of the group G by $\hat\alpha$, of the group H by $\hat a$ and L by $\hat A$. In order to promote this action to a full theory of first lepton family coupled to electroweak gauge fields, Higgs field, and gravity, we again introduce the appropriate simplicity constraint, as follows

$$
\begin{aligned}
S = \int & B_{\hat{\alpha}} \wedge \mathcal{F}^{\hat{\alpha}} + e_{\hat{a}} \wedge \mathcal{G}^{\hat{a}} + D_{\hat{A}} \wedge \mathcal{H}^{\hat{A}} \\
& + \left(B_{\hat{\alpha}} - C_{\hat{\alpha}}{}^{\hat{\beta}} M_{cd\hat{\beta}} e^c \wedge e^d \right) \wedge \lambda^{\hat{\alpha}} - \left(\gamma_{\hat{A}} - e^a \wedge e^b \wedge e^c C_{\hat{A}}{}^{\hat{B}} M_{abc\hat{B}} \right) \wedge \lambda^{\hat{A}} \\
& + \zeta^{ab}{}_{\hat{\alpha}} \wedge \left(M_{ab}{}^{\hat{\alpha}} \varepsilon^{cdef} e_c \wedge e_d \wedge e_e \wedge e_f - F^{\hat{\alpha}} \wedge e_c \wedge e_d \right) \\
& + \zeta^{ab}{}_{\hat{A}} \wedge \left(M_{abc}{}^{\hat{A}} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - F^{\hat{A}} \wedge e_a \wedge e_b \right) \\
& - \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \left(Y_{\hat{A}\hat{B}\hat{C}} D^{\hat{A}} D^{\hat{B}} D^{\hat{C}} + M_{\hat{A}\hat{B}} D^{\hat{A}} D^{\hat{B}} + L_{\hat{A}\hat{B}\hat{C}\hat{D}} D^{\hat{A}} D^{\hat{B}} D^{\hat{C}} D^{\hat{D}} \right) \\
& - 4\pi i \, l_p^2 \, \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c D_{\hat{A}} T^{d\hat{A}}{}_{\hat{B}} D^{\hat{B}} \,,
\end{aligned}
\tag{4.7}
$$

where:

$$
\begin{aligned}
& B_{\hat{\alpha}} = \begin{bmatrix} B_{ab} & B_I & B \end{bmatrix}, \qquad \mathcal{F}^{\hat{\alpha}} = \begin{bmatrix} R_{ab} & F_I & F \end{bmatrix}^T, \qquad D_{\hat{A}} = \begin{bmatrix} \psi^{\alpha}{}_{\tilde{L}} & \bar{\psi}_{\dot{\alpha}\tilde{L}} & \psi^{\alpha}{}_R & \bar{\psi}_{\dot{\alpha}R} & \phi_{\tilde{a}} \end{bmatrix}, \\
& \mathcal{H}^{\hat{A}} = \begin{bmatrix} (\overrightarrow{\nabla}\gamma_{\tilde{L}})_{\alpha} & (\bar{\gamma}_{\tilde{L}}\overleftarrow{\nabla})^{\dot{\alpha}} & (\overrightarrow{\nabla}\gamma_{\tilde{R}})_{\alpha} & (\bar{\gamma}_{\tilde{R}}\overleftarrow{\nabla})^{\dot{\alpha}} & \mathrm{d}\gamma_{\tilde{a}} \end{bmatrix}^T, \quad \gamma_{\hat{A}} = \begin{bmatrix} \gamma^{\alpha}{}_{\tilde{L}} & \bar{\gamma}_{\dot{\alpha}\tilde{L}} & \gamma^{\alpha}{}_{\tilde{R}} & \bar{\gamma}_{\dot{\alpha}\tilde{R}} & \gamma_{\tilde{a}} \end{bmatrix}, \\
& \lambda^{\hat{\alpha}} = \begin{bmatrix} -\lambda^{ab} & \lambda^I & \lambda \end{bmatrix}^T, \qquad \zeta^{cd}{}_{\hat{\alpha}} = \begin{bmatrix} 0 & \zeta^{cd}{}_I & \zeta^{cd} \end{bmatrix}, \qquad \zeta^{ab}{}_{\hat{A}} = \begin{bmatrix} \zeta^{ab} & 0 & 0 \end{bmatrix}, \\
& \lambda^{\hat{A}} = \begin{bmatrix} \lambda_{\alpha L} & \bar{\lambda}^{\dot{\alpha}}{}_L & \lambda_{\alpha R} & \bar{\lambda}^{\dot{\alpha}}{}_R & \lambda^{\tilde{a}} \end{bmatrix}^T, \qquad M_{cd\hat{\alpha}} = \begin{bmatrix} \varepsilon_{abcd} & M_{cdI} & M_{cd} \end{bmatrix}, \\
& M_{abc\hat{A}} = \begin{bmatrix} \varepsilon_{abcd}\sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}{}_L & \varepsilon_{abcd}\bar{\sigma}^{d\dot{\alpha}\beta}\psi_{\beta L} & \varepsilon_{abcd}\sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}{}_R & \varepsilon_{abcd}\bar{\sigma}^{d\dot{\alpha}\beta}\psi_{\beta R} & M_{abc\tilde{a}} \end{bmatrix}.
\end{aligned}
$$

The matrices $C^{\hat{\alpha}}{}_{\hat{\beta}}$, $C^{\hat{A}}{}_{\hat{B}}$, $M_{\hat{A}\hat{B}}$, $Y_{\hat{A}\hat{B}\hat{C}}$, $L_{\hat{A}\hat{B}\hat{C}\hat{D}}$ and $T^{d\hat{A}}{}_{\hat{B}}$ are constant matrices, and carry the information about gauge coupling constants, mass of the Higgs field, Yukawa couplings and mixing angles, Higgs self-coupling constant and torsion coupling, respectively.

5 Conclusions

Let us summarize the results of the paper. In section 2 we have given a short reminder of the BF theory and described how one can use it to construct the action for general relativity (the well known Plebanski model), and the action for the Yang-Mills theory in flat spacetime, in a novel way. Passing on to higher gauge theory, we have reviewed the formalism of 2-groups and the corresponding $2BF$ theory, using it again to construct the action for general relativity (a model first described in [12]), and the unified action of general relativity and Yang-Mills theory, both naturally described using the 2-group formalism. With this background material in hand, in section 3 we have used the idea of a categorical ladder yet again, generalizing the $2BF$ theory to $3BF$ theory, with the underlying structure of a 3-group instead of a 2-group. This has led us to the main insight that the *scalar and fermion fields can be specified using a gauge group*, namely the third gauge group, denoted L, present in the 2-crossed module corresponding to a given 3-group. This has allowed us to single out specific gauge groups corresponding to the Klein-Gordon, Dirac, Weyl and Majorana fields, and to construct the relevant constrained $3BF$ actions that describe all these fields coupled to gravity in the standard way.

The obtained results represent the fundamental building blocks for the construction of the complete Standard Model of elementary particles coupled to Einstein-Cartan gravity as a $3BF$ action with suitable simplicity constraints, as demonstrated in section 4. In this way, we can complete the first step of the spinfoam quantization programme for the complete theory of gravity and all matter fields, as specified in the Introduction. This is a clear improvement over the ordinary spinfoam models based on an ordinary constrained BF theory.

In addition to this, the gauge group which determines the matter spectrum of the theory is a completely novel structure, not present in the Standard Model. This new gauge group stems from the 3-group structure of the theory, so it is not surprising that it is invisible in the ordinary formulation of the Standard Model, since the latter does not use any 3-group structure in an explicit way. In this paper, we have discussed the choices of this group which give rise to all relevant matter fields, and these can simply be directly multiplied to give the group corresponding to the full Standard Model, encoding the quark and lepton families and all other structure of the matter spectrum. However, the true potential of the matter gauge group lies in a possibility of nontrivial unification of matter fields, by choosing it to be something other than the ordinary product of its component groups. For example, instead of choosing $\mathbb{R}^8(\mathbb{G})$ for the Dirac field, one can try a noncommutative SU(3) group, which also contains 8 generators, but its noncommutativity requires that the maps δ and $\{\_,\_\}$ be nontrivial, in order to satisfy the axioms of a 2-crossed module. This, in turn, leads to a distinction between 3-curvature and fake 3-curvature, which can have consequences for the dynamics of the theory. In this way, by studying nontrivial choices of a 3-group, one can construct various different 3-group-unified models of gravity and matter fields, within the context of higher gauge theory. This idea resembles the ordinary grand unification programme within the framework of the standard gauge theory, where one constructs various different models of vector fields by making various choices for the Yang-Mills gauge group. The detailed discussion of these 3-group unified models is left for future work.

As far as the spinfoam quantization programme is concerned, having completed the step 1 (as outlined in the Introduction), there is a clear possibility to complete the steps 2 and 3 as well. First, the fact that the full action is written completely in terms of differential forms of various degrees, allows us to adapt it to a triangulated spacetime manifold, in the sense of Regge calculus. In particular, all fields and their field strengths present in the $3BF$ action can be naturally associated to the appropriate d-dimensional simplices of a 4-dimensional triangulation, by matching 0-forms to vertices, 1-forms to edges, etc. This leads us to the following table:

d	triangulation	dual triangulation	form	fields	field strengths
0	vertex	4-polytope	0-form	ϕ, $\psi_{\tilde{\alpha}}$, $\bar{\psi}^{\tilde{\alpha}}$	
1	edge	3-polyhedron	1-form	ω^{ab}, A^I, e^a	
2	triangle	face	2-form	β^a, B^{ab}	R^{ab}, F^I, T^a
3	tetrahedron	edge	3-form	γ, $\gamma_{\tilde{\alpha}}$, $\bar{\gamma}^{\tilde{\alpha}}$	$\mathcal{G}^a$
4	4-simplex	vertex	4-form		$\mathcal{H}$, $\mathcal{H}_{\tilde{\alpha}}$, $\bar{\mathcal{H}}^{\tilde{\alpha}}$

Once the classical Regge-discretized topological $3BF$ action is constructed, one can attempt to construct a state sum Z which defines the path integral for the theory. The topological nature of the pure $3BF$ action, together with the underlying structure of the 3-group, should ensure that such a state sum Z is a topological invariant, in the sense that it is triangulation independent. Unfortunately, in order to perform this step precisely, one needs a generalization of the Peter-Weyl and Plancharel theorems to 2-groups and 3-groups, a mathematical result that is presently still missing. The purpose of the Peter-Weyl theorem is to provide a decomposition of a function on a group into a sum over the corresponding irreducible representations, which ultimately specifies the appropriate spectrum of labels for the d-simplices in the triangulation, fixing the domain of values for the fields living on those d-simplices. In the case of 2-groups and especially 3-groups, the representation theory has not been developed well enough to allow for such a construction, with a consequence of the missing Peter-Weyl theorem for 2-groups and 3-groups. However, until the theorem is proved, we can still try to *guess* the appropriate structure of the irreducible representations of the 2- and 3-groups, as was done for example in [12], leading to the so-called *spincube model* of quantum gravity.

Finally, if we remember that for the purpose of physics we are not really interested in a topological theory, but instead in one which contains local propagating degrees of freedom, we are therefore not really engaged in constructing a topological invariant Z, but rather a state sum which describes nontrivial dynamics. In particular, we need to impose the simplicity constraints onto the state sum Z, which is the step 3 of the spinfoam quantization programme. In light of that, one of the main motivations and also main results of our paper was to rewrite the action for gravity and matter in a way that explicitly distinguishes the topological sector from the simplicity constraints. Imposing the constraints is therefore straightforward in the context of a 3-group gauge theory, and completing this step would ultimately lead us to a state sum corresponding to a tentative theory of quantum gravity with matter. This is also a topic for future work.

In the end, let us also mention that aside from the unification and quantization programmes, there is also a plethora of additional studies one can perform with the constrained $3BF$ action, such as the analysis of the Hamiltonian structure of the theory (suitable for a potential canonical quantization programme), the idea of imposing the simplicity constraints using a spontaneous symmetry breaking mechanism, and finally a detailed study of the mathematical structure and properties of the simplicity constraints. This list is of course not conclusive, and there may be many more interesting related topics to study in both physics and mathematics.

Acknowledgments

The authors would like to thank Aleksandar Miković, Jeffrey Morton, John Baez, Roger Picken and John Huerta for helpful discussions, comments, and suggestions. This work was supported by the project ON171031 of the Ministry of Education, Science and Technological Development (MPNTR) of the Republic of Serbia, and partially by the bilateral scientific cooperation between Austria and Serbia through the project "Causality in Quantum

Mechanics and Quantum Gravity - 2018-2019", no. 451-03-02141/2017-09/02, supported by the Federal Ministry of Science, Research and Economy (BMWFW) of the Republic of Austria, and the Ministry of Education, Science and Technological Development (MPNTR) of the Republic of Serbia.

A Category theory, 2-groups and 3-groups

Definition 1 (Pre-crossed module and crossed module) *A pre-crossed module* $(H \stackrel{\partial}{\to} G\,,\triangleright)$ *of groups* G *and* H*, is given by a group map* $\partial : H \to G$*, together with a left action* $\triangleright$ *of* G *on* H*, by automorphisms, such that for each* $h_1\,,h_2 \in H$ *and* $g \in G$ *the following identity hold:*

$$g\partial h g^{-1} = \partial(g \triangleright h)\,.$$

In a pre-crossed module the **Peiffer commutator** *is defined as:*

$$\langle h_1\,,h_2\rangle_{\rm p} = h_1 h_2 h_1^{-1} \partial(h_1) \triangleright h_2^{-1}\,.$$

A pre-crossed module is said to be a **crossed module** *if all of its Peiffer commutators are trivial, which is to say that*

$$(\partial h) \triangleright h' = h h' h^{-1}\,,$$

i.e. the **Peiffer identity** *is satisfied.*

Definition 2 (2-crossed module) *A 2-crossed module* $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G,\, \triangleright,\, \{-,-\})$ *is given by three groups* G*,* H *and* L*, together with maps* ∂ *and* δ *such that:*

$$L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,,$$

where $\partial\delta = 1$*, an action* $\triangleright$ *of the group* G *on all three groups, and an* G*-equivariant map called the* **Peiffer lifting***:*

$$\{-\,,-\} : H \times H \to L\,.$$

The following identities are satisfied:

1. *The maps* ∂ *and* δ *are* G*-equivariant, i.e. for each* $g \in G$ *and* $h \in H$*:*

   $$g \triangleright \partial(h) = \partial(g \triangleright h)\,, \qquad g \triangleright \delta(l) = \delta(g \triangleright l)\,,$$

   the action of the group G *on the groups* H *and* L *is a smooth left action by automorphisms, i.e. for each* $g, g_1, g_2 \in G$*,* $h_1, h_2 \in H$*,* $l_1, l_2 \in L$ *and* $e \in H, L$*:*

   $$g_1 \triangleright (g_2 \triangleright e) = (g_1 g_2) \triangleright e\,, \quad g \triangleright (h_1 h_2) = (g \triangleright h_1)(g \triangleright h_2)\,, \quad g \triangleright (l_1 l_2) = (g \triangleright l_1)(g \triangleright l_2)\,,$$

   and the Peiffer lifting is G*-equivariant, i.e. for each* h_1*,* $h_2 \in H$ *and* $g \in G$*:*

   $$g \triangleright \{h_1\,,h_2\} = \{g \triangleright h_1,\, g \triangleright h_2\}\,;$$

2. *the action of the group* G *on itself is via conjugation, i.e. for each* $g\,,g_0 \in G$*:*

   $$g \triangleright g_0 = g\, g_0\, g^{-1}\,;$$

3. *In a 2-crossed module the structure* $(L \xrightarrow{\delta} H, \rhd')$ *is a crossed module, with action of the group* H *on the group* L *is defined for each* $h \in H$ *and* $l \in L$ *as:*

$$h \rhd' l = l\,\{\delta(l)^{-1},\, h\}\,,$$

but $(H \xrightarrow{\partial} G\,, \rhd)$ *may not be one, and the Peiffer identity does not necessary hold. However, when* ∂ *is chosen to be trivial and group* H *Abelian, the Peiffer identity is satisfied, i.e. for each* $h,\, h' \in H$*:*

$$\delta(h) \rhd h' = h\, h'\, h^{-1}\,;$$

4. $\delta(\{h_1, h_2\}) = \langle h_1\,, h_2\rangle_{\rm p}, \qquad \forall h_1, h_2 \in H,$

5. $[l_1, l_2] = \{\delta(l_1)\,, \delta(l_2)\}, \qquad \forall l_1\,, l_2 \in L.$ *Here, the notation* $[l, k] = lkl^{-1}k^{-1}$ *is used;*

6. $\{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\}\partial(h_1) \rhd \{h_2, h_3\}, \qquad \forall h_1, h_2, h_3 \in H;$

7. $\{h_1, h_2 h_3\} = \{h_1, h_2\}\{h_1, h_3\}\{\langle h_1, h_3\rangle_{\rm p}^{-1}, \partial(h_1) \rhd h_2\}, \qquad \forall h_1, h_2, h_3 \in H;$

8. $\{\delta(l), h\}\{h, \delta(l)\} = l(\partial(h) \rhd l^{-1}), \qquad \forall h \in H\,, \quad \forall l \in L.$

Definition 3 (Differential pre-crossed module, differential crossed module) *A differential pre-crossed module* $(\mathfrak{h} \xrightarrow{\partial} \mathfrak{g}\,, \rhd)$ *of algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$ *is given by a Lie algebra map* $\partial : \mathfrak{h} \to \mathfrak{g}$ *together with an action* $\rhd$ *of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *such that for each* $\underline{h} \in \mathfrak{h}$ *and* $\underline{g} \in \mathfrak{g}$*:*

$$\partial(\underline{g} \rhd \underline{h}) = [\underline{g}, \partial(\underline{h})]\,.$$

The action $\rhd$ *of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *is on left by derivations, i.e. for each* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ *and each* $\underline{g} \in \mathfrak{g}$*:*

$$\underline{g} \rhd [\underline{h}_1,\, \underline{h}_2] = [\underline{g} \rhd \underline{h}_1,\, \underline{h}_2] + [\underline{h}_1,\, \underline{g} \rhd \underline{h}_2]\,.$$

In a differential pre-crossed module, the Peiffer commutators are defined for each $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ *as:*

$$\langle \underline{h}_1,\, \underline{h}_2\rangle_{\rm p} = [\underline{h}_1, \underline{h}_2] - \partial(\underline{h}_1) \rhd \underline{h}_2\,.$$

The map $(\underline{h}_1,\, \underline{h}_2) \in \mathfrak{h} \times \mathfrak{h} \to \langle \underline{h}_1,\, \underline{h}_2\rangle_{\rm p} \in \mathfrak{h}$ *is bilinear* $\mathfrak{g}$*-equivariant map called the* **Peiffer paring***, i.e. all* $\underline{h}_1\,, \underline{h}_2 \in \mathfrak{h}$ *and* $\underline{g} \in \mathfrak{g}$ *satisfy the following identity:*

$$\underline{g} \rhd \langle \underline{h}_1\,, \underline{h}_2\rangle_{\rm p} = \langle \underline{g} \rhd \underline{h}_1\,, \underline{h}_2\rangle + \langle \underline{h}_1\,, \underline{g} \rhd \underline{h}_2\rangle_{\rm p}\,.$$

A differential pre-crossed module is said to be a **differential crossed module** *if all of its Peiffer commutators vanish, which is to say that for each* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$*:*

$$\partial(\underline{h}_1) \rhd \underline{h}_2 = [\underline{h}_1,\, \underline{h}_2]\,.$$

Definition 4 (Differential 2-crossed module) *A differential 2-crossed module is given by a complex of Lie algebras:*

$$\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}\,,$$

together with left action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{h}$, $\mathfrak{l}$, by derivations, and on itself via adjoint representation, and a $\mathfrak{g}$-equivariant bilinear map called the **Peiffer lifting***:*

$$\{-\,,\,-\}:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{l}$$

Fixing the basis in algebra $T_A\in\mathfrak{l}$, $t_a\in\mathfrak{h}$ and $\tau_\alpha\in\mathfrak{g}$:

$$[T_A,T_B]=f_{AB}{}^C\,T_C\,,\qquad [t_a,t_b]=f_{ab}{}^c\,t_c\,,\qquad [\tau_\alpha,\tau_\beta]=f_{\alpha\beta}{}^\gamma\,\tau_\gamma\,,$$

one defines the maps ∂ and δ as:

$$\partial(t_a)=\partial_a{}^\alpha\,\tau_\alpha\,,\qquad \delta(T_A)=\delta_A{}^a\,t_a\,,$$

and action of $\mathfrak{g}$ on the generators of $\mathfrak{l}$, $\mathfrak{h}$ and $\mathfrak{g}$ is, respectively:

$$\tau_\alpha\triangleright T_A=\triangleright_{\alpha A}{}^B\,T_B\,,\qquad \tau_\alpha\triangleright t_a=\triangleright_{\alpha a}{}^b\,t_b\,,\qquad \tau_\alpha\triangleright\tau_\beta=\triangleright_{\alpha\beta}{}^\gamma\,\tau_\gamma\,.$$

Note that when η is $\mathfrak{g}$-valued differential form and ω is $\mathfrak{l}$, $\mathfrak{h}$ or $\mathfrak{g}$ valued differential form the previous action is defined as:

$$\eta\triangleright\omega=\eta^\alpha\wedge\omega^A\triangleright_{\alpha A}{}^B\,T_B\,,\qquad \eta\triangleright\omega=\eta^\alpha\wedge\omega^a\triangleright_{\alpha a}{}^b\,t_b\,,\qquad \eta\triangleright\omega=\eta^\alpha\wedge\omega^\beta f_{\alpha\beta}{}^\gamma\,\tau_\gamma\,.$$

The coefficients $X_{ab}{}^A$ are introduced as:

$$\{t_a,\,t_b\}=X_{ab}{}^AT_A\,.$$

The following identities are satisfied:

1. *In the differential crossed module $(L\overset{\delta}{\to}H\,,\triangleright')$ the action $\triangleright'$ of $\mathfrak{h}$ on $\mathfrak{l}$ is defined for each $\underline{h}\in\mathfrak{h}$ and $\underline{l}\in\mathfrak{l}$ as:*

   $$\underline{h}\triangleright'\underline{l}=-\{\delta(\underline{l}),\,\underline{h}\}\,,$$

   or written in the basis where $t_a\triangleright' T_A=\triangleright'{}_{aA}{}^BT_B$ the previous identity becomes:

   $$\triangleright'{}_{aA}{}^B=-\delta_A{}^bX_{ba}{}^B\,;$$

2. *The action of $\mathfrak{g}$ on itself is via adjoint representation:*

   $$\triangleright_{\alpha\beta}{}^\gamma=f_{\alpha\beta}{}^\gamma\,;$$

3. *The action of $\mathfrak{g}$ on $\mathfrak{h}$ and $\mathfrak{l}$ is equivariant, i.e. the following identities are satisfied:*

   $$\partial_a{}^\beta f_{\alpha\beta}{}^\gamma=\triangleright_{\alpha a}{}^b\partial_b{}^\gamma\,,\qquad \delta_A{}^a\triangleright_{\alpha a}{}^b=\triangleright_{\alpha A}{}^B\delta_B{}^b\,;$$

4. *The Peiffer lifting is $\mathfrak{g}$-equivariant, i.e. for each $\underline{h}_1,\underline{h}_2\in\mathfrak{h}$ and $\underline{g}\in\mathfrak{g}$:*

   $$\underline{g}\triangleright\{\underline{h}_1,\underline{h}_2\}=\{\underline{g}\triangleright\underline{h}_1,\underline{h}_2\}+\{\underline{h}_1,\,\underline{g}\triangleright\underline{h}_2\}\,,$$

   or written in the basis:

   $$X_{ab}{}^B\triangleright_{\alpha B}{}^A=\triangleright_{\alpha a}{}^cX_{cb}{}^A+\triangleright_{\alpha b}{}^cX_{ac}{}^A\,;$$

5. $\delta(\{\underline{h}_1, \underline{h}_2\}) = \langle \underline{h}_1, \underline{h}_2 \rangle_{\mathrm{p}}$, $\quad \forall \underline{h}_1, \underline{h}_2 \in \mathfrak{h}$, *i.e.*

$$X_{ab}{}^{A}\delta_A{}^{c} = f_{ab}{}^{c} - \partial_a{}^{\alpha} \rhd_{\alpha b}{}^{c} \,;$$

6. $[\underline{l}_1, \underline{l}_2] = \{\delta(\underline{l}_1), \delta(\underline{l}_2)\}$, $\quad \forall \underline{l}_1, \underline{l}_2 \in \mathfrak{l}$, *i.e.*

$$f_{AB}{}^{C} = \delta_A{}^{a}\delta_B{}^{b}X_{ab}{}^{C} \,;$$

7. $\{[\underline{h}_1, \underline{h}_2], \underline{h}_3\} = \partial(\underline{h}_1) \rhd \{\underline{h}_2, \underline{h}_3\} + \{\underline{h}_1, [\underline{h}_2, \underline{h}_3]\} - \partial(\underline{h}_2) \rhd \{\underline{h}_1, \underline{h}_3\} - \{\underline{h}_2, [\underline{h}_1, \underline{h}_3]\}$, $\forall \underline{h}_1, \underline{h}_2, \underline{h}_3 \in \mathfrak{h}$, *i.e.*

$$\{[\underline{h}_1, \underline{h}_2], \underline{h}_3\} = \{\partial(\underline{h}_1) \rhd \underline{h}_2, \underline{h}_3\} - \{\partial(\underline{h}_2) \rhd \underline{h}_1, \underline{h}_3\} - \{\underline{h}_1, \delta\{\underline{h}_2, \underline{h}_3\}\} + \{\underline{h}_2, \delta\{\underline{h}_1, q, \underline{h}_3\}\},$$

$$f_{ab}{}^{d}X_{dc}{}^{B} = \partial_a{}^{\alpha}X_{bc}{}^{A} \rhd_{\alpha A}{}^{B} + X_{ad}{}^{B}f_{bc}{}^{d} - \partial_b{}^{\alpha} \rhd_{\alpha A}{}^{B}X_{ac}{}^{A} - X_{bd}{}^{B}f_{ac}{}^{d} \,;$$

8. $\{\underline{h}_1, [\underline{h}_2, \underline{h}_3]\} = \{\delta\{\underline{h}_1, \underline{h}_2\}, \underline{h}_3\} - \{\delta\{\underline{h}_1, \underline{h}_3\}, \underline{h}_2\}$, $\quad \forall \underline{h}_1, \underline{h}_2, \underline{h}_3 \in \mathfrak{h}$, *i.e.*

$$X_{ad}{}^{A}f_{bc}{}^{d} = X_{ab}{}^{B}\delta_B{}^{d}X_{dc}{}^{A} - X_{ac}{}^{B}\delta_B{}^{d}X_{db}{}^{A} \,;$$

9. $\{\delta(\underline{l}), \underline{h}\} + \{\underline{h}, \delta(\underline{l})\} = -\partial(\underline{h}) \rhd \underline{l}$, $\quad \forall \underline{l} \in \mathfrak{l}$, $\quad \forall \underline{h} \in \mathfrak{h}$, *i.e.*

$$\delta_A{}^{a}X_{ab}{}^{B} + \delta_A{}^{a}X_{ba}{}^{B} = -\partial_b{}^{\alpha} \rhd_{\alpha A}{}^{B} \,.$$

Note that the property 6. implies that either trivial map δ or the trivial Peiffer lifting imply that L is an Abelian group. Conversely, if L is Abelian, property 6. implies that either the map δ or the Peiffer lifting is trivial, or both.

In the case of an Abelian group H and trivial map ∂, among the aforementioned properties the only non-trivial remaining are:

1. $\delta\{\underline{h}_1, \underline{h}_2\} = 0$, $\quad \forall \underline{h}_1, \underline{h}_2 \in \mathfrak{h}$;
2. $[\underline{l}_1, \underline{l}_2] = \{\delta(\underline{l}_1), \delta(\underline{l}_2)\}$, $\quad \forall \underline{l}_1, \underline{l}_2 \in \mathfrak{l}$;
3. $\{\delta(\underline{l}), \underline{h}\} = -\{\underline{h}, \delta(\underline{l})\}$, $\quad \forall \underline{h} \in \mathfrak{h}$, $\quad \forall \underline{l} \in \mathfrak{l}$.

A reader intrested in more details about 3-groups is referred to [25].

B The construction of gauge-invariant actions for 3BF theory

Symmetric bilinear invariant nondegenerate forms are defined as:

$$\langle T_A, T_B \rangle_{\mathfrak{l}} = g_{AB} \,, \qquad \langle t_a, t_b \rangle_{\mathfrak{h}} = g_{ab} \,, \qquad \langle \tau_\alpha, \tau_\beta \rangle_{\mathfrak{g}} = g_{\alpha\beta} \,.$$

They satisfy the following properties:

- $\langle \_, \_ \rangle_{\mathfrak{g}}$ is G-invariant:

$$\langle g\tau_\alpha g^{-1}, g\tau_\beta g^{-1} \rangle_{\mathfrak{g}} = \langle \tau_\alpha, \tau_\beta \rangle_{\mathfrak{g}} \,, \quad \forall g \in G \,;$$

- $\langle \_ , \_ \rangle_{\mathfrak{h}}$ is G-invariant:

$$\langle g \triangleright t_a \,, g \triangleright t_b \rangle_{\mathfrak{h}} = \langle t_a \,, t_b \rangle_{\mathfrak{h}} \,, \quad \forall g \in G \,,$$

  and, when $(H \xrightarrow{\partial} G \,, \triangleright)$ is a crossed module, consequently H-invariant:

$$\langle h t_a h^{-1} \,, h t_b h^{-1} \rangle_{\mathfrak{h}} = \langle \partial(h) \triangleright t_a \,, \partial(h) \triangleright t_b \rangle_{\mathfrak{h}} = \langle t_a \,, t_b \rangle_{\mathfrak{h}} \,, \quad \forall h \in H \,;$$

- $\langle \_ , \_ \rangle_{\mathfrak{l}}$ is G-invariant:

$$\langle g \triangleright T_A \,, g \triangleright T_B \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall g \in G \,,$$

  and in the case when the Peiffer lifting or the map δ is trivial consequently H-invariant:

$$\langle h \triangleright' T_A \,, h \triangleright' T_B \rangle_{\mathfrak{l}} = \langle T_A - \{\delta(T_A), h\} \,, T_B - \{\delta(T_B), h\} \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall h \in H \,.$$

  From the H-invariance of $\langle \_ , \_ \rangle_{\mathfrak{l}}$ and properties of a crossed module $(L \xrightarrow{\delta} H \,, \triangleright')$ follows L-invariance:

$$\langle l T_A l^{-1} \,, l T_B l^{-1} \rangle_{\mathfrak{l}} = \langle \delta(l) \triangleright' T_A \,, \delta(l) \triangleright' T_B \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall l \in L \,.$$

From the invariance of the bilinear forms follows the existence of gauge-invariant topological $3BF$ action of the form:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle \mathcal{C} \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle \mathcal{D} \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,, \tag{B.1}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4 \,, \mathfrak{g})$, $C \in \mathcal{A}^1(\mathcal{M}_4 \,, \mathfrak{h})$ and $D \in \mathcal{A}^0(\mathcal{M}_4 \,, \mathfrak{l})$ are Lagrange multipliers, and $\mathcal{F} \in \mathcal{A}^2(\mathcal{M}_4 \,, \mathfrak{g})$, $\mathcal{G} \in \mathcal{A}^3(\mathcal{M}_4 \,, \mathfrak{h})$ and $\mathcal{H} \in \mathcal{A}^4(\mathcal{M}_4 \,, \mathfrak{l})$ are curvatures defined as in (3.1). Written in the basis:

$$\begin{aligned}
\mathcal{F} &= \frac{1}{2} \mathcal{F}^{\alpha}{}_{\mu\nu} \tau_\alpha \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \qquad \mathcal{G} = \frac{1}{3!} \mathcal{G}^{a}{}_{\mu\nu\rho} t_a \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \,, \\
\mathcal{H} &= \frac{1}{4!} \mathcal{H}^{A}{}_{\mu\nu\rho\sigma} T_A \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \,,
\end{aligned}$$

the coefficients are:

$$\begin{aligned}
\mathcal{F}^{\alpha}{}_{\mu\nu} &= \partial_\mu \alpha^\alpha{}_\nu - \partial_\nu \alpha^\alpha{}_\mu + f_{\beta\gamma}{}^{\alpha} \alpha^\beta{}_\mu \alpha^\gamma{}_\nu - \beta^a{}_{\mu\nu} \partial_a{}^\alpha \,, \\
\mathcal{G}^{a}{}_{\mu\nu\rho} &= \partial_\mu \beta^a{}_{\nu\rho} + \partial_\nu \beta^a{}_{\rho\mu} + \partial_\rho \beta^a{}_{\mu\nu} \\
&\quad + \alpha^\alpha{}_\mu \beta^b{}_{\nu\rho} \triangleright_{\alpha b}{}^a + \alpha^\alpha{}_\nu \beta^b{}_{\rho\mu} \triangleright_{\alpha b}{}^a + \alpha^\alpha{}_\rho \beta^b{}_{\mu\nu} \triangleright_{\alpha b}{}^a - \gamma^A{}_{\mu\nu\rho} \delta_A{}^a \,, \\
\mathcal{H}^{A}{}_{\mu\nu\rho\sigma} &= \partial_\mu \gamma^A{}_{\nu\rho\sigma} - \partial_\nu \gamma^A{}_{\rho\sigma\mu} + \partial_\rho \gamma^A{}_{\sigma\mu\nu} - \partial_\sigma \gamma^A{}_{\mu\nu\rho} \\
&\quad + 2\beta^a{}_{\mu\nu} \beta^b{}_{\rho\sigma} X_{\{ab\}}{}^A - 2\beta^a{}_{\mu\rho} \beta^b{}_{\nu\sigma} X_{\{ab\}}{}^A + 2\beta^a{}_{\mu\sigma} \beta^b{}_{\nu\rho} X_{\{ab\}}{}^A \\
&\quad + \alpha^\alpha{}_\mu \gamma^B{}_{\nu\rho\sigma} \triangleright_{\alpha B}{}^A - \alpha^\alpha{}_\nu \gamma^B{}_{\rho\sigma\mu} \triangleright_{\alpha B}{}^A + \alpha^\alpha{}_\rho \gamma^B{}_{\sigma\mu\nu} \triangleright_{\alpha B}{}^A - \alpha^\alpha{}_\sigma \gamma^B{}_{\mu\nu\rho} \triangleright_{\alpha B}{}^A \,.
\end{aligned}$$

Note that the wedge product $A \wedge B$ when A is a 0-form and B is a p-form is defined as $A \wedge B = \frac{1}{p!} A B_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge x^{\mu_p}$.

JHEP10(2019)222

Given G-invariant symmetric non-degenerate bilinear forms in $\mathfrak{g}$ and $\mathfrak{h}$, one can define a bilinear antisymmetric map $\mathcal{T} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ by the rule:

$$\langle \mathcal{T}(\underline{h}_1, \underline{h}_2)\,, \underline{g} \rangle_{\mathfrak{g}} = -\langle \underline{h}_1, \underline{g} \rhd \underline{h}_2 \rangle_{\mathfrak{h}}, \qquad \forall \underline{h}_1, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{g} \in \mathfrak{g}\,.$$

See [17] for more properties and the construction of $2BF$ invariant topological action using this map. To define $3BF$ invariant topological action one has to first define a bilinear antisymmetric map $\mathcal{S} : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{g}$ by the rule:

$$\langle \mathcal{S}(\underline{l}_1, \underline{l}_2), \underline{g} \rangle_{\mathfrak{g}} = -\langle \underline{l}_1, \underline{g} \rhd \underline{l}_2 \rangle_{\mathfrak{l}}\,, \qquad \forall \underline{l}_1, \forall \underline{l}_2 \in \mathfrak{l}\,, \quad \forall \underline{g} \in \mathfrak{g}\,.$$

Note that $\langle \_\,, \_\rangle_{\mathfrak{g}}$ is non-degenerate and

$$\langle \underline{l}_1, \underline{g} \rhd \underline{l}_2 \rangle_{\mathfrak{l}} = -\langle \underline{g} \rhd \underline{l}_1, \underline{l}_2 \rangle_{\mathfrak{l}} = -\langle \underline{l}_2, \underline{g} \rhd \underline{l}_1 \rangle_{\mathfrak{l}}\,, \qquad \forall \underline{g} \in \mathfrak{g}, \quad \forall \underline{l}_1, \underline{l}_2 \in \mathfrak{l}\,.$$

Morever, given $g \in G$ and $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$ one has:

$$\mathcal{S}(g \rhd \underline{l}_1, g \rhd \underline{l}_2) = g\, \mathcal{S}(\underline{l}_1, \underline{l}_2)\, g^{-1}\,,$$

since for each $\underline{g} \in \mathfrak{g}$ and $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$:

$$\begin{aligned}\langle \underline{g}, g^{-1} \mathcal{S}(g \rhd \underline{l}_1\,, g \rhd \underline{l}_2) g \rangle_{\mathfrak{g}} &= \langle g \underline{g} g^{-1}, \mathcal{S}(g \rhd \underline{l}_1, g \rhd \underline{l}_2) \rangle_{\mathfrak{g}} \\ &= -\langle (g\, \underline{g}\, g^{-1}) \rhd g \rhd \underline{l}_1, g \rhd \underline{l}_2 \rangle_{\mathfrak{l}} \\ &= -\langle \underline{g} \rhd \underline{l}_1\,, \underline{l}_2 \rangle_{\mathfrak{l}} = \langle \underline{g}\,, \mathcal{S}(\underline{l}_1, \underline{l}_2) \rangle_{\mathfrak{g}}\,,\end{aligned}$$

where the following mixed relation has been used:

$$g \rhd (\underline{g} \rhd \underline{l}) = (g\, \underline{g}\, g^{-1}) \rhd g \rhd \underline{l}\,. \tag{B.2}$$

We thus have the following identity:

$$\mathcal{S}(\underline{g} \rhd \underline{l}_1, \underline{l}_2) + \mathcal{S}(\underline{l}_1, \underline{g} \rhd \underline{l}_2) = [\underline{g}, \mathcal{S}(\underline{l}_1, \underline{l}_2)]\,.$$

As far as the bilinear antisymmetric map $\mathcal{S} : l \times l \to g$, one can write it in the basis:

$$\mathcal{S}(T_A, T_B) = \mathcal{S}_{AB}{}^{\alpha} \tau_\alpha\,,$$

so that the defining relation for $\mathcal{S}$ becomes the relation:

$$\mathcal{S}_{AB}{}^{\alpha} g_{\alpha\beta} = -\rhd_{\alpha[B}{}^{C} g_{A]C}\,.$$

Given two $\mathfrak{l}$-valued forms η and ω, one can define a $\mathfrak{g}$-valued form:

$$\omega \wedge^{\mathcal{S}} \eta = \omega^A \wedge \eta^B \mathcal{S}_{AB}{}^{\alpha} \tau_\alpha\,.$$

Now one can define the transformations of the Lagrange multipliers under L-gauge transformations (3.15).

Further, to define the transformations of the Lagrange multipliers under H-gauge transformations one needs to define the bilinear map $\mathcal{X}_1 : \mathfrak{l} \times \mathfrak{h} \to \mathfrak{h}$ by the rule:

$$\langle \mathcal{X}_1(\underline{l},\, \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}} = -\langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}\,,$$

and bilinear map $\mathcal{X}_2 : \mathfrak{l} \times \mathfrak{h} \to \mathfrak{h}$ by the rule:

$$\langle \mathcal{X}_2(\underline{l},\, \underline{h}_2),\, \underline{h}_1\rangle_{\mathfrak{h}} = -\langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}\,.$$

As far as the bilinear maps $\mathcal{X}_1$ and $\mathcal{X}_2$ one can define the coefficients in the basis as:

$$\mathcal{X}_1(T_A, t_a) = \mathcal{X}_{1Aa}{}^b\, t_b\,, \qquad \mathcal{X}_2(T_A, t_a) = \mathcal{X}_{2Aa}{}^b\, t_b\,.$$

When written in the basis the defining relations for the maps $\mathcal{X}_1$ and $\mathcal{X}_2$ become:

$$\mathcal{X}_{1Ab}{}^c g_{ac} = -X_{ba}{}^B g_{AB}\,, \qquad \mathcal{X}_{2Ab}{}^c g_{ac} = -X_{ab}{}^B g_{AB}\,.$$

Given $\mathfrak{l}$-valued differential form ω and $\mathfrak{h}$-valued differential form η, one defines a $\mathfrak{h}$-valued form as:

$$\omega \wedge^{\mathcal{X}_1} \eta = \omega^A \wedge \eta^a \mathcal{X}_{1Aa}{}^b t_b\,, \qquad \omega \wedge^{\mathcal{X}_2} \eta = \omega^A \wedge \eta^a \mathcal{X}_{2Aa}{}^b t_b\,.$$

Given any $g \in G$, $\underline{l} \in \mathfrak{l}$ and $\underline{h} \in \mathfrak{h}$ one has:

$$\mathcal{X}_1(g \rhd \underline{l},\, g^{-1} \rhd \underline{h}) = g \rhd \mathcal{X}_1(\underline{l},\, \underline{h})\,, \qquad \mathcal{X}_2(g \rhd \underline{l},\, g \rhd \underline{h}) = g^{-1} \rhd \mathcal{X}_2(\underline{l},\, \underline{h})\,,$$

since for each $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ and $\underline{l} \in \mathfrak{l}$:

$$\begin{aligned}\langle \underline{h}_2,\, g^{-1} \rhd \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{h}_1)\rangle_{\mathfrak{h}} &= \langle g \rhd \underline{h}_2,\, \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{h}_1)\rangle_{\mathfrak{h}} = \langle g \rhd \underline{l},\, \{g \rhd \underline{h}_1,\, g \rhd \underline{h}_2\}\rangle_{\mathfrak{l}} \\ \langle g \rhd \underline{l},\, g \rhd \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}} &= \langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}} = \langle \underline{h}_2,\, \mathcal{X}_1(l,\, \underline{h}_1)\rangle_{\mathfrak{h}}\,,\end{aligned}$$

and similarly for $\mathcal{X}_2$. Finaly, one needs to define a trilinear map $\mathcal{D} : \mathfrak{h} \times \mathfrak{h} \times \mathfrak{l} \to \mathfrak{g}$ by the rule:

$$\langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l}),\, \underline{g}\rangle_{\mathfrak{g}} = -\langle \underline{l},\, \{\underline{g} \rhd \underline{h}_1,\, \underline{h}_2\}\rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}, \quad \forall \underline{g} \in \mathfrak{g}\,,$$

One can define the coefficients of the trilinear map as:

$$\mathcal{D}(t_a,\, t_b,\, T_A) = \mathcal{D}_{abA}{}^\alpha \tau_\alpha\,,$$

and the defining relation for the map $\mathcal{D}$ expressed in terms of coefficients becomes:

$$\mathcal{D}_{abA}{}^\beta g_{\alpha\beta} = - \rhd_{\alpha a}{}^c X_{cb}{}^B g_{AB}\,.$$

Given two $\mathfrak{h}$-valued forms ω and η, and $\mathfrak{l}$-valued form ξ, the g-valued form is given by the formula:

$$\omega \wedge^{\mathcal{D}} \eta \wedge^{\mathcal{D}} \xi = \omega^a \wedge \eta^b \wedge \xi^A \mathcal{D}_{abA}{}^\beta \tau_\beta\,.$$

The following compatibility relation between the maps $\mathcal{X}_1$ and $\mathcal{D}$ hold:

$$\langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l}),\, \underline{g}\rangle_{\mathfrak{g}} = \langle \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}, \quad \forall \underline{g} \in \mathfrak{g}\,, \tag{B.3}$$

JHEP10(2019)222

which one can prove valid from the defining relations in terms of the coefficients. One can demonstrate that for each $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$ and $g \in G$:

$$\mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l}) = g\, \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})\, g^{-1}\,,$$

since for each $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$, $\underline{g} \in \mathfrak{g}$ and $g \in G$:

$$\begin{aligned}
\langle g^{-1}\mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l})g,\, \underline{g}\rangle_{\mathfrak{g}} &= \langle \mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l}),\, g\underline{g}g^{-1}\rangle_{\mathfrak{g}} \\
&= \langle \mathcal{X}_1(g \rhd \underline{l},\, g\underline{g}g^{-1} \rhd g \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{g} \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle g \rhd \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})\,,\underline{g}\rangle_{\mathfrak{g}}\,,
\end{aligned}$$

where the relation (B.2) and the compatibility relation (B.3) were used. We thus have for each $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$ and $\underline{g} \in \mathfrak{g}$ the following identity:

$$\mathcal{D}(\underline{g} \rhd \underline{h}_1,\, \underline{h}_2,\, \underline{l}) + \mathcal{D}(\underline{h}_1,\, \underline{g} \rhd \underline{h}_2,\, \underline{l}) + \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{g} \rhd \underline{l}) = [\underline{g},\, \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})]\,.$$

Now one can define the transformations of the Lagrange multipliers under H-gauge transformations as in (3.14).

C The equations of motion for the Weyl and Majorana fields

The action for the Weyl spinor field coupled to gravity is given by (3.72). The variation of this action with respect to the variables B_{ab}, λ^{ab}, γ_α, $\bar{\gamma}^{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, e^a, β^a and ω^{ab} one obtains the complete set of equations of motion, as follows:

$$\begin{aligned}
R^{ab} - \lambda^{ab} &= 0\,,\\
B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd}e^c \wedge e^d &= 0\,,\\
\nabla\psi_\alpha + \lambda_\alpha &= 0\,,\\
\nabla\bar{\psi}^{\dot{\alpha}} + \bar{\lambda}^{\dot{\alpha}} &= 0\,,\\
-\gamma_\alpha + \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}} &= 0\,,\\
-\bar{\gamma}^{\dot{\alpha}} + \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta &= 0\,,\\
\nabla\gamma_\alpha - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}} &= 0\,,\\
\nabla\bar{\gamma}^{\dot{\alpha}} - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta}\lambda_\beta &= 0\,,\\
\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{i}{2}\varepsilon_{abcd}e^b \wedge e^c \wedge (\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta + \lambda^\alpha\sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}})&\\
-8\pi i l_p^2 \varepsilon_{abcd}e^b\beta^c\big(\psi^\alpha(\sigma^d)_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}\big) &= 0\,,
\end{aligned}$$

JHEP10(2019)222

$$\nabla e_a - 4\pi l_p^2 \varepsilon_{abcd} e^b \wedge e^c \wedge (\bar{\psi}_{\dot\alpha} \bar{\sigma}^{d\dot\alpha\beta} \psi_\beta) = 0\,,$$
$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} - \frac{1}{2}\gamma \sigma^{ab}{}_\alpha{}^\beta \psi_\beta - \frac{1}{2}\bar{\gamma}_{\dot\alpha} \bar{\sigma}^{ab\dot\alpha}{}_{\dot\beta} \bar{\psi}^{\dot\beta} = 0\,.$$

In the case of the Majorana field, one adds the mass term (3.75) to the action (3.72). Then, the variation of the action with respect to B_{ab}, ψ^{ab}, γ^α, $\bar{\gamma}_{\dot\alpha}$, λ_α, $\bar{\lambda}^{\dot\alpha}$, ψ_α, $\bar{\psi}_I^{\dot\alpha}$, e^a, β^a and ω_{ab} gives the equations of motion for the Majorana case, as follows:

$$R^{ab} - \lambda^{ab} = 0\,,$$
$$B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd} e^c \wedge e^d = 0\,,$$
$$-\nabla\psi_\alpha + \lambda_\alpha = 0\,,$$
$$-\nabla\bar{\psi}^{\dot\alpha} + \lambda^{\dot\alpha} = 0\,,$$
$$\gamma^\alpha - \frac{i}{6}\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \bar{\psi}_{\dot\beta} (\bar{\sigma}^d)^{\dot\beta\alpha} = 0\,,$$
$$\bar{\gamma}_{\dot\alpha} - \frac{i}{6}\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \psi^\beta (\sigma^d)_{\beta\dot\alpha} = 0\,,$$
$$\nabla\gamma^\alpha + \frac{i}{6}\varepsilon_{abcd}\lambda^{\dot\beta} \wedge e^a \wedge e^b \wedge e^c (\sigma^d)^\alpha{}_{\dot\beta} - \frac{1}{6}\, m\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \psi^\alpha$$
$$-4i\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c \bar{\psi}_{\dot\beta} (\bar{\sigma}^d)^{\dot\beta\alpha} = 0\,,$$
$$\nabla\bar{\gamma}_{\dot\alpha} + \frac{i}{6}\varepsilon_{abcd}\lambda_\beta \wedge e^a \wedge e^b \wedge e^c (\bar{\sigma}^d)_{\dot\alpha}{}^\beta - \frac{1}{6} m\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \psi_{\dot\alpha}$$
$$-4i\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c \psi^\beta (\sigma^d)_{\beta\dot\alpha} = 0\,,$$
$$\nabla\beta^a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{i}{2}\varepsilon_{abcd}\lambda_\alpha \wedge e^b \wedge e^c \bar{\psi}_{\dot\beta} (\bar{\sigma}^d)^{\dot\beta\alpha} + \frac{i}{2}\varepsilon_{abcd}\lambda^{\dot\alpha} \wedge e^b \wedge e^c \psi^\beta (\sigma^d)_{\beta\dot\alpha}$$
$$-\frac{1}{3} m\varepsilon_{abcd} e^b \wedge e^c \wedge e^d (\psi^\alpha\psi_\alpha + \bar{\psi}_{\dot\alpha}\bar{\psi}^{\dot\alpha}) - 8\pi i l_p^2 \varepsilon_{abcd} e^b \beta^c \big(\psi^\alpha (\sigma^d)_{\alpha\dot\beta} \bar{\psi}^{\dot\beta}\big) = 0\,,$$
$$\nabla e_a - 4i\pi l_p^2 \varepsilon_{abcd} e^b \wedge e^c \big(\psi^\alpha (\sigma^d)_{\alpha\dot\beta} \bar{\psi}^{\dot\beta}\big) = 0\,,$$
$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} - \frac{1}{2}\psi^\alpha (\sigma^{ab})_\alpha{}^\beta \gamma_\beta - \frac{1}{2}\bar{\psi}_{\dot\alpha} (\bar{\sigma}^{ab})^{\dot\alpha}{}_{\dot\beta} \bar{\gamma}^{\dot\beta} = 0\,.$$

References

[1] C. Rovelli, *Quantum gravity*, Cambridge University Press, Cambridge, U.K. (2004).

[2] C. Rovelli and F. Vidotto, *Covariant loop quantum gravity*, Cambridge University Press, Cambridge, U.K. (2014).

[3] T. Thiemann, *Modern canonical quantum general relativity*, Cambridge University Press, Cambridge, U.K. (2007).

[4] G. Ponzano and T. Regge, *Semiclassical limit of Racah coefficients*, in *Spectroscopic and group theoretical methods in physics*, F. Block ed., North Holland, Amsterdam, The Netherlands (1968).

[5] J.W. Barrett and L. Crane, *Relativistic spin networks and quantum gravity*, J. Math. Phys. **39** (1998) 3296 [gr-qc/9709028] [INSPIRE].

[6] J.W. Barrett and L. Crane, *A Lorentzian signature model for quantum general relativity*, Class. Quant. Grav. **17** (2000) 3101 [gr-qc/9904025] [INSPIRE].

[7] H. Ooguri, *Topological lattice models in four-dimensions*, Mod. Phys. Lett. **A 7** (1992) 2799 [hep-th/9205090] [INSPIRE].

[8] J. Engle, E. Livine, R. Pereira and C. Rovelli, *LQG vertex with finite Immirzi parameter*, Nucl. Phys. **B 799** (2008) 136 [arXiv:0711.0146] [INSPIRE].

[9] L. Freidel and K. Krasnov, *A new spin foam model for* 4*d gravity*, Class. Quant. Grav. **25** (2008) 125018 [arXiv:0708.1595] [INSPIRE].

[10] E. Bianchi, M. Han, C. Rovelli, W. Wieland, E. Magliaro and C. Perini, *Spinfoam fermions*, Class. Quant. Grav. **30** (2013) 235023 [arXiv:1012.4719] [INSPIRE].

[11] J.C. Baez and J. Huerta, *An invitation to higher gauge theory*, Gen. Rel. Grav. **43** (2011) 2335 [arXiv:1003.4485] [INSPIRE].

[12] A. Miković and M. Vojinović, *Poincaré* 2*-group and quantum gravity*, Class. Quant. Grav. **29** (2012) 165003 [arXiv:1110.4694] [INSPIRE].

[13] M. Celada, D. González and M. Montesinos, *BF gravity*, Class. Quant. Grav. **33** (2016) 213001 [arXiv:1610.02020] [INSPIRE].

[14] C. Rovelli, *Zakopane lectures on loop gravity*, PoS(QGQGS2011)003 (2011) [arXiv:1102.3660] [INSPIRE].

[15] J.F. Plebanski, *On the separation of Einsteinian substructures*, J. Math. Phys. **18** (1977) 2511 [INSPIRE].

[16] F. Girelli, H. Pfeiffer and E.M. Popescu, *Topological higher gauge theory — from BF to BFCG theory*, J. Math. Phys. **49** (2008) 032503 [arXiv:0708.3051] [INSPIRE].

[17] J.F. Martins and A. Miković, *Lie crossed modules and gauge-invariant actions for* 2*-BF theories*, Adv. Theor. Math. Phys. **15** (2011) 1059 [arXiv:1006.0903] [INSPIRE].

[18] L. Crane and M.D. Sheppeard, 2*-categorical Poincaré representations and state sum applications*, math.QA/0306440 [INSPIRE].

[19] M. Vojinović, *Causal dynamical triangulations in the spincube model of quantum gravity*, Phys. Rev. **D 94** (2016) 024058 [arXiv:1506.06839] [INSPIRE].

[20] A. Miković, *Spin-cube models of quantum gravity*, Rev. Math. Phys. **25** (2013) 1343008 [arXiv:1302.5564] [INSPIRE].

[21] A. Miković and M.A. Oliveira, *Canonical formulation of Poincaré BFCG theory and its quantization*, Gen. Rel. Grav. **47** (2015) 58 [arXiv:1409.3751] [INSPIRE].

[22] A. Miković, M.A. Oliveira and M. Vojinovic, *Hamiltonian analysis of the BFCG theory for a generic Lie* 2*-group*, arXiv:1610.09621 [INSPIRE].

[23] A. Miković, M.A. Oliveira and M. Vojinovic, *Hamiltonian analysis of the BFCG formulation of general relativity*, Class. Quant. Grav. **36** (2019) 015005 [arXiv:1807.06354] [INSPIRE].

[24] J.F. Martins and R. Picken, *The fundamental Gray* 3*-groupoid of a smooth manifold and local* 3*-dimensional holonomy based on a* 2*-crossed module*, Differ. Geom. Appl. J. **29** (2011) 179 [arXiv:0907.2566] [INSPIRE].

[25] W. Wang, *On* 3*-gauge transformations,* 3*-curvatures and Gray-categories*, J. Math. Phys. **55** (2014) 043506 [arXiv:1311.3796] [INSPIRE].

Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

Entanglement-induced deviation from the geodesic motion in quantum gravity

Francisco Pipa,[a,b] **Nikola Paunković**[c,d] **and Marko Vojinović**[e]

[a]Department of Philosophy, University of Kansas, 1445 Jayhawk Blvd., Wescoe Hall, Lawrence, KS 66045-7590, U.S.A.
[b]Departamento de Física, Instituto Superior Técnico, Universidade de Lisboa, Avenida Rovisco Pais 1049-001, Lisboa, Portugal
[c]Instituto de Telecomunicações, Av. Rovisco Pais 1, 1049-001, Lisbon, Portugal
[d]Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal
[e]Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia

E-mail: franciscosapipa@gmail.com, npaunkov@math.tecnico.ulisboa.pt, vmarko@ipb.ac.rs

Received August 6, 2019
Accepted August 29, 2019
Published September 26, 2019

Abstract. We study the derivation of the effective equation of motion for a pointlike particle in the framework of quantum gravity. Just like the geodesic motion of a classical particle is a consequence of classical field theory coupled to general relativity, we introduce the similar notion of an effective equation of motion, but starting from an abstract quantum gravity description. In the presence of entanglement between gravity and matter, quantum effects give rise to modifications of the geodesic trajectory, primarily as a consequence of the interference between various coherent states of the gravity-matter system. Finally, we discuss the status of the weak equivalence principle in quantum gravity and its possible violation due to the nongeodesic motion.

Keywords: gravity, quantum field theory on curved space, quantum gravity phenomenology

ArXiv ePrint: 1801.03207

https://doi.org/10.1088/1475-7516/2019/09/057

JCAP09(2019)057

Contents

1 Introduction

The formulation of the theory of quantum gravity (QG) is one of the most fundamental open problems in modern theoretical physics. In models of QG, as in any quantum theory, superpositions of states are allowed. In a tentative "theory of everything", which includes both gravity and matter at a fundamental quantum level, superpositions of product gravity-matter states are particularly interesting. Entangled states are highly nonclassical, and as such are especially relevant because they give rise to a drastically different behavior of matter from what one would expect based on classical intuition, as confirmed by numerous examples from the standard quantum mechanics (QM). Therefore, it is interesting to study such states in the context of a QG coupled to matter, in particular the Schrödinger cat-like states. Moreover, a recent study [1] suggests that physically allowed states of a gravity-matter system are generically entangled due to gauge invariance, providing additional motivation for our study.

In standard QM, entanglement is generically a consequence of the interaction. Nevertheless, there exist situations which give rise to entanglement even without interaction. For example, the Pauli exclusion principle in the case of identical particles generates entanglement without an interaction, giving rise to an effective force (also called the "exchange interaction"). We investigate in detail whether an entanglement between gravity and matter

JCAP09(2019)057

could also be described as a certain type of an effective interaction, and if so, what are its aspects and details. In order to study this problem, we analyze the motion of a free test particle in a gravitational field. In general relativity (GR), this motion is described by a geodesic trajectory. However, we show that in the presence of the gravity-matter entanglement, the resulting effective interaction causes a deviation from a classical geodesic trajectory. In particular, we generalize the standard derivation of a geodesic equation from the case of classical gravity to the case of a full QG model, and derive the equation of motion for a particle which contains a non-geodesic term, reflecting the presence of the entanglement-induced effective interaction. The effects we discuss are purely quantum with respect to both gravity and matter, unlike previous studies of quantum matter in classical curved spacetime [2–5].

As a consequence of the modified equation of motion for a particle, we also discuss the status of the equivalence principle in the context of QG, and a possible violation of its weak flavor.

The paper is organized as follows. Section 2 is devoted to a review of the derivation of the geodesic equation in classical gravity, particularly in GR. The multipole formalism is employed and the geodesic equation for a particle is derived from the covariant conservation of the stress-energy tensor. In section 3 we generalize this procedure and derive our main results. Subsection 3.1 contains the general setup, the abstract quantum gravity framework that will be used, and the main assumptions. In subsection 3.2 we discuss the effective covariant conservation equation, which receives a correction to the classical one, due to the quantum gravity effects. In subsection 3.3 we put everything together and derive our main result — the effective equation of motion for a point particle, with the leading quantum correction. In subsection 3.4 we discuss the consistency of the assumptions that enter the approximation scheme used to derive the effective equation of motion. Section 4 is devoted to the discussion of the consequences of our results in the context of the weak equivalence principle. For the purpose of clarity, in subsection 4.1 we first provide the definitions of various flavors of the equivalence principle. Then, in subsection 4.2 we discuss the status of the equivalence principle in the context of quantum gravity and the results obtained in section 3. Subsection 4.3 provides further analysis of universality and equality between inertial and gravitational masses, in the context of the Newtonian approximation. Finally, section 5 contains our conclusions, discussion of the results and possible lines of further research. In the Appendix we give a short review of the multipole formalism used in the main text, with some mathematical details.

Our notation and conventions are as follows. We will work in the natural system of units in which $c = \hbar = 1$ and $G = l_p^2$, where l_p is the Planck length and G is the Newton's gravitational constant. By convention, the metric of spacetime will have the spacelike Lorentz signature $(-,+,+,+)$. The spacetime indices are denoted with lowercase Greek letters $\mu,\nu,\dots$ and take the values $0,1,2,3$. These can be split into the timelike index 0 and the spacelike indices denoted with lowercase Latin letters $i,j,k,\dots$ which take the values $1,2,3$. The Lorentz-invariant metric tensor is denoted as $\eta_{\mu\nu}$. Quantum operators always carry a hat, $\hat{\phi}(x)$, $\hat{g}(x)$, etc. The parentheses around indices indicate symmetrization with respect to those indices, while brackets indicate antisymmetrization:

$$A_{(\mu\nu)} \equiv \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right) , \qquad A_{[\mu\nu]} \equiv \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right) .$$

Finally, we will systematically denote the values of functions with parentheses, $f(x)$, while functionals will be denoted with brackets, $F[\phi]$.

JCAP09(2019)057

2 Geodesic equation in general relativity

In the context of the classical theories of gravity, like GR, the question of deriving the geodesic equation for a particle has initially been studied by Einstein, Infeld and Hoffmann [6], Mathisson [7], Lubánski [8], Fock [9], and others. Slightly later, the question was revisited in the seminal paper by Papapetrou [10], with generalizations followed by a number of authors [11–22], developing the so-called *multipole formalism*, see the appendix A. Recently, the multipole formalism has been reformulated in a manifestly covariant language and extended from pointlike objects to strings, membranes and further to p-branes, with general equations of motion studied in Riemann and Riemann-Cartan spaces [23–28]. Today, the multipole formalism and the resulting classes of effective equations of motion have found applications in a wide range of topics, from string theory [29] to cosmology [30] to blackbrane dynamics [31–33] to elasticity and the studies of the shape of red blood cells in biological systems [34].

In this section we will demonstrate the application of the multipole formalism in its crudest *single pole* approximation, and employ it to derive the geodesic equation of motion for a point particle in classical Riemannian spacetime. The results presented in this section are well known in the literature, and illustrate the derivation procedure of the geodesic motion for a point particle. After reviewing the standard results in this section, in section 3 the same procedure will be utilized to study the quantum gravity case.

The derivation procedure is based on two main assumptions. The first assumption is that the matter fields have internal dynamics such that they form particle-like kink solutions which are stable (i.e., non-decaying) across the spacetime regions under consideration. If that is the case, one can employ the multipole formalism and expand the stress-energy tensor into a series of derivatives of the Dirac δ function as (see the appendix A for details):

$$T^{\mu\nu}(x) = \int_{\mathcal{C}} d\tau \left[B^{\mu\nu}(\tau) \frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}} + \nabla_\rho \left(B^{\mu\nu\rho}(\tau) \frac{\delta^{(4)}(x-z(\tau))}{\sqrt{-g}} \right) + \dots \right] . \tag{2.1}$$

Here we assume that the stress-energy tensor of matter fields has nonzero value only near some timelike curve $\mathcal{C}$ represented by parametric equations $x^\mu = z^\mu(\tau)$, where τ is a parameter counting the points along the curve $\mathcal{C}$. In that case, the B-coefficients in the δ series will be smaller and smaller with each new term in the series. We introduce a series of smallness scales for the coefficients,

$$B^{\mu\nu} \sim \mathcal{O}_0 \,, \qquad B^{\mu\nu\rho} \sim \mathcal{O}_1 \,, \qquad B^{\mu\nu\rho\sigma} \sim \mathcal{O}_2 \,, \qquad \dots$$

such that one can consider the multipole scales to behave as

$$\mathcal{O}_0 \gg \mathcal{O}_1 \gg \mathcal{O}_2 \gg \dots \tag{2.2}$$

Next we choose to work in the so-called *single pole* approximation, in which all quantities of order $\mathcal{O}_1$ and higher can be neglected. It is also assumed that the typical radius of curvature of spacetime near the curve $\mathcal{C}$ will be large enough not to interfere in the internal dynamics of the matter fields along $\mathcal{C}$ and break the kink configuration apart. Physically speaking, the sequence of inequalities (2.2) states that one can systematically approximate the full solution of the matter field equations of motion by neglecting various degrees of freedom which describe the "size" and "shape" of the kink compared to its orbital motion (i.e., motion along the curve $\mathcal{C}$). Given this setup, in the single pole approximation the matter fields are in a configuration that looks like a point particle traveling along a worldline curve $\mathcal{C}$, and terms of order $\mathcal{O}_1$ and higher can be dropped from the stress-energy tensor, giving:

$$T^{\mu\nu}(x) = \int_{\mathcal{C}} d\tau \, B^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \,. \tag{2.3}$$

The second assumption is the validity of the local Poincaré invariance for the matter field equations. Namely, the classical action which describes the gravity-matter system can be generally written as

$$S[g, \phi] = S_G[g] + S_M[g, \phi] \,,$$

where g and ϕ denote gravitational and matter degrees of freedom, respectively, and it is generally considered to feature local Poincaré invariance. Our assumption is that the matter action S_M and the gravitational action S_G are invariant even taken separately. If this is the case, the Noether theorem gives us the covariant conservation of the stress-energy tensor of matter fields,

$$\nabla_\nu T^{\mu\nu} = 0 \,. \tag{2.4}$$

Taken together, assumptions (2.3) and (2.4) are sufficient to establish two results:

(a) that the parametric functions $z(\tau)$ of the curve $\mathcal{C}$ satisfy the geodesic equation,

$$\frac{d^2 z^\lambda(\tau)}{d\tau^2} + \mathit{\Gamma}^\lambda{}_{\mu\nu} \frac{dz^\mu(\tau)}{d\tau} \frac{dz^\nu(\tau)}{d\tau} = 0 \,, \tag{2.5}$$

where $\mathit{\Gamma}^\lambda{}_{\mu\nu}$ is the Christoffel connection for the background spacetime metric $g_{\mu\nu}$, and

(b) that the leading order coefficient $B^{\mu\nu}(\tau)$ in the stress-energy tensor for the particle has the form

$$B^{\mu\nu}(\tau) = m \, u^\mu(\tau) u^\nu(\tau) \,, \tag{2.6}$$

where $m \in \mathbb{R}\backslash\{0\}$ is an arbitrary constant parameter, while u^μ is the normalized tangent vector to the curve $\mathcal{C}$,

$$u^\mu \equiv \frac{dz^\mu(\tau)}{d\tau} \,, \qquad u^\mu u^\nu g_{\mu\nu} = -1 \,.$$

In order to demonstrate these two statements, we start from (2.4), contract it with an arbitrary test function $f_\mu(x)$ of compact support, and integrate over the whole spacetime,

$$\int_{\mathcal{M}_4} d^4x \sqrt{-g} \, f_\mu \nabla_\nu T^{\mu\nu} = 0 \,.$$

Then we perform the partial integration to move the covariant derivative from the stress-energy tensor to the test function. The boundary term vanishes since the test function has compact support, giving

$$\int_{\mathcal{M}_4} d^4x \sqrt{-g} \, T^{\mu\nu} \nabla_\nu f_\mu = 0 \,.$$

Then we substitute (2.3), switch the order of integrations and perform the integral over spacetime $\mathcal{M}_4$, ending up with

$$\int_{\mathcal{C}} d\tau B^{\mu\nu} \nabla_\nu f_\mu = 0 \,. \tag{2.7}$$

JCAP09(2019)057

The spacetime covariant derivative of the test function can be split into a component tangent to the curve $\mathcal{C}$ and a component orthogonal to it, in the following way. Using the identity

$$\delta^{\lambda}_{\mu} = -u^{\lambda}u_{\mu} + P^{\lambda}_{\perp\mu}\,, \tag{2.8}$$

where $-u^{\lambda}u_{\mu}$ and $P^{\lambda}_{\perp\mu}$ are projectors along u^{μ} and orthogonal to u^{μ}, respectively, we rewrite the derivative of f_{ν} as

$$\nabla_{\nu} f_{\mu} = -u_{\nu}\nabla f_{\mu} + f^{\perp}_{\nu\mu}\,, \tag{2.9}$$

where $\nabla \equiv u^{\lambda}\nabla_{\lambda}$ is the covariant derivative in the direction of the curve $\mathcal{C}$, while $f^{\perp}_{\nu\mu} \equiv P^{\lambda}_{\perp\nu}\nabla_{\lambda}f_{\mu}$ is a quantity orthogonal to the curve $\mathcal{C}$ with respect to its first index. Substituting (2.9) into (2.7), and performing another partial integration, we find

$$\int_{\mathcal{C}} d\tau \left[f_{\mu}\nabla(B^{\mu\nu}u_{\nu}) + B^{\mu\nu} f^{\perp}_{\nu\mu} \right] = 0\,,$$

where the boundary term again vanishes due to the compact support of the test function.

Given that the values of f_{μ} and $f^{\perp}_{\nu\mu}$ are both arbitrary and mutually independent along the curve $\mathcal{C}$, the coefficients multiplying them must each be zero. The first term gives us

$$\nabla(B^{\mu\nu}u_{\nu}) = 0\,, \tag{2.10}$$

while the second term, knowing that $f^{\perp}_{\nu\mu}$ is orthogonal to the curve $\mathcal{C}$ in its first index, gives

$$B^{\mu\nu}P^{\lambda}_{\perp\nu} = 0\,. \tag{2.11}$$

Focus first on (2.11). Knowing that $B^{\mu\nu}$ is symmetric, we can use (2.8) to decompose it into orthogonal and parallel components with respect to its two indices,

$$B^{\mu\nu} = B^{\mu\nu}_{\perp} + B^{\mu}_{\perp}u^{\nu} + B^{\nu}_{\perp}u^{\mu} + Bu^{\mu}u^{\nu}\,,$$

where $B^{\mu\nu}_{\perp}$, $B^{\mu}_{\perp}$ and B are unknown coefficients, the first two being orthogonal to the curve $\mathcal{C}$ in all their indices. Substituting this expansion into (2.11), one finds that

$$B^{\mu\nu}_{\perp} = 0\,, \qquad B^{\mu}_{\perp} = 0\,,$$

leaving the scalar B as the only nonzero component of $B^{\mu\nu}$. Changing the notation from B to m, one obtains

$$B^{\mu\nu}(\tau) = m(\tau)u^{\mu}u^{\nu}\,. \tag{2.12}$$

This equation looks very similar to (2.6) but is still not equivalent to it, since the coefficient $m(\tau)$ is still not known to be a constant.

Next, focus on (2.10). Substituting (2.12), it reduces to

$$\nabla(mu^{\mu}) = 0\,. \tag{2.13}$$

Projecting onto the tangent direction u_{μ} and using the identity $u_{\mu}\nabla u^{\mu} = 0$, one obtains

$$\nabla m \equiv \frac{dm}{d\tau} = 0\,, \tag{2.14}$$

establishing that the parameter m is actually a constant. Given this, equation (2.13) reduces to

$$\nabla u^{\mu} = 0\,. \tag{2.15}$$

Remembering that $\nabla \equiv u^\lambda \nabla_\lambda$ and expanding the covariant derivative, we see that this is the geodesic equation (2.5). Finally, (2.14) and (2.12) together give (2.6), which completes the proof of statements (a) and (b).

There are three general remarks one should make regarding the above procedure. The first remark is about the physical interpretation and properties of the free parameter m. Namely, it can be given the interpretation of the total mass of the particle — substituting (2.6) into the stress-energy tensor (2.3) and integrating the T^{00} component over the volume of the spatial hypersurface orthogonal to u^μ, one can easily verify that the total rest-energy of the matter fields at a given time is equal to m. Note, however, that the sign of m is not fixed to be positive. This is not surprising, since the covariant conservation equation (2.4) and the stress-energy tensor (2.3) do not contain any information (or assumption) about the positivity of energy. Instead, the positive energy condition $m > 0$ has to be established from the full matter field equations, which take into account the internal dynamics of the matter fields that make up the particle.

The second remark is about the metric $g_{\mu\nu}$ of the background geometry. When discussing the motion of a particle, the background geometry is usually assumed to be fixed, and backreaction of the gravitational field of the particle itself is not taken into account, leading to the notion of a "test particle". However, ignoring the backreaction is not a necessary assumption. Namely, one can take the full stress-energy tensor of the matter fields which form the kink solution (as opposed to the approximate single pole stress-energy tensor (2.3)), put it as a source into the Einstein's field equations and solve for the metric $g_{\mu\nu}$. The resulting metric does include the backreaction, and can then be reinserted into the geodesic equation for the particle. Note that this procedure is self-consistent, since the geodesic motion of the particle is a consequence of the covariant conservation equation (2.4) which is in turn itself a consequence of Einstein's field equations. Also note that the metric $g_{\mu\nu}$ obtained in this way does not necessarily give rise to the black hole geometry in the neighborhood of the particle. This is because the Schwarzschild radius of the kink may be (and usually is) much smaller than the scale $\mathcal{O}_1$ which defines the precision of the single pole approximation (2.3). A simple example would be the motion of a planet around the Sun — in the single pole approximation, the radius of the planet (itself far larger than the planet's gravitational radius) is considered to be of the order $\mathcal{O}_1$ and the planet is treated as a pointlike object, but the spacetime metric used in the geodesic equation can still take into account the planet's gravitational field in addition to the field of the Sun.

The third remark is about going beyond the single pole approximation. This has been studied in detail in the literature [10–21, 25–28], so here we merely point out the main physical interpretation. Namely, keeping the second term in the multipole expansion (2.1) physically amounts to giving the particle a nonzero "thickness", in the sense that its internal angular momentum can be considered nonzero. In the resulting equation of motion for the particle, this angular momentum couples to the spacetime curvature tensor, giving rise to a deviation from the geodesic motion. This can intuitively be understood as an effect of tidal forces acting across the scale of the kink's width, pushing it off the geodesic trajectory. Similarly, including quadrupole and higher order terms in (2.1) takes into account additional internal degrees of freedom of the kink, which also couple to spacetime geometry and produce a further deviation from geodesic motion.

The above review of the multipole formalism, and its application to the derivation of the geodesic equation in GR, will be used in the next section to discuss the corrections to the motion of a particle stemming from quantum gravity. As we shall see, these quantum

JCAP09(2019)057

corrections will give rise to additional terms in the effective equation of motion for a particle, pushing it slightly off the geodesic trajectory, even in the single pole approximation.

3 Geodesic equation in quantum gravity

In this section we discuss the motion of a particle within the framework of quantum gravity. The exposition is structured into four parts — first, we introduce the abstract quantum gravity formalism, and give some technical details about the description of the states. In the second part, we discuss the quantum version of the covariant conservation equation of the stress-energy tensor. In the third part we adapt the derivation presented in section 2 to the quantum formalism, and obtain the effective equation of motion for the particle. Finally, in the fourth part we discuss the self-consistency assumptions that go into the calculation.

3.1 Preliminaries and the setup

We work in the so-called generic abstract quantum gravity setup, as follows. Starting from the Heisenberg picture for the description of quantum systems, we assume that gravitational degrees of freedom are described by some gravitational field operators $\hat{g}(x)$, while matter degrees of freedom are described by matter field operators $\hat{\phi}(x)$, where x represents the coordinates of some point on a 4-dimensional spacetime manifold $\mathcal{M}_4$. Both sets of operators have their corresponding canonically conjugate momentum operators, $\hat{\pi}_g(x)$ and $\hat{\pi}_\phi(x)$, such that the usual canonical commutation relations hold. The total (kinematical) Hilbert space of the theory is $\mathcal{H}_{\text{kin}} = \mathcal{H}_G \otimes \mathcal{H}_M$, where the gravitational and matter Hilbert spaces $\mathcal{H}_G$ and $\mathcal{H}_M$ are spanned by the bases of eigenvectors for the operators $\hat{g}$ and $\hat{\phi}$, respectively. The total state of the system, $|\Psi\rangle \in \mathcal{H}_{\text{kin}}$, does not depend on x, in line with the Heisenberg picture framework.

There are several important points that need to be emphasized regarding the above setup. First, we do not explicitly state what are the fundamental degrees of freedom $\hat{g}$ for the gravitational field. They can be chosen in many different ways, giving rise to different models of quantum gravity. Since we aim to present the analysis of geodesic motion which is model-independent, we refrain from specifying what are the fundamental degrees of freedom $\hat{g}$. Instead, we merely assume that the operators describing the spacetime geometry, i.e., the metric, connection, curvature, etc., depend somehow on $\hat{g}$ and $\hat{\pi}_g$, and are expressible as operator functions in terms of them:

$$\hat{g}_{\mu\nu} = \hat{g}_{\mu\nu}(\hat{g}, \hat{\pi}_g)\,, \qquad \hat{\Gamma}^\lambda{}_{\mu\nu} = \hat{\Gamma}^\lambda{}_{\mu\nu}(\hat{g}, \hat{\pi}_g)\,, \qquad \hat{R}^\lambda_{\mu\nu\rho} = \hat{R}^\lambda_{\mu\nu\rho}(\hat{g}, \hat{\pi}_g)\,, \qquad \dots$$

When discussing these geometric operators, for simplicity we will usually not explicitly write their $(\hat{g}, \hat{\pi}_g)$-dependence.

Second, in order for any operator function to be well defined, some operator ordering has to be assumed. However, since we aim to work in an abstract model-independent QG formalism, we do not choose any particular ordering, but merely assume that one such ordering has been fixed. In a similar fashion, we also simply assume that all operators and spaces are well defined, convergent, and otherwise specified in enough mathematical detail to have a well defined and unique QG model. In a nutshell, our calculations are formal, in the sense that one should be able to repeat them in a detailed fashion if one is given a specific model of QG. This also means that our analysis and results should not depend on any of these details, but are rather common to a large class of QG models, and are based only on very few assumptions given above.

JCAP09(2019)057

Third, we employ a natural distinction between gravitational and matter degrees of freedom. Namely, whereas geometric operators such as metric, curvature, and so on, depend only on $\hat{g}$ and $\hat{\pi}_g$, matter operators like field strengths, stress-energy tensor, etc., will generically be operator functions of both $\hat{g}$, $\hat{\pi}_g$, and the fundamental matter degrees of freedom $\hat{\phi}$ and $\hat{\pi}_\phi$. In other words, we assume that the separation between gravity and matter present in the classical theory, described by an action of the form

$$S_{\text{total}}[g,\phi] = S_{\text{gravity}}[g] + S_{\text{matter}}[g,\phi]\,,$$

remains present also in the full quantum regime. That is to say, we assume that one can construct a theory of quantum gravity without matter fields, using only gravitational degrees of freedom g, so that this theory gives sourceless Einstein's equations of GR in the classical limit. Once such a pure-QG model has been constructed, we assume one can couple matter ϕ to it without changing the structure of the gravitational sector, obtaining the full QG model which features Einstein's equations with appropriate matter sources in the classical limit. While we do not consider this to be a strong assumption, we feel that it is nevertheless important to spell it out explicitly, since there may exist some QG models which fail to satisfy it, and our analysis may be inapplicable to such models.

After the introduction of the above conceptual setup, we turn to some more practical details. For the purpose of discussing geodesic motion, we are mostly interested in the effective classical theory of the abstract QG introduced above. To that end, the main objects of attention are *classical states* of gravity and matter, denoted by $|\Psi\rangle \in \mathcal{H}_G \otimes \mathcal{H}_M$. By classical, we mean that the "effective classical" values for the metric tensor and the matter stress-energy tensor, given by the expectation values of the corresponding operators

$$g_{\mu\nu} = \langle\Psi|\hat{g}_{\mu\nu}|\Psi\rangle\,, \qquad T_{\mu\nu} = \langle\Psi|\hat{T}_{\mu\nu}|\Psi\rangle \tag{3.1}$$

satisfy classical Einstein equations of the GR. A recent study suggests that physical states of gravity and matter are generically entangled [1]. For our analysis, we do not need to assume that the overall gravity-matter state is separable, and thus we will work with a generic state $|\Psi\rangle$ (see appendix B for the discussion of the separable case).

For the purpose of our paper, we will consider a *toy example state*, defined as

$$|\boldsymbol{\Psi}\rangle = \alpha|\Psi\rangle + \beta|\tilde{\Psi}\rangle\,, \tag{3.2}$$

where $|\tilde{\Psi}\rangle$ is some other classical state analogous to $|\Psi\rangle$, but giving different expectation values for the classical metric and stress-energy tensors:

$$\tilde{g}_{\mu\nu} = \langle\tilde{\Psi}|\hat{g}_{\mu\nu}|\tilde{\Psi}\rangle\,, \qquad \tilde{T}_{\mu\nu} = \langle\tilde{\Psi}|\hat{T}_{\mu\nu}|\tilde{\Psi}\rangle\,. \tag{3.3}$$

One can see that our toy-example state (3.2) is a Schrödinger-cat type of state, describing a coherent superposition of two classical configurations of gravitational and matter fields. It will become evident later on that qualitative conclusions of the paper do not depend on the fact that (3.2) has precisely two terms in the sum. Choosing the state with three, four or more terms will lead to analogous conclusions, although quantitative details of the computation may become technically more involved.

Given that (3.2) is a Schrödinger-cat type of state, there are some phenomenological restrictions on the values of the independent parameters β and $S \equiv \langle\Psi|\tilde{\Psi}\rangle$. Namely, in the ordinary experimental situations we basically never observe this kind of states, which means that the overall entangled state $|\boldsymbol{\Psi}\rangle$ looks pretty much like a classical state, say the state $|\Psi\rangle$. In other words, we want the fidelity between these two states to be large, $F(|\Psi\rangle, |\boldsymbol{\Psi}\rangle) = |\langle\Psi|\boldsymbol{\Psi}\rangle| \approx 1$. From (3.2) we obtain

$$\langle\Psi|\boldsymbol{\Psi}\rangle = \alpha + \beta S \equiv \kappa\,.$$

Define $|\Psi^{\perp}\rangle$, such that

$$|\tilde{\Psi}\rangle = S|\Psi\rangle + \epsilon|\Psi^{\perp}\rangle\,,$$

$\langle\Psi|\Psi^{\perp}\rangle = 0$ and $\epsilon = \sqrt{1-|S|^2}$. Thus,

$$F^2 = |\langle\Psi|\boldsymbol{\Psi}\rangle|^2 = 1 - \eta^2\,, \tag{3.4}$$

where we introduce the small parameter

$$\eta \equiv \beta\epsilon\,.$$

Here, we have used the normalization condition for the entangled state (3.2),

$$\langle\boldsymbol{\Psi}|\boldsymbol{\Psi}\rangle = \alpha^2 + \beta^2 + 2\alpha\beta\,\mathrm{Re}(S) = 1\,.$$

Given the above definitions for κ and η, we can rewrite the total state (3.2) as

$$|\boldsymbol{\Psi}\rangle = \kappa|\Psi\rangle + \eta|\Psi^{\perp}\rangle\,, \tag{3.5}$$

where $|\kappa|^2 = F^2 = 1-\eta^2$. The physical requirement of large fidelity implies that we study the limit $|\kappa| \approx 1$, $\eta \to 0$. We will therefore systematically expand the expectation values of all operators into power series in η, up to order $\mathcal{O}(\eta^2)$.

At this point we can evaluate the expectation values for the metric and stress-energy operators in the state (3.5), obtaining

$$\boldsymbol{g}_{\mu\nu} = \langle\boldsymbol{\Psi}|\hat{g}_{\mu\nu}|\boldsymbol{\Psi}\rangle = (1-\eta^2)g_{\mu\nu} + \eta^2\langle\Psi^{\perp}|\hat{g}_{\mu\nu}|\Psi^{\perp}\rangle + 2\eta\,\mathrm{Re}\left(\kappa\langle\Psi^{\perp}|\hat{g}_{\mu\nu}|\Psi\rangle\right)\,, \tag{3.6}$$

$$\boldsymbol{T}_{\mu\nu} = \langle\boldsymbol{\Psi}|\hat{T}_{\mu\nu}|\boldsymbol{\Psi}\rangle = (1-\eta^2)T_{\mu\nu} + \eta^2\langle\Psi^{\perp}|\hat{T}_{\mu\nu}|\Psi^{\perp}\rangle + 2\eta\,\mathrm{Re}\left(\kappa\langle\Psi^{\perp}|\hat{T}_{\mu\nu}|\Psi\rangle\right)\,. \tag{3.7}$$

It is easy to see that interference terms from the above expressions are generically nonvanishing. Indeed, even if, say, $\kappa\langle\Psi^{\perp}|\hat{g}_{\mu\nu}|\Psi\rangle$ were purely imaginary, a simple change of relative phase between $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ would give rise to a nontrivial real part. Namely, given a fixed choice of $|\tilde{\Psi}\rangle$, the set of choices for $|\Psi\rangle$ for which the interference term is purely imaginary is of measure zero compared to the full set of possible phase shifts of $|\Psi\rangle$. An analogous argument applies for $\kappa\langle\Psi^{\perp}|\hat{T}_{\mu\nu}|\Psi\rangle$ as well. For a detailed analysis, see appendix C.

Let us denote the metric and stress-energy interference terms as $\bar{g}_{\mu\nu}$ and $\bar{T}_{\mu\nu}$, respectively. Since we want to expand (3.6) and (3.7) into power series in η up to linear order, we can write

$$\bar{g}_{\mu\nu} \equiv 2\,\mathrm{Re}\left(\kappa\langle\Psi^{\perp}|\hat{g}_{\mu\nu}|\Psi\rangle\right) = h_{\mu\nu} + \mathcal{O}(\eta)\,, \tag{3.8}$$

$$\bar{T}_{\mu\nu} \equiv 2\,\mathrm{Re}\left(\kappa\langle\Psi^{\perp}|\hat{T}_{\mu\nu}|\Psi\rangle\right) = t_{\mu\nu} + \mathcal{O}(\eta)\,. \tag{3.9}$$

Here, $h_{\mu\nu}$ and $t_{\mu\nu}$ are η-independent parts of $\bar{g}_{\mu\nu}$ and $\bar{T}_{\mu\nu}$. Thus, we can finally write:

$$\boldsymbol{g}_{\mu\nu} = g_{\mu\nu} + \eta\,h_{\mu\nu} + \mathcal{O}(\eta^2)\,, \tag{3.10}$$

$$\boldsymbol{T}_{\mu\nu} = T_{\mu\nu} + \eta\,t_{\mu\nu} + \mathcal{O}(\eta^2)\,. \tag{3.11}$$

JCAP09(2019)057

In what follows, we will refer to the classical state $|\Psi\rangle$ as the *dominant state*, while the other classical state $|\tilde{\Psi}\rangle$ will be called the *sub-dominant state*. To justify this terminology, recall the above requirement (3.4) that the overall entangled state $|\boldsymbol{\Psi}\rangle$ looks like the classical state $|\Psi\rangle$, i.e., $F^2 = 1 - \eta^2$, with the parameter $\eta \equiv \beta\epsilon$ being small. Therefore, in the case $\beta \to 0$ and ϵ finite, the state $|\tilde{\Psi}\rangle$ enters (3.2) with a very small contribution, and is thus sub-dominant. On the other hand, in the case when β is finite and $\epsilon \to 0$, the states $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ are essentially indistinguishable, and their roles can be exchanged, as either can be considered sub-dominant to the other. By convention, we choose $|\tilde{\Psi}\rangle$ to again play the role of the sub-dominant state.

While in any quantum theory entangled states are allowed, note that when considering a product state of the gravity-matter system (i.e., the case $\eta = 0$), there is a danger that such a state may fail to be gauge invariant, as argued in [1]. So we need to introduce at least a small sub-dominant state, in order to ensure the gauge invariance of the total state. The simplest possible candidate state which describes the classical physics sufficiently well, and simultaneously stands a chance of being gauge invariant, is the genuinely entangled state (3.2), with $\beta \neq 0$ and $|\tilde{\Psi}\rangle \neq |\Psi\rangle$, leading to η being very small, but nonzero.

Regarding the effective entangled metric and stress-energy tensors (3.10) and (3.11), it is important to stress that they do not satisfy classical Einstein's equations of GR. Namely, we assume that Einstein's equations are separately satisfied by the metric and stress-energy tensors (3.1) coming from the classical state $|\Psi\rangle$, and by the metric and stress-energy tensors (3.3) coming from the other classical state $|\tilde{\Psi}\rangle$, as two different classical solutions:

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) = 8\pi l_p^2\, T_{\mu\nu}\,, \qquad R_{\mu\nu}(\tilde{g}) - \frac{1}{2}\tilde{g}_{\mu\nu}R(\tilde{g}) = 8\pi l_p^2\, \tilde{T}_{\mu\nu}\,.$$

However, due to the nonlinearity of Einstein's equations, and due to the presence of the interference terms $h_{\mu\nu}$ and $t_{\mu\nu}$ in (3.10) and (3.11), quantities $\boldsymbol{g}_{\mu\nu}$ and $\boldsymbol{T}_{\mu\nu}$ do not satisfy Einstein's equations, as long as $\eta \neq 0$. This leads us to the following physical interpretation. First, it is natural to expand all quantities as corrections to the dominant classical configuration $(g_{\mu\nu}, T_{\mu\nu})$, including the equation of motion for a point particle. Second, as we shall see in the remainder of the text, given that $(\boldsymbol{g}_{\mu\nu}, \boldsymbol{T}_{\mu\nu})$ contains quantum gravity corrections through the interference terms, the presence of these quantum corrections in (3.10) and (3.11) will introduce an "effective force" term into the effective equation of motion for the particle. Finally, this effective force term will be pushing the particle off the geodesic trajectory defined by the classical dominant metric $g_{\mu\nu}$.

3.2 Effective covariant conservation equation

After the discussion of the general QG setup and the state (3.2), we move on to the discussion of the quantum analog of the covariant conservation equation (2.4). As in the classical theory, our basic assumption is that the matter sector of our QG model features local Poincaré invariance, i.e., that this symmetry is preserved at the quantum level. This assumption gives rise to a Gupta-Bleuler-like condition on the physical states, in the form

$$\langle \boldsymbol{\Psi} | \hat{\nabla}_\nu \hat{T}^{\mu\nu} | \boldsymbol{\Psi} \rangle = 0\,, \tag{3.12}$$

where $\hat{\nabla}_\mu$ is the covariant derivative operator, defined by promoting the metric appearing in the Christoffel symbols into a corresponding operator. In general, the action of the stress-energy operator on the state $|\boldsymbol{\Psi}\rangle$ can be written[1] as

$$\hat{T}^{\mu\nu}|\boldsymbol{\Psi}\rangle = \boldsymbol{T}^{\mu\nu}|\boldsymbol{\Psi}\rangle + \boldsymbol{\Delta T}^{\mu\nu}|\boldsymbol{\Psi}^{\perp}\rangle \,, \tag{3.13}$$

where $\boldsymbol{T}^{\mu\nu}$ and $\boldsymbol{\Delta T}^{\mu\nu}$ are the expectation value and the uncertainty of the operator $\hat{T}^{\mu\nu}$ in the state $|\boldsymbol{\Psi}\rangle$, respectively,

$$\boldsymbol{T}^{\mu\nu} \equiv \langle\boldsymbol{\Psi}|\hat{T}^{\mu\nu}|\boldsymbol{\Psi}\rangle \,, \qquad \boldsymbol{\Delta T}^{\mu\nu} \equiv \sqrt{\langle\boldsymbol{\Psi}|\left(\hat{T}^{\mu\nu}\right)^2|\boldsymbol{\Psi}\rangle - \left(\langle\boldsymbol{\Psi}|\hat{T}^{\mu\nu}|\boldsymbol{\Psi}\rangle\right)^2} \,,$$

while $|\boldsymbol{\Psi}^{\perp}\rangle$ is some state orthogonal to $|\boldsymbol{\Psi}\rangle$. Note that the equation of the form (3.13) is completely general, holding for any stress-energy operator acting on an arbitrary state. Substituting (3.13) into (3.12), we obtain

$$\boldsymbol{\nabla}_{\nu}\boldsymbol{T}^{\mu\nu} + \langle\boldsymbol{\Psi}|\hat{\nabla}_{\nu}|\boldsymbol{\Psi}^{\perp}\rangle\boldsymbol{\Delta T}^{\mu\nu} = 0 \,, \tag{3.14}$$

where $\boldsymbol{\nabla}_{\nu}$ is the expectation value of the operator $\hat{\nabla}_{\nu}$,

$$\boldsymbol{\nabla}_{\nu} \equiv \langle\boldsymbol{\Psi}|\hat{\nabla}_{\nu}|\boldsymbol{\Psi}\rangle \,.$$

At this point we need to make one more assumption. Namely, we assume that the error scale of the single pole approximation is bigger than the uncertainty of the stress-energy operator, $\boldsymbol{\Delta T}^{\mu\nu}$. Symbolically,

$$\mathcal{O}_1 \gtrsim \boldsymbol{\Delta T}^{\mu\nu} \,. \tag{3.15}$$

This means that in the single pole approximation we do not see the effects of the quantum fluctuations of matter fields. Intuitively, this is a reasonable assumption in most cases. For example, in the case of the kink solution describing the hydrogen atom, the scale on which one can detect quantum fluctuations (i.e., the Lamb shift effects) is much smaller than the size of the atom itself (i.e., the radius of the first Bohr orbit). Therefore, we expect that if our single pole approximation ignores the size of the atom itself, it also ignores the corresponding quantum fluctuations. An analogous assumption is made in relation to the uncertainty of the metric operator $\hat{g}_{\mu\nu}$,

$$\mathcal{O}_1 \gtrsim \boldsymbol{\Delta g}_{\mu\nu} \,, \tag{3.16}$$

given that the quantum gravity fluctuations can arguably also be ignored in the single pole approximation.

Applying (3.15) to (3.14), in the single pole approximation the second term can be dropped, leading to the effective classical covariant conservation equation,

$$\boldsymbol{\nabla}_{\nu}\boldsymbol{T}^{\mu\nu} = 0 \,. \tag{3.17}$$

In a similar fashion, one can employ (3.16) to drop the off-diagonal components in the Christoffel symbol operators, leading to an effective classical expression

$$\boldsymbol{\Gamma}^{\lambda}{}_{\mu\nu} = \frac{1}{2}\boldsymbol{g}^{\lambda\sigma}\left(\partial_{\mu}\boldsymbol{g}_{\sigma\nu} + \partial_{\nu}\boldsymbol{g}_{\sigma\mu} - \partial_{\sigma}\boldsymbol{g}_{\mu\nu}\right) \,, \tag{3.18}$$

where $\boldsymbol{g}_{\mu\nu} \equiv \langle\boldsymbol{\Psi}|\hat{g}_{\mu\nu}|\boldsymbol{\Psi}\rangle$ is the effective classical metric and $\boldsymbol{g}^{\mu\nu}$ is its inverse matrix.

[1] Given any self-adjoint operator $\hat{A}$ and any state $|\Psi\rangle$, one can write

$$\hat{A}|\Psi\rangle = a|\Psi\rangle + b|\Psi^{\perp}\rangle \,,$$

where $\langle\Psi|\Psi^{\perp}\rangle \equiv 0$ and $a, b \in \mathbb{C}$. Multiplying this equation by $\langle\Psi|$ and by $\langle\Psi|\hat{A}$ from the left, one easily obtains that a and b are the expectation value and the uncertainty of the operator $\hat{A}$ in the state $|\Psi\rangle$, respectively.

With effective classical expressions (3.17) and (3.18) in hand, we can now employ (3.10) and (3.11) to expand them into the dominant and correction parts. First we use (3.10) and $\boldsymbol{g}_{\mu\lambda}\boldsymbol{g}^{\lambda\nu} = \delta^\nu_\mu$ to find the inverse entangled metric $\boldsymbol{g}^{\mu\nu} = g^{\mu\nu} - \eta g^{\mu\rho} g^{\nu\sigma} h_{\rho\sigma} + \mathcal{O}(\eta^2)$, and then substitute into (3.18) to obtain

$$\boldsymbol{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + \frac{\eta}{2} g^{\lambda\sigma} \left(\nabla_\mu h_{\sigma\nu} + \nabla_\nu h_{\sigma\mu} - \nabla_\sigma h_{\mu\nu} \right) + \mathcal{O}(\eta^2) \,, \tag{3.19}$$

where the Christoffel symbols in ordinary ∇_μ are defined with respect to the dominant classical metric $g_{\mu\nu}$. Then, expanding (3.17) into the form

$$\partial_\nu \boldsymbol{T}^{\mu\nu} + \boldsymbol{\Gamma}^\mu{}_{\sigma\nu} \boldsymbol{T}^{\sigma\nu} + \boldsymbol{\Gamma}^\nu{}_{\sigma\nu} \boldsymbol{T}^{\mu\sigma} = 0 \,,$$

we substitute (3.11) and (3.19), and after a bit of algebra we rewrite it as:

$$\nabla_\nu T^{\mu\nu} + \eta \left[\nabla_\nu t^{\mu\nu} + T^{\sigma\nu} \left(\nabla_\sigma h^\mu{}_\nu - \frac{1}{2} \nabla^\mu h_{\nu\sigma} \right) + \frac{1}{2} T^{\mu\sigma} \nabla_\sigma h^\nu{}_\nu \right] + \mathcal{O}(\eta^2) = 0 \,. \tag{3.20}$$

This equation is the one we sought out — it represents the analog of the classical covariant conservation equation (2.4), while taking into account the interference terms between the two classical states in (3.2), approximated to the linear order in η.

As a final step, (3.20) can be rewritten in a more compact form. For convenience, introduce the following shorthand notation (see our conventions from the last paragraph of the Introduction),

$$F^\mu{}_{\nu\sigma} \equiv \nabla_{(\sigma} h^\mu{}_{\nu)} - \frac{1}{2} \nabla^\mu h_{\nu\sigma} \,, \tag{3.21}$$

and also note that

$$F^\nu{}_{\nu\sigma} = \frac{1}{2} \nabla_\sigma h^\nu{}_\nu + \frac{1}{2} \nabla_\nu h^\nu{}_\sigma - \frac{1}{2} \nabla^\nu h_{\nu\sigma} = \frac{1}{2} \nabla_\sigma h^\nu{}_\nu \,,$$

so that, dropping the term $\mathcal{O}(\eta^2)$, equation (3.20) is rewritten as:

$$\nabla_\nu \left(T^{\mu\nu} + \eta t^{\mu\nu} \right) + 2\eta F^{(\mu}{}_{\nu\sigma} T^{\nu)\sigma} = 0 \,. \tag{3.22}$$

The equation (3.22) represents the effective classical covariant conservation law for the stress-energy tensor, with the included quantum correction, represented to first order in η. It is the starting point for the remainder of our analysis, and replaces equation (2.4) in the derivation of the equation of motion for a point particle.

Finally, note that the quantum correction term in (3.22) has two distinct parts — one part comes from the quantum correction to the dominant classical stress-energy tensor, i.e., the interference term $t^{\mu\nu}$, while the other part comes from the quantum correction to the dominant classical metric, i.e., the interference term $h_{\mu\nu}$. This latter quantum correction enters through the Christoffel connection terms present in the covariant derivative. As we shall see in the next subsection, its presence will be crucial for the "force term" in the equation of motion for the particle, responsible for the deviation from the classical geodesic trajectory.

3.3 Effective equation of motion

We are now ready to derive the equation of motion for a particle in the single pole approximation, using the technique presented in section 2. However, instead of (2.4), we start from the effective covariant conservation law (3.22), which contains the quantum correction terms. Throughout, we assume the following relation of scales,

$$\mathcal{O}(\eta) > \mathcal{O}_1 \geq \mathcal{O}(\eta^2)\,.$$

In other words, we assume that the quantum correction terms linear in η are not smaller than the width of our particle, since otherwise one could simply ignore them and recover the classical geodesic motion for the particle.

Repeating the method of section 2, we begin by contracting (3.22) with an arbitrary test function $f_\mu(x)$ of compact support, and integrating over the whole spacetime,

$$\int_{\mathcal{M}_4} d^4x\sqrt{-g}\, f_\mu \Big[\nabla_\nu \left(T^{\mu\nu} + \eta t^{\mu\nu}\right) + 2\eta F^{(\mu}{}_{\nu\sigma} T^{\nu)\sigma}\Big] = 0\,.$$

We then perform the partial integration to move the covariant derivative from the stress-energy tensors to the test function. As before, the boundary term vanishes since the test function has compact support, giving

$$\int_{\mathcal{M}_4} d^4x\sqrt{-g}\,\Big[-\left(T^{\mu\nu} + \eta t^{\mu\nu}\right)\nabla_\nu f_\mu + 2\eta F^{(\mu}{}_{\nu\sigma} T^{\nu)\sigma} f_\mu\Big] = 0\,. \tag{3.23}$$

Now we need to model the dominant and correction parts of the stress-energy tensor. For the dominant part, it is straightforward to assume the single pole approximation, as was done in the classical case:

$$T^{\mu\nu}(x) = \int_{\mathcal{C}} d\tau B^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}}\,. \tag{3.24}$$

Regarding the correction term, we also use the single pole approximation,

$$t^{\mu\nu}(x) = \int_{\mathcal{C}} d\tau \bar{B}^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}}\,, \tag{3.25}$$

but one should note that in the case of $t^{\mu\nu}$ it is less obvious why this approximation is adequate, and requires some justification. However, in order to focus on the derivation of the particle equation of motion, for the moment we simply adopt (3.25), and postpone the analysis and the meaning of this approximation for subsection 3.4.

Then we substitute (3.24) and (3.25) into (3.23), switch the order of integrations and perform the integral over spacetime $\mathcal{M}_4$, ending up with

$$\int_{\mathcal{C}} d\tau \Big[-\left(B^{\mu\nu} + \eta\bar{B}^{\mu\nu}\right)\nabla_\nu f_\mu + 2\eta F^{(\mu}{}_{\nu\sigma} B^{\nu)\sigma} f_\mu\Big] = 0\,. \tag{3.26}$$

The next step is to employ the identity (2.9) to separate the tangential and orthogonal components of the derivative of the test function. Substituting it into (3.26), and performing another partial integration, we find

$$\int_{\mathcal{C}} d\tau \Big[\left(B^{\mu\nu} + \eta\bar{B}^{\mu\nu}\right) f^{\perp}_{\nu\mu} + \Big[\nabla\left(B^{\mu\nu}u_\nu + \eta\bar{B}^{\mu\nu}u_\nu\right) - 2\eta F^{(\mu}{}_{\nu\sigma} B^{\nu)\sigma}\Big] f_\mu\Big] = 0\,,$$

where the boundary term again vanishes due to the compact support of the test function.

After these transformations, we make use of the same argument that both f_μ and $f^\perp_{\nu\mu}$ are arbitrary and mutually independent along the curve $\mathcal{C}$, concluding that the coefficients multiplying them must each be zero. The first term gives us

$$\nabla\left(B^{\mu\nu}u_\nu + \eta\bar{B}^{\mu\nu}u_\nu\right) - 2\eta F^{(\mu}{}_{\nu\sigma}B^{\nu)\sigma} = 0\,, \tag{3.27}$$

while the second term, knowing that $f^\perp_{\nu\mu}$ is orthogonal to the curve $\mathcal{C}$ in its first index, gives

$$\left(B^{\mu\nu} + \eta\bar{B}^{\mu\nu}\right)P^\lambda_{\perp\nu} = 0\,.$$

As in the previous case, given that $B^{\mu\nu}$ and $\bar{B}^{\mu\nu}$ are symmetric, one can decompose them into tangential and orthogonal components using (2.8), and then from (2.11) read off that all orthogonal components must be zero, concluding that

$$B^{\mu\nu} + \eta\bar{B}^{\mu\nu} = (B + \eta\bar{B})u^\mu u^\nu \equiv m(\tau)u^\mu u^\nu\,, \tag{3.28}$$

where again we emphasized that the parameter m may depend on the particle's proper time τ.

Next, substituting this into (3.27) and neglecting the term $\mathcal{O}(\eta^2)$, we obtain

$$\nabla\left(mu^\mu\right) + \eta m u^\sigma\left(F^\mu{}_{\nu\sigma}u^\nu + F^\nu{}_{\nu\sigma}u^\mu\right) = 0\,. \tag{3.29}$$

Projecting onto the tangent direction u_μ and using the identity $u_\mu\nabla u^\mu = 0$, one obtains

$$\nabla m = \eta m u^\sigma\left(u^\nu u_\lambda F^\lambda{}_{\nu\sigma} - F^\nu{}_{\nu\sigma}\right)\,, \tag{3.30}$$

establishing that, in contrast to the classical case, here the parameter m fails to be constant. Substituting this back into the equation (3.29), after some simple algebra we obtain

$$\nabla u^\mu + \eta u^\nu u^\sigma P^\mu_{\perp\lambda}F^\lambda{}_{\nu\sigma} = 0\,,$$

where the parameter m cancels out of the equation. As a final step, introducing the shorthand notation $F^\mu_{\perp\nu\sigma} \equiv P^\mu_{\perp\lambda}F^\lambda{}_{\nu\sigma}$, we can rewrite the equation of motion in its final form

$$\nabla u^\mu + \eta u^\nu u^\sigma F^\mu_{\perp\nu\sigma} = 0\,. \tag{3.31}$$

The presence of the orthogonal projector in the second term should not be surprising. Namely, since acceleration must always be orthogonal to the velocity, the second term in the equation must also be orthogonal to velocity, and this is guaranteed by the presence of the orthogonal projector.

Equations (3.28), (3.30) and (3.31) are the main result of this paper, and we discuss them in turn. Equation (3.28) determines the structure of the stress-energy tensor describing the point particle, as a function of tangent vectors of its world line and a scalar parameter $m(\tau)$. Formally, it has the same form as its classical counterpart (2.12), and provisionally the parameter m may be even called *effective mass*. Namely, in the rest frame of the particle, integration of the $\boldsymbol{T}^{00}$ component of the entangled stress-energy tensor over the 3-dimensional spatial hypersurface can be interpreted as the total rest-energy of the kink configuration of fields that represents the particle. This terminology is of course provisional, since all these notions are merely a part of the semiclassical approximation of the full quantum gravity description.

JCAP09(2019)057

Equation (3.30) determines the proper time evolution of the parameter $m(\tau)$. In contrast to the classical case, where $m(\tau)$ was determined by (2.14) to be a constant, here we see that its time derivative is proportional to (covariant derivatives of) the interference term $h_{\mu\nu}$ between the dominant and sub-dominant classical geometry, via (3.21). If one puts $\eta = 0$, (3.30) reduces to the classical case, as expected. The interference between the two geometries gives rise to an effective force that is responsible for the change in time of the particle's effective mass. Since the particle is (effectively) not isolated, its total energy is therefore not conserved, in the sense of equation (3.30).

Finally, and most importantly, equation (3.31) represents the effective equation of motion of the particle, determining its world line. It has the form of the classical geodesic equation (2.15) with an additional correction term proportional to η and to the interference term $h_{\mu\nu}$. This additional term represents an *effective force*, pushing the particle off the classical geodesic trajectory. It is analogous to the notion of the "exchange interaction" force in molecular physics, in the region where the wavefunctions of the two electrons in a molecule overlap.

In our case, however, the force term is determined by the interference between the two classical spacetime and matter configurations superposed in the state (3.2), and in particular by the off-diagonal components of the metric operator $\hat{g}_{\mu\nu}$, see (3.8). It is thus a *pure quantum gravity effect*, a consequence of the nontrivial structure of the metric operator. Of course, the detailed properties and the magnitude of the force term depend on the choice of the two classical gravity-matter configurations and on the details of the quantization of the gravitational field.

3.4 Consistency of the approximation scheme

Regarding the analysis and the derivation of the effective equation of motion for a particle discussed in the previous subsection, there is one issue that we should reflect on. It is related to the additional consistency conditions that stem from our assumption that the quantum correction to the stress-energy tensor is approximated with a single pole term (3.25).

Namely, the two stress-energy tensors that enter the derivation of the effective equation of motion — the classical dominant stress-energy tensor $T^{\mu\nu}$, and the interference stress-energy tensor $t^{\mu\nu}$ — can in general be written in the single pole approximation as:

$$T^{\mu\nu} = \int_{\mathcal{C}} d\tau \, B^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} + \mathcal{O}_1(T) \,, \tag{3.32}$$

$$t^{\mu\nu} = \int_{\mathcal{C}} d\tau \, \bar{B}^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} + \mathcal{O}_1(t) \,. \tag{3.33}$$

Note that we have introduced two different $\mathcal{O}_1$ scales, one for each tensor. This is because, although we assume that both can be expanded into the δ series around the same curve $\mathcal{C}$, each tensor may have different "width", or in other words, the two configurations of matter fields may be such that they can be well approximated with a single pole term up to a priori two different $\mathcal{O}_1$ scales. In particular, if one chooses the $\mathcal{O}_1$ scale to write $T^{\mu\nu}$ in a single pole approximation, $\mathcal{O}_1 = \mathcal{O}_1(T)$, it is not obvious that $t^{\mu\nu}$ can also be approximated by a single pole term, compared to the same scale, and vice versa. Therefore, there is an assumption about the relationship between scales that we have made when we used expressions (3.24) and (3.25) in the derivation of the effective equation of motion.

Looking at the structure of the equation (3.23) into which (3.24) and (3.25) have been substituted, the consistency condition for the approximation scheme can be written as

$$\mathcal{O}_1 \equiv \mathcal{O}_1(T) \geq \eta \mathcal{O}_1(t) \,. \tag{3.34}$$

In particular, if this inequality were not valid, the dipole term in (3.33) would contribute to (3.23) with a magnitude comparable to the single pole term of (3.32), and it would be inconsistent to ignore it in the derivation of the effective equation of motion.

The consistency condition (3.34) can be rewritten into a more explicit form. Substituting (3.33) and (3.9) into (3.34), we get

$$\mathcal{O}_1(T) \geq \eta \left[2\,\mathrm{Re}\left(\langle \Psi^\perp | \hat{T}_{\mu\nu} | \Psi \rangle \right) - \int_{\mathcal{C}} d\tau\, \bar{B}^{\mu\nu}(\tau) \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \right] .$$

In addition, one can use (3.28) and (3.32) to eliminate the coefficient $\bar{B}^{\mu\nu}$ in favor of $T^{\mu\nu}$ and $m(\tau)$, which are arguably more observable, obtaining

$$\mathcal{O}_1(T) \geq 2\eta\,\mathrm{Re}\left(\langle \Psi^\perp | \hat{T}_{\mu\nu} | \Psi \rangle \right) + T^{\mu\nu} - \int_{\mathcal{C}} d\tau\, m(\tau) u^\mu u^\nu \frac{\delta^{(4)}(x - z(\tau))}{\sqrt{-g}} \,. \tag{3.35}$$

This inequality should be interpreted as follows. Given an explicit model of quantum gravity, and within it an explicit configuration of matter fields that make up a particle, one can estimate all three quantities on the right-hand side of (3.35), namely the off-diagonal components of the stress-energy operator, its expectation value in the dominant classical state, and the total mass of the particle, respectively. Then, the consistency condition (3.35) gives a lower bound on the scale $\mathcal{O}_1$, which represents an estimate of the error when discussing the effective equation of motion for the particle. In other words, the equation of motion can be considered to be approximately valid only across scales much larger than the $\mathcal{O}_1$ scale, bounded from below by inequality (3.35).

Finally, if one needs better precision than the scale determined by (3.35), one should take into account the dipole term in (3.33) and rederive a more precise form of the equation of motion. Still better precision would be obtained by including the dipole term in (3.32), which would amount to the equation of motion in the full pole-dipole approximation, and so on.

4 Status of the weak equivalence principle

In light of the results of section 3, it is important to discuss the status of the equivalence principle (EP). Throughout the literature, one can find various different formulations of EP, in various flavors such as weak, medium-strong, strong, and so on (see [35, 36] for a review, and [2–5, 37, 38] for various examples). Often these formulations and flavors are interpretation-dependent, and it is not always clear whether they are mutually equivalent or not, and what are the underlying assumptions and definitions used to express them.

Needless to say, such situation is less than satisfactory [35, 36], and in order to circumvent it, in this section we opt to specify one particular definition of the weak and strong equivalence principles (WEP and SEP, respectively) and to use this definition in the rest of the text. We do not aspire to claim that our definition is either equivalent to, or in any sense better than, other definitions present in the literature, and may not even correspond to the usual usage of the terminology. But for the purpose of clarity, it is prudent to fix one definition and stick to it. Therefore, in light of the results obtained in section 3, in this section we discuss the status of WEP defined as below.

4.1 Definition and flavors of the equivalence principle

The purpose of the equivalence principle is to prescribe the coupling of matter to gravity [39]. Its precise formulation therefore depends on the particular choice of the gravitational and matter degrees of freedom which one uses to describe matter and gravity. For the purpose of this paper, we assume that the classical limit of quantum gravity corresponds to general relativity, which means that in this limit the fundamental gravitational degrees of freedom give rise to a nonflat spacetime metric. Given any choice of the gravitational degrees of freedom that belong to this class, in the classical framework one can formulate the equivalence principle as a two-step recipe to couple matter to gravity (we will discuss the quantum framework in subsection 4.2).

Start from the classical equation of motion for matter degrees of freedom in flat spacetime, written symbolically as

$$\mathcal{D}_{\text{flat}}[\phi, \eta_{\mu\nu}] = 0\,, \tag{4.1}$$

where ϕ denotes the matter degrees of freedom, $\eta_{\mu\nu}$ is the Minkowski metric, while $\mathcal{D}_{\text{flat}}$ is an appropriate functional describing the equation of motion for ϕ in flat spacetime and is assumed to be local. Given this equation of motion, couple it to gravity as follows:

1. Rewrite the equation of motion in a manifestly diffeomorphism-invariant form, typically by a change of variables to a generic curvilinear coordinate system,

$$\mathcal{D}_{\text{curvilinear}}[\phi, g^{(0)}_{\mu\nu}] = 0\,,$$

where $g^{(0)}_{\mu\nu}$ is still the flat spacetime metric, appropriately transformed from $\eta_{\mu\nu}$, and similarly for $\mathcal{D}_{\text{curvilinear}}$.

2. Promote the curvilinear equation of motion to the equation of motion in curved spacetime by replacing the flat spacetime metric $g^{(0)}_{\mu\nu}$ with an arbitrary metric $g_{\mu\nu}$,

$$\mathcal{D}_{\text{curvilinear}}[\phi, g_{\mu\nu}] = 0\,,$$

thereby specifying the equation of motion for matter coupled to gravity.

The first step describes the matter equation of motion from a perspective of a generic curvilinear (or "arbitrarily accelerated") coordinate system, reflecting the principle of *general relativity*. The second step simply promotes that same equation to curved spacetime as it stands, with no additional coupling of any kind. This can be loosely formulated as a statement of *local equivalence between gravity and acceleration*, which is how the EP historically got its name. Also, note that these two steps operationally correspond to the standard *minimal coupling* prescription [39].

It is important to stress the *local* nature of EP, which manifests itself in the assumption that the initial equation of motion (4.1) is local, and that the EP essentially does not change it at all, at any given point in spacetime. This has one important implication — the gravitational degrees of freedom manifest themselves only through *nonlocal measurements*, as tidal effects induced by spacetime curvature. We will return to this point and comment more on it later in the text.

Depending on the further specification of the matter degrees of freedom, one can distinguish between various flavors of the EP. For example, if one talks about the mechanics of point particles, one can start from the Newton's first law of motion, which states that

JCAP09(2019)057

in the absence of any forces, a particle has a straight-line trajectory in Minkowski spacetime. According to the step 1 above, the differential equation for a straight line in a generic curvilinear coordinate system is the geodesic equation,

$$\frac{d^2 z^\lambda}{d\tau^2} + \Gamma^\lambda_{(0)\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\mu}{d\tau} = 0\,,$$

where the index (0) on the Christoffel symbol indicates that it is calculated using the metric $g^{(0)}_{\mu\nu}$, which is obtained by a curvilinear coordinate transformation from the Minkowski metric $\eta_{\mu\nu}$. Then, according to step 2, one again writes the same equation, only dropping the requirement of flat spacetime metric,

$$\frac{d^2 z^\lambda}{d\tau^2} + \Gamma^\lambda{}_{\mu\nu} \frac{dz^\mu}{d\tau} \frac{dz^\mu}{d\tau} = 0\,,$$

so that this time the Christoffel symbol is calculated using an arbitrary metric $g_{\mu\nu}$, and now encodes the interaction with the gravitational degrees of freedom. So one starts from the Newton's first law of motion for a particle in the absence of the gravitational field, and ends up with a geodesic equation in the presence of the gravitational field. We define this flavor of the EP as the *weak equivalence principle* (WEP).

Instead of mechanical particles, one can study matter degrees of freedom described by a field theory. For example, if one starts from the equation of motion for a single real scalar field,

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu - m)\,\phi = 0\,,$$

according to the step 1 of the EP, one can rewrite it in a general curvilinear coordinate system as

$$\left(g^{\mu\nu}_{(0)}\nabla_\mu\nabla_\nu - m\right)\phi = 0\,,$$

where the Christoffel symbol inside the covariant derivative is again calculated using the flat-space metric $g^{(0)}_{\mu\nu}$. Then, according to step 2 of the EP, this equation is promoted to curved spacetime as it stands, leading to

$$(g^{\mu\nu}\nabla_\mu\nabla_\nu - m)\,\phi = 0\,,$$

where now the covariant derivative is given with respect to an arbitrary metric $g_{\mu\nu}$ describing curved spacetime. Thus one arrives to the equation of motion for a scalar field coupled to gravity. We define this flavor of the EP as the *strong equivalence principle* (SEP).

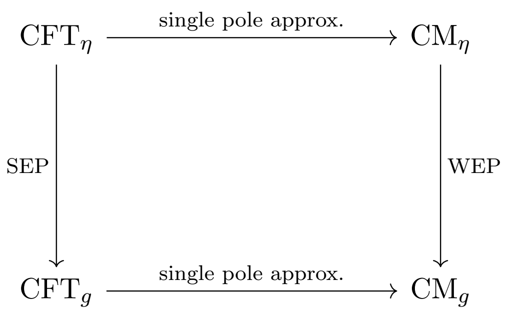
So in short, WEP is a statement about mechanical systems such as particles and small bodies, while SEP is a statement about fields. We emphasize again that the above definitions may or may not correspond to what is known in other literature as WEP and SEP, depending on the particular source one compares our definitions to. For example, one can often find a definition of WEP as a statement about equality of inertial and gravitational masses. As another example, one can also find a definition of WEP as Galileo's statement that the acceleration of a particle due to the gravitational field is independent of the particle's internal details such as mass or chemical composition, a property also called *universality*, emphasizing the fact that gravitation interacts with all types of particles in the same way. For an excellent review of the various formulations and flavors of EP present in the literature, see [35].

In relation to these alternative formulations of WEP, one should note two comments. First, while the notion of "gravitational mass" may be useful in the context of Newtonian theory, in frameworks such as GR it is not useful, since the source in Einstein's equations is the whole stress-energy tensor, rather than any particular mass-like parameter. This renders any definition of WEP which relies on the notion of the gravitational mass unsuitable for analysis in a fundamental QG framework. Second, one can argue (see for example [35]) that the property of universality is implicitly present even without gravitational interaction, in the Newton's first law of motion. Namely, the first Newton's law can be formulated more precisely as follows: in the absence of any forces, a particle has a straight-line trajectory in Minkowski spacetime, *regardless of its internal details such as mass or chemical composition*. The Newton's first law is never spelled out in this way in textbooks, making room for a point of view that universality has something to do with gravity or the EP. However, if one accepts our definition of WEP given above, it is more natural to say that universality is a property of Newtonian mechanics, and is merely *being preserved* by the WEP when one lifts the straight-line equation of motion to curved spacetime. So from this point of view, one should arguably say that WEP is merely *compatible* with universality, rather than equivalent to it.

Given all these reasons, and despite the fact that these alternative definitions of WEP may be suitable in various other contexts, they are not quite adequate for the analysis given in this paper. We therefore choose to retain our own definition of WEP, while the principles of universality and equality between inertial and gravitational mass will be called as such. They are discussed in more detail in subsection 4.3.

4.2 Equivalence principle and quantum theory

Adopting the above definitions of WEP and SEP, it is important to discuss their relationship. From the perspective of the classical field theory (CFT), the notion of a particle can be introduced as a localized kink-like configuration of matter fields, described as a solution of the (usually quite complicated) matter field equations. One can then employ the apparatus of multipole formalism and describe the evolution of this kink configuration in the single pole approximation, as was discussed in section 2. Using this method, one can recover the equation of motion for a particle in classical mechanics (CM) as an approximation of the field theory. Moreover, all this can be done before or after the application of the EP, leading to the following diagram:



Here the indices η and g indicate that equations of motion in a given theory are written in flat and in curved spacetime, respectively.

The question whether this diagram commutes is nontrivial. Namely, on one hand, one can start from a flat-space classical field theory, approximate it to derive the equations of motion for a particle in flat-space classical mechanics, and then invoke WEP to reach classical mechanics coupled to gravity. On the other hand, one can first invoke SEP to couple matter to gravity at the field theory level, and then approximate it to derive the equation of motion for a particle in curved spacetime. A priori, there is no guarantee that one will reach the same equation of motion for a particle in curved spacetime using both methods.

It is in fact the existence of the local Poincaré symmetry that leads to the commutativity of the diagram. Namely, as was discussed in section 2, in the curved spacetime local Poincaré symmetry gives rise to the covariant conservation equation for the stress-energy tensor of matter fields, and this is all one needs to reach the geodesic equation as an equation of motion for the particle, in the sense that one does not need to know the full matter field equations in curved spacetime. This establishes the ⟨SEP → single pole⟩ path of the diagram. On the other hand, in flat spacetime one can also perform the calculation of section 2, this time using the ordinary (noncovariant) conservation equation for the stress-energy tensor, which is a consequence not of the local, but rather of the global Poincaré invariance of Minkowski spacetime. Repeating the calculation of section 2 with the symbolic substitutions $g \to \eta$ and $\nabla \to \partial$, it is not hard to conclude that one will obtain the equation of motion for a straight line in flat spacetime, again without knowing all details of the full matter field equations in flat spacetime. Then, applying WEP as discussed in subsection 4.1, one reaches the geodesic equation in curved spacetime. This establishes the ⟨single pole → WEP⟩ path of the diagram, concluding that the resulting equation of motion for the particle is the same in both cases, i.e., that the diagram commutes.

Let us also note that, going beyond the single pole approximation, WEP is known to be violated, with SEP remaining valid. For example, in the pole-dipole approximation, it is well known that the analogous diagram

$$
\begin{array}{ccc}
\mathrm{CFT}_\eta & \xrightarrow{\text{pole-dipole approx.}} & \mathrm{CM}_\eta \\
\Big\downarrow{\scriptstyle \mathrm{SEP}} & & \Big\downarrow{\scriptstyle \mathrm{WEP}} \\
\mathrm{CFT}_g & \xrightarrow{\text{pole-dipole approx.}} & \mathrm{CM}_g
\end{array}
$$

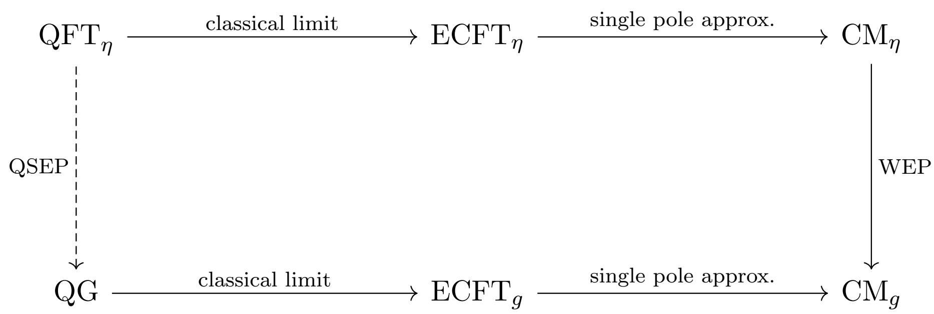
fails to commute. Namely, the ⟨SEP → pole-dipole⟩ path leads to an effective equation of motion for the particle in which there is an explicit coupling of the particle's total angular momentum to the spacetime curvature [10]. On the other hand, the ⟨pole-dipole → WEP⟩ path produces the equation of motion without the curvature term. Thus, in the pole-dipole approximation, WEP fails to reproduce the correct equation of motion, since the particle is coupled to gravity in a nonminimal way, in spite of the fact that the fields which make up the particle are still minimally coupled to gravity, in line with SEP. Of course, this situation is to be expected, given that in the pole-dipole approximation the particle is no longer completely pointlike, and the coupling of the angular momentum to the curvature can be understood as a tidal effect of gravity across the "width" of the particle. On the other hand, one can instead argue that it would be wrong to apply WEP to the pole-dipole equation of motion for a particle. Namely, despite the fact that the latter is formally still local, it describes an object that is "less-than-perfectly pointlike", in the sense that its stress-energy tensor is proportional not only to a δ function but also to its derivative. From that point of view, one should not be allowed to apply the two-step prescription of EP defined above. Either way, the bottom line is that one can either declare WEP as violated or as inapplicable beyond the single pole

JCAP09(2019)057

approximation, but it cannot be declared as valid. This results in the noncommutativity of the above diagram.

Let us now turn to the quantum theory. Starting first from some quantum field theory (QFT_η) which describes the fundamental matter fields in Minkowski spacetime, one can take its classical limit, giving rise to some effective classical field theory ($ECFT_\eta$). Then, assuming that the latter features kink solutions, one can describe those using the single pole approximation, leading to classical mechanics (CM_η) of the corresponding particles. Finally, applying WEP one couples those particles to gravity. The resulting equation of motion will always be a geodesic equation, assuming that the initial QFT and all subsequent approximations respect the global Poincaré invariance of Minkowski spacetime. This symmetry guarantees the conservation of the stress-energy tensor of the matter fields throughout the sequence of approximations, leading invariably to the geodesic equation of motion for the particle.

On the other hand, it is arguably more appropriate to take an alternative, more fundamental route — start from some fundamental quantum gravity (QG) model, and take the classical limit leading to some effective classical field theory ($ECFT_g$) for both matter and gravity. Then, again assuming that this theory features kink solutions, employ the single pole approximation to obtain the classical mechanics for the particle in the gravitational field (CM_g). Note that this is in fact precisely the program that was performed in section 3, leading to the non-geodesic equation of motion (3.31) for the particle. In effect, one can conclude that the following diagram fails to commute:



As a side comment, note that the dashed QSEP arrow represents some hypothetical map leading from a QFT in Minkowski spacetime to a full-blown model of QG, according to a notion that might be called a "quantum strong equivalence principle". It is unclear whether such a principle exists or not, let alone what its formulation is supposed to be, even if one is given precisely defined models of QFT and QG in question. We introduce it here simply for completeness, speculating that such a notion should exist, as a generalization of SEP from classical to quantum physics. It is also convenient to introduce it, in order to close the diagram and discuss its commutativity.

It is important to stress the reason why this diagram does not commute. Recalling the details of section 3, the local Poincaré symmetry is assumed to be respected at the fundamental level of QG and onwards, just like in the classical case. Moreover, the single pole approximation is used, avoiding any nonminimal coupling of the tidal forces that may be present. And yet, in spite of all that, the resulting equation of motion is not a geodesic. Looking at the equation of motion (3.31), the reason for this is the nontrivial interference between classical states describing two classical configurations of matter, and more importantly, of gravity. In other words, the deviation from the geodesic motion is a *pure quantum gravity effect* — it is not present in the classical case, nor in the case of quantum matter in

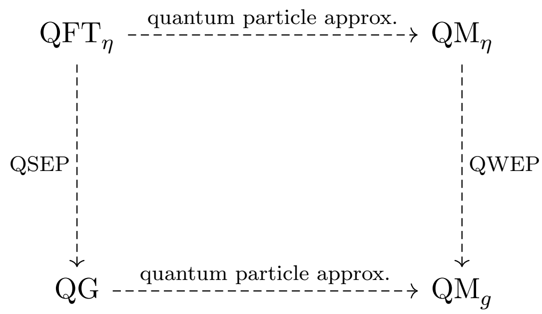
JCAP09(2019)057

classical Minkowski spacetime. A testimony of this fact is the quantum correction term for the metric (3.8), which features off-diagonal matrix elements of the metric operator $\hat{g}_{\mu\nu}$:

$$h_{\mu\nu} = 2\,\mathrm{Re}\left(\kappa\langle\Psi^{\perp}|\hat{g}_{\mu\nu}|\Psi\rangle\right) + \mathcal{O}(\eta)\,.$$

In this sense, due to the noncommutativity of the above diagram, one can argue that (within the discussed framework) *quantum gravity violates the weak equivalence principle.* Nevertheless, we would like to stress that our discussion regarding both strong and weak equivalence principles, based on the above prescription from subsection 4.1, is inherently *classical.* Indeed, in steps 1 and 2 which define the implementation of EP, one considers classical equations of motion. In our case, such definition suffices, as our entangled state (3.2) consists of a dominant and a sub-dominant term. Thus, we could expand our entangled equations (3.10) and (3.11) around the dominant classical terms, and discuss WEP in such a scenario. In fact, according to the definition of WEP, in general one can discuss its violation only *with respect to* some (perhaps unspecified, but assumed) classical spacetime metric. In our case, this role is played by the dominant classical metric $g_{\mu\nu}$.

In the more general case of superpositions of states which are more equally weighted, $\alpha \approx \beta$, and which consist of almost orthogonal states, $\langle\Psi|\tilde{\Psi}\rangle \approx 0$, one cannot single out a preferred classical metric, and therefore the classical definitions of SEP and WEP are inapplicable in this regime. Therefore, both equivalence principles ought to be extended to their respective *quantum* domains, denoted QSEP and QWEP respectively, in the sense of the following diagram:



Note that here all arrows are dashed, indicating the speculative nature of all these maps. Also, QM$_g$ represents a hypothetical theory of quantum particles coupled to a quantum gravitational field.

In this highly quantum regime ($\alpha \approx \beta$ and $\langle\Psi|\tilde{\Psi}\rangle \approx 0$), one could try to define the quantum weak equivalence principle (QWEP) in terms of the classical WEP, applied separately to each "branch" in the superposition. As long as the two "branches" $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ are themselves classical states, corresponding to the respective solutions of Einstein's equations, such a definition might seem suitable. Note that this approach is compatible with the notion of a superposed observer (see recent work [40] and the references therein). However, the formulation of the quantum strong and weak equivalence principles for the case of generic non-classical quantum states is an open question, outside of the scope of the current work.

Finally, the quantum version of the single pole approximation, called "quantum particle approximation" in the diagram above, is also not well defined — neither conceptually nor technically. Essentially, the whole diagram represents merely a speculation about the prescriptions which ought to map between the respective theories. In addition, like in the

JCAP09(2019)057

previous cases, the commutativity of the diagram (i.e., the violation of QWEP, given the validity of QSEP) would also be an open question. In some sense, the QSEP would represent a "true" equivalence principle, while QWEP would be a particle-like approximate image of QSEP. Being approximate, QWEP could possibly be violated in some cases, giving rise to noncommutativity of the diagram.

4.3 Universality, gravitational and inertial mass

In light of the results of section 3, in addition to the discussion of WEP violation, it is also important to discuss the status of the principle of universality, and the principle of equality between inertial and gravitational masses. In order to discuss them, it is instructive to study the Newtonian limit of the effective equation of motion (3.31), as follows.

We define the Newtonian limit in the standard way [39] — by assuming small spacetime curvature, nonrelativistic motion, and ignoring the backreaction of the particle on the background spacetime geometry. These approximations are implemented in the following way. First, ignoring the backreaction of the particle allows us to choose the dominant classical metric $g_{\mu\nu}$ as specified by the Newtonian line element

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + dx^2 + dy^2 + dz^2 \,, \tag{4.2}$$

where $x^\mu \equiv (t, x, y, z)$ are spacetime coordinates, M is the mass of the gravitational source, $r \equiv \sqrt{x^2 + y^2 + z^2}$, and $G \equiv l_p^2$ is Newton's gravitational constant. We will discuss the motion of a test-particle in this background, given by the effective equation of motion (3.31). Second, the assumption of nonrelativistic motion of the particle allows us to neglect its spacelike velocity,

$$u^k \equiv \frac{dz^k}{d\tau} \approx 0 \,,$$

leaving only the timelike component $u^0 \equiv dz^0/d\tau$ nonzero (the position of the particle $z^\mu(\tau)$ should not be confused with the label for the third spatial coordinate $z \equiv x^3$). Finally, the assumption of small spacetime curvature allows us to neglect all terms of order $\mathcal{O}(M^2)$ and higher.

Given this setup, one can easily calculate all nonzero Christoffel symbols corresponding to the dominant metric, obtaining:

$$\mathit{\Gamma}^0{}_{0k} = \mathit{\Gamma}^0{}_{k0} = \mathit{\Gamma}^k{}_{00} = \frac{GM}{r^3} x^k \,, \qquad k \in \{1, 2, 3\} \,.$$

One can then employ them to write the time and space components for the particle's effective equation of motion (3.31). Using (2.8) and (3.21), after some straightforward algebra, the time component of the equation of motion reduces to

$$\frac{d^2 z^0(\tau)}{d\tau^2} = 0 \,,$$

owing to the normalization condition $u^\mu u^\nu g_{\mu\nu} = -1$ and the presence of the orthogonal projector in (3.31). Using convenient initial conditions, this equation can be integrated to make an identification between the proper time τ and the time component of the particle's parametric equation of trajectory $x^\mu = z^\mu(\tau)$ as

$$t = z^0(\tau) = \tau \,,$$

reflecting the notion of global universal time of Newtonian theory. Using this result, one can show that the space components of the particle's equation of motion obtain the following form (note that the spacelike indices can be raised and lowered at will, since the spatial part of the metric (4.2) is a unit matrix):

$$\frac{d^2 z^k}{d\tau^2} + \frac{GM}{r^3} z^k + \eta \left[\partial_0 h_{0k} - \frac{1}{2} \partial_k h_{00} - \frac{GM}{r^3} z^j h_{jk} \right] = 0 \,. \tag{4.3}$$

Note that here r has been evaluated at the position of the particle, $r = \sqrt{z^k z_k}$, and similarly for the gradients of h_{0k} and h_{00}. The first two terms in the equation come from the classical geodesic part ∇u^k in (3.31), while the third term is the quantum correction, coming from the effective force term $\eta u^\nu u^\sigma F^k_{\perp \nu\sigma}$.

The most important aspect of equation (4.3) is the similarity between the second term of the classical part and the final term of the quantum correction. The spacelike components h_{jk} can be separated into the trace and traceless part,

$$h_{jk} \equiv \frac{1}{3} h^i{}_i \, \delta_{jk} + \tilde{h}_{ij} \,, \qquad \tilde{h}^k{}_k \equiv 0 \,,$$

and the trace can be grouped together with the classical term, giving

$$\frac{d^2 z^k}{d\tau^2} + \frac{GM}{r^3} z^k \left(1 - \frac{1}{3} \eta h^i{}_i \right) + \eta \left[\partial_0 h_{0k} - \frac{1}{2} \partial_k h_{00} - \frac{GM}{r^3} z^j \tilde{h}_{jk} \right] = 0 \,. \tag{4.4}$$

Finally, multiplying the whole equation (4.4) with an arbitrary positive number, called the particle's *inertial mass* and denoted m_I, it takes the form of the Newton's second law of motion,

$$m_I \frac{d^2 z^k}{d\tau^2} = -m_I \left(1 - \frac{1}{3} \eta h^i{}_i \right) \frac{GM}{r^3} z^k - \eta m_I \left[\partial_0 h_{0k} - \frac{1}{2} \partial_k h_{00} - \frac{GM}{r^3} z^j \tilde{h}_{jk} \right] . \tag{4.5}$$

One can recognize two force terms on the right-hand side. The second term is of purely quantum origin, and represents the effective force acting on the particle ultimately due to the presence of the quantum state $|\tilde{\Psi}\rangle$ in (3.2). It has a non-Newtonian form, in the sense that none of its parts can be grouped together with the first force term, as was done with the trace part. The first force term, however, can be recognized as the classical Newton's gravitational force law, provided that one defines the ratio between the *gravitational mass* m_G and the *inertial mass* m_I of the particle as

$$\frac{m_G}{m_I} \equiv \left(1 - \frac{1}{3} \eta h^i{}_i \right) . \tag{4.6}$$

At this point we are ready to discuss the principles of universality and of the equality between gravitational and inertial masses. To begin with, it is obvious from (4.6) that the gravitational mass is equal to the inertial mass only up to a quantum correction term. This term contains the trace of spatial components of the metric interference tensor $h_{\mu\nu}$, defined by equation (3.8), from which we obtain:

$$h^i{}_i = 2\delta^{ij} \, \mathrm{Re} \left(\kappa \langle \Psi^\perp | \hat{g}_{ij} | \Psi \rangle \right) + \mathcal{O}(\eta) \,. \tag{4.7}$$

JCAP09(2019)057

It is crucial to notice that, in addition to the dependence of the off-diagonal matrix element of the metric operator, this expression also depends on the matter fields (which are present in $|\Psi\rangle$ and $|\Psi^{\perp}\rangle$), including the particle itself. Therefore, the term in the parentheses in (4.6) cannot be reabsorbed into the constants G and M, since these describe the external source of gravity which should remain independent of the properties of the test particle. Thus, the only possibility to cast the first force term in (4.5) into the form of the Newton's law of gravitation, is to define the ratio between the gravitational and the inertial mass as in (4.6). As a consequence, the principle of equality between gravitational and inertial masses is violated by the presence of the correction term coming from quantum gravity.

A similar argument can be made regarding the principle of universality. One may cancel away the inertial mass from the Newton's law (4.5), returning to (4.4) which describes the acceleration of the particle in the presence of an external gravitational field. Again, the presence of (4.7) in the classical gravitational acceleration term guarantees that this term depends not only on the external gravitational source, but also on the structure of the test particle itself. Moreover, the remaining quantum correction terms also depend on $h_{\mu\nu}$, and therefore they too carry information about the internal structure of the particle. In this sense, test particles described by different matter configurations may therefore display different accelerations, given the same background gravitational field. This means that the principle of universality is violated by the presence of the quantum gravity correction terms.

As a final comment, we should also note that m_I (and consequently m_G as well) is a completely free parameter in the Newtonian setup, and should be determined by the interactions of nongravitational type. In particular, the Newtonian framework does not allow us to connect m_I, m_G with the effective mass m of the particle, discussed in the context of (3.28) and (3.30). This is because the total rest-energy of a particle is an inherently relativistic concept, not defined in Newtonian mechanics. On the other hand, if one goes to the relativistic framework, the notions of inertial and gravitational masses become ill-defined, since gravitational interaction cannot be described anymore by a mere force law in the Newtonian sense. Therefore, the relationship between m on one side, and both m_I, m_G on the other side, remains undefined.

5 Conclusions

5.1 Summary of the results

In this paper, we have discussed the effective motion of a point particle within the framework of quantum gravity, in particular the case where both matter and gravity are in a quantum superposition of the Schrödinger cat type. In section 2 we gave a recapitulation of the results of the classical theory, introducing the multipole formalism framework and illustrating the derivation of the geodesic equation for the motion of a particle in GR. Section 3 was devoted to the generalization of these results to the realm of the full quantum gravity. In subsection 3.1 we introduced the abstract quantum gravity framework, discussed the model of the superposition of two classical states, and established the main assumptions for the derivation of the effective equation of motion. In subsection 3.2 we have analyzed in detail the quantum version of the equation for the covariant conservation of stress-energy tensor, which is a crucial ingredient in the derivation of the effective equation of motion. The explicit derivation of the equation of motion itself was then given in subsection 3.3, giving rise to the main results of the paper — the equation for the stress-energy kernel (3.28), the equation for the time-evolution of the particle's mass (3.30), and the effective equation of motion for the particle (3.31). Most importantly, the effective equation of motion turns out to contain a non-geodesic term, giving rise to an effective force acting on the particle, as a consequence of the interference terms between the two classical states of the gravity-matter system. The last subsection 3.4 discusses the self-consistency of the assumptions used in the above analysis, giving rise to the equation (3.35) for the error estimate of the single pole approximation scale.

In light of the nongeodesic motion established in section 3, it is important to discuss it in the context of the equivalence principle. This topic was taken up in section 4. After we have defined various flavors of the equivalence principle in subsection 4.1, the main analysis was presented in subsection 4.2, discussing a possible violation of (various forms of) the weak equivalence principle, as a consequence of the nongeodesic correction to the equation of motion (3.31). Also, given the inherently classical nature of the equivalence principle, we have also speculated on possible generalizations to the quantum realm, introducing the notions of the quantum strong and weak equivalence principles, albeit without giving explicit statements about their definitions. Finally, in subsection 4.3 we have discussed the notions of universality and equality between inertial and gravitational masses in the context of quantum gravity, by studying the Newtonian limit of the equation of motion (3.31). This analysis gave a clear interpretation that both universality and the equality between gravitational and inertial masses are violated in our context, corroborating the conclusions of the abstract analysis of the EP given in subsection 4.2.

5.2 Discussion of the results

By far the most interesting topic to discuss in the context of the equation of motion (3.31) is how to estimate the magnitude of the nongeodesic term. As far as the analysis of this paper goes, we can only say that this term is very small, given that it is proportional to η, which is in turn bounded from above by phenomenological argument that we do not observe superpositions of the gravitational field in nature. However, aside from this qualitative argument, in order to estimate the actual magnitude of the nongeodesic term one would need to go beyond the abstract quantum gravity formalism, and construct an explicit quantum gravity model coupled to matter fields, find some explicit kink solutions of the matter sector, and then calculate the overlap terms and the off-diagonal interference terms of the metric operator. Of course, any estimate obtained in such a way would be model-dependent. We consider this to be a feature of the abstract quantum gravity approach, since the magnitude of the nongeodesic term represents one way to operationally distinguish between different QG models. In other words, one could use equation (3.31) to experimentally test and compare these models, at least in principle. Probably the most obvious such test would employ equation (4.6) which relates the gravitational and inertial mass of the particle.

One result that was not discussed in detail is the nonconservation law for the effective mass of the particle, (3.30). However, it is not really surprising that the particle's total rest energy fails to be constant in the presence of gravity-matter entanglement. As (3.30) tells us, the nonconservation is actually a consequence of the additional effective force, which is itself a consequence of the quantum interference between two classical geometries and matter states. Nevertheless, it would indeed be interesting to study the mass nonconservation in more detail.

It is also important to discuss the generalization of our results from the case of the superposition of two classical states to many classical states. In particular, one could discuss the case where the state $|\tilde{\Psi}\rangle$ in (3.2) is not a single classical state, but a superposition of

JCAP09(2019)057

many classical states,

$$|\tilde{\Psi}\rangle = \sum_i \gamma_i |\Psi_i\rangle \,.$$

As long as we maintain the assumption that the fidelity $F(|\Psi\rangle, |\boldsymbol{\Psi}\rangle) \approx 1$, it is straightforward to see that all our results and conclusions still hold in the generic case. Therefore, there is no substantial difference in the analysis of a state which is a superposition of two classical states, compared to the analysis of a superposition of many classical states, as long as one of them is dominant while all others are sub-dominant. Note that in this case, even when β is finite and $\epsilon \to 0$, the role of the metrics generated by $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ cannot be exchanged anymore, as the latter generically does not satisfy Einstein's equations. This fixes the choice of $|\Psi\rangle$ as the dominant state. A detailed quantitative description is technically more complicated, but qualitatively all results will hold for both types of states.

5.3 Future lines of research

One of the main lines of future work would be to perform a similar analysis as was done in this paper, but keeping the η^2 terms. This would naturally include the sub-dominant effective metric and stress-energy (3.3), giving qualitatively new insight into the notion of quantum superpositions of two classical geometries. That analysis might provide clues about the properties of quantum gravity which could arguably hold even in the equal-weight superpositions of two classical states, defined by the choice $\alpha \approx \beta \approx 1/\sqrt{2}$ in (3.2).

Alternatively, one could repeat the analysis of this paper, but in a pole-dipole approximation. This would also lead to novel effects, one of which might be a coupling of various quantum interference terms to the spacetime curvature and the angular momentum of the particle, generalizing the classical pole-dipole equation of motion [10].

Also, given that the multipole formalism is also applicable to Riemann-Cartan spacetimes [19–22], the analysis of this paper could be generalized to include coupling of quantum interference terms to spacetime torsion and the spin of the particle.

Finally, one could further discuss a more general setup in which the off-diagonal terms in the covariant conservation equation (3.14) are not ignored, in the sense of going beyond the approximations (3.15) and (3.16).

In addition to all of the above, one important line of research would be to study possible connections to experiments. First, one should study the counterpart of the so-called *geodesic deviation equation*. Namely, in GR, the geodesic motion as such is not observable, as a consequence of the equivalence principle. As we have emphasized in subsection 4.1, the EP dictates that the only way to observe gravitational degrees of freedom is via *nonlocal measurements*, which are not encoded in the geodesic equation. Therefore, what one can actually observe is the change in the relative separation of two nearby geodesic trajectories, due to the tidal effects. This is in turn described by the geodesic deviation equation, which explicitly features the Riemann curvature tensor. In our case, the equation of motion (3.31) is not a geodesic, but is still local in character, in the sense that it does contain gravitational degrees of freedom at the given point, but still it does not combine gravitational degrees of freedom of two or more points. Thus, one ought to compare the trajectories of two nearby particles, both following a trajectory determined by (3.31). The equation governing the separation between two particles in such a setup would be a counterpart to the geodesic deviation equation of GR with a corresponding quantum correction term. It should be derived and studied in detail, in order to better understand what effects could be in principle directly experimentally observable.

JCAP09(2019)057

Second, one could also test our results by measuring the violation of the universality and of the equality of the gravitational and the inertial mass in the semiclassical Newtonian limit.

The above list of possible topics for further research is of course not exhaustive — one can probably study various additional aspects and topics related to this work, in particular giving more precise meaning to the notions of the quantum strong and weak equivalence principles.

Acknowledgments

NP acknowledges the support of SQIG (Security and Quantum Information Group), the Instituto de Telecomunicações (IT) Research Unit, Ref. UID/EEA/50008/2019, funded by Fundação para a Ciência e Tecnologia (FCT), project H2020 SPARTA, and the FCT projects Confident PTDC/EEI-CTP/4503/2014, QuantumMining POCI-01-0145-FEDER-031826 and Predict PTDC/CCI-CIF/29877/2017, supported by the European Regional Development Fund (FEDER) through the Competitiveness and Internationalization Operational Programme (COMPETE 2020), and by the Regional Operational Program of Lisbon, as well as the bilateral scientific cooperation between Portugal and Serbia through the project "Noise and measurement errors in multi-party quantum security protocols", no. 451-03-01765/2014-09/04 supported by the Foundation for Science and Technology (FCT), Portugal, and the Ministry of Education, Science and Technological Development of the Republic of Serbia. FCT grant CEECIND/04594/2017 is also acknowledged.

MV was supported by the project ON171031 of the Ministry of Education, Science and Technological Development of the Republic of Serbia, by the bilateral scientific cooperation between Portugal and Serbia through the project "Quantum Gravity and Quantum Integrable Models — 2015-2016", no. 451-03-01765/2014-09/24 supported by the Foundation for Science and Technology (FCT), Portugal, and the Ministry of Education, Science and Technological Development of the Republic of Serbia, by the bilateral scientific cooperation between Austria and Serbia through the project "Causality in Quantum Mechanics and Quantum Gravity — 2018-2019", no. 451-03-02141/2017-09/02 supported by the Austrian Academy of Sciences (ÖAW), Austria, and the Ministry of Education, Science and Technological Development of the Republic of Serbia, and by the "JESH Incoming 2016" grant of the Austrian Academy of Sciences (ÖAW).

NP and MV would also like to thank the Erwin Schrödinger International Institute for Mathematics and Physics (ESI Vienna), for the warm hospitality during the workshop "Quantum Physics and Gravity" (29 May – 30 June 2017) and partial support.

A Short review of the multipole formalism

In this appendix we give a short review of the multipole formalism, providing some basic motivation for its introduction and a few elementary properties. A more rigorous treatment and more details can be found in [25].

The multipole formalism revolves around the idea of expanding a function into a series of derivatives of the Dirac δ function, or δ series for short. Perhaps the easiest way to understand the δ series is to introduce it as a Fourier transform of a power series. For example, given a real-valued function $f(x)$, one can write it as a Fourier transform of $\tilde{f}(k)$ as

$$f(x) = \int_{\mathbb{R}} dk\, \tilde{f}(k)\, e^{ikx} \,. \tag{A.1}$$

In principle, we can expand $\tilde{f}(k)$ into power series as

$$\tilde{f}(k) = \sum_{n=0}^{\infty} c_n k^n \,,$$

where c_n are some coefficients, substitute the expansion back into (A.1), and integrate term by term. Using the identity

$$k^n e^{ikx} = (-i)^n \frac{\partial^n}{\partial x^n} e^{ikx}$$

and the integral representation of the Dirac δ function

$$\delta(x) = \frac{1}{2\pi} \int_{\mathbb{R}} dk\, e^{ikx} \,,$$

we obtain

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n \int_{\mathbb{R}} dk\, k^n\, e^{ikx} = \sum_{n=0}^{\infty} (-i)^n c_n \frac{d^n}{dx^n} \int_{\mathbb{R}} dk\, e^{ikx} \\ &= \sum_{n=0}^{\infty} 2\pi(-i)^n c_n \frac{d^n}{dx^n} \delta(x) \equiv \sum_{n=0}^{\infty} b_n \frac{d^n}{dx^n} \delta(x) \,. \end{aligned}$$

In the last step, we have merely renamed the coefficients in the expansion.

The above example is the most elementary construction of the δ series, providing some intuition. It is straightforward to see that one can generalize the procedure to perform the expansion around an arbitrary point z instead of zero, such that

$$f(x) = \sum_{n=0}^{\infty} b_n \frac{d^n}{dx^n} \delta(x-z) \,.$$

The coefficients b_n can be evaluated using the inverse formula,

$$b_n = \frac{(-1)^n}{n!} \int_{\mathbb{R}} dx\, (x-z)^n f(x) \,, \tag{A.2}$$

and are usually called *n-th order moments* of the function $f(x)$. From (A.2) one sees that the δ series is well defined for every function $f(x)$, which falls off to zero faster than any power of x at both infinities.

Let us study an instructive example. Let the function $f(x)$ be an ordinary Gaussian, peaked around the point x_0,

$$f(x) = \frac{1}{\sqrt{\pi}} e^{-(x-x_0)^2} \,.$$

One can evaluate the coefficients in the corresponding δ series using (A.2) to obtain:

$$b_n = \begin{cases} \displaystyle\sum_{k=0}^{n/2} \frac{(z-x_0)^{n-2k}}{4^k\, k!\, (n-2k)!} & \text{for even } n \,, \\ \displaystyle -\sum_{k=0}^{(n-1)/2} \frac{(z-x_0)^{n-2k}}{4^k\, k!\, (n-2k)!} & \text{for odd } n \,. \end{cases}$$

JCAP09(2019)057

It is important to note the following property — if the expansion point z does not coincide with the peak of the Gaussian, x_0, the magnitude of the coefficients b_n in general grows with n. For example, if $z - x_0 = 2$, we have

$$f(x) = \delta(x-z) - 2\frac{d}{dx}\delta(x-z) + \frac{9}{4}\frac{d^2}{dx^2}\delta(x-z) - \frac{11}{6}\frac{d^3}{dx^3}\delta(x-z) + \frac{115}{96}\frac{d^4}{dx^4}\delta(x-z) + \dots$$

However, if the expansion point coincides with the peak, $z - x_0 = 0$, the magnitude of the coefficients falls off as n grows:

$$f(x) = \delta(x-z) + \frac{1}{4}\frac{d^2}{dx^2}\delta(x-z) + \frac{1}{32}\frac{d^4}{dx^4}\delta(x-z) + \frac{1}{384}\frac{d^6}{dx^6}\delta(x-z) + \dots$$

From this simple example one can infer an important property of δ series — the coefficients b_n decrease as n grows, if the expansion point is near the peak of the function $f(x)$. Turning the argument around, if we require that the coefficients decrease with n,

$$|b_n| > |b_{n+1}|\,, \qquad \forall n \in \mathbb{N}_0\,,$$

this places *a restriction on the possible values of the expansion point z*. This is the crucial property of the δ series, and is being used to define the "position of the particle" which corresponds to a distribution of matter fields described by a localized function $f(x)$.

Also, assuming that the expansion point z has been chosen to be near the peak of the function, the decreasing nature of the coefficients b_n allows one to approximate the function $f(x)$ by a truncated series. This formalizes the intuitive idea that if one looks at some localized distribution of matter fields from "far away", it will look roughly as a point particle. The truncation point then quantifies the amount of "internal structure" that is known about $f(x)$. One can therefore study the function $f(x)$ at various approximation levels: the *single pole* approximation,

$$f(x) \sim b_0 \delta(x-z)\,,$$

the *pole-dipole* approximation,

$$f(x) \sim b_0 \delta(x-z) + b_1 \frac{d}{dx}\delta(x-z)\,,$$

the *pole-dipole-quadrupole* approximation,

$$f(x) \sim b_0 \delta(x-z) + b_1 \frac{d}{dx}\delta(x-z) + b_2 \frac{d^2}{dx^2}\delta(x-z)\,,$$

and so on.

It is completely straightforward to generalize the δ series to three (or more) dimensions, with the δ series of a function $f(\vec{x})$ around the point $\vec{z}$ defined as

$$f(\vec{x}) = \sum_{n=0}^{\infty} b_n^{i_1 \dots i_n} \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_n}} \delta^{(3)}(\vec{x} - \vec{z})\,. \tag{A.3}$$

Here the indices $i_1, \dots i_n$ take values 1, 2 and 3, and the inverse formula for the coefficients is

$$b_n^{i_1 \dots i_n} = \frac{(-1)^n}{n!} \int_{\mathbb{R}^3} d^3x\, (x^{i_1} - z^{i_1}) \dots (x^{i_n} - z^{i_n}) f(\vec{x})\,. \tag{A.4}$$

For example, in electrostatics, one can expand the charge density $\rho(\vec{x})$ localized around the point $\vec{z} = 0$ as

$$\rho(\vec{x}) = b_0 \delta^{(3)}(\vec{x}) + b_1^i \frac{\partial}{\partial x^i} \delta^{(3)}(\vec{x}) + \dots$$

According to (A.4), the coefficients are

$$b_0 = \int_{\mathbb{R}^3} d^3x \, \rho(\vec{x}) \equiv Q \,, \qquad \vec{b}_1 = -\int_{\mathbb{R}^3} d^3x \, \vec{x} \rho(\vec{x}) \equiv -\vec{p} \,,$$

where we recognize the total charge Q and the electrostatic dipole moment $\vec{p}$ of the source. Thus we have

$$\rho(\vec{x}) = Q \delta^{(3)}(\vec{x}) - \vec{p} \cdot \nabla \delta^{(3)}(\vec{x}) + \dots$$

Substituting the δ series expansion of $\rho(\vec{x})$ into the formula for the electrostatic potential,

$$\varphi(\vec{r}) = \int_{\mathbb{R}^3} d^3x \, \frac{\rho(\vec{x})}{|\vec{r} - \vec{x}|} \,,$$

and evaluating the integral, one obtains the familiar expression for the multipole expansion in electrostatics [41]:

$$\varphi(\vec{r}) = \frac{Q}{|\vec{r}|} + \frac{\vec{p} \cdot \vec{r}}{|\vec{r}|^3} + \dots$$

This example also illustrates what type of approximation is achieved with the truncation of the δ series.

Next we generalize to time-dependent functions. If the function $f(\vec{x})$ evolves in time, while remaining localized in space, one can expand it into δ series by choosing the most convenient reference point $z(t)$ at each moment of time,

$$f(\vec{x}, t) = \sum_{n=0}^{\infty} b^{i_1 \dots i_n}(t) \frac{\partial}{\partial x^{i_1}} \dots \frac{\partial}{\partial x^{i_n}} \delta^{(3)}(\vec{x} - \vec{z}(t)) \,, \tag{A.5}$$

where $t \in \mathbb{R}$ is a time variable, and the coefficients b are now time-dependent. Then one can introduce the proper time τ, and use the identity

$$\int_{\mathbb{R}} d\tau \, \delta(t - \tau) = 1$$

to rewrite (A.5) in a 4-dimensional manifestly Lorentz-invariant form

$$f(x) = \int_{\mathbb{R}} d\tau \sum_{n=0}^{\infty} b^{\mu_1 \dots \mu_n}(\tau) \partial_{\mu_1} \dots \partial_{\mu_n} \delta^{(4)}(x - z(\tau)) \,, \tag{A.6}$$

where we have relabeled $(\vec{x}, t) \equiv x$, introduced $z^0(\tau) = \tau$, used shorthand notation $\partial_\mu \equiv \partial / \partial x^\mu$, and defined $b^0 = b^{00} = b^{000} = \dots = 0$, since the time derivatives do not actually appear in (A.5). The introduction of these auxiliary timelike components of the b-coefficients, demanded by Lorentz invariance, gives rise to an additional gauge symmetry of the expansion coefficients, since only the "spatial" components carry nontrivial information about the function $f(x)$. This additional gauge symmetry is called *extra symmetry 1*, and is studied in detail in [25].

JCAP09(2019)057

Finally, one can make one more generalization, and introduce the notion of a δ series around a p-brane, a $(p+1)$-dimensional submanifold living in a D-dimensional spacetime manifold. Namely, we have seen that one can expand a function into a δ series around a point and around a one-dimensional line (equations (A.3) and (A.6), respectively). Generalizing in that direction, one can introduce the world-trajectory of a p-dimensional object through D-dimensional spacetime $\mathcal{M}$, with parametric equations $x^\mu = z^\mu(\xi^a)$ describing the trajectory as a $(p+1)$-dimensional submanifold $\Sigma \subset \mathcal{M}$. Here $\mu \in \{0, \dots, D-1\}$ and $a \in \{0, \dots p\}$, where x^μ are coordinates on $\mathcal{M}$ while ξ^a are intrinsic coordinates on Σ. Then, given a function $f(x)$ whose support is localized near the submanifold Σ, one can write its δ series expansion around Σ in a fully diffeomorphism- and reparametrization-invariant way as:

$$f(x) = \int_\Sigma d^{p+1}\xi \sqrt{-\gamma} \sum_{n=0}^{\infty} \nabla_{\mu_1} \dots \nabla_{\mu_n} \left[B^{\mu_1 \dots \mu_n}(\xi) \frac{\delta^{(D)}(x - z(\xi))}{\sqrt{-g}} \right] . \tag{A.7}$$

Here γ is the determinant of the induced metric $\gamma_{ab} = g_{\mu\nu} u^\mu_a u^\nu_b$ on Σ, where $g_{\mu\nu}$ is the metric on $\mathcal{M}$ and $u^\mu_a \equiv \partial z^\mu / \partial \xi^a$ are the tangent vectors of Σ. Note that, in order to ensure the correct tensorial behavior, the B-coefficients have been moved inside the action of the covariant derivatives. Namely, despite the fact that the covariant derivatives act with respect to x and B's do not depend on x, covariant derivatives still act nontrivially on B's with the connection terms. For similar reasons, the term $\sqrt{-g}$ has been introduced to combine with the δ function into a quantity which transforms as a scalar under diffeomorphisms. Its introduction amounts merely to a suitable redefinition of B's and does not modify the δ series in any nontrivial way.

The fully general δ series (A.7) has been studied in detail in [25]. For the purpose of the discussion given in the main text of this paper, we are interested in the case of a particle, i.e., a $(p = 0)$-brane, moving along a 1-dimensional timelike curve $\mathcal{C}$ which is a submanifold of the $(D = 4)$-dimensional spacetime $\mathcal{M}$. In this case, there is only one intrinsic coordinate on $\mathcal{C}$, denoted $\xi^0 \equiv \tau$, only one tangent vector

$$u^\mu_0 \equiv \frac{\partial z^\mu(\xi)}{\partial \xi^0} = \frac{d z^\mu(\tau)}{d\tau} = u^\mu ,$$

while the induced metric tensor is a 1×1 matrix $\gamma_{00} = g_{\mu\nu} u^\mu_0 u^\nu_0$. The parametrization of the curve $\mathcal{C}$ with the coordinate τ can be chosen to fix the reparametrization gauge symmetry via the gauge-fixing condition $\gamma_{00} = -1$, which is actually the natural normalization of the tangent vector, $g_{\mu\nu} u^\mu u^\nu = -1$. Finally, one can then apply the δ series expansion (A.7) to the stress-energy tensor $T^{\mu\nu}(x)$ of the matter fields as

$$T^{\mu\nu}(x) = \int_{\mathcal{C}} d\tau \sum_{n=0}^{\infty} \nabla_{\rho_1} \dots \nabla_{\rho_n} \left[B^{\mu\nu\rho_1 \dots \rho_n}(\tau) \frac{\delta^{(D)}(x - z(\tau))}{\sqrt{-g}} \right] .$$

Note that the coefficients B now carry two additional indices inherited from the stress-energy tensor. In the single pole approximation, one drops all terms in the sum except the $n = 0$ term, truncating the series to the form

$$T^{\mu\nu}(x) = \int_{\mathcal{C}} d\tau \, B^{\mu\nu}(\tau) \frac{\delta^{(D)}(x - z(\tau))}{\sqrt{-g}} ,$$

as used in the main text.

B Separable classical states

As mentioned in the main text, a recent study suggests that physical states of gravity and matter are generically entangled [1]. In this appendix, we analyze a simple, yet possibly intriguing, consequence of the assumption that the overall classical gravity-matter state can be approximated by (or indeed is) the product of the gravity and the matter classical states, $|\Psi\rangle = |g\rangle \otimes |\phi\rangle$, where $|g\rangle \in \mathcal{H}_G$ and $|\phi\rangle \in \mathcal{H}_M$ are classical states for the gravity and the matter sector, respectively (and analogously for $|\tilde{\Psi}\rangle$).

To begin with we introduce the overlaps as follows:

$$S_G \equiv \langle g|\tilde{g}\rangle\,, \qquad S_M \equiv \langle \phi|\tilde{\phi}\rangle\,, \qquad S \equiv \langle \Psi|\tilde{\Psi}\rangle = S_G S_M\,.$$

Note that, since in (3.2) only the relative phase between $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$ is important, we can reabsorb the phases of the coefficients α and β into $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$, respectively. In this way, we have $\alpha, \beta \in \mathbb{R}$, while only the overlap S between the two coherent states carries the information about the relative phase, and is therefore complex. Moreover, since S is a product between S_G and S_M, the phase of S can be distributed between S_G and S_M in an arbitrary way. A convenient choice is to have the phase in the matter sector, so that $S_G \in \mathbb{R}$ and $S_M \in \mathbb{C}$. Next, we can decompose $|\tilde{g}\rangle$ and $|\tilde{\phi}\rangle$ into parts proportional to and orthogonal to $|g\rangle$ and $|\phi\rangle$, respectively,

$$|\tilde{g}\rangle = S_G|g\rangle + \epsilon_G|g^{\perp}\rangle\,, \qquad |\tilde{\phi}\rangle = S_M|\phi\rangle + \epsilon_M|\phi^{\perp}\rangle\,, \tag{B.1}$$

where $\langle g|g^{\perp}\rangle \equiv 0$, $\langle \phi|\phi^{\perp}\rangle \equiv 0$, and

$$\epsilon_G \equiv \sqrt{1-(S_G)^2}\,, \qquad \epsilon_M \equiv \sqrt{1-|S_M|^2}\,.$$

Note that $\epsilon_G, \epsilon_M \in \mathbb{R}$. Additionally, one can use (B.1) to rewrite $|\tilde{\Psi}\rangle$ into the form

$$|\tilde{\Psi}\rangle = S|\Psi\rangle + \epsilon|\Psi^{\perp}\rangle\,,$$

where

$$\epsilon = \sqrt{\epsilon_M^2 + \epsilon_G^2 - \epsilon_M^2\epsilon_G^2}\,,$$

and

$$|\Psi^{\perp}\rangle = \frac{\epsilon_M S_G}{\epsilon}|g\rangle \otimes |\phi^{\perp}\rangle + \frac{\epsilon_G S_M}{\epsilon}|g^{\perp}\rangle \otimes |\phi\rangle + \frac{\epsilon_G\epsilon_M}{\epsilon}|g^{\perp}\rangle \otimes |\phi^{\perp}\rangle\,. \tag{B.2}$$

Note that in the cases when S_G and S_M are large (and consequently ϵ_G and ϵ_M are small), we can neglect the final term from (B.2), obtaining the Schmidt form of the "orthogonal correction" of the state $|\tilde{\Psi}\rangle$, with respect to $|\Psi\rangle$. It is interesting to observe that such a state is always necessarily entangled, as its entanglement entropy is always bigger than zero. In other words, to obtain a nearby classical product state of gravity and matter $|\tilde{\Psi}\rangle$, one has to perturb the original (classical product) state $|\Psi\rangle$ with an entangled state $|\Psi^{\perp}\rangle \approx \epsilon^{-1}\epsilon_M S_G|g\rangle \otimes |\phi^{\perp}\rangle + \epsilon^{-1}\epsilon_G S_M|g^{\perp}\rangle \otimes |\phi\rangle$.

C Phase of interference terms

In this appendix, we analyze the expressions for the expectation values of the metric and the stress-energy tensors in the entangled state (3.5), given by (3.6) and (3.7), respectively. We show that their third, interference, terms are generically different from zero, and thus contain non-trivial contributions linear in η. Since the two terms have the same form, we will consider the case of the metric operator only.

By writing $\kappa = |\kappa|e^{i\varphi_\kappa} = Fe^{i\varphi_\kappa}$ and $\langle\Psi^\perp|\hat{g}_{\mu\nu}|\Psi\rangle = |\langle\Psi^\perp|\hat{g}_{\mu\nu}|\Psi\rangle|e^{i\varphi_g}$, the third term of (3.6) has the form

$$2\eta\,\mathrm{Re}\left(\kappa\langle\Psi^\perp|\hat{g}_{\mu\nu}|\Psi\rangle\right) = 2\eta F|\langle\Psi^\perp|\hat{g}_{\mu\nu}|\Psi\rangle|\cos(\varphi_\kappa+\varphi_g)\,. \tag{C.1}$$

In case $\varphi_\kappa+\varphi_g = \pm\pi/2$, i.e., the interference term is zero, then for *any other generic* choice of $|\Psi'\rangle = e^{i\delta}|\Psi\rangle$, we have that $\varphi'_\kappa+\varphi'_g \neq \pm\pi/2$.

Indeed, changing $|\Psi\rangle \to |\Psi'\rangle = e^{i\delta}|\Psi\rangle$ induces the change of the other classical state

$$|\tilde{\Psi}\rangle = S|\Psi\rangle + \epsilon|\Psi^\perp\rangle \;\longrightarrow\; |\tilde{\Psi}'\rangle = S|\Psi'\rangle + \epsilon|\Psi^\perp\rangle = S'|\Psi\rangle + \epsilon|\Psi^\perp\rangle$$

with $S' = Se^{i\delta} = |S|e^{i(\varphi_s+\delta)}$ (where $S = |S|e^{i\varphi_s}$), but the orthogonal state $|\Psi^\perp\rangle$ *does not* change. Thus, the phase of the matrix element from (C.1) changes to $\varphi'_g = \varphi_g+\delta$. On the other hand, the phase of κ changes to $\varphi'_\kappa = \varphi_\kappa+\tilde{\delta}$ (note that $\tilde{\delta}$ is a function of δ, see below). Since

$$\kappa = \alpha+\beta|S|e^{i\varphi_s} = |\kappa|e^{i\varphi_\kappa}\,,$$
$$\kappa' = \alpha+\beta|S|e^{i(\varphi_s+\delta)} = |\kappa|e^{i\varphi'_\kappa}\,,$$

it is obvious that for a generic choice of the parameters, i.e., in but a discrete number of points, we have $\tilde{\delta} = \varphi'_k-\varphi_k \neq -\delta$, obtaining $\varphi'_\kappa+\varphi'_g = (\varphi_s+\varphi_\kappa)+(\delta+\tilde{\delta}) = \pm\pi/2+(\delta+\tilde{\delta}) \neq \pm\pi/2$.

Thus, the linear correction to (3.6), and to (3.7) as well, is zero only for a discrete number of the relative phases between the classical states $|\Psi\rangle$ and $|\tilde{\Psi}\rangle$. Otherwise, it is generically non-trivial.

References

[1] N. Paunković and M. Vojinović, *Gauge protected entanglement between gravity and matter*, *Class. Quant. Grav.* **35** (2018) 185015 [arXiv:1702.07744] [INSPIRE].

[2] P. Chowdhury, D. Home, A.S. Majumdar, S.V. Mousavi, M.R. Mozaffari and S. Sinha, *Strong quantum violation of the gravitational weak equivalence principle by a non-Gaussian wave packet*, *Class. Quant. Grav.* **29** (2012) 025010 [arXiv:1107.5159] [INSPIRE].

[3] M. Zych and Č. Brukner, *Quantum formulation of the Einstein Equivalence Principle*, *Nature Phys.* **14** (2018) 1027 [arXiv:1502.00971] [INSPIRE].

[4] G. Rosi et al., *Quantum test of the equivalence principle for atoms in superpositions of internal energy eigenstates*, *Nature Commun.* **8** (2017) 5529 [arXiv:1704.02296] [INSPIRE].

[5] S. Longhi, *Equivalence principle and quantum mechanics: quantum simulation with entangled photons*, *Opt. Lett.* **43** (2018) 226 [arXiv:1712.02054].

[6] A. Einstein, L. Infeld and B. Hoffmann, *The Gravitational equations and the problem of motion*, *Annals Math.* **39** (1938) 65.

[7] M. Mathisson, *Neue mechanik materieller systemes*, *Acta Phys. Polon.* **6** (1937) 163 [INSPIRE].

[8] J. Lubánski, *Neue Bewegungsgleichungen materieller Systeme in Minkowskischer Welt*, *Acta Phys. Polon.* **6** (1937) 356.

[9] V.A. Fock, *Sur le mouvement des masses finie d'après la théorie de gravitation einsteinienne*, *J. Phys. USSR* **1** (1939) 81.

[10] A. Papapetrou, *Spinning test particles in general relativity. 1.*, Proc. Roy. Soc. Lond. **A 209** (1951) 248 [INSPIRE].

[11] J. Weyssenhoff and A. Raabe, *Relativistic dynamics of spin-fluids and spin-particles*, Acta Phys. Polon. **9** (1947) 7 [INSPIRE].

[12] W. Tulczyjev, *Motion of multipole particles in general relativity theory*, Acta Phys. Polon. **18** (1959) 393.

[13] A.H. Taub, *Motion of Test Bodies in General Relativity*, J. Math. Phys. **5** (1964) 112.

[14] G. Dixon, *A covariant multipole formalism for extended test bodies in general relativity*, Nuovo Cim. **34** (1964) 317.

[15] G. Dixon, *Classical theory of charged particles with spin and the classical limit of the Dirac equation*, Nuovo Cim. **38** (1965) 1616.

[16] W.G. Dixon, *Dynamics of extended bodies in general relativity. I. Momentum and angular momentum*, Proc. Roy. Soc. Lond. **A 314** (1970) 499 [INSPIRE].

[17] W.G. Dixon, *Dynamics of extended bodies in general relativity. II. Moments of the charge-current vector*, Proc. Roy. Soc. Lond. **A 319** (1970) 509 [INSPIRE].

[18] G. Dixon, *The definition of multipole moments for extended bodies*, Gen. Rel. Grav. **4** (1973) 199.

[19] P.B. Yasskin and S.J. Stoeger, William R., *Propagation Equations for Test Bodies With Spin and Rotation in Theories of Gravity With Torsion*, Phys. Rev. **D 21** (1980) 2081 [INSPIRE].

[20] K. Nomura, T. Shirafuji and K. Hayashi, *Spinning test particles in space-time with torsion*, Prog. Theor. Phys. **86** (1991) 1239 [INSPIRE].

[21] K. Nomura, T. Shirafuji and K. Hayashi, *Semiclassical particles with arbitrary spin in the Riemann-Cartan space-time*, Prog. Theor. Phys. **87** (1992) 1275 [INSPIRE].

[22] M. Vasilić and M. Vojinović, *Zero-size objects in Riemann-Cartan spacetime*, JHEP **08** (2008) 104 [arXiv:0807.0596] [INSPIRE].

[23] C. Aragone and S. Deser, *String Dynamics from Energy Momentum Conservation*, Nucl. Phys. **B 92** (1975) 327 [INSPIRE].

[24] M. Vasilić and M. Vojinović, *Classical string in curved backgrounds*, Phys. Rev. **D 73** (2006) 124013 [gr-qc/0610014] [INSPIRE].

[25] M. Vasilić and M. Vojinović, *Classical spinning branes in curved backgrounds*, JHEP **07** (2007) 028 [arXiv:0707.3395] [INSPIRE].

[26] M. Vasilić and M. Vojinović, *Single-Pole Interaction of the Particle with the String*, SIGMA **4** (2008) 019 [arXiv:0802.1655] [INSPIRE].

[27] M. Vasilić and M. Vojinović, *Interaction of the particle with the string in pole-dipole approximation*, Fortsch. Phys. **56** (2008) 542 [arXiv:1010.1852] [INSPIRE].

[28] M. Vasilić and M. Vojinović, *Spinning branes in Riemann-Cartan spacetime*, Phys. Rev. **D 78** (2008) 104002 [arXiv:1010.1861] [INSPIRE].

[29] M. Vasilić and M. Vojinović, *Test membranes in Riemann-Cartan spacetimes*, Phys. Rev. **D 81** (2010) 024025 [arXiv:0812.4694] [INSPIRE].

[30] M. Vasilić, *Classical dynamics of brane-world extended objects*, Phys. Rev. **D 81** (2010) 124029 [INSPIRE].

[31] J. Armas, J. Camps, T. Harmark and N.A. Obers, *The Young Modulus of Black Strings and the Fine Structure of Blackfolds*, JHEP **02** (2012) 110 [arXiv:1110.4835] [INSPIRE].

[32] J. Armas, J. Gath and N.A. Obers, *Black Branes as Piezoelectrics*, Phys. Rev. Lett. **109** (2012) 241101 [arXiv:1209.2127] [INSPIRE].

[33] J. Armas, J. Gath and N.A. Obers, *Electroelasticity of Charged Black Branes*, *JHEP* **10** (2013) 035 [arXiv:1307.0504] [INSPIRE].

[34] J. Armas, *How Fluids Bend: the Elastic Expansion for Higher-Dimensional Black Holes*, *JHEP* **09** (2013) 073 [arXiv:1304.7773] [INSPIRE].

[35] E. Okon and C. Callender, *Does Quantum Mechanics Clash with the Equivalence Principle — and Does it Matter?*, *Eur. J. Phil. Sci.* **1** (2011) 133 [arXiv:1008.5192] [INSPIRE].

[36] E. Di Casola, S. Liberati and S. Sonego, *Nonequivalence of equivalence principles*, *Am. J. Phys.* **83** (2015) 39 [arXiv:1310.7426] [INSPIRE].

[37] A. Accioly and R. Paszko, *Conflict Between the Classical Equivalence Principle and Quantum Mechanics*, *Adv. Studies Theor. Phys.* **3** (2009) 65.

[38] H.R. Brown and J. Read, *Clarifying possible misconceptions in the foundations of general relativity*, *Am. J. Phys.* **84** (2016) 327 [arXiv:1512.09253] [INSPIRE].

[39] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman and Co., San Francisco, U.S.A. (1973).

[40] F. Giacomini, E. Castro-Ruiz and Č. Brukner, *Quantum mechanics and the covariance of physical laws in quantum reference frames*, *Nature Commun.* **10** (2019) 494 [arXiv:1712.07207] [INSPIRE].

[41] J.D. Jackson, *Classical Electrodynamics*, John Wiley and Sons Inc., New York, U.S.A. (1999).

Classical and Quantum Gravity

PAPER

Hamiltonian analysis of the *BFCG* formulation of general relativity

To cite this article: A Mikovi *et al* 2019 *Class. Quantum Grav.* **36** 015005

IOP Publishing Classical and Quantum Gravity

Class. Quantum Grav. **36** (2019) 015005 (26pp) https://doi.org/10.1088/1361-6382/aaee25

Hamiltonian analysis of the *BFCG* formulation of general relativity

A Miković[1,2], M A Oliveira[2] and M Vojinović[3]

[1] Departamento de Matemática, Universidade Lusófona de Humanidades e Tecnologias, Av. do Campo Grande 376, 1749-024 Lisboa, Portugal
[2] Grupo de Física Matemática, Faculdade de Ciências da Universidade de Lisboa, Campo Grande, Edifício C6, 1749-016 Lisboa, Portugal
[3] Group for Gravitation, Particles and Fields, Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia

E-mail: amikovic@ulusofona.pt, masm.oliveira@gmail.com and vmarko@ipb.ac.rs

Received 17 July 2018, revised 29 October 2018
Accepted for publication 5 November 2018
Published 3 December 2018

Abstract
We perform the complete Hamiltonian analysis of the BFCG action for general relativity. We determine all the constraints of the theory and classify them into the first-class and the second-class constraints. We also show how the canonical formulation of BFCG general relativity reduces to the Einstein–Cartan and triad canonical formulations. The reduced phase space analysis also gives a 2-connection which is suitable for the construction of a spin-foam basis which will be a categorical generalization of the spin-network basis from loop quantum gravity.

Keywords: BFCG model, Poincaré 2-group, general relativity, Hamiltonian analysis, algebra of constraints, spin-cube model, spin-foam model

1. Introduction

Among the fundamental problems of modern theoretical physics, by far the most prominent one is the construction of the tentative theory of quantum gravity (QG). There are many approaches to QG, one of which is called loop quantum gravity (LQG), see [1–4]. As with any other physical system, the quantization of the gravitational field can be performed either canonically, using the Hamiltonian framework, or covariantly, using the Lagrangian, i.e. the path integral framework. Within the LQG approach, in the canonical framework [2] one chooses the connection variables and their momenta as fundamental fields for gravity, and uses them to construct an appropriate physical Hilbert space, giving rise to the spin-network states. In the covariant framework, one puts the connection variables onto a spacetime triangulation, see [3, 4], and uses this construction to define a path integral for gravity, giving rise to the spin-foam (SF) models.

The *BFCG* formulation of GR [5] was invented in order to find a categorical generalization of the SF models. A categorical generalization of a SF model is called a spin-cube model, since the path integral is based on a colored 3-complex where the colors are the representations of a 2-group [5, 6]. The 2-group, see [7] for a review and references, replaces the Lorentz group, and becomes the fundamental algebraic structure. The reason for introducing spin-cube models was that the SF models have two problems. One problem is that the classical limit of a SF model is described by the area-Regge action [6, 8]. The second problem is that the fermions cannot be coupled to a SF model [5]. These two problems are caused by the fact that the tetrads are absent from the Plebanski action, see [3, 9–11], which is used as the classical action to build the SF amplitudes. The *BFCG* action for GR is a categorical generalization of the Plebanski action, and the *BFCG* action contains both the B field and the tetrads [5].

The path integral quantization of *BFCG* GR reduces to the Regge path integral [6]. However, in the case of the canonical quantization, it is not known what kind of theories can be obtained. It was argued in [12] that a spin-foam basis should exist, as a categorical generalization of the spin-network basis from LQG, but in order to rigorously prove such a statement, one needs a canonical formulation of the *BFCG* GR theory. The canonical analysis of *BFCG* GR action is much more complicated than the canonical analysis of the Einstein–Hilbert action. One can see what kind of canonical analysis will be necessary from the canonical analysis of simpler but related actions given by the unconstrained *BFCG* action [13] or the Einstein–Cartan action [14].

In this paper we present the Hamiltonian analysis of the *BFCG* GR theory in full detail. Despite being straightforward, the calculations involved are quite nontrivial, so it is important to perform the full analysis in a systematic manner. Due to the amount of material presented, subsequent topics such as quantization schemes and similar have been postponed for future work, while the present paper deals only with the canonical structure of the classical theory.

The paper is organized as follows. In section 2 we give an overview of the *BFCG* GR action, discuss the Lagrange equations of motion, and prepare for the Hamiltonian analysis. The first part of the Hamiltonian analysis is done in section 3. We evaluate the conjugate momenta for the fields, obtain the primary constraints and construct the Hamiltonian of the theory. Then we impose consistency conditions on on all constraints in turn, giving rise to a full set of primary, secondary and tertiary constraints, along with some determined Lagrange multipliers. Section 4 is devoted to the second part of the Hamiltonian analysis—the separation of the constraints into first and second class, computing their algebra, and determining the number of physical degrees of freedom. Building on these results, in section 5 we discuss various avenues for the elimination of the second class constraints from the theory, gauge fixing conditions and the analysis of the first class constraints, and the resulting possible reductions of the phase space of the theory. Section 6 contains our concluding remarks, discussion of the results and future lines of research. The appendix contains four sections with a lot of technical details about the calculations performed in the main text.

Our notation and conventions are as follows. The spacetime indices are denoted with lowercase Greek alphabet letters from the middle of the alphabet $\lambda, \mu, \nu, \rho, \ldots$ and take the values $0, 1, 2, 3$. When discussing the foliation of spacetime into space and time, the spacetime indices are split as $\mu = (0, i)$, where the lowercase indices from the middle of the Latin alphabet $i, j, k, \ldots$ take only spacelike values $1, 2, 3$. The Poincaré group indices are denoted with lowercase letters from the beginning of the Latin alphabet, $a, b, c, \ldots$ and take the values $0, 1, 2, 3$, while their spacelike counterparts are denoted by the lower-case Greek letters from the beginning of the alphabet $\alpha, \beta, \ldots$, and take the values $1, 2, 3$. The group indices are raised and lowered with the Minkowski metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$. Capital Latin indices $A, B, C, \ldots$ represent multi-index notation, and are used to count the second class constraints, fields and momenta, and various other objects, depending on the context. Antisymmetrization is denoted with the square brackets around the indices with the $1/2$ factor, $X_{[ab]} \equiv (X_{ab} - X_{ba})/2$. In order to simplify the notation involving Poisson brackets, we will adopt the following convention. The left quantity in every Poisson bracket is assumed to be evaluated at the point $x = (t, \vec{x})$, while the right quantity at the point $y = (t, \vec{y})$. In addition, we use the shorthand notation for the 3-dimensional Dirac delta function $\delta^{(3)} \equiv \delta^{(3)}(\vec{x} - \vec{y})$. For example, an expression

$$\{ U^{\alpha}(t, \vec{x}) \, , \, V^{\beta}(t, \vec{y}) \} = W^{\alpha\beta}(t, \vec{x}) \delta^{(3)}(\vec{x} - \vec{y}) + Z^{\alpha\beta i}(t, \vec{x}) \, \partial_i \delta^{(3)}(\vec{x} - \vec{y}), \qquad (1)$$

where $\partial_i = \partial/\partial x^i$, can be written more compactly as

$$\{ U^{\alpha} \, , \, V^{\beta} \} = W^{\alpha\beta} \delta^{(3)} + Z^{\alpha\beta i} \, \partial_i \delta^{(3)}, \qquad (2)$$

usually without any ambiguity. In the rare ambiguous cases, the expressions will be written more explicitly. This notation will be used systematically unless stated otherwise.

2. *BFCG* action for GR

Given a Lie group $\mathcal{G}$ and its Lie algebra $\mathfrak{g}$, and the $\mathfrak{g}$-valued connection one-form A on a spacetime manifold $\mathcal{M}$, the BF action (see [15] for a review and applications to gravity)

$$S_{BF} = \int_{\mathcal{M}} \langle B \wedge F \rangle_{\mathfrak{g}}, \qquad (3)$$

describes the dynamics of flat connections, where $F = \mathrm{d}A + A \wedge A$ is the curvature two-form. B is a $\mathfrak{g}$-valued Lagrange multiplier two-form and $\langle , \rangle_{\mathfrak{g}}$ represents the invariant nondegenerate symmetric bilinear form in $\mathfrak{g}$. The BF theory relevant for the construction of spin-foam models is based on the Lorentz group $SO(3,1)$. A categorical generalization of the BF theory is based on the concept of a strict 2-group, which is a pair of groups $(\mathcal{G}, \mathcal{H})$ with certain maps between them (see [7] for details). The corresponding theory of flat 2-connections is called the *BFCG* theory [16, 17], and its dynamics is given by the action

$$S_{BFCG} = \int_{\mathcal{M}} \left[\langle B \wedge F \rangle_{\mathfrak{g}} + \langle C \wedge G \rangle_{\mathfrak{h}} \right]. \qquad (4)$$

The second term in (4) consists of a $\mathfrak{h}$-valued one-form Lagrange multiplier C, and a curvature three-form $G = \mathrm{d}\beta + A \wedge \beta$ for the $\mathfrak{h}$-valued two-form β, where $\mathfrak{h}$ is the Lie algebra of the group $\mathcal{H}$. The pair (A, β) is called the 2-connection for the 2-group, while the pair (F, G) is the corresponding 2-curvature. The $\langle , \rangle_{\mathfrak{h}}$ is the invariant nondegenerate symmetric bilinear form in $\mathfrak{h}$, which is $\mathfrak{g}$-invariant.

The Poincaré 2-group, defined by $\mathcal{G} = SO(3,1)$ and $\mathcal{H} = \mathbb{R}^4$, is relevant for GR since the Einstein equations can be obtained from a constrained *BFCG* action [5], given by

$$S_{GR} = \int_{\mathcal{M}} \left[\langle B \wedge R \rangle_{\mathfrak{g}} + \langle e \wedge G \rangle_{\mathfrak{h}} - \langle \phi \wedge (B - \star(e \wedge e)) \rangle_{\mathfrak{g}} \right] . \qquad (5)$$

Here we have relabeled $C \equiv e$ and $F \equiv R$, since in the case of the Poincaré 2-group these fields have the interpretation of the tetrad field and the curvature two-form for the spin connection $A \equiv \omega$. The $\mathfrak{g}$-valued two-form ϕ is an additional Lagrange multiplier, featuring in the simplicity constraint term. The $\star$ is the Hodge dual operator for the Minkowski space.

The action (5) can be written as

$$S_{GR} = \int_{\mathcal{M}} \left[B_{ab} \wedge R^{ab} + e^a \wedge G_a - \phi^{ab} \wedge \left(B_{ab} - \varepsilon_{abcd}\, e^c \wedge e^d \right) \right], \tag{6}$$

where the curvatures R^{ab} and G^a are given by

$$R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \tag{7}$$

$$G^a = \nabla \beta^a \equiv \mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b. \tag{8}$$

The action (6) can even be extended to include the cosmological constant, and it is related to the MacDowell–Mansouri action [18–22], see appendix E for details.

It is convenient to introduce the torsion 2-form

$$T^a = \nabla e^a \equiv \mathrm{d}e^a + \omega^a{}_b \wedge e^b, \tag{9}$$

so that one can rewrite the action as

$$S_{PGT} = \int_{\mathcal{M}} \left[B_{ab} \wedge R^{ab} + \beta^a \wedge T_a - \phi^{ab} \wedge \left(B_{ab} - \varepsilon_{abcd}\, e^c \wedge e^d \right) \right] \tag{10}$$

by using the integration by parts. The action (10) is a constrained *BF* action for the Poincaré group, since the tetrads and the spin connection can be considered as components of a Poincaré group connection, while the curvature and the torsion are the components of the Poincaré group curvature [12]. This equivalence of a Poincaré gauge theory formulation to a 2-group gauge theory formulation is specific to 4 spacetime dimensions only.

The relationship between the topological, unconstrained versions of the actions (6) and (10) has been discussed in detail in [13]. There, a real parameter ξ was introduced to interpolate between the two actions, the full Hamiltonian analysis was performed, and the implications of the parameter ξ for the structure of the resulting phase space were studied in detail. It is noteworthy that the actions (6) and (10) differ from the actions discussed in [13] only by the presence of the simplicity constraint term, which is the same for both actions and does not contain any time derivatives. Therefore, the presence of the simplicity constraint does not change any results of [13] pertaining to the ξ parameter, and all conclusions related to ξ given in [13] carry over unmodified to the constrained actions (6) and (10) discussed in this paper. Given this situation, we opt not to introduce and discuss the ξ parameter again in this paper, and refer the reader to [13] instead.

It is clear that the actions (6) and (10) give rise to the same set of equations of motion, since these do not depend on the boundary. Taking the variation of (6) with respect to all the variables, one obtains

$$\delta B : R^{ab} - \phi^{ab} = 0, \tag{11}$$

$$\delta\beta : T^a = 0, \tag{12}$$

$$\delta e : G_a + 2\varepsilon_{abcd}\, \phi^{bc} \wedge e^d = 0, \tag{13}$$

$$\delta\omega : \nabla B^{ab} - e^{[a} \wedge \beta^{b]} = 0, \tag{14}$$

$$\delta\phi : B_{ab} - \varepsilon_{abcd}\, e^c \wedge e^d = 0, \tag{15}$$

where the covariant exterior derivative of B^{ab} is defined as

$$\nabla B^{ab} \equiv \mathrm{d}B^{ab} + \omega^a{}_c \wedge B^{cb} + \omega^b{}_c \wedge B^{ac}. \tag{16}$$

One can simplify the equations of motion in the following way. Taking the covariant exterior derivative of (15) and using (12) one obtains $\nabla B^{ab} = 0$. Substituting this into (14) one further obtains $e^{[a} \wedge \beta^{b]} = 0$. Under the assumption that $\det(e^a{}_\mu) \neq 0$, it follows that $\beta^a = 0$ (see appendix in [5] for proof), and therefore also $G^a = 0$. As a consequence, we see that the equations of motion (11)–(15) are equivalent to the following system:

- the equation that determines the multiplier ϕ^{ab} in terms of curvature,

$$\phi^{ab} = R^{ab}, \tag{17}$$

- the equation that determines the multiplier B^{ab} in terms of tetrads,

$$B_{ab} = \varepsilon_{abcd}\, e^c \wedge e^d, \tag{18}$$

- the equation that determines β^a,

$$\beta^a = 0, \tag{19}$$

- the equation for the torsion,

$$T^a = 0, \tag{20}$$

- and the Einstein field equation,

$$\varepsilon_{abcd}\, R^{bc} \wedge e^d = 0. \tag{21}$$

Finally, for the convenience of the Hamiltonian analysis, we need to rewrite both the action and the equations of motion in a local coordinate frame. Choosing $\mathrm{d}x^\mu$ as basis one-forms, we can expand the fields in the standard fashion:

$$e^a = e^a{}_\mu \mathrm{d}x^\mu, \qquad \omega^{ab} = \omega^{ab}{}_\mu \mathrm{d}x^\mu, \tag{22}$$

$$B^{ab} = \frac{1}{2} B^{ab}{}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \quad \beta^a = \frac{1}{2}\beta^a{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \quad \phi^{ab} = \frac{1}{2}\phi^{ab}{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{23}$$

Similarly, the field strengths for ω, e and β are

$$\begin{aligned} R^{ab} &= \frac{1}{2} R^{ab}{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \\ T^a &= \frac{1}{2} T^a{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \\ G^a &= \frac{1}{6} G^a{}_{\mu\nu\rho}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho. \end{aligned} \tag{24}$$

Using the relations (7)–(9), we can write the component equations

$$\begin{aligned} R^{ab}{}_{\mu\nu} &= \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^a{}_{c\mu}\omega^{cb}{}_\nu - \omega^a{}_{c\nu}\omega^{cb}{}_\mu, \\ T^a{}_{\mu\nu} &= \partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu, \\ G^a{}_{\mu\nu\rho} &= \partial_\mu \beta^a{}_{\nu\rho} + \partial_\nu \beta^a{}_{\rho\mu} + \partial_\rho \beta^a{}_{\mu\nu} + \omega^a{}_{b\mu}\beta^b{}_{\nu\rho} + \omega^a{}_{b\nu}\beta^b{}_{\rho\mu} + \omega^a{}_{b\rho}\beta^b{}_{\mu\nu}. \end{aligned} \tag{25}$$

Substituting expansions (22)–(24) into the action, we obtain

$$S = \int_{\mathcal{M}} \mathrm{d}^4 x\, \varepsilon^{\mu\nu\rho\sigma} \left[\frac{1}{4} B_{ab\mu\nu} R^{ab}{}_{\rho\sigma} + \frac{1}{6} e_{a\mu} G^a{}_{\nu\rho\sigma} - \frac{1}{4}\phi^{ab}{}_{\mu\nu} \left(B_{ab\rho\sigma} - 2\varepsilon_{abcd}\, e^c{}_\rho e^d{}_\sigma \right) \right]. \tag{26}$$

Assuming that the spacetime manifold has the topology $\mathcal{M} = \Sigma \times \mathbb{R}$, where Σ is a 3-dimensional spacelike hypersurface, from the above action we can read off the Lagrangian, which is the integral of the Lagrangian density over the hypersurface Σ:

$$L = \int_{\Sigma} \mathrm{d}^3 x \, \varepsilon^{\mu\nu\rho\sigma} \left[\frac{1}{4} B_{ab\mu\nu} R^{ab}{}_{\rho\sigma} + \frac{1}{6} e_{a\mu} G^{a}{}_{\nu\rho\sigma} - \frac{1}{4} \phi^{ab}{}_{\mu\nu} \left(B_{ab\rho\sigma} - 2\varepsilon_{abcd} \, e^{c}{}_{\rho} e^{d}{}_{\sigma} \right) \right]. \tag{27}$$

Finally, the component form of equations of motion (17)–(21) is:

$$\begin{aligned} &\phi^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu}, \qquad && B_{ab\mu\nu} = 2\varepsilon_{abcd} \, e^{c}{}_{\mu} e^{d}{}_{\nu}, \\ &\beta^{a}{}_{\mu\nu} = 0, && T^{a}{}_{\mu\nu} = 0, \\ &\varepsilon^{\lambda\mu\nu\rho} \varepsilon_{abcd} \, R^{bc}{}_{\mu\nu} e^{d}{}_{\rho} = 0. && \end{aligned} \tag{28}$$

3. Hamiltonian analysis

Now we turn to the Hamiltonian analysis. A detailed review of the general formalism can be found in [14], chapter V. In addition, a good pedagogical example of the Hamiltonian analysis which is relevant for our case is the topological *BFCG* gravity [13].

3.1. Primary constraints and the Hamiltonian

As a first step, we calculate the momenta π corresponding to the field variables $B^{ab}{}_{\mu\nu}$, $\phi^{ab}{}_{\mu\nu}$, $e^{a}{}_{\mu}$, $\omega^{ab}{}_{\mu}$ and $\beta^{a}{}_{\mu\nu}$. Differentiating the action (26) with respect to the time derivative of the appropriate fields, we obtain the momenta as follows:

$$\begin{aligned} \pi(B)_{ab}{}^{\mu\nu} &= \frac{\delta S}{\delta \partial_0 B^{ab}{}_{\mu\nu}} = 0, \\ \pi(\phi)_{ab}{}^{\mu\nu} &= \frac{\delta S}{\delta \partial_0 \phi^{ab}{}_{\mu\nu}} = 0, \\ \pi(e)_{a}{}^{\mu} &= \frac{\delta S}{\delta \partial_0 e^{a}{}_{\mu}} = 0, \\ \pi(\omega)_{ab}{}^{\mu} &= \frac{\delta S}{\delta \partial_0 \omega^{ab}{}_{\mu}} = \varepsilon^{0\mu\nu\rho} B_{ab\nu\rho}, \\ \pi(\beta)_{a}{}^{\mu\nu} &= \frac{\delta S}{\delta \partial_0 \beta^{a}{}_{\mu\nu}} = -\varepsilon^{0\mu\nu\rho} e_{a\rho}. \end{aligned} \tag{29}$$

None of the momenta can be solved for the corresponding 'velocities', so they all give rise to primary constraints:

$$\begin{aligned} P(B)_{ab}{}^{\mu\nu} &\equiv \pi(B)_{ab}{}^{\mu\nu} \approx 0, \\ P(\phi)_{ab}{}^{\mu\nu} &\equiv \pi(\phi)_{ab}{}^{\mu\nu} \approx 0, \\ P(e)_{a}{}^{\mu} &\equiv \pi(e)_{a}{}^{\mu} \approx 0, \\ P(\omega)_{ab}{}^{\mu} &\equiv \pi(\omega)_{ab}{}^{\mu} - \varepsilon^{0\mu\nu\rho} B_{ab\nu\rho} \approx 0, \\ P(\beta)_{a}{}^{\mu\nu} &\equiv \pi(\beta)_{a}{}^{\mu\nu} + \varepsilon^{0\mu\nu\rho} e_{a\rho} \approx 0. \end{aligned} \tag{30}$$

The weak, on-shell equality is denoted '$\approx$', as opposed to the strong, off-shell equality which is denoted by the usual symbol '$=$'.

Next we introduce the fundamental simultaneous Poisson brackets between the fields and their conjugate momenta,

$$\begin{aligned} \{ B^{ab}{}_{\mu\nu} \,,\, \pi(B)_{cd}{}^{\rho\sigma} \} &= 4\delta^a_{[c}\delta^b_{d]}\delta^\rho_{[\mu}\delta^\sigma_{\nu]}\delta^{(3)}, \\ \{ \phi^{ab}{}_{\mu\nu} \,,\, \pi(\phi)_{cd}{}^{\rho\sigma} \} &= 4\delta^a_{[c}\delta^b_{d]}\delta^\rho_{[\mu}\delta^\sigma_{\nu]}\delta^{(3)}, \\ \{ e^a{}_\mu \,,\, \pi(e)_b{}^\nu \} &= \delta^a_b\delta^\nu_\mu\delta^{(3)}, \\ \{ \omega^{ab}{}_\mu \,,\, \pi(\omega)_{cd}{}^\nu \} &= 2\delta^a_{[c}\delta^b_{d]}\delta^\nu_\mu\delta^{(3)}, \\ \{ \beta^a{}_{\mu\nu} \,,\, \pi(\beta)_b{}^{\rho\sigma} \} &= 2\delta^a_b\delta^\rho_{[\mu}\delta^\sigma_{\nu]}\delta^{(3)}, \end{aligned} \tag{31}$$

and we employ them to calculate the algebra of primary constraints,

$$\begin{aligned} \{ P(B)^{abjk} \,,\, P(\omega)_{cd}{}^i \} &= 4\varepsilon^{0ijk}\delta^a_{[c}\delta^b_{d]}\delta^{(3)}, \\ \{ P(e)^{ak} \,,\, P(\beta)_b{}^{ij} \} &= -\varepsilon^{0ijk}\delta^a_b\delta^{(3)}, \end{aligned} \tag{32}$$

while all other Poisson brackets vanish.

Next we construct the canonical, on-shell Hamiltonian:

$$\begin{aligned} H_c = \int_\Sigma \mathrm{d}^3\vec{x} \Big[&\frac{1}{4}\pi(B)_{ab}{}^{\mu\nu}\partial_0 B^{ab}{}_{\mu\nu} + \frac{1}{4}\pi(\phi)_{ab}{}^{\mu\nu}\partial_0\phi^{ab}{}_{\mu\nu} + \pi(e)_a{}^\mu\partial_0 e^a{}_\mu \\ &+\frac{1}{2}\pi(\omega)_{ab}{}^\mu\partial_0\omega^{ab}{}_\mu + \frac{1}{2}\pi(\beta)_a{}^{\mu\nu}\partial_0\beta^a{}_{\mu\nu} \Big] - L. \end{aligned} \tag{33}$$

The factors $1/4$ and $1/2$ are introduced to prevent overcounting of variables. Using (25) and (27), one can rearrange the expressions such that all velocities are multiplied by primary constraints, and therefore vanish from the Hamiltonian. After some algebra, the resulting expression can be written as

$$\begin{aligned} H_c = -\int_\Sigma &\mathrm{d}^3\vec{x}\,\varepsilon^{0ijk}\Big[\frac{1}{2}B_{ab0i}\left(R^{ab}{}_{jk} - \phi^{ab}{}_{jk}\right) + e^a{}_0\left(\frac{1}{6}G_{aijk} + \varepsilon_{abcd}\,\phi^{bc}{}_{ij}e^d{}_k\right) \\ &+\frac{1}{2}\beta_{a0k}T^a{}_{ij} + \frac{1}{2}\omega_{ab0}\left(\nabla_i B^{ab}{}_{jk} - e^a{}_i\beta^b{}_{jk}\right) - \frac{1}{2}\phi^{ab}{}_{0i}\left(B_{abjk} - 2\varepsilon_{abcd}\,e^c{}_j e^d{}_k\right)\Big], \end{aligned} \tag{34}$$

up to a boundary term. The canonical Hamiltonian does not depend on any momenta, but only on fields and their spatial derivatives. Finally, introducing Lagrange multipliers λ for each of the primary constraints, we construct the total, off-shell Hamiltonian:

$$\begin{aligned} H_T = H_c + \int_\Sigma \mathrm{d}^3\vec{x} \Big[&\frac{1}{4}\lambda(B)^{ab}{}_{\mu\nu}P(B)_{ab}{}^{\mu\nu} + \frac{1}{4}\lambda(\phi)^{ab}{}_{\mu\nu}P(\phi)_{ab}{}^{\mu\nu} \\ &+\lambda(e)^a{}_\mu P(e)_a{}^\mu + \frac{1}{2}\lambda(\omega)^{ab}{}_\mu P(\omega)_{ab}{}^\mu + \frac{1}{2}\lambda(\beta)^a{}_{\mu\nu}P(\beta)_a{}^{\mu\nu}\Big]. \end{aligned} \tag{35}$$

3.2. Consistency procedure

We proceed with the calculation of the consistency requirements for the constraints. The consistency requirement is that the time derivative of each constraint (or equivalently its Poisson bracket with the total Hamiltonian (35)) must vanish on-shell. This requirement can either give rise to a new constraint, or determine some multiplier, or be satisfied identically. In our case, the consistency requirements give rise to a complicated chain structure, depicted in the following diagram:

$$
\begin{array}{lclclcl}
P(\beta)_a{}^{0i} & \xrightarrow{1} & S(T)^{ai} & \xrightarrow{15} & T(eR\phi)^{ai} & \multirow{2}{*}{$\Big]\xrightarrow{16}$} & T(eR\phi)^{ab}{}_k \\
P(e)_a{}^{i} & \xrightarrow{11} & S(eR\phi)^{ai} & \text{———} & \text{———} & & \\
P(B)_{ab}{}^{0i} & \xrightarrow{2} & S(R\phi)^{abi} & \xrightarrow{13} & \lambda(\phi)_{abij} & & \big\downarrow 17 \\
P(\phi)_{ab}{}^{ij} & \xrightarrow{3} & S(Bee)^{abij} & \xrightarrow{14} & \lambda(B)_{ab0i} & & \lambda(\phi)^{ab}{}_{0i} \\
P(B)_{ab}{}^{ij} & \xrightarrow{4} & \lambda(\omega)_{abi} & & & & \\
P(\beta)_a{}^{ij} & \xrightarrow{5} & \lambda(e)_{ai} & & & & \\
P(\omega)_{ab}{}^{i} & \xrightarrow{6} & \lambda(B)_{abij} & & & & \\
P(\phi)_{ab}{}^{0i} & \xrightarrow{7} & S(Bee)^{abi} & \Big]\xrightarrow{8} & T(\beta)^a{}_{\mu\nu} & \xrightarrow{9} & \lambda(\beta)^a{}_{\mu\nu} \\
P(\omega)_{ab}{}^{0} & \text{———} & \text{———} & & & & \\
P(e)_a{}^{0} & \xrightarrow{10} & S(eR)^{a} & \xrightarrow{12} & 0 & &
\end{array}
$$

Here every arrow represents one consistency requirement, and numbers on the arrows denote the order in which we will discuss them. Steps 8 and 16 involve multiple constraints simultaneously, and will require special consideration. Primary, secondary and tertiary constraints are denoted as P, S and T, respectively.

We begin by discussing consistency conditions 1–7,

$$
\begin{array}{l}
\dot{P}(\beta)_a{}^{0i} \approx 0, \quad \dot{P}(B)_{ab}{}^{0i} \approx 0, \quad \dot{P}(\phi)_{ab}{}^{ij} \approx 0, \quad \dot{P}(\phi)_{ab}{}^{0i} \approx 0, \\
\dot{P}(B)_{ab}{}^{ij} \approx 0, \quad \dot{P}(\beta)_a{}^{ij} \approx 0, \quad \dot{P}(\omega)_{ab}{}^{i} \approx 0.
\end{array} \tag{36}
$$

Calculating the corresponding Poisson brackets with the total Hamiltonian, these give rise to the following secondary constraints,

$$
\begin{array}{lcl}
S(T)^{ai} & \equiv & \varepsilon^{0ijk} T^a{}_{jk} \approx 0, \\
S(R\phi)^{abi} & \equiv & \varepsilon^{0ijk} \left(R^{ab}{}_{jk} - \phi^{ab}{}_{jk} \right) \approx 0, \\
S(Bee)^{abij} & \equiv & \varepsilon^{0ijk} \left(B^{ab}{}_{0k} - 2\varepsilon^{abcd}\, e_{c0} e_{dk} \right) \approx 0, \\
S(Bee)^{abi} & \equiv & \varepsilon^{0ijk} \left(B^{ab}{}_{jk} - 2\varepsilon^{abcd}\, e_{cj} e_{dk} \right) \approx 0,
\end{array} \tag{37}
$$

and determine the following multipliers,

$$
\begin{array}{lcl}
\lambda(\omega)^{ab}{}_i & \approx & \nabla_i \omega^{ab}{}_0 + \phi^{ab}{}_{0i}, \\
\lambda(e)^a{}_i & \approx & \nabla_i e^a{}_0 - \omega^a{}_{b0} e^b{}_i, \\
\lambda(B)^{ab}{}_{ij} & \approx & 4\varepsilon^{abcd} \left(\nabla_{[i} e_{c0} - \omega_{cf0} e^f{}_{[i} \right) e_{dj]} + e^{[a}{}_0 \beta^{b]}{}_{ij} - 2e^{[a}{}_{[i} \beta^{b]}{}_{0j]}.
\end{array} \tag{38}
$$

In step 8 we discuss the consistency conditions

$$
\dot{S}(Bee)^{abi} \approx 0, \qquad \dot{P}(\omega)_{ab}{}^0 \approx 0, \tag{39}
$$

simultaneously. Calculating the time derivatives, we obtain

$$
\varepsilon^{0ijk} \left(e^{[a}{}_0 \beta^{b]}{}_{jk} - 2e^{[a}{}_j \beta^{b]}{}_{0k} \right) \approx 0, \qquad \varepsilon^{0ijk}\, e^{[a}{}_i \beta^{b]}{}_{jk} \approx 0, \tag{40}
$$

which can be jointly written as a covariant equation

$$
\varepsilon^{\mu\nu\rho\sigma}\, e^{[a}{}_\nu \beta^{b]}{}_{\rho\sigma} \approx 0. \tag{41}
$$

With the assumption that $\det(e^a{}_\mu) \neq 0$, this can be solved for β^a, giving a set of very simple tertiary constraints:

$$T(\beta)^a{}_{\mu\nu} \equiv \beta^a{}_{\mu\nu} \approx 0. \tag{42}$$

At this point we can immediately analyze the consistency step 9 as well. Taking the time derivative of (42), one easily determines the corresponding multipliers,

$$\lambda(\beta)^a{}_{\mu\nu} \approx 0. \tag{43}$$

Next, in steps 10 and 11, from the consistency conditions for the remaining two primary constraints,

$$\dot{P}(e)_a{}^0 \approx 0, \qquad \dot{P}(e)_a{}^i \approx 0, \tag{44}$$

we obtain two new secondary constraints,

$$\begin{aligned} S(eR)_a &\equiv \varepsilon^{0ijk}\varepsilon_{abcd} e^b{}_i R^{cd}{}_{jk} \approx 0, \\ S(eR\phi)_a{}^i &\equiv \varepsilon^{0ijk}\varepsilon_{abcd}\left(e^b{}_0 R^{cd}{}_{jk} - 2e^b{}_j \phi^{cd}{}_{0k}\right) \approx 0. \end{aligned} \tag{45}$$

In step 12 we need to discuss the consistency condition for the constraint $S(eR)_a$. After a straightforward but tedious calculation, one eventually ends up with the following expression:

$$\dot{S}(eR)_a = \nabla_i S(eR\phi)_a{}^i + \omega^b{}_{a0} S(eR)_b + 2\varepsilon_{abcd}\phi^{cd}{}_{0k} S(T)^{bk}, \tag{46}$$

up to terms proportional to primary constraints. Since the time derivative is already expressed as a linear combination of constraints, the consistency condition is trivially satisfied, which is denoted with a zero in the diagram above.

Moving on to steps 13–15, the consistency conditions

$$\dot{S}(R\phi)^{abi} \approx 0, \qquad \dot{S}(Bee)^{abij} \approx 0, \qquad \dot{S}(T)^{ai} \approx 0, \tag{47}$$

determine the multipliers

$$\begin{aligned} \lambda(\phi)^{ab}{}_{jk} &\approx 2\omega^{[a}{}_{c0} R^{b]c}{}_{jk} + 2\nabla_{[j}\phi^{ab}{}_{0k]}, \\ \lambda(B)_{ab0k} &\approx 2\varepsilon_{abcd}\left[e^d{}_k \lambda(e)^c{}_0 - e^d{}_0 \nabla_k e^c{}_0 + \omega^c{}_{f0} e^d{}_0 e^f{}_k\right], \end{aligned} \tag{48}$$

and another tertiary constraint

$$T(eR\phi)^{ai} \equiv \varepsilon^{0ijk}\left(R^{ab}{}_{jk} e_{b0} + 2\phi^{ab}{}_{0j} e_{bk}\right) \approx 0. \tag{49}$$

Now we turn to step 16. At this point there are only two constraints, $T(eR\phi)^{ai}$ and $S(eR\phi)^{ai}$, whose consistency conditions have not been discussed yet. To this end, note that these two constraints can be rewritten into a very similar form,

$$\begin{aligned} S(eR\phi)_a{}^i &= \varepsilon_{abcd}\varepsilon^{0ijk}\left(e^b{}_0 R^{cd}{}_{jk} - 2e^b{}_j \phi^{cd}{}_{0k}\right), \\ T(eR\phi)_a{}^i &= \eta_{ac}\eta_{bd}\varepsilon^{0ijk}\left(e^b{}_0 R^{cd}{}_{jk} - 2e^b{}_j \phi^{cd}{}_{0k}\right), \end{aligned} \tag{50}$$

where the identical expression in parentheses is contracted with ε_{abcd} in the first constraint and with $\eta_{ac}\eta_{bd}$ in the second. This suggests that we should discuss their consistency conditions simultaneously. As suggested in the diagram above, we will first rewrite these 24 constraints (50) into a system of $18 + 6$ constraints (to be denoted $T(eR\phi)_{abk}$ and $T(eR\phi)_{jk}$ respectively) as follows. Given that the tetrad $e^a{}_\mu$ is nondegenerate, we can freely multiply the constraints with it and split the index μ into space and time components. The $\mu = 0$ part is

$$
\begin{aligned}
e^a{}_0 S(eR\phi)_a{}^i &= -2\varepsilon_{abcd}\varepsilon^{0ijk} e^a{}_0 e^b{}_j \phi^{cd}{}_{0k}, \\
e^a{}_0 T(eR\phi)_a{}^i &= -2\eta_{ac}\eta_{bd}\varepsilon^{0ijk} e^a{}_0 e^b{}_j \phi^{cd}{}_{0k},
\end{aligned} \tag{51}
$$

where the curvature terms have automatically vanished, while the $\mu = m$ part is

$$
\begin{aligned}
e^a{}_m S(eR\phi)_a{}^i &= e^a{}_m \varepsilon_{abcd}\varepsilon^{0ijk} \left(e^b{}_0 R^{cd}{}_{jk} - 2e^b{}_j \phi^{cd}{}_{0k} \right), \\
e^a{}_m T(eR\phi)_a{}^i &= e^a{}_m \eta_{ac}\eta_{bd}\varepsilon^{0ijk} \left(e^b{}_0 R^{cd}{}_{jk} - 2e^b{}_j \phi^{cd}{}_{0k} \right).
\end{aligned} \tag{52}
$$

The system of 18 constraints (52) can be shown to be equivalent to the following constraint:

$$
T(eR\phi)^{ab}{}_k \equiv \phi^{ab}{}_{0k} - e^f{}_0 R^{cd}{}_{ij} F^{abij}{}_{fcdk}, \tag{53}
$$

where $F^{abij}{}_{fcdk}$ is a complicated function of $e^a{}_i$ only. The proof that the system (52) is equivalent to (53) is given in appendix C, and the explicit expression for $F^{abij}{}_{fcdk}$ is given in equation (C.27). Second, introducing the shorthand notation $K_{abcd} \in \{\varepsilon_{abcd}, \eta_{ac}\eta_{bd}\}$ and using (53), we define

$$
T(eR\phi)^i \equiv -2K_{abcd}\varepsilon^{0ijk} e^a{}_0 e^b{}_j e^f{}_0 R^{gh}{}_{mn} F^{cdmn}{}_{fghk}, \tag{54}
$$

which represents a set of $3+3=6$ constraints equivalent to (51). However, a straightforward and meticulous (albeit very long) calculation shows that the expression (54) is already a linear combination of known constraints and Bianchi identities, and is thus already weakly equal to zero. Therefore, $T(eR\phi)^i$ is not a new independent constraint, and its consistency condition is automatically satisfied.

Summing up the step 16, we have replaced the set of constraints (50) by an equivalent set (53). It thus follows that the consistency conditions for $S(eR\phi)_a{}^i$ and $T(eR\phi)_a{}^i$ are equivalent to the consistency condition for $T(eR\phi)^{ab}{}_k$. Consequently, in step 17, we find that the consistency condition

$$
\dot{T}(eR\phi)^{ab}{}_k \approx 0 \tag{55}
$$

determines the multiplier $\lambda(\phi)^{ab}{}_{0k}$ as

$$
\begin{aligned}
\lambda(\phi)^{ab}{}_{0k} \approx{}& \lambda(e)^f{}_0 R^{cd}{}_{ij} F^{abij}{}_{fcdk} + 2e^f{}_0 \left[R^c{}_{hij}\omega^{hd}{}_0 + \nabla_i \phi^{cd}{}_{0j} \right] F^{abij}{}_{fcdk} \\
&+ e^f{}_0 R^{cd}{}_{ij} \frac{\partial F^{abij}{}_{fcdk}}{\partial e^h{}_m} \left(\nabla_m e^h{}_0 - \omega^h{}_{g0} e^g{}_m \right).
\end{aligned} \tag{56}
$$

This concludes the consistency procedure for all constraints.

3.3. Results

Let us sum up the results of the consistency procedure. We have determined the full set of constraints and multipliers as follows: the primary constraints are

$$
P(B)_{ab}{}^{\mu\nu}, \qquad P(\phi)_{ab}{}^{\mu\nu}, \qquad P(\beta)_a{}^{\mu\nu}, \qquad P(\omega)_{ab}{}^{\mu}, \qquad P(e)_a{}^{\mu}, \tag{57}
$$

and they have 36, 36, 24, 24 and 16 components, respectively, or 136 in total. The secondary constraints are

$$
S(T)^{ai}, \qquad S(R\phi)^{abi}, \qquad S(Bee)^{abij}, \qquad S(Bee)^{abi}, \qquad S(eR)^a, \tag{58}
$$

and they have $12+18+18+18+4=70$ components in total. The tertiary constraints are

$$
T(\beta)^a{}_{\mu\nu}, \qquad T(eR\phi)^{ab}{}_i \tag{59}
$$

and they have $24 + 18 = 42$ components. In addition, the determined multipliers are

$$\lambda(B)^{ab}{}_{\mu\nu}, \qquad \lambda(\phi)^{ab}{}_{\mu\nu}, \qquad \lambda(\beta)^{a}{}_{\mu\nu}, \qquad \lambda(\omega)^{ab}{}_{i}, \qquad \lambda(e)^{a}{}_{i}, \tag{60}$$

and they have $36 + 36 + 24 + 18 + 12 = 126$ components. Finally, there are 10 remaining undetermined multipliers,

$$\lambda(\omega)^{ab}{}_{0}, \qquad \lambda(e)^{a}{}_{0}. \tag{61}$$

In total, there are $C = 136 + 70 + 42 = 248$ constraints, 126 determined and 10 undetermined multipliers, the latter corresponding to the 10 parameters of the local Poincaré symmetry of the action.

4. The physical degrees of freedom

Once we have found all the constraints in the theory, we need to classify each constraint as a first-class or a second-class constraint. While some of the second class constraints can be identified from (32), the classification is not easy since constraints are unique only up to linear combinations. The most efficient way to tabulate all first class constraints is to substitute all determined multipliers into the total Hamiltonian (35) and rewrite it in the form

$$H_T = \int \mathrm{d}^3\vec{x} \left[\frac{1}{2}\lambda(\omega)^{ab}{}_0 \, \Phi(\omega)_{ab} + \lambda(e)^a{}_0 \, \Phi(e)_a + \frac{1}{2}\omega^{ab}{}_0 \, \Phi(T)_{ab} + e^a{}_0 \, \Phi(R)_a \right]. \tag{62}$$

The quantities Φ are linear combinations of the constraints, and they must all be of the first class, since the total Hamiltonian weakly commutes with all constraints. Written in terms of the primary and the secondary constraints, the first-class constraints are given by

$$\begin{aligned}
\Phi(\omega)^{ab} &= P(\omega)^{ab0}, \\
\Phi(e)_a &= P(e)_a{}^0 + \frac{1}{2} R^{cd}{}_{ij} F^{fbij}{}_{acdk} P(\phi)_{fb}{}^{0k} + \varepsilon_{abcd} e^b{}_k P(B)^{cd0k}, \\
\Phi(T)^{ab} &= 4\varepsilon^{abcd} e_{ci} S(T)_d{}^i - \nabla_i S(Bee)^{abi} + \varepsilon^{0ijk} e^{[a}{}_i T(\beta)^{b]}{}_{jk} \\
&\quad + 2\varepsilon^{abcd} e^f{}_i e_{cj} P(B)_{fd}{}^{ij} - \nabla_i P(\omega)^{abi} + 2e^{[a}{}_i P(e)^{b]i} \\
&\quad - R^{[ac}{}_{ij} P(\phi)_c{}^{b]ij}, \\
\Phi(R)_a &= -S(eR)_a + R^c{}_{hij} \omega^{hd}{}_0 F^{fbij}{}_{acdk} P(\phi)_{fb}{}^{0k} \\
&\quad + R^{cd}{}_{ij} \frac{\partial F^{fbij}{}_{acdk}}{\partial e^h{}_m} \left(\nabla_m e^h{}_0 - \omega^h{}_{g0} e^g{}_m \right) P(\phi)_{fb}{}^{0k} \\
&\quad - \varepsilon^{0ijk} \nabla_i T(\beta)_{ajk} + \varepsilon_{abcd} e^b{}_i \nabla_j P(B)^{cdij} - \nabla_i P(e)_a{}^i \\
&\quad + \varepsilon_{abcd} \left(\nabla_k e^b{}_0 - \omega^b{}_{f0} e^f{}_k \right) P(B)^{cd0k} \\
&\quad + \frac{1}{2} R^{cd}{}_{ij} F^{fbij}{}_{acdk} \left[S(Bee)_{fb}{}^k + P(\omega)_{fb}{}^k + \nabla_m P(\phi)_{fb}{}^{km} \right. \\
&\qquad \left. - 2\nabla_m \left(e^e{}_0 F^{ghmk}{}_{efbn} P(\phi)_{gh}{}^{0n} \right) \right].
\end{aligned} \tag{63}$$

The constraints (63) are the first-class constraints in the theory. The remaining constraints are of the second class

$$
\begin{array}{lll}
\chi(T)^{ai} & = & S(T)^{ai}, \\
\chi(R\phi)^{abi} & = & S(R\phi)^{abi}, \\
\chi(Bee)^{abij} & = & S(Bee)^{abij}, \\
\chi(Bee)^{abi} & = & S(Bee)^{abi}, \\
\chi(\beta)^{a}{}_{\mu\nu} & = & T(\beta)^{a}{}_{\mu\nu}, \\
\chi(eR\phi)^{ab}{}_{i} & = & T(eR\phi)^{ab}{}_{i}.
\end{array}
\qquad
\begin{array}{lll}
\chi(B)_{ab}{}^{\mu\nu} & = & P(B)_{ab}{}^{\mu\nu}, \\
\chi(\phi)_{ab}{}^{\mu\nu} & = & P(\phi)_{ab}{}^{\mu\nu}, \\
\chi(\beta)_{a}{}^{\mu\nu} & = & P(\beta)_{a}{}^{\mu\nu}, \\
\chi(\omega)_{ab}{}^{i} & = & P(\omega)_{ab}{}^{i}, \\
\chi(e)_{a}{}^{i} & = & P(e)_{a}{}^{i}.
\end{array}
\tag{64}
$$

Note that $\chi(\beta)_a{}^{\mu\nu}$ and $\chi(\beta)^a{}_{\mu\nu}$ are different constraints, despite similar notation. Of course, there is no possibility of confusion since we will never raise or lower spacetime indices of these constraints in the rest of this paper. Also, note that despite the fact that there are 12 components of $\chi(T)^{ai}$, only 6 of them can be considered second class, since the other 6 are part of the first class constraint $\Phi(T)^{ab}$.

At this point we can count the physical degrees of freedom. Given a field theory with N fields whose canonical formulation possesses F first-class constraints, one can gauge fix F fields. The second-class constraints do not generate any gauge symmetries and S second-class constraints are equivalent to vanishing of $S/2$ fields and $S/2$ canonically conjugate momenta. Hence the number of independent (physical) fields is given by

$$n = N - F - \frac{S}{2}\,. \tag{65}$$

The number of field components for each of the fundamental fields is

$\omega^{ab}{}_{\mu}$	$\beta^{a}{}_{\mu\nu}$	$e^{a}{}_{\mu}$	$B^{ab}{}_{\mu\nu}$	$\phi^{ab}{}_{\mu\nu}$
24	24	16	36	36

which gives the total $N = 136$. The number of components of the first class constraints is

$\Phi(e)_a$	$\Phi(\omega)_{ab}$	$\Phi(R)^a$	$\Phi(T)^{ab}$
4	6	4	6

which gives the total of $F = 20$. Similarly, the number of components for the second class constraints is

$\chi(R\phi)^{abi}$	$\chi(Bee)^{abij}$	$\chi(Bee)^{abi}$	$\chi(\beta)^{a}{}_{\mu\nu}$	$\chi(eR\phi)^{ab}{}_{i}$
18	18	18	24	18

and

$\chi(B)_{ab}{}^{\mu\nu}$	$\chi(\phi)_{ab}{}^{\mu\nu}$	$\chi(\beta)_{a}{}^{\mu\nu}$	$\chi(\omega)_{ab}{}^{i}$	$\chi(e)_{a}{}^{i}$	$\chi(T)^{ai}$
36	36	24	18	12	$12-6$

where we have denoted that only 6 of the total 12 components of $\chi(T)^{ai}$ are independent. Thus the total number of independent second class constraints is $S = 228$. This number can also be deduced as the difference between the previously counted total number of constraints $C = 248$ and the number of first class constraints $F = 20$.

Finally, substituting N, F and S into (65), we obtain:

$$n = 136 - 20 - \frac{228}{2} = 2. \tag{66}$$

We conclude that the theory has two physical degrees of freedom, as expected for general relativity.

At this point it is convenient to rewrite the last term in (62) in the traditional ADM form. This is done by projecting the constraint $\Phi(R)_a$ onto the hypersurface Σ and its orthogonal direction. Using the inverse tetrad $e^\mu{}_a$, define the unit vector n_a orthogonal to Σ as

$$n_a \equiv \frac{e^0{}_a}{\sqrt{-g^{00}}} \tag{67}$$

where $g^{00} \equiv \eta^{ab} e^0{}_a e^0{}_b$ is the time-time component of the inverse metric $g^{\mu\nu}$. The vector n_a is thus normalized, $n_a n^a = -1$, and we can define the orthogonal and parallel projectors with respect to Σ as

$$P^a_{\perp b} \equiv -n^a n_b, \qquad P^a_{\| b} \equiv \delta^a_b + n^a n_b. \tag{68}$$

One can then employ these projectors to rewrite the final term in (62) as

$$\begin{aligned} e^a{}_0 \Phi(R)_a &= e^a{}_0 \left(P^b_{\perp a} + P^b_{\| a} \right) \Phi(R)_b \\ &= -e^a{}_0 n_a n^b \Phi(R)_b + e^a{}_0 P^b_{\| a} \left(e^\mu{}_b e^c{}_\mu \right) \Phi(R)_c \\ &= \left[e^a{}_0 n_a \right] \left[-n^b \Phi(R)_b \right] + \left[e^a{}_0 P^b_{\| a} e^i{}_b \right] \left[e^c{}_i \Phi(R)_c \right] + \left[e^a{}_0 P^b_{\| a} e^0{}_b \right] \left[e^c{}_0 \Phi(R)_c \right] \\ &= N\mathcal{H}_\perp + N^i \mathcal{D}_i. \end{aligned} \tag{69}$$

Note that the final term in the second-to-last equality drops out because $P^b_{\| a} e^0{}_b = \sqrt{-g^{00}} P^b_{\| a} n_b \equiv 0$. In the last equality we have introduced the well known ADM lapse and shift functions,

$$N \equiv e^a{}_0 n_a = \frac{1}{\sqrt{-g^{00}}}, \qquad N^i \equiv e^a{}_0 P^b_{\| a} e^i{}_b = -\frac{g^{0i}}{g^{00}}, \tag{70}$$

and we have split the constraint $\Phi(R)_a$ into the scalar constraint and 3-diffeomorphism constraint,

$$\mathcal{H}_\perp \equiv -n^b \Phi(R)_b, \qquad \mathcal{D}_i \equiv e^c{}_i \Phi(R)_c. \tag{71}$$

The constraints $\Phi(T)^{ab}$ are equivalent to the local Lorentz constraints $\mathcal{J}^{ab}$, which generate the local Lorentz transformations, and together with the 10 momentum constraints $\Phi(\omega)^{ab}$ and $\Phi(e)_a$, one can use the scalar constraint $\mathcal{H}_\perp$ and the 3-diffeomorphism constraint $\mathcal{D}_i$ to find the Poisson bracket algebra of the first-class constraints. This algebra takes the form

$$\begin{aligned} \{ \mathcal{J}^{ab}(x) \,,\, \mathcal{J}^{cd}(y) \} &= \frac{1}{2} \left[\eta^{a[c} \mathcal{J}^{d]b}(x) - \eta^{b[c} \mathcal{J}^{d]a}(x) \right] \delta^{(3)}, \\ \{ \mathcal{D}_i(x) \,,\, \mathcal{D}_j(y) \} &= \left[\mathcal{D}_i(x) + \mathcal{D}_i(y) \right] \partial_j \delta^{(3)} + R^{ab}{}_{ij}(x) \mathcal{J}_{ab}(x)\, \delta^{(3)}, \\ \{ \mathcal{D}_i(x) \,,\, \mathcal{H}_\perp(y) \} &= \left[\mathcal{H}_\perp(x) + \mathcal{H}_\perp(y) \right] \partial_i \delta^{(3)} + R^{ab}{}_{i0}(x) \mathcal{J}_{ab}(x)\, \delta^{(3)}, \\ \{ \mathcal{H}_\perp(x) \,,\, \mathcal{H}_\perp(y) \} &= \left[\tilde{g}^{ij}(x) D_j(x) + \tilde{g}^{ij}(y) D_j(y) \right] \partial_i \delta^{(3)}, \end{aligned} \tag{72}$$

while all other first-class Poisson brackets are zero, see [23]. Here it is assumed that $x \equiv (t, \vec{x})$, $y \equiv (t, \vec{y})$ and $\delta^{(3)} \equiv \delta^{(3)}(\vec{x} - \vec{y})$, while $\tilde{g}^{ij}$ is the $3D$ inverse metric, defined in appendix B.

The Poisson brackets between the second class constraints and the Poisson brackets between the first and the second class constraints can be calculated, but we do not give their explicit form because we do not need these Poisson brackets for the purposes of this paper. Their generic structure is given by

$$\{ \chi_I(x) \, , \, \chi_J(y) \} = \Delta_{IJ}(x,y) + \tilde{\Delta}_{IJ}(x,y), \tag{73}$$

and

$$\begin{aligned} \{ \Phi_A(x) \, , \, \chi_I(y) \} \quad = \quad & f_{AI}{}^B(x,y)\,\Phi_B(x) + \tilde{f}_{AI}{}^B(x,y)\,\Phi_B(y) \\ & + f_{AI}{}^J(x,y)\,\chi_J(x) + \tilde{f}_{AI}{}^J(x,y)\,\chi_J(y). \end{aligned} \tag{74}$$

If we denote all the fields collectively as $\theta^N = (e^a{}_\mu, \omega^{ab}{}_\mu, \beta^a{}_{\mu\nu}, B^{ab}{}_{\mu\nu}, \phi^{ab}{}_{\mu\nu})$ and their corresponding momenta as $\pi_N = (\pi(e)_a{}^\mu, \pi(\omega)_{ab}{}^\mu, \pi(\beta)_a{}^{\mu\nu}, \pi(B)_{ab}{}^{\mu\nu}, \pi(\phi)_{ab}{}^{\mu\nu})$, we can denote Δ and f as generalized functions of the type

$$F(\theta(x), \pi(x))\delta^{(3)} + F^i(\theta(x), \pi(x))\,\partial_i\delta^{(3)} + \cdots$$

so that all the coefficients are evaluated at the point x, while $\tilde{\Delta}$ and $\tilde{f}$ as

$$F(\theta(y), \pi(y))\delta^{(3)} + F^i(\theta(y), \pi(y))\,\partial_i\delta^{(3)} + \cdots$$

so that all the coefficients are evaluated at the point y.

5. The phase space reductions

The results of the Hamiltonian analysis imply that the $BFCG$ GR action (6) can be written as

$$\begin{aligned} S_0 \quad = \quad & \int_{t_1}^{t_2} \mathrm{d}t \int_\Sigma \mathrm{d}^3x \left[\pi_N \dot{\theta}^N - \lambda(e)^a{}_0 \Phi(e)_a - \frac{1}{2}\lambda(\omega)^{ab}{}_0 \Phi(\omega)_{ab} \right. \\ & \left. - e^a{}_0 \Phi(R)_a - \frac{1}{2}\omega^{ab}{}_0 \Phi(T)_{ab} - \mu^L \chi_L \right], \end{aligned} \tag{75}$$

where χ_L counts over the set of all second-class constraints (64), while μ^L are Lagrange multipliers for the second-class constraints.

This action can be reduced to an action for a smaller number of canonical variables by partially solving some of the constraints. Solving M first-class constraints $\phi_m = 0$ requires that we make M gauge-fixing conditions $G_m = 0$, such that $\{G_m, G_{m'}\} = 0$ and $\det\{G_m, \phi_{m'}\} \neq 0$. We can then solve the equations $\phi_m = 0$ for the momenta $\pi(G_m)$. The simplest way to do this is to chose G_m to be a set of M coordinates θ_m, and then to solve the corresponding M first-class constraints $\phi_m = 0$ for the momenta π_m. As far as the second-class constraints are concerned, we can solve $2K$ of them for K coordinates and their K momenta.

It is not difficult to see that one can solve the following 192 second-class constraints

$$\begin{array}{lcll} \chi(B)_{ab}{}^{\mu\nu} & \equiv & \pi(B)_{ab}{}^{\mu\nu} & \approx \; 0, \\ \chi(\phi)_{ab}{}^{\mu\nu} & \equiv & \pi(\phi)_{ab}{}^{\mu\nu} & \approx \; 0, \\ \chi(\beta)^a{}_{\mu\nu} & \equiv & \beta^a{}_{\mu\nu} & \approx \; 0, \\ \chi(\beta)_a{}^{\mu\nu} & \equiv & \pi(\beta)_a{}^{\mu\nu} + \varepsilon^{0\mu\nu\rho} e_{a\rho} & \approx \; 0, \\ \chi(Bee)^{abij} & \equiv & \varepsilon^{0ijk}\left(B^{ab}{}_{0k} - 2\varepsilon^{abcd} e_{c0} e_{dk}\right) & \approx \; 0, \\ \chi(Bee)^{abi} & \equiv & \varepsilon^{0ijk}\left(B^{ab}{}_{jk} - 2\varepsilon^{abcd} e_{cj} e_{dk}\right) & \approx \; 0, \\ \chi(R\phi)^{abi} & \equiv & \varepsilon^{0ijk}\left(R^{ab}{}_{jk} - \phi^{ab}{}_{jk}\right) & \approx \; 0, \\ \chi(eR\phi)^{ab}{}_i & \equiv & \phi^{ab}{}_{0i} - e^f{}_0 R^{cd}{}_{jk} F^{abjk}{}_{fcdi} & \approx \; 0, \end{array} \tag{76}$$

for (B, β, ϕ) and their momenta. This will give (B, β, ϕ) and their momenta as functions of the canonical coordinates $(e, \omega, \pi(e), \pi(\omega))$ so that one obtains a reduced phase-space (RPS) theory described by the action

$$S_1 = \int d^4x \left[\pi(e)_a{}^\mu \, \dot{e}^a{}_\mu + \frac{1}{2}\pi(\omega)_{ab}{}^\mu \, \dot{\omega}^{ab}{}_\mu - \lambda(e)^a{}_0 \tilde{\Phi}(e)_a - \frac{1}{2}\lambda(\omega)^{ab}{}_0 \tilde{\Phi}(\omega)_{ab} \right.$$
$$\left. - e^a{}_0 \tilde{\Phi}(R)_a - \frac{1}{2}\omega^{ab}{}_0 \tilde{\Phi}(T)_{ab} - \mu^L \tilde{\chi}_L \right], \tag{77}$$

where $\tilde{C}$ denotes a constraint C on the RPS $(e, \omega, \pi(e), \pi(\omega))$. There are still 20 first-class constraints, namely $\tilde{\Phi}(\omega)^{ab}$, $\tilde{\Phi}(e)_a$, $\tilde{\Phi}(T)^{ab}$, $\tilde{\Phi}(R)_a$, and 36 second-class constraints $\tilde{\chi}_L = (\tilde{\chi}(e)_a{}^i, \tilde{\chi}(\omega)_{ab}{}^i, \tilde{\chi}(T)^{ai})$ on the RPS, so that S_1 is equivalent to the Hamiltonian form of the Einstein–Cartan action [14].

One would like to understand a reduction of S_1 to an action for the triads and spatial spin connections $(e^\alpha{}_i, \omega^{\alpha\beta}{}_i)$. This can be done by gauge fixing $e^a{}_0 = 0$ and solving the corresponding momenta from $\tilde{\Phi}(e)_a = 0$. One can also gauge fix $\omega^{ab}{}_0 = 0$ and eliminate the corresponding momenta from $\tilde{\Phi}(\omega)^{ab} = 0$, as well as to set $e^{\underline{0}}{}_i = 0$ and eliminate the corresponding momenta from $\tilde{\Phi}(T)^{\underline{0}\alpha} = 0$. Note that here we have split the group indices into space and time components, $a = (\underline{0}, \alpha)$ where $\alpha = 1, 2, 3$, see appendix B for details and the notation.

As far as the second-class constraints $\tilde{\chi}_L$ are concerned, one can eliminate $\omega^{\underline{0}\,\alpha}{}_i$ and the corresponding momenta from

$$\tilde{\chi}(\omega)_{\underline{0}\alpha}{}^i = 0, \qquad \tilde{\chi}(e)_a{}^i = 0, \qquad \tilde{\chi}(T)^{\underline{0}i} = 0. \tag{78}$$

Note that there are 24 constraints in (78), but there are six relations among them, so that we have only 18 independent constraints.

Solving the constraints (78) leads to a RPS based on $(e^\alpha{}_i, \omega^{\alpha\beta}{}_i) \cong (e^\alpha{}_i, \omega^\alpha{}_i)$ and their momenta. However, there are still 7 first-class constraints

$$\tilde{\Phi}(R)^a = 0, \qquad \tilde{\Phi}(T)^{\alpha\beta} = 0, \tag{79}$$

and 18 second-class constraints

$$\tilde{\chi}(T)^{\alpha i} = 0, \qquad \tilde{\chi}(\omega)_{\alpha\beta}{}^i = 0. \tag{80}$$

The corresponding action is given by

$$S_2 = \int d^4x \left[\pi(e)_\alpha{}^i \dot{e}^\alpha{}_i + \pi(\omega)_\alpha{}^i \dot{\omega}^\alpha{}_i - N\tilde{\mathcal{H}}_\perp - N^i \tilde{\mathcal{D}}_i - \frac{1}{2}\omega^{\alpha\beta}{}_0 \tilde{\mathcal{J}}_{\alpha\beta} - \mu^L \tilde{\chi}_L \right], \tag{81}$$

where $\tilde{\chi}_L = (\tilde{\chi}(T)^{\alpha i}, \tilde{\chi}(\omega)_{\alpha\beta}{}^i)$ and $\omega^\alpha{}_i \equiv \frac{1}{2}\varepsilon^{\alpha\beta\gamma}\omega_{\beta\gamma i}$.

We can further eliminate $\omega^\alpha{}_i$ and their momenta from the 18 second-class constraints (80) so that one obtains a RPS based on $(e, \pi(e))$ variables and the action

$$S_3 = \int d^4x \left[\pi(e)_\alpha{}^i \dot{e}^\alpha{}_i - N\tilde{\mathcal{H}}_\perp - N^i \tilde{\mathcal{D}}_i - \frac{1}{2}\omega^{\alpha\beta}{}_0 \tilde{\mathcal{J}}_{\alpha\beta} \right]. \tag{82}$$

This action corresponds to the triad Hamiltonian formulation of general relativity. The ADM formulation is obtained by using the $3D$ metric $g_{ij} \equiv e^a{}_i e_{aj} = e^\alpha{}_i e_{\alpha j}$ and the corresponding momenta. The ADM variables are invariant under the local rotations generated by $\tilde{\mathcal{J}}^{\alpha\beta}$, so that the corresponding action is given by

$$S_4 = \int \mathrm{d}^4x \left[\pi(g)^{ij}\,\dot{g}_{ij} - N\tilde{\mathcal{H}}_\perp - N^i\tilde{\mathcal{D}}_i\right], \tag{83}$$

where $\mathcal{H}_\perp$ and $\mathcal{D}_i$ are the ADM constraints.

6. Conclusions

We found all the constraints and determined the Lagrange multipliers for the *BFCG* GR action (6). We also determined the total Hamiltonian (62), the first-class constraints (63), the second-class constraints (64) and the algebra of the constraints (72)–(74). The obtained constraints also give the correct number of the physical DOF, see (66). We also showed how the other known canonical formulations of GR, namely Einstein–Cartan, triad and ADM, arise from the canonical formulation of *BFCG* GR by performing the RPS analysis. This analysis also gave a new canonical formulation for GR, namely the action S_2, which is based on the reduced phase space of triads and $SO(3)$ connections and their canonically conjugate momenta.

Since the main motivation for finding a canonical formulation of the *BFCG* GR theory is the construction of a spin-foam basis which will be a categorical generalization of the spin-network basis from LQG, then the results of the RPS analysis in section 5 are of great importance for this goal. Namely, in order to construct such a spin-foam basis one needs a 2-connection (A, β) for the Euclidean 2-group $(SO(3), \mathbb{R}^3)$ on the spatial manifold Σ, see [12]. This makes the RPS space $(e^\alpha{}_i, \omega^{\alpha\beta}{}_i, \pi(e)_\alpha{}^i, \pi(\omega)_{\alpha\beta}{}^i)$ and the corresponding action S_2 a natural starting point for the canonical quantization. Furthermore, this RPS provides a natural 2-connection on Σ

$$(A^{\alpha\beta}{}_i\,, \beta^\alpha{}_{ij}) = (\omega^{\alpha\beta}{}_i, \epsilon_{ijk}\tilde{e}^{k\alpha}\,), \tag{84}$$

where $\tilde{e}^k{}_\alpha$ are the inverse triads.

Hence one can use the 2-holonomy invariants for the 2-connection (84) associated to embedded 2-graphs in Σ, see [24], in order to construct the wavefunctions corresponding to the spin-foam basis. However, the existence of the second-class constraints χ_m will complicate the task of obtaining the physical Hilbert space. One can avoid the second-class constraints by using the Dirac brackets, but this may produce non-canonical commutators among the fields and their canonical momenta. If one wants to preserve the Heisenberg algebra of the canonical variables, then one can use the Gupta–Bleuler quantization approach, where the second-class constraints would be imposed weakly, as $\langle\Psi|\hat{\chi}_m|\Psi\rangle = 0$.

A simpler approach to the problem of second-class constraints in quantum theory is to solve classically the second-class constraints χ_m, which is equivalent to using the $(e^\alpha{}_i, \pi(e)_\alpha{}^i)$ RPS and the action S_3. Then the spin connection $\omega^{\alpha\beta}{}_i$ becomes a function of the triads and the components of the 2-connection (84) will still commute as operators, so that a spin-foam basis can be constructed, and the e-representation will be the most convenient for this.

Note that in the triad formulation of GR the Ashtekar variables can be defined via a series of canonical transformations,

$$(e^\alpha{}_i, \pi(e)_\alpha{}^i) \to (\tilde{e}^i{}_\alpha\,, \pi(\tilde{e})_i{}^\alpha) \to \Big(f(\zeta)E^i{}_\alpha = |e|_3\,\tilde{e}^i{}_\alpha\,, A^\alpha{}_i = \omega(e)^\alpha{}_i + \frac{\zeta}{|e|_3}\pi(\tilde{e})_i{}^\alpha\Big), \tag{85}$$

where $|e|_3 = \det(e^\alpha{}_i)$ and for $\zeta = \sqrt{-1}$, $f(\zeta) = 1$ [25] while for $\zeta \in \mathbb{R}$, $f(\zeta) = \zeta$ [26]. Then one can define the spin-network basis by using spin-network graphs and the associated holonomies for the connection A, see [1]. This suggests that the Ashtekar variables could be also a natural starting point for the construction of a spin-foam basis. However, the corresponding 2-connection components

$$A^{\alpha\beta}{}_i = \epsilon^{\alpha\beta\gamma} A_{\gamma i}, \qquad \beta^\alpha{}_{ij} = \epsilon_{ijk} E^{k\alpha}, \tag{86}$$

will not commute as operators and one has to use again the 2-connection (84).

Let us also note that the results obtained about the Hamiltonian structure of the theory can be important if one considers minisuperspace or midisuperspace models of quantum gravity, as is commonly done in the context of cosmology. For example, in Loop Quantum Cosmology (for a review, see [27–30] and references therein), one typically performs some type of symmetry reduction or gauge fixing prior to quantization, and then considers a resulting quantum-mechanical model of the Universe. However, in this work we have discussed only pure gravity, without matter fields. For this reason, our results are not directly applicable in the context of cosmology, since cosmological models without matter fields are not realistic. Repeating our analysis with included matter fields therefore represents an interesting avenue for further research.

Acknowledgments

MV would like to thank Prof Milovan Vasilić for discussion and helpful suggestions.

AM was partially supported by the FCT projects PEst-OE/MAT/UI0208/2013 and EXCL/MAT-GEO/0222/2012. MAO was supported by the FCT grant SFRH/BD/ 79285/2011. MV was supported by the FCT project PEst-OE/MAT/UI0208/2013, the FCT grant SFRH/BPD/46376/2008, the bilateral project 'Quantum Gravity and Quantum Integrable Models—2015–2016' number 451-03-01765/2014-09/24 between Portugal and Serbia, and partially by the project ON171031 of the Ministry of Education, Science and Technological Development, Serbia.

Appendix A. Bianchi identities

Recalling the definitions of the torsion and curvature 2-forms,

$$T^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b, \qquad R^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}, \tag{A.1}$$

one can take the exterior derivative of T^a and R^a, and use the property $\mathrm{dd} \equiv 0$ to obtain the following two identities:

$$\begin{aligned} \nabla T^a &\equiv \mathrm{d}T^a + \omega^a{}_b \wedge T^b = R^a{}_b \wedge e^b, \\ \nabla R^{ab} &\equiv \mathrm{d}R^{ab} + \omega^a{}_c \wedge R^{cb} + \omega^b{}_c \wedge R^{ac} = 0. \end{aligned} \tag{A.2}$$

These two identities are universally valid for torsion and curvature, and are called Bianchi identities. By expanding all quantities into components as

$$T^a = \frac{1}{2} T^a{}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \qquad R^{ab} = \frac{1}{2} R^{ab}{}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{A.3}$$

$$e^a = e^a{}_\mu \mathrm{d}x^\mu, \qquad \omega^{ab} = \omega^{ab}{}_\mu \mathrm{d}x^\mu, \tag{A.4}$$

and using the formula $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = \varepsilon^{\mu\nu\rho\sigma} \mathrm{d}^4 x$, one can rewrite the Bianchi identities in component form as

$$\varepsilon^{\lambda\mu\nu\rho} \left(\nabla_\mu T^a{}_{\nu\rho} - R^a{}_{b\mu\nu} e^b{}_\rho \right) = 0, \tag{A.5}$$

and

$$\varepsilon^{\lambda\mu\nu\rho}\nabla_{\mu}R^{ab}{}_{\nu\rho}=0. \tag{A.6}$$

For the purpose of Hamiltonian analysis, one can split the Bianchi identities into those which do not feature a time derivative and those that do. The time-independent pieces are obtained by taking $\lambda=0$ components:

$$\varepsilon^{0ijk}\left(\nabla_{i}T^{a}{}_{jk}-R^{a}{}_{bij}e^{b}{}_{k}\right)=0, \tag{A.7}$$

$$\varepsilon^{0ijk}\nabla_{i}R^{ab}{}_{jk}=0. \tag{A.8}$$

These identities are valid as off-shell, strong equalities for every spacelike slice in spacetime, and can be enforced in all calculations involving the Hamiltonian analysis. The time-dependent pieces are obtained by taking $\lambda=i$ components:

$$\varepsilon^{0ijk}\left(\nabla_{0}T^{a}{}_{jk}-2\nabla_{j}T^{a}{}_{0k}-2R^{a}{}_{b0j}e^{b}{}_{k}-R^{a}{}_{bjk}e^{b}{}_{0}\right)=0, \tag{A.9}$$

and

$$\varepsilon^{0ijk}\left(\nabla_{0}R^{ab}{}_{jk}-2\nabla_{j}R^{ab}{}_{0k}\right)=0. \tag{A.10}$$

Due to the fact that they connect geometries of different spacelike slices in spacetime, they cannot be enforced off-shell. Instead, they can be derived from the Hamiltonian equations of motion of the theory.

In light of the Bianchi identities, we should note that the action (6) features three more fields, β^{a}, B^{ab} and ϕ^{ab}, which also have field strengths G^{a}, ∇B^{ab}, $\nabla\phi^{ab}$, and for which one can similarly derive Bianchi-like identities,

$$\begin{aligned}
&\nabla G^{a}=R^{a}{}_{b}\wedge\beta^{b},\\
&\nabla^{2}B^{ab}=R^{a}{}_{c}\wedge B^{cb}+R^{b}{}_{c}\wedge B^{ac},\\
&\nabla^{2}\phi^{ab}=R^{a}{}_{c}\wedge\phi^{cb}+R^{b}{}_{c}\wedge\phi^{ac}.
\end{aligned} \tag{A.11}$$

However, due to the fact that all three fields are two-forms, in 4-dimensional spacetime these identities will be single-component equations, with no free spacetime indices,

$$\varepsilon^{\lambda\mu\nu\rho}\left(\frac{2}{3}\nabla_{\lambda}G^{a}{}_{\mu\nu\rho}-R^{a}{}_{b\mu\nu}\beta^{b}{}_{\nu\rho}\right)=0, \tag{A.12}$$

and similarly for $\nabla^{2}B^{ab}$ and $\nabla^{2}\phi^{ab}$. Therefore, these equations necessarily feature time derivatives of the fields, and do not have a purely spatial counterpart to (A.7) and (A.8). In this sense, like the time-dependent pieces of the Bianchi identities, they do not enforce any restrictions in the sense of the Hamiltonian analysis, but can instead be derived from the equations of motion and expressions for the Lagrange multipliers.

Appendix B. Inverse tetrad and metric

We perform the split of the group indices into space and time components as $a=(\underline{0},\alpha)$ where $\alpha=1,2,3$, and write the tetrad $e^{a}{}_{\mu}$ as a $1+3$ matrix

$$e^a{}_\mu = \left[\begin{array}{c|c} e^0{}_0 & e^0{}_m \\ \hline e^\alpha{}_0 & e^\alpha{}_m \end{array}\right]. \tag{B.1}$$

Then the inverse tetrad $e^\mu{}_b$ can be expressed in terms of the 3D inverse tetrad $\tilde{e}^m{}_\beta$ as

$$e^\mu{}_b = \left[\begin{array}{c|c} \frac{1}{\sigma} & -\frac{1}{\sigma}\tilde{e}^m{}_\beta\, e^0{}_m \\ \hline -\frac{1}{\sigma}\tilde{e}^m{}_\alpha\, e^\alpha{}_0 & \tilde{e}^m{}_\beta + \frac{1}{\sigma}\left(\tilde{e}^m{}_\alpha\, e^\alpha{}_0\right)\left(\tilde{e}^k{}_\beta\, e^0{}_k\right) \end{array}\right], \tag{B.2}$$

where

$$\sigma \equiv e^0{}_0 - e^0{}_k \tilde{e}^k{}_\alpha\, e^\alpha{}_0 \tag{B.3}$$

is the 1×1 Schur complement [31] of the 4×4 matrix $e^a{}_\mu$. By definition, the 3D tetrad satisfies the identities

$$e^\alpha{}_m \tilde{e}^m{}_\beta = \delta^\alpha_\beta, \qquad e^\alpha{}_m \tilde{e}^n{}_\alpha = \delta^n_m. \tag{B.4}$$

In addition, if we denote $e \equiv \det e^a{}_\mu$ and $e_3 \equiv \det e^\alpha{}_m$, the Schur complement σ satisfies the Schur determinant formula

$$e = \sigma e_3, \tag{B.5}$$

which can be proved as follows.

Given any square matrix divided into blocks as

$$\Delta = \left[\begin{array}{c|c} A & B \\ \hline C & M \end{array}\right] \tag{B.6}$$

such that A and M are square matrices and M has an inverse, we can use the Aitken block diagonalization formula [31]

$$\left[\begin{array}{c|c} I & -BM^{-1} \\ \hline 0 & I \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & M \end{array}\right] \left[\begin{array}{c|c} I & 0 \\ \hline -M^{-1}C & I \end{array}\right] = \left[\begin{array}{c|c} S & 0 \\ \hline 0 & M \end{array}\right], \tag{B.7}$$

where

$$S = A - BM^{-1}C \tag{B.8}$$

is called the Schur complement of the matrix Δ. The Aitken formula can be written in the compact form

$$P\Delta Q = S \oplus M, \tag{B.9}$$

where P and Q are the above triangular matrices. Taking the determinant, we obtain

$$\det P \det \Delta \det Q = \det S \det M. \tag{B.10}$$

Since the determinant of a triangular matrix is the product of its diagonal elements, we have $\det P = \det Q = 1$, which then gives the famous Schur determinant formula:

$$\det \Delta = \det S \det M. \tag{B.11}$$

Now, performing the $1+3$ block splitting of the tetrad matrix $\Delta = [e^a{}_\mu]_{4\times4}$, we obtain the Schur complement $S = [\sigma]_{1\times1}$, while $M = [e^\alpha{}_m]_{3\times3}$. The Schur determinant formula then gives

$$e = \sigma e_3, \tag{B.12}$$

which completes the proof.

Similarly to the tetrad, one can perform a $1+3$ split of the metric $g_{\mu\nu}$,

$$g_{\mu\nu} = \left[\begin{array}{c|c} g_{00} & g_{0j} \\ \hline g_{i0} & g_{ij} \end{array}\right]. \tag{B.13}$$

The inverse metric $g^{\mu\nu}$ can be expressed in terms of the $3D$ inverse metric $\tilde{g}^{ij}$ as

$$g^{\mu\nu} = \left[\begin{array}{c|c} \dfrac{1}{\rho} & -\dfrac{1}{\rho}\tilde{g}^{in}g_{0i} \\ \hline -\dfrac{1}{\rho}\tilde{g}^{mj}g_{0j} & \tilde{g}^{mn} + \dfrac{1}{\rho}\left(\tilde{g}^{mj}g_{0j}\right)\left(\tilde{g}^{in}g_{0i}\right) \end{array}\right], \tag{B.14}$$

where

$$\rho \equiv g_{00} - g_{0i}\tilde{g}^{ij}g_{0j} \tag{B.15}$$

is the 1×1 Schur complement of $g_{\mu\nu}$. By definition, the $3D$ metric satisfies the identity

$$g_{ij}\tilde{g}^{jk} = \delta_i^k. \tag{B.16}$$

In addition, if we denote $g \equiv \det g_{\mu\nu}$ and $g_3 \equiv \det g_{ij}$, the Schur complement ρ satisfies the Schur determinant formula

$$g = \rho g_3. \tag{B.17}$$

The components of the metric can of course be written in terms of the components of the tetrad,

$$g_{\mu\nu} = \eta_{ab}e^a{}_\mu e^b{}_\nu. \tag{B.18}$$

Regarding the inverse metric, the only nontrivial identity is between $\tilde{g}^{ij}$ and $\tilde{e}^i{}_\alpha$. Introducing the convenient notation $e_\alpha \equiv \tilde{e}^i{}_\alpha\, e^0{}_i$, it reads:

$$\tilde{g}^{ij} = \tilde{e}^i{}_\alpha\,\tilde{e}^j{}_\beta\,\left[\eta^{\alpha\beta} + \frac{e^\alpha e^\beta}{1 - e_\gamma e^\gamma}\right]. \tag{B.19}$$

The relationship between determinants and Schur complements is:

$$g = -e^2, \qquad g_3 = (e_3)^2\left(1 - e_\alpha e^\alpha\right), \qquad \rho = \frac{\sigma^2}{e_\alpha e^\alpha - 1}. \tag{B.20}$$

Finally, there is one more useful identity,

$$g_{0j}\tilde{g}^{ij} = \tilde{e}^i{}_\alpha\, e^\alpha{}_0 - \frac{\sigma}{1 - e_\beta e^\beta}\tilde{e}^i{}_\alpha\, e^\alpha, \tag{B.21}$$

which can be easily proved with some patient calculation and the other identities above.

Appendix C. Solving the system of equations

In order to show that the constraints (52) are equivalent to the constraint (53), we proceed as follows. Introducing the shorthand notation $K_{abcd} \in \{\varepsilon_{abcd}, \eta_{ac}\eta_{bd}\}$, we can rewrite (52) in a convenient form

$$e^a{}_m K_{abcd} \varepsilon^{0ijk} \left(e^b{}_0 R^{cd}{}_{jk} - 2 e^b{}_j \phi^{cd}{}_{0k} \right) \approx 0. \tag{C.1}$$

Next we multiply it with the Levi-Civita symbol ε_{0iln} in order to cancel the ε^{0ijk}, relabel the index $m \to i$ and obtain

$$K_{abcd} \left(e^a{}_i e^b{}_j \phi^{cd}{}_{0k} - e^a{}_i e^b{}_k \phi^{cd}{}_{0j} \right) \approx K_{abcd} e^a{}_i e^b{}_0 R^{cd}{}_{jk}. \tag{C.2}$$

The antisymmetrization in jk indices can be eliminated by writing each equation three times with cyclic permutations of indices ijk, then adding the first two permutations and subtracting the third. This gives:

$$K_{abcd} e^a{}_i e^b{}_j \phi^{cd}{}_{0k} \approx K_{abcd} e^a{}_0 \left[\frac{1}{2} e^b{}_k R^{cd}{}_{ij} - e^b{}_{[i} R^{cd}{}_{j]k} \right]. \tag{C.3}$$

Introducing the shorthand notation P_{ijk} and Q_{ijk} for the expression on the right-hand side as

$$\begin{aligned} P_{ijk} &\equiv \eta_{ac}\eta_{bd} e^a{}_0 \left[\frac{1}{2} e^b{}_k R^{cd}{}_{ij} - e^b{}_{[i} R^{cd}{}_{j]k} \right], \\ Q_{ijk} &\equiv \varepsilon_{abcd} e^a{}_0 \left[\frac{1}{2} e^b{}_k R^{cd}{}_{ij} - e^b{}_{[i} R^{cd}{}_{j]k} \right], \end{aligned} \tag{C.4}$$

our system can be rewritten as

$$\eta_{ac}\eta_{bd} e^a{}_i e^b{}_j \phi^{cd}{}_{0k} \approx P_{ijk}, \qquad \varepsilon_{abcd} e^a{}_i e^b{}_j \phi^{cd}{}_{0k} \approx Q_{ijk}. \tag{C.5}$$

This system consists of 18 equations for the 18 variables $\phi^{ab}{}_{0k}$. We look for a solution in the form

$$\phi^{cd}{}_{0k} = A^{cdmn} P_{mnk} + B^{cdmn} Q_{mnk}, \tag{C.6}$$

where the coefficients A^{cdmn} and B^{cdmn} are to be determined, for arbitrarily given values of P_{ijk} and Q_{ijk}. Substituting (C.6) into (C.5) we obtain

$$\begin{aligned} \left[\eta_{ac}\eta_{bd} e^a{}_i e^b{}_j A^{cdmn} - \delta_i^{[m} \delta_j^{n]} \right] P_{mnk} + \left[\eta_{ac}\eta_{bd} e^a{}_i e^b{}_j B^{cdmn} \right] Q_{mnk} &\approx 0, \\ \left[\varepsilon_{abcd} e^a{}_i e^b{}_j A^{cdmn} \right] P_{mnk} + \left[\varepsilon_{abcd} e^a{}_i e^b{}_j B^{cdmn} - \delta_i^{[m} \delta_j^{n]} \right] Q_{mnk} &\approx 0. \end{aligned} \tag{C.7}$$

Since P_{mnk} and Q_{mnk} are considered arbitrary, the expressions in the brackets must vanish, giving the following equations for A^{cdmn},

$$\eta_{ac}\eta_{bd} e^a{}_i e^b{}_j A^{cdmn} \approx \delta_i^{[m} \delta_j^{n]}, \qquad \varepsilon_{abcd} e^a{}_i e^b{}_j A^{cdmn} \approx 0, \tag{C.8}$$

and for B^{cdmn},

$$\eta_{ac}\eta_{bd} e^a{}_i e^b{}_j B^{cdmn} \approx 0, \qquad \varepsilon_{abcd} e^a{}_i e^b{}_j B^{cdmn} \approx \delta_i^{[m} \delta_j^{n]}. \tag{C.9}$$

Focus first on (C.8). The first equation can be rewritten in the form

$$e_{ci} e_{dj} A^{cdmn} \approx \delta_i^{[m} \delta_j^{n]}, \tag{C.10}$$

and we want to rewrite the second equation in a similar form as well. In order to do that, we need to get rid of the Levi-Civita symbol on the left-hand side, by virtue of the identity

$$\det(e_{a\mu})\varepsilon_{abcd} = \varepsilon^{\mu\nu\rho\sigma} e_{a\mu} e_{b\nu} e_{c\rho} e_{d\sigma}. \tag{C.11}$$

Noting that $\det(e_{a\mu}) = \det(\eta_{ab} e^b{}_\mu) = -\det(e^a{}_\mu) = -e$ and introducing the metric $g_{\mu\nu} \equiv e^a{}_\mu e_{a\nu}$, we can multiply this identity with $e^a{}_i e^b{}_j$ to obtain:

$$\varepsilon_{abcd} e^a{}_i e^b{}_j = -\frac{1}{e}\varepsilon^{\mu\nu\rho\sigma} g_{\mu i} g_{\nu j} e_{c\rho} e_{d\sigma}. \tag{C.12}$$

Substituting this into the second equation in (C.8) gives

$$\varepsilon^{\mu\nu\rho\sigma} g_{\mu i} g_{\nu j} e_{c\rho} e_{d\sigma} A^{cdmn} \approx 0. \tag{C.13}$$

Next we expand the ρ and σ indices into space and time components as $\rho = (0, k)$ and $\sigma = (0, l)$ to obtain

$$2\varepsilon^{\mu\nu 0l} g_{\mu i} g_{\nu j} e_{c0} e_{dl} A^{cdmn} + \varepsilon^{\mu\nu kl} g_{\mu i} g_{\nu j} e_{ck} e_{dl} A^{cdmn} \approx 0. \tag{C.14}$$

The second term on the left can be evaluated using (C.10), which gives:

$$2\varepsilon^{\mu\nu 0l} g_{\mu i} g_{\nu j} e_{c0} e_{dl} A^{cdmn} + \varepsilon^{\mu\nu mn} g_{\mu i} g_{\nu j} \approx 0. \tag{C.15}$$

The Levi-Civita symbol in the first term is nonzero only if $\mu\nu$ are spatial indices, so we can write

$$2\varepsilon^{rs0l} g_{ri} g_{sj} e_{c0} e_{dl} A^{cdmn} + \varepsilon^{\mu\nu mn} g_{\mu i} g_{\nu j} \approx 0. \tag{C.16}$$

At this point we need to introduce $3D$ inverse metric, $\tilde{g}^{ij}$, and to split the group indices into $3+1$ form $a = (\underline{0}, \alpha)$ where $\alpha = 1, 2, 3$, see appendix B. Multiplying (C.16) with two inverse spatial metrics and another Levi-Civita symbol, we can finally rewrite it as:

$$e_{c0} e_{di} A^{cdmn} \approx g_{0j} \tilde{g}^{j[m} \delta_i^{n]}. \tag{C.17}$$

The goal of all these transformations was to rewrite the system (C.8) into the form

$$e_{ci} e_{dj} A^{cdmn} \approx \delta_i^{[m} \delta_j^{n]}, \qquad e_{c0} e_{di} A^{cdmn} \approx g_{0j} \tilde{g}^{j[m} \delta_i^{n]}. \tag{C.18}$$

At this point we can expand the group indices on the left-hand side into $3+1$ form, to obtain:

$$e_{\gamma i} e_{\delta j} A^{\gamma\delta mn} + \left(e^{\underline{0}}{}_j e_{\delta i} - e^{\underline{0}}{}_i e_{\delta j}\right) A^{\underline{0}\delta mn} \approx \delta_i^{[m} \delta_j^{n]}, \tag{C.19}$$

$$e_{\gamma 0} e_{\delta j} A^{\gamma\delta mn} + \left(e^{\underline{0}}{}_j e_{\delta 0} - e^{\underline{0}}{}_0 e_{\delta j}\right) A^{\underline{0}\delta mn} \approx g_{0k} \tilde{g}^{k[m} \delta_j^{n]}. \tag{C.20}$$

Now we multiply (C.19) with $\tilde{e}^i{}_\alpha \, e^\alpha{}_0$ and subtract it from (C.20). The first terms on the left cancel, and (C.20) becomes

$$-\sigma e_{\delta j} A^{\underline{0}\delta mn} \approx g_{0k} \tilde{g}^{k[m} \delta_j^{n]} - \tilde{e}^{[m}{}_\alpha \, \delta_j^{n]} e^\alpha{}_0, \tag{C.21}$$

where σ is the 1×1 Schur complement matrix of the tetrad $e^a{}_\mu$ (see appendix B). Multiplying with another inverse $3D$ tetrad and using the identity (B.21), we finally obtain the first half of the coefficients A:

$$A^{\underline{0}\alpha mn} \approx \frac{1}{1 - e^\gamma e_\gamma} \tilde{e}^{[m}{}_\delta \, \tilde{e}^{n]\alpha} \, e^\delta. \tag{C.22}$$

Finally, substituting this back into (C.19) and multiplying with two more inverse 3*D* tetrads we obtain the second half of the coefficients A:

$$A^{\alpha\beta mn} \approx \tilde{e}^{[m\alpha}\,\tilde{e}^{n]\beta} + \frac{e^{\delta}}{1-e_{\gamma}e^{\gamma}}\left[e^{\alpha}\tilde{e}^{[m}{}_{\delta}\,\tilde{e}^{n]\beta} - e^{\beta}\tilde{e}^{[m}{}_{\delta}\,\tilde{e}^{n]\alpha}\right]. \tag{C.23}$$

Next we turn to the system (C.9) for coefficients B. The method to solve it is completely analogous to the above method of solving (C.8), and we will not repeat all the steps, but rather only quote the final result:

$$B^{0\beta mn} \approx \frac{1}{4}\varepsilon^{0\beta\gamma\delta}\left[\tilde{e}^{m}{}_{\gamma}\,\tilde{e}^{n}{}_{\delta} + 2\tilde{e}^{[m}{}_{\alpha}\,\tilde{e}^{n]}{}_{\delta}\,\frac{e^{\alpha}e_{\gamma}}{1-e_{\epsilon}e^{\epsilon}}\right], \tag{C.24}$$

and

$$B^{\alpha\beta mn} \approx \frac{1}{2}\frac{1}{1-e_{\epsilon}e^{\epsilon}}\,\varepsilon^{0\alpha\beta\gamma}\tilde{e}^{[m}{}_{\gamma}\,\tilde{e}^{n]}{}_{\delta}\,e^{\delta}. \tag{C.25}$$

To conclude, by determining the A and B coefficients in (C.6) we have managed to solve the original system of equations (C.1) for $\phi^{ab}{}_{0k}$. Substituting (C.4) into (C.6) the expression for $\phi^{ab}{}_{0k}$ can be arranged into the form

$$\phi^{ab}{}_{0k} \approx e^{f}{}_{0}R^{cd}{}_{mn}F^{abmn}{}_{fcdk}, \tag{C.26}$$

where

$$\begin{aligned} F^{abmn}{}_{fcdk} \equiv \frac{1}{2}\Big[& A^{abmn}\eta_{fc}e_{dk} - 2A^{abim}\eta_{fc}e_{di}\delta^{n}_{k} \\ & + B^{abmn}\varepsilon_{fhcd}e^{h}{}_{k} - 2B^{abim}\varepsilon_{fhcd}e^{h}{}_{i}\delta^{n}_{k}\Big], \end{aligned} \tag{C.27}$$

and coefficients A and B are specified by (C.22)–(C.25). Note that (C.27) depends only on $e^{a}{}_{i}$ components of the metric (in a very complicated way), while the dependence of $\phi^{ab}{}_{0k}$ on $e^{a}{}_{0}$ and $\omega^{ab}{}_{i}$ is factored out in (C.26).

Appendix D. Levi-Civita identity

The identity for the Levi-Civita symbol in 4 dimensions used in the main text is:

$$A_{[a}\varepsilon_{b]cdf}C^{c}D^{d}F^{f} = -\frac{1}{2}\varepsilon_{abcd}A_{f}\left[C^{d}D^{f}F^{c} + C^{c}D^{d}F^{f} + C^{f}D^{c}F^{d}\right]. \tag{D.1}$$

The proof goes as follows. Denote the left-hand side of the identity as

$$K_{ab} \equiv A_{[a}\varepsilon_{b]cdf}C^{c}D^{d}F^{f} \tag{D.2}$$

and take the dual to obtain:

$$\varepsilon^{aba'b'}K_{ab} = \varepsilon^{aba'b'}\varepsilon_{bcdf}A_{a}C^{c}D^{d}F^{f}. \tag{D.3}$$

Next expand the product of two Levi-Civita symbols into Kronecker deltas and use them to contract the vectors A, C, D and F:

$$\varepsilon^{aba'b'}K_{ab} = 2\left[(A\cdot D)F^{[a'}C^{b']} + (A\cdot F)C^{[a'}D^{b']} + (A\cdot C)D^{[a'}F^{b']}\right]. \tag{D.4}$$

Now take the dual again, i.e. contract with $\varepsilon_{a'b'cd}$ to obtain

$$\begin{aligned} -4K_{cd} &= \varepsilon_{a'b'cd}\varepsilon^{aba'b'}K_{ab} \\ &= 2\varepsilon_{a'b'cd}\left[(A\cdot D)F^{[a'}C^{b']} + (A\cdot F)C^{[a'}D^{b']} + (A\cdot C)D^{[a'}F^{b']}\right]. \end{aligned} \tag{D.5}$$

Finally, multiply by $-1/4$ and relabel the indices to obtain

$$K_{ab} = -\frac{1}{2}\varepsilon_{abcd}A_f\left[C^dD^fF^c + C^cD^dF^f + C^fD^cF^d\right], \tag{D.6}$$

which proves the identity.

Appendix E. Relation between the *BFCG* and the MacDowell–Mansouri models

Given that the constrained *BFCG* action (6) is equivalent to GR, it is a straightforward exercise to include a cosmological constant term:

$$\begin{aligned} S_{GR\Lambda} &= \int_{\mathcal{M}}\Big[B_{ab}\wedge R^{ab} + e^a\wedge G_a - \phi^{ab}\wedge\left(B_{ab} - \varepsilon_{abcd}\,e^c\wedge e^d\right) \\ &\qquad -\frac{\Lambda}{6}\,\varepsilon_{abcd}\,e^a\wedge e^b\wedge e^c\wedge e^d\Big], \end{aligned} \tag{E.1}$$

Working out the corresponding equations of motion, one obtains the same set (17)–(20) as for the action (6), except for the Einstein field equation (21) which is modified into

$$\varepsilon_{abcd}\left(R^{bc} - \frac{\Lambda}{3}\,e^b\wedge e^c\right)\wedge e^d = 0, \tag{E.2}$$

which can in turn be rewritten into the standard component form

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda\,g_{\mu\nu} = 0. \tag{E.3}$$

Here the parameter $\Lambda\in\mathbb{R}$ is the cosmological constant.

It is interesting to note that one can obtain the MacDowell–Mansouri action for GR [18–22] from the action (E.1). In particular, the relationship between (E.1) and the MacDowell–Mansouri action is analogous to the relationship between the Palatini and Einstein–Hilbert actions, respectively, as we shall now demonstrate. To this end, first add and subtract a term $\zeta B_{ab}\wedge e^a\wedge e^b$ to (E.1), where $\zeta = \pm 1$, and rewrite it in the form

$$\begin{aligned} S_{GR\Lambda} = \int_{\mathcal{M}}\Big[&B_{ab}\wedge\left(R^{ab} - \zeta e_a\wedge e_b\right) + e^a\wedge G_a - \phi^{ab}\wedge\left(B_{ab} - \varepsilon_{abcd}\,e^c\wedge e^d\right) \\ &+ e^a\wedge e^b\wedge\left(\zeta B_{ab} - \frac{\Lambda}{6}\varepsilon_{abcd}\,e^c\wedge e^d\right)\Big]. \end{aligned} \tag{E.4}$$

Next we perform the partial integration over the $e^a\wedge G_a$ term, and rewrite the action as

$$\begin{aligned} S_{GR\Lambda} = \int_{\mathcal{M}}\Big[&B_{ab}\wedge\left(R^{ab} - \zeta e_a\wedge e_b\right) + \beta^a\wedge\nabla e_a - \phi^{ab}\wedge\left(B_{ab} - \varepsilon_{abcd}\,e^c\wedge e^d\right) \\ &+ e^a\wedge e^b\wedge\left(\zeta B_{ab} - \frac{\Lambda}{6}\varepsilon_{abcd}\,e^c\wedge e^d\right)\Big]. \end{aligned} \tag{E.5}$$

Now we want to eliminate the Lagrange multiplier ϕ^{ab} from the action. This is performed in analogy with the way the Palatini action is transformed into the Einstein–Hilbert action—we take the variation of the action with respect to ϕ^{ab} to obtain the corresponding equation of motion, and then substitute this equation back into the action. The equation of motion is algebraic rather than differential,

$$B_{ab} = \varepsilon_{abcd} e^c \wedge e^d, \tag{E.6}$$

which suggests that no propagating degrees of freedom will be lost upon substituting it back into the action. So we solve it for the product of two tetrads,

$$e^a \wedge e^b = -\frac{1}{4} \varepsilon^{abcd} B_{cd}, \tag{E.7}$$

and substitute it back into (E.5), eliminating the product of the tetrads from all terms except the first one, to obtain:

$$S = \int_{\mathcal{M}} \left[B_{ab} \wedge \left(R^{ab} - \zeta e_a \wedge e_b \right) + \beta^a \wedge \nabla e_a + \frac{\Lambda - 6\zeta}{24} \varepsilon_{abcd} B^{ab} \wedge B^{cd} \right]. \tag{E.8}$$

Note that the term containing ϕ^{ab} has vanished from the action, while the final term has been transformed into the $B \wedge B$ term.

Finally, to see that (E.8) is actually the MacDowell–Mansouri action, introduce the following change of variables:

$$B^{AB} \equiv \left[\begin{array}{c|c} B^{ab} & \frac{\beta^a}{2} \\ \hline -\frac{\beta^b}{2} & 0 \end{array} \right], \qquad A^{AB} \equiv \left[\begin{array}{c|c} \omega^{ab} & e^a \\ \hline -e^b & 0 \end{array} \right], \tag{E.9}$$

and

$$F^{AB} \equiv \mathrm{d}A^{AB} + A^A{}_C \wedge A^{CB} = \left[\begin{array}{c|c} R^{ab} - \zeta e^a \wedge e^b & \nabla e^a \\ \hline -\nabla e^b & 0 \end{array} \right], \qquad V^A \equiv \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \end{array} \right]. \tag{E.10}$$

These represent the 5-dimensional 2-form B^{AB}, connection 1-form A^A, its field strength 2-form F^{AB} and a 0-form V^A. The capital Latin indices take values $0, 1, 2, 3, 5$, and we can also introduce the 5-dimensional Levi-Civita symbol ε_{ABCDE}, which is related to the ordinary 4-dimensional one as $\varepsilon_{abcd5} \equiv \varepsilon_{abcd}$. Using all this, the action (E.8) can be rewritten into the form

$$S = \int_{\mathcal{M}} \left[B_{AB} \wedge F^{AB} + \frac{\Lambda - 6\zeta}{24} B^{ab} \wedge B^{cd} \varepsilon_{ABCDE} V^E \right], \tag{E.11}$$

which is manifestly covariant with respect to the action of the groups $SO(4, 1)$ or $SO(3, 2)$, depending on the choice of $\zeta = \pm 1$, which enters the 5-dimensional metric

$$\eta_{AB} \equiv \left[\begin{array}{cccc|c} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ \hline & & & & \zeta \end{array} \right], \tag{E.12}$$

where the off-diagonal values are assumed to be zero. The action (E.11) is precisely the BF-formulation of the MacDowell–Mansouri action [18–22], as we have set out to demonstrate.

ORCID iDs

A Miković https://orcid.org/0000-0002-0009-1411
M A Oliveira https://orcid.org/0000-0001-9548-4906
M Vojinović https://orcid.org/0000-0001-6977-4870

References

[1] Rovelli C 2004 *Quantum Gravity* (Cambridge: Cambridge University Press)
[2] Thiemann T 2007 *Introduction to Modern Canonical Quantum General Relativity* (Cambridge: Cambridge University Press)
[3] Perez A 2013 *Living Rev. Relativ.* **16** 3
[4] Rovelli C and Vidotto F 2014 *Covariant Loop Quantum Gravity* (Cambridge: Cambridge University Press)
[5] Miković A and Vojinović M 2012 *Class. Quantum Grav.* **29** 165003
[6] Miković A 2013 *Rev. Math. Phys.* **25** 1343008
[7] Baez J C and Huerta J 2011 *Gen. Relativ. Gravit.* **43** 2335
[8] Miković A and Vojinović M 2011 *Class. Quantum Grav.* **28** 225004
[9] Plebanski J F 1977 *J. Math. Phys.* **12** 2511
[10] De Pietri R and Freidel L 1999 *Class. Quantum Grav.* **16** 2187
[11] Livine R E and Oriti D 2002 *Phys. Rev.* D **65** 044025
[12] Miković A and Oliveira M A 2015 *Gen. Relativ. Gravit.* **47** 58
[13] Miković A, Oliveira M A and Vojinović M 2016 *Class. Quantum Grav.* **33** 065007
[14] Blagojević M 2002 *Gravitation and Gauge Symmetries* (Bristol: Institute of Physics Publishing)
[15] Celada M, González D and Montesinos M 2016 *Class. Quantum Grav.* **33** 213001
[16] Girelli F, Pfeiffer H and Popescu E M 2008 *J. Math. Phys.* **49** 032503
[17] Faria Martins J and Miković A 2011 *Adv. Theor. Math. Phys.* **15** 1059
[18] MacDowell S and Mansouri F 1977 *Phys. Rev. Lett.* **38** 739
MacDowell S and Mansouri F 1977 *Phys. Rev. Lett.* **38** 1376 (erratum)
[19] Smolin L 2000 *Phys. Rev.* D **61** 084007
[20] Ling Y and Smolin L 2001 *Phys. Rev.* D **63** 064010
[21] Smolin L and Starodubtsev A 2003 General relativity with a topological phase: an action principle (arXiv:hep-th/0311163)
[22] Wise D K 2010 *Class. Quantum Grav.* **27** 155010
[23] Blagojević M 2001 *Ann. Phys.* **10** 367
[24] Faria Martins J and Picken R 2008 A cubical set approach to 2-bundles with connection and Wilson surfaces (arXiv:0808.3964)
[25] Ashtekar A 1987 *Phys. Rev.* D **36** 1587
[26] Barbero G J F 1995 *Phys. Rev.* D **51** 5507
[27] Ashtekar A and Singh P 2011 *Class. Quantum Grav.* **28** 213001
[28] Singh P 2014 *Bull. Astron. Soc. India* **42** 121
[29] Navascués B E, Martín-Benito M and Marugán G A M 2016 *Int. J. Mod. Phys.* D **25** 1642007
[30] Wilson-Ewing E 2017 *C. R. Phys.* **18** 207
[31] Zhang F 2005 *The Schur Complement and its Applications* (New York: Springer)

Classical and Quantum Gravity

PAPER

Gauge protected entanglement between gravity and matter

To cite this article: N Paunkovi and M Vojinovi 2018 *Class. Quantum Grav.* **35** 185015

IOP Publishing Classical and Quantum Gravity
Class. Quantum Grav. 35 (2018) 185015 (27pp) https://doi.org/10.1088/1361-6382/aad7f1

Gauge protected entanglement between gravity and matter

N Paunković[1,2,3] and M Vojinović[4]

[1] Instituto de Telecomunicações, Avenida Rovisco Pais 1, 1049-001 Lisbon, Portugal
[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Avenida Rovisco Pais 1, 1049-001 Lisbon, Portugal
[3] CeFEMA, Instituto Superior Técnico, Universidade de Lisboa, Avenida Rovisco Pais 1, 1049-001 Lisbon, Portugal
[4] Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia

E-mail: npaunkov@math.tecnico.ulisboa.pt and vmarko@ipb.ac.rs

Received 14 May 2018, revised 1 August 2018
Accepted for publication 3 August 2018
Published 29 August 2018

Abstract

We show that, as a consequence of the local Poincaré symmetry, gravity and matter fields have to be entangled, unless the overall action is carefully fine-tuned. First, we present a general argument, applicable to any particular theory of quantum gravity with matter, by performing the analysis in the abstract nonperturbative canonical framework, demonstrating the nonseparability of the scalar constraint, thus promoting the entangled states as the physical ones. Also, within the covariant framework, using a particular toy model, we show explicitly that the Hartle–Hawking state in the Regge model of quantum gravity is entangled. Our result is potentially relevant for the quantum-to-classical transition, taken within the framework of the decoherence programme: due to the gauge symmetry requirements, the matter does not decohere, it is by default decohered by gravity. Generically, entanglement is a consequence of interaction. This new entanglement could potentially, in form of an 'effective interaction', bring about corrections to the weak equivalence principle, further confirming that spacetime as a smooth four-dimensional manifold is an emergent phenomenon. Finally, the existence of the gauge-protected entanglement between gravity and matter could be seen as a criterion for a plausible theory of quantum gravity, and in the case of perturbative quantisation approaches, a confirmation of the persistence of the manifestly broken gauge symmetry.

Keywords: quantum entanglement, quantum gravity, diffeomorphism invariance, scalar constraint

1. Introduction

The unsolved problems of formulating quantum theory of gravity (QG) and interpreting quantum mechanics (QM) are arguably the two most prominent ones of the modern theoretical physics. So far, most of the approaches to solve the two were studied independently. Indeed, the majority of the interpretations of QM do not involve explicit dynamical effects (with notable exceptions of the spontaneous collapse and the de Broglie–Bohm theories), while the researchers from the QG community often just adopt some particular interpretation of QM, assuming that it contains no unresolved issues. Nevertheless, the two problems share a number of similar unsolved questions and counter-intuitive features. A prominent example is nonlocality: entanglement-based nonlocality in QM, as well as the anticipated explicit dynamical nonlocality in QG (a consequence of quantum superpositions of different gravitational fields, i.e. different spacetimes and their respective causal orders). Another prominent issue relevant for both standard QM and QG is the quantum-to-classical transition and the related measurement problem.

In relation to the latter, decoherence is in QM the standard approach to the emergence of classicality: due to huge complexity of macroscopic ('classical') systems and the surrounding environment (bath), the (*for all practical purposes*) inevitable interaction between the two leads to the entanglement and the loss of coherence. While technically this is completely within the standard QM, when coupled with additional assumptions, such as the many-world interpretation (likely to be the predominant within the community working on decoherence and quantum-to-classical transition), the decoherence offers a possible solution to the measurement problem. In an alternative approach, problems with quantising gravity led to the half century old idea of gravitationally induced objective collapse of the wave function [1] (for an overview, see for example [2], chapter III.B): roughly speaking, due to the position uncertainty of massive bodies, which are the sources of gravitational field, the latter exhibits quantum fluctuations that decohere the matter, forcing it (or, rather both the matter and gravity) to collapse in a well defined (classical) state. Without invoking objective collapse, decoherence of quantum matter by purely classical gravity was studied in [3, 4]. In the context of perturbative quantum gravity, the topic of gravitationally induced decoherence of matter, taken purely within the scope of standard QM (i.e. in the same fashion in which macroscopic bodies decohere due to inevitable interaction with surrounding photons, neutrinos, microwave background radiation, etc), became recently an intensive field of research [5], see also [6] and the references therein for decoherence in the context of cosmological inflation. In addition, a lot of research focuses on entanglement induced by the presence of horizons in curved spacetime, in approaches based on the holography conjecture and in the studies of the black hole information problem [7] (for a review, see recent lecture notes [8]). In particular, these approaches study the entanglement between the degrees of freedom (*both* gravitational and matter) on the two sides of the horizon.

In this paper we study the entanglement between gravitational and matter fields, in the context of an abstract nonperturbative theory of quantum gravity, as well as on the example of the Hartle–Hawking state in the Regge quantum gravity model, and show that the two fields should *always* be entangled. Our approach is different from the standard one, studied in the perturbative framework: instead of 'for all practical purposes' inevitable fast interaction-induced decoherence from initially product states between two sub-systems [5, 9–13], we show that the gauge symmetry requirements (coming in particular from the local Poincaré symmetry) secure the entangled states between matter and gravity as physical states. We call the latter the *gauge-protected* decoherence, in contrast to the *dynamical* decoherence of the former. In addition, unlike the horizon-based studies, we discuss the entanglement between the gravitational and the matter degrees of freedom, rather than between the two specially chosen regions of spacetime.

Our analysis rests on two main assumptions. First, we assume the validity of the local Poincaré symmetry at the quantum level. In the classical field theory, the local Poincaré symmetry is a formalisation of the principle of general relativity, which is one of the foundational principles of Einstein's theory of gravity. It is therefore natural to assume that this gauge symmetry exists at the quantum level as well. Second, at the classical level we assume the validity of the equivalence principle, which is also the main ingredient of Einstein's general relativity. In particular, we assume its 'strong' version, namely that the equivalence principle applies to all matter fields (i.e. all non-gravitational fields) present in nature.

Given these two assumptions, we focus on the general nonperturbative abstract canonical quantisation of the gravitational and matter fields, thus giving a generic model-independent argument for a theory of quantum gravity with matter. We analyse the consequences of the local Poincaré symmetry-enforced scalar, 3-diffeomorphism and local Lorentz constraints on the structure of the total Hilbert space of the theory. Namely, since the physical states must be invariant with respect to the gauge symmetry, the constraints induce the Gupta–Bleuler-like conditions on the state vectors. Based on the equivalence principle, we then show that the particular non-separable form of the scalar constraint renders typical product states non-invariant. Thus, it eliminates the product states from the physical Hilbert space of the quantum theory, unless the interaction between gravity and matter is specifically designed to circumvent the non-invariance of product states. In this way, the local Poincaré symmetry protects the existence of entanglement between the gravitational and matter fields.

In order to verify our results obtained within the abstract canonical framework, we also study the covariant (i.e. path integral) quantisation. In particular, knowing that the Hartle–Hawking state [14] satisfies the scalar constraint, and is therefore an element of the physical Hilbert space, we explicitly test whether the matter and gravitational fields are entangled for this state vector. We perform the calculation in the Regge quantum gravity model, since it is one of the simplest models which provide an explicit definition of the gravitational path integral with matter, and show that the gravitational and matter fields are indeed entangled for the Hartle–Hawking state constructed on a simple toy example triangulation.

Therefore, our analysis shows that either gravity and matter fields are indeed entangled, or there exists an additional, unknown property of the action, implementing the fine tuning needed to allow for the invariance of separable states.

The paper is organised as follows. Section 2 is divided into three subsections. The first is devoted to the recapitulation of the Hamiltonian structure of Poincaré gauge theories. The second outlines the procedure of nonperturbative canonical quantisation of constrained systems and its application to the case of gravity with matter fields. In the third subsection we use those results to show that the scalar constraint suppresses the existence of separable states of a matter-gravity system. In section 3, we present a standard entanglement criterion for pure bipartite quantum states and discuss it, within the framework of the path integral quantisation, for the case of the Hartle–Hawking state of quantum fields of gravity and matter. In section 4, we first introduce the Regge model of quantum gravity, and then apply it to evaluate the entanglement criterion for the Hartle–Hawking state, demonstrating that gravity and matter are indeed entangled in this state. Finally, in section 5 we present the summary of the results, their discussion, and possible future lines of research.

It is important to stress that the gauge-protected entanglement is not an automatic consequence of the universal coupling between gravity and matter, or the fact that matter fields are always defined over some background spacetime geometry. For example, in perturbative gravity approach, it is quite possible to write the separable state between gravity and matter as

$$|\Psi\rangle = |g\rangle \otimes |\phi\rangle,$$

where $|g\rangle$ is the graviton state vector, while $|\phi\rangle$ is the state vector of a scalar particle (both vectors obtained by acting with graviton and scalar creation operators on the Minkowski vacuum state $|0\rangle \equiv |0\rangle_G \otimes |0\rangle_M$). The reason why such a state can be considered legitimate is that local Poincaré symmetry is explicitly broken in the perturbative gravity approach, with both matter and gravity being treated as spin-zero and spin-two fields, respectively, living on a Minkowski spacetime manifold. A similar situation arises in perturbative string theory, where local Poincaré symmetry is also manifestly broken. However, in quantum gravity models where the local Poincaré symmetry is not violated, our analysis shows that a generic product state between gravity and matter would fail to be gauge invariant. Thus, the gauge-protected entanglement between gravity and matter is a nontrivial statement and a consequence of local Poincaré symmetry, rather than an automatic property of matter fields living on a spacetime manifold.

Our notation and conventions are as follows. We will work in the natural system of units in which $c = \hbar = 1$ and $G = l_p^2$, where l_p is the Planck length. By convention, the metric of spacetime will have the spacelike Lorentz signature $(-,+,+,+)$. The spacetime indices are denoted with lowercase Greek letters $\mu, \nu, \ldots$ and take the values $0, 1, 2, 3$. The spatial part of these, taking values $1, 2, 3$, will be denoted with lowercase Latin letters $i, j, \ldots$ from the middle of the alphabet. The $SO(3,1)$ group indices will be denoted with the lowercase Latin letters $a, b, \ldots$ from the beginning of the alphabet, and take the values $0, 1, 2, 3$. The Lorentz-invariant metric tensor is denoted as η_{ab}. The capital Latin indices $A, B, \ldots$ count the field components in a particular representation of the $SO(3,1)$ group, and take the values from 1 up to the dimension of that representation. Quantum operators will always carry a hat, $\hat{\phi}(x)$, $\hat{g}(x)$, etc. Finally, we will systematically denote the values of functions with parentheses, $f(x)$, while functionals will be denoted with brackets, $F[\phi]$.

2. Entanglement from the scalar constraint

This section is dedicated to the analysis of the constraints imposed by the relativity and equivalence principles. In section 2.1 we briefly recapitulate the classical Hamiltonian structure of gravitational interaction, followed by a short review of canonical quantisation, presented in section 2.2. After that, in section 2.3 we present the main result of our paper: we show that the scalar constraint, and possibly the 3-diffeomorphism constraint, bring about the generic entanglement between gravity and matter.

2.1. Hamiltonian structure of Poincaré gauge theories

We begin with a short review of the Hamiltonian structure of gravitational interaction, based on the local Poincaré symmetry. This subsection is aimed to be only a review of the main results, so we will skip all proofs and derivations. The details of the Hamiltonian structure for Poincaré gauge theories (PGT) can be found in many textbooks, see for example [15], chapter V, and the references therein.

We will assume a foliation of spacetime into space and time, with the spacetime topology $\mathcal{M}_4 = \Sigma_3 \times \mathbb{R}$, where Σ_3 is the $3D$ hypersurface. For the purpose of generality, we will describe the gravitational field as $g(x)$ and matter fields as $\phi(x)$, without specifying their exact field content, except in examples. A typical example would be the Einstein–Cartan gravity coupled to a Dirac matter field, so that the choice of fundamental gravitational fields g would be the tetrads $e^a{}_\mu(x)$ and the spin connection $\omega^{ab}{}_\mu(x)$, while the choice for the fundamental matter field ϕ would be a Dirac fermion field $\psi(x)$. However, other choices for g and ϕ are also possible, for example the metric tensor $g_{\mu\nu}$ for gravity and the electromagnetic potential A^μ for matter, etc. Since our analysis is largely independent of such choices, we will stick to the abstract notation g and ϕ, assuming that one can apply our analysis to each particular concrete choice of fundamental fields.

Given the above notation, we will assume that the action of the theory can be written as

$$S[g,\phi] = S_G[g] + S_M[g,\phi], \tag{1}$$

where $S_G[g]$ is the action of the pure gravitational field, while $S_M[g,\phi]$ is the action of the matter fields coupled to gravity. Since the spacetime metric must both be a function of the gravitational field g and is always present in the definition of the dynamics of matter fields, the action for the matter fields cannot contain terms independent of g. This elementary fact is the crux of our main argument below, and is justified by the equivalence principle, which dictates how matter couples to gravity.

To a large extent, we also do not need to specify the details of the actions $S_G[g]$ and $S_M[g,\phi]$. We will only assume that the action (1) belongs to the PGT class of theories, i.e. that it is invariant with respect to local Poincaré group $P(4) = \mathbb{R}^4 \ltimes SO(3,1)$. Every theory belonging to the PGT class has the Hamiltonian with the following general structure [15]:

$$H = \int_{\Sigma_3} \mathrm{d}^3\vec{x} \left[N\mathcal{C} + N^i\mathcal{C}_i + N^{ab}\mathcal{C}_{ab} \right], \tag{2}$$

up to a 3-divergence. Here N, N^i and N^{ab} are Lagrange multipliers, the first two of which are commonly known as lapse and shift functions. The quantities $\mathcal{C}$, $\mathcal{C}_i$ and $\mathcal{C}_{ab}$ are usually known as the scalar constraint, 3-diffeomorphism constraint, and the local Lorentz constraint (sometimes also called the Gauss constraint), respectively. They are a (g,ϕ)-field representation of the 10 generators of the Poincaré group $P(4)$, in particular the time translation generator, the three space translation generators, and six local Lorentz generators (rotations and boosts). Note that the Hamiltonian (2) is always a linear combination of these constraints.

The constraints in (2) have the structure similar to the structure of the gravity-matter action (1), namely

$$\begin{aligned}
\mathcal{C} &= \mathcal{C}^G(g,\pi_g) + \mathcal{C}^M(g,\pi_g,\phi,\pi_\phi), \\
\mathcal{C}_i &= \mathcal{C}_i^G(g,\pi_g) + \mathcal{C}_i^M(g,\pi_g,\phi,\pi_\phi), \\
\mathcal{C}_{ab} &= \mathcal{C}_{ab}^G(g,\pi_g) + \mathcal{C}_{ab}^M(g,\pi_g,\phi,\pi_\phi),
\end{aligned} \tag{3}$$

where π_g and π_ϕ are the momenta canonically conjugated to the fields g and ϕ, respectively, defined as functional derivatives of the action with respect to the time-derivatives of the fields,

$$\pi_g(x) = \frac{\delta S}{\delta \partial_0 g(x)}, \qquad \pi_\phi(x) = \frac{\delta S}{\delta \partial_0 \phi(x)}.$$

The general dependence (3) on the fields and momenta reflects the corresponding dependence in (1).

The exact forms of the gravitational terms of the constraints, namely $\mathcal{C}^G$, $\mathcal{C}_i^G$ and $\mathcal{C}_{ab}^G$, will be immaterial for our main argument presented in the section 2.3 below. In contrast, the structure of the matter terms $\mathcal{C}^M$, $\mathcal{C}_i^M$ and $\mathcal{C}_{ab}^M$ will be crucial, so we discuss it here in more detail. Choose a matter field such that it transforms according to some specific irreducible transformation of the Poincaré group, and denote it as $\phi^A(x)$, where the capital index A counts the field components in that representation. Then the 3-diffeo constraint $\mathcal{C}_i^M$ and the Gauss constraint $\mathcal{C}_{ab}^M$ are given as

$$\begin{array}{rcl} \mathcal{C}_i^M(g,\pi_g,\phi,\pi_\phi) & = & \pi_{\phi A} \nabla_i{}^A{}_B \phi^B, \\ \mathcal{C}_{ab}^M(g,\pi_g,\phi,\pi_\phi) & = & \pi_{\phi A} (M_{ab})^A{}_B \phi^B, \end{array} \tag{4}$$

where $\nabla_i{}^A{}_B$ is a covariant derivative for the irreducible representation according to which the field ϕ transforms, while $(M_{ab})^A{}_B$ is the representation of the generator M_{ab} of the Lorentz group $SO(3,1)$ in the same representation. In general, the covariant derivative depends on the spacetime metric or connection, which is a function of the fundamental gravitational fields g, and possibly their momenta π_g. The Lorentz group generators, on the other hand, do not depend on the spacetime geometry, so the Gauss constraint is actually independent of g and π_g, and we can write $\mathcal{C}_{ab}^M(g,\pi_g,\phi,\pi_\phi) = \mathcal{C}_{ab}^M(\phi,\pi_\phi)$.

In order to illustrate the two constraints, we will write (4) for the scalar and Dirac fields, as the most elementary examples. In the case of the scalar field, we write $\phi^A(x) = \varphi(x)$, where the index A takes only a single value. The covariant derivative acts on the scalar field as an ordinary derivative, while the representation of the Lorentz generators is trivial, so we can write

$$\mathcal{C}_i^M(\varphi,\pi_\varphi) = \pi_\varphi \partial_i \varphi, \qquad \mathcal{C}_{ab}^M(\varphi,\pi_\varphi) = \pi_\varphi \varphi. \tag{5}$$

We see that in the case of the scalar field, both constraints are independent of the gravitational fields and their momenta. In the case of the Dirac fields, we write $\phi^A(x) = (\psi^A(x), \bar{\psi}^A(x))$, where the index A now represents the spinorial index, and we will omit writing it. The covariant derivative acts on the Dirac field in the standard way,

$$\begin{array}{rcl} \overrightarrow{\nabla}_\mu \psi & \equiv & \partial_\mu \psi + \dfrac{1}{2}\omega^{ab}{}_\mu \sigma_{ab}\psi, \\ \bar{\psi}\overleftarrow{\nabla}_\mu & \equiv & \partial_\mu \bar{\psi} - \dfrac{1}{2}\omega^{ab}{}_\mu \bar{\psi}\sigma_{ab}, \end{array} \tag{6}$$

where $\omega^{ab}{}_\mu$ is the spin connection, $\sigma_{ab} = \frac{1}{4}[\gamma_a,\gamma_b]$, and γ_a are the standard Dirac gamma-matrices satisfying the anticommutation relation $\{\gamma_a,\gamma_b\} = -2\eta_{ab}$. The representation of the Lorentz generators for the case of the Dirac field is $M_{ab} = \sigma_{ab}$. Denoting the conjugate momentum for ψ as $\bar{\pi}$ and conjugate momentum for $\bar{\psi}$ as π, we can write the constraints (4) as:

$$\begin{array}{rcl} \mathcal{C}_i^M(\omega,\psi,\bar{\pi},\bar{\psi},\pi) & = & \bar{\pi}\overrightarrow{\nabla}_i\psi + (\bar{\psi}\overleftarrow{\nabla}_i)\pi, \\ \mathcal{C}_{ab}^M(\psi,\bar{\pi},\bar{\psi},\pi) & = & \bar{\pi}\sigma_{ab}\psi - \bar{\psi}\sigma_{ab}\pi. \end{array} \tag{7}$$

Note that here, unlike in the scalar field example, the 3-diffeo constraint contains the spin connection $\omega^{ab}{}_\mu$, which is a part of the gravitational field $g = (e^a{}_\mu, \omega^{ab}{}_\mu)$ for the Einstein–Cartan gravity.

In contrast to the 3-diffeo and Gauss constraints (4), the scalar constraint $\mathcal{C}^M$ has a more complicated form,

$$\mathcal{C}^M(g,\pi_g,\phi,\pi_\phi) = \pi_{\phi A}\nabla_\perp{}^A{}_B\phi^B - \frac{1}{N}\mathcal{L}_M(g,\pi_g,\phi,\pi_\phi), \tag{8}$$

where the matter Lagrangian density is defined via

$$S_M[g,\phi] = \int \mathrm{d}^4x\, \mathcal{L}_M(g,\partial g,\phi,\partial\phi),$$

and $\nabla_\perp \equiv n^\mu \nabla_\mu$ is the covariant derivative in the direction of the timelike vector n^μ orthogonal to the spacelike hypersurface Σ_3. The vector n^μ obviously depends on the spacetime metric $g_{\mu\nu}$, and is thus a function of the fundamental gravitational fields g.

There are several things to note regarding the scalar constraint (8). First, it is clear that $N\mathcal{C}^M$ is the Legendre transformation of the Lagrangian density $\mathcal{L}_M$ with respect to the 'velocity' $N\nabla_\perp \phi$. Second, in contrast to the constraints (4), which depend only on the symmetry transformation properties of the fields, the form of the scalar constraint (8) depends also on the choice of the matter Lagrangian density $\mathcal{L}_M$, and is therefore described by the dynamics of the matter fields coupled to gravity. And third, the scalar constraint $\mathcal{C}^M$ always necessarily depends on the gravitational fields g, in contrast to the 3-diffeo constraint which may or may not depend on g, and the Gauss constraint which never depends on g. As we already suggested above, this is because the Lagrangian of the matter fields coupled to gravity always contains the gravitational degrees of freedom, courtesy of the equivalence principle.

Let us illustrate this dependence of $\mathcal{C}^M$ on the gravitational fields g in the case of the Dirac field. The action for the Dirac field $\phi = (\psi, \bar{\psi})$ coupled to the gravitational fields $g = (e^a{}_\mu, \omega^{ab}{}_\mu)$ is given as

$$S_M[e,\omega,\psi,\bar{\psi}] = \int \mathrm{d}^4x\, e \left(\frac{\mathrm{i}}{2} \bar{\psi}\gamma^a e^\mu{}_a \overleftrightarrow{\nabla}_\mu \psi - m\bar{\psi}\psi \right), \tag{9}$$

where e is the determinant of the tetrad $e^a{}_\mu$, while $e^\mu{}_a$ is the inverse tetrad. In addition, $\overleftrightarrow{\nabla}_\mu \equiv \overrightarrow{\nabla}_\mu - \overleftarrow{\nabla}_\mu$, and the covariant derivatives $\overrightarrow{\nabla}_\mu$ and $\overleftarrow{\nabla}_\mu$ act to the right and to the left as defined in (6), from which one can see that the action also explicitly depends on the connection $\omega^{ab}{}_\mu$. From the action one can read off the Lagrangian density, and calculate the scalar constraint (8) as

$$\mathcal{C}^M(e,\omega,\psi,\bar{\psi}) = -\frac{e}{N} \left(\frac{\mathrm{i}}{2} \bar{\psi}\gamma^a e^\mu{}_a \left(\delta^\mu_\nu + n^\mu n_\nu\right) \overleftrightarrow{\nabla}_\nu \psi - m\bar{\psi}\psi \right).$$

Note that the quantity $\delta^\mu_\nu + n^\mu n_\nu$ is a projector to the hypersurface Σ_3.

2.2. Canonical quantisation

Having discussed the Hamiltonian structure of the action (1), we now pass on to a short description of the canonical quantisation of the theory. The quantisation of an arbitrary physical system with constraints is performed in the standard way, using the Dirac's procedure [16, 17] (see [15] for a review). One begins by classifying all constraints of the theory into the first and the second class. The second class constraints are then eliminated by passing from the Poisson brackets to the Dirac brackets. The first class constraints remain and represent the generators of the gauge symmetry. In general, the Hamiltonian of the theory can be written as

$$H = H_0 + \lambda^A \mathcal{C}_A, \tag{10}$$

where λ^A are Lagrange multipliers, $\mathcal{C}_A$ are first class constraints, and H_0 is the part of the Hamiltonian which describes the evolution of the physical degrees of freedom. Given all this, the quantisation is performed in the Heisenberg picture, promoting fundamental fields $\phi(x)$ to quantum mechanical operators $\hat{\phi}(x)$, and introducing the state vectors $|\Psi\rangle \in \mathcal{H}_{\mathrm{kin}}$, where $\mathcal{H}_{\mathrm{kin}}$ is the kinematical Hilbert space of the theory. The Dirac brackets between the fields and their momenta are then promoted to the commutators of the corresponding operators. The Hamiltonian, being a functional of the fields and momenta, also becomes an operator, providing the usual Heisenberg equations of motion for the field operators,

$$\mathrm{i}\frac{\partial\hat{\phi}(x)}{\partial t} = [\hat{\phi}(x), \hat{H}].$$

Finally, the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ is projected onto its gauge invariant subspace $\mathcal{H}_{\text{phys}}$, by requiring that every state vector $|\Psi\rangle \in \mathcal{H}_{\text{phys}}$ is annihilated by the generators of the gauge symmetry group,

$$\hat{\mathcal{C}}_A|\Psi\rangle = 0.$$

In quantum electrodynamics these conditions are known as Gupta–Bleuler quantisation conditions [18, 19]. This requirement ensures that the gauge symmetry of the classical theory remains to be a symmetry of the quantum theory as well.

Of course, one cannot hope to implement the above quantisation programme in full detail for the general action (1), especially without the detailed specification of the fundamental degrees of freedom that define the theory. Instead, we assume that the quantisation programme has been carried out in detail, and that all quantities we will write are well defined. This approach has one important feature and one important drawback. The feature is generality—our main argument for the inevitable entanglement between gravity and matter, to be presented in section 2.3, should hold for every particular quantum theory constructed in the above way, as it does not actually depend on the details of the quantisation. The drawback is abstractness—in using such a general formalism and making a flat assumption that all details are well defined, we lose the capability to provide any concrete examples. That said, in section 4 we discuss one rigorously defined example of a theory of quantum gravity with matter (Regge quantum gravity), and demonstrate the entanglement between gravity and matter fields. Unlike the canonical quantisation discussed in this section, that example will be done in the framework of the path integral quantisation.

Keeping this disclaimer in mind, we proceed along the lines outlined above and perform the canonical quantisation. The most prominent property of our model is the structure of the Hilbert space of the theory. The initial kinematical Hilbert space $\mathcal{H}_{\text{kin}} = \mathcal{H}_G \otimes \mathcal{H}_M$ has a natural product structure between the gravitational and matter Hilbert spaces, since we have two sets of fields, $\hat{g}$ and $\hat{\phi}$, corresponding to gravity and matter, respectively. Thus, we have a naturally preferred bipartite physical system, because gravitational and matter degrees of freedom can be fully distinguished from each other. Second, in order to preserve the Poincaré gauge symmetry of the theory at the quantum level, we have to pass from the kinematical Hilbert space to the gauge invariant, physical Hilbert space $\mathcal{H}_{\text{phys}}$. By definition, a state $|\Psi\rangle \in \mathcal{H}_{\text{kin}}$ is an element of $\mathcal{H}_{\text{phys}}$ iff it satisfies

$$\begin{aligned} \hat{\mathcal{C}}_{ab}|\Psi\rangle &\equiv \left[\mathcal{C}^G_{ab}(\hat{g},\hat{\pi}_g) + \mathcal{C}^M_{ab}(\hat{\phi},\hat{\pi}_\phi)\right]|\Psi\rangle = 0, \\ \hat{\mathcal{C}}_{i}|\Psi\rangle &\equiv \left[\mathcal{C}^G_{i}(\hat{g},\hat{\pi}_g) + \mathcal{C}^M_{i}(\hat{g},\hat{\pi}_g,\hat{\phi},\hat{\pi}_\phi)\right]|\Psi\rangle = 0, \end{aligned} \tag{11}$$

and

$$\hat{\mathcal{C}}|\Psi\rangle \equiv \left[\mathcal{C}^G(\hat{g},\hat{\pi}_g) + \mathcal{C}^M(\hat{g},\hat{\pi}_g,\hat{\phi},\hat{\pi}_\phi)\right]|\Psi\rangle = 0. \tag{12}$$

As stated above, we assume that the operators $\hat{\mathcal{C}}_{ab}$, $\hat{\mathcal{C}}_i$ and $\hat{\mathcal{C}}$ are well defined, that operator ordering choice has been fixed, as well as all other necessary technical choices, in order for the expressions above to make sense mathematically.

We argue that, due to these constraint equations, there are no states in $\mathcal{H}_{\text{phys}}$ which can be written as product states of the form $|\Psi_G\rangle \otimes |\Psi_M\rangle$, where $|\Psi_G\rangle \in \mathcal{H}_G$ and $|\Psi_M\rangle \in \mathcal{H}_M$, i.e. the states in $\mathcal{H}_{\text{phys}}$ are entangled. We focus on the scalar constraint (12), while the constraints

(11) are either irrelevant or redundant for our analysis. This main argument of our paper is presented in the next subsection.

2.3. Entanglement

Given a state vector $|\Psi\rangle \in \mathcal{H}_{\rm kin} = \mathcal{H}_G \otimes \mathcal{H}_M$, it is an element of the physical Hilbert space $\mathcal{H}_{\rm phys}$ if it satisfies the Gauss and 3-diffeo constraints (11) and the scalar constraint (12). Choosing the eigenbases $\{|g\rangle\}$ and $\{|\phi\rangle\}$ of the quantum field operators $\hat{g}$ and $\hat{\phi}$, respectively, we can work in the so-called field representation, defined as

$$\begin{array}{rclcrcl} \langle g|\hat{g} & = & g\langle g|, & \qquad & \langle g|\hat{\pi}_g & = & -\mathrm{i}\dfrac{\delta}{\delta g}\langle g|, \\ \langle \phi|\hat{\phi} & = & \phi\langle \phi|, & & \langle \phi|\hat{\pi}_\phi & = & -\mathrm{i}\dfrac{\delta}{\delta \phi}\langle \phi|. \end{array} \tag{13}$$

Acting on (12) with $\langle g,\phi| \equiv \langle g| \otimes \langle \phi|$ from the left, the scalar constraint becomes a functional partial differential equation of Wheeler–DeWitt type:

$$\left[\mathcal{C}_G\left(g, -\mathrm{i}\tfrac{\delta}{\delta g}\right) + \mathcal{C}_M\left(g, -\mathrm{i}\tfrac{\delta}{\delta g}, \phi, -\mathrm{i}\tfrac{\delta}{\delta \phi}\right)\right]\Psi[g,\phi] = 0, \tag{14}$$

where $\Psi[g,\phi] \equiv \langle g,\phi|\Psi\rangle$ is the wavefunctional of the combined gravity-matter system. We now try to look for a separable state, in the form $|\Psi\rangle = |\Psi_G\rangle \otimes |\Psi_M\rangle$, where $|\Psi_G\rangle \in \mathcal{H}_G$ and $|\Psi_M\rangle \in \mathcal{H}_M$, as a solution of this equation. Using the field representation (13), we write the wavefunctional $\Psi[g,\phi]$ as

$$\begin{array}{rcl} \Psi[g,\phi] & \equiv & \langle g,\phi|\Psi\rangle \\ & = & (\langle g| \otimes \langle \phi|)\,(|\Psi_G\rangle \otimes |\Psi_M\rangle) \\ & = & \langle g|\Psi_G\rangle\langle \phi|\Psi_M\rangle \\ & \equiv & \Psi_G[g]\Psi_M[\phi]. \end{array} \tag{15}$$

Equation (14) can have separable solutions $\Psi[g,\phi] = \Psi_G[g]\Psi_M[\phi]$ if the functional differential operator $\mathcal{C}_M$ can be written as a product of two operators, denoted $\mathcal{K}_G$ and $\mathcal{K}_M$, depending only on $(g, \frac{\delta}{\delta g})$ and on $(\phi, \frac{\delta}{\delta \phi})$, respectively,

$$\mathcal{C}_M\left(g, -\mathrm{i}\tfrac{\delta}{\delta g}, \phi, -\mathrm{i}\tfrac{\delta}{\delta \phi}\right) = \mathcal{K}_G\left(g, \tfrac{\delta}{\delta g}\right)\mathcal{K}_M\left(\phi, \tfrac{\delta}{\delta \phi}\right). \tag{16}$$

If such operators $\mathcal{K}_G$ and $\mathcal{K}_M$ exist so that (16) holds, the scalar constraint equation (14) can be rewritten as

$$\Psi_M[\phi]\,\mathcal{C}_G\left(g, -\mathrm{i}\tfrac{\delta}{\delta g}\right)\Psi_G[g] = -\left[\mathcal{K}_G\left(g, \tfrac{\delta}{\delta g}\right)\Psi_G[g]\right]\left[\mathcal{K}_M\left(\phi, \tfrac{\delta}{\delta \phi}\right)\Psi_M[\phi]\right].$$

Dividing this with $\Psi_M[\phi]\,\mathcal{K}_G\left(g, \frac{\delta}{\delta g}\right)\Psi_G[g]$, assuming it is well-defined, we obtain

$$\frac{1}{\mathcal{K}_G\left(g, \frac{\delta}{\delta g}\right)\Psi_G[g]}\,\mathcal{C}_G\left(g, -\mathrm{i}\tfrac{\delta}{\delta g}\right)\Psi_G[g] = -\frac{1}{\Psi_M[\phi]}\,\mathcal{K}_M\left(\phi, \tfrac{\delta}{\delta \phi}\right)\Psi_M[\phi] = A,$$

where A is a constant, since the terms on the left and the right of the first equality depend on different sets of variables. Therefore, the above equation splits into two independent equations,

$$\begin{array}{rcl} \left[\mathcal{C}_G\left(g, -\mathrm{i}\tfrac{\delta}{\delta g}\right) - A\,\mathcal{K}_G\left(g, \tfrac{\delta}{\delta g}\right)\right]\Psi_G[g] & = & 0, \\ \left[\mathcal{K}_M\left(\phi, \tfrac{\delta}{\delta \phi}\right) + A\right]\Psi_M[\phi] & = & 0, \end{array} \tag{17}$$

which are to be solved independently for $\Psi_G[g]$ and $\Psi_M[\phi]$, thus providing a separable solution of (14).

The whole procedure above rests on the assumption (16) that the matter part $\mathcal{C}_M$ of the scalar constraint operator can be written as a product of two operators $\mathcal{K}_G$ and $\mathcal{K}_M$. Our main argument is to demonstrate that the assumption (16) is never satisfied for the usual matter fields, due to the universal nature of the coupling of gravity to matter, ultimately dictated by the equivalence principle. Namely, given the structure of the classical scalar constraint for matter (8), the corresponding operator can be written as

$$\mathcal{C}^M(\hat{g}, \hat{\pi}_g, \hat{\phi}, \hat{\pi}_\phi) = \hat{\pi}_{\phi A} \hat{\nabla}_\perp{}^A{}_B \hat{\phi}^B - \frac{1}{N} \mathcal{L}_M(\hat{g}, \hat{\pi}_g, \hat{\phi}, \hat{\pi}_\phi), \tag{18}$$

where a certain ordering of the operators is assumed. The constraint (18) features the operator-valued matter Lagrangian $\mathcal{L}_M$. Therefore, in order to demonstrate that $\mathcal{C}_M$ does not satisfy the separability criterion (16) it is enough to demonstrate that the matter Lagrangian does not satisfy it. This can be done on a case-by-case basis, for each particular matter field. Invoking the equivalence principle, we can write the operator-valued Lagrangian for the scalar field coupled to gravity as

$$\mathcal{L}_M(\hat{g}, \hat{\varphi}, \partial\hat{\varphi}) = \frac{1}{2} \hat{e} \left[\hat{g}^{\mu\nu} (\partial_\mu \hat{\varphi})(\partial_\nu \hat{\varphi}) - m^2 \hat{\varphi}^2 + U(\hat{\varphi}) \right],$$

where $\hat{e}$ is the square-root of the minus determinant operator of the metric tensor,

$$\hat{e} \equiv \left[\frac{1}{4!} \varepsilon^{\alpha\beta\gamma\delta} \varepsilon^{\mu\nu\rho\sigma} \hat{g}_{\alpha\mu} \hat{g}_{\beta\nu} \hat{g}_{\gamma\rho} \hat{g}_{\delta\sigma} \right]^{\frac{1}{2}},$$

and U is some interaction potential of the scalar field. Ignoring the multiplicative factor $\hat{e}$ that acts only on $\mathcal{H}_G$, the Lagrangian is a sum of two types of terms: the kinetic term, containing the inverse metric $\hat{g}^{\mu\nu}$, and the mass and potential terms not featuring the gravitational field in any form. The sum cannot therefore be factored into the form $\mathcal{K}_G(\hat{g})\mathcal{K}_M(\hat{\phi}, \partial\hat{\phi})$, since the Lagrangian is not a homogeneous function of the gravitational degrees of freedom. Even in the case of the massless free scalar field, i.e. when $m = 0$ and $U = 0$, the kinetic term is a sum of several different components of the metric and the derivatives of the scalar field,

$$\hat{g}^{00}(\partial_0\hat{\varphi})(\partial_0\hat{\varphi}) + \hat{g}^{01}(\partial_0\hat{\varphi})(\partial_1\hat{\varphi}) + \hat{g}^{12}(\partial_1\hat{\varphi})(\partial_2\hat{\varphi}) + \dots$$

and this still cannot be factored into a product of two operators $\mathcal{K}_G$ and $\mathcal{K}_M$.

In the case of the Dirac field, again invoking the equivalence principle, the operator-valued Lagrangian is given by (9),

$$\mathcal{L}_M(\hat{e}, \hat{\omega}, \hat{\psi}, \hat{\bar{\psi}}) = \hat{e} \left(\frac{\mathrm{i}}{2} \hat{\bar{\psi}} \gamma^a \hat{e}^\mu{}_a \overset{\leftrightarrow}{\hat{\nabla}}_\mu \hat{\psi} - m \hat{\bar{\psi}} \hat{\psi} \right).$$

Like in the case of the scalar field, the kinetic and mass terms in the Lagrangian depend differently on the gravitational fields $\hat{e}^a{}_\mu$ and $\hat{\omega}^{ab}{}_\mu$, and $\mathcal{L}_M$ cannot be factored. Moreover, the kinetic term itself cannot be factored, since it is a sum of two terms (see equations (6)), only one of which contains the spin connection $\hat{\omega}^{ab}{}_\mu$.

Next, the operator-valued Lagrangian for the electromagnetic field coupled to gravity has the form

$$\mathcal{L}_M(\hat{g}, \hat{A}, \partial\hat{A}) = -\frac{1}{4} \hat{e}\, \hat{g}^{\mu\rho} \hat{g}^{\nu\sigma} \hat{F}_{\mu\nu} \hat{F}_{\rho\sigma},$$

where $\hat{F}_{\mu\nu} \equiv \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$. Applying the same argument as in the case of the free massless scalar field, this Lagrangian also cannot be factored into the form $\mathcal{K}_G \mathcal{K}_M$. The same argument also applies to the case of the non-Abelian Yang–Mills Lagrangians.

Summing up, given the ways the matter fields are coupled to gravity, based on the equivalence principle, we conclude that the separability criterion (16) is never satisfied for the physically relevant cases of scalar, spinor and vector fields. Therefore, according to the discussion above, the scalar constraint (12) should not admit separable state vectors into $\mathcal{H}_{\text{phys}}$.

Regarding the above analysis, it is important to emphasize the following. Namely, one should note that it is in principle possible for equation (14) to have product state solutions (15) despite the fact that it does not satisfy the separability criterion (16). In other words, the criterion (16) is a sufficient condition for the existence of product state solutions of (14), but it is not necessary, so its violation does not strictly imply the absence of product state solutions. Nevertheless, given the arguably highly complex structure of equation (14)—meaning that it represents a nonlinear functional partial differential equation of at least second order in g and ϕ—it is natural to regard any potential product state solutions as completely accidental. Moreover, it is questionable if the boundary conditions required for such solutions correspond to any realistic physical situation in nature, i.e. they could be irrelevant for realistic physics. Due to all these arguments, the existence of product state solutions, in spite of the violation of the separability criterion (16), is in our opinion an extraordinary claim, and as such requires extraordinary evidence. In other words, the burden of proof is in fact with the statement that any product state solution exists, rather than the opposite. Consequently, product states (15) are generically not elements of $\mathcal{H}_{\text{phys}}$, and even if one can prove that there exist some product states which do happen to belong to $\mathcal{H}_{\text{phys}}$, such states would arguably be completely accidental, with questionable relevance for physics. Otherwise, if there exists a whole class of separable states which solve (14) despite the violation of the criterion (16), there must be some deep eluding property of the scalar constraint equation, which is both completely unknown and very interesting to study.

Finally, while it turns out that the analysis of the scalar constraint equation (12) is sufficient for our conclusions, let us briefly mention the status of the remaining two constraint equations (11). First, the Gauss constraint $\hat{\mathcal{C}}_{ab}$ obviously admits separable state vectors. On the other hand, the situation with the 3-diffeo constraint $\hat{\mathcal{C}}_i$ is more complicated, and the conclusion depends on the type of the field. For example, in the case of the scalar field, from (5) we read that $\hat{\mathcal{C}}_i^M$ depends only on the scalar field and its momentum, which means that the constraint equation does admit separable state vectors. However, in the case of the Dirac field, from (7) we read that $\hat{\mathcal{C}}_i^M$ depends on the spin connection in addition to the Dirac field, and this dependence is not homogeneous in the spin connection, see (6). Thus, the 3-diffeo constraint equation does not admit separable state vectors. However, the behaviour of the Gauss and 3-diffeo constraint equations is redundant for our argument, since the scalar constraint equation (12) already suppresses separable state vectors for all fields, due to the dynamical form of the coupling of matter to gravity. Therefore, our initial assumption of local Poincaré symmetry can be weakened to the localisation of its translational subgroup, while the generators of the local Lorentz subgroup are irrelevant for our argument.

3. Entanglement in the path integral framework

In the previous section we have discussed the gauge-protected entanglement within the framework of the canonical quantisation of the gravitational field with matter. In this section, we focus instead on the path integral framework of quantisation. We analyse the entanglement on the example of the Hartle–Hawking state, which is known to satisfy all constraints of the theory. In the next section, we are going to apply the results of this section to the concrete case of Regge quantum gravity.

First, we discuss an entanglement criterion for the case of pure overall state of the gravity and matter fields. We begin with a brief recapitulation of basic results from the standard QM and quantum information theory. A pure bipartite state $|\Psi\rangle_{12} \in \mathcal{H}_1 \otimes \mathcal{H}_2$ of systems 1 and 2 can be written in the Schmidt bi-orthogonal form (see, for example [20]):

$$|\Psi\rangle_{12} = \sum_i \sqrt{r_i} |\alpha_i\rangle_1 \otimes |\beta_i\rangle_2, \tag{19}$$

where $\{|\alpha_i\rangle_1\}$ and $\{|\beta_i\rangle_2\}$ are two sets of mutually orthogonal states from $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The partial sub-system states are then given as

$$\hat{\rho}_1 = \sum_i r_i |\alpha_i\rangle_1 \otimes \langle\alpha_i|_1, \tag{20}$$

for the system 1, and analogously for the system 2. Squaring $\hat{\rho}_1$, we have

$$\hat{\rho}_1^2 = \sum_i r_i^2 |\alpha_i\rangle_1 \otimes \langle\alpha_i|_1. \tag{21}$$

If the overall state $|\Psi\rangle_{12}$ is separable (i.e. a simple product state), the above sum in (20) will be trivial, consisting of a single projector onto the ray $|\alpha_1\rangle_1 \otimes \langle\alpha_1|_1$, with $r_1 = 1$. Thus, we have that $\hat{\rho}_1^2 = \hat{\rho}_1$, or simply, $\mathrm{Tr}\,\hat{\rho}_1^2 = \mathrm{Tr}\,\hat{\rho}_1 = 1$. In case the state $|\Psi\rangle_{12}$ is entangled, the sum (20) will consist of more than just one term, resulting in $(\forall\ i)\ r_i < 1$. Therefore, $(\forall\ i)\ r_i^2 < r_i$, and we finally have

$$\mathrm{Tr}\,\hat{\rho}_1^2 = \sum_i r_i^2 < \sum_i r_i = \mathrm{Tr}\,\hat{\rho}_1 = 1. \tag{22}$$

Due to the symmetry of the Schmidt form (19), the same is valid for the system 2 (for the formal proof of the above entanglement criterion (22), see for example [20]).

After this recapitulation of the standard results from QM, we proceed with the analysis of the bipartite system of the gravity (G) and matter (M) fields, applying the above entanglement criterion (22) to the case of quantum fields. For simplicity, we omit the subscripts G and M for pure states of gravity and matter, respectively.

Let $\mathcal{H}_{\mathrm{kin}} = \mathcal{H}_G \otimes \mathcal{H}_M$ be the combined kinematical gravity-matter Hilbert space. Denote the bases in $\mathcal{H}_G$ and $\mathcal{H}_M$ as $\{|g\rangle\}$ and $\{|\phi\rangle\}$, respectively. These are the eigenbases of the corresponding quantum field operators $\hat{g}$ and $\hat{\phi}$, evaluated on the $3D$ boundary $\Sigma_3 = \partial\mathcal{M}_4$ of the $4D$ spacetime manifold $\mathcal{M}_4$. The general state vector $|\Psi\rangle \in \mathcal{H}_{\mathrm{kin}}$ of the gravity-matter system can then be written as

$$|\Psi\rangle = \int \mathcal{D}g \int \mathcal{D}\phi\, \Psi[g,\phi]\, |g\rangle \otimes |\phi\rangle, \tag{23}$$

where $\Psi[g,\phi] = \langle g,\phi|\Psi\rangle$ is called the wavefunctional (in analogy to wavefunction from quantum mechanics), and the functional integrals over gravitational degrees of freedom g and matter degrees of freedom ϕ are assumed to be well defined in some way (in section 4 we present an explicit example of this). The bases $\{|g\rangle\}$ and $\{|\phi\rangle\}$ are assumed to be orthonormal, satisfying

$$\langle g|g'\rangle = \delta[g-g'], \qquad \langle\phi|\phi'\rangle = \delta[\phi-\phi'], \tag{24}$$

where the Dirac delta functional is assumed to satisfy the formal functional integral identities

$$\int \mathcal{D}g\, F[g]\, \delta[g-g'] = F[g'],$$
$$\int \mathcal{D}\phi\, F[\phi]\, \delta[\phi-\phi'] = F[\phi'], \tag{25}$$

for any functionals $F[g]$ and $F[\phi]$ belonging to some suitable relevant class.

From the state (23) one can construct a reduced density matrix $\hat{\rho}_M$ for matter fields, by taking the partial trace over gravitational degrees of freedom of the full density matrix $\hat{\rho} \equiv |\Psi\rangle \otimes \langle\Psi|$, as

$$\hat{\rho}_M = \mathrm{Tr}_G\, \hat{\rho} = \int \mathcal{D}g\, \langle g| \Big(|\Psi\rangle \otimes \langle\Psi| \Big) |g\rangle.$$

Substituting (23) we get

$$\hat{\rho}_M = \int \mathcal{D}g \int \mathcal{D}g' \int \mathcal{D}\phi' \int \mathcal{D}g'' \int \mathcal{D}\phi'' \\ \Psi^*[g',\phi']\, \Psi[g'',\phi'']\, \langle g| \Big(|g''\rangle \otimes |\phi''\rangle \otimes \langle g'| \otimes \langle\phi'| \Big) |g\rangle.$$

Using (24) and (25), the expression for the reduced density matrix can be evaluated to

$$\hat{\rho}_M = \int \mathcal{D}g \int \mathcal{D}\phi' \int \mathcal{D}\phi''\, \Psi^*[g,\phi'] \Psi[g,\phi'']\, |\phi''\rangle \otimes \langle\phi'|. \tag{26}$$

Taking the square and using (24) and (25) again, one obtains

$$\hat{\rho}_M^2 = \int \mathcal{D}g \int \mathcal{D}g' \int \mathcal{D}\phi' \int \mathcal{D}\phi'' \int \mathcal{D}\phi''' \\ \Psi^*[g,\phi']\, \Psi[g,\phi'']\, \Psi^*[g',\phi''']\, \Psi[g',\phi']\, |\phi''\rangle \otimes \langle\phi'''|.$$

Finally, taking the trace over matter fields,

$$\mathrm{Tr}_M\, \hat{\rho}_M^2 = \int \mathcal{D}\phi\, \langle\phi|\hat{\rho}_M^2|\phi\rangle,$$

we get

$$\mathrm{Tr}_M\, \hat{\rho}_M^2 = \int \mathcal{D}g \int \mathcal{D}g' \int \mathcal{D}\phi \int \mathcal{D}\phi'\, \Psi^*[g,\phi']\, \Psi[g,\phi]\, \Psi^*[g',\phi]\, \Psi[g',\phi']. \tag{27}$$

Now we want to evaluate (27) for one specific state, namely the Hartle–Hawking state, denoted $|\Psi_{\mathrm{HH}}\rangle$. This state is known to satisfy the scalar constraint equation (12), see [14], and thus belongs to the physical Hilbert space $\mathcal{H}_{\mathrm{phys}}$. Our aim is to demonstrate that the Hartle–Hawking state is nonseparable, and the strategy is to argue that $\mathrm{Tr}_M\, \hat{\rho}_M^2 < 1$ for $\hat{\rho} = |\Psi_{\mathrm{HH}}\rangle \otimes \langle\Psi_{\mathrm{HH}}|$. The Hartle–Hawking state is defined by specifying the wavefunctional $\Psi[g,\phi]$ in (23) as

$$\Psi_{\mathrm{HH}}[g,\phi] = \mathcal{N} \int \mathcal{D}G \int \mathcal{D}\Phi\, \mathrm{e}^{\mathrm{i}S_{\mathrm{tot}}[g,\phi,G,\Phi]}. \tag{28}$$

Here $\mathcal{N}$ is a normalisation constant, the variables G and Φ (denoted with the capital letters) live in the bulk spacetime $\mathcal{M}_4$, while g and ϕ (denoted with lowercase letters) live on the boundary $\Sigma_3 = \partial\mathcal{M}_4$, as before. The path integrals are taken over the bulk while keeping the boundary fields constant. Finally, the total action functional S_{tot} has the following structure

$$S_{\text{tot}}[g, \phi, G, \Phi] = S_G[g, G] + S_M[g, \phi, G, \Phi], \tag{29}$$

where S_G is the action for the gravitational field (for example the Einstein–Hilbert action with a cosmological constant), while S_M is the action for the matter fields coupled to gravity—hence its dependence on both the gravitational and matter fields. See [14] for details on the construction of the expression (28).

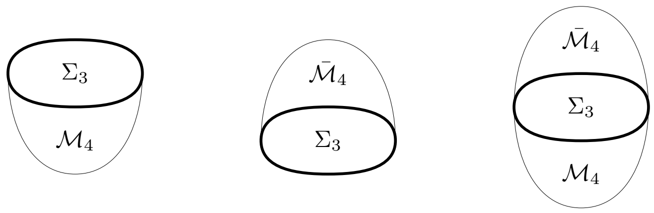
In order to analyse the expression (27) more efficiently, it is convenient to introduce the following quantity,

$$Z[\phi, \phi'] \equiv \int \mathcal{D}g \, \Psi_{\text{HH}}[g, \phi] \Psi^*_{\text{HH}}[g, \phi'], \tag{30}$$

which represents the matrix element of the reduced density matrix $\hat{\rho}_M$. Namely, by evaluating (26) for the Hartle–Hawking state, one obtains

$$\hat{\rho}_M = \int \mathcal{D}\phi \int \mathcal{D}\phi' \, Z[\phi, \phi'] \, |\phi\rangle \otimes \langle \phi'|. \tag{31}$$

In addition, $Z[\phi, \phi']$ has an important geometric structure. Namely, one can consider two copies of the spacetime manifold $\mathcal{M}_4$, where the boundary Σ_3 of the first copy features the fields g, ϕ, while the boundary of the second copy features the fields g, ϕ', i.e. such that the gravitational field g is the same, while matter fields ϕ and ϕ' are different on the boundaries. Then one takes the second copy of $\mathcal{M}_4$, inverts it with respect to the boundary Σ_3 (the result is denoted as $\bar{\mathcal{M}}_4$), and glues it to the first copy along the common boundary, to obtain a manifold $\mathcal{M}_4 \cup \bar{\mathcal{M}}_4$, which has no boundary. This can be illustrated by the following diagrams:



The quantity $Z[\phi, \phi']$ is then obtained by integrating over all gravitational degrees of freedom, and all bulk matter degrees of freedom, weighted by the kernel $e^{iS_{\text{tot}}}$ of the Hartle–Hawking wavefunction (28). This construction is important because the trace of $Z[\phi, \phi']$ is the state sum of the gravitational and matter fields over the manifold $\mathcal{M}_4 \cup \bar{\mathcal{M}}_4$:

$$\int \mathcal{D}\phi \, Z[\phi, \phi] = Z \equiv \int \mathcal{D}G \int \mathcal{D}\Phi \, e^{iS[G,\Phi]}. \tag{32}$$

Here, $S[G, \Phi]$ is the total gravity-matter action similar to (29), defined over the manifold $\mathcal{M}_4 \cup \bar{\mathcal{M}}_4$, and thus features no boundary fields. From (30)–(32) it is then easy to see that the normalization of the state sum, $Z = 1$, and simultaneously the normalization of the reduced density matrix, $\text{Tr}\, \hat{\rho}_M = 1$, i.e.

$$\int \mathcal{D}\phi \, Z[\phi, \phi] = 1, \tag{33}$$

are equivalent to the normalisation of the Hartle–Hawking state, $\langle \Psi_{\text{HH}} | \Psi_{\text{HH}} \rangle = 1$. Finally, from the definition (30) it is easy to see that $Z[\phi, \phi']$ is self-adjoint,

$$Z[\phi, \phi'] = Z^*[\phi', \phi],$$

as the matrix elements of the density matrix $\hat{\rho}_M$ are supposed to be.

Returning to the evaluation of (27) for the Hartle–Hawking state, one can use (30) to rewrite it into the compact form

$$\mathrm{Tr}_M\,\hat{\rho}_M^2 = \int \mathcal{D}\phi \int \mathcal{D}\phi' \, \left| Z[\phi,\phi'] \right|^2 . \tag{34}$$

At this point the general analysis cannot proceed any further, since the right-hand side cannot be evaluated explicitly without specifying the details of the theory. The calculation will therefore proceed further in the next section, where we consider one detailed model of quantum gravity with matter.

Despite the inability to evaluate the integral (34) in the general case, one can give a qualitative argument that the result is not equal to one, leading to the nonseparability of the Hartle–Hawking state. Namely, given the definition (28) of the Hartle–Hawking state, it is easy to see that it essentially depends on two quantities—the normalisation constant $\mathcal{N}$, and the choice of the action S_{tot}. The normalisation constant is fixed by the requirement that (33) holds. This leaves the value of the integral (34) depending solely on the choice of the classical action of the theory. It is qualitatively straightforward to see that different choices of the action will lead to different values of $\mathrm{Tr}_M\,\hat{\rho}_M^2$, so any generic choice of S_{tot} is likely to give $\mathrm{Tr}_M\,\hat{\rho}_M^2 < 1$. A tentative choice for (29) would be the Einstein–Hilbert action for S_G and the Standard Model of elementary particle physics for S_M, based on the gauge group $SU(3) \times SU(2) \times U(1)$. However, we know that the Standard Model action is incomplete, for example due to the fact that dark matter is not included in the description. Therefore, the choice of the classical action is a sort of a moving target, and it is unlikely that any candidate action we choose will give $\mathrm{Tr}_M\,\hat{\rho}_M^2 = 1$. In this sense, one can only conclude that in a generic case the Hartle–Hawking state is nonseparable, supporting the abstract argument from section 2.

Finally, let us note that our assumption of local Poincaré gauge symmetry implies that we are discussing the Lorentzian path integral formulation of the theory. In contrast, within the Euclidean approach, the Hartle–Hawking state has some problematic characteristics, see for example [21] and references therein.

4. Regge quantum gravity example

In this section we will present a short review of the Regge quantum gravity model coupled to scalar matter, and then use this model to evaluate (34) for the Hartle–Hawking state. The Regge quantum gravity model is intimately connected to the covariant loop quantum gravity research framework [22, 23], its generalisations [24–26], and various related research areas [27, 28] (see also [29] for an interesting connection to the noncommutative geometry approach in the 3D case). Nevertheless, it can be introduced and studied as a simple standalone model of quantum gravity independent of any other context, as was done in [30], where some preliminary results regarding the entanglement in the Hartle–Hawking state have been announced.

4.1. Formalism of Regge quantum gravity

The Regge quantum gravity model is arguably the simplest toy-model of quantum gravity constructed by providing a rigorous definition for the gravitational path integral, generically denoted as

$$Z_G = \mathcal{N} \int \mathcal{D}g\, \mathrm{e}^{\mathrm{i}S_{\mathrm{EH}}[g]}, \tag{35}$$

where $S_{\mathrm{EH}}[g]$ is the Einstein–Hilbert action for general relativity. The construction of the path integral follows Feynman's original idea of path integral definition-by-discretisation. We begin by passing from a smooth $4D$ spacetime manifold $\mathcal{M}_4$ to a piecewise-linear $4D$ manifold, most commonly a triangulation $T(\mathcal{M}_4)$. This structure naturally features 4-simplices σ as basic building blocks, which themselves consist of tetrahedra τ, triangles Δ, edges ϵ and vertices v. The invariant quantities associated to these objects are the 4-volume of the 4-simplex ${}^{(4)}V_\sigma$, the 3-volume of the tetrahedron ${}^{(3)}V_\tau$, the area of the triangle A_Δ and the length of the edge l_ϵ, respectively, while the vertices do not have nontrivial quantities assigned to them.

It is important to emphasise that the edge lengths are most fundamental of all these quantities, since one can always uniquely express ${}^{(4)}V_\sigma$, ${}^{(3)}V_\tau$ and A_Δ as functions of l_ϵ. For example, the most well-known is the Heron formula for the area of a triangle in terms of its three edge lengths,

$$A_\Delta(l) = \sqrt{s(s-l_1)(s-l_2)(s-l_3)}, \qquad s \equiv \frac{l_1+l_2+l_3}{2},$$

where the three edges $\epsilon = 1, 2, 3$ belong to the triangle Δ.

Given a spacetime triangulation, the Einstein–Hilbert action of general relativity,

$$S_{\mathrm{EH}}[g] = -\frac{1}{16\pi l_p^2}\int_{\mathcal{M}_4} \mathrm{d}^4x\sqrt{-g}\,R(g),$$

can be reformulated in terms of edge lengths of the triangulation as the Regge action

$$S_R[l] = -\frac{1}{8\pi l_p^2}\sum_{\Delta\in T(\mathcal{M}_4)} A_\Delta(l)\delta_\Delta(l),$$

where δ_Δ is the so-called deficit angle at triangle Δ, measuring the amount of spacetime curvature around Δ. See [31] and [27] for details and a review.

Once the classical action for general relativity has been adapted to a piecewise-linear manifold structure, we can take the edge lengths of the edges in the triangulation as the fundamental degrees of freedom of the theory, and define the gravitational path integral (35) as:

$$Z_G = \mathcal{N}\int_D \prod_{\epsilon\in T(\mathcal{M}_4)} \mathrm{d}l_\epsilon\,\mu(l)\mathrm{e}^{\mathrm{i}S_R[l]}. \tag{36}$$

Here $\mathcal{N}$ is a normalisation constant, while $\mu(l)$ is the measure term which ensures the convergence of the state sum Z_G. For the purpose of this paper, we choose the exponential measure

$$\mu(l) = \exp\left(-\frac{1}{L_\mu^4}\sum_{\sigma\in T(\mathcal{M}_4)} {}^{(4)}V_\sigma(l)\right), \tag{37}$$

where $L_\mu > 0$ is a constant and a free parameter of the model (see [32–34] for motivation and analysis). Note that the sum of the 4-volumes of all 4-simplices gives the total 4-volume of the triangulation $T(\mathcal{M}_4)$, and will sometimes be denoted simply as V_4.

The choice of edge lengths as the fundamental gravitational degrees of freedom in (36) determines the integration domain D as a subset of the Cartesian product $(\mathbb{R}_0^+)^E$, where E is the total number of edges in $T(\mathcal{M}_4)$, while $\mathbb{R}_0^+$ is the maximum integration domain of each individual edge length. We should note that D is a strict subset of $(\mathbb{R}_0^+)^E$ due to the presence of triangle inequalities which must be satisfied for all triangles, tetrahedra and 4-simplices in a given triangulation.

Once we have defined the gravitational path integral (35) via the state sum (36), it is straightforward to generalise this definition to the situation which includes matter fields. For simplicity, we will discuss only a single real scalar field φ, although it is not a problem to include other fields as well. The path integral we are interested in can be denoted as

$$Z_{G+M} = \mathcal{N} \int \mathcal{D}g \int \mathcal{D}\varphi \, e^{iS_{\rm tot}[g,\varphi]}, \tag{38}$$

where $S_{\rm tot}[g,\varphi]$ is the sum of the Einstein–Hilbert action and the action for the scalar field in curved spacetime,

$$\begin{aligned} S_{\rm tot}[g,\varphi] \quad = \quad & -\frac{1}{16\pi l_p^2} \int_{\mathcal{M}_4} d^4x \sqrt{-g}\, R(g) \\ & + \frac{1}{2} \int_{\mathcal{M}_4} d^4x \sqrt{-g} \left[g^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) + m^2 \varphi^2 + U(\varphi) \right], \end{aligned}$$

where $U(\varphi)$ is a self-interaction potential of the scalar field. The corresponding lattice version of this action is given as

$$\begin{aligned} S_{\rm tot}[l,\varphi] \quad = \quad & -\frac{1}{8\pi l_p^2} \sum_{\Delta \in T(\mathcal{M}_4)} A_\Delta(l) \delta_\Delta(l) + \\ & + \frac{1}{2} \sum_{\sigma \in T(\mathcal{M}_4)} {}^{(4)}V_\sigma(l) g^{\mu\nu}_{(\sigma)}(l) \partial\varphi_\mu \partial\varphi_\nu \\ & + \frac{1}{2} \sum_{v \in T(\mathcal{M}_4)} {}^{(4)}V^*_v(l) \left[m^2 \varphi_v^2 + U(\varphi_v) \right]. \end{aligned} \tag{39}$$

Here, a value of the scalar field $\varphi_v \in \mathbb{R}$ is assigned to each vertex $v \in T(\mathcal{M}_4)$. Given any 4-simplex $\sigma \in T(\mathcal{M}_4)$, one can label its five vertices as $0, 1, 2, 3, 4$, and then define a skew-coordinate system taking the vertex 4 as the origin and edges $4-0$, $4-1$, $4-2$, $4-3$, respectively as coordinate lines for coordinates x^μ, $\mu \in \{0, 1, 2, 3\}$. In these coordinates, the derivative $\partial_\mu \varphi$ is replaced by the finite difference between the values of the field at the vertex $v = \mu$ and at the coordinate origin of the 4-simplex σ (divided by the distance between them),

$$\partial\varphi_\mu \equiv \frac{\varphi_\mu - \varphi_4}{l_{\mu 4}}.$$

In addition, the metric tensor between vertices μ and ν is given in terms of edge lengths as

$$g^{(\sigma)}_{\mu\nu}(l) \equiv \frac{l_{\mu 4}^2 + l_{\nu 4}^2 - l_{\mu\nu}^2}{2 l_{\mu 4} l_{\nu 4}},$$

while $g^{\mu\nu}_{(\sigma)}(l)$ is its inverse matrix. Finally, ${}^{(4)}V^*_v(l)$ is the 4-volume of the 4-cell surrounding the vertex v in the Poincaré dual lattice of the triangulation $T(\mathcal{M}_4)$.

After we have defined the classical action on $T(\mathcal{M}_4)$, we finally proceed to define the path integral (38) as the state sum:

$$Z_{G+M} = \mathcal{N} \int \prod_{\epsilon \in T(\mathcal{M}_4)} dl_\epsilon \, \mu(l) \int \prod_{v \in T(\mathcal{M}_4)} d\varphi_v \, e^{iS_{\rm tot}[l,\varphi]}. \tag{40}$$

Here, the domain of integration for the scalar field is the Cartesian product $\mathbb{R}^V$, where V is the total number of vertices in the triangulation.

The state sum (40) defines one concrete QG model, called the Regge quantum gravity model. While it goes without saying that this is just a toy model, it is nevertheless a realistic one, since it is finite and has a correct semiclassical continuum limit (see [32] for proofs). Therefore it can be used to study various aspects of quantum gravity, including the entanglement between gravity and matter fields, as we discuss next.

4.2. Calculation of the trace formula

Having formulated the Regge quantum gravity model and having the state sum (40) in hand, we can proceed to study the entanglement between gravity and matter, in particular by evaluating the expression for the trace of $\hat{\rho}_M^2$ given by equation (34). In order to evaluate it, we first need to formulate the Hartle–Hawking state (28) in the framework of Regge quantum gravity model, then work out the matrix elements of the reduced density matrix (30), and finally plug them into (34) to obtain a number. If this number is different from 1, we can conclude that the Hartle–Hawking state features entanglement between the gravitational and matter fields.

We begin by formulating the Hartle–Hawking state (28). Consider a 4-manifold $\mathcal{M}_4$ with a nontrivial boundary $\Sigma_3 = \partial\mathcal{M}_4$, such that the triangulation $T(\mathcal{M}_4)$ induces a triangulation $T(\Sigma_3)$ on the boundary. In this sense we can distinguish the vertices, edges, areas, and tetrahedra which belong to the boundary triangulation $T(\Sigma_3)$ (from now on shortly called 'boundary', and denoted as ∂T), from the vertices, edges, areas, tetrahedra and 4-simplices belonging to $T(\mathcal{M}_4)$ but not to $T(\Sigma_3)$ (from now on shortly called 'bulk', and denoted as T). Since the Regge quantum gravity model encodes gravitational degrees of freedom as lengths of the edges, and matter degrees of freedom as real numbers attached to vertices, we can easily split them into boundary variables l_ϵ, φ_v and bulk variables L_ϵ, Φ_v, where we maintain our previous convention to denote the bulk variables with capital letters and boundary variables with lowercase letters.

Given the bulk and the boundary, we use the formulation of the Regge quantum gravity state sum (40) to write down the Hartle–Hawking wavefunction as

$$\Psi_{\text{HH}}[l,\varphi] = \mathcal{N} \int \prod_{\epsilon\in T} \mathrm{d}L_\epsilon\, \mu(l,L) \int \prod_{v\in T} \mathrm{d}\Phi_v\, \mathrm{e}^{\mathrm{i}S_{\text{tot}}[l,\varphi,L,\Phi]}. \tag{41}$$

Next we want to construct the matrix elements of the reduced density matrix (30). To this end, we need two copies of the Hartle–Hawking state: one with matter fields φ_v on the boundary ∂T of the bulk T, and the other with matter fields φ'_v on the boundary ∂T of the bulk $\bar{T}$ defined as the mirror-reflection of T with respect to the boundary ∂T. This mirror-reflection gives rise to an additional overall minus sign in the action (39) which is then cancelled by the complex conjugation of the imaginary unit in the exponent of the second Hartle–Hawking wavefunction in (30). Integrating over the boundary edge lengths, we end up with:

$$Z[\varphi,\varphi'] = |\mathcal{N}|^2 \int \prod_{\epsilon\in T\cup\bar{T}\cup\partial T} \mathrm{d}L_\epsilon\, \mu(L) \int \prod_{v\in T\cup\bar{T}} \mathrm{d}\Phi_v\, \mathrm{e}^{\mathrm{i}S_{\text{tot}}[\varphi,\varphi',L,\Phi]}. \tag{42}$$

Note that all edge lengths are being integrated over in the 'total' triangulation $T\cup\bar{T}\cup\partial T$ (and we have thus denoted them all with a capital letter L for simplicity). In contrast, the scalar field is being integrated only over the two bulks $T\cup\bar{T}$, while the boundary scalar field values φ, φ' remain fixed on two identical copies of the boundary ∂T. Also, note that

$$S_{\text{tot}}[\varphi,\varphi',L,\Phi] \equiv S_{\text{tot}}[\varphi,L,\Phi]\Big|_{T\cup\partial T} + S_{\text{tot}}[\varphi',L,\Phi]\Big|_{\bar{T}\cup\partial T},$$

where the boundary edge lengths l have been relabelled as L and reabsorbed into the set of bulk edge lengths.

The next step one should perform is to take the trace of (42) and equate it to 1 as in (33), in order to make sure that the Hartle–Hawking wavefunction (41) is properly normalised. This leads to the equation

$$|\mathcal{N}|^2 \int \prod_{\epsilon \in T\cup\bar{T}\cup\partial T} \mathrm{d}L_\epsilon \, \mu(L) \int \prod_{v\in T\cup\bar{T}\cup\partial T} \mathrm{d}\Phi_v \, \mathrm{e}^{\mathrm{i}S_{\mathrm{tot}}[L,\Phi]} = 1,$$

which determines the normalisation constant $\mathcal{N}$ up to an overall phase factor. Note that the boundary scalar fields φ have been integrated over and consequently reabsorbed into the bulk variables Φ, similarly to edge lengths L. Both the integration over L and the integration over Φ is now being performed over the 'total' triangulation $T\cup\bar{T}\cup\partial T$ which has no boundary.

As the final step of the construction of the trace formula (34), we substitute (42) and $\mathcal{N}$ into it, to obtain:

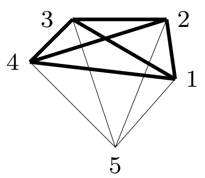
$$\mathrm{Tr}_M \, \hat{\rho}_M^2 = \frac{\displaystyle\int \prod_{v\in\partial T} \mathrm{d}\varphi_v \int \prod_{v\in\partial T} \mathrm{d}\varphi'_v \left| \int \prod_{\epsilon\in T\cup\bar{T}\cup\partial T} \mathrm{d}L_\epsilon \, \mu(L) \int \prod_{v\in T\cup\bar{T}} \mathrm{d}\Phi_v \, \mathrm{e}^{\mathrm{i}S_{\mathrm{tot}}[\varphi,\varphi',L,\Phi]} \right|^2}{\displaystyle\left(\int \prod_{\epsilon\in T\cup\bar{T}\cup\partial T} \mathrm{d}L_\epsilon \, \mu(L) \int \prod_{v\in T\cup\bar{T}\cup\partial T} \mathrm{d}\Phi_v \, \mathrm{e}^{\mathrm{i}S_{\mathrm{tot}}[L,\Phi]} \right)^2}. \tag{43}$$

This is the final expression we set out to derive. It represents a concrete realisation of the trace formula (34), it is completely well defined, and can in principle be evaluated. In practice, though, for a generic choice of the triangulation, this expression is very hard to evaluate even numerically. Therefore, in what follows we shall enforce some very hard approximations in order to make it more manageable for study. Nevertheless, by looking at the structure of the numerator and the denominator, one can already see that the two expressions can be equal to each other only in some very special cases, if at all. However, the dependence of the action S_{tot} on the boundary and bulk variables is such that one cannot rely on any special mathematical properties of the action which could help make the final result be 1, for a generic choice of the spacetime triangulation. In this sense, we can conjecture already at this level that in generic cases we have

$$\mathrm{Tr}_M \, \hat{\rho}_M^2 < 1,$$

as we wanted to demonstrate.

But in order to give a more convincing argument, let us study a special case and try to evaluate this trace to the very end. The simplest possible example of a triangulation T is a single 4-simplex. Labelling its vertices as $1, 2, 3, 4, 5$, we can depict it with a following diagram:



The 4-simplex has five boundary tetrahedra, namely

$$\tau_{1234}, \qquad \tau_{1235}, \qquad \tau_{1245}, \qquad \tau_{1345}, \qquad \tau_{2345}.$$

The first tetrahedron, τ_{1234}, is depicted with thick edges, and we will choose it to be the boundary ∂T. Since we do not want the four remaining tetrahedra to belong to the boundary, we will glue them onto each other in pairs, as

$$\tau_{1235} \equiv \tau_{1245}, \qquad \tau_{1345} \equiv \tau_{2345}.$$

This means that every point in τ_{1235} is identified with the corresponding point in τ_{1245}, and similarly with the other pair of tetrahedra. In this way we obtain a manifold with a nontrivial topology, but described with only five vertices and one boundary tetrahedron. In order for this gluing to be consistent, the gravitational and matter degrees of freedom living on $T \cup \partial T$ must satisfy the following constraints:

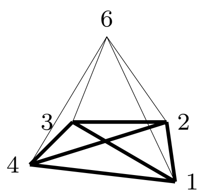
$$\begin{gathered} l_{14} = l_{23} = l_{24} = l_{13}, \qquad L_{25} = L_{15}, \qquad L_{45} = L_{35}, \\ \varphi_2 = \varphi_1, \qquad \varphi_4 = \varphi_3. \end{gathered} \tag{44}$$

This leaves us with the following independent degrees of freedom living on the 4-simplex:

$$l_{12}, \qquad l_{13}, \qquad L_{15}, \qquad l_{34}, \qquad L_{35}, \qquad \varphi_1, \qquad \varphi_3, \qquad \Phi_5,$$

where we have denoted the bulk degrees of freedom with capital letters and boundary degrees of freedom with lowercase letters. The 4-simplex diagram above is the graphical representation of the Hartle–Hawking wavefunction $\Psi_{\mathrm{HH}}[l, \varphi]$ (41).

Next we construct $\bar{T}$. Since the boundary tetrahedron ∂T defines a single 3-dimensional hypersurface, there is precisely one axis in 4-dimensional space which is orthogonal to ∂T. Performing the reflection of T with respect to ∂T is therefore identical to reversing the orientation of this orthogonal axis. In this way we construct another 4-simplex, with vertices labeled $1, 2, 3, 4, 6$ and depicted as



One can see that the main difference between the 4-simplex σ_{12346} and the previously constructed 4-simplex σ_{12345} is that the vertex 6 is on the 'opposite side' of the tetrahedron τ_{1234} as compared to the vertex 5 of σ_{12345}.

Like we did for σ_{12345}, we again want to glue the boundary tetrahedra pairwise, so that only the tetrahedron τ_{1234} remains as the boundary $\partial\bar{T}$. The pairwise gluing of tetrahedra

$$\tau_{1236} \equiv \tau_{1246}, \qquad \tau_{1346} \equiv \tau_{2346}$$

gives rise to the constraints

$$\begin{gathered} l_{14} = l_{23} = l_{24} = l_{13}, \qquad L_{26} = L_{16}, \qquad L_{46} = L_{36}, \\ \varphi'_2 = \varphi'_1, \qquad \varphi'_4 = \varphi'_3, \end{gathered}$$

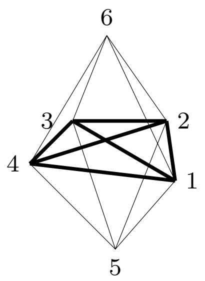
where only the constraints containing the vertex 6 are additional to (44). This leaves us with the following independent degrees of freedom living on σ_{12346}:

$$l_{12}, \qquad l_{13}, \qquad L_{16}, \qquad l_{34}, \qquad L_{36}, \qquad \varphi'_1, \qquad \varphi'_3, \qquad \Phi_6.$$

As noted in the general discussion leading to equation (43), the matter degrees of freedom on the boundary of T are different than the corresponding degrees of freedom living on the boundary of $\bar{T}$, despite the fact that the boundary is identical, $\partial\bar{T} \equiv \partial T$. To that end, we have added a prime to φ in the above equations. Like for the 4-simplex σ_{12345}, the diagram of the

4-simplex σ_{12346} above is the graphical representation of the (complex-conjugate) Hartle–Hawking wavefunction $\Psi^*_{\text{HH}}[l, \varphi']$.

At this point we are ready to glue T and $\bar{T}$ along the common boundary ∂T, to obtain the manifold $T \cup \bar{T} \cup \partial T$ which has no boundary. It is depicted on the diagram below.



It consists of two 4-simplices σ_{12345} and σ_{12346} constructed above and glued along the common tetrahedron τ_{1234}. The full set of independent gravitational degrees of freedom is

$$l_{12}, \qquad l_{13}, \qquad l_{34}, \qquad L_{15}, \qquad L_{16}, \qquad L_{35}, \qquad L_{36},$$

while the independent matter degrees of freedom are

$$\varphi_1, \qquad \varphi_3, \qquad \varphi'_1, \qquad \varphi'_3, \qquad \Phi_5, \qquad \Phi_6.$$

This diagram is the graphical representation for the matrix element $Z[\varphi, \varphi']$ of the reduced density matrix $\hat{\rho}_M$ (see equations (42) and (30)).

Applying the general trace formula (43) to our case then gives

$$\text{Tr}_M \, \hat{\rho}_M^2 = \frac{\displaystyle\int \text{d}\varphi_1 \text{d}\varphi_3 \text{d}\varphi'_1 \text{d}\varphi'_3 \left| \int \text{d}^7 L \, \mu(L) \int \text{d}\Phi_5 \text{d}\Phi_6 \, \text{e}^{\text{i}S_{\text{tot}}[\varphi, \varphi', L, \Phi]} \right|^2}{\displaystyle\left(\int \text{d}^7 L \, \mu(L) \int \text{d}^4 \Phi \, \text{e}^{\text{i}S_{\text{tot}}[L, \Phi]} \right)^2}, \tag{45}$$

where

$$\text{d}^7 L \equiv \text{d}l_{12} \text{d}l_{13} \text{d}l_{34} \text{d}L_{15} \text{d}L_{16} \text{d}L_{35} \text{d}L_{36},$$

and

$$\text{d}^4 \Phi \equiv \text{d}\varphi_1 \text{d}\varphi_3 \text{d}\Phi_5 \text{d}\Phi_6.$$

Note that the action in the denominator is evaluated using $\varphi'_1 = \varphi_1$ and $\varphi'_3 = \varphi_3$, as explained in the general discussion above. In order to make the equation (45) fully explicit, we need to choose the values of the free parameters in the classical action (39) and the measure (37). The parameters of the action are the Planck length l_p, the mass m of the scalar field, and the self-interaction potential $U(\varphi)$. For the purpose of this example, the simplest possible choice is the free massless scalar field, so that we have

$$l_p = 10^{-35} \text{ m}, \qquad m = 0, \qquad U(\varphi) = 0.$$

Second, the measure (37) contains a single free parameter L_μ. This parameter can be connected to the value of the effective cosmological constant Λ, via the relation

$$\Lambda = \frac{l_p^2}{2L_\mu^4},$$

see [32–34] for details. Taking the observed value $\Lambda = 10^{-52}$ m^{-2} (also often quoted as a dimensionless product $\Lambda l_p^2 = 10^{-122}$), we obtain

$$L_\mu = 10^{-5} \text{ m}.$$

Using these numeric values of the parameters, the right-hand side of (45) is fully specified, and can be evaluated using a computer. However, in order to render the calculation more manageable, for the purpose of this paper we instead choose to evaluate (45) with $L_\mu = 10^{-33}$ m, which corresponds to a larger cosmological constant, $\Lambda l_p^2 = 10^{-8}$, to speed up the convergence of the Monte-Carlo integration method. The result is strictly less than one,

$$\mathrm{Tr}_M \, \hat{\rho}_M^2 = 0.977 \pm 0.002,$$

as we had set out to demonstrate. Note that, although close to one, the above result is: (i) strictly smaller than one (within the computational error); (ii) obtained within extremely simplified toy model whose system consists of only two 4-simplices of spacetime. Thus, our result can serve as a proof of principle that gravity-matter entanglement is always present. The total amount of such entanglement in realistic models, as well as its spatial distribution, remains to be further explored. Namely, note that even though the approximation of product gravity-matter states has been up to now successfully applied, the *overall* entanglement between the two systems, considered within complex realistic situations/models, does not at all have to be small, nor its effects negligible. Indeed, the standard entanglement that is considered to cause the decoherence of matter by the environment and the quantum-to-classical transition has profoundly striking effects, despite the fact of being difficult to characterise, evaluate and manipulate.

5. Conclusions

5.1. Summary of the results

We analyse the quantum gravity coupled to the most common matter fields (namely, scalar, spinor and vector fields), and show that the gravity and matter are generically entangled, as a consequence of the nonseparability of the scalar constraint $\mathcal{C}$, and in some cases the 3-diffeo constraint $\mathcal{C}_i^M$. Thus, simple separable gravity-matter product states are excluded from the physical Hilbert space, unless the constraint equations feature some deep unknown property which allows for the invariance of a whole class of product states. We demonstrate this in two different ways: (i) within the general abstract nonperturbative canonical formalism, by directly analysing the mathematical structure of the constraints, and (ii) within the path integral formalism, by directly checking for entanglement of the Hartle–Hawking state in the Regge model of quantum gravity.

5.2. Discussion of the results

This *gauge-protected* decoherence due to the entanglement (in contrast to the standard *'for all practical purposes'* dynamical one) offers a possibly deeper fundamental explanation of the long-standing problem of the quantum-to-classical transition: the matter does not *decohere*, it is by default *decohered*.

Any potential entanglement, either dynamical or gauge-protected one, depends on the details of the coupling between matter and gravity. For the purpose of this paper, the coupling is prescribed by the strong equivalence principle, which states that the equations of motion for all matter fields must locally be identical to the equations of motion for those fields in flat spacetime. This is implemented by choosing the action for matter fields with minimal coupling prescription, and employed in both the canonical and the path integral frameworks. We should stress that the validity of the strong equivalence principle is a sufficient, but potentially not a necessary assumption for our main result. Namely, it is plausible that nonminimal coupling choices, involving explicit spacetime curvature terms in the matter Lagrangian, could also lead to the conclusion that entanglement between gravity and matter is unavoidable. However, it is also possible that one could come up with some particular complicated choice of nonminimal coupling which does admit some nonentagled states. In order to avoid complicating the analysis with such cases, given that nonminimal coupling between gravity and matter has absolutely no experimental evidence in its favor so far, we have chosen to assume the validity of the strong equivalence principle throughout the paper.

In standard QM entanglement is a generic consequence of the interaction. Nevertheless, there exist alternative mechanisms for creating it, such as the indistinguishability of identical particles, leading to effective 'exchange interactions'. This new gauge-protected gravity-matter entanglement can thus introduce additional 'effective interaction', which can possibly result in corrections to Einstein's weak equivalence principle (see for example [35]).

It is interesting to note that a possible peculiar impact of the quantised gravity to the whole decoherence programme was already inferred in Zurek's seminal paper [36], where on page 1520 the author writes: (the assumption of pairwise interactions) 'is customary and clear, even though it may prevent one from even an approximate treatment of the gravitational interaction beyond its Newtonian pairwise form'. Our result confirms Zurek's disclaimer—gravity (environment $\mathcal{E}$) is generically entangled with the *whole* matter (both the system $\mathcal{S}$ and the apparatus $\mathcal{A}$), that way allowing for non-trivial tripartite system-apparatus-environment *effective* interaction of the form $\mathcal{H}_{\mathcal{SAE}}$, explicitly excluded in [36]. In other words, the environment (spacetime) interaction with the matter could potentially disturb the system-apparatus correlations, thus violating the stability criterion of a faithful measurement (see [37], p 1271).

As a consequence of generic gravity-matter entanglement, the effective interaction between gravity and matter forbids the existence of a single background spacetime. Thus, when concerning quantum effects of gravity, one cannot talk of 'matter in a point of space', confirming the conjecture that spacetime is an 'emergent phenomenon'. In contrast to this, Penrose argues that spacetime, seen as a (four-dimensional) differentiable manifold, does not support superpositions of massive bodies and the corresponding (relative) states of gravity (i.e. the gravity-matter entanglement), leading to the objective collapse onto the product states of matter and (classical) spacetime [38]. Our result can therefore be treated as a possible criterion for a plausible candidate theory of quantum gravity.

Finally, not allowing product states between the matter and gravity is in tune with the relational approach to physics [22], in particular to quantum gravity (note that the original name for the many-world interpretation of QM was the *'Relative State' Formulation of Quantum Mechanics* [39]). See also [40] for an interesting treatment of relative state and decoherence approaches.

5.3. Relation to common quantum gravity research programs

In order to discuss our results in the context of various quantum gravity research programs, note that the gauge-protected entanglement between gravity and matter should exist in any model of quantum gravity with matter which respects local Poincaré symmetry. In this sense, various approaches to quantum gravity can be classified into four distinct categories.

(i) The first category represents models which explicitly respect (or at least aim to respect) local Poincaré symmetry. These include nonperturbative string theory/M-theory [41–43], loop quantum gravity [22, 23], Wheeler–DeWitt quantization [44, 45], and similar approaches.
(ii) The second category represents models in which local Poincaré symmetry is explicitly broken. These include perturbative quantum gravity [46], petrurbative string theory [43], causal dynamical triangulations approach [28], doubly-special relativity models [47], Hořava–Lifshitz gravity [48], various nonrelativistic quantization proposals, and so on.
(iii) The third category represents models in which it is not clear whether local Poincaré symmetry is broken or not. For example, in the asymptotic safety approach [49] this may depend on the properties of the fixed point. In noncommutative geometry [50, 51] it depends on the particular choice of the algebra. In higher-derivative theories and theories with propagating torsion [52] it may depend on various details of the model, etc.
(iv) Finally, the fourth category represents models which have not been developed enough to allow for coupling of matter fields. In models like entropic gravity [53, 54] and causal set theory [55, 56], it is not obvious how to couple matter fields to gravity, and whether this coupling would violate local Poincaré invariance or not.

It should be clear that our results apply to the first category of quantum gravity models, while for other three categories it either does not apply, or it is an open question. We should also state that the validity of local Poincaré symmetry is ultimately an experimental question, one over which various quantum gravity proposals may disagree.

In relation to the previous comment, it is worthwhile to also discuss the impact of possible anomalies to the gauge protected entanglement. As we have discussed in the final paragraph of section 2, the entanglement is a consequence of the scalar constraint $\hat{\mathcal{C}}$, see (12), and for certain types of matter fields also of the 3-diffeo constraint $\hat{\mathcal{C}}_i$ in (11), while the local Lorentz constraint $\hat{\mathcal{C}}_{ab}$ in (11) does not require entanglement. From this one can see that if the theory features anomalies due to the breaking of the 4D diffeomorphism symmetry, one cannot impose $\hat{\mathcal{C}}$ and $\hat{\mathcal{C}}_i$ as the Gupta–Bleuler-like conditions on the Hilbert space of the theory, and thus all subsequent results regarding the entanglement are void. In short, there cannot be any gauge protected entanglement if there is no relevant gauge symmetry to begin with. Nevertheless, if the theory features anomalies due to the breaking of the local Lorentz or any internal symmetries, while maintaining diffeomoprhism symmetry at the quantum level, the gauge protected entanglement will not be influenced by the anomaly.

5.4. Future lines of research

One of the main lines of future work would be to perform a detailed numerical analysis of $\mathrm{Tr}\,\hat{\rho}_M^2$ and the von Neumann entropy $S(\hat{\rho}_M)$ for the Hartle–Hawking state (either within the Regge, or some other QG model). The latter quantity, called the *entropy of entanglement*, represents the measure of the entanglement in *pure* and *bipartite* states [57], in our case between gravity and matter in the Hartle–Hawking state. The precise numerical deviation of the $\mathrm{Tr}\,\hat{\rho}_M^2$ from its maximal value 1 could indicate in which cases this new entanglement has relevant physical consequences. This way, it would be possible to determine the boundaries of validity of the assumption of the product gravity-matter states of the form $|G\rangle|M\rangle$, which has been up to now used in numerous studies (analogously to the case of determining the regimes in which two coherent states become effectively orthogonal). In connection to this, one could analyse in more detail quantitatively to what extent the gauge-protected gravity-matter entanglement constrains the existence of macroscopic superpositions, and its effect to the quantum-to-classical transition (see the related work [3, 4, 9, 58–60]).

Further, studying the structure of the gauge-imposed entanglement for a tripartite system of gravity-matter-EM fields might bring qualitatively new effects. Unlike the case of pure bipartite states, where any two entangled states could be obtained from each other by local operations and classical communication (LOCC), thus forming a single class of entangled states and providing a unique measure of entanglement, the multipartite entanglement has a more complex structure. Indeed, in the tripartite case, in addition to the trivial classes of purely bipartite entanglement, say, $|a\rangle(|b_1c_1\rangle + |b_2c_2\rangle)$, genuine tripartite entanglement consists of a number of inequivalent classes of entangled states: in the simplest case of three qubits we have two classes of tripartite entanglement, represented by the states $|GHZ\rangle = (|0\,0\,0\rangle + |1\,1\,1\rangle)/\sqrt{2}$ and $|W\rangle = (|0\,0\,1\rangle + |0\,1\,0\rangle + |100\rangle)/\sqrt{3}$, which cannot be obtained from each other by the means of LOCC, but as soon as neither of the subsystems is a qubit, there exist infinitely many inequivalent classes [61].

It would also be interesting to see how other QG candidates incorporate the general gravity constraints regarding the entanglement with matter, in particular the string theory. Namely, perturbative string theory is formulated by manifestly breaking the gauge symmetry (a consequence of perturbative expansion of the gravitational field). The existence of the gravity-matter entanglement in, say Hartle–Hawking state, would then present a strong argument that the gauge symmetry can be restored in a tentative nonperturbative formulation of string theory. In connection to this, one could analyse the entanglement between different spacetime regions induced by the gauge-protected gravity-matter entanglement, and compare it to that present in theories based on the AdS/CFT correspondence and the holographic principle [7, 8]. Namely, entanglement is a property of a quantum state *with respect* to a particular factorisation of a composite system into its factor sub-systems. To illustrate this, consider a particle in a two-dimensional plane. Given orthogonal axes x and y of a 2D plane, the Hilbert space of the system is given by $\mathcal{H} = \mathcal{H}_x \otimes \mathcal{H}_y$, and the equal spatial superposition (for simplicity, we omit the overall normalisation constant) $|\varphi\rangle \sim (|a\rangle_x + |b\rangle_x)|0\rangle_y$, with $a, b \in \mathbb{R}$, is clearly separable, with respect to the given factorisation of $\mathcal{H}$. Nevertheless, with respect to *any other* factorisation of $\mathcal{H}$, defined by any other axes X and Y inducing the Hilbert-space factorisation $\mathcal{H} = \mathcal{H}_X \otimes \mathcal{H}_Y$, the system is entangled. As an example, for axes X and Y obtained by rotating x and y by $-\pi/4$, the same state of the system is maximally entangled, $|\varphi\rangle \sim (|a/\sqrt{2}\rangle_X |a/\sqrt{2}\rangle_Y + |b/\sqrt{2}\rangle_X |b/\sqrt{2}\rangle_Y)$ (for the entanglement in the second quantisation formalism, and its dependence on the choice of fundamental modes, see for example [62]). Following the above example, one might expect that the existence of the entanglement between gravity and matter would induce the entanglement between two generic spacetime regions (each containing a portion of both gravitational and matter degrees of freedom). Possible relationship between this, gauge-protected entanglement, and that present as a consequence of assumptions that do not explicitly rely on the existence of local Poincaré symmetry (holography and the AdS/CFT correspondence) would indicate interesting fundamental connections that could help breaching the long-standing gap between quantum mechanics and general relativity.

Finally, detecting gravity-matter entanglement in the experiment might not be that far from the reach of the current or the near-future technology, see [63] for a recent proposal of testing gravitational decoherence. Proposing, and possibly performing, experiments to distinguish different contributions of the gravitational interaction to the decoherence of matter, in particular the generic one based on the gauge symmetry constraints, presents a relevant direction of further research.

Acknowledgments

We would like to thank Rafael Sorkin for helpful suggestions and comments.

MV was supported by the project ON171031 of the Ministry of Education, Science and Technological Development of the Republic of Serbia, and the bilateral scientific cooperation between Portugal and Serbia through the project 'Quantum Gravity and Quantum Integrable Models—2015–2016', no. 451-03-01765/2014-09/24 supported by the Foundation for Science and Technology (FCT), Portugal, and the Ministry of Education, Science and Technological Development of the Republic of Serbia. NP acknowledges the support of SQIG—Security and Quantum Information Group, the IT project QbigD funded by FCT PEst-OE/EEI/LA0008/2013 and UID/EEA/50008/2013 and the bilateral scientific cooperation between Portugal and Serbia through the project 'Noise and measurement errors in multiparty quantum security protocols', no. 451-03-01765/2014-09/04 supported by the Foundation for Science and Technology (FCT), Portugal, and the Ministry of Education, Science and Technological Development of the Republic of Serbia.

The authors would also like to thank the Erwin Schrödinger International Institute for Mathematics and Physics (ESI Vienna), for the warm hospitality during the workshop 'Quantum Physics and Gravity' (29 May–30 June 2017) and partial support.

ORCID iDs

N Paunković https://orcid.org/0000-0002-9345-4321
M Vojinović https://orcid.org/0000-0001-6977-4870

References

[1] Karolyhazy F 1966 *Nuovo Cimento* A **42** 390
[2] Bassi A, Lochan K, Satin S, Singh T P and Ulbricht H 2013 *Rev. Mod. Phys.* **85** 471
[3] Pikovski I, Zych M, Costa F and Brukner Č 2011 *Nat. Commun.* **2** 505
[4] Pikovski I, Zych M, Costa F and Brukner Č 2015 *Nat. Phys.* **11** 668
[5] Oniga T and Wang C H T 2016 *Phys. Rev.* D **93** 044027
[6] Liu J, Sou C M and Wang Y 2016 *J. High Energy Phys.* JHEP10(2016)072
[7] Dong X 2016 *Nat. Commun.* **7** 12472
[8] Raamsdonk M V 2017 *Lectures on Gravity and Entanglement* (Oxford: World Scientific Publishing) pp 297–351
[9] Kay B S 1998 *Class. Quantum Grav.* **15** L89
[10] Kok P and Yurtsever U 2003 *Phys. Rev.* D **68** 085006
[11] Hamma A, Markopoulou F, Lloyd S, Caravelli F, Severini S and Markström K 2010 *Phys. Rev.* D **81** 104032
[12] Hamma A and Markopoulou F 2011 *New J. Phys.* **13** 095006
[13] Bassi A, Grossardt A and Ulbricht H 2017 *Class. Quantum Grav.* **34** 193002
[14] Hartle J B and Hawking S W 1983 *Phys. Rev.* D **28** 2960
[15] Blagojević M 2002 *Gravitation and Gauge Symmetries* (Bristol: Institute of Physics Publishing)
[16] Dirac P A M 1958 *Proc. R. Soc.* A **246** 333
[17] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University)
[18] Gupta S 1950 *Proc. Phys. Soc.* A **63** 681
[19] Bleuler K 1950 *Helv. Phys. Acta* **23** 567
[20] Nielsen M and Chuang I 2004 *Quantum Computation and Quantum Information* (*Cambridge Series on Information and the Natural Sciences*) (Cambridge: Cambridge University Press)
[21] Feldbrugge J, Lehners J L and Turok N 2018 *Phys. Rev.* D **97** 023509
[22] Rovelli C 2004 *Quantum Gravity* (Cambridge: Cambridge University Press)

[23] Rovelli C and Vidotto F 2014 *Covariant Loop Quantum Gravity: an Elementary Introduction to Quantum Gravity and Spinfoam Theory* (Cambridge: Cambridge University Press)
[24] Miković A and Vojinović M 2012 *Class. Quantum Grav.* **29** 165003
[25] Miković A 2013 *Rev. Math. Phys.* **25** 1343008
[26] Vojinović M 2016 *Phys. Rev.* D **94** 024058
[27] Hamber H W 2009 *Gen. Relativ. Gravit.* **41** 817
[28] Ambjørn J, Görlich A, Jurkiewicz J and Loll R 2012 *Phys. Rep.* **519** 127
[29] Freidel L and Livine E R 2006 *Phys. Rev. Lett.* **96** 221301
[30] Paunković N and Vojinović M 2016 *J. Phys.: Conf. Ser.* **701** 012035
[31] Regge T 1961 *Nuovo Cimento* **19** 558
[32] Miković A 2017 *Adv. Theor. Math. Phys.* **21** 631
[33] Miković A and Vojinović M 2015 *Europhys. Lett.* **110** 40008
[34] Miković A and Vojinović M 2014 Cosmological constant in a quantum gravity theory for a piecewise-linear spacetime (arXiv:1407.1124)
[35] Pipa F, Paunković N and Vojinović M 2018 Entanglement-induced deviation from the geodesic motion in quantum gravity (arXiv:1801.03207)
[36] Zurek W H 1981 *Phys. Rev.* D **24** 1516
[37] Schlosshauer M 2005 *Rev. Mod. Phys.* **76** 1267
[38] Penrose R 1996 *Gen. Relativ. Gravit.* **28** 581
[39] Everett H 1957 *Rev. Mod. Phys.* **29** 454
[40] DeWitt B S 2003 *The Global Approach to Quantum Field Theory* vol 1 (Oxford: Oxford University Press)
[41] Duff M J 1996 *Int. J. Mod. Phys.* A **11** 5623
[42] Lambert N 2012 *Ann. Rev. Nucl. Part. Sci.* **62** 285
[43] Becker K, Becker M and Schwarz J H 2007 *String Theory and M-Theory* (Cambridge: Cambridge University Press)
[44] DeWitt B S 1967 *Phys. Rev.* **160** 1113
[45] Arnowitt R, Deser S and Misner C W 2008 *Gen. Relativ. Gravit.* **40** 1997
[46] Feynman R, Morinigo F, Wagner W, Hatfield B and Pines D 2002 *Feynman Lectures on Gravitation* (Boulder, CO: Westview Press)
[47] Amelino-Camelia G 2012 Doubly-special relativity: facts, myths and some key open issues *Recent Developments in Theoretical Physics* ed S Ghosh and G Kar (Singapore: World Scientific)
[48] Hořava P 2009 *Phys. Rev.* D **79** 084008
[49] Niedermaier M and Reuter M 2006 *Living Rev. Relativ.* **9** 5
[50] Connes A 1994 *Noncommutative Geometry* (San Diego, CA: Academic)
[51] Sakellariadou M 2013 *J. Phys.: Conf. Ser.* **442** 012015
[52] Hehl F W, McCrea J D, Mielke E W and Ne'eman Y 1995 *Phys. Rep.* **258** 1
[53] Verlinde E 2011 *J. High Energy Phys.* JHEP04(2011)029
[54] Jacobson T 1995 *Phys. Rev. Lett.* **75** 1260
[55] Henson J 2009 The causal set approach to Quantum Gravity *Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter* ed D Oriti (Cambridge: Cambridge University Press)
[56] Bombelli L, Lee J, Meyer D and Sorkin R D 1987 *Phys. Rev. Lett.* **59** 521
[57] Bennett C H, Bernstein H J, Popescu S and Schumacher B 1996 *Phys. Rev.* A **53** 2046
[58] Peres A and Rosen N 1964 *Phys. Rev.* **135** B1486
[59] Baym G and Ozawa T 2009 *Proc. Natl Acad. Sci.* **106** 3035
[60] Brukner Č 2017 On the quantum measurement problem *Quantum [Un]Speakables II: Half a Century of Bell's Theorem* ed R Bertlmann and A Zeilinger (Berlin: Springer) pp 95–117
[61] Dür W, Vidal G and Cirac J I 2000 *Phys. Rev.* A **62** 062314
[62] Vedral V 2003 *Open Phys.* **1** 289
[63] Pfister C, Kaniewski J, Tomamichel M, Mantri A, Schmucker R, McMahon N, Milburn G and Wehner S 2016 *Nat. Commun.* **7** 13022

symmetry

MDPI

Article

Hamiltonian Analysis for the Scalar Electrodynamics as 3*BF* Theory

Tijana Radenković *,† and Marko Vojinović †

Institute of Physics, University of Belgrade, Pregrevica 118, 11080 Belgrade, Serbia; vmarko@ipb.ac.rs
* Correspondence: rtijana@ipb.ac.rs
† These authors contributed equally to this work.

Received: 13 March 2020; Accepted: 6 April 2020; Published: 14 April 2020

Abstract: The higher category theory can be employed to generalize the *BF* action to the so-called 3*BF* action, by passing from the notion of a gauge group to the notion of a gauge 3-group. The theory of scalar electrodynamics coupled to Einstein–Cartan gravity can be formulated as a constrained 3*BF* theory for a specific choice of the gauge 3-group. The complete Hamiltonian analysis of the 3*BF* action for the choice of a Lie 3-group corresponding to scalar electrodynamics is performed. This analysis is the first step towards a canonical quantization of a 3*BF* theory, an important stepping stone for the quantization of the complete scalar electrodynamics coupled to Einstein–Cartan gravity formulated as a 3*BF* action with suitable simplicity constraints. It is shown that the resulting dynamic constraints eliminate all propagating degrees of freedom, i.e., the 3*BF* theory for this choice of a 3-group is a topological field theory, as expected.

Keywords: Hamiltonian analysis; higher gauge theory; BF theory; topological theory; scalar electrodynamics

1. Introduction

The vast majority of physics community agrees that the quantum theory of gravity is necessary, even if they disagree on the quantization approach. The theory of loop quantum gravity is one of the well-formulated possible candidates for the desired theory of quantum gravity [1–3]. There are two approaches within the theory—the canonical and the covariant quantization method. The covariant quantization method is focused on obtaining a generating functional, by considering a triangulated spacetime manifold and defining the functional as a state sum over all configurations of a field living on simplices of the triangulation [2].

One of the key tools in the covariant quantization approach is the so-called *BF* theory. Given a Lie group G and its corresponding Lie algebra $\mathfrak{g}$, one considers a $\mathfrak{g}$-valued connection 1-form A, and its corresponding field strength 2-form $F \equiv \mathrm{d}A + A \wedge A$. Multiplying F with a $\mathfrak{g}$-valued Lagrange multiplier 2-form B and integrating over a four-dimensional spacetime manifold $\mathcal{M}$, one obtains the action of the *BF* theory,

$$S_{BF}[A,B] = \int_{\mathcal{M}} \langle B \wedge F \rangle_{\mathfrak{g}} \,,$$

where $\langle \_ , \_ \rangle_{\mathfrak{g}}$ is a G-invariant non-degenerate symmetric bilinear form. The *BF* theory derives its name from the symbols B and F for the Lagrange multiplier and the field strength present in the action. As it is defined, the *BF* theory is topological, containing no local propagating degrees of freedom. Therefore, for the purpose of building physically relevant actions, attention usually focuses not on the pure *BF* theory, but rather on the theory with constraints. The constrained *BF* models are based on deformations of the *BF* theory [4], by adding constraints to the topological *BF* action that promote some of the gauge degrees of freedom into physical ones. The well known example is the Plebanski model for general relativity [5]. Constrained BF models represent a starting point in the spinfoam approach to the construction of quantum gravity models [2].

The main shortcoming of building a quantum gravity model using a BF theory is the fact that it is very hard, if not impossible, to write the action for matter fields (specifically scalar and fermion fields) in the form of a constrained BF theory. Thus, the spinfoam quantization method is limited to pure gravity, and the problem of consistently coupling matter fields to gravity in this framework becomes highly nontrivial. One of the proposed ways to circumvent this issue is to generalize the notion of a BF theory using the mathematical apparatus of higher category theory.

The higher category theory [6] can be employed to generalize the BF action to the so-called nBF action, by passing from the notion of a gauge group to the notion of a gauge n-group (for a comprehensive review of n-groups see for example [7], and also Appendix C). Specifically, the notion of a 3-group in the framework of higher category theory is introduced as a 3-category with only one object where all the morphisms, 2-morphisms and 3-morphisms are invertible. Based on this generalization, recently a constrained $3BF$ action has been introduced, which describes the full Standard Model coupled to Einstein–Cartan gravity [8].

As a first step to the study of the Hamiltonian structure of such theories, in this work, we discuss the simplest nontrivial toy example, namely the theory of scalar electrodynamics coupled to gravity. The standard way to define scalar electrodynamics coupled to gravity is by the action:

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left[-\frac{1}{16\pi l_p^2} R - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + g^{\mu\nu} \nabla_\mu \phi^* \nabla_\nu \phi - m^2 \phi^* \phi \right] . \tag{1}$$

Here, $g_{\mu\nu}$ is the spacetime metric, $g \equiv \det(g_{\mu\nu})$ is its determinant, R is the corresponding curvature scalar, and l_p is the Planck length, its square being equal to the Newton's gravitational constant, $l_p^2 = G$, in the natural system of units $\hbar = c = 1$. The total covariant derivative ∇_μ of the complex scalar field ϕ is defined as $\nabla_\mu \phi = (\partial_\mu + iqA_\mu)\phi$, and thus coupled to the electromagnetic potential A_μ via the coupling constant q (the electric charge of the field ϕ). See Appendix A for more detailed notation. In the next section, we will reformulate this model as a classically equivalent constrained $3BF$ theory for a specific choice of the gauge 3-group. Moreover, for reasons of simplicity, in the Hamiltonian analysis, we will focus only on the topological sector, disregarding the simplicity constraints. The Hamiltonian structure of the theory is important for various reasons, primarily for the canonical quantization program.

The layout of the paper is as follows. In Section 2, we introduce the 3-group structure corresponding to the theory of scalar electrodynamics coupled to Einstein–Cartan gravity and the corresponding constrained $3BF$ action. Section 3 contains the Hamiltonian analysis for the topological, $3BF$ sector of the action, with the resulting first-class and second-class constraints present in the theory, and their mutual Poisson brackets. In Section 4, we analyze the Bianchi identities that the first-class constraints satisfy, which enforce restrictions in the sense of Hamiltonian analysis, and reduce the number of independent first-class constraints present in the theory. Section 5 focuses on the counting of the dynamical degrees of freedom present in the theory, based on the results from Sections 3 and 4. Encouraged by these results, in Section 6, we construct the generator of the gauge symmetries for the topological theory and we find the form variations of all variables and their canonical momenta. Finally, Section 7 is devoted to the discussion of the results and the possible future lines of research. The Appendices contain various technical details.

The notation and conventions are as follows. The local Lorentz indices are denoted by the Latin letters $a, b, c, \dots$, take values $0, 1, 2, 3$, and are raised and lowered using the Minkowski metric η_{ab} with signature $(-, +, +, +)$. Spacetime indices are denoted by the Greek letters $\mu, \nu, \dots$, and are raised and lowered by the spacetime metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ are the tetrad fields. The inverse tetrad is denoted as $e^\mu{}_a$, so that the standard orthogonality conditions hold: $e^a{}_\mu e^\mu{}_b = \delta^a_b$ and $e^a{}_\mu e^\nu{}_a = \delta^\nu_\mu$. When needed, spacetime indices will be split into time and space indices, denoted with a 0 and lowcase Latin indices $i, j, \dots$, respectively. All other indices that appear in the paper are dependent on the context, and their usage is explicitly defined in the text where they appear. The antisymmetrization over two indices is introduced with the factor one half that is $A_{[a_1|a_2\dots a_{n-1}|a_n]} = \frac{1}{2}\left(A_{a_1a_2\dots a_{n-1}a_n} - A_{a_na_2\dots a_{n-1}a_1}\right)$, and the total antisymmetrization is introduced as $A_{[a_1\dots a_n]} = \frac{1}{n!}\sum_{\sigma\in S_n}(-1)^{\mathrm{sign}(\sigma)} A_{a_{\sigma(1)}\dots a_{\sigma(n)}}$.

2. Scalar Electrodynamics as a Constrained 3*BF* Action

Let us begin by providing a short introduction into the construction and structure of a 3*BF* theory, after which we will impose appropriate simplicity constraints, in order to obtain the equations of motion for scalar electrodynamics coupled to gravity.

As was discussed in detail in [8], one formulates a topological 3*BF* action by specifying a particular gauge Lie 3-group. It has been proved that any strict 3-group is equivalent to a 2-crossed module [9,10]. A gauge theory for the manifold $\mathcal{M}_4$ and 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,,\triangleright\,,\{\_\,,\_\})$ can be constructed for the following choice of the three Lie groups as:

$$G = SO(3,1)\times U(1)\,,\quad H = \mathbb{R}^4\,,\quad L = \mathbb{R}^2\,.$$

The maps ∂ and δ are chosen to be trivial. The action of the algebra $\mathfrak{g}$ on $\mathfrak{h}$ and $\mathfrak{l}$ is chosen as:

$$\begin{array}{ll} M_{ab}\triangleright P_c = \triangleright_{ab,c}{}^{d}\,P_d = \delta_{[a|}{}^{d}\eta_{|b]c}\,P_d = \eta_{[b|c}\,P_{|a]}\,, & T\triangleright P_a = 0\,, \\ M_{ab}\triangleright P_A = 0\,, & T\triangleright P_A = \triangleright_A{}^B\,P_B \end{array} \tag{2}$$

where M_{ab} denote the six generators of $\mathfrak{so}(3,1)$, T is the sole generator of $\mathfrak{u}(1)$, P_a are the four generators of $\mathbb{R}^4$ and P_A are the two generators of $\mathbb{R}^2$. In the previous expression, the action of the algebra $\mathfrak{u}(1)$ on the algebra $\mathbb{R}^2$ is defined via

$$\triangleright_A{}^B = iq\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

The action of the algebra $\mathfrak{g}$ on itself is by definition given via the adjoint representation and, for the choice $\mathfrak{g} = \mathfrak{so}(3,1)\times\mathfrak{u}(1)$, one obtains

$$\begin{gathered} M_{ab}\triangleright M_{cd} = \triangleright_{ab\,,cd}{}^{ef}\,M_{ef} = f_{ab\,,cd}{}^{ef}\,M_{ef} = \eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}\,, \\ M_{ab}\triangleright T = 0\,,\quad T\triangleright M_{ab} = 0\,,\quad T\triangleright T = 0\,, \end{gathered} \tag{3}$$

as the consequence of the direct product structure and the Abelian nature of the subgroup $U(1)$. The Peiffer lifting

$$\{\_\,,\_\}: H\times H\to L$$

is also trivial, i.e., all the coefficients $X_{ab}{}^A$ are equal to zero:

$$\{P_a\,,P_b\} \equiv X_{ab}{}^A T_A = 0\,. \tag{4}$$

Given Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$, one can introduce a 3-connection (α,β,γ) given by the algebra-valued differential forms $\alpha\in\mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$, $\beta\in\mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$ and $\gamma\in\mathcal{A}^3(\mathcal{M}_4\,,\mathfrak{l})$. The corresponding fake 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is then defined as:

$$\mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,,\qquad \mathcal{G} = \mathrm{d}\beta + \alpha\wedge^{\triangleright}\beta - \delta\gamma\,,\qquad \mathcal{H} = \mathrm{d}\gamma + \alpha\wedge^{\triangleright}\gamma + \{\beta\wedge\beta\}\,, \tag{5}$$

see [9,10] for details. For this specific choice of a 3-group, where $\alpha = \omega + A$, given by the algebra-valued differential forms $\omega \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{so}(3,1))$, $A \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{u}(1))$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathbb{R}^4)$ and $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,,\mathbb{R}^2)$, the corresponding 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is defined as

$$\begin{aligned} \mathcal{F} &= R^{ab}M_{ab} + FT &&= \left(\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\right)M_{ab} + \mathrm{d}A\;T\,, \\ \mathcal{G} &= \mathcal{G}^a P_a &&= \left(\mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b\right)P_a\,, \\ \mathcal{H} &= \mathcal{H}^A P_A &&= \left(\mathrm{d}\gamma^A + \rhd_B{}^A A \wedge \gamma^B\right)P_A\,. \end{aligned} \tag{6}$$

Note that the connection ω^{ab} is not present in the last expression, as follows from the definition of the action $\rhd$ and the Peiffer lifting $\{\_\,,\_\}$, see Equations (2) and (4):

$$\begin{aligned} \mathcal{H} &= \mathrm{d}\gamma + \alpha \wedge^{\rhd} \gamma + \{\beta \wedge \beta\} \\ &= \mathrm{d}\gamma^A P_A + (\omega^{ab}M_{ab} + AT) \wedge^{\rhd} (\gamma^A P_A) \\ &= \mathrm{d}\gamma^A P_A + \omega^{ab} \wedge \gamma^A M_{ab} \rhd P_A + A \wedge \gamma^A T \rhd P_A \\ &= \mathrm{d}\gamma^A P_A + A \wedge \gamma^A \rhd_A{}^B P_B \\ &= (\mathrm{d}\gamma^A + \rhd_B{}^A A \wedge \gamma^B) P_A\,. \end{aligned} \tag{7}$$

The coefficients of the differential 2-forms F and R^{ab}, 3-form $\mathcal{G}$, and 4-form $\mathcal{H}$ are:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\,, \\ R^{ab}{}_{\mu\nu} &= \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^a{}_{c\mu}\omega^{cb}{}_\nu - \omega^a{}_{c\nu}\omega^{cb}{}_\mu\,, \\ \mathcal{G}^a{}_{\mu\nu\rho} &= \partial_\mu \beta^a{}_{\nu\rho} + \partial_\nu \beta^a{}_{\rho\mu} + \partial_\rho \beta^a{}_{\mu\nu} + \omega^a{}_{b\mu}\,\beta^b{}_{\nu\rho} + \omega^a{}_{b\nu}\,\beta^b{}_{\rho\mu} + \omega^a{}_{b\rho}\,\beta^b{}_{\mu\nu}\,, \\ \mathcal{H}^A{}_{\mu\nu\rho\sigma} &= \partial_\mu \gamma^A{}_{\nu\rho\sigma} - \partial_\nu \gamma^A{}_{\rho\sigma\mu} + \partial_\rho \gamma^A{}_{\sigma\mu\nu} - \partial_\sigma \gamma^A{}_{\mu\nu\rho} \\ &\quad + \rhd_B{}^A A_\mu \gamma^B{}_{\nu\rho\sigma} - \rhd_B{}^A A_\nu \gamma^B{}_{\rho\sigma\mu} + \rhd_B{}^A A_\rho \gamma^B{}_{\sigma\mu\nu} - \rhd_B{}^A A_\sigma \gamma^B{}_{\mu\nu\rho}\,. \end{aligned} \tag{8}$$

Now, one can define a gauge invariant $3BF$ action as:

$$S_{3BF} = \int_{\mathcal{M}_4} \Big(\langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \Big)\,, \tag{9}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{so}(3,1))$, $C \in \mathcal{A}^1(\mathcal{M}_4\,,\mathbb{R}^4)$ and $D \in \mathcal{A}^0(\mathcal{M}_4\,,\mathbb{R}^2)$ are Lagrange multipliers. The forms $\langle \_\,,\_ \rangle_{\mathfrak{g}}$, $\langle \_\,,\_ \rangle_{\mathfrak{h}}$ and $\langle \_\,,\_ \rangle_{\mathfrak{l}}$ are G-invariant bilinear symmetric nondegenerate forms on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, respectively, defined as

$$\langle M_{ab}\,, M_{cd} \rangle_{\mathfrak{g}} = g_{ab,cd}\,, \quad \langle T\,, T \rangle_{\mathfrak{g}} = 1\,, \quad \langle M_{ab}\,, T \rangle_{\mathfrak{g}} = 0\,, \quad \langle P_a\,, P_b \rangle_{\mathfrak{h}} = g_{ab}\,, \quad \langle P_A\,, P_B \rangle_{\mathfrak{l}} = g_{AB}\,,$$

where

$$g_{ab,cd} = \eta_{a[c|}\eta_{b|d]}\,, \qquad g_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad g_{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Identifying the Lagrange multiplier C^a as the tetrad field e^a, and the Lagrange multiplier D^A as the doublet of scalar fields ϕ^A,

$$\phi = \phi^A P_A = \phi P_1 + \phi^* P_2\,,$$

based on their transformation properties as discussed in [8,11], the Lagrangian of the action (9) obtains the form:

$$S_{3BF} = \int_{\mathcal{M}_4} \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} \big(\frac{1}{4} B^{ab}{}_{\mu\nu} R^{cd}{}_{\rho\sigma} g_{ab,cd} + \frac{1}{4} B_{\mu\nu} F_{\rho\sigma} + \frac{1}{3!} e^a{}_\mu \mathcal{G}^b{}_{\nu\rho\sigma} g_{ab} + \frac{1}{4!} \phi^A \mathcal{H}^B{}_{\mu\nu\rho\sigma} g_{AB} \big) . \quad (10)$$

Varying the action with respect to all the variables, one obtains the equations of motion:

varied variable	equation of motion	varied variable	equation of motion
δB^{ab}	$R_{ab} = 0$	δB	$F = 0$
$\delta\omega^{ab}$	$\nabla B_{ab} - e_{[a\|} \wedge \beta_{\|b]} = 0$	δA	$\mathrm{d}B + \phi_A \rhd_B{}^A \gamma^B = 0$
δe^a	$\mathcal{G}_a = 0$	$\delta\beta^a$	$\nabla e_a = 0$
$\delta\phi^A$	$\nabla\gamma_A = 0$	$\delta\gamma^A$	$\nabla\phi_A = 0$

(11)

Since one is interested in the doublet of scalar fields ϕ^A of mass m and charge q minimally coupled to gravity and electromagnetic field, we impose additional simplicity constraint terms to the topological action (9), in order to obtain the appropriate equations of motion equivalent to the equations of motion for the action (1):

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B \wedge F + e_a \wedge \nabla\beta^a + \phi_A \nabla\gamma^A \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda^A \wedge \left(\gamma_A - \frac{1}{2} H_{abcA} e^a \wedge e^b \wedge e^c \right) + \Lambda^{abA} \wedge \left(H_{abcA} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \nabla\phi_A \wedge e_a \wedge e_b \right) \\ & + \lambda \wedge \left(B - \frac{12}{q} M_{ab} e^a \wedge e^b \right) + \zeta^{ab} \left(M_{ab} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b \right) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi_A \phi^A \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d . \end{aligned} \quad (12)$$

For the notation used here and the equations of motion obtained by varying the action (12), see Appendix A.

The dynamical degrees of freedom are the tetrad fields e^a, the scalar doublet ϕ^A, and the electromagnetic potential A, while the remaining variables are algebraically determined in terms of them, as shown in Appendix A. The equation of motion for the field ϕ^A reduces to the covariant Klein-Gordon equation for the scalar field,

$$\left(\nabla_\mu \nabla^\mu - m^2 \right) \phi_A = 0 . \quad (13)$$

The differential equation of motion for the field A is:

$$\nabla_\mu F^{\mu\nu} = j^\nu , \qquad j^\mu \equiv \frac{1}{2} \left(\nabla^\nu \phi^A \rhd^B{}_A \phi_B - \phi_A \rhd_B{}^A \nabla^\nu \phi^B \right) = iq \left(\nabla\phi^* \phi - \phi^* \nabla\phi \right) . \quad (14)$$

Finally, the equation of motion for e^a becomes:

$$\begin{aligned} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R &= 8\pi l_p^2\, T^{\mu\nu} , \\ T^{\mu\nu} \equiv \nabla^\mu \phi_A \nabla^\nu \phi^A - \frac{1}{2} g^{\mu\nu} \left(\nabla_\rho \phi_A \nabla^\rho \phi^A + m^2 \phi_A \phi^A \right) &- \frac{1}{4q} \left(F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} + 4 F^{\mu\rho} F_\rho{}^\nu \right) . \end{aligned} \quad (15)$$

3. The Hamiltonian Analysis

The Hamiltonian analysis of the constrained $3BF$ action (12) for scalar electrodynamics is exceedingly complicated to study. A testament to this is the level of complexity of the constrained $2BF$ formulation of general relativity [12], which is merely one sector in the action (12). Therefore, in this paper, we will limit ourselves to the topological sector of the theory, namely the unconstrained $3BF$ theory (9), which consists of the terms in the first row of Equation (12), and is written in full detail in Equation (10). One should be aware that this restriction changes various properties of the theory. Namely, the simplicity constraints (everything but the first row in Equation (12)) substantially modify the dynamics of the theory—they increase the number of local propagating degrees of freedom of the theory, a property that was known since the original Plebanski model [5]. On the other hand, the unconstrained $3BF$ theory (9) is important even in its own right, and the Hamiltonian analysis may give important insight into the structure of both the unconstrained and the constrained theory.

In what follows, the complete Hamiltonian analysis for the action (9) is presented, see [13] for an overview and a comprehensive introduction of the Hamiltonian analysis. The Hamiltonian analysis for a $2BF$ action is performed in [12,14–16].

Under the standard assumption that the spacetime manifold is globally hyperbolic, $\mathcal{M}_4 = \mathbb{R} \times \Sigma_3$, the Lagrangian of the action (9) has the form:

$$L_{3BF} = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \epsilon^{\mu\nu\rho\sigma} \Big(\frac{1}{4} B^{ab}{}_{\mu\nu} R^{cd}{}_{\rho\sigma}\, g_{ab,cd} + \frac{1}{4} B_{\mu\nu} F_{\rho\sigma} + \frac{1}{3!} e^a{}_\mu\, \mathcal{G}^b{}_{\nu\rho\sigma}\, g_{ab} + \frac{1}{4!} \phi^A \mathcal{H}^B{}_{\mu\nu\rho\sigma}\, g_{AB} \Big) . \tag{16}$$

The canonical momentum $\pi(q)$ corresponding for the canonical coordinate q from the set of all variables in the theory, $q \in \{B^{ab}{}_{\mu\nu}, \omega^{ab}{}_\mu, B_{\mu\nu}, A_\mu, e^a{}_\mu, \beta^a{}_{\mu\nu}, \phi^A, \gamma^A{}_{\mu\nu\rho}\}$, is obtained as a derivative of the Lagrangian with respect to the appropriate velocity,

$$\pi(q) \equiv \frac{\delta L}{\delta \partial_0 q} ,$$

giving:

$$\begin{array}{lcllcl} \pi(B)_{ab}{}^{\mu\nu} & = & 0 , & \pi(\omega)_{ab}{}^{\mu} & = & \epsilon^{0\mu\nu\rho} B_{ab\nu\rho} , \\ \pi(B)^{\mu\nu} & = & 0 , & \pi(A)^{\mu} & = & \frac{1}{2}\epsilon^{0\mu\nu\rho} B_{\nu\rho} , \\ \pi(e)_a{}^{\mu} & = & 0 , & \pi(\beta)_a{}^{\mu\nu} & = & -\epsilon^{0\mu\nu\rho} e_{a\rho} , \\ \pi(\phi)_A & = & 0 , & \pi(\gamma)_A{}^{\mu\nu\rho} & = & \epsilon^{0\mu\nu\rho} \phi_A . \end{array} \tag{17}$$

Since these momenta cannot be inverted for the time derivatives of the variables, they all give rise to primary constraints:

$$\begin{array}{lcllcl} P(B)_{ab}{}^{\mu\nu} & \equiv & \pi(B)_{ab}{}^{\mu\nu} \approx 0 , & P(\omega)_{ab}{}^{\mu} & \equiv & \pi(\omega)_{ab}{}^{\mu} - \epsilon^{0\mu\nu\rho} B_{ab\nu\rho} \approx 0 , \\ P(B)^{\mu\nu} & \equiv & \pi(B)^{\mu\nu} \approx 0 , & P(A)^{\mu} & \equiv & \pi(A)^{\mu} - \frac{1}{2}\epsilon^{0\mu\nu\rho} B_{\nu\rho} \approx 0 , \\ P(e)_a{}^{\mu} & \equiv & \pi(e)_a{}^{\mu} \approx 0 , & P(\beta)_a{}^{\mu\nu} & \equiv & \pi(\beta)_a{}^{\mu\nu} + \epsilon^{0\mu\nu\rho} e_{a\rho} \approx 0 , \\ P(\phi)_A & \equiv & \pi(\phi)_A \approx 0 , & P(\gamma)_A{}^{\mu\nu\rho} & \equiv & \pi(\gamma)_A{}^{\mu\nu\rho} - \epsilon^{0\mu\nu\rho} \phi_A \approx 0 . \end{array} \tag{18}$$

Here, the symbol "$\approx$" denotes the so-called "weak" equality, i.e., the equality that holds on a subspace of the phase space determined by the constraints, while the equality that holds for any point of the phase space is referred to as the "strong" equality and it is denoted by the symbol "$=$". The expressions "on-shell" and "off-shell" are used for weak and strong equalities, respectively, and henceforth will be used in this paper.

The fundamental Poisson brackets are defined as:

$$
\begin{aligned}
\{ B^{ab}{}_{\mu\nu}(x) \,,\, \pi(B)_{cd}{}^{\rho\sigma}(y) \} &= 4\delta^a{}_{[c}\delta^b{}_{d]}\delta^\rho{}_{[\mu}\delta^\sigma{}_{\nu]}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ \omega^{ab}{}_{\mu}(x) \,,\, \pi(\omega)_{cd}{}^{\nu}(y) \} &= 2\delta^a{}_{[c}\delta^b{}_{d]}\delta^\nu{}_{\mu}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ B_{\mu\nu}(x) \,,\, \pi(B)^{\rho\sigma}(y) \} &= 2\delta^\rho{}_{[\mu}\delta^\sigma{}_{\nu]}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ A_{\mu}(x) \,,\, \pi(A)^{\nu}(y) \} &= \delta^\nu{}_{\mu}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ e^{a}{}_{\mu}(x) \,,\, \pi(e)_{b}{}^{\nu}(y) \} &= \delta^a{}_{b}\delta^\nu{}_{\mu}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ \beta^{a}{}_{\mu\nu}(x) \,,\, \pi(\beta)_{b}{}^{\rho\sigma}(y) \} &= 2\delta^a{}_{b}\,\delta^\rho{}_{[\mu}\delta^\sigma{}_{\nu]}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ \phi^{A}(x) \,,\, \pi(\phi)_{B}(y) \} &= \delta^A{}_{B}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ \gamma^{A}{}_{\mu\nu\rho}(x) \,,\, \pi(\gamma)_{B}{}^{\alpha\beta\gamma}(y) \} &= 3!\delta^A{}_{B}\,\delta^\alpha{}_{[\mu}\delta^\beta{}_{\nu}\delta^\gamma{}_{\rho]}\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{19}
$$

Using these relations, one can calculate the algebra between the primary constraints,

$$
\begin{aligned}
\{ P(B)^{abjk}(x) \,,\, P(\omega)_{cd}{}^{i}(y) \} &= 4\epsilon^{0ijk}\,\delta^a{}_{[c}\delta^b{}_{d]}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ P(B)^{jk}(x) \,,\, P(A)^{i}(y) \} &= \epsilon^{0ijk}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ P(e)^{ak} \,,\, P(\beta)_{b}{}^{ij}(y) \} &= -\epsilon^{0ijk}\,\delta^a{}_{b}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{ P(\phi)^{A}(x) \,,\, P(\gamma)_{B}{}^{ijk}(y) \} &= \epsilon^{0ijk}\,\delta^A{}_{B}\,\delta^{(3)}(\vec{x}-\vec{y})\,,
\end{aligned}
\tag{20}
$$

while all other Poisson brackets vanish. The canonical on-shell Hamiltonian is defined by

$$
\begin{aligned}
H_c = \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big[&\frac{1}{4}\pi(B)_{ab}{}^{\mu\nu}\,\partial_0 B^{ab}{}_{\mu\nu} + \frac{1}{2}\pi(\omega)_{ab}{}^{\mu}\,\partial_0\omega^{ab}{}_{\mu} + \frac{1}{2}\pi(B)^{\mu\nu}\,\partial_0 B_{\mu\nu} + \pi(A)^{\mu}\,\partial_0 A_{\mu} \\
&+ \pi(e)_{a}{}^{\mu}\,\partial_0 e^{a}{}_{\mu} + \frac{1}{2}\pi(\beta)_{a}{}^{\mu\nu}\,\partial_0\beta^{a}{}_{\mu\nu} + \pi(\phi)_{A}\,\partial_0 D^{A} + \frac{1}{3!}\pi(\gamma)_{A}{}^{\mu\nu\rho}\,\partial_0\gamma^{A}{}_{\mu\nu\rho} \Big] - L\,.
\end{aligned}
\tag{21}
$$

Rewriting the Hamiltonian (21) such that all the velocities are multiplied by the first class constraints and therefore in an on-shell quantity they drop out, one obtains:

$$
\begin{aligned}
H_c = -\int_{\Sigma_3} \mathrm{d}^3\vec{x}\,\epsilon^{0ijk} \Big[&\frac{1}{2}B_{ab0i}\,R^{ab}{}_{jk} + \frac{1}{2}B_{0i}F_{jk} + \frac{1}{6}e_{a0}\,\mathcal{G}^{a}{}_{ijk} + \beta^{a}{}_{0i}\nabla_j e_{ak} \\
&+ \frac{1}{2}\omega^{ab}{}_{0}\Big(\nabla_i B_{ab\,jk} - e_{[a|i}\,\beta_{|b]jk}\Big) + \frac{1}{2}A_0\Big(\partial_i B_{jk} + \frac{1}{3}\,\phi_A \rhd_B{}^{A}\,\gamma^{B}{}_{ijk}\Big) + \frac{1}{2}\gamma^{A}{}_{0ij}\nabla_k\phi_A \Big]\,.
\end{aligned}
\tag{22}
$$

This expression does not depend on any of the canonical momenta and it contains only the fields and their spatial derivatives. By adding a Lagrange multiplier λ for each of the primary constraints we can build the off-shell Hamiltonian, which is given by:

$$
\begin{aligned}
H_T = H_c + \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Big[&\frac{1}{4}\lambda(B)^{ab}{}_{\mu\nu}P(B)_{ab}{}^{\mu\nu} + \frac{1}{2}\lambda(\omega)^{ab}{}_{\mu}P(\omega)_{ab}{}^{\mu} + \frac{1}{2}\lambda(B)_{\mu\nu}P(B)^{\mu\nu} + \lambda(A)_{\mu}P(A)^{\mu} \\
&+ \lambda(e)^{a}{}_{\mu}P(e)_{a}{}^{\mu} + \frac{1}{2}\lambda(\beta)^{a}{}_{\mu\nu}P(\beta)_{a}{}^{\mu\nu} + \lambda(\phi)^{A}P(\phi)_{A} + \frac{1}{3!}\lambda(\gamma)^{A}{}_{\mu\nu\rho}P(\gamma)_{A}{}^{\mu\nu\rho} \Big]\,.
\end{aligned}
\tag{23}
$$

Since the primary constraints must be preserved in time, one must impose the following requirement:

$$
\dot{P} \equiv \{ P \,,\, H_T \} \approx 0\,,
\tag{24}
$$

for each primary constraint P. By using the consistency condition (24) for the primary constraints $P(B)_{ab}{}^{0i}$, $P(\omega)_{ab}{}^{0}$, $P(B)^{0i}$, $P(A)^{0}$, $P(e)_a{}^{0}$, $P(\beta)_a{}^{0i}$, and $P(\gamma)_A{}^{0ij}$,

$$\begin{array}{llll} \dot{P}(B)_{ab}{}^{0i} \approx 0\,, & \dot{P}(\omega)_{ab}{}^{0} \approx 0\,, & \dot{P}(B)^{0i} \approx 0\,, & \dot{P}(A)^{0} \approx 0\,, \\ \dot{P}(e)_a{}^{0} \approx 0\,, & \dot{P}(\beta)_a{}^{0i} \approx 0\,, & \dot{P}(\gamma)_A{}^{0ij} \approx 0\,, & \end{array} \tag{25}$$

one obtains the secondary constraints $\mathcal{S}$,

$$\begin{array}{lcllcl} \mathcal{S}(R)_{ab}{}^{i} & \equiv & \epsilon^{0ijk} R_{abjk} \approx 0\,, & \mathcal{S}(\nabla B)_{ab} & \equiv & \epsilon^{0ijk}\left(\nabla_i B_{ab\,jk} - e_{[a|i}\,\beta_{|b]\,jk}\right) \approx 0\,, \\ \mathcal{S}(F)^{i} & \equiv & \frac{1}{2}\epsilon^{0ijk} F_{jk} \approx 0\,, & \mathcal{S}(\nabla B) & \equiv & \frac{1}{2}\epsilon^{0ijk}\left(\partial_i B_{jk} + \frac{1}{3}\,\phi_A \rhd_B{}^{A}\,\gamma^{B}{}_{ijk}\right) \approx 0\,, \\ \mathcal{S}(\mathcal{G})_a & \equiv & \frac{1}{6}\epsilon^{0ijk}\mathcal{G}_{aijk} \approx 0\,, & \mathcal{S}(\nabla e)_a{}^{i} & \equiv & \epsilon^{0ijk}\nabla_j e_{ak} \approx 0\,, \\ \mathcal{S}(\nabla\phi)_A{}^{ij} & \equiv & \epsilon^{0ijk}\nabla_k \phi_A \approx 0\,, & & & \end{array} \tag{26}$$

while in the case of $P(B)_{ab}{}^{jk}$, $P(\omega)_{ab}{}^{k}$, $P(B)^{jk}$, $P(A)^{k}$, $P(e)_a{}^{k}$, $P(\beta)_a{}^{jk}$, $P(\phi)_A$ and $P(\gamma)_A{}^{ijk}$ the consistency conditions

$$\begin{array}{llll} \dot{P}(B)_{ab}{}^{jk} \approx 0\,, & \dot{P}(\omega)_{ab}{}^{k} \approx 0\,, & \dot{P}(B)^{jk} \approx 0\,, & \dot{P}(A)^{k} \approx 0\,, \\ \dot{P}(e)_a{}^{k} \approx 0\,, & \dot{P}(\beta)_a{}^{jk} \approx 0\,, & \dot{P}(\phi)_A \approx 0\,, & \dot{P}(\gamma)_A{}^{ijk} \approx 0\,, \end{array} \tag{27}$$

determine the following Lagrange multipliers:

$$\begin{array}{lcllcl} \lambda(\omega)_{ab}{}^{i} & \approx & \nabla^i\,\omega_{ab\,0}\,, & \lambda(B)^{ij} & \approx & 2\partial^{[i|}\,B^{0|j]} + \gamma_A{}^{0ij} \rhd_B{}^{A}\,\phi^B\,, \\ \lambda(A)^{i} & \approx & \partial^i\,A_0\,, & \lambda(\beta)_a{}^{ij} & \approx & 2\nabla^{[i|}\,\beta_a{}^{0|j]} - \omega_{ab}{}^{0}\,\beta^{b\,ij}\,, \\ \lambda(\phi)^{A} & \approx & A^0 \rhd{}^{A}{}_{B}\,\phi^B\,, & \lambda(e)_a{}^{i} & \approx & \nabla^i\,e_a{}^{0} - \omega_a{}^{b\,0}\,e_b{}^{i}\,, \\ \lambda(B)_{ab}{}^{ij} & \approx & \multicolumn{4}{l}{2\nabla^{[i|}B_{ab}{}^{0|j]} + e_{[a|\,0}\beta_{|b]}{}^{ij} - 2e_{[a|}{}^{[i|}\beta_{|b]}{}^{0|j]} + 2\omega_{[a|}{}^{c}B_{|b]}{}^{c\,ij}\,,} \\ \lambda(\gamma)_A{}^{ijk} & \approx & \multicolumn{4}{l}{-A^0 \rhd_A{}^{B}\,\gamma_B{}^{ijk} + \nabla^i\gamma_A{}^{0jk} - \nabla^j\gamma_A{}^{0ik} + \nabla^k\gamma_A{}^{0ij}\,.} \end{array} \tag{28}$$

Note that the consistency conditions leave the Lagrange multipliers

$$\lambda(B)^{ab}{}_{0i}\,, \qquad \lambda(\omega)^{ab}{}_{0}\,, \qquad \lambda(B)_{0i}\,, \qquad \lambda(A)_{0}\,, \qquad \lambda(e)^{a}{}_{0}\,, \qquad \lambda(\beta)^{a}{}_{0i}\,, \qquad \lambda(\gamma)^{A}{}_{0ij} \tag{29}$$

undetermined. The consistency conditions of the secondary constraints do not produce new constraints, since one can show that

$$\begin{array}{lclcl} \dot{\mathcal{S}}(R)^{abi} & = & \{\mathcal{S}(R)^{abi}\,,\,H_T\} & = & \omega^{[a|}{}_{c0}\,\mathcal{S}(R)^{c|b]i}\,, \\ \dot{\mathcal{S}}(\nabla B) & = & \{\mathcal{S}(\nabla B),\,H_T\} & = & -\rhd_B{}^{A}\,\gamma^{B}{}_{0ij}\,\mathcal{S}(\nabla\phi)_A{}^{ij}\,, \\ \dot{\mathcal{S}}(\mathcal{G})^{a} & = & \{\mathcal{S}(\mathcal{G})^{a}\,,\,H_T\} & = & \beta_{b0k}\,\mathcal{S}(R)^{abk} - \omega^{ab}{}_{0}\,\mathcal{S}(\mathcal{G})_b\,, \\ \dot{\mathcal{S}}(\nabla e)_a{}^{i} & = & \{\mathcal{S}(\nabla e)_a{}^{i}\,,\,H_T\} & = & e^{b}{}_{0}\,\mathcal{S}(R)_{ab}{}^{i} - \omega_a{}^{b}{}_{0}\,\mathcal{S}(\nabla e)_b{}^{i}\,, \\ \dot{\mathcal{S}}(\nabla\phi)_A{}^{ij} & = & \{\mathcal{S}(\nabla\phi)_A{}^{ij}\,,\,H_T\} & = & A_0 \rhd_A{}^{B}\mathcal{S}(\nabla\phi)_B{}^{ij}\,, \\ \dot{\mathcal{S}}(F)^{i} & = & \{\mathcal{S}(F)^{i}\,,\,H_T\} & = & 0\,, \\ \dot{\mathcal{S}}(\nabla B)_{ab} & = & \{\mathcal{S}(\nabla B)_{ab}\,,\,H_T\} & = & \mathcal{S}(R)_{[a|c}{}^{k}\,B^{c}{}_{|b]0k} + \omega_{[a|}{}^{c}{}_{0}\mathcal{S}(\nabla B)_{|b]c} \\ & & & & -\beta_{[a|0k}\,\mathcal{S}(\nabla e)_{|b]}{}^{k} + e_{[a|0}\,\mathcal{S}(\mathcal{G})_{|b]}\,. \end{array} \tag{30}$$

Then, the total Hamiltonian can be written as

$$
\begin{aligned}
H_T \quad = \quad \int_{\Sigma_3} \mathrm{d}^3\vec{x} \quad \Big[& \frac{1}{2}\lambda(B)_{ab}{}^{0i}\,\Phi(B)^{ab}{}_i + \frac{1}{2}\lambda(\omega)_{ab}{}^{0}\,\Phi(\omega)^{ab} + \lambda(B)^{0i}\,\Phi(B)_i + \lambda(A)^0\,\Phi(A) \\
& + \lambda(e)_a{}^0\,\Phi(e)^a + \lambda(\beta)_a{}^{0i}\,\Phi(\beta)^a{}_i + \frac{1}{2}\lambda(\gamma)_A{}^{0ij}\Phi(\gamma)^A{}_{ij} \\
& - \frac{1}{2}B_{ab0i}\,\Phi(R)^{abi} - \frac{1}{2}\omega_{ab0}\,\Phi(\nabla B)^{ab} - B_{0i}\,\Phi(F)^i - A_0\,\Phi(\nabla B) \\
& - e_{a0}\,\Phi(\mathcal{G})^a - \beta_{a0i}\,\Phi(\nabla e)^{ai} - \frac{1}{2}\gamma_{A0ij}\,\Phi(\nabla\phi)^{Aij} \Big] \,,
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
\Phi(B)^{ab}{}_i &= P(B)^{ab}{}_{0i}\,, & \Phi(\gamma)^A{}_{ij} &= P(\gamma)^A{}_{0ij}\,, \\
\Phi(\omega)^{ab} &= P(\omega)^{ab}{}_0\,, & \Phi(F)^i &= \mathcal{S}(F)^i - \partial_j P(B)^{ij}\,, \\
\Phi(B)_i &= P(B)_{0i}\,, & \Phi(R)^{abi} &= \mathcal{S}(R)^{abi} - \nabla_j P(B)^{ab\,ij}\,, \\
\Phi(A) &= P(A)_0\,, & \Phi(\mathcal{G})^a &= \mathcal{S}(\mathcal{G})^a + \nabla_i P(e)^{a\,i} - \tfrac{1}{4}\,\beta_{b\,ij}\,P(B)^{ab\,ij}\,, \\
\Phi(e)^a &= P(e)^a{}_0\,, & \Phi(\nabla e)^{a\,i} &= \mathcal{S}(\nabla e)^{a\,i} - \nabla_j P(\beta)^{a\,ij} + \tfrac{1}{2}\,e_{b\,j}\,P(B)^{ab\,ij}\,, \\
\Phi(\beta)^a{}_i &= P(\beta)^a{}_{0i}\,, & \Phi(\nabla\phi)^{A\,ij} &= \mathcal{S}(\nabla\phi)^{A\,ij} + \nabla_k P(\gamma)^{A\,ijk} - \rhd_B{}^A\,\phi^B\,P(B)^{ij}\,, \\
\Phi(\nabla B) &= \mathcal{S}(\nabla B) + \partial_i P(A)^i + \frac{1}{3!}\,\gamma^A{}_{ijk}\,\rhd_A{}^B\,P(\gamma)_B{}^{ijk} - \phi_A\,\rhd_B{}^A\,P(\phi)^B\,, \\
\Phi(\nabla B)^{ab} &= \mathcal{S}(\nabla B)^{ab} + \nabla_i P(\omega)^{abi} + B^{[a|}{}_{c\,ij}\,P(B)^{c|b]\,ij} - 2e^{[a|}{}_i\,P(e)^{|b]\,i} - \beta^{[a|}{}_{ij}\,P(\beta)^{|b]\,ij}\,,
\end{aligned}
\tag{32}
$$

are the first-class constraints, while

$$
\begin{aligned}
&\chi(B)_{ab}{}^{jk} = P(B)_{ab}{}^{jk}\,, && \chi(B)^{jk} = P(B)^{jk}\,, && \chi(e)_a{}^i = P(e)_a{}^i\,, && \chi(\phi)_A = P(\phi)_A\,, \\
&\chi(\omega)_{ab}{}^i = P(\omega)_{ab}{}^i\,, && \chi(A)^i = P(A)^i\,, && \chi(\beta)_a{}^{ij} = P(\beta)_a{}^{ij}\,, && \chi(\gamma)_A{}^{ijk} = P(\gamma)_A{}^{ijk}\,,
\end{aligned}
\tag{33}
$$

are the second-class constraints.

The PB algebra of the first-class constraints is given by:

$$
\begin{aligned}
\{\,\Phi(\mathcal{G})^a(x)\,,\,\Phi(\nabla e)_b{}^i(y)\,\} &= -\,\Phi(R)^a{}_b{}^i(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\mathcal{G})^a(x)\,,\,\Phi(\nabla B)_{bc}(y)\,\} &= 2\delta^a{}_{[b|}\,\Phi(\mathcal{G})_{|c]}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla e)^a{}_i(x)\,,\,\Phi(\nabla B)_{bc}(y)\,\} &= 2\delta^a{}_{[b|}\Phi(\nabla e)_{|c]i}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(R)^{abi}(x)\,,\,\Phi(\nabla B)_{cd}(y)\,\} &= -4\delta^{[a|}{}_{[c}\,\Phi(R)^{|b]}{}_{d]}{}^i(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\Phi(\nabla B)_{cd}(y)\,\} &= -4\delta^{[a|}{}_{[c|}\,\Phi(\nabla B)^{|b]}{}_{|d]}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\,\Phi(\nabla B)(x)\,,\,\Phi(\nabla\phi)_A{}^{ij}(y)\,\} &= -2\rhd^B{}_A\;\Phi(\nabla\phi)_B{}^{ij}(x)\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{34}
$$

The PB algebra between the first and the second-class constraints is given by:

$$
\begin{aligned}
\{\, \Phi(R)^{abi}(x)\,,\, \chi(\omega)_{cd}{}^{j}(y)\, \} &= 4\,\delta^{[a|}{}_{[c|}\, \chi(B)^{|b]}{}_{|d]}{}^{ij}(x)\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\mathcal{G})^{a}(x)\,,\, \chi(\omega)_{cd}{}^{i}(y)\, \} &= 2\,\delta^{a}{}_{[c|}\, \chi(e)_{|d]}{}^{i}(x)\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\mathcal{G})^{a}(x)\,,\, \chi(\beta)_{c}{}^{jk}(y)\, \} &= -\frac{1}{2}\,\chi(B)^{a}{}_{c}{}^{jk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\nabla e)^{ai}(x)\,,\, \chi(\omega)_{cd}{}^{j}(y)\, \} &= -2\,\delta^{a}{}_{[c|}\, \chi(\beta)_{|d]}{}^{ij}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\nabla e)^{ai}(x)\,,\, \chi(e)_{b}{}^{j}(y)\, \} &= \frac{1}{2}\,\chi(B)^{a}{}_{b}{}^{ij}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\nabla B)^{ab}(x)\,,\, \chi(\omega)_{cd}{}^{i}(y)\, \} &= 4\,\delta^{[a|}{}_{[c|}\, \chi(\omega)_{|d]}{}^{|b]i}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\nabla B)(x)\,,\, \chi(A)^{i}(y)\, \} &= 2\,\chi(A)^{i}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\nabla B)^{ab}(x)\,,\, \chi(\beta)_{c}{}^{jk}(y)\, \} &= -2\delta^{[a|}{}_{c}\,\chi(\beta)^{|b]jk}\,\delta^{(3)}(x-y)\,, \\
\{\, \Phi(\nabla B)(x)\,,\, \chi(\gamma)_{A}{}^{ijk}(y)\, \} &= \rhd_{A}{}^{B}\,\chi(\gamma)_{B}{}^{ijk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\nabla B)^{ab}(x)\,,\, \chi(B)_{cd}{}^{jk}(y)\, \} &= 4\,\delta^{[a|}{}_{[c}\,\chi(B)_{d]}{}^{|b]jk}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\nabla B)^{ab}(x)\,,\, \chi(e)_{a}{}^{i}(y)\, \} &= -2\delta^{[a|}{}_{c}\,\chi(e)^{|b]i}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\nabla B)(x)\,,\, \chi(\phi)_{A}(y)\, \} &= -\rhd^{B}{}_{A}\,\chi(\phi)_{B}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\nabla\phi)^{Aij}(x)\,,\, \chi(A)^{k}(y)\, \} &= -\rhd_{B}{}^{A}\,\chi(\gamma)^{Bijk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\
\{\, \Phi(\nabla\phi)^{Aij}(x)\,,\, \chi(\phi)_{B}(y)\, \} &= -\rhd_{B}{}^{A}\,\chi(B)^{ij}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{35}
$$

The PB algebra between the second-class constraints has already been calculated, and is given in Equations (20).

4. The Bianchi Identities

In order to calculate the number of degrees of freedom in the theory, one needs to make use of the *Bianchi identities* (BI), as well as additional, *generalized Bianchi identities* (GBI) that are an analogue of the ordinary BI for the additional fields present in the theory.

One uses BI associated with the 1-form fields ω^{ab} and e^{a}, as well as the GBI for the 1-form A. Namely, the corresponding 2-form curvatures

$$
R^{ab} = \mathrm{d}\omega^{ab} + \omega^{a}{}_{c} \wedge \omega^{cb}\,, \qquad T^{a} = \mathrm{d}e^{a} + \omega^{a}{}_{b} \wedge e^{b}\,, \qquad F = \mathrm{d}A\,, \tag{36}
$$

satisfy the following identities:

$$
\epsilon^{\lambda\mu\nu\rho}\,\nabla_{\mu}R^{ab}{}_{\nu\rho} = 0\,, \tag{37}
$$

$$
\epsilon^{\lambda\mu\nu\rho}\left(\nabla_{\mu}T^{a}{}_{\nu\rho} - R^{ab}{}_{\mu\nu}\,e_{b\rho}\right) = 0\,, \tag{38}
$$

$$
\epsilon^{\lambda\mu\nu\rho}\,\nabla_{\mu}F_{\nu\rho} = 0\,. \tag{39}
$$

Choosing the free index to be time coordinate $\lambda = 0$, these indentities, as the time-independent parts of the Bianchi identities, become the off-shell restrictions in the sense of the Hamiltonian analysis. On the other hand, choosing the free index to be a spatial coordinate, one obtains time-dependent pieces of the Bianchi identities, which do not enforce any restrictions, but can instead be derived as a consequence of the Hamiltonian equations of motion.

There are also GBI associated with the 2-form fields B^{ab}, B and β^a. The corresponding 3-form curvatures are given by

$$S^{ab} = \mathrm{d}B^{ab} + 2\omega^{[a|}{}_c \wedge B^{c\,|b]}\,, \qquad P = \mathrm{d}B\,, \qquad G^a = \mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b\,. \tag{40}$$

Differentiating these expressions, one obtains the following GBI:

$$\epsilon^{\lambda\mu\nu\rho}\left(\frac{1}{3}\nabla_\lambda\, S^{ab}{}_{\mu\nu\rho} - R^{[a|\,c}{}_{\lambda\mu}\, B_c{}^{|b]}{}_{\nu\rho}\right) = 0\,, \tag{41}$$

$$\epsilon^{\lambda\mu\nu\rho}\partial_\lambda\, P_{\mu\nu\rho} = 0\,, \tag{42}$$

$$\epsilon^{\lambda\mu\nu\rho}\left(\frac{2}{3}\nabla_\lambda\, G^a{}_{\mu\nu\rho} - R^{ab}{}_{\lambda\mu}\,\beta_{b\,\nu\rho}\right) = 0\,. \tag{43}$$

However, in four-dimensional spacetime, these identities will be single-component equations, with no free spacetime indices, and therefore necessarily feature time derivatives of the fields. Thus, they do not impose any off-shell restictions on the canonical variables.

Finally, there is also GBI associated with the 0-form ϕ. The corresponding 1-form curvature is:

$$Q^A = \mathrm{d}\phi^A + \rhd_B{}^A\, A \wedge \phi^B\,, \tag{44}$$

so that the GBI associated with this curvature is:

$$\epsilon^{\lambda\mu\nu\rho}\left(\nabla_\nu Q^A{}_\rho - \frac{1}{2}\rhd_B{}^A\, F_{\nu\rho}\phi^B\right) = 0. \tag{45}$$

This GBI consists of 12 component equations, corresponding to six possible choices of the free antisymmetrized spacetime indices $\lambda\mu$, and the 2 possible choices of the free group index A. However, not all of these 12 identities are independent. This can be seen by taking the derivative of the Equation (45) and obtaining eight identities of the form

$$\rhd_B{}^A\,\epsilon^{\lambda\mu\nu\rho}\,\partial_\mu\, F_{\nu\rho}\,\phi^B = 0\,, \tag{46}$$

which are automatically satisfied because of the GBI (39). One concludes there are only four independent identities (45). Now, fixing the value $\lambda = 0$, one obtains the time-independent components of both Equations (45) and (46),

$$\epsilon^{0ijk}\left(\nabla_j Q^A{}_k - \frac{1}{2}\rhd_B{}^A\, F_{jk}\phi^B\right) = 0\,, \tag{47}$$

and

$$\rhd_B{}^A\,\epsilon^{0ijk}\,\partial_i\, F_{jk}\,\phi^B = 0\,. \tag{48}$$

Of these, there are six components in Equation (47), but, because of the two components of Equation (48), there are overall only four independent GBI relevant for the Hamiltonian analysis.

5. Number of Degrees of Freedom

Let us now show that the structure of the constraints implies that there are no local degrees of freedom (DoF) in a 3BF theory. In the general case, if there are N initial fields in the theory and there are F independent first-class constraints per space point and S independent second-class constraints per space point, then the number of local DoF, i.e., the number of independent field components, is given by

$$n = N - F - \frac{S}{2}\,. \tag{49}$$

Equation (49) is a consequence of the fact that S second-class constraints are equivalent to vanishing of $S/2$ canonical coordinates and $S/2$ of their momenta. The F first-class constraints are equivalent to vanishing of F canonical coordinates, and since the first-class constraints generate the gauge symmetries, we can impose F gauge-fixing conditions for the corresponding F canonical momenta. Consequently, there are $2N - 2F - S$ independent canonical coordinates and momenta and therefore $2n = 2N - 2F - S$, giving rise to Equation (49).

In our case, N can be determined from the Table 1, giving rise to a total of $N = 120$ canonical coordinates. Similarly, the number of independent components for the second class constraints is determined by the Table 2, so that $S = 70$.

Table 1. The number of components for all fields present in the theory.

$\omega^{ab}{}_{\mu}$	A_{μ}	$\beta^{a}{}_{\mu\nu}$	$\gamma^{A}{}_{\mu\nu\rho}$	$B^{ab}{}_{\mu\nu}$	$B_{\mu\nu}$	$e^{a}{}_{\mu}$	ϕ^{A}
24	4	24	8	36	6	16	2

Table 2. The number of components for the second class constraints present in the theory.

$\chi(B)_{ab}{}^{jk}$	$\chi(B)^{jk}$	$\chi(e)_{a}{}^{i}$	$\chi(\phi)_{A}$	$\chi(\omega)_{ab}{}^{i}$	$\chi(A)^{i}$	$\chi(\beta)_{a}{}^{ij}$	$\chi(\gamma)_{A}{}^{ijk}$
18	3	12	2	18	3	12	2

The first-class constraints are not all independent because of BI and GBI. To see that, take the derivative of $\Phi(R)^{abi}$ to obtain

$$\nabla_i \Phi(R)^{abi} = \varepsilon^{0ijk} \nabla_i R^{ab}{}_{jk} + \frac{1}{2} R^{c[a|}{}_{ij} P(B)_c{}^{|b]ij} \,. \tag{50}$$

The first term on the right-hand side is zero off-shell because $\epsilon^{ijk}\, \nabla_i R^{ab}{}_{jk} = 0$, which is a $\lambda = 0$ component of the BI (37). The second term on the right-hand side is also zero off-shell, since it is a product of two constraints,

$$R^{c[a|}{}_{ij}\, P(B)_c{}^{|b]ij} \equiv \frac{1}{2} \epsilon_{0ijk} \mathcal{S}(R)^{c[a|k}\, P(B)_c{}^{|b]ij} = 0 \,. \tag{51}$$

Therefore, we have the off-shell identity

$$\nabla_i \Phi(R)^{abi} = 0 \,, \tag{52}$$

which means that six components of $\Phi(R)^{abi}$ are not independent of the others. In an analogous fashion, taking the derivative of $\Phi(F)^i$, one obtains

$$\partial_i \Phi(F)^i = \varepsilon^{0ijk}\, \partial_i F_{jk} + \frac{1}{2}\, F_{ij}\, P(B)^{ij} \,. \tag{53}$$

The first term on the right-hand side is zero off-shell because $\epsilon^{ijk}\, \partial_i F_{jk} = 0$, which is a $\lambda = 0$ component of the GBI (37). The second term on the right-hand side is also zero off-shell, since it is a product of two constraints,

$$F_{ij}\, P(B)^{ij} \equiv \frac{1}{2} \epsilon_{0ijk}\, \mathcal{S}(F)^k\, P(B)^{ij} = 0 \,. \tag{54}$$

Therefore, we have the off-shell identity

$$\partial_i \Phi(F)^i = 0 \,, \tag{55}$$

which means that one component of $\Phi(F)^i$ is not independent of the others. Similarly, one can demonstrate that

$$\nabla_i\Phi(\nabla e)_a{}^i - \frac{1}{2}\,\Phi(R)_{ab}{}^i\,e^b{}_i + \frac{1}{4}\epsilon^{0ijk}\mathcal{S}(R)_{abk}\,P(\beta)^b{}_{ij} = \frac{1}{2}\epsilon^{0ijk}\left(\nabla_i T_{ajk} - R_{ab\,ij}\,e^b{}_k\right) . \tag{56}$$

The right-hand side of the Equation (56) is the $\lambda = 0$ component of the BI (38), so that Equation (56) gives the relation:

$$\nabla_i\Phi(\nabla e)_a{}^i - \frac{1}{2}\,\Phi(R)_{ab}{}^i\,e^b{}_i = 0\,, \tag{57}$$

where we have omitted the term that is the product of two constraints. This relation means that four components of the constraints $\Phi(\nabla e)_a{}^i$ and $\Phi(R)_{ab}{}^i$ can be expressed in terms of the rest. Finally, one can also demonstrate that

$$\begin{aligned}\nabla_i\Phi(\nabla\phi)_A{}^{ij} &- \frac{1}{2}\epsilon_{0ikl}\,\rhd_A\;\mathcal{S}(F)^l\,\chi(\gamma)_B{}^{ijk} + \rhd^B{}_A\,\phi_B\,\Phi(F)^j \\ &+ \frac{1}{2}\epsilon_{0ilm}\,\rhd^B{}_A\,P(B)^{ij}\,\mathcal{S}(\nabla\phi)_B{}^{lm} = \epsilon^{0ijk}\left(\nabla_i Q_{Ak} + \frac{1}{2}\rhd^B{}_A\,F_{ik}\,\phi_B\right) ,\end{aligned} \tag{58}$$

which gives

$$\nabla_i\Phi(\nabla\phi)_A{}^{ij} + \frac{1}{2}\rhd^B{}_A\,\phi_B\,\Phi(F)^j = 0\,, \tag{59}$$

for $\lambda = 0$ component of the GBI (45), where we have again used that the product of two contraints is zero off-shell. This relation suggests that six components of two first-class constraints, $\Phi(\nabla\phi)_A{}^{ij}$ and $\Phi(F)^j$, are not independent of the others. However, in the previous section, we have discussed that only four of these six identities are mutually independent, which means that we have only four independent identities (59). A rigorous proof of this statement entails the evaluation of the corresponding Wronskian, and is left for future work.

Taking into account all of the above indentites (52), (55), (57), and (59), we can finally evaluate the total number of independent first-class constraints. From the Table 3, one can see that the total number of components of the first-class constraints is given by $F^* = 100$. However, the number of independent components of the first-class constraints is $F = 85$, obtained by subtracting the six relations (52), one relation (55), four relations (57) and four relations (59).

Table 3. The number of components for the first class constraints present in the theory. The identities (52), (55), (57), and (59) reduce the number of components which are independent. This reduction is explicitly denoted in the table.

$\Phi(B)_{ab}{}^i$	$\Phi(B)^i$	$\Phi(e)_a$	$\Phi(\omega)_{ab}$	$\Phi(A)$	$\Phi(\beta)_a{}^i$	$\Phi(\gamma)_A{}^{ij}$	$\Phi(R)_{ab}{}^i$	$\Phi(F)^i$	$\Phi(\mathcal{G})_a$	$\Phi(\nabla e)_a{}^i$	$\Phi(\nabla B)_{ab}$	$\Phi(\nabla B)$	$\Phi(\nabla\phi)_A{}^{ij}$
18	3	4	6	1	12	6	$18-6$	$3-1$	4	$12-4$	6	1	$6-4$

Therefore, substituting all the obtained results into Equation (49), one gets

$$n = 120 - 85 - \frac{70}{2} = 0, \tag{60}$$

which means that there are no propagating DoF in a $3BF$ theory described by the action (10).

6. Generator of the Gauge Symmetry

Based on the results of the Hamiltonian analysis of the action (10), it can also be interesting to calculate the generator of the complete gauge symmetry of the action. The gauge generator of the theory is obtained by using the Castellani's procedure (see Chapter V in [13] for details of the procedure), and one gets the following result (see Appendix B for details of the calculation):

$$
\begin{aligned}
G \;=\; \int_{\Sigma_3} \mathrm{d}^3\vec{x}\bigg(& \frac{1}{2}(\nabla_0\epsilon^{ab}{}_i)\Phi(B)_{ab}{}^i - \frac{1}{2}\epsilon^{ab}{}_i\Phi(R)_{ab}{}^i + \frac{1}{2}(\nabla_0\epsilon^{ab})\Phi(\omega)_{ab} - \frac{1}{2}\epsilon^{ab}\Phi(\nabla B)_{ab} \\
& + (\partial_0\epsilon_i)\Phi(B)^i - \epsilon_i\Phi(F)^i + (\partial_0\epsilon)\Phi(A) - \epsilon\Phi(\nabla B) \\
& + (\nabla_0\epsilon^a)\Phi(e)_a - \epsilon^a\Phi(\mathcal{G})_a + (\nabla_0\epsilon^a{}_i)\Phi(\beta)_a{}^i - \epsilon^a{}_i\Phi(\nabla e)_a{}^i \\
& + \frac{1}{2}(\nabla_0\epsilon^A{}_{ij})\Phi(\gamma)_A{}^{ij} - \frac{1}{2}\epsilon^A{}_{ij}\Phi(\nabla\phi)_A{}^{ij} \\
& + \epsilon^{ab}\left(\beta_{[a|0i}P(\beta)_{|b]}{}^i + e_{[a|0}P(e)_{|b]} + B_{[a|c0i}P(B)^c{}_{|b]}{}^i\right) - \epsilon\,\gamma_{A0ij} \rhd_B{}^A P(\gamma)^{Bij} \\
& + \epsilon^a\beta_{b0i}P(B)^{abi} + \epsilon^a{}_i\, e_{b0}P(B)_a{}^{bi}\bigg) .
\end{aligned}
\tag{61}
$$

Here, $\epsilon^{ab}{}_i$, ϵ^{ab}, ϵ_i, ϵ, ϵ^a, $\epsilon^a{}_i$ and $\epsilon^A{}_{ij}$ are the independent parameters of the gauge transformations.

Furthermore, one can employ the gauge generator to calculate the form-variations for all canonical coordinates and their corresponding momenta, by computing the Poisson bracket of the chosen variable $A(t,\vec{x})$ and the generator (61):

$$
\delta_0 A(t,\vec{x}) = \{A(t,\vec{x})\,, G\}\,. \tag{62}
$$

The results are given as follows:

$$
\begin{aligned}
\delta_0\omega^{ab}{}_0 &= \nabla_0\epsilon^{ab}\,, & \delta_0\pi(\omega)_{ab}{}^0 &= -2\epsilon_{[a|}{}^c{}_i\pi(B)_{c|b]}{}^{0i} - 2\epsilon_{[a|}{}^c\pi(\omega)_{c|b]}{}^0\,, \\
& & &\quad +2\epsilon_{[a|}\pi(e)_{|b]}{}^0 + 2\epsilon_{[a|i}\pi(\beta)_{|b]}{}^{0i}\,, \\
\delta_0\omega^{ab}{}_i &= \nabla_i\epsilon^{ab}\,, & \delta_0\pi(\omega)_{ab}{}^i &= -2\epsilon_{[a|}{}^c{}_j\,\pi(B)_{c|b]}{}^{ij} - 2\epsilon_{[a|}{}^c{}_i\,\pi(\omega)_{|b]c}{}^i \\
& & &\quad +2\epsilon_{[a|}\,\pi(e)_{|b]i} + 2\epsilon_{[a|j}\pi(\beta)_{|b]}{}^{ij} \\
& & &\quad +2\epsilon^{0ijk}\,\nabla_{[j|}\epsilon_{ab|k]} + \epsilon^{0ijk}\epsilon_{[a|}\beta_{|b]jk}\,, \\
\delta_0 B^{ab}{}_{0i} &= \nabla_0\epsilon^{ab}{}_i + \epsilon^{[a|}{}_i e^{|b]}{}_0 & \delta_0\pi(B)_{ab}{}^{0i} &= 2\epsilon_{[a|c}\,\pi(B)_{|b]}{}^{ci}\,, \\
&\quad +2\epsilon^{[a|c}B^{|b]}{}_{c0i} + \epsilon^{[a|}\beta^{|b]}{}_{0i}\,, & & \\
\delta_0 B^{ab}{}_{ij} &= 2\nabla_{[i|}\epsilon^{ab}{}_{|j]} + 2\epsilon^{[a|c}B^{|b]}{}_{cij} & \delta_0\pi(B)_{ab}{}^{ij} &= 2\epsilon_{[a|c}\,\pi(B)_{|b]}{}^{cij}\,, \\
&\quad +2\epsilon^{[a|}{}_{[i}e^{|b]}{}_{j]} + \epsilon^{[a|}\beta^{|b]}{}_{ij}\,, & & \\
\delta_0 A_0 &= \partial_0\epsilon\,, & \delta_0\pi(A)^0 &= -\tfrac{1}{2}\epsilon^A{}_{ij} \rhd^B{}_A\,\pi(\gamma)_B{}^{0ij}\,, \\
\delta_0 A_i &= \partial_i\epsilon\,, & \delta_0\pi(A)^i &= \epsilon^{0ijk}\partial_j\epsilon_k - \tfrac{1}{2}\epsilon^A{}_{jk} \rhd^B{}_A\,\pi(\gamma)_B{}^{ijk}\,, \\
\delta_0 B_{0i} &= \partial_0\epsilon_i\,, & \delta_0\pi(B)^{0i} &= 0\,, \\
\delta_0 B_{ij} &= 2\,\partial_{[i|}\epsilon_{|j]} + \epsilon^A{}_{ij} \rhd^B{}_A\,\phi_B\,, & \delta_0\pi(B)^{ij} &= -\epsilon^{0ijk}\partial_k\epsilon\,, \\
\delta_0\beta^a{}_{0i} &= \nabla_0\epsilon^a{}_i - \epsilon^{ab}\beta_{b0i}\,, & \delta_0\pi(\beta)_a{}^{0i} &= -\epsilon_{ab}\pi(\beta)^{b0i} + \tfrac{1}{2}\epsilon^b\pi(B)_{ab}{}^{0i}\,, \\
\delta_0\beta^a{}_{ij} &= 2\nabla_{[i|}\epsilon^a{}_{|j]} - \epsilon^{ab}\,\beta_{bij}\,, & \delta_0\pi(\beta)_a{}^{ij} &= -\epsilon_{ab}\,\pi(\beta)^{bij} + \tfrac{1}{2}\epsilon^b\,\pi(B)_{ab}{}^{ij} \\
& & &\quad -\epsilon^{0ijk}\,\nabla_k\epsilon^a\,, \\
\delta_0 e^a{}_0 &= \nabla_0\epsilon^a - \epsilon^{ab}\,e_{b0}\,, & \delta_0\pi(e)_a{}^0 &= -\epsilon_{ab}\,\pi(e)^{b0} + \tfrac{1}{2}\epsilon^b{}_i\,\pi(B)_{ab}{}^{0i}\,, \\
\delta_0 e^a{}_i &= \nabla_i\epsilon^a - \epsilon^{ab}\,e_{bi}\,, & \delta_0\pi(e)_a{}^i &= -\epsilon_{ab}\,\pi(e)^{bi} + \epsilon^{0ijk}\Big(\nabla_{[j|}\epsilon_{a|k]} + \epsilon_{ab}\beta^{bjk}\Big) \\
& & &\quad +\tfrac{1}{2}\epsilon^b{}_j\,\pi(B)_{ab}{}^{ij}\,,
\end{aligned}
$$

$$
\begin{array}{rclrcl}
\delta_0\gamma^A{}_{0ij} & = & \nabla_0\epsilon^A{}_{ij} - \epsilon\,\gamma^B{}_{0ij} \rhd^A{}_B\,, & \delta_0\pi(\gamma)_A{}^{0ij} & = & \epsilon \rhd^B{}_A\,\pi(\gamma)_B{}^{0ij}\,, \\
\delta_0\gamma^A{}_{ijk} & = & -\,\epsilon\,\gamma^B{}_{ijk} \rhd_B{}^A + \nabla_i\epsilon^A{}_{jk} & \delta_0\pi(\gamma)_A{}^{ijk} & = & \epsilon \rhd_A{}^B \left(\pi(\gamma)_B{}^{ijk} + \epsilon^{0ijk}\,\phi_B\right)\,, \\
 & & -\nabla_j\epsilon^A{}_{ik} + \nabla_k\epsilon^A{}_{ij}\,, & & & \\
\delta_0\phi^A & = & \epsilon\,\phi^B \rhd^A{}_B\,, & \delta_0\pi(\phi)_A & = & -\epsilon \rhd^B{}_A\,\pi(\phi)_B + \frac{1}{3!}\,\epsilon\,\epsilon^{0ijk} \rhd^B{}_A\,\gamma_{Bijk} \\
 & & & & & -\frac{1}{2} \rhd_{A\,B}\,\epsilon^B{}_{ij}\,\pi(B)^{ij} - \frac{1}{2}\epsilon^{0ijk}\,\nabla_i\epsilon^A{}_{jk}\,,
\end{array}
\tag{63}
$$

These transformations are an extension of the form-variations in the case of the Poincaré 2-group obtained in [17].

7. Conclusions

Let us summarize the results of the paper. In Section 2, we have demonstated in detail how to use the idea of a categorical ladder to introduce the 3-group structure corresponding to the theory of scalar electrodynamics coupled to Einstein–Cartan gravity. We have introduced the topological $3BF$ action corresponding to this choice of a 3-group, as well as the constrained $3BF$ action which gives rise to the standard equations of motion for the scalar electrodynamics. In order to perform the canonical quantization of this theory, the complete Hamiltonian analysis of the full theory with constraints has to be performed, but the important step towards this goal is the Hamiltonian analysis of the topological $3BF$ action. This has been done in Section 3. Here, the first-class and second-class constraints of the theory, as well as their Poisson brackets, have been obtained. In Section 4, we have discussed the Bianchi identities and also the generalized Bianchi identities, since they enforce restrictions in the sense of Hamiltonian analysis, and reduce the number of independent first-class constraints present in the theory. With this background material in hand, in Section 5, the counting of the dynamical degrees of freedom present in the theory has been performed and it was established that the considered $3BF$ action is a topological theory, i.e., the diffeomorphism invariant theory without any propagating degrees of freedom. In Section 6, we have constructed the generator of the gauge symmetries for the theory, and we found the form-variations for all the variables and their canonical momenta.

The results obtained in this paper represent the straightforward generalization of Hamiltonian analysis done in [15] for the Poincaré 2-group, and a first example of the Hamiltonian analysis of a $3BF$ action. The fact that the theory was found to be topological is nontrivial, since it relies on the existence of the generalized Bianchi identities, which have been identified for the first time. In addition to that, it was demonstrated that the algebra of constraint closes, which is an important consistency check for the theory. There is another very interesting aspect of the constraint algebra. Namely, one can recognize, looking at the structure of Equations (34) that the subalgebra generated by the first-class constraint $\Phi(\nabla\phi)_A{}^{ij}$ is in fact an *ideal* of the constraint algebra because the Poisson bracket between this constraint and all other constraints is again proportional to that constraint. It is curious that precisely the constraint $\Phi(\nabla\phi)_A{}^{ij}$ is the only one related to the Lie group L from the 3-group, according to its index structure, and also that the structure constant of the ideal is determined by the action $\rhd$ of the group G on L. Let us also note that the action $\rhd$ appears as well in the structure constants of the algebra between the first-class and second-class constraints.

The results of this work open several avenues for future research. From the point of view of mathematics, the relationship between the algebraic structures mentioned above should be understood in more detail. More generally, one should understand the correspondence between the gauge group generated by the generator (61) and the 3-group structure used to define the theory. This is not viable in the special case of the 3-group discussed in this work, but instead needs to be done in the case of a generic 3-group, where homomorphisms δ and ∂ and the Peiffer lifting $\{\_,\_\}$ are nontrivial. From the point of view of physics, the obtained results represent the fundamental building blocks for the construction of the quantum theory of scalar electrodynamics coupled to gravity, as well as a convenient model to discuss before proceeding to the Hamiltonian analysis and canonical quantization of the full Standard Model coupled to gravity, formulated as a $3BF$ action with suitable constraints [8]. Both the Hamiltonian analysis of constrained $3BF$ models and the corresponding canonical quantization programme need to be further developed in order to achieve these goals. Our work is a first step in this direction.

Finally, let us note in the end that the above list of topics for future research is by no means complete, and there are potentially many other interesting topics that can be studied in this context.

Author Contributions: Investigation, T.R. and M.V; methodology, T.R. and M.V.; writing–original draft preparation, T.R.; writing–review and editing, M.V. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the project ON171031 of the Ministry of Education, Science and Technological Development (MPNTR) of the Republic of Serbia, and partially by the bilateral scientific cooperation between Austria and Serbia through the project "Causality in Quantum Mechanics and Quantum Gravity-2018-2019", No. 451-03-02141/2017-09/02, supported by the Federal Ministry of Science, Research and Economy (BMWFW) of the Republic of Austria, and the Ministry of Education, Science and Technological Development (MPNTR) of the Republic of Serbia.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

LQG	Loop Quantum Gravity
BI	Bianchi Identities
GBI	Generalized Bianchi Identities
DoF	Degrees of Freedom
PB	Poisson Bracket

Appendix A. The Equations of Motion for the Scalar Electrodynamics

The action of scalar electrodynamics coupled to Einstein–Cartan gravity is given in the form (12):

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B \wedge F + e_a \wedge \nabla \beta^a + \phi_A \, \nabla \gamma^A \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda^A \wedge \left(\gamma_A - \frac{1}{2} H_{abcA} e^a \wedge e^b \wedge e^c \right) + \Lambda^{abA} \wedge \left(H_{abcA} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \nabla \phi_A \wedge e_a \wedge e_b \right) \\ & + \lambda \wedge \left(B - \frac{12}{q} M_{ab} e^a \wedge e^b \right) + \zeta^{ab} \left(M_{ab} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b \right) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi_A \, \phi^A \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,. \end{aligned} \tag{A1}$$

Varying the total action (12) with respect to the variables B_{ab}, B, ω_{ab}, β_a, λ_{ab}, Λ^{abA}, γ^A, λ^A, H_{abcA}, ζ^{ab}, M_{ab}, λ, A, ϕ^A and e^a, one obtains the equations of motion:

$$R^{ab} - \lambda^{ab} = 0 \,, \tag{A2}$$

$$F + \lambda = 0 \,, \tag{A3}$$

$$\nabla B^{ab} - e^{[a|} \wedge \beta^{|b]} = 0 \,, \tag{A4}$$

$$\nabla e^a = 0 \,, \tag{A5}$$

$$B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d = 0 \,, \tag{A6}$$

$$H_{abcA}\varepsilon^{cdef}e_d \wedge e_e \wedge e_f - \nabla\phi_A \wedge e_a \wedge e_b = 0\,, \tag{A7}$$

$$\nabla\phi_A - \lambda_A = 0\,, \tag{A8}$$

$$\gamma_A - \frac{1}{2}H_{abcA}e^a \wedge e^b \wedge e^c = 0\,, \tag{A9}$$

$$-\frac{1}{2}\lambda^A \wedge e^a \wedge e^b \wedge e^c + \varepsilon^{cdef}\Lambda^{abA} \wedge e_d \wedge e_e \wedge e_f = 0\,, \tag{A10}$$

$$M_{ab}\varepsilon_{cdef}e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b = 0\,, \tag{A11}$$

$$-\frac{12}{q}\lambda \wedge e^a \wedge e^b + \zeta^{ab}\varepsilon_{cdef}e^c \wedge e^d \wedge e^e \wedge e^f = 0\,, \tag{A12}$$

$$B - \frac{12}{g}M_{ab}e^a \wedge e^b = 0\,, \tag{A13}$$

$$-\,\mathrm{d}B + \mathrm{d}(\zeta^{ab}e_a \wedge e_b) - \phi_A \rhd_B{}^A\gamma^B - \Lambda^{abA} \rhd^B{}_A\,\phi_B \wedge e_a \wedge e_b = 0\,, \tag{A14}$$

$$\nabla\gamma_A - \nabla(\Lambda^{ab}{}_A \wedge e_a \wedge e_b) - \frac{1}{4!}m^2\,\phi_A\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d = 0\,, \tag{A15}$$

$$\begin{aligned}
&\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{3}{2}H_{abcA}\lambda^A \wedge e^b \wedge e^c + 3H^{defA}\varepsilon_{abcd}\Lambda_{efA} \wedge e^b \wedge e^c \\
&- 2\Lambda_{abA} \wedge \nabla\phi^A \wedge e^b - 2\frac{1}{4!}m^2\phi_A\;\phi^A\varepsilon_{abcd}e^b \wedge e^c \wedge e^d \\
&- \frac{24}{q}M_{ab}\lambda \wedge e^b + 4\zeta^{ef}M_{ef}\varepsilon_{abcd}e^b \wedge e^c \wedge e^d - 2\zeta_{ab}F \wedge e^b = 0\,.
\end{aligned} \tag{A16}$$

The dynamical degrees of freedom are the tetrad fields e^a, the scalar field ϕ^A, and the electromagnetic potential A, while the remaining variables are algebraically determined in terms of them. Specifically, Equations (A2)–(A13) give

$$\begin{gathered}
\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,, \qquad \gamma^A{}_{\mu\nu\rho} = -\frac{1}{2e}\varepsilon_{\mu\nu\rho\sigma}\,\nabla^\sigma\phi^A\,, \\
\Lambda^{abA}{}_\mu = \frac{1}{12e}g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\nabla_\nu\phi^A\,e^a{}_\rho e^b{}_\sigma\,, \qquad \beta^a{}_{\mu\nu} = 0\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \\
H^{abcA} = \frac{1}{6e}\varepsilon^{\mu\nu\rho\sigma}\,\nabla_\mu\phi^A\,e^a{}_\nu e^b{}_\rho e^c{}_\sigma\,, \qquad \lambda^A{}_\mu = \nabla_\mu\phi^A\,, \\
\lambda_{\mu\nu} = F_{\mu\nu}\,, \quad B_{\mu\nu} = -\frac{1}{2eq}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,, \\
M^{ab} = -\frac{1}{4e}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,e^a{}_\rho e^b{}_\sigma\,, \quad \zeta^{ab} = \frac{1}{4eq}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,e^a{}_\rho e^b{}_\sigma\,.
\end{gathered} \tag{A17}$$

Note that from the Equations (A4)–(A6) it follows that $\beta^a = 0$, as in the pure gravity case. The equation of motion (A15) reduces to the covariant Klein–Gordon equation for the scalar field coupled to the electromagnetic potential A,

$$\left(\nabla_\mu\nabla^\mu - m^2\right)\phi_A = 0\,. \tag{A18}$$

From Equation (A14), we obtain the differential equation of motion for the field A:

$$\nabla_\mu F^{\mu\nu} = j^\nu\,, \qquad j^\mu \equiv \frac{1}{2}\left(\nabla^\nu\phi^A \rhd^B{}_A\phi_B - \phi_A \rhd_B{}^A\nabla^\nu\phi^B\right) = iq\left(\nabla\phi^*\,\phi - \phi^*\nabla\phi\right). \tag{A19}$$

Finally, the equation of motion (A16) for e^a becomes:

$$\begin{aligned} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2 \, T^{\mu\nu} \,, \\ T^{\mu\nu} \equiv \nabla^\mu \phi_A \, \nabla^\nu \phi^A - \frac{1}{2} g^{\mu\nu} \left(\nabla_\rho \phi_A \, \nabla^\rho \phi^A + m^2 \phi_A \, \phi^A \right) - \frac{1}{4q} \left(F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} + 4 F^{\mu\rho} F_\rho{}^\nu \right) \,. \end{aligned} \tag{A20}$$

The system of Equations (A2)–(A16) is equivalent to the system of Equations (A17)–(A20).

Appendix B. The Calculation of the Gauge Generator

The gauge generator of the theory is obtained by the standard Castellani procedure (see [13] for an introduction). One starts from the generic form for the generator,

$$\begin{aligned} G \;=\; & \int_{\Sigma_3} \partial^3 \vec{x} \Big(\frac{1}{2} (\partial_0 \epsilon^{ab}{}_i) G_{1ab}{}^i + \frac{1}{2} \epsilon^{ab}{}_i G_{0ab}{}^i + \frac{1}{2} (\partial_0 \epsilon^{ab}) G_{1ab} + \frac{1}{2} \epsilon^{ab} G_{0ab} \\ & + (\partial_0 \epsilon_i) G_1{}^i + \epsilon_i G_0{}^i + (\partial_0 \epsilon) G_1 + \epsilon G_0 \\ & + (\partial_0 \epsilon^a) G_{1a} + \epsilon^a G_{0a} + (\partial_0 \epsilon^a{}_i) G_{1a}{}^i + \epsilon^a{}_i G_{0a}{}^i \\ & + \frac{1}{2} (\partial_0 \epsilon^A{}_{ij}) G_{1A}{}^{ij} + \frac{1}{2} \epsilon^A{}_{ij} G_{0A}{}^{ij} \Big) \,, \end{aligned} \tag{A21}$$

where the generators G_0 and G_1 are obtained by the standard prescription [13]:

$$\begin{aligned} G_1 = C_{PFC} \,, \\ G_0 + \{ \, G_1 \,, \, H_T \, \} = C_{PFC} \,, \\ \{ \, G_0 \,, \, H_T \, \} = C_{PFC} \,, \end{aligned} \tag{A22}$$

where C_{PFC} is a primary first-class constraint. For example, one choses $G_{1ab}{}^i = \Phi(B)_{ab}{}^i$. From the conditions

$$\begin{aligned} G_{0ab}{}^i + \{ \, \Phi(B)_{ab}{}^i \,, \, H_T \, \} = G_{0ab}{}^i + \Phi(R)_{ab}{}^i = C_{PFC} \,, \\ \{ \, G_{0ab}{}^i \,, \, H_T \, \} = C_{PFC}{}^* = \{ \, C_{PFC} - \Phi(R)_{ab}{}^i \,, \, H_T \, \} \,, \end{aligned} \tag{A23}$$

we solve for $G_{0ab}{}^i$ by determining C_{PFC} from the second equation. Evaluating one PB, one can reexpress the second equation in the form:

$$\{ \, C_{PFC} \,, \, H_T \, \} = C_{PFC}{}^* + 2\omega_{[a|}{}^d{}_0 \Phi(R)_{|b]d}{}^i = \{ \, 2\omega_{[a|}{}^d{}_0 P(B)_{|b]d}{}^i \,, \, H_T \, \} \,. \tag{A24}$$

From the second equality, we recognize that

$$C_{PFC} = 2\omega_{[a|}{}^d{}_0 P(B)_{|b]d}{}^i \,, \tag{A25}$$

which can then be substituted into the first condition above, giving

$$G_{0ab}{}^i = 2\omega_{[a|}{}^d{}_0 \Phi(B)_{|b]d}{}^i - \Phi(R)_{ab}{}^i \,. \tag{A26}$$

One thus obtains

$$\frac{1}{2} (\partial_0 \epsilon^{ab}{}_i) (G_1)_{ab}{}^i + \frac{1}{2} \epsilon^{ab}{}_i G_{0ab}{}^i = \frac{1}{2} \nabla_0 \epsilon^{ab}{}_i \Phi(B)_{ab}{}^i - \frac{1}{2} \epsilon^{ab}{}_i \Phi(R)_{ab}{}^i \,.$$

The other G_0 and G_1 terms are obtained in a similar way, and the generator (61) is derived.

Appendix C. Introduction to 3-Groups

The notion of a 3-*group* is usually introduced in the framework of higher category theory [6]. In category theory, every *group* can be understood as a *category* which has only one element, and morphisms which are all invertible. The group elements are then individual morphisms that map the category element to itself, while the group operation is the categorical composition of the morphisms. In such a case, the axioms of the category guarantee the validity of all axioms of a group. This kind of construction can be generalized to 2-*groups,* 3-*groups* and, in general, *n-groups*. Namely, a 2-group is by definition a 2-category which has only one element, and whose morphisms and 2-morhisms (i.e., morphisms between morphisms) are invertible. Similarly, a 3-group is by definition a 3-category which has only one element, while its morphisms, 2-morphisms, and 3-morphisms are invertible.

The above definition of a 3-group is very abstract, and while theoretically very important, in itself not very useful for practical calculations and applications in physics. Fortunately, there is a theorem of equivalence between 3-groups and the so-called 2-*crossed modules*, which are algebraic structures with more familiar properties [9,10]. For the applications in physics, attention focuses on the so-called *strict Lie* 3-*groups*, and their corresponding differential (Lie algebra) structure, which corresponds to the *differential Lie* 2-*crossed module*. Let us therefore give a brief overview of the latter.

A *differential Lie* 2-*crossed module* $(\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}, \triangleright, \{\_\,,\_\})$ is given by three Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, maps $\delta : \mathfrak{l} \to \mathfrak{h}$ and ∂: $\mathfrak{h} \to \mathfrak{g}$, together with a map called the Peiffer lifting,

$$\{\_\,,\_\} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{l}\,, \tag{A27}$$

and an action $\triangleright$ of the algebra $\mathfrak{g}$ on all three algebras.

Let us introduce the bases in the three algebras, $\tau_\alpha \in \mathfrak{g}$, $t_a \in \mathfrak{h}$ and $T_A \in \mathfrak{l}$, and structure constants in those bases, as follows:

$$[\tau_\alpha\,,\tau_\beta] = f_{\alpha\beta}{}^{\gamma}\tau_\gamma\,, \quad [t_a\,,t_b] = f_{ab}{}^{c}t_c\,, \quad [T_A\,T_B] = f_{AB}{}^{C}T_C\,. \tag{A28}$$

Now, the maps ∂ and δ can be written as

$$\partial(t_a) = \partial_a{}^{\alpha}\,\tau_\alpha\,, \qquad \delta(T_A) = \delta_A{}^{a}\,t_a\,, \tag{A29}$$

and the action of the algebra $\mathfrak{g}$ on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ as:

$$\tau_\alpha \triangleright \tau_\beta = \triangleright_{\alpha\beta}{}^{\gamma}\,\tau_\gamma\,, \quad \tau_\alpha \triangleright t_a = \triangleright_{\alpha a}{}^{b}\,t_b\,, \quad \tau_\alpha \triangleright T_A = \triangleright_{\alpha A}{}^{B}\,T_B\,. \tag{A30}$$

Finally, the Peiffer lifting can be encoded into coefficients $X_{ab}{}^{A}$ as:

$$\{t_a,\,t_b\} = X_{ab}{}^{A}\,T_A\,. \tag{A31}$$

A differential Lie 2-crossed module has the following properties (we write all equations in the abstract and their corresponding component forms, side by side):

1. The action of the algebra $\mathfrak{g}$ on itself is via the adjoint representation, i.e., $\forall g, g_1 \in \mathfrak{g}$:

$$g \triangleright g_1 = [g, g_1]\,, \qquad \triangleright_{\alpha\beta}{}^{\gamma} = f_{\alpha\beta}{}^{\gamma}\,. \tag{A32}$$

2. The action of the algebra $\mathfrak{g}$ on algebras $\mathfrak{h}$ and $\mathfrak{l}$ is $\mathfrak{g}$-equivariant, i.e., $\forall g \in \mathfrak{g}$, $h \in \mathfrak{h}$, $l \in \mathfrak{l}$:

$$\partial(g \triangleright h) = g \triangleright \partial(h)\,, \qquad \partial_a{}^{\beta}\,f_{\alpha\beta}{}^{\gamma} = \triangleright_{\alpha a}{}^{b}\,\partial_b{}^{\gamma}\,, \tag{A33}$$

$$\delta(g \triangleright l) = g \triangleright \delta(l)\,, \qquad \delta_A{}^{a}\,\triangleright_{\alpha a}{}^{b} = \triangleright_{\alpha A}{}^{B}\,\delta_B{}^{b}\,. \tag{A34}$$

3. The Peiffer lifting is a $\mathfrak{g}$-equivariant map, i.e., for every $g \in \mathfrak{g}$ and $h_1, h_2 \in \mathfrak{h}$:

$$g \triangleright \{h_1, h_2\} = \{g \triangleright h_1, h_2\} + \{h_1, g \triangleright h_2\}\,, \qquad X_{ab}{}^{B} \triangleright_{\alpha B}{}^{A} = \triangleright_{\alpha a}{}^{c} X_{cb}{}^{A} + \triangleright_{\alpha b}{}^{c} X_{ac}{}^{A}\,. \tag{A35}$$

4. For every $h_1,\, h_2 \in \mathfrak{h}$, the following identity holds:

$$\delta(\{h_1,\, h_2\}) = [h_1\,, h_2] - \partial(h_1) \triangleright h_2\,, \qquad X_{ab}{}^{A}\, \delta_A{}^{c} = f_{ab}{}^{c} - \partial_a{}^{\alpha} \triangleright_{\alpha b}{}^{c}\,. \tag{A36}$$

5. For all $l_1,\, l_2 \in \mathfrak{l}$, the following identity holds:

$$[l_1,\, l_2] = \{\delta(l_1),\, \delta(l_2)\}\,, \qquad f_{AB}{}^{C} = \delta_A{}^{a}\, \delta_B{}^{b}\, X_{ab}{}^{C}\,. \tag{A37}$$

6. For all $h_1, h_2, h_3 \in \mathfrak{h}$:

$$\begin{aligned} \{[h_1,\, h_2],\, h_3\} = &\ \partial(h_1) \triangleright \{h_2,\, h_3\} + \{h_1,\, [h_2,\, h_3]\} - \partial(h_2) \triangleright \{h_1,\, h_3\} - \{h_2,\, [h_1,\, h_3]\}\,, \\ f_{ab}{}^{d}\, X_{dc}{}^{B} = &\ \partial_a{}^{\alpha}\, X_{bc}{}^{A} \triangleright_{\alpha A}{}^{B} + X_{ad}{}^{B}\, f_{bc}{}^{d} - \partial_b{}^{\alpha} \triangleright_{\alpha A}{}^{B}\, X_{ac}{}^{A} - X_{bd}{}^{B}\, f_{ac}{}^{d}\,. \end{aligned} \tag{A38}$$

7. For all $h_1, h_2, h_3 \in \mathfrak{h}$:

$$\begin{aligned} \{h_1,\, [h_2,\, h_3]\} = &\ \{\delta\,\{h_1,\, h_2\}\,, h_3\} - \{\delta\,\{h_1,\, h_3\}\,,\, h_2\}\,, \\ X_{ad}{}^{A}\, f_{bc}{}^{d} = &\ X_{ab}{}^{B}\, \delta_B{}^{d}\, X_{dc}{}^{A} - X_{ac}{}^{B}\, \delta_B{}^{d} X_{db}{}^{A}\,. \end{aligned} \tag{A39}$$

8. For all $l \in \mathfrak{l}$ and $\forall h \in \mathfrak{h}$:

$$\{\delta(l),\, h\} + \{h,\, \delta(l)\} = -\partial(h) \triangleright l\,, \qquad 2\, \delta_A{}^{a}\, X_{\{ab\}}{}^{B} = -\partial_b{}^{\alpha} \triangleright_{\alpha A}{}^{B}\,. \tag{A40}$$

Finally, when dealing with various algebra valued differential forms, one multiplies them as differential forms using the ordinary wedge product $\wedge$, and simultaneously as algebra elements using one of maps defined above. For example, the product with an action $\wedge^{\triangleright}$ of the $\mathfrak{g}$-valued n-form ρ on the $\mathfrak{h}$-valued m-form η is defined as:

$$\begin{aligned} \rho \wedge^{\triangleright} \eta = &\ \frac{1}{n!m!}\, \rho^{\alpha}{}_{\mu_1 \dots \mu_m}\, \eta^{a}{}_{\nu_1 \dots \nu_n}\, \tau_\alpha \triangleright t_a\, \mathrm{d}x^{\mu_1} \wedge \dots \mathrm{d}x^{\mu_m} \wedge \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_n} \\ = &\ \frac{1}{n!m!}\, \rho^{\alpha}{}_{\mu_1 \dots \mu_m}\, \eta^{a}{}_{\nu_1 \dots \nu_n}\, \triangleright_{\alpha a}{}^{b} t_b\, \mathrm{d}x^{\mu_1} \wedge \dots \mathrm{d}x^{\mu_m} \wedge \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_n}\,. \end{aligned} \tag{A41}$$

References

1. Rovelli, C. *Quantum Gravity*; Cambridge University Press: Cambridge, UK, 2004.
2. Rovelli, C.; Vidotto, F. *Covariant Loop Quantum Gravity*; Cambridge University Press: Cambridge, UK, 2014.
3. Thiemann, T. *Modern Canonical Quantum General Relativity*; Cambridge University Press: Cambridge, UK, 2007.
4. Celada, M.; González, D.; Montesinos, M. BF gravity. *Class. Quant. Grav.* **2016**, *33*, 213001. [CrossRef]
5. Plebanski, J.F. On the separation of Einsteinian substructures. *J. Math. Phys.* **1977**, *12*, 2511. [CrossRef]
6. Baez, J.C.; Huerta, J. An Invitation to Higher Gauge Theory. *Gen. Relativ. Gravit.* **2011**, *43*, 2335–2392. [CrossRef]
7. Garzón, A.R.; Miranda, J.G. Serre homotopy theory in subcategories of simplicial groups. *J. Pure Appl. Algebra* **2000**, *147*, 107–123. [CrossRef]
8. Radenković, T.; Vojinović, M. Higher Gauge Theories Based on 3-groups. *J. High Energy Phys.* **2019**, *10*, 222. [CrossRef]
9. Martins, J.F.; Picken, R. The fundamental Gray 3-groupoid of a smooth manifold and local three-dimensional holonomy based on a 2-crossed module. *Differ. Geom. Appl. J.* **2011**, *29*, 179–206. [CrossRef]
10. Wang, W. On 3-gauge transformations, 3-curvature and Gray-categories. *J. Math. Phys.* **2014**, *55*, 043506. [CrossRef]

11. Miković, A.; Vojinović, M. Poincare 2-group and quantum gravity. *Class. Quant. Grav.* **2012**, *29*, 165003. [CrossRef]
12. Miković, A.; Oliveira, M.A.; Vojinović, M. Hamiltonian analysis of the BFCG formulation of General Relativity. *Class. Quantum Gravity* **2019**, *36*, 015005. [CrossRef]
13. Blagojević, M. *Gravitation and Gauge Symmetries*; Institute of Physics Publishing: Bristol, UK, 2002.
14. Miković, A.; Oliveira, M.A.; Vojinović, M. Hamiltonian analysis of the BFCG theory for a generic Lie 2-group. *arXiv* **2016**, arXiv:1610.09621.
15. Miković, A.; Oliveira, M.A.; Vojinović, M. Hamiltonian analysis of the *BFCG* theory for the Poincaré 2-group. *Class. Quantum Gravity* **2016**, *33*, 065007. [CrossRef]
16. Miković, A.; Oliveira, M.A. Canonical formulation of Poincare BFCG theory and its quantization. *Gen. Relat. Gravity* **2015**, *47*, 58. [CrossRef]
17. Oliveira, M.A. The *BFCG* Theory and Canonical Quantization of Gravity. *arXiv* **2018**, arXiv:1801.04818.

Physics AUC, vol. 30 (part II), 74-84 (2020) **PHYSICS AUC**

Quantum gravity and elementary particles from higher gauge theory

Tijana Radenković[1,*] and Marko Vojinović[1,†]

[1]Institute of Physics, University of Belgrade,

Pregrevica 118, 11080 Belgrade, Serbia

*E-mail: rtijana@ipb.ac.rs †E-mail: vmarko@ipb.ac.rs

Abstract

We give a brief overview how to couple general relativity to the Standard Model of elementary particles, within the higher gauge theory framework, suitable for the spinfoam quantization procedure. We begin by providing a short review of all relevant mathematical concepts, most notably the idea of a categorical ladder, 3-groups and generalized parallel transport. Then, we give an explicit construction of the algebraic structure which describes the full Standard Model coupled to Einstein-Cartan gravity, along with the classical action, written in the form suitable for the spinfoam quantization procedure. We emphasize the usefulness of the 3-group concept as a superior tool to describe gauge symmetry, compared to an ordinary Lie group, as well as the possibility to employ this new structure to classify matter fields and study their spectrum, including the origin of fermion families.

1 Introduction

The quantization of the gravitational field is one of the most fundamental open problems of modern theoretical physics. Since the inceptions of general relativity (GR) and quantum field theory (QFT), many attempts have been made over the years to unify the two into a self-consistent description of gravitational and matter fields as basic building blocks of nature. Some of the attempts have developed into vast research areas, such as String Theory, Loop Quantum Gravity, Causal Set Theory, and so on. One of the prominent approaches is Loop Quantum Gravity (LQG) [1, 2], which has branched into the canonical and covariant frameworks, the latter known as the *spinfoam* approach [3].

The spinfoam approach to the quantization of the gravitational field revolves around the idea of providing a precise mathematical definition to the Feynman path integral for the gravitational field,

$$Z = \int \mathcal{D}g \; e^{iS_{GR}[g]} \,,$$

where g denotes the gravitational degrees of freedom, and $S_{GR}[g]$ is the GR action expressed in terms of variables g. The strategy of defining the path integral can be roughly expressed in three main steps, called the *spinfoam quantization procedure*:

1. Choose convenient variables g and rewrite the classical action in the form

$$S_{GR}[g] = S_{\text{topological}}[g] + S_{\text{simp}}[g] \,, \tag{1}$$

where the first term represents a topological theory (with no propagating degrees of freedom), while the second term corresponds to the so-called *simplicity constraint* terms, whose purpose is to transform the full action into a realistic non-topological action with propagating degrees of freedom.

2. Employ the methods of topological quantum field theory (TQFT) to define the path integral for the topological part of the action. This is typically implemented by passing from a smooth spacetime manifold to a simplicial complex (triangulation), and writing the path integral in the form of a discrete state sum,

$$Z = \sum_g \prod_v \mathcal{A}_v(g) \prod_\epsilon \mathcal{A}_\epsilon(g) \prod_\Delta \mathcal{A}_\Delta(g) \prod_\tau \mathcal{A}_\tau(g) \prod_\sigma \mathcal{A}_\sigma(g) \,.$$

   Here g represents the gravitational field variables living on the vertices v, edges ϵ, triangles Δ, terahedra τ, and 4-simplices σ of the simplicial complex, describing its geometry, while the corresponding amplitudes $\mathcal{A}_v(g), \dots, \mathcal{A}_\sigma(g)$ are chosen to render the whole state sum Z independent of the particular choice of the triangulation of the spacetime manifold.

3. Enforce the simplicity constraints of the theory by a suitable deformation of the amplitudes $\mathcal{A}$ and the set of independent variables g, thereby obtaining a modified state sum Z which corresponds to one possible rigorous definition of the realistic gravitational path integral.

Since its inception, the spinfoam quantization procedure has been formulated and implemented for various choices of the classical action, leading to a plethora of *spinfoam models* of quantum gravity, starting from the Ponzano-Regge model for $3D$ gravity [4], and leading up to the currently most sophisticated EPRL/FK model for the realistic $4D$ case [5, 6]. However, one property common to all spinfoam models is the fact that they all describe pure gravity, without matter fields. This is due to the common choice of the classical action — it is the well known BF theory [7], which is usually defined for the Lorentz group $SO(3,1)$, with some form of the simplicity constraint terms. The prototype description of GR in this form is the Plebanski action [8]. The reason why matter fields are absent from all such models lies in the fact that the BF action does not feature tetrad fields at the fundamental level. Instead, the tetrads appear as a consequence of classical equations of motion, and are thus inherently classical, on-shell quantities. This renders the approach based on the BF theory incapable of adding matter fields at the quantum level, since matter is coupled to gravity using precisely the tetrad fields.

The issue of the absence of the tetrad fields at the fundamental level has been successfully resolved in [9], where a categorical generalization has been made, and the $2BF$ action (introduced in [10, 11]) has been employed to build an action for GR, featuring tetrads explicitly in the topological sector of the action. The categorical generalization is based on a concept of a *categorical ladder*, an abstraction scheme introducing a chain of new objects: from categories to 2-categories to 3-categories and so forth. This powerful mathematical language gave rise to the idea that the notion of gauge symmetry in physics may be described by objects other than Lie groups. The new approach is called *higher gauge theory* (HGT), see [12] for an introduction. In the context of the spinfoam quantization procedure, HGT has been successfully applied to build a quantum gravity model, based on the Poincaré 2-group [13] as a gauge symmetry structure, and the corresponding $2BF$ action, leading to the so-called *spincube model* of quantum gravity [9]. Having the tetrads as fundamental fields in the $2BF$ action, the new model could be extended to include matter fields in a straightforward way. Nevertheless, the matter field action does not have the form analogous to (1), which renders the steps 2 and 3 of the spinfoam quantization procedure moot, since they can be applied only to the gravitational sector of the theory.

Thus, a natural need appeared to generalize the theory once more, in order to include the matter fields into the topological sector of the theory, in a similar way that was done to include the tetrad fields. The basic idea was to pass from the notion of a 2-group to a notion of a 3-group as a mathematical descriptor of gauge symmetry [12, 14, 15], giving rise to a topological $3BF$ action. With suitable simplicity constraint terms added, a $3BF$ action perfectly fits together all fields necessary for a unified description of quantum gravity coupled to matter fields — it features tetrads, spin connection, gauge fields, scalar fields and fermions. The explicit construction was done in [16], where the full Standard Model (SM) coupled to GR in the Einstein-Cartan formulation was rewritten in the form (1), suitable for the implementation of the spinfoam quantization procedure and building a full quantum theory. This demostrates the power and expressiveness of the HGT approach, and it provides us with novel mathematical tools to study the algebraic properties of the matter sector of the SM, in analogy to the gauge field sector which is being described in terms of ordinary Lie groups. In this paper we will review the essential properties of the new approach.

The layout of the paper is the following. In section 2 we give a brief introduction to the category theory, categorical ladder, and the notion of n-groups. Our attention focuses on 3-groups, in particular their representation in terms of 2-crossed modules. Section 3 reviews the construction and general properties of the $3BF$ action, and its relationship with the 3-group structure. Then, in section 4 we apply this developed formalism to construct the *Standard Model 3-group*, and explicitly build the action for the Standard Model coupled to Einstein-Cartan gravity in the form of the $3BF$ action with suitable simplicity constraints. Section 5 contains our concluding remarks.

2 Category theory and 3-groups

Let us begin by giving a short introduction to the category theory, and in particular the notion of *category theory ladder*, a concept used in higher gauge theory to generalize the notion of gauge symmetry. A nice introduction to this topic can be found in [12] and further technical details in [14, 15].

A category $\mathcal{C} = (Obj, Mor)$ is a structure which has objects and morphisms between them,

$$X, Y, Z, \dots \in Obj \,, \qquad f, g, h, \dots \in Mor \,,$$

where

$$f : X \to Y, \quad g : Z \to X, \quad h : X \to Y, \dots$$

such that certain rules are respected, like the associativity of composition of morphisms, and similar. Similarly, a 2-category $\mathcal{C}_2 = (Obj, Mor_1, Mor_2)$ is a structure which has objects, morphisms between them, and morphisms between morphisms, called 2-morphisms,

$$X, Y, Z, \dots \in Obj \,, \qquad f, g, h, \dots \in Mor_1 \,, \qquad \alpha, \beta, \dots \in Mor_2 \,,$$

where

$$f : X \to Y, \quad g : Z \to X, \quad h : X \to Y, \dots \qquad \alpha : f \to h \,, \dots$$

such that similar rules about compositions are respected. Then, a 3-category $\mathcal{C}_3 = (Obj, Mor_1, Mor_2, Mor_3)$ additionally has morphisms between 2-morphisms, called 3-morphisms,

$$\Theta, \Phi, \dots \in Mor_3\,, \qquad \Theta : \alpha \to \beta\,, \dots$$

again with a certain set of axioms about compositions of various n-morphisms. One can further generalize these structures to introduce 4-categories, n-categories, ∞-categories, etc. The process of raising the "dimensionality" of a categorical structure is called a *categorical ladder*.

It is useful to understand other algebraic structures as special cases of categories. As a particularly important example, the algebraic structure of a *group* is a special case of a category — it is a category with only one object, while all morphisms (i.e., group elements) are invertible. It is straightforward to verify that axioms of a group follow from this definition and the axioms of a category. Any group can be represented in this way, for example finite groups, Lie groups, and so on.

The notion of a categorical ladder then provides us with a natural way to introduce novel, more general algebraic structures, by extending the above definition to 2-categories, 3-categories, etc. In particular,

- a 2-group is a 2-category with only one object, while all 1-morphisms and 2-morphisms are invertible;
- a 3-group is a 3-category with only one object, while all 1-morphisms, 2-morphisms and 3-morphisms are invertible.

It is important to emphasize that an n-group is not a particular type of group. Instead, it is a different algebraic structure, which shares some of the features of groups, but is governed by a qualitatively different set of axioms.

The framework of higher gauge theory is centered around the idea that gauge symmetries in physics can be better described using these alternative algebraic structures than using the ordinary Lie groups. To that end, our attention will mostly focus on the so-called Lie 3-groups and their corresponding Lie 3-algebras. While the abstract definition in terms of n-category theory is particularly appealing from the conceptual point of view, for applications in physics there exists a more practical way to talk about 3-group. Namely, every strict Lie 3-group is known to be equivalent to a so-called 2-*crossed module*, defined as an exact sequence of three Lie groups G, H and L,

$$L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \tag{2}$$

and equipped with two "boundary homomorphisms" δ and ∂, an action $\triangleright$ of G onto G, H and L,

$$\triangleright : G \times G \to G\,, \qquad \triangleright : G \times H \to H\,, \qquad \triangleright : G \times L \to L\,,$$

and a bracket operation called *Peiffer lifting* over H to L,

$$\{\_\,,\_\} : H \times H \to L\,.$$

Certain set of axioms is assumed to hold true among all these maps. In particular, for all $g \in G$, $h \in H$ and $l \in L$, we have:

- the axiom stating that (2) is an exact sequence,

$$\partial\delta = 1_G\,, \tag{3}$$

- the axiom specifying that the action of G onto itself is conjugation,

$$g \triangleright g_0 = g\, g_0\, g^{-1}\,, \tag{4}$$

- the axioms stating that the action of G on H and L is equivariant with respect to homomorphisms ∂ and δ and the Peiffer lifting,

$$\begin{array}{lcl} g \triangleright \partial h & = & \partial(g \triangleright h)\,, \\ g \triangleright \delta l & = & \delta(g \triangleright l)\,, \\ g \triangleright \{h_1, h_2\} & = & \{g \triangleright h_1, g \triangleright h_2\}\,, \end{array} \tag{5}$$

- and finally the axioms determining the properties of the Peiffer lifting,

$$\begin{array}{lcl} \delta\{h_1, h_2\} & = & h_1 h_2 h_1^{-1} (\partial h_1) \triangleright h_2^{-1}\,, \\ \{\delta l_1, \delta l_2\} & = & l_1 l_2 l_1^{-1} l_2^{-1}\,, \\ \{h_1 h_2, h_3\} & = & \{h_1, h_2 h_3 h_2^{-1}\}\, \partial h_1 \triangleright \{h_2, h_3\}\,, \\ \{\delta l, h\}\,\{h, \delta l\} & = & l(\partial h \triangleright l^{-1})\,. \end{array} \tag{6}$$

Since it is constructed from three Lie groups, a Lie 3-group has a corresponding Lie 3-algebra, also called a *differential* 2*-crossed module*,

$$\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}\,,$$

where $\mathfrak{l}$, $\mathfrak{h}$, $\mathfrak{g}$ are Lie algebras of L, H, G, the maps δ, ∂, $\triangleright$ and $\{\_\,,\_\}$ are inherited from the 3-group via natural linearization, and finally, the set of corresponding axioms applies. In addition to all this, Lie algebras have their own usual Lie structure — the generators,

$$T_A \in \mathfrak{l}\,, \qquad t_a \in \mathfrak{h}\,, \qquad \tau_\alpha \in \mathfrak{g}$$

the corresponding structure constants,

$$[T_A, T_B] = f_{AB}{}^C T_C\,, \qquad [t_a, t_b] = f_{ab}{}^c t_c\,, \qquad [\tau_\alpha, \tau_\beta] = f_{\alpha\beta}{}^\gamma \tau_\gamma\,,$$

and G-invariant nondegenerate symmetric bilinear forms (for example Killing forms),

$$\langle T_A, T_B \rangle_{\mathfrak{l}} = g_{AB}\,, \qquad \langle t_a, t_b \rangle_{\mathfrak{h}} = g_{ab}\,, \qquad \langle \tau_\alpha, \tau_\beta \rangle_{\mathfrak{g}} = g_{\alpha\beta}\,.$$

The main purpose of the 3-group structure is to *generalize the notion of parallel transport* from curves to surfaces to volumes. Namely, given a 4-dimensional manifold $\mathcal{M}$, one defines a 3-connection (α, β, γ) as a triple of 3-algebra-valued differential forms,

$$\begin{array}{lcll} \alpha & = & \alpha^\alpha{}_\mu(x)\, \tau_\alpha\, \mathbf{d}x^\mu & \in \Lambda^1(\mathcal{M}, \mathfrak{g})\,, \\ \beta & = & \dfrac{1}{2} \beta^a{}_{\mu\nu}(x)\, t_a\, \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu & \in \Lambda^2(\mathcal{M}, \mathfrak{h})\,, \\ \gamma & = & \dfrac{1}{3!} \gamma^A{}_{\mu\nu\rho}(x)\, T_A\, \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \wedge \mathbf{d}x^\rho & \in \Lambda^3(\mathcal{M}, \mathfrak{l})\,. \end{array}$$

Then one can introduce the line, surface and volume holonomies,

$$g = \mathcal{P}\mathrm{exp} \int_{\mathcal{P}_1} \alpha \,, \qquad h = \mathcal{S}\mathrm{exp} \int_{\mathcal{S}_2} \beta \,, \qquad l = \mathcal{V}\mathrm{exp} \int_{\mathcal{V}_3} \gamma \,,$$

and corresponding curvature forms,

$$\begin{array}{rcl} \mathcal{F} & = & \mathbf{d}\alpha + \alpha \wedge \alpha - \partial\beta \,, \\ \mathcal{G} & = & \mathbf{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma \,, \\ \mathcal{H} & = & \mathbf{d}\gamma + \alpha \wedge^{\triangleright} \gamma - \{\beta \wedge \beta\} \,. \end{array}$$

The 3-group structure ensures that all these quantities are well defined, in particular the surface- and volume-ordered exponentials and the respective holonomies.

3 Higher gauge theories

The basic idea behind the higher gauge theory approach is to employ the structure of n-groups as a mathematical representation of gauge symmetries in physics, generalizing the ordinary notion of gauge symmetry described via a Lie group. Namely, in ordinary gauge theory, the prototype action functional was the so-called BF action [7], based on a chosen gauge group G. In the HGT approach, one generalizes the BF action in accord with the chosen n-group structure, leading to the nBF action. For the case of 3-groups, one defines a $3BF$ action as:

$$S_{3BF} = \int_{\mathcal{M}} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,.$$

Here B, C, and D are Lagrange multipliers, in particular a $\mathfrak{g}$-valued 2-form, an $\mathfrak{h}$-valued 1-form, and an $\mathfrak{l}$-valued 0-form, respectively.

As in the case of a BF theory, one can demonstrate that $3BF$ theory is a topological gauge theory, having no local propagating degrees of freedom. Nevertheless, it can be transformed into a physically relevant action by adding the so-called *simplicity constraint terms* to the action, changing the dynamical structure of the theory. The prototype of this procedure is represented by transforming the topological BF theory based on the Lorentz group $SO(3,1)$ into a Plebanski action [8], which describes general relativity.

One can even do more, and provide a physical interpretation of the Lagrange multipliers C and D in the $3BF$ action, as follows:

- the $\mathfrak{h}$-valued 1-form C can be interpreted as the tetrad field, if $H = \mathbb{R}^4$ is the spacetime translation group,

$$C \to e = e^a{}_\mu(x)\, t_a \, \mathbf{d}x^\mu \,,$$

- the $\mathfrak{l}$-valued 0-form D can be interpreted as the set of real-valued matter fields, given some Lie group L,

$$D \to \phi = \phi^A(x)\, T_A \,.$$

An interested reader can see [16] for further details.

4 The Standard Model

One natural question that can be asked is what choice of a 3-group can be relevant for physics. There are various answers to this question, but perhaps the most illustrative example is a choice of the 3-group which reproduces the Standard Model of elementary particles, coupled to general relativity in the Einstein-Cartan version. This is called the *Standard Model 3-group*, and in the remainder of this section we will demonstrate how it can be constructed, step by step.

The first step is to specify the groups G and H as the usual Lorentz, internal, and translational symmetries:

$$G = SO(3,1) \times SU(3) \times SU(2) \times U(1) \,, \qquad H = \mathbb{R}^4 \,.$$

Note that the Poincaré group has been broken into the separate Lorentz and translational parts, and these have been associated with two different groups within the 3-group structure.

The next step is to define the homomorphisms δ and ∂, as well as the Peiffer lifting, to be trivial,

$$\delta l = 1_H = 0 \,, \qquad \partial \vec{v} = 1_G \,,$$

and

$$\{\vec{u}, \vec{v}\} = 1_L \,,$$

for all $l \in L$ and $\vec{u}, \vec{v} \in H$. Additionally, we define the action of the group G on H via vector representation for the $SO(3,1)$ sector and via trivial representation for the $SU(3) \times SU(2) \times U(1)$ sector. Finally, the choice of the group L and the action of G on L will be discussed below. But already now one can verify that all axioms (3)–(6) are satisfied, thus making sure that these choices represent one genuine 3-group.

The next step is to choose the group L. One general property of L that can be determined immediately comes from the second axiom in (6). Namely, due to the trivial choices for the Peiffer lifting and the homomorphism δ, the axiom implies that L must be Abelian. Aside from this, the choice of the group L is guided by physical requirements, as follows.

Begin by rewriting the $3BF$ action in the form

$$S_{3BF} = \int_{\mathcal{M}} B^\alpha \wedge \mathcal{F}^\beta g_{\alpha\beta} + e^a \wedge \mathcal{G}^b g_{ab} + \phi^A \mathcal{H}^B g_{AB} \,.$$

Since the group G is a direct product of the Lorentz and internal groups, the corresponding indices α of G split according to this structure, as $\alpha = (ab\,,\, i)$, leading to the corresponding splitting of the connection α and its curvature $\mathcal{F}$,

$$\alpha = \omega^{ab} J_{ab} + A^i \tau_i \,, \qquad \mathcal{F} = R^{ab} J_{ab} + F^i \tau_i \,.$$

Here ω^{ab} is the ordinary spin connection 1-form, J_{ab} are Lorentz generators, while A^i are internal gauge potential 1-forms and τ_i the generators of $SU(3)\times SU(2)\times U(1)$. Also, R^{ab} and F^i are the Riemann curvature and gauge field strength 2-forms, respectively. Also, given that the action of $SO(3,1)$ onto $H = \mathbb{R}^4$ is via vector representation, and given that the bilinear symmetric nondegenerate form for H must be G-invariant, the only available choice is

$$g_{ab} = \eta_{ab} \equiv \text{diag}(-1, +1, +1, +1) \,.$$

Finally, given that the matter fields are elements in the Lie algebra $\mathfrak{l}$ of the group L, namely $\phi = \phi^A T_A$, we observe that there should be precisely one real-valued field $\phi^A(x)$ for each generator $T_A \in \mathfrak{l}$. This information allows us to determine the dimension of the algebra $\mathfrak{l}$, by counting the total number of real-valued components of all matter fields in the Standard Model. The matter fields have two sectors — fermions and the Higgs.

The number of the real-valued components of all fermion fields can be counted according to the following scheme:

$$\left.\begin{array}{cccc} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L & \begin{pmatrix} u_r \\ d_r \end{pmatrix}_L & \begin{pmatrix} u_g \\ d_g \end{pmatrix}_L & \begin{pmatrix} u_b \\ d_b \end{pmatrix}_L \\ & & & \\ (\nu_e)_R & (u_r)_R & (u_g)_R & (u_b)_R \\ & & & \\ (e^-)_R & (d_r)_R & (d_g)_R & (d_b)_R \end{array}\right\} = 16 \ \frac{\text{Weyl spinors}}{\text{family}} \times$$

$$\times 3 \text{ families } \times 4 \ \frac{\text{real-valued fields}}{\text{Weyl spinor}} \ = 192 \text{ real-valued fields } \phi^A \,.$$

Similarly, the Higgs sector gives us:

$$\left.\begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix}\right\} = 2 \text{ complex scalar fields } = 4 \text{ real-valued fields } \phi^A \,.$$

This suggests the structure for L in the form:

$$L = L_{\text{fermion}} \times L_{\text{Higgs}} \,, \qquad \dim L_{\text{fermion}} = 192 \,, \qquad \dim L_{\text{Higgs}} = 4 \,.$$

The structure of L can be further understood by looking at the action of the gauge group G on various components of fields ϕ^A. This is fixed by the choice of the action of G on L, chosen as follows. Given that G is constructed from Lorentz and internal gauge symmetry groups, the action $\triangleright : G \times L \to L$ specifies the transformation properties of each real-valued field ϕ^A with respect to those symmetries. For example, if we look at a Weyl spinor u_b that sits in the doublet

$$\begin{pmatrix} u_b \\ d_b \end{pmatrix}_L ,$$

the action $g \triangleright u_b$ (where $g \in SO(3,1) \times SU(3) \times SU(2) \times U(1)$) encodes that u_b consists of 4 real-valued fields which transform as:

- a left-handed spinor with respect to $SO(3,1)$,
- as a "blue" component of the fundamental representation of $SU(3)$,
- and as "isospin $+\frac{1}{2}$" of the left doublet with respect to $SU(2) \times U(1)$.

The action $\triangleright : G \times L \to L$ similarly defines the transformation properties for all other fermions in the theory, as well as for the Higgs field.

From such a definition of the action $\triangleright$, one can observe that G acts on L in precisely the same way across the three fermion families. This implies that L_{fermion} can be written as

$$L_{\text{fermion}} = L_{\text{1st family}} \times L_{\text{2nd family}} \times L_{\text{3rd family}} \,, \qquad \dim L_{k\text{-th family}} = 64 \,.$$

Ultimately, given that the components of Weys spinors mutually anticommute, given that the group L is Abelian, and given that it has the structure and dimension as given above, we can fix the choice of the group L which corresponds to the Standard Model as

$$L = \mathbb{R}^4(\mathbb{C}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \,,$$

where $\mathbb{G}$ is the algebra of Grassmann numbers. This completes the construction of the Standard Model 3-group.

The final step in specifying the theory is to spell out its classical action. As was previously discussed, the action has the form of a $3BF$ action, with the addition of appropriate simplicity constraints which will transform it into a non-topological theory, i.e., a theory with local propagating degrees of freedom. The choice of the Standard Model 3-group completely fixes the structure of the $3BF$ action, and the only thing left to do is to add the appropriate simplicity constraints. The details of the construction of these terms is given in detail in [16], and will not be repeated here. We will only quote the result,

$$S_{SM+EC} = S_{3BF} + S_{\text{simp}} \,,$$

where

$$S_{3BF} = \int B_{\hat{\alpha}} \wedge \mathcal{F}^{\hat{\alpha}} + e_{\hat{a}} \wedge \mathcal{G}^{\hat{a}} + \phi_{\hat{A}} \wedge \mathcal{H}^{\hat{A}} \,,$$

and

$$\begin{aligned} S_{\text{simp}} = \left(B_{\hat{\alpha}} - C_{\hat{\alpha}}{}^{\hat{\beta}} M_{cd\hat{\beta}} e^c \wedge e^d\right) \wedge \lambda^{\hat{\alpha}} - \left(\gamma_{\hat{A}} - e^a \wedge e^b \wedge e^c C_{\hat{A}}{}^{\hat{B}} M_{abc\hat{B}}\right) \wedge \lambda^{\hat{A}} \\ -4\pi i\, l_p^2\, \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c \phi_{\hat{A}} T^{d\hat{A}}{}_{\hat{B}} \phi^{\hat{B}} \\ +\zeta^{ab}{}_{\hat{\alpha}} \wedge \left(M_{ab}{}^{\hat{\alpha}} \varepsilon^{cdef} e_c \wedge e_d \wedge e_e \wedge e_f - F^{\hat{\alpha}} \wedge e_c \wedge e_d\right) \\ +\zeta^{ab}{}_{\hat{A}} \wedge \left(M_{abc}{}^{\hat{A}} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - F^{\hat{A}} \wedge e_a \wedge e_b\right) \\ -\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \ \left(\Lambda + M_{\hat{A}\hat{B}} \phi^{\hat{A}} \phi^{\hat{B}} + Y_{\hat{A}\hat{B}\hat{C}} \phi^{\hat{A}} \phi^{\hat{B}} \phi^{\hat{C}} + L_{\hat{A}\hat{B}\hat{C}\hat{D}} \phi^{\hat{A}} \phi^{\hat{B}} \phi^{\hat{C}} \phi^{\hat{D}}\right) \,. \end{aligned}$$

See [16] for details and notation.

By varying the action with respect to all variables, and with a little technical effort, one can demonstrate that the corresponding equations of motion are precisely the classical equations of the Standard Model, coupled to general relativity in the Einstein-Cartan formulation.

5 Conclusions

Let us summarize the results of the paper. In section 2 we have given a short introduction into the category theory, introduced the notions of categorical ladder and n-categories, and in the resulting framework, provided a definition for the notion of an n-group. Our attention focused on the case of 3-groups, which are relevant for applications in physics, and the equivalent notion of a 2-crossed module, which is more convenient for practical applications. Section 3 was devoted to introducing the higher gauge theory formalism and the $3BF$ action corresponding to a choice of a 3-group, as a generalization of the well known BF action in terms of the categorical ladder. Also, we have interpreted the additional Lagrange multipliers appearing in the $3BF$ action as the tetrad and matter fields, providing the setup for the application in physics. This application was then demonstrated in detail in section 4, where the Standard Model 3-group has been defined, and utilized to construct a physically relevant constrained $3BF$ action, which is classically equivalent to the Standard Model of elementary particles coupled to general relativity in the Einstein-Cartan formulation. This is the main result, which successfully establishes the first step of the spinfoam quantization procedure, and opens up a possibility of straightforward implementation of the second and third steps, hopefully leading to a full model of quantum gravity with matter.

It should be noted that the most important feature of the higher gauge theory framework is its ability to treat gravity, gauge fields, fermions and scalar fields on completely equal footing, describing all of them via the underlying algebraic structure of a 3-group. The 3-group also provides us with a natural geometric description of a generalized notion of parallel transport, namely along a surface and along a volume, in addition to the standard notion of parallel transport along a curve. This relationship opens up a possibility for a fully geometric interpretation of all fields present in physics.

Moreover, just as the gauge group dictates the number and properties of gauge fields in Yang-Mills theories, the sector of the 3-group described by the Lie group L determines the number and properties of the fermion and scalar fields. This fact enables us to classify the spectrum of matter fields in terms of group theory, generalizing the constructions present in the Standard Model, where only gauge fields are classified in such terms. The choice of the group L thus opens up novel avenues for research on the unification of all fields, and specifically the origin of particle families, Higgs and fermion sectors, and so on.

Finally, the higher gauge theory framework may have applications in other areas of physics and mathematics as well, and various possible research directions are yet to be explored.

Acknowledgments. The authors have been supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia.

References

[1] C. Rovelli, *Quantum Gravity*, Cambridge University Press, Cambridge (2004).

[2] T. Thiemann, *Modern Canonical Quantum General Relativity*, Cambridge University Press, Cambridge (2007).

[3] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity*, Cambridge University Press, Cambridge (2014).

[4] G. Ponzano and T. Regge, *Spectroscopic and Group Theoretical Methods in Physics*, edited by F. Block, North Holland, Amsterdam (1968).

[5] J. Engle, E. R. Livine, R. Pereira and C. Rovelli, *Nucl. Phys.* **B799**, 136 (2008), `arXiv:0711.0146`.

[6] L. Freidel and K. Krasnov, *Class. Quant. Grav.* **25**, 125018 (2008), `arXiv:0708.1595`.

[7] M. Celada, D. González and M. Montesinos, *Class. Quant. Grav.* **33**, 213001 (2016), `arXiv:1610.02020`.

[8] J. F. Plebanski, *J. Math. Phys.* **18**, 2511 (1977).

[9] A. Miković and M. Vojinović, *Class. Quant. Grav.* **29**, 165003 (2012), arXiv:1110.4694.

[10] F. Girelli, H. Pfeiffer and E. M. Popescu, *Jour. Math. Phys.* **49**, 032503 (2008), arXiv:0708.3051.

[11] J. F. Martins and A. Miković, *Adv. Theor. Math. Phys.* **15**, 1059 (2011), arXiv:1006.0903.

[12] J. C. Baez and J. Huerta, *Gen. Relativ. Gravit.* **43**, 2335 (2011), arXiv:1003.4485.

[13] L. Crane and M. D. Sheppeard, arXiv:math/0306440.

[14] J. F. Martins and R. Picken, *Differ. Geom. Appl. Jour.* **29**, 179 (2011), arXiv:0907.2566.

[15] W. Wang, *Jour. Math. Phys.* **55**, 043506 (2014), arXiv:1311.3796.

[16] T. Radenković and M. Vojinović, *JHEP* **10**, 222 (2019), arXiv:1904.07566.

Construction and examples of higher gauge theories*

Tijana Radenković†
Institute of Physics, University of Belgrade,
Pregrevica 118, 11080 Belgrade, Serbia

Marko Vojinović‡
Institute of Physics, University of Belgrade,
Pregrevica 118, 11080 Belgrade, Serbia

Abstract

We provide several examples of higher gauge theories, constructed as generalizations of a BF model to $2BF$ and $3BF$ models with constraints. Using the framework of higher category theory, we introduce appropriate 2-groups and 3-groups, and construct the actions for the corresponding constrained $2BF$ and $3BF$ theories. In this way, we can construct actions which describe the correct dynamics of Yang-Mills, Klein-Gordon, Dirac, Weyl, and Majorana fields coupled to Einstein-Cartan gravity. Each action is naturally split into a topological sector and a sector with simplicity constraints. The properties of the higher gauge group structure opens up a possibility of a nontrivial unification of all fields.

1. Introduction

The quantization of the gravitational field is one of the fundamental open problems in modern physics. There are various approaches to this problem, some of which have developed into vast research frameworks. One of such frameworks is the Loop Quantum Gravity approach, which aims to establish a nonperturbative quantization of gravity, both canonically and covariantly [1, 2, 3]. The covariant approach is slightly more general, and

---

* This work was supported by the project ON171031 of the Ministry of Education, Science and Technological Development (MPNTR) of the Republic of Serbia, and partially by the bilateral scientific cooperation between Austria and Serbia through the project "Causality in Quantum Mechanics and Quantum Gravity - 2018-2019", no. 451-03-02141/2017-09/02, supported by the Federal Ministry of Science, Research and Economy (BMWFW) of the Republic of Austria, and the Ministry of Education, Science and Technological Development (MPNTR) of the Republic of Serbia.

† e-mail address: rtijana@ipb.ac.rs

‡ e-mail address: vmarko@ipb.ac.rs

focuses on providing a possible rigorous definition of the path integral for the gravitational field,

$$Z = \int \mathcal{D}g \, e^{iS[g]} \,. \tag{1}$$

This is done by considering a triangulation of a spacetime manifold, and defining the path integral as a discrete state sum of the gravitational field configurations living on the simplices in the triangulation. This quantization technique is known as the *spinfoam* quantization method, and is performed via the following three steps:

(1) one writes the classical action $S[g]$ as a constrained BF action;

(2) one uses the Lie group structure, underlying the topological sector of the action, to define a triangulation-independent state sum Z;

(3) one imposes the simplicity constraints on the state sum, promoting it into a triangulation-dependent state sum, which serves as a definition for the path integral (1).

So far, this quantization prescription has been implemented for various choices of the gravitational action, of the Lie group, and of the spacetime dimension. For example, in 3 dimensions, historically the first spinfoam model is known as the Ponzano-Regge model [4]. In 4 dimensions there are multiple models, depending on the choice of the Lie group and the way one imposes the simplicity constraints [5, 6, 7, 8, 9]. While these models do give a definition for the gravitational path integral, none of them are able to consistently include matter fields. Including the matter fields has so far had limited success [10], mainly due to the absence of the tetrad fields from the topological sector of the theory.

In order to resolve this issue, a new approach has been developed, using the framework of *higher gauge theory* (see [11] for a review). In particular, one uses the idea of a *categorical ladder* to generalize the BF action (based on a Lie group) into a $2BF$ action (based on the so-called 2-group structure). A suitable choice of the *Poincaré 2-group* introduces the needed tetrad fields into the topological sector of the action [12]. While this result opened up a possibility to couple matter fields to gravity, the matter fields could not be naturally expressed using the underlying algebraic structure of a 2-group, rendering the spinfoam quantization method inapplicable. Namely, the matter sector could indeed be added to the classical action, but could not be expressed itself as a constrained $2BF$ theory, which means that the steps 1–3 above could not be performed for the matter sector of the action, but only for gravity.

This final issue has recently been resolved in [13], by passing from the 2-group structure to the 3-group structure, generalizing the action one step further in the categorical ladder. This generalization naturally gives rise to the so-called $3BF$ action, which turns out to be suitable for a unified description of both gravity and matter fields. The steps of the categorical ladder and their corresponding structures are summarized as follows:

categorical structure	algebraic structure	linear structure	topological action	degrees of freedom
Lie group	Lie group	Lie algebra	BF theory	gauge fields
Lie 2-group	Lie crossed module	differential Lie crossed module	$2BF$ theory	tetrad fields
Lie 3-group	Lie 2-crossed module	differential Lie 2-crossed module	$3BF$ theory	scalar and fermion fields

The purpose of this paper is to give a systematic overview of the constructions of classical BF, $2BF$ and $3BF$ actions, both pure and constrained, in order to demonstrate the categorical ladder procedure and the construction of higher gauge theories. In other words, we focus on the step 1 of the spinfoam quantization programme.

The layout of the paper is as follows. Section 2 deals with models based on a BF theory. First we discuss the pure, topological BF theory, and then pass on to the the physically more interesting Yang-Mills theory in Minkowski spacetime and the Plebanski formulation of general relativity. In Section 3 we study the first step in the categorical ladder, namely models based on the $2BF$ theory. After introducing the pure $2BF$ theory, we study the relevant formulation of general relativity [12], and then the coupled Einstein-Yang-Mills theory. Then, in Section 4 we perform the second step in the categorical ladder, passing on to models based on the $3BF$ theory. After the introduction of the pure $3BF$ model, we construct constrained $3BF$ actions for the cases of Klein-Gordon, Dirac, Weyl and Majorana fields, all coupled to the Einstein-Cartan gravity in the standard way. As we shall see, the scalar and fermion fields will be *naturally associated to a new gauge group*, generalizing the purpose of a gauge group in the Yang-Mills theory, which opens up a possibility of an algebraic classification of matter fields. Finally, Section 5 contains a discussion and conclusions.

The notation and conventions are as follows. The local Lorentz indices are denoted by the Latin letters $a, b, c, \dots$, take values $0, 1, 2, 3$, and are raised and lowered using the Minkowski metric η_{ab} with signature $(-,+,+,+)$. Spacetime indices are denoted by the Greek letters $\mu, \nu, \dots$, and are raised and lowered by the spacetime metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ are the tetrad fields. The inverse tetrad is denoted as $e^\mu{}_a$. All other indices that appear in the paper are dependent on the context, and their usage is explicitly defined in the text where they appear. We work in the natural system of units where $c = \hbar = 1$, and $G = l_p^2$, where l_p is the Planck length.

2. BF theory

We begin with a short review of BF theories. See [14, 15, 16] for additional information.

2.1. Pure BF theory

Given a Lie group G, and denoting its corresponding Lie algebra as $\mathfrak{g}$, one introduces the pure BF action as follows (we limit ourselves to the physically relevant case of 4-dimensional spacetime manifolds $\mathcal{M}_4$):

$$S_{BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} \,. \tag{2}$$

Here, $\mathcal{F} \equiv \mathrm{d}\alpha + \alpha \wedge \alpha$ is the curvature 2-form for the algebra-valued connection 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$, and $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ is a Lagrange multiplier 2-form, while $\langle \_\,, \_ \rangle_{\mathfrak{g}}$ denotes a G-invariant bilinear symmetric nondegenerate form.

One can see from (2) that the action is diffeomorphism invariant, and it is also gauge invariant with respect to G, provided that B transforms as a scalar with respect to G.

Varying the action (2) with respect to B^β and α^β, where the index β is the group G index (which counts the generators of $\mathfrak{g}$), one obtains the following equations of motion,

$$\mathcal{F}^\beta = 0\,, \qquad \nabla B^\beta \equiv \mathrm{d}B^\beta + f_{\gamma\delta}{}^\beta \alpha^\gamma \wedge B^\delta = 0\,, \tag{3}$$

where $f_{\gamma\delta}{}^\beta$ are the structure constants of the Lie group G. From the first equation of motion, one immediately sees that α is a flat connection, meaning that $\alpha = 0$ up to gauge transformations. Given this, the second equation of motion implies that B is constant. Therefore, there are no local propagating degrees of freedom, and the theory is called *topological*.

2.2. Yang-Mills theory

In physics one is usually interested in theories which are not topological, i.e., which have local propagating degrees of freedom. As a rule of thumb, one recognizes that the theory does have local propagating degrees of freedom if one of the equations of motion is a second-order partial differential equation, usually featuring a D'Alambertian operator $\Box$ in some form. In order to transform the pure BF action into such a theory, one adds an additional term to the action, commonly called the *simplicity constraint*. The resulting action is called a *constrained BF theory*. A nice example is the Yang-Mills theory for the $SU(N)$ group in Minkowski spacetime, which can be rewritten as a constrained BF theory in the following way:

$$\begin{aligned} S \;=\; & \int B_I \wedge F^I + \lambda^I \wedge \Big(B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b \Big) \\ & + \zeta^{abI} \Big(M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - g_{IJ} F^J \wedge \delta_a \wedge \delta_b \Big) \,. \end{aligned} \tag{4}$$

Here $F \equiv \mathrm{d}A + A \wedge A$ is again the curvature 2-form for the connection $A \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{su}(N))$, and $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{su}(N))$ is the Lagrange multiplier

2-form. The Killing form $g_{IJ} \equiv \langle \tau_I \, , \tau_J \rangle_{\mathfrak{su}(N)} \propto f_{IK}{}^L f_{JL}{}^K$ is used to raise and lower the indices $I, J, \dots$ which count the generators of $SU(N)$, while $f_{IJ}{}^K$ are the structure constants for the $\mathfrak{su}(N)$ algebra. In addition to the topological $B \wedge F$ term, there are also two simplicity constraint terms present, featuring two Lagrange multipliers, a 2-form λ^I and a 0-form ζ^{abI}. The 0-form M_{abI} is also a Lagrange multiplier, while g is the coupling constant for the Yang-Mills theory.

Finally, δ^a is a nondynamical 1-form, such that there exists a global coordinate frame in which its components are equal to the Kronecker symbol $\delta^a{}_\mu$ (hence the notation δ^a). The 1-form δ^a plays the role of a background field, and defines the global spacetime metric, via the equation

$$\eta_{\mu\nu} = \eta_{ab} \delta^a{}_\mu \delta^b{}_\nu \,, \tag{5}$$

where $\eta_{ab} \equiv \operatorname{diag}(-1, +1, +1, +1)$ is the Minkowski metric. Since the coordinate system is global, the spacetime manifold $\mathcal{M}_4$ is understood to be flat. The indices $a, b, \dots$ are local Lorentz indices, taking values $0, \dots, 3$. Note that the field δ^a has all the properties of the tetrad 1-form e^a in the flat Minkowski spacetime. Also note that the action (4) is manifestly diffeomorphism invariant and gauge invariant with respect to $SU(N)$, but not background independent, due to the presence of δ^a.

Varying the action (4) with respect to the variables ζ^{abI}, M_{abI}, A^I, B_I, and λ^I, respectively (but not with respect to the background field δ^a), we obtain the equations of motion:

$$M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - F_I \wedge \delta_a \wedge \delta_b = 0 \,, \tag{6}$$

$$-\frac{12}{g} \lambda^I \wedge \delta^a \wedge \delta^b + \zeta^{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f = 0 \,, \tag{7}$$

$$-\mathrm{d}B_I + f_{JI}{}^K B_K \wedge A^J + \mathrm{d}(\zeta^{ab}{}_I \delta_a \wedge \delta_b) - f_{JI}{}^K \zeta^{ab}{}_K \delta_a \wedge \delta_b \wedge A^J = 0 \,, \tag{8}$$

$$F_I + \lambda_I = 0 \,, \tag{9}$$

$$B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b = 0 \,, \tag{10}$$

From the equations (6), (7), (9) and (10) one obtains the multipliers as algebraic functions of the field strength $F^I{}_{\mu\nu}$ for the dynamical field A^I:

$$\begin{aligned} M_{abI} &= \frac{1}{48} \varepsilon_{abcd} F_I{}^{cd} \,, \qquad & \zeta^{abI} &= \frac{1}{4g} \varepsilon^{abcd} F^I{}_{cd} \,, \\ \lambda_{Iab} &= F_{Iab} \,, & B_{Iab} &= \frac{1}{2g} \varepsilon_{abcd} F_I{}^{cd} \,. \end{aligned} \tag{11}$$

Here we used the notation $F_{Iab} = F_{I\mu\nu}\delta_a{}^\mu \delta_b{}^\nu$, and similarly for other variables, where we exploited the fact that $\delta^a{}_\mu$ is invertible. Using these equations and the differential equation (8) one obtains the equation of motion for gauge field $A^I{}_\mu$,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0\,. \tag{12}$$

This is precisely the classical equation of motion for the free Yang-Mills theory. Note that this is a second-order partial differential equation for the field $A^I{}_\mu$, and moreover contains the $\Box$ operator in the first term.

In addition to the Yang-Mills theory, one can easily extend the action (4) in order to describe the massive vector field and obtain the Proca equation of motion. This is done by adding a mass term

$$-\frac{1}{4!} m^2 A_{I\mu} A^I{}_\nu \eta^{\mu\nu} \varepsilon_{abcd} \delta^a \wedge \delta^b \wedge \delta^c \wedge \delta^d \tag{13}$$

to the action (4). Of course, this term explicitly breaks the $SU(N)$ gauge symmetry of the action.

2.3. Plebanski general relativity

The second example of the constrained BF theory is the Plebanski action for general relativity [16, 14]. Using the Lorentz group $SO(3,1)$ as a gauge group, one constructs a constrained BF action as

$$S = \int_{\mathcal{M}_4} B_{ab} \wedge R^{ab} + \phi_{abcd} B^{ab} \wedge B^{cd}\,. \tag{14}$$

Here R^{ab} is the curvature 2-form for the spin connection ω^{ab}, B_{ab} is the usual Lagrange multiplier 2-form, while ϕ_{abcd} is the additional Lagrange multiplier 0-form multiplying the term $B^{ab} \wedge B^{cd}$ to form a simplicity constraint. It can be shown that the variation of this action with respect to B_{ab}, ω^{ab} and ϕ_{abcd} gives rise to the equations of motion of vacuum general relativity. However, in this model the tetrad fields appear only as a solution of the simplicity constraint equation of motion $B^{ab} \wedge B^{cd} = 0$. Therefore, being intrinsically on-shell objects, the tetrad fields are not present in the action itself and cannot be quantized. This renders the Plebanski model unsuitable for coupling of matter fields to gravity [10, 12, 20]. Nevertheless, regarded as a model for pure gravity, the Plebanski model has been successfully quantized in the context of spinfoam models [8, 9, 1, 2].

3. $2BF$ theory

In this section we perform the first step of the *categorical ladder*, generalizing the algebraic notion of a group to the notion of a 2-group. This leads to the generalization of the BF theory to the $2BF$ theory, also sometimes called $BFCG$ theory [11, 17, 18, 19].

3.1. Pure $2BF$ theory

In order to circumvent the issue of tetrad fields not being present in the Plebanski action, in the context of higher category theory [11] a recent promising approach has been developed [12, 21, 22, 23, 20, 24]. As an essential ingredient, let us first give a short review of the 2-group formalism.

Within the framework of category theory, the group as an algebraic structure can be understood as a category with only one object and invertible morphisms [11]. Additionally, the notion of a category can be generalized to the so-called *higher categories*, which have not only objects and morphisms, but also 2-morphisms (morphisms between morphisms), and so on. This process of generalization is called the *categorical ladder*. Using this process, one can introduce the notion of a 2*-group* as a 2-category consisting of only one object, where all the morphisms and all 2-morphisms are invertible. It has been shown that every strict 2-group is equivalent to a *crossed module* $(H \stackrel{\partial}{\to} G\,,\rhd)$, see [13] for detailed definitions. Here G and H are groups, ∂ is a homomorphism from H to G, while $\rhd : G \times H \to H$ is an action of G on H.

Similarly to the case of an ordinary Lie group G which has a naturally associated notion of a connection α, giving rise to a BF theory, the 2-group structure has a naturally associated notion of a 2-connection $(\alpha\,,\beta)$, described by the usual $\mathfrak{g}$-valued 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$ and an $\mathfrak{h}$-valued 2-form $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$, where $\mathfrak{h}$ is a Lie algebra of the Lie group H. The 2-connection gives rise to the so-called *fake* 2*-curvature* $(\mathcal{F},\mathcal{G})$, given as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,, \qquad \mathcal{G} = d\beta + \alpha\wedge^{\rhd}\beta\,. \tag{15}$$

Here $\alpha\wedge^{\rhd}\beta$ means that α and β are multiplied as forms using $\wedge$, and simultaneously multiplied as algebra elements using $\rhd$, see [13]. The curvature pair $(\mathcal{F},\mathcal{G})$ is called "fake" because of the presence of the additional term $\partial\beta$ in the definition of $\mathcal{F}$ [11].

Using the structure of a 2-group, or equivalently the crossed module, one can generalize the BF action to the so-called $2BF$ action, defined as follows [17, 18]:

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}} + \langle C\wedge\mathcal{G}\rangle_{\mathfrak{h}}\,. \tag{16}$$

Here the 2-form $B \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{g})$ and the 1-form $C \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{h})$ are Lagrange multipliers. Also, $\langle \_\,,\_\rangle_{\mathfrak{g}}$ and $\langle \_\,,\_\rangle_{\mathfrak{h}}$ denote the G-invariant bilinear symmetric nondegenerate forms for the algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. As a consequence of the axiomatic structure of a crossed module (see [13]), the bilinear form $\langle \_\,,\_\rangle_{\mathfrak{h}}$ is H-invariant as well. See [17, 18] for review and references.

Similarly to the BF action, the $2BF$ action is also topological, which can be seen from equations of motion. Varying with respect to B^{α} and C^{a} one obtains

$$\mathcal{F}^{\alpha} = 0\,, \qquad \mathcal{G}^{a} = 0\,, \tag{17}$$

where indices a count the generators of the group H. Varying with respect to α^α and β^a one obtains the equations for the multipliers,

$$\mathrm{d}B_\alpha + f_{\alpha\beta}{}^\gamma B_\gamma \wedge \alpha^\beta - \rhd_{\alpha a}{}^b C_b \wedge \beta^a = 0\,, \tag{18}$$

$$\mathrm{d}C_a - \partial_a{}^\alpha B_\alpha + \rhd_{\alpha a}{}^b C_b \wedge \alpha^\alpha = 0\,. \tag{19}$$

We can again see that the equations of motion are only first-order and have only very simple solutions (note that this is not a sufficient argument for the absence of local propagating degrees of freedom — a counterexample is the Dirac equation, being a first-order partial differential equation which *does* have propagating degrees of freedom). One can additionally use the Hamiltonian analysis to rigorously demonstrate that there are no local propagating degrees of freedom [22, 23]. Thus the $2BF$ theory is also topological.

3.2. General relativity

An important example of a crossed module structure is a vector space V equipped with an isometry group O. Namely, V can be regarded as an Abelian Lie group with addition as a group operation, so that a representation of O on V is an action $\rhd$ of O on the group V, giving rise to the crossed module $(V \xrightarrow{\partial} O\,, \rhd)$, where the homomorphism ∂ is chosen to be trivial (it maps every element of V into a unit of O).

We can employ this construction to introduce the *Poincaré 2-group*. One constructs a crossed module by choosing

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,. \tag{20}$$

The map ∂ is trivial, while $\rhd$ is a natural action of $SO(3,1)$ on $\mathbb{R}^4$, defined by the equation

$$M_{ab} \rhd P_c = \eta_{[bc} P_{a]}\,, \tag{21}$$

where M_{ab} and P_a are the generators of groups $SO(3,1)$ and $\mathbb{R}^4$, respectively. The action $\rhd$ of $SO(3,1)$ on itself is given via conjugation. At the level of the algebra, conjugation reduces to the action via the adjoint representation, so that

$$M_{ab} \rhd M_{cd} = [\, M_{ab}\,,\, M_{cd}\,] \equiv \eta_{ad} M_{bc} - \eta_{ac} M_{bd} + \eta_{bc} M_{ad} - \eta_{bd} M_{ac}\,. \tag{22}$$

The 2-connection (α, β) is given by the algebra-valued differential forms

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \tag{23}$$

where ω^{ab} is called the spin connection. The corresponding 2-curvature in this case is given by

$$\begin{aligned} \mathcal{F} &= (\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}) M_{ab} &&\equiv R^{ab} M_{ab}\,, \\ \mathcal{G} &= (\mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b) P_a &&\equiv \nabla \beta^a P_a \equiv G^a P_a\,, \end{aligned} \tag{24}$$

Note that, since ∂ is trivial, the fake curvature is the same as ordinary curvature. Introducing the bilinear forms

$$\langle M_{ab}\,, M_{cd}\rangle_{\mathfrak{g}} = \eta_{a[c}\eta_{bd]}\,, \qquad \langle P_a\,, P_b\rangle_{\mathfrak{h}} = \eta_{ab}\,, \tag{25}$$

one can show that 1-forms C^a transform in the same way as the tetrad 1-forms e^a under the Lorentz transformations and diffeomorphisms, so the fields C^a can be identified with the tetrads. Then one can rewrite the pure $2BF$ action (16) for the Poincaré 2-group as

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a\,. \tag{26}$$

Note that the above step of recognizing that $C^a \equiv e^a$ was crucial, since we now see that the tetrad fields are explicitly present in the $2BF$ action for the Poincaré 2-group.

In order to promote (26) to an action for general relativity, we add a convenient simplicity constraint term:

$$S = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c \wedge e_d\right). \tag{27}$$

Here λ_{ab} is a Lagrange multiplier 2-form associated to the simplicity constraint term, and l_p is the Planck length. Note that the term "simplicity constraint" derives its name from the fact that the constraint imposes the property of *simplicity* on B^{ab} — a 2-form is said to be *simple* if it can be written as an exterior product of two 1-forms.

Varying the action (27) with respect to B_{ab}, e_a, ω_{ab}, β_a and λ_{ab}, we obtain the following equations of motion:

$$R_{ab} - \lambda_{ab} = 0\,, \tag{28}$$

$$\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d = 0\,, \tag{29}$$

$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} = 0\,, \tag{30}$$

$$\nabla e_a = 0\,, \tag{31}$$

$$B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c \wedge e_d = 0\,. \tag{32}$$

Given this system of equations, all fields can be algebraically determined in terms of the tetrads ${e^a}_\mu$, as follows. From the equations (31) and (32) we obtain that $\nabla B^{ab} = 0$, from which it follows, using the equation (30), that

$e_{[a} \wedge \beta_{b]} = 0$. Assuming that the tetrads are nondegenerate, $e \equiv \det(e^a{}_\mu) \neq 0$, it can be shown that this is equivalent to $\beta^a = 0$ [12]. Therefore, from the equations (28), (30), (31) and (32) we obtain

$$\lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu} \,, \quad \beta^a{}_{\mu\nu} = 0 \,, \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu \,, \quad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu \,. \tag{33}$$

Here the Ricci rotation coefficients are defined as

$$\triangle^{ab}{}_\mu \equiv \frac{1}{2}(c^{abc} - c^{cab} + c^{bca}) e_{c\mu} \,, \tag{34}$$

where

$$c^{abc} = e^\mu{}_b e^\nu{}_c \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu \right) \,. \tag{35}$$

The last equation establishes that the spin connection 1-form ω^{ab} is expressed as a function of the tetrads, which then implies the same for the curvature 2-form R^{ab}. Finally, the remaining equation (29) then reduces to

$$\varepsilon_{abcd} R^{bc} \wedge e^d = 0 \,, \tag{36}$$

which is nothing but the vacuum Einstein field equation,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \,.$$

Therefore, the action (27) is classically equivalent to general relativity.

3.3. Einstein-Yang-Mills theory

As we have already mentioned above, the main advantage of the action (27) over the Plebanski model lies in the fact that the tetrad fields are explicitly present in the topological sector of the action. This allows one to couple matter fields in a straightforward way [12]. However, one can do even more [13], and couple the $SU(N)$ Yang-Mills fields to gravity within a unified framework of 2-group formalism.

Namely, we can modify the Poincaré 2-group structure to include the $SU(N)$ gauge group, as follows. We choose the two Lie groups as

$$G = SO(3,1) \times SU(N) \,, \qquad H = \mathbb{R}^4 \,, \tag{37}$$

and we define the action $\triangleright$ of the group G in the following fashion. As in the case of the Poincaré 2-group, it acts on itself via conjugation. Next, it acts on H such that the $SO(3,1)$ subgroup acts on $\mathbb{R}^4$ via the vector representation (21), while the action of the $SU(N)$ subgroup is trivial,

$$\tau_I \triangleright P_a = 0 \,, \tag{38}$$

where τ_I are the $SU(N)$ generators. The map ∂ also remains trivial, as before. The form of the 2-connection (α, β) now reflects the structure of the group G,

$$\alpha = \omega^{ab} M_{ab} + A^I \tau_I \,, \qquad \beta = \beta^a P_a \,, \tag{39}$$

where A^I is the gauge connection 1-form. Next, the curvature for α then becomes

$$\mathcal{F} = R^{ab} M_{ab} + F^I \tau_I \,, \qquad F^I \equiv \mathrm{d}A^I + f_{JK}{}^I A^J \wedge A^K \,. \tag{40}$$

The curvature for β remains the same as before, because of (38). Finally, the product structure of the group G implies that its Killing form $\langle \_ , \_ \rangle_{\mathfrak{g}}$ reduces to the Killing forms for the $SO(3,1)$ and $SU(N)$, along with the identity $\langle M_{ab} \,, \tau_I \rangle_{\mathfrak{g}} = 0$.

Given a crossed module defined in this way, its corresponding pure $2BF$ action (16) becomes

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a \,, \tag{41}$$

where $B^I \in \mathcal{A}^2(\mathcal{M}_4 \,, \mathfrak{su}(N))$ is the new Lagrange multiplier. The action (41) is topological, and again we add appropriate simplicity constraint terms, in order to transform it into action with nontrivial dynamics. The constraint giving rise to gravity is the same as in (27), while the constraint for the gauge fields is given as in the action (4) with the substitution $\delta^a \to e^a$. Putting everything together, we obtain:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a \\ & - \lambda_{ab} \wedge \Big(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \Big) + \lambda^I \wedge \Big(B_I - \frac{12}{g} M_{abI} e^a \wedge e^b \Big) \\ & + \zeta^{abI} \Big(M_{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - g_{IJ} F^J \wedge e_a \wedge e_b \Big) \,. \end{aligned} \tag{42}$$

It is crucial to note that the Yang-Mills simplicity constraints in (42) are obtained from the Yang-Mills action (4) by substituting the nondynamical background field δ^a from (4) with a dynamical field e^a. The relationship between these fields has already been hinted at in the equation (5), which describes the connection between δ^a and the flat spacetime metric $\eta_{\mu\nu}$. Once promoted to e^a, this field becomes dynamical due to the presence of gravitational terms, while the equation (5) becomes the usual relation between the tetrad and the metric,

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \,, \tag{43}$$

further confirming the identification $C^a = e^a$. Moreover, the total action (42) now becomes background independent, as expected in general relativity. All this is a consequence of the fact that the tetrad field is explicitly present in the topological sector of the action (27), and represents a clear improvement over the Plebanski model.

Taking the variations of the action (42) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, ζ^{abI}, M_{abI}, B_I, λ^I, A^I, and e^a, we obtain equations of motion. Similarly as before, all variables can be algebraically expressed as functions of A^I and e^a and their derivatives:

$$
\begin{gathered}
\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \beta_{a\mu\nu} = 0\,, \qquad \omega_{ab\mu} = \triangle_{ab\mu}\,, \qquad \lambda_{abI} = F_{abI}\,, \\
B_{\mu\nu I} = -\frac{e}{2g}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}{}_I\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \\
M_{abI} = -\frac{1}{4eg}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma\,, \qquad \zeta^{abI} = \frac{1}{4eg}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma\,.
\end{gathered}
\tag{44}
$$

In addition, we obtain two differential equations — An equation for A^I,

$$
\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + \Gamma^\rho{}_{\lambda\rho}F^{I\lambda\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0\,, \tag{45}
$$

where $\Gamma^\lambda{}_{\mu\nu}$ is the standard Levi-Civita connection, and an equation for e^a,

$$
R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \tag{46}
$$

where

$$
T^{\mu\nu} \equiv -\frac{1}{4g}\left(F_{\rho\sigma}{}^I F^{\rho\sigma}{}_I g^{\mu\nu} + 4F^{\mu\rho}{}_I F_\rho{}^{\nu I}\right)\,. \tag{47}
$$

In this way, we see that both gravity and gauge fields can be successfully represented within a unified framework of higher gauge theory, based on a 2-group structure. A generalization from $SU(N)$ Yang-Mills case to more complicated cases such as $SU(3)\times SU(2)\times U(1)$ is completely straightforward.

4. $3BF$ theory

While the structure of a 2-group can successfully describe both gravitational and gauge fields, unfortunately it cannot accommodate other matter fields, such as scalars or fermions. In order to remedy this drawback, we make one further step in the categorical ladder, passing from the notion of a 2-group to the notion of a 3-group. As it turns out, the 3-group structure is excellent for the description of all fields that are present in the Standard Model, coupled to gravity. Moreover, a 3-group contains one more gauge group, which is novel and corresponds to the choice of the scalar and fermion fields present in the theory. This is an unexpected and beautiful result, not present in ordinary gauge theory.

As before, we will begin by introducing the notion of a 3-group, and constructing the corresponding $3BF$ action. Afterwards, we will modify this action by adding appropriate simplicity constraints, giving rise to theories with expected nontrivial dynamics. Along the way, we shall see that scalar and fermion fields are being treated pretty much on an equal footing with gravity and gauge fields.

4.1. Pure $3BF$ theory

Similarly to the concepts of a group and a 2-group, one can introduce the notion of a 3-group in the framework of higher category theory, as a 3-category with only one object where all the morphisms, 2-morphisms and 3-morphisms are invertible. Also, in the same way as a 2-group is equivalent to a crossed module, it was proved that a strict 3-group is equivalent to a 2-*crossed module* [25].

A Lie 2-crossed module, denoted as $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$, is an algebraic structure specified by three Lie groups G, H and L, together with the homomorphisms δ and ∂, an action $\triangleright$ of the group G on all three groups, and a G-equivariant map

$$\{\_\,,\_\} : H \times H \to L\,.$$

called the Peiffer lifting. The maps ∂, δ, $\triangleright$ and the Peiffer lifting satisfy certain axioms, so that the resulting structure is equivalent to a 3-group [13].

Like in the cases of BF and $2BF$ actions, we can introduce a gauge invariant topological $3BF$ action over the manifold $\mathcal{M}_4$ for a given 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$. Denoting $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ as Lie algebras corresponding to the groups G, H and L, respectively, one can introduce a 3-connection (α, β, γ) given by the algebra-valued differential forms $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{h})$ and $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,, \mathfrak{l})$. The corresponding fake 3-curvature $(\mathcal{F}\,, \mathcal{G}\,, \mathcal{H})$ is then defined as

$$\begin{gathered} \mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \\ \mathcal{H} = \mathrm{d}\gamma + \alpha \wedge^{\triangleright} \gamma + \{\beta \wedge \beta\}\,, \end{gathered} \tag{48}$$

see [25, 26] for details. Note that γ is a 3-form, while its corresponding field strength $\mathcal{H}$ is a 4-form, necessitating that the spacetime manifold be at least 4-dimensional. Then, a $3BF$ action is defined as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}}\,, \tag{49}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$, $C \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ and $D \in \mathcal{A}^0(\mathcal{M}_4, \mathfrak{l})$ are Lagrange multipliers. Note that in precisely 4 spacetime dimensions the Lagrange multiplier D corresponding to $\mathcal{H}$ is a 0-form, i.e. a scalar function. The functionals $\langle \_ , \_ \rangle_{\mathfrak{g}}$, $\langle \_ , \_ \rangle_{\mathfrak{h}}$ and $\langle \_ , \_ \rangle_{\mathfrak{l}}$ are G-invariant bilinear symmetric non-degenerate forms on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, respectively. Under certain conditions, the forms $\langle \_ , \_ \rangle_{\mathfrak{h}}$ and $\langle \_ , \_ \rangle_{\mathfrak{l}}$ are also H-invariant and L-invariant.

One can see that varying the action with respect to the variables B^α, C^a and D^A (where indices A count the generators of the group L), one obtains the equations of motion

$$\mathcal{F}^\alpha = 0\,, \qquad \mathcal{G}^a = 0\,, \qquad \mathcal{H}^A = 0\,, \tag{50}$$

while varying with respect to α^α, β^a, γ^A one obtains

$$\mathrm{d}B_\alpha + f_{\alpha\beta}{}^{\gamma} B_\gamma \wedge \alpha^\beta - \rhd_{\alpha a}{}^{b} C_b \wedge \beta^a + \rhd_{\alpha B}{}^{A} D_A \wedge \gamma^B = 0\,, \tag{51}$$

$$\mathrm{d}C_a - \partial_a{}^{\alpha} B_\alpha + \rhd_{\alpha a}{}^{b} C_b \wedge \alpha^\alpha + 2X_{\{ab\}}{}^{A} D_A \wedge \beta^b = 0\,, \tag{52}$$

$$\mathrm{d}D_A - \rhd_{\alpha A}{}^{B} D_B \wedge \alpha^\alpha + \delta_A{}^{a} C_a = 0\,. \tag{53}$$

4.2. Klein-Gordon theory

Now we proceed to demonstrate that one can use the 3-group structure and the corresponding $3BF$ theory to describe the Klein-Gordon field coupled to general relativity. We begin by specifying a 2-crossed module, which is used to construct the topological $3BF$ theory, and then we impose appropriate simplicity constraints to obtain the desired equations of motion.

We specify a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \rhd\,, \{\_ , \_\})$, as follows. The groups are given as

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}\,. \tag{54}$$

The group G acts on itself via conjugation, on H via the vector representation, and on L via the trivial representation. This specifies the definition of the action $\rhd$. The map ∂ is chosen to be trivial, as before. The map δ is also trivial, that is, every element of L is mapped to the identity element of H. Finally, the Peiffer lifting is trivial as well, mapping every ordered pair of elements in H to an identity element in L. This specifies one concrete 2-crossed module which, as we shall see below, corresponds to gravity and one real scalar field.

Given this choice of a 2-crossed module, the 3-connection $(\alpha\,, \beta\,, \gamma)$ takes the form

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma \mathbb{I}\,, \tag{55}$$

where $\mathbb{I}$ is the sole generator of the Lie group $\mathbb{R}$. Since the homomorphisms ∂ and δ are trivial, as well as the Peiffer lifting, the fake 3-curvature (48) reduces to the ordinary 3-curvature,

$$\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla \beta^a P_a\,, \qquad \mathcal{H} = \mathrm{d}\gamma\,, \tag{56}$$

where we used the fact that G acts trivially on L, that is, $M_{ab} \triangleright \mathbb{I} = 0$. This means that the 3-form γ transforms as a scalar with respect to Lorentz symmetry. Consequently, its Lagrange multiplier D also transforms as a scalar, since it also belongs to the algebra $\mathfrak{l}$. Since D is also a 0-form, it transforms as a scalar with respect to diffeomorphisms as well. In other words, D completely behaves as a real scalar field, so we relabel it into more traditional notation, $D \equiv \phi$, and write the pure $3BF$ action (49) as:

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \phi \, \mathrm{d}\gamma \,, \tag{57}$$

where the bilinear form for L is $\langle \mathbb{I}, \mathbb{I} \rangle_{\mathfrak{l}} = 1$.

The existence of a scalar field in the $3BF$ action is a crucial property of a 3-group in a 4-dimensional spacetime, just like identifying the Lagrange multiplier C^a with a tetrad field e^a was a crucial property of the $2BF$ action and the Poincaré 2-group. We can also see that the choice of the third gauge group, L, dictates the number and the structure of the matter fields present in the action. In this case, $L = \mathbb{R}$ implies that we have only one real scalar field, corresponding to a single generator $\mathbb{I}$ of $\mathbb{R}$. The trivial nature of the action $\triangleright$ of $SO(3,1)$ on $\mathbb{R}$ implies that ϕ transforms as a scalar field. Finally, the scalar field appears in the topological sector of the action, making the quantization procedure feasible.

As in the case of BF and $2BF$ theories, we need to add appropriate simplicity constraints to the action (57). In order to obtain the Klein-Gordon field ϕ of mass m coupled to gravity in the standard way, the action takes the form:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \phi \, \mathrm{d}\gamma \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda \wedge \left(\gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c \right) \\ & + \Lambda^{ab} \wedge \left(H_{abc} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b \right) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi^2 \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,. \end{aligned} \tag{58}$$

The first row is the topological sector (57), the second row is the familiar simplicity constraint for gravity from the action (27), the third and fourth rows contain the new simplicity constraints featuring the Lagrange multiplier 1-forms λ and Λ^{ab} and the 0-form H_{abc}, while the fifth row is the mass term for the scalar field.

The variation of (58) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, Λ_{ab}, γ, λ, H_{abc}, ϕ and e^a gives us the equations of motion. As before, all variables can be algebraically expressed in terms of the tetrads e^a and the scalar field ϕ:

$$\begin{gathered}\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \omega^{ab}{}_{\mu} = \triangle^{ab}{}_{\mu}\,, \qquad \gamma_{\mu\nu\rho} = -\frac{e}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^{\sigma}\phi\,, \\ \beta^{a}{}_{\mu\nu} = 0\,, \qquad \Lambda^{ab}{}_{\mu} = \frac{1}{12e}g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\partial_{\nu}\phi e^{a}{}_{\rho}e^{b}{}_{\sigma}\,, \qquad \lambda_{\mu} = \partial_{\mu}\phi\,, \\ H^{abc} = \frac{1}{6e}\varepsilon^{\mu\nu\rho\sigma}\partial_{\mu}\phi e^{a}{}_{\nu}e^{b}{}_{\rho}e^{c}{}_{\sigma}\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^{c}{}_{\mu}e^{d}{}_{\nu}\,.\end{gathered} \tag{59}$$

The equations of motion for e^a and ϕ, however, are differential equations. The equation for the scalar field becomes the covariant Klein-Gordon equation,

$$\left(\nabla_{\mu}\nabla^{\mu} - m^2\right)\phi = 0\,, \tag{60}$$

while the equation for the tetrads is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \tag{61}$$

where

$$T^{\mu\nu} \equiv \partial^{\mu}\phi\partial^{\nu}\phi - \frac{1}{2}g^{\mu\nu}\left(\partial_{\rho}\phi\partial^{\rho}\phi + m^2\phi^2\right) \tag{62}$$

is the stress-energy tensor for a single real scalar field.

4.3. Einstein-Cartan-Dirac theory

In order to describe the Dirac field coupled to Einstein-Cartan gravity, we follow the same procedure as for the case of the scalar field, but now we choose the 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$ in a different way, as follows. The groups are:

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^8(\mathbb{G})\,, \tag{63}$$

where $\mathbb{G}$ is the algebra of complex Grassmann numbers. The maps ∂, δ and the Peiffer lifting are trivial, as before. The action of the group G on itself is given via conjugation, on H via vector representation, and on L via spinor representation, in the following way. Denoting the 8 generators of the Lie group $\mathbb{R}^8(\mathbb{G})$ as P_α and P^α, where the index α takes the values $1,\dots,4$, the action $\triangleright$ of G on L is thus given explicitly as

$$M_{ab} \triangleright P_{\alpha} = \frac{1}{2}(\sigma_{ab})^{\beta}{}_{\alpha}P_{\beta}\,, \qquad M_{ab} \triangleright P^{\alpha} = -\frac{1}{2}(\sigma_{ab})^{\alpha}{}_{\beta}P^{\beta}\,, \tag{64}$$

where $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, and γ_a are the usual Dirac matrices, satisfying the anticommutation rule $\{\gamma_a\,, \gamma_b\} = -2\eta_{ab}$.

As in the case of the scalar field, the choice of the group L dictates the matter content of the theory, while the action $\triangleright$ of G on L specifies its transformation properties.

Let us now proceed to construct the $3BF$ action. The 3-connection $(\alpha\,,\beta\,,\gamma)$ takes the form

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma^\alpha P_\alpha + \bar{\gamma}_\alpha P^\alpha\,, \tag{65}$$

while the 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is given as

$$\begin{aligned} &\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla \beta^a P_a\,,\\ \mathcal{H} &= \Big(\mathrm{d}\gamma^\alpha + \frac{1}{2}\omega^{ab}(\sigma_{ab})^\alpha{}_\beta \gamma^\beta\Big) P_\alpha + \Big(\mathrm{d}\bar{\gamma}_\alpha - \frac{1}{2}\omega^{ab}\bar{\gamma}_\beta (\sigma_{ab})^\beta{}_\alpha\Big) P^\alpha \\ &\equiv (\overset{\rightarrow}{\nabla}\gamma)^\alpha P_\alpha + (\bar{\gamma}\overset{\leftarrow}{\nabla})_\alpha P^\alpha\,, \end{aligned} \tag{66}$$

where we have used (64). The bilinear form $\langle \_\,,\_\rangle_{\mathfrak{l}}$ is defined via its action on the generators:

$$\begin{aligned} \langle P_\alpha\,, P_\beta\rangle_{\mathfrak{l}} &= 0\,, \qquad \langle P^\alpha\,, P^\beta\rangle_{\mathfrak{l}} = 0\,,\\ \langle P_\alpha\,, P^\beta\rangle_{\mathfrak{l}} &= -\delta_\alpha^\beta\,, \qquad \langle P^\alpha\,, P_\beta\rangle_{\mathfrak{l}} = \delta^\alpha_\beta\,. \end{aligned} \tag{67}$$

Note that the bilinear form defined in this way is antisymmetric, rather than symmetric, when it acts on the generators. The reason for this is the following. For general $A, B \in \mathfrak{l}$, we want the bilinear form to be symmetric. Expanding A and B into components, we can write

$$\langle A\,, B\rangle_{\mathfrak{l}} = A^I B^J g_{IJ}\,, \qquad \langle B\,, A\rangle_{\mathfrak{l}} = B^J A^I g_{JI}\,. \tag{68}$$

Since we require the bilinear form to be symmetric, the two expressions must be equal. However, since the coefficients in $\mathfrak{l}$ are Grassmann numbers, we have $A^I B^J = -B^J A^I$, so it follows that $g_{IJ} = -g_{JI}$. Hence the antisymmetry of (67) — it compensates for the anticommutativity property of the Grassman coefficients, making the bilinear form symmetric for general algebra elements $A, B \in \mathfrak{l}$.

Now we employ the action $\triangleright$ of G on L to determine the transformation properties of the Lagrange multiplier D in (49). Indeed, the choice of the group L dictates that D contains 8 independent complex Grassmannian matter fields as its components. Moreover, due to the fact that D is a 0-form and that it transforms according to the spinorial representation of $SO(3,1)$, we can identify its components with the Dirac bispinor fields, and write

$$D = \psi^\alpha P_\alpha + \bar{\psi}_\alpha P^\alpha\,. \tag{69}$$

This is again an illustration of the fact that information about the structure of the matter sector in the theory is specified by the choice of the group L

in the 2-crossed module, and its transformation properties with respect to the Lorentz group are fixed by the action $\triangleright$.

Given all of the above, we write the corresponding pure $3BF$ action as:

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + (\bar{\gamma}\overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla}\gamma)^\alpha \,. \tag{70}$$

In order to obtain the action that gives us the dynamics of Einstein-Cartan theory of gravity coupled to a Dirac field, we add the following simplicity constraints:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + (\bar{\gamma}\overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla}\gamma)^\alpha \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & - \lambda^\alpha \wedge \left(\bar{\gamma}_\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar{\psi}\gamma^d)_\alpha \right) \\ & + \bar{\lambda}_\alpha \wedge \left(\gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^\alpha \right) \\ & - \frac{1}{12} m \,\bar{\psi}\psi \,\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d + 2\pi i l_p^2 \,\bar{\psi}\gamma_5\gamma^a\psi \,\varepsilon_{abcd} e^b \wedge e^c \wedge \beta^d. \end{aligned} \tag{71}$$

Similarly to the previous case of the scalar field, we recognize the topological sector in the first row, the gravitational simplicity constraint in the second row, while the third and fourth rows contain the new simplicity constraints for the Dirac field, featuring the Lagrange multiplier 1-forms λ^α and $\bar{\lambda}_\alpha$. The fifth row contains the mass term for the Dirac field, and a term which ensures the correct coupling between the torsion and the spin of the Dirac field. In particular, we want to obtain

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a \,, \tag{72}$$

as one of the equations of motion, where

$$s_a = i\varepsilon_{abcd} e^b \wedge e^c \bar{\psi}\gamma_5\gamma^d\psi \tag{73}$$

is the Dirac spin 2-form. Of course, other alternative coupling choices are possible, but we choose this one since this is the traditional coupling most often discussed in textbooks.

The variation of the action (71) with respect to B_{ab}, λ^{ab}, $\bar{\gamma}_\alpha$, γ^α, λ^α, $\bar{\lambda}_\alpha$, $\bar{\psi}_\alpha$, ψ^α, e^a, β^a and ω^{ab}, again gives us equations of motion, which can be algebraically solved for all fields as functions of e^a, ψ and $\bar{\psi}$:

$$
\begin{gathered}
B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu \,, \qquad \lambda^\alpha{}_\mu = (\overrightarrow{\nabla}_\mu \psi)^\alpha \,, \qquad \bar{\lambda}_{\alpha\mu} = (\bar{\psi}\overleftarrow{\nabla}_\mu)_\alpha \,, \\
\bar{\gamma}_{\alpha\mu\nu\rho} = i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho (\bar{\psi}\gamma^d)_\alpha \,, \qquad \gamma^\alpha{}_{\mu\nu\rho} = -i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho (\gamma^d \psi)^\alpha \,, \\
\beta^a{}_{\mu\nu} = 0 \,, \qquad \lambda_{ab\mu\nu} = R_{ab\mu\nu} \,, \qquad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu + K^{ab}{}_\mu \,.
\end{gathered} \tag{74}
$$

Here $K^{ab}{}_\mu$ is the contorsion tensor, constructed in the standard way from the torsion tensor. In addition, we also obtain

$$
T_a \equiv \nabla e_a = 2\pi l_p^2 s_a \,, \tag{75}
$$

which is precisely the desired equation (72) for the torsion. Finally, the differential equations of motion for ψ and $\bar{\psi}$ are the standard covariant Dirac equation,

$$
(i\gamma^a e^\mu{}_a \overrightarrow{\nabla}_\mu - m)\psi = 0 \,, \tag{76}
$$

and its conjugate,

$$
\bar{\psi}(i\overleftarrow{\nabla}_\mu e^\mu{}_a \gamma^a + m) = 0 \,, \tag{77}
$$

where $e^\mu{}_a$ is the inverse tetrad. The differential equation of motion for e^a is

$$
R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2 \, T^{\mu\nu} \,, \tag{78}
$$

where

$$
T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\gamma^a \overleftrightarrow{\nabla}^\nu e^\mu{}_a \psi - \frac{1}{2}g^{\mu\nu}\bar{\psi}\Big(i\gamma^a\overleftrightarrow{\nabla}_\rho e^\rho{}_a - 2m\Big)\psi \,, \tag{79}
$$

Here, we used the notation $\overleftrightarrow{\nabla} = \overrightarrow{\nabla} - \overleftarrow{\nabla}$. As expected, the equations of motion (75), (76), (77) and (78) are precisely the equations of motion of the Einstein-Cartan-Dirac theory.

4.4. Weyl and Majorana fields coupled to Einstein-Cartan gravity

As is well known, the Dirac fermions are not an irreducible representation of the Lorentz group, and one can rewrite them as left-chiral and right-chiral irreducible Weyl fermion fields. Hence, it is useful to construct the 2-crossed module and a constrained $3BF$ action for left and right Weyl spinors. For simplicity, we will discuss only the left-chiral spinor field (the right-chiral can be studied analogously). Additionally, we can also describe Majorana fermions using the same formalism, the only difference being the presence of an additional mass term in the Majorana action.

We soecify a 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,,\rhd\,,\{\_\,,\_\})$, in a way similar to the Dirac case, as follows. The groups are:

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^4(\mathbb{G})\,. \tag{80}$$

The maps ∂, δ and the Peiffer lifting are trivial. The action $\rhd$ of the group G on G, H and L is given in the same way as for the Dirac case, whereas the spinorial representation reduces to

$$M_{ab} \rhd P^\alpha = \frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta\,, \qquad M_{ab} \rhd P_{\dot\alpha} = \frac{1}{2}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha} P_{\dot\beta}\,, \tag{81}$$

where $\sigma^{ab} = -\bar\sigma^{ab} = \frac{1}{4}(\sigma^a\bar\sigma^b - \sigma^b\bar\sigma^a)$, for $\sigma^a = (1,\vec\sigma)$ and $\bar\sigma^a = (1,-\vec\sigma)$, in which $\vec\sigma$ denotes the set of three Pauli matrices. The four generators of the group L are denoted as P^α and $P_{\dot\alpha}$, where the Weyl indices $\alpha,\dot\alpha$ take values $1,2$.

The 3-connection $(\alpha\,,\beta\,,\gamma)$ takes the form

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma_\alpha P^\alpha + \bar\gamma^{\dot\alpha} P_{\dot\alpha}\,, \tag{82}$$

while the 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is

$$\begin{aligned} &\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,,\\ \mathcal{H} &= \big(\mathrm{d}\gamma_\alpha + \frac{1}{2}\omega^{ab}(\sigma^{ab})^\beta{}_\alpha \gamma_\beta\big)P^\alpha + \big(\mathrm{d}\bar\gamma^{\dot\alpha} + \frac{1}{2}\omega_{ab}(\bar\sigma^{ab})^{\dot\alpha}{}_{\dot\beta}\bar\gamma^{\dot\beta}\big)P_{\dot\alpha}\\ &\equiv (\overrightarrow{\nabla}\gamma)_\alpha P^\alpha + (\bar\gamma\overleftarrow{\nabla})^{\dot\alpha} P_{\dot\alpha}\,. \end{aligned} \tag{83}$$

The Lagrange multiplier D now contains as coefficients the spinor fields ψ_α and $\bar\psi^{\dot\alpha}$,

$$D = \psi_\alpha P^\alpha + \bar\psi^{\dot\alpha} P_{\dot\alpha}\,, \tag{84}$$

and the bilinear form $\langle\_\,,\_\rangle_\mathfrak{l}$ for the group L is

$$\begin{aligned} \langle P^\alpha\,, P^\beta\rangle_\mathfrak{l} = \varepsilon^{\alpha\beta}\,, \qquad \langle P_{\dot\alpha}\,, P_{\dot\beta}\rangle_\mathfrak{l} = \varepsilon_{\dot\alpha\dot\beta}\,,\\ \langle P^\alpha\,, P_{\dot\beta}\rangle_\mathfrak{l} = 0\,, \qquad \langle P_{\dot\alpha}\,, P^\beta\rangle_\mathfrak{l} = 0\,, \end{aligned} \tag{85}$$

where $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot\alpha\dot\beta}$ are the usual two-dimensional antisymmetric Levi-Civita symbols.

The pure $3BF$ action (49) now becomes

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + e_a \wedge \nabla\beta^a + \psi^\alpha \wedge (\overrightarrow{\nabla}\gamma)_\alpha + \bar\psi_{\dot\alpha}\wedge(\bar\gamma\overleftarrow{\nabla})^{\dot\alpha}\,. \tag{86}$$

In order to obtain the suitable equations of motion for the Weyl spinors, we again introduce appropriate simplicity constraints, to obtain:

$$
\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \psi^\alpha \wedge (\overrightarrow{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}} \wedge (\bar{\gamma}\overleftarrow{\nabla})^{\dot{\alpha}} \\
& - \lambda_{ab} \wedge (B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c \wedge e_d) \\
& - \lambda^\alpha \wedge (\gamma_\alpha + \frac{i}{6}\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}}) \\
& - \bar{\lambda}_{\dot{\alpha}} \wedge (\bar{\gamma}^{\dot{\alpha}} + \frac{i}{6}\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta) \\
& - 4\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\bar{\psi}_{\dot{\alpha}} \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta) \,.
\end{aligned}
\tag{87}
$$

The new simplicity constraints, in the third and fourth rows, feature the Lagrange multiplier 1-forms λ_α and $\bar{\lambda}^{\dot{\alpha}}$. Also, in analogy to the coupling between the spin and the torsion in Einstein-Cartan-Dirac theory, the term in the fifth row is chosen to ensure that the coupling between the Weyl spin tensor

$$s_a \equiv i\varepsilon_{abcd} e^b \wedge e^c \; \psi^\alpha \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} \tag{88}$$

and torsion is given as:

$$T_a = 4\pi l_p^2 s_a \,. \tag{89}$$

The action for the Majorana field is precisely the same, but for an additional mass term in the action:

$$-\frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d (\psi^\alpha \psi_\alpha + \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}) \,. \tag{90}$$

The variation of the action (87) with respect to the variables B_{ab}, λ^{ab}, γ_α, $\bar{\gamma}^{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, e^a, β^a and ω^{ab} gives us the equations of motion, which can be algebraically solved for all variables as functions of ψ_α, $\bar{\psi}^{\dot{\alpha}}$ and e^a:

$$
\begin{gathered}
\beta^a{}_{\mu\nu} = 0 \,, \quad \lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu} \,, \quad \lambda_{\alpha\mu} = \nabla_\mu \psi_\alpha \,, \quad \bar{\lambda}^{\dot{\alpha}}{}_\mu = \nabla_\mu \bar{\psi}^{\dot{\alpha}} \,, \\
B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu \,, \quad \omega_{ab\mu} = \triangle_{ab\mu} + K_{ab\mu} \,, \\
\gamma_{\alpha\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} \,, \quad \bar{\gamma}^{\dot{\alpha}}{}_{\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta \,.
\end{gathered}
\tag{91}
$$

In addition, one also obtains (89). Finally, the differential equations of motion for the spinor and tetrad fields are

$$\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta = 0 \,, \qquad \sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} = 0 \,, \tag{92}$$

and

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2 \, T^{\mu\nu} \,, \tag{93}$$

where

$$\begin{aligned} T^{\mu\nu} \;\; \equiv \;\; & \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} \\ & -\frac{1}{2} g^{\mu\nu} \Big(i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi} \Big) \,. \end{aligned} \tag{94}$$

Here we have suppressed the spinor indices, for simplicity. In the case of the Majorana field, the equations of motion (91) remain the same. The equations of motion for ψ_α and $\bar{\psi}^{\dot{\alpha}}$ obtain the additional mass term,

$$i\sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} - m\psi_\alpha = 0 \,, \qquad i\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta - m\bar{\psi}^{\dot{\alpha}} = 0 \,, \tag{95}$$

while the stress-energy tensor becomes

$$\begin{aligned} T^{\mu\nu} \;\; \equiv \;\; & \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} \\ & -g^{\mu\nu}\frac{1}{2} \left[i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi} - \frac{1}{2} m \left(\psi\psi + \bar{\psi}\bar{\psi} \right) \right] \,. \end{aligned} \tag{96}$$

5. Conclusions

Let us summarize the results of the paper. In Section 2 we have introduced the BF theory and discussed models based on constrained BF action, in particular the Yang-Mills theory in Minkowski spacetime and the Plebanski formulation of general relativity. Section 3 was devoted to the first step in the categorical ladder and the $2BF$ theory. After introducing the notions of a 2-group, a crossed module, and the corresponding $2BF$ theory, we have studied the $2BF$ formulation of general relativity and the Einstein-Yang-Mills theory. Then, in Section 4 we have performed one more step in the categorical ladder, and introduced the notions of a 3-group, 2-crossed module, and the $3BF$ theory. This structure was employed to construct the constrained $3BF$ actions for the cases of Klein-Gordon, Dirac, Weyl and Majorana fields, each coupled to the Einstein-Cartan gravity in the standard way. In those descriptions, it turned out that the scalar and fermion fields are associated to a *new gauge group*, similar to the gauge fields being associated to a gauge group in the Yang-Mills theory. This opens up a possibility of a classification of matter fields based on an algebraic structure of a 3-group.

All the obtained results serve to complete the first step of the spinfoam quantization programme, as outlined in the Introduction. This paves the way to the study of steps 2 and 3 of the programme. Namely, the full action for gravity, gauge fields and matter is written completely in the langulage of differential forms, which can be easily adapted to a triangulated spacetime manifold, in the sense of Regge calculus. This can be seen in the following table:

d	triangulation	dual triangulation	form	fields	field strengths
0	vertex	4-polytope	0-form	ϕ, $\psi_{\tilde{\alpha}}$, $\bar{\psi}^{\tilde{\alpha}}$	
1	edge	3-polyhedron	1-form	ω^{ab}, A^I, e^a	
2	triangle	face	2-form	β^a, B^{ab}	R^{ab}, F^I, T^a
3	tetrahedron	edge	3-form	γ, $\gamma_{\tilde{\alpha}}$, $\bar{\gamma}^{\tilde{\alpha}}$	$\mathcal{G}^a$
4	4-simplex	vertex	4-form		$\mathcal{H}$, $\mathcal{H}_{\tilde{\alpha}}$, $\bar{\mathcal{H}}^{\tilde{\alpha}}$

This data can be utilized to construct a Regge-discretized topological $3BF$ action, and from that a state sum Z, giving rise to a rigorous definition of the path integral

$$Z = \int \mathcal{D}g \int \mathcal{D}\phi \; e^{iS[g,\phi]} \,, \tag{97}$$

which is a generalization of (1) in the sense that it adds matter fields (including the gauge boson sector) to gravity at the quantum level. Being a topological theory, and given the underlying structure of the 3-group, a pure $3BF$ action ought to ensure the topological invariance of the state sum Z, i.e., Z should be triangulation independent. This step, however, requires the generalizations of the Peter-Weyl and Plancharel theorems to 2-groups and 3-groups, which are unfortunately still missing (though there are some attempts to circumvent them at least in the 2-group case [27, 28]). Namely, the purpose of the Peter-Weyl and Plancharel theorems is to provide a decomposition of a function on a group into a sum over the corresponding irreducible representations, which then specifies the spectrum of labels for the simplices in the triangulation, and fixes the domain of values for the fields living on those simplices. In the absence of the two theorems, one can still try to *guess* the irreducible representations of the 2- and 3-groups, as was done for example in the *spincube model* of quantum gravity [12], or to try to construct the state sum using other techniques, as was done in [27, 28]).

Of course, when building a realistic theory, we are not interested in a topological theory, but instead in one which contains local propagating degrees of freedom. Thus the state sum Z need not be a topological invariant. This is obtained via the step 3 of the spinfoam quantization programme, by imposing the simplicity constraints on Z. The classical actions discussed in this paper manifestly distinguish the topological sector from the simplicity constraints, which have been explicitly determined. Imposing them should thus be a straightforward procedure for a given Z. Completing this programme would ultimately lead us to a tentative state sum describing both gravity and matter at a quantum level, which is a topic for future research.

In addition to the construction of a full quantum theory of gravity, there are also many additional possible studies of the classical constrained $3BF$ action. For example, a Hamiltonian analysis of the theory could be interesting for the canonical quantization programme, and some work has begun in this area [29]. Also, it is worth looking into the idea of imposing the simplicity constraints using a spontaneous symmetry breaking mechanism. Finally, one can also study in more depth the mathematical structure and properties of the simplicity constraints. The list is not conclusive, and there may be many other interesting topics to study.

References

[1] C. Rovelli, *Quantum Gravity*, Cambridge University Press, Cambridge (2004).

[2] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity*, Cambridge University Press, Cambridge (2014).

[3] T. Thiemann, *Modern Canonical Quantum General Relativity*, Cambridge University Press, Cambridge (2007).

[4] G. Ponzano and T. Regge, *Semiclassical limit of Racah coefficients*, *Spectroscopic and Group Theoretical Methods in Physics*, edited by F. Block, North Holland, Amsterdam (1968).

[5] J. W. Barrett and L. Crane, *Relativistic spin networks and quantum gravity*, *J. Math. Phys.* **39**, 3296 (1998), `arXiv:gr-qc/9709028`.

[6] J. W. Barrett and L. Crane, *A Lorentzian Signature Model for Quantum General Relativity*, *Class. Quant. Grav.* **17**, 3101 (2000), `arXiv:gr-qc/9904025`.

[7] H. Ooguri, *Topological Lattice Models in Four Dimensions*, *Mod. Phys. Lett. A* **7**, 279 (1992), `arXiv:hep-th/9205090`.

[8] J. Engle, E. R. Livine, R. Pereira and C. Rovelli, *LQG vertex with finite Immirzi parameter*, *Nucl. Phys.* **B799**, 136 (2008), `arXiv:0711.0146`.

[9] L. Freidel and K. Krasnov, *A New Spin Foam Model for 4d Gravity*, *Class. Quant. Grav.* **25**, 125018 (2008), `arXiv:0708.1595`.

[10] E. Bianchi, M, Han, E. Magliaro, C. Perini, C. Rovelli and W. Wieland, *Spinfoam fermions*, *Class. Quantum Grav.* **30**, 235023 (2013), `arXiv:1012.4719`.

[11] J. C. Baez and J. Huerta, *An Invitation to Higher Gauge Theory*, *Gen. Relativ. Gravit.* **43**, 2335 (2011), `arXiv:1003.4485`.

[12] A. Miković and M. Vojinović, *Poincare 2-group and quantum gravity*, *Class. Quant. Grav.* **29**, 165003 (2012), `arXiv:1110.4694`.

[13] T. Radenković and M. Vojinović, *Higher Gauge Theories Based on 3-groups*, *JHEP* **10**, 222 (2019), `arXiv:1904.07566`.

[14] M. Celada, D. González and M. Montesinos, *BF gravity*, *Class. Quant. Grav.* **33**, 213001 (2016), `arXiv:1610.02020`.

[15] C. Rovelli, *Zakopane lectures on loop gravity,* `arXiv:1102.3660`.

[16] J. F. Plebanski, *On the separation of Einsteinian substructures*, *J. Math. Phys.* **18**, 2511 (1977).

[17] F. Girelli, H. Pfeiffer and E. M. Popescu, *Topological Higher Gauge Theory - from BF to BFCG theory*, *Jour. Math. Phys.* **49**, 032503 (2008), `arXiv:0708.3051`.

[18] J. F. Martins and A. Miković, *Lie crossed modules and gauge-invariant actions for 2-BF theories*, *Adv. Theor. Math. Phys.* **15**, 1059 (2011), arXiv:1006.0903.

[19] L. Crane and M. D. Sheppeard, *2-categorical Poincare Representations and State Sum Applications*, arXiv:math/0306440.

[20] M. Vojinović, *Causal Dynamical Triangulations in the Spincube Model of Quantum Gravity*, *Phys. Rev. D* **94**, 024058 (2016), arXiv:1506.06839.

[21] A. Miković, *Spin-cube Models of Quantum Gravity*, *Rev. Math. Phys.* **25**, 1343008 (2013), arXiv:1302.5564.

[22] A. Miković and M. A. Oliveira, *Canonical formulation of Poincare BFCG theory and its quantization*, *Gen. Relativ. Gravit.* **47**, 58 (2015), arXiv:1409.3751.

[23] A. Miković, M. A. Oliveira and M. Vojinović, *Hamiltonian analysis of the BFCG theory for a generic Lie 2-group*, arXiv:1610.09621.

[24] A. Miković, M. A. Oliveira and M. Vojinović, *Hamiltonian analysis of the BFCG formulation of General Relativity*, *Class. Quant. Grav.* **36**, 015005 (2019), arXiv:1807.06354.

[25] J. F. Martins and R. Picken, *The fundamental Gray 3-groupoid of a smooth manifold and local 3-dimensional holonomy based on a 2-crossed module*, *Differ. Geom. Appl. Journal* **29**, 179 (2011), arXiv:0907.2566.

[26] W. Wang, *On 3-gauge transformations, 3-curvature and Gray-categories*, *Jour. Math. Phys.* **55**, 043506 (2014), arXiv:1311.3796.

[27] A. Baratin and L. Freidel, *A 2-categorical state sum model*, *J. Math. Phys.* **56**, 011705 (2015), arXiv:1409.3526.

[28] S. K. Asante, B. Dittrich, F. Girelli, A. Riello and P. Tsimiklis, *Quantum geometry from higher gauge theory*, arXiv:1908.05970.

[29] T. Radenković and M. Vojinović, *Hamiltonian Analysis for the Scalar Electrodynamics as 3BF Theory*, *Symmetry* **12**, 620 (2020), arXiv:2004.06901.

Quantum gravity for piecewise flat spacetimes*

Aleksandar Miković†
Departamento de Matemática, Universidade Lusófona
Av. do Campo Grande, 376, 1749-024 Lisboa, Portugal

Marko Vojinović‡
Institute of Physics, Pregrevica 118, 11080 Belgrade, Serbia

Abstract

We describe a theory of quantum gravity which is based on the assumption that the spacetime structure at small distances is given by a piecewise linear (PL) 4-manifold corresponding to a triangulation of a smooth 4-manifold. The fundamental degrees of freedom are the edge lengths of the triangulation. One can work with finitely many edge lengths, so that the corresponding Regge path integral can be made finite by using an appropriate path-integral measure. The semi-classical limit is computed by using the effective action formalism, and the existence of a semi-classical effective action restricts the choice of the path-integral measure. The classical limit is given by the Regge action, so that one has a quantum gravity theory for a piecewise-flat general relativity. By using the effective action formalism we show that the observed value of the cosmological constant can be recovered from the effective cosmological constant. When the number of 4-simplices in the spacetime triangulation is large, then the PL effective action is well approximated by a quantum field theory effective action with a physical cutoff determined by the smallest edge length.

1. Introduction

The standard approach to the problem of constructing a quantum gravity (QG) theory [1, 2] can be described as the following problem. Let M be a smooth 4-manifold, of topology $\Sigma \times I$, where Σ is a 3-manifold and I an interval from $\mathbf{R}$. Let g be a Minkowski-signature metric on M and Φ a set of matter fields on M. Then the goal is to find a triple $(\hat{g}, \hat{\Phi}, \hat{U})$, where $\hat{g}$ and $\hat{\Phi}$ represent Hermitian operators parametrized by the points of M,

* This work has been supported by GFMUL (Mathematical Physics Group at University of Lisbon) and the project ON 171031 of the Ministry of Education, Science and Technological Development, Serbia.

† e-mail address: amikovic@ulusofona.pt

‡ e-mail address: vmarko@ipb.ac.rs

acting in some Hilbert space $\mathcal{H}$, while $\hat{U}$ is a unitary evolution operator parametrized by I, such that the the classical limit ($\hbar \to 0$) of the quantum time-evolution is equivalent to the Einstein equations.

The best known example of this approach is Loop Quantum Gravity (LQG), see [3] for a recent review and references. In the LQG case, the Hilbert space $\mathcal{H}$ is only known to be a subset of a non-separable Hilbert space and $\hat{U}$ can be constructed only for a triangulation $T(M)$ of M, so that it is not clear what is the classical limit. Note that in the standard QG approach, the structure of M is not changed after the quantization, and it is well known that this is the main source of the difficulties for a quantization of gravity [1, 2]. This leads us to an alternative approach where M is replaced by a quantum spacetime $\widehat{M}$. The obvious choice would be a non-commutative manifold based on M, like in the case of noncommutative geometry (NCG) [4], where the coordinates of M become elements of a noncommutative algebra. Another choice is made in the superstring theory [5], where the coordinates of M become coordinates of the loop manifold $\mathcal{L}M$ and new Grassmann (anticommuting) coordinates are added, so that $\widehat{M}$ is a loop super manifold.

In this paper we would like to present the case when $\widehat{M} = T(M)$, see [12, 13]. This is clearly a much simpler choice for $\widehat{M}$ than the one made in NCG or in the superstring theory, but the price paid is that the spacetime triangulation becomes a physical structure. However, the PL manifold $T(M)$ looks like the smooth manifold M when the number of 4-simplices is large. Also, by using $T(M)$ one reduces the infinite number of the degrees of freedom (DOF) for g and Φ to a finite number, which then simplifies the quantization.

Note that Regge was the first to use $T(M)$ in order to define the path integral for general relativity (GR) [6], see [7] for a modern review. However, in Regge's approach the triangulation was an auxiliary structure and had to be removed via the smooth limit $T(M) \to M$. However, obtaining the smooth limit in the Regge approach is a difficult problem. The same applies to the case of spin-foam models of LQG, which can be only defined when the spacetime is a PL manifold. In causal dynamical triangulations (CDT) approach [8], $T(M)$ is also used to define the path integral, but it is also considered an auxiliary structure. Obtaining the smooth limit in CDT is proposed by performing a sum over the triangulations.

2. PL gravity path integral

Let $T(M)$ be a regular[1] triangulation of a smooth 4-manifold M. We will assign positive numbers L_ϵ to the edges ϵ of $T(M)$. If we think of an L_ϵ as a distance between two vertices of $T(M)$ induced by some metric, then we

[1]Any two k-simplices of $T(M)$ cannot have more than one common $(k-1)$-simplex, where $k = 1, 2, 3, 4$.

can define a constant metric in each 4-simplex σ

$$g_{kl}^{(\sigma)} = \frac{L_{0k}^2 + L_{0l}^2 - L_{kl}^2}{2L_{0k}L_{0l}}\,, \qquad 1 \le k,l \le 4\,, \tag{1}$$

where the indices $0,1,2,3,4$ denote the vertices of a 4-simplex σ. Hence we replace a smooth metric g on M by a PL version (1). We want that the PL metric has the Minkowski signature, and this can be ensured by requiring that L_ϵ satisfy the triangle inequalities for the triangles which belong to one of the tetrahedrons of σ, for example the tetrahedron $(1,2,3,4)$, while the L_ϵ of the triangles $(0,i,j)$ must not satisfy the triangle inequalities.

Having all $L_\epsilon > 0$ means that all triangles in $T(M)$ are spacelike. For $M = \Sigma \times I$ manifolds, this gives an accordion-like triangulation (triangulation of a cylinder). A more natural triangulation is to take a finite number of spacelike slices $T_k(\Sigma)$ which are linked by timelike edges such that each 4-simplex has a spacelike tetrahedron in T_k and a vertex in T_{k-1} or in T_{k+1}. This class of triangulations is used in CDT models [8]. We will then require that the L_ϵ of T_k satisfy the triangle inequalities, while a timelike edge will be assigned an imaginary length iL_ϵ. Hence the labels of the edges of timelike triangles will not satisfy the triangle inequalities and the metric (1) will have the correct signature.

The curvature scalar R will be concentrated on the triangles and R will be given by the deficit angle divided by the area of the dual face. Hence in each σ we have a flat metric (1) so that we can say the corresponding PL metric is a piecewise-flat metric.

Note that an L_ϵ label represents a proper length, so that L_ϵ is invariant under the local Lorentz transformations in each 4-simplex. We will also have $(L_\epsilon)^2 > 0$ for a spacelike edge, while $(L_\epsilon)^2 < 0$ for a timelike edge.

The Einstein-Hilbert action for the PL metric (1) becomes the Regge action

$$S_{Rc} = \frac{1}{G_N}\sum_{\Delta=1}^{F} A_\Delta(L)\theta_\Delta(L) + \Lambda_c V_4(L)\,, \tag{2}$$

where G_N is the Newton constant, $A_\Delta(L)$ is the area of a triangle $\Delta \in T(M)$ and θ_Δ is the deficit angle. Λ_c is the cosmological constant and V_4 is the 4-volume of $T(M)$. See [8] how to define (2) when the timelike triangles are present. Note that the Regge action describes a theory with a finite number of DOF when Σ is compact, while in the case when Σ is non-compact, we can restrict L_ϵ to be non-zero only in a ball $B \subset \Sigma$.

One can also couple the matter fields to a Regge PL metric and the corresponding smooth actions will become the PL actions for a finite number of matter DOF. For example, a scalar field will be defined by the values of the field at the vertices of $T(M)$, which is equivalent to a PL function on the 4-polytopes of the dual triangulation.

In the case of a scalar field matter, the Regge path integral will be given by the following $(E+V)$-dimensional integral

$$Z = \int_{D_E} \mu(L)\, d^E L \int_{\mathbf{R}^V} \prod_{\nu=1}^{V} d\phi_\nu \, e^{i[S_{Rc}(L)+S_m(L,\phi)]/\hbar} \,, \tag{3}$$

where E is the number of the edges in $T(M)$ and V is the number of the vertices in $T(M)$ [13]. S_m is the PL form of the scalar-field action and the integration region D_E is a subset of $\mathbf{R}_+^E$ where the triangle inequalities hold. The measure μ has to be chosen such that it makes Z finite. The matter PI measure is taken to be trivial and we will assume that the matter path integral is finite. This is true, because the matter path integral will be given by a finite product of the integrals of the type

$$I(\alpha,\beta) = \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2 - \beta x^4} \,, \tag{4}$$

where $\alpha, \beta \in \mathbf{C}$. Since I is convergent for $\alpha, \beta > 0$, the analytic continuation $I(i\alpha, i\beta)$ will be finite.

Note that in the standard Regge formulation the spacetime metric is of the Euclidean signature. This was done in analogy to the QFT case where the Euclidean signature improves the convergence of the path integral (3). However, in the QG case this does not help, because the scalar curvature also changes the sign in the Euclidean case and can be unbounded. Actually, the Lorentzian integral has better convergence properties, which can be seen on a toy example $R(x) = \alpha\, x^2$ where $x \in \mathbf{R}_+$ and α is a constant different from zero. Then $Z_E = \int_0^\infty dx\, e^{-R(x)}$ is convergent only for $\alpha > 0$ while $Z_L = \int_0^\infty dx\, e^{iR(x)}$ is convergent for any sign of α. The presence of imaginary edge lengths and imaginary angles in the Lorentzian case is not a problem, since all the geometric quantities can be defined [8].

Finding the smooth limit $T(M) \to M$ for Z is a difficult problem. However, there is a promising approach, based on the Wilson renormalization group [7]. In this approach one considers Z as function of the dimensionless couplings γ e λ

$$\gamma = l_0^2/(G_N\hbar) = l_0^2/l_P^2 \,, \quad \lambda = l_0^4 \Lambda_c/\hbar = l_0^4/(L_c^2 l_P^2) \,,$$

where $L_c^2 = G_N/\Lambda_c$ and l_0 is an arbitrary length. One then looks for a critical point $P_0 = (\gamma_0, \lambda_0)$ where the second derivatives of Z diverge so that there is a second-order phase transition. At the critical point the correlation length diverges, so that a transition to the smooth phase occurs. However, the problem with this approach is that at P_0 the perturbation theory does not apply, so that the calculation has to be done by using numerical methods. Also the semiclassical limit $l_P^2 \to 0$ corresponds to a strong coupling region $\gamma \to \infty$ and $\lambda \to \infty$ so that it is difficult to determine it analytically.

However, the easiest way to determine the semiclassical limit in a QG theory defined by a path integral is to use the effective action, see [9, 10, 11, 12, 13]. Namely, the effective action can be calculated analytically in the $\hbar \to 0$ limit. Also the PI measure $\mu(L)$ has to be such that allows a semiclassical expansion for the effective action for large L_ϵ. This gives us an additional constraint on the choice of $\mu(L)$.

3. Effective action for PL quantum gravity

We will assume that $T(M)$ is the fundamental spacetime structure, i.e. the spacetime is a *piecewise linear* 4-manifold $T(M)$ with a flat metric in each cell (4-simplex σ). If N is the number of cells of $T(M)$, then for $N \gg 1$, $T(M)$ will look like the smooth manifold M on a scale much larger than the maximal edge length.

By an appropriate choice of the measure μ the integral $Z(T(M))$ can be made finite. Since $T(M)$ is the physical spacetime, there is no need to define the smooth limit $T(M) \to M$. Instead, we need a large-N approximation for the observables. This is analogous to the fluid dynamics situation where on the scales much larger than the inter-molecular distance we can approximate the molecular velocities as a smooth field and use the Navier-Stokes equations.

We will determine the semiclassical limit of PL quantum gravity by using the effective action. It can be computed by using the effective action equation in the limit $L_\epsilon \gg l_P = \sqrt{G_N \hbar}$.

Let us recall first the effective action definition from quantum field theory (QFT). Let ϕ be a real scalar field on M and let

$$S(\phi) = \frac{1}{2}\int_M d^4x \sqrt{|g|}\left[g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}\omega^2\phi^2 - \lambda\,\phi^4\right],$$

be a flat-spacetime action. The effective action $\Gamma(\phi)$ can be determined from the following integro-differential equation

$$e^{i\Gamma(\phi)/\hbar} = \int \mathcal{D}h \exp\left[\frac{i}{\hbar}S(\phi+h) - \frac{i}{\hbar}\int_M d^4x\,\frac{\delta\Gamma}{\delta\phi(x)}h(x)\right], \qquad (5)$$

see [14, 15].

Note that a generic solution $\Gamma(\phi)$ is a function with values in $\mathbf{C}$. The Wick rotation is used to obtain a real-valued function $\Gamma(\phi)$. This is done by solving first the EA equation in the Euclidean spacetime

$$e^{-\Gamma_E(\phi)/\hbar} = \int \mathcal{D}h \exp\left[-\frac{1}{\hbar}S_E(\phi+h) + \frac{1}{\hbar}\int_M d^4x\,\frac{\delta\Gamma_E}{\delta\phi(x)}h(x)\right]. \qquad (6)$$

Then $x_0 = -it$ is inserted into a solution $\Gamma_E(\phi)$, where (x_0, x_k) are the spacetime coordinates, so that

$$\Gamma(\phi) = i\Gamma_E(\phi)|_{x_0=-it}\,.$$

However, the Wick rotation cannot be used in quantum gravity, since in many problems of interest, introducing a flat background metric does not make sense. One way to resolve this difficulty is to use the fact that the Wick rotation in QFT is equivalent to

$$\varGamma(\phi) \to Re\,\varGamma(\phi) + Im\,\varGamma(\phi)\,, \tag{7}$$

see [9, 10]. This prescription is convenient for quantum gravity because it does not involve a background metric, nor a system of coordinates.

In the case of PL quantum gravity without matter, the effective action (EA) equation is given by

$$e^{i\varGamma(L)/l_P^2} = \int_{D_E(L)} d^E x\,\mu(L+x) e^{iS_{Rc}(L+x)/l_P^2 - i\sum_{\epsilon=1}^E \varGamma'_\epsilon(L)x_\epsilon/l_P^2}\,, \tag{8}$$

where $l_P^2 = G_N\hbar$ and $D_E(L)$ is a subset of $\mathbf{R}^E$ obtained by translating D_E by a vector $-L$ [12]. Note that $D_E(L) \subseteq [-L_1,\infty)\times\cdots\times[-L_E,\infty)$.

We will look for a semiclassical solution

$$\varGamma(L) = S_{Rc}(L) + l_P^2\varGamma_1(L) + l_P^4\varGamma_2(L) + \cdots\,,$$

where $L_\epsilon \gg l_P$ and

$$|\varGamma_n(L)| \gg l_P^2|\varGamma_{n+1}(L)|\,.$$

When $L_\epsilon\to\infty$, then $D_E(L)\to\mathbf{R}^E$ and

$$e^{i\varGamma(L)/l_P^2} \approx \int_{\mathbf{R}^E} d^E x\,\mu(L+x) e^{iS_{Rc}(L+x)/l_P^2 - i\sum_{\epsilon=1}^E \varGamma'_\epsilon(L)x_\epsilon/l_P^2}\,. \tag{9}$$

Actually, one can use the equation (9) to determine $\varGamma(L)$ for large L when μ falls off sufficiently quickly [12]. The reason is that

$$D_E(L) \approx [-L_1,\infty)\times\cdots\times[-L_E,\infty)\,,$$

for $L_\epsilon\to\infty$, so that the relevant behaviour is captured by the following one-dimensional integral

$$\int_{-L}^{\infty} dx\, e^{-zx^2/l_P^2 - wx} = \sqrt{\pi}\,l_P \exp\Big[-\frac{1}{2}\log z + l_P^2\frac{w^2}{4z}$$

$$+ l_P\frac{e^{-z\bar{L}^2/l_P^2}}{2\sqrt{\pi z}\bar{L}}\left(1 + O(l_P^2/z\bar{L}^2)\right)\Big]\,,$$

where $\bar{L} = L + l_P^2\frac{w}{2z}$ and $Re\,z = -(\log\mu)''$. The non-analytic terms in $\hbar$ will be absent if

$$\lim_{L\to\infty} e^{-z\bar{L}^2/l_P^2} = 0 \Leftrightarrow (\log\mu)'' < 0 \text{ for } L\to\infty\,.$$

Hence the perturbative solution exists for the exponentially damped measures and it will be given by the equation (9).

For $D_E(L) = \mathbf{R}^E$ and $\mu(L)$ a constant, the perturbative solution is given by the EA diagrams

$$\Gamma_1 = \frac{i}{2} Tr \log S''_{Rc} , \quad \Gamma_2 = \langle S_3^2 G^3 \rangle + \langle S_4 G^2 \rangle ,$$

and

$$\Gamma_3 = \langle S_3^4 G^6 \rangle + \langle S_3^2 S_4 G^5 \rangle + \langle S_3 S_5 G^4 \rangle + \langle S_4^2 G^4 \rangle + \langle S_6 G^3 \rangle , ...$$

where $G = i(S''_{Rc})^{-1}$ is the propagator and $S_n = iS_{Rc}^{(n)}/n!$ for $n > 2$, are the vertex weights, see [15, 12]. The contractions $\langle X \cdots Y \rangle$ are the sums over the repeated DOF indices

$$\langle X \cdots Y \rangle = \sum_{k,...,l} X_{k...l} \cdots Y_{k...l} \quad .$$

When $\mu(L)$ is not a constant, then the perturbative solution is given by

$$\Gamma(L) = \bar{S}_{Rc}(L) + l_P^2 \bar{\Gamma}_1(L) + l_P^4 \bar{\Gamma}_2(L) + \cdots ,$$

where

$$\bar{S}_{Rc} = S_{Rc} - i l_P^2 \log \mu ,$$

while $\bar{\Gamma}_n$ is given by the sum of n-loop EA diagrams with $\bar{G}$ propagators and $\bar{S}_n$ vertex weights [12].

Therefore

$$\Gamma_1 = -i \log \mu + \frac{i}{2} Tr \log S''_{Rc} ,$$

$$\Gamma_2 = \langle S_3^2 G^3 \rangle + \langle S_4 G^2 \rangle + Res[l_P^{-4} Tr \log \bar{G}] ,$$

$$\Gamma_3 = \langle S_3^4 G^6 \rangle + \cdots + \langle S_6 G^3 \rangle + Res[l_P^{-6} Tr \log \bar{G}]$$

$$+ Res[l_P^{-6} \langle \bar{S}_3^2 \bar{G}^3 \rangle] + Res[l_P^{-6} \langle \bar{S}_4 \bar{G}^2 \rangle] ,$$

see [12].

Since the PI measure μ has to vanish exponentially for large edge lengths, a natural choice is

$$\mu(L) = \exp\left(-V_4(L)/(L_0)^4 \right) , \tag{10}$$

where L_0 is a length parameter [12]. Since $\log \mu(L) = O\left((L/L_0)^4 \right)^2$ then for $L_\epsilon > L_c$ and

$$L_0 > \sqrt{l_P \, L_c} , \tag{11}$$

[2]The notation $f(x_1, ..., x_n) = O(x^\alpha)$ means that $f(\lambda x_1, ..., \lambda x_n) = O(\lambda^\alpha)$ for $\lambda \to \infty$.

where $L_c^{-2} = \Lambda_c$, we get the following large-L asymptotics [13, 16]

$$\Gamma_1(L) = O(L^4/L_0^4) + \log O(L^2/L_c^2) + \log\theta(L) + O(L_c^2/L^2) \qquad (12)$$

and

$$\Gamma_{n+1}(L) = O\left((L_c^2/L^4)^n\right) + L_{0c}^{-2n} O\left((L_c^2/L^2)\right) , \qquad (13)$$

where $L_{0c} = L_0^2/L_c$.

4. Effective cosmological constant

The asymptotics (12) and (13) imply that the series

$$\Gamma(L) = \sum_{n\geq 0} (l_P)^{2n}\Gamma_n(L)$$

is semiclassical (SC) for $L_\epsilon \gg l_P$ and $L_0 \gg \sqrt{l_P\, L_c}$.

Let $\Gamma \to \Gamma/G_N$ so that $S_{eff} = (Re\,\Gamma + Im\,\Gamma)/G_N$. The effective action is then given by

$$S_{eff} = \frac{S_{Rc}}{G_N} + \frac{l_P^2}{G_N L_0^4} V_4 + \frac{l_P^2}{2G_N} Tr\log S''_{Rc} + O(l_P^4)\,,$$

for $L_\epsilon \gg l_P$. Hence the $O(\hbar)$, or the one-loop, cosmological constant (CC) for pure gravity is given by

$$\Lambda = \Lambda_c + \frac{l_P^2}{L_0^4} = \Lambda_c + \Lambda_{qg}\,. \qquad (14)$$

One can show that the one-loop cosmological constant is exact because there are no $O(L^4)$ terms beyond the one-loop order [13, 16]. This is a consequence of the large-L asymptotics

$$\log \bar{S}''_{Rc}(L) = \log O(L^2/\bar{L}_c^2) + \log\theta(L) + O(\bar{L}_c^2/L^2)$$

$$\bar{\Gamma}_{n+1}(L) = O\left((\bar{L}_c^2/L^4)^n\right) ,$$

where $\bar{L}_c^2 = L_c^2\left[1 + i l_P^2 (L_c^2/L_0^4)\right]^{-1/2}$.

Hence the one-loop formula (14) is exact in the case of pure gravity. If $\Lambda_c = 0$, the observed value of Λ is obtained for $L_0 \approx 10^{-5}m$ so that $l_P^2\Lambda \approx 10^{-122}$ [12]. Note that $L_0 \approx 10^{-5}m$ is consistent with the requirement that $L_0 \gg l_P$, which replaces the SC condition $L_0 \gg \sqrt{L_c l_P}$ when $\Lambda_c = 0$.

The formula (14) is intriguing but unrealistic, since there is matter in the universe. In order to obtain a realistic expression for the effective CC, we need to study the EA equation with matter. This study also requires the understanding of the emergence of the smooth spacetime from a PL manifold $T(M)$. If $T(M)$ has a large number of the edges ($E \gg 1$) then the following approximations are valid

$$S_R(L) \approx \frac{1}{2}\int_M d^4x\sqrt{|g|}\,R(g)\,, \tag{15}$$

and

$$\Lambda_c V_4(L) \approx \Lambda_c \int_M d^4x\sqrt{|g|} = \Lambda_c\, V_M\,, \tag{16}$$

where $|g| = |\det g|$. These are the standard formulas of the Regge calculus and they nicely illustrate how the PL manifold $T(M)$ with many 4-simplices can be approximated by a smooth manifold M with a smooth (differentiable) metric g.

Similarly, the effective action $\Gamma(L)$ will be approximated by a QFT effective action $\Gamma^*(g)$, where g is a smooth metric on M. Let L_K be a minimal length in a triangulation, so that $L_\epsilon \geq L_K$ and let $L_K \gg l_P$. When $E \gg 1$ the following approximation is valid

$$Tr\log S_R''(L) \approx \int_M d^4x\sqrt{|g|}\left[aR^2 + bR_{\mu\nu}R^{\mu\nu}\right]\log(K/K_0)\,, \tag{17}$$

where $R_{\mu\nu}$ is the Ricci tensor, and a, b, K_0 are some constants.

The formula (17) follows from the fact that a PL function on a lattice with a cell size L_K can be written as a Fourier integral over a compact region $|q| \leq \pi/L_K$ where q is the wave vector[3]. Hence the PL trace-log term can be approximated by using the QFT formulation of GR with a momentum cutoff $K = 2\pi\hbar/L_K$.

The effect of the matter on the CC can be studied by introducing a scalar field on M

$$S_m(g,\phi) = \frac{1}{2}\int_M d^4x\sqrt{|g|}\left[g^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi - U(\phi)\right]\,, \tag{18}$$

where $U = \frac{1}{2}\omega^2\phi^2 + \lambda\,\phi^4$.

On a PL manifold $T(M)$ the action (18) becomes

$$S_m = \frac{1}{2}\sum_\sigma V_\sigma(L)\sum_{k,l} g_\sigma^{kl}(L)\,\phi_k'\,\phi_l' - \frac{1}{2}\sum_p V_p^*(L)\,U(\phi_p)\,,$$

where $\phi_k' = (\phi_k - \phi_0)/L_{0k}$ and $k, l, 0$ are vertices in a 4-simplex σ, p labels the vertices of $T(M)$ and V^* is the volume of the dual cell. Then the total classical action of gravity plus matter on $T(M)$ is given by

$$S(L,\phi) = \frac{1}{G_N} S_{Rc}(L) + S_m(L,\phi)\,.$$

[3]This region is known as the first Brillouin zone.

The corresponding EA equation is given by

$$e^{\frac{i}{l_P^2}\Gamma(L,\phi)} = \int_{D_E(L)} d^E l \int_{\mathbf{R}^V} d^V\chi$$
$$\exp\left[\frac{i}{l_P^2}\left(\bar{S}(L+l,\phi+\chi) - \sum_\epsilon \frac{\partial\Gamma}{\partial L_\epsilon}\, l_\epsilon - \sum_p \frac{\partial\Gamma}{\partial\phi_p}\,\chi_p\right)\right], \quad (19)$$

where $\bar{S} = S_{Rc} - il_P^2\log\mu + G_N S_m$, see [13].

We will look for a perturbative solution

$$\Gamma(L,\phi) = S(L,\phi) + l_P^2\Gamma_1(L,\phi) + l_P^4\Gamma_2(L,\phi) + \cdots,$$

and require it to be semiclassical for $L_\epsilon \gg l_P$ and $|\sqrt{G_N}\,\phi| \ll 1$. This can be checked on the $E=1$ toy model

$$S(L,\phi) = (L^2 + L^4/L_c^2)\theta(L) + L^2\theta(L)\phi^2(1+\omega^2L^2+\lambda\phi^2L^2),$$

where $\theta(L)$ is a homogeneous function of degree zero.

It is not difficult to see that

$$\Gamma(L,\phi) = \Gamma_g(L) + \Gamma_m(L,\phi),$$

and

$$\Gamma_m(L,\phi) = V_4(L)\,U_{eff}(\phi)$$

for constant ϕ where $U_{eff}(0)=0$. Furthermore,

$$\Gamma_g(L) = \Gamma_{pg}(L) + \Gamma_{mg}(L),$$

where Γ_{pg} is the pure gravity contribution and Γ_{mg} is the matter induced contribution.

In the smooth-manifold approximation one has

$$\Gamma_{mg}(L) \approx \Lambda_m V_M + \Omega_m(R,K),$$

where $K = 2\pi\hbar/L_K$ is the momentum cutoff. One can show that

$$\Omega_m = \Omega_1 l_P^2 + O(l_P^4)$$

and

$$\begin{aligned}\Omega_1(R,K) &= a_1K^2\int_M d^4x\sqrt{|g|}\,R \\ &+ \log(K/\omega)\int_M d^4x\sqrt{|g|}\Big[a_2R^2 + a_3R^{\mu\nu}R_{\mu\nu} \\ &\qquad + a_4R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + a_5\nabla^2R\Big] \quad (20)\\ &+ O(1/K^2),\end{aligned}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann curvature tensor, see [13].

The effective CC will be then given as

$$\Lambda = \Lambda_c + \Lambda_{qg} + \Lambda_m \,,$$

where Λ_{qg} is given by (14). Note that the matter contribution to CC can be approximated by a sum

$$\Lambda_m \approx \sum_\gamma v(\gamma, K) \tag{21}$$

where $v(\gamma, K)$ is a one-particle irreducible vacuum Feynman diagram for the field-theory action S_m in flat spacetime with the cutoff K. One can show that

$$\begin{aligned} \sum_\gamma v(\gamma, K) \;\; &\approx \;\; l_P^2 K^4 \Big[c_1 \ln(K^2/\omega^2) + \sum_{n\geq 2} c_n (\bar\lambda)^{n-1} (\ln(K^2/\omega^2))^{n-2} \\ &+ \;\; \sum_{n\geq 4} d_n (\bar\lambda)^{n-1} (K^2/\omega^2)^{n-3} \Big] \,, \end{aligned} \tag{22}$$

for $K \gg \omega$, where $\bar\lambda = l_P^2 \lambda$, see [16]. Therefore one has a highly divergent sum of matter vacuum-energy contributions to the cosmological constant when $K \to \infty$. This is the famous cosmological constant problem which appears in any QFT formulation of quantum gravity.

However, in the PL formulation of quantum gravity (PLQG), the QFT which produces the infinite sum in (22) is just an approximation. The fundamental theory has finitely many DOF so that the exact solution of the EA equation will give a finite and cutoff-independent value for Λ. Therefore

$$\Lambda_m = V(\omega^2, \lambda, l_P^2) \,, \tag{23}$$

and

$$\Lambda = \pm \frac{1}{L_c^2} + \frac{l_P^2}{2L_0^4} + V(\omega 2, \lambda, l_P^2) \,. \tag{24}$$

The equation (24) can be used to fix the free parameters L_0 and L_c. By equating Λ with the experimentally observed value, we obtain

$$\lambda = x + y + \lambda_m \tag{25}$$

where $\lambda = l_P^2 \Lambda \approx 10^{-122}$, $x = \pm\, l_P^2/L_c^2$, $y = l_P^4/2L_0^4$ and $\lambda_m = l_P^2 V$. The equation (25) has infinitely many solutions, but we also have to impose the condition for the existence of the semi-classical limit (11). This gives the restriction

$$0 < y < 2|x| \,. \tag{26}$$

The value of λ_m is not known, but for any value of λ_m the equation (25) has infinitely many solutions which obey the restriction (26). Note that the solution $x = -\lambda_m$ and $y = \lambda$, which was proposed in [13], will be acceptable if $|\lambda_m| > \lambda/2$. This solution is special because it gives a value for L_0 which is independent of the value of λ_m, $L_0 \approx 10^{-5}$m. This is the same value which was obtained in the case of pure PL gravity without the cosmological constant [12].

5. The CC problem in quantum gravity

The formula (24) for the exact effective cosmological constant is an essential ingredient for the resolution of the CC problem from QFT in the context of a QG theory. The result (24) can be better understood if we recall the definition of the CC problem given by Polchinski [17]. According to this definition, the CC problem in a QG theory has two parts:

1) show that the observed CC value is in the CC spectrum,

2) explain why the CC takes the observed value.

The meaning of the first part (P1) of the CC problem is obvious if the cosmological constant is represented by an operator. In the case when one has a quantum corrected expression of the classical CC value, one has to show that there are values of the free parameters which give the observed CC value. The PLQG theory clearly solves P1, while the second part (P2) of the CC problem cannot be addressed by the current formalism. The reason is that one has to generalise the standard formalism of quantum mechanics in order to provide a mechanism for a selection of a wavefunction of the universe with a particular value of the cosmological constant.

Note that demonstrating P1 is a highly non-trivial task in any QG theory. The problem P1 has been addressed so far only in PLQG theory and in string theory. In the string theory case there are only plausibility arguments that P1 is true [18, 19]. The CC spectrum in string theory is discrete with $O(10^{500})$ values [18]. Although positive CC values are not natural in string theory, a mechanism for their appearance was provided in [19]. Hence it is plausible to assume that the CC spectrum is sufficiently dense around zero such that the observed value is sufficiently close to some CC spectrum value.

The second part of the CC problem has been only addressed in string theory. This is the multiverse proposal, see [21], and the assumption is that there are many universes, each having a fixed CC value from the CC spectrum. We live in the universe with the CC value $\Lambda_c l_P^2 \approx 10^{-122}$, because this is the value that allows formation of galaxies, planets and life, see [20] for the anthropic determination of the CC value.

Note that there are many proposals for P2 which are not derived from a QG theory, but instead it is assumed that a certain effective action exists such that its equations of motion give the required CC value, see for example [22].

6. Conclusions

The PLQG theory is a theory of quantum gravity which has finitely many degrees of freedom and no infinities. The underlying spacetime structure is a PL manifold $T(M)$ and the smooth spacetime M is recovered as an approximation valid when the number of 4-simplices is large and at a length scale much larger than the typical edge length. The smooth spacetime approximation is analogous to the smooth vector field approximation for the molecular velocities in a fluid.

The PLQG theory is defined by the Regge path integral with a non-trivial measure. The measure is chosen such that it gives a finite path integral, and also it has to admit a semi-classical solution of the effective action equation. These criteria select the exponentially vanishing measures for large edge lengths, and a simple and natural choice for the measure is (10). This measure simplifies the analysis of the effective cosmological constant and one can obtain the formula (24) for the exact effective CC, i.e. to all orders in $\hbar$. The two free parameters in (24) can be consistently chosen such that the observed CC value is obtained. This is an important requirement for any QG theory and PLQG is the only existing QG theory where this property has been demonstrated explicitly.

Another nice property of the PLQG theory is that the effective action Γ can be approximated by a QFT effective action Γ^* when the number of 4-simplices in $T(M)$ is large. Γ^* can be calculated by using the perturbative QFT for GR with matter and with a momentum cutoff K, when $L_\epsilon \geq L_K \gg l_P$. Hence the minimal edge length L_K in the triangulation determines the momentum cutoff K and

$$\Gamma(L_1, \cdots, L_E, \phi_1, \cdots, \phi_V) \approx \Gamma^*(g(x), \phi(x), K)\,, \tag{27}$$

for $E \gg 1$ and $V \gg 1$.

The QFT approximation (27) will be still valid for $L_K \leq l_P$, but in this case Γ^* cannot be calculated by the perturbative QFT methods. Instead, one has to use a non-perturbative method to solve the EA equation. The existence of the QFT approximation (27) implies that one can obtain the running of the elementary particle masses and the coupling constants with K, see for example the equation (20).

Note that the effective action only makes sense for the spacetimes which are given by the direct product of a 3-manifold with an interval. In order to study the quantum cosmology questions, one needs to consider 4-manifolds of general topology, which is different from $\Sigma\times I$ topology. When $M \neq \Sigma\times I$, the concept of the effective action cannot be used. However, the Hartle-Hawking (HH) wavefunction [23] can be defined for any $T(M)$ by using the PLQG path integral (3). By choosing a triangulation for a manifold

$$M \cup (\Sigma \times I)\ , \quad \partial M = \Sigma\,,$$

one can describe a Big-Bang quantum cosmology with an initial HH state, which evolves by the evolution operator defined by the PLQG path integral for the $T(\Sigma \times I)$ part of the spacetime. It is then plausible to assume that the effective dynamics which corresponds to the time evolution of the HH state will be given by the PLQG effective action, defined by the equation (19).

References

[1] C. Isham, Lect. Notes Phys. 434 (1994) 1-21.

[2] C. Isham, Lect. Notes Phys. 434 (1994) 150-169.

[3] D.W. Chiou, Int. J. Mod. Phys. D, Vol. 24, No. 1 (2015) 1530005, `arxiv:1412.4362`.

[4] A. Connes, *Noncommutative geometry*, Academic Press, San Diego (1994).

[5] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press (1987).

[6] T. Regge, Nuovo Cim. 19 (1961) 558-571.

[7] H. W. Hamber, Gen. Rel. Grav. 41 (2009) 817, `arXiv:0901.0964`.

[8] J. Ambjorn, A. Goerlich, J. Jurkiewicz, R. Loll, Phys. Rept. 519 (2012) 127-210.

[9] A. Miković and M. Vojinović, Class. Quant. Grav. 28 (2011) 225004.

[10] A. Miković and M. Vojinović, J. Phys.: Conf. Ser. 360 (2012) 012049.

[11] A. Miković, Rev. Math. Phys. 25 (2013) 1343008.

[12] A. Miković, Adv. Theor. Math. Phys. 21 (2017) 631.

[13] A. Miković and M. Vojinović, EPL 110 (2015) 40008.

[14] V.P. Nair, *Quantum Field Theory - A Modern Perspective*, Springer (2005).

[15] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial Markets*, World Scientific (2006).

[16] A. Miković and M. Vojinović, Cosmological constant in a quantum gravity theory for a piecewise-linear spacetime, `arxiv:1407.1124`.

[17] J. Polchinski, The cosmological constant and the string landscape, *The Quantum Structure of Space and Time: Proceedings of the 23rd Solvay Conference on Physics*. Brussels, Belgium. 1 - 3 December 2005, D. Gross, M. Nenneaux and A. Servin (eds), 2007, `arXiv:hep-th/0603249`.

[18] R. Bousso and J. Polchinski, JHEP 0006 (2000) 006.

[19] S. Kachru, R. Kalosh, A. Linde and S. Trivedi, Phys. Rev. D 68 (2003) 046005.

[20] S. Weinberg, Rev. Mod. Phys. 61 (1989) 1.

[21] L. Susskind, The anthropic landscape of string theory, *Universe or Multiverse?*, B. Carr (ed.) Cambridge University Press (2007) 247-266, `arxiv:hep-th/0302219`.

[22] G. D'Amico, N. Kaloper, A. Padilla, et al. JHEP 1709 (2017) 74.

[23] J.B. Hartle and S.W. Hawking, Phys. Rev. D 28 (1983) 2960-2975.

P r o c e e d i n g s

of the

10th MATHEMATICAL PHYSICS MEETING:
School and Conference on
Modern Mathematical Physics

September 9 – 14, 2019, Belgrade, Serbia

Editors

B. Dragovich, I. Salom and M. Vojinović

Institute of Physics
Belgrade, 2020
SERBIA

Autor: Grupa autora

Naslov: 10th MATHEMATICAL PHYSICS MEETING: SCHOOL AND CONFERENCE ON MODERN MATHEMATICAL PHYSICS (Deseti naučni skup iz matematičke fizike: škola i konferencija iz savremene matematičke fizike)

Izdavač: Institut za fiziku, Beograd, Srbija

Izdanje: Prvo izdanje
(SFIN year XXXIII Series A: Conferences, No. A1 (2020))

Štampa: Zemunplast, Beograd

Tiraž: 150
ISBN: 978-86-82441-51-9

1. Dragović Branko
Matematička fizika-Zbornici

CIP – Каталогизација у публикацији
Народна библиотека Србије, Београд

51-7:53(082)

MATHEMATICAL Physics Meeting: School and Conference on Modern Mathematical Physics (10 ; 2019 ; Beograd)
Proceedings of the 10th Mathematical Physics Meeting: School and Conference on Modern Mathematical Physics, September 9-14, 2019, Belgrade, Serbia / [organizers Institute of Physics, Belgrade ... [et al.]] ; editors B. [Branko] Dragovich, I. [Igor] Salom and M. [Marko] Vojinović. - 1. izd. - Belgrade : Institute of Physics, 2020 (Beograd : Zemunplast). - XII, 383 str. : ilustr. ; 28 cm. - (SFIN ; year 33. Ser. A, Conferences, ISSN 0354-9291 ; n° A1, (2020))

Nasl. u kolofonu: Deseti naučni skup iz matematičke fizike: škola i konferencija iz savremene matematičke fizike. - Tiraž 150. - Str. VII: Preface / editors. - Napomene i bibliografske reference uz radove. - Bibliografija uz svaki rad.

ISBN 978-86-82441-51-9

1. Dragović, Branko, 1945- [уредник] [аутор додатног текста]
а) Математичка физика -- Зборници

COBISS.SR-ID 13561865

10th MATHEMATICAL PHYSICS MEETING:
School and Conference on Modern Mathematical Physics

Belgrade, September 9 - 14, 2019

Organizers

- **Institute of Physics, Belgrade, Serbia**
- **Faculty of Mathematics, Belgrade, Serbia**
- **Mathematical Institute, Belgrade, Serbia**
- **Faculty of Science, Kragujevac, Serbia**
- **Serbian Academy of Sciences and Arts, Serbia**

Co-organizers

- **Institute of Nuclear Sciences “Vinča”, Belgrade, Serbia**
- **Institute of Physics, Kragujevac, Serbia**
- **Faculty of Physics, Belgrade, Serbia**
- **Faculty of Sciences, Novi Sad, Serbia**
- **School of Electrical Engineering, Belgrade, Serbia**
- **SEENET-MTP, Niš, Serbia**

International Advisory Committee

International Organizing Committee

Vladimir Dobrev (Sofia, Bulgaria)
Radu Constantinescu (Craiova, Romania)
Branko Dragovich (Belgrade, Serbia)
Goran Đorđević (Niš, Serbia)
Zoran Rakić (Belgrade, Serbia)
Đorđe Šijački (Belgrade, Serbia)
Marko Vojinović (Belgrade, Serbia)
Igor Volovich (Moscow, Russia)

Local Organizing Committee

Branko Dragovich (Co-Chairman, Institute of Physics, Belgrade)
Marko Vojinović (Co-Chairman, Institute of Physics, Belgrade)
Sanja Ćirković (Institute of Physics, Belgrade)
Ljubica Davidović (Institute of Physics, Belgrade)
Ivan Dimitrijević (Faculty of Mathematics, Belgrade, Serbia)
Ilija Ivanišević (Institute of Physics, Belgrade)
Danijel Obrić (Institute of Physics, Belgrade)
Tijana Radenković (Institute of Physics, Belgrade)
Igor Salom (Institute of Physics, Belgrade)
Dejan Simić (Institute of Physics, Belgrade)

Sponsors

- **Ministry of Education, Science and Technological Development, Republic of Serbia**

- **Telekom Srbija**

- **Open Access Journal "Symmetry" (MDPI, Switzerland)**

Media Partner

- **Open Access Journal "Entropy" (MDPI, Switzerland)**

P R E F A C E

This volume contains some reviews and original research contributions, which are related to the **10th Mathematical Physics Meeting: School and Conference on Modern Mathematical Physics**, organized by the Institute of Physics, Belgrade (Serbia), September 9–14, 2019. The programme of this meeting was mainly oriented towards some recent developments in gravity and cosmology, string and quantum field theory, and some relevant mathematical methods. We hope that articles presented here will be valuable literature not only for the participants of this meeting but also for many other PhD students and researchers in modern mathematical and theoretical physics. We are grateful to all authors for writing their contributions for these proceedings.

The previous nine meetings in this series of schools and conferences on modern mathematical physics were also held in Serbia: Sokobanja 2001, Kopaonik 2002, Zlatibor 2004, Belgrade 2006, 2008, 2010, 2012, 2014, and 2017. The corresponding proceedings of all these meetings were published by the Institute of Physics Belgrade, and are available in the printed form as well as online at the websites. According to an agreement with the journal Symmetry, several papers are published in the special issue "Selected Papers: 10th Mathematical Physics Meeting".

This jubilary tenth meeting took place at two different venues — the opening and the first day of lectures was held in the grand lecture hall of the Serbian Academy of Sciences and Arts, while the lectures for the remaining five days were held at the Mathematical Institute. Both venues are located in Belgrade downtown, across the road of each other. We hope that all attendees of this meeting will recall it as a useful and pleasant event, and will wish to participate again in the future.

We wish to thank all lecturers and other speakers for their interesting and valuable talks. We also thank all participants for their active participation. Financial support of our sponsors, *Ministry of Education, Science and Technological Development of the Republic of Serbia, Belgrade; Telekom Srbija; Open access journal "Symmetry"*, and the support of our media partner, *Open access journal "Entropy"*, were very significant for realization of this activity.

April 2020

E d i t o r s

B. Dragovich
I. Salom
M. Vojinović


СУВЕНИРИ - SOUVENIRS

CONTENTS

Review and Research Works

P r o c e e d i n g s

of the

9th MATHEMATICAL PHYSICS MEETING: School and Conference on Modern Mathematical Physics

September 18 – 23, 2017, Belgrade, Serbia

Editors

B. Dragovich, I. Salom and M. Vojinović

Institute of Physics
Belgrade, 2018
SERBIA

Autor: Grupa autora

Naslov: 9th MATHEMATICAL PHYSICS MEETING: SCHOOL AND CONFERENCE ON MODERN MATHEMATICAL PHYSICS (Deveti naučni skup iz matematičke fizike: škola i konferencija iz savremene matematičke fizike)

Izdavač: Institut za fiziku, Beograd, Srbija

Izdanje: Prvo izdanje
(SFIN year XXXI Series A: Conferences No. A1 (2018))

Štampa: Dual Mode, Beograd

Tiraž: 150
ISBN: 978-86-82441-48-9

1. Dragović Branko
Matematička fizika-Zbornici

CIP- Каталогизација у публикацији
Народна библиотека Србије

51-73:53(082)

SCHOOL and Conference on Modern Mathematical Physics (9 ; 2017 ; Beograd)
Proceedings of the 9th Mathematical Physics Meeting: School and Conference on Modern Mathematical Physics, September 18-23, 2017, Belgrade, Serbia / [organizers Institute of Physics, Belgrade ... et al.] ; editors B. [Branko] Dragovich, I. [Igor] Salom and M. [Marko] Vojinović. - 1. izd. - Belgrade : Institute of Physics, 2018 (Beograd : Dual Mode). - VIII, 324 str. : ilustr. ; 25 cm. - (Sveske fizičkih nauka : SFIN ; year 31. Ser. A, Conferences, ISSN 0354-9291 ; n° A1, (2018))

U kolofonu: Deveti naučni skup iz matematičke fizike: škola i konferencija iz savremene matematičke fizike. - Tiraž 150. - Str. V: Preface / editors. - Napomene i bibliografske reference uz tekst. - Bibliografija uz svaki rad.

ISBN 978-86-82441-48-9

1. Dragović, Branko, 1945- [уредник] [аутор додатног текста] 2. Institut za fiziku (Beograd)

а) Математичка физика - Зборници

COBISS.SR-ID 265404172

9th MATHEMATICAL PHYSICS MEETING: School and Conference on Modern Mathematical Physics

Belgrade, September 18 - 23, 2017

Organizers

- **Institute of Physics, Belgrade, Serbia**
- **Mathematical Institute, Belgrade, Serbia**
- **Faculty of Mathematics, Belgrade, Serbia**

International Advisory Committee

International Organizing Committee

Vladimir Dobrev (Sofia, Bulgaria)
Branko Dragovich (Belgrade, Serbia)
Goran Djordjevic (Nis, Serbia)
Zoran Rakic (Belgrade, Serbia)
Igor Volovich (Moscow, Russia)

Local Organizing Committee

Branko Dragovich (Co-Chairman, Institute of Physics, Belgrade)
Igor Salom (Co-Chairman, Institute of Physics, Belgrade)
Sanja Ćirković (Institute of Physics, Belgrade)
Ljubica Davidović (Institute of Physics, Belgrade)
Marko Vojinović (Institute of Physics, Belgrade)

Sponsors

- **Ministry of Education, Science and Technological Development, Republic of Serbia**
- **The Abdus Salam International Centre for Theoretical Physics**

P R E F A C E

This volume contains some reviews and original research contributions, which are related to the **9th Mathematical Physics Meeting: School and Conference on Modern Mathematical Physics**, organized by the Institute of Physics, Belgrade (Serbia), September 18–23, 2017. The programme of this meeting was mainly oriented towards some recent developments in gravity and cosmology, string and quantum field theory, and some relevant mathematical methods. We hope that articles presented here will be valuable literature not only for the participants of this meeting but also for many other PhD students and researchers in modern mathematical and theoretical physics. We are grateful to all authors for writing their contributions for these proceedings.

The previous eight meetings in this series of schools and conferences on modern mathematical physics were also held in Serbia: Sokobanja, 13–25 August 2001; Kopaonik, 1–12 September 2002; Zlatibor, 20–31 August 2004; Belgrade, 3–14 September 2006; Belgrade, 6–17 July 2008; Belgrade, 14–23 September 2010; Belgrade, 9–19 September 2012; and Belgrade, 24–31 August 2014. The corresponding proceedings of all these meetings were published by the Institute of Physics, Belgrade, and are available in the printed form as well as online at the websites.

This ninth meeting took place at the Mathematical Institute, Serbian Academy of Sciences and Arts, located in Belgrade downtown. We hope that all attending this meeting will recall it as a useful and pleasant event, and will wish to participate again in the future.

We wish to thank all lecturers and other speakers for their interesting and valuable talks. We also thank all participants for their active participation. Financial support of our sponsors, *Ministry of Education, Science and Technological Development of the Republic of Serbia, Belgrade; and the Abdus Salam International Centre for Theoretical Physics* was very significant for realization of this activity.

April 2018

E d i t o r s

B. Dragovich

I. Salom

M. Vojinović

CONTENTS

Review and Research Works