Научном већу Института за физику Београд

Београд, 13. јул 2020.

Предмет:

Молба за покретање поступка за избор у звање истраживач сарадник

Молим Научно веће Института за физику у Београду да покрене поступак за мој избор у звање истраживач сарадник.

У прилогу достављам:

1. мишљење руководиоца лабораторије са предлогом чланова комисије за избор у звање;
2. стручну биографију;
3. преглед научне активности;
4. списак објављених научних радова и њихове копије;
5. потврда о уписаним докторским студијама;
6. копију диплама основних и мастер академских студија;
7. уверење о прихваћеној теми докторске дисертације.

Са поштовањем,

Тијана Раденковић

истраживач приправник

Тијана Раденковић

Београд, 13. јул 2020. године

Предмет: Мишљење руководиоца лабораторије о избору Тијане Раденковић у звање истраживач сарадник

Тијана Раденковић је била ангажована на пројекту основних истраживања Министарства просвете, науке и технолошког развоја Републике Србије ОН 171031, под називом „*Физичке импликације модификованог просторвремена*“. Запослена је у групи за Гравитацију, честице и поља Института за физику у Београду од априла 2018. године, када је изабрана у звање истраживач приправник. Ради на теми конструисања уједињене теорије квантне гравитације и материје Стандардног Модела у контексту математичке теорије виших категорија под руководством др Марка Војиновића. С обзиром да испуњава све предвиђене услове у складу са Правилником о поступку, начину вредновања и квантитативном исказивању научноистраживачких резултата Министарства просвете науке и технолошког развоја, сагласан сам са покретањем поступка и предлажем избор Тијане Раденковић у звање истраживач сарадник.

За састав комисије за избор Тијане Раденковић у звање истраживач сарадник предлажем:

1) др Марко Војиновић, виши научни сарадник, Институт за физику у Београду,
2) др Бранислав Цветковић, научни саветник, Институт за физику у Београду,
3) проф др Маја Бурић, редовни професор, Физички факултет Универзитета у Београду.

Бранислав Цветковић

др Бранислав Цветковић

научни саветник Института за физику

руководилац групе за Гравитацију, честице и поља

Биографија Тијане Раденковић

Тијана Раденковић је рођена 21.3.1992. године у Београду, где је завршила основну школу и Математичку гимназију. Основне академске студије на Физичком факултету Универзитета у Београду, смер Теоријска и експериментална физика, започела је 2011. године и завршила јула 2016. године са просечном оценом 9,33. Мастер академске студије на истом факултету, смер Теоријска и експериментална физика, завршила је октобра 2017. године са просечном оценом 9,33, одбранивши мастер рад на тему „Квантна гравитација на део-по-део равним многострукостима".
Мастер рад је урађен под руководством др Марка Војиновића, вишег научног сарадника Института за физику у Београду. Мастер теза награђена је наградом "Проф. Љубомир Ћирковић" за најбољу мастер тезу током школске 2017/2018 године.
Новембра 2017. године уписала је докторске академске студије на Физичком факултету Универзитета у Београду, ужа научна област квантна поља, честице и гравитација. Под руководством др Марка Војиновића ради на темама везаним за уједињење теорије квантне гравитације са материјом Стандардног модела, у оквиру математичке теорије виших категорија. Од априла 2017. године Тијана Раденковић је запослена у Институту за физику у Београду као истраживач приправник у групи за Гравитацију, честице и поља, чији је руководилац др Бранислав Цветковић. Учествовала је на пројекту основних истраживања „Физичке импликације модификованог просторвремена" (ОН171031) Министарства просвете, науке и технолошког развоја Републике Србије, којим је руководила проф др Маја Бурић.
До сада је похађала неколико школа за докторанде и учествовала на конференцијама, међу којима су: 10th MATHEMATICAL PHYSICS MEETING: School and Conference on Modern Mathematical Physics, Belgrade, Serbia (2019); Twistors and Loops Meeting "Théorie des twisteurs et gravitation quantique à boucles", Marseille, France (2019); BS2019: SEENET-MTP Balkan School on High Energy and Particle Physics: Theory and Phenomenology, Ioannina, Greece (2019); "Quantum Gravity in Paris" conference, Paris, France (2019); Workshop on Gravity and String Theory "New ideas for unsolved problems III" Zlatibor, Serbia (2018); CERN-SEENET-MTP PhD Training Program "High Energy and Particle Physics: Theory and Phenomenology", Niš, Serbia (2018); Summer School on High Energy Physics, Petnica, Serbia (2018); Workshop on Gravity, Holography, Strings and Noncommutative Geometry, Belgrade, Serbia (2018); CERN-SEENET-MTP PhD Training Program "New Trends in High Energy Theory", Sofia, Bulgaria (2017); School and Conference on Modern Mathematical Physics, Belgrade, Serbia (2017).
На Колегијуму докторских студија Физичког факултета Универзитета у Београду, одржаном 1. 7. 2020. године, одобрена је њена тема докторске дисертације под насловом „Квантна гравитација и више градијентне теорије", а за ментора је одређен др Марко Војиновић.
До сада, Тијана Раденковић има један рад објављен у часопису категорије М21, један рад објављен у часопису категорије М22, као и једно саопштење са међународног скупа штампано у целини (М33).

Списак објављених радова

Радови у врхунским међународним часописима (категорија М21)

T. Radenković and M. Vojinović, Higher Gauge Theories Based on 3-groups, JHEP **10**, 222 (2019), arXiv:1904.07566.

Радови у истакнутим међународним часописима (категорија М22)

T. Radenković and M. Vojinović, Hamiltonian Analysis for the Scalar Electrodynamics as 3BF Theory, Symmetry **12**, 620 (2020), arXiv:2004.06901.

Саопштења са међународног скупа штампана у целини (категорија М33)

T. Radenković and M. Vojinović, Construction and examples of higher gauge Theories, SFIN XXXIII, 251 (2020), arXiv:2005.09404.

Преглед научне активности Тијане Раденковић

Тијана Раденковић се у свом научном раду бави проблемима квантне гравитације и њеног уједињења са осталим фундаменталним силама.

Досадашњи научно истраживачки рад Тијане Раденковић, може се класификовати у следеће основне правце:

1. Формулисање $2BF$, односно $3BF$ дејства, са везама за Јанг-Милсово, Клајн-Гордоново, Дираково, Вајлово и Мајорана поље, у интеракцији са Ајнштајн-Картановом гравитацијом у облику прилагођеном за спровођење процедуре коваријантне квантизације.
2. Први корак канонске квантизационе процедуре: Хамилтонова анализа тополошког $3BF$ дејства.

1. Формулисање $2BF$, односно $3BF$ дејства, са везама за Јанг-Милсово, Клајн-Гордоново, Дираково, Вајлово и Мајорана поље, у интеракцији са Ајнштајн-Картановом гравитацијом у облику прилагођеном за спровођење процедуре коваријантне квантизације (квантизациона процедура спинске пене)

Посматрана је генерализација BF теорије у формализму теорије категорија - тзв. $2BF$, односно $3BF$ теорија, са одговарајућом 2- групом, односно 3- групом, градијентних симетрија. Конструисано је $2BF$ дејство које даје одговарајућу динамику за Јанг-Милсово поље које интерагује са гравитацијом, као и одговарајућа $3BF$ дејства која описују Клајн-Гордоново, Дираково, Вајлово и Мајорана поље, у интеракцији са Ајнштајн-Картановом гравитацијом. Дејство је написано у облику збира тополошког дела и сектора са везама, прилагођено за спровођење коваријантне квантизационе процедуре карактеристичне за моделе спинске пене. Преписан је целокупан Стандардни модел у овом облику и препозната је нова група симетрије која одређује спектар материје присутне у теорији.

2. Први корак канонске квантизационе процедуре: Хамилтонова анализа тополошког $3BF$ дејства

Хамилтонова анализа теорије је неопходан први корак канонске квантизационе процедуре, који нам дозвољава да формулишемо квантну теорију за системе који поседују градијентну симетрију. Урађена је Хамилтонова анализа $3BF$ дејства које одговара тополошком сектору скаларне електродинамике у интеракцији са Ајнштајн-Картановом гравитацијом. Добијене су везе прве класе и везе друге класе присутне у теорији, алгебра веза у теорији, као и генератор градијентних трансформација и варијације форми варијабла и њихових конјугованих импулса које одговарају овим трансформацијама симетрије.

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Higher gauge theories based on 3-groups

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Abstract: We study the categorical generalizations of a BF theory to $2BF$ and $3BF$ theories, corresponding to 2-groups and 3-groups, in the framework of higher gauge theory. In particular, we construct the constrained $3BF$ actions describing the correct dynamics of Yang-Mills, Klein-Gordon, Dirac, Weyl, and Majorana fields coupled to Einstein-Cartan gravity. The action is naturally split into a topological sector and a sector with simplicity constraints, adapted to the spinfoam quantization programme. In addition, the structure of the 3-group gives rise to a novel gauge group which specifies the spectrum of matter fields present in the theory, just like the ordinary gauge group specifies the spectrum of gauge bosons in the Yang-Mills theory. This allows us to rewrite the whole Standard Model coupled to gravity as a constrained $3BF$ action, facilitating the nonperturbative quantization of both gravity and matter fields. Moreover, the presence and the properties of this new gauge group open up a possibility of a nontrivial unification of all fields and a possible explanation of fermion families and all other structure in the matter spectrum of the theory.

Keywords: Models of Quantum Gravity, Topological Field Theories, Gauge Symmetry, Beyond Standard Model

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Contents

1 Introduction

The quantization of the gravitational field is one of the most prominent open problems in modern theoretical physics. Within the Loop Quantum Gravity framework, one can study the nonperturbative quantization of gravity, both canonically and covariantly, see [1–3] for an overview and a comprehensive introduction. The covariant approach focuses on the definition of the path integral for the gravitational field,

$$Z = \int \mathcal{D}g \; e^{iS[g]} \,, \tag{1.1}$$

by considering a triangulation of a spacetime manifold, and defining the path integral as a discrete state sum of the gravitational field configurations living on the simplices in the triangulation. This quantization technique is known as the *spinfoam* quantization method, and roughly goes along the following lines:

1. first, one writes the classical action $S[g]$ as a topological BF action plus a simplicity constraint,

2. then one uses the algebraic structure (a Lie group) underlying the topological sector of the action to define a triangulation-independent state sum Z,

3. and finally, one imposes the simplicity constraints on the state sum, promoting it into a path integral for a physical theory.

This quantization prescription has been implemented for various choices of the action, the Lie group, and the spacetime dimension. For example, in 3 dimensions, the prototype spinfoam model is known as the Ponzano-Regge model [4]. In 4 dimensions there are multiple models, such as the Barrett-Crane model [5, 6], the Ooguri model [7], and the most sophisticated EPRL/FK model [8, 9]. All these models aim to define a viable theory of quantum gravity, with variable success. However, virtually all of them are focused on pure gravity, without matter fields. The attempts to include matter fields have had limited success [10], mainly because the mass terms could not be expressed in the theory due to the absence of the tetrad fields from the BF sector of the theory.

In order to resolve this issue, a new approach has been developed, using the categorical generalization of the BF action, within the framework of *higher gauge theory* (see [11] for a review). In particular, one uses the idea of a categorical ladder to promote the BF action, which is based on some Lie group, into a $2BF$ action, which is based on the so-called 2-group structure. If chosen in a suitable way, the 2-group structure should hopefully introduce the tetrad fields into the action. This approach has been successfully implemented [12], rewriting the action for general relativity as a constrained $2BF$ action, such that the tetrad fields are present in the topological sector. This result opened up a possibility to couple all matter fields to gravity in a straightforward way. Nevertheless, the matter fields could not be naturally expressed using the underlying algebraic structure of a 2-group, rendering the spinfoam quantization method only half-implementable, since the matter sector of the classical action could not be expressed as a topological term plus a simplicity constraint, which means that the steps 2 and 3 above could not be performed for the matter sector of the action.

We address this problem in this paper. As we will show, it turns out that it is necessary to perform one more step in the categorical ladder, generalizing the underlying algebraic structure from a 2-group to a 3-group. This generalization then naturally gives rise to the so-called $3BF$ action, which proves to be suitable for a unified description of both gravity and matter fields. The steps of the categorical ladder can be conveniently summarized in the following table:

categorical structure	algebraic structure	linear structure	topological action	degrees of freedom
Lie group	Lie group	Lie algebra	BF theory	gauge fields
Lie 2-group	Lie crossed module	differential Lie crossed module	$2BF$ theory	tetrad fields
Lie 3-group	Lie 2-crossed module	differential Lie 2-crossed module	$3BF$ theory	scalar and fermion fields

Once the suitable gauge 3-group has been specified and the corresponding $3BF$ action constructed, the most important thing that remains, in order to complete the step 1 of the spinfoam quantization programme, is to impose appropriate simplicity constraints onto the degrees of freedom present in the $3BF$ action, so that we obtain the desired classical dynamics of the gravitational and matter fields. Then one can proceed with steps 2 and 3 of the spinfoam quantization, hopefully ending up with a viable model of quantum gravity and matter.

In this paper, we restrict our attention to the first of the above steps: we will construct a constrained $3BF$ action for the cases of Klein-Gordon, Dirac, Weyl and Majorana fields, as well as Yang-Mills and Proca vector fields, all coupled to the Einstein-Cartan gravity in the standard way. This construction will lead us to an unexpected novel result. As we shall see, the scalar and fermion fields will be *naturally associated to a new gauge group*, generalizing the notion of a gauge group in the Yang-Mills theory, which describes vector bosons. This new group opens up a possibility to use it as an algebraic way of classifying matter fields, describing the structures such as quark and lepton families, and so on. The insight into the existence of this new gauge group is the consequence of the categorical ladder and is one of the main results of the paper. However, given the complexity of the algebraic properties of 3-groups, we will restrict ourselves only to the reconstruction of the already known theories, such as the Standard Model (SM), in the new framework. In this sense, any potential explanation of the spectrum of matter fields in the SM will be left for future work.

The layout of the paper is as follows. In subsection 2.1 we will give a short overview of the constrained BF actions, including the well-known example of the Plebanski action for general relativity, and a completely new example of the Yang-Mills theory rewritten as a constrained BF model. In the subsection 2.2 we also introduce the formalism of the constrained $2BF$ actions, reviewing the example of general relativity as a constrained $2BF$ action, first introduced in [12]. In addition, we will demonstrate how to couple gravity in a natural way within the formalism of 2-groups. Section 3 contains the main results of the paper and is split into 4 subsections. The subsection 3.1 introduces the formalism of 3-groups, and the definition and properties of a $3BF$ action, including the three types of gauge transformations. The subsection 3.2 focuses on the construction of a constrained $3BF$ action which describes a single real scalar field coupled to gravity. It provides the most elementary example of the insight that matter fields correspond to a gauge group. Encouraged by these results, in the subsection 3.3 we construct the constrained $3BF$ action for the Dirac field coupled to gravity and specify its gauge group. Finally, the subsection 3.4 deals with the construction of the constrained $3BF$ action for the Weyl and Majorana fields coupled to gravity, thereby covering all types of fields potentially relevant for the Standard Model and beyond. After the construction of all building blocks, in section 4 we apply the results of sections 2 and 3 to construct the constrained $3BF$ action corresponding to the full Standard Model coupled to Einstein-Cartan gravity. Finally, section 5 is devoted to the discussion of the results and the possible future lines of research. The appendices contain some mathematical reminders and technical details.

The notation and conventions are as follows. The local Lorentz indices are denoted by the Latin letters $a, b, c, \ldots$, take values $0, 1, 2, 3$, and are raised and lowered using the

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Minkowski metric η_{ab} with signature $(-,+,+,+)$. Spacetime indices are denoted by the Greek letters $\mu, \nu, \ldots$, and are raised and lowered by the spacetime metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ are the tetrad fields. The inverse tetrad is denoted as $e^\mu{}_a$. All other indices that appear in the paper are dependent on the context, and their usage is explicitly defined in the text where they appear. A lot of additional notation is defined in appendix A. We work in the natural system of units where $c = \hbar = 1$, and $G = l_p^2$, where l_p is the Planck length.

2 *BF* and 2*BF* models, ordinary gauge fields and gravity

Let us begin by giving a short review of BF and $2BF$ theories in general. For additional information on these topics, see for example [11, 13–18].

2.1 *BF* theory

Given a Lie group G and its corresponding Lie algebra $\mathfrak{g}$, one can introduce the so-called BF action as

$$S_{BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} \,. \tag{2.1}$$

Here, $\mathcal{F} \equiv \mathrm{d}\alpha + \alpha \wedge \alpha$ is the curvature 2-form for the algebra-valued connection 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$ on some 4-dimensional spacetime manifold $\mathcal{M}_4$. In addition, $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ is a Lagrange multiplier 2-form, while $\langle \_ , \_ \rangle_{\mathfrak{g}}$ denotes the G-invariant bilinear symmetric nondegenerate form.

From the structure of (2.1), one can see that the action is diffeomorphism invariant, and it is usually understood to be gauge invariant with respect to G. In addition to these properties, the BF action is topological, in the following sense. Varying the action (2.1) with respect to B^β and α^β, where the index β counts the generators of $\mathfrak{g}$ (see appendix A for notation and conventions), one obtains the equations of motion of the theory,

$$\mathcal{F} = 0\,, \qquad \nabla B \equiv \mathrm{d}B + \alpha \wedge B = 0\,. \tag{2.2}$$

From the first equation of motion, one immediately sees that α is a flat connection, which then together with the second equation of motion implies that B is constant. Therefore, there are no local propagating degrees of freedom in the theory, and one then says that the theory is topological.

Usually, in physics one is interested in theories which are nontopological, i.e., which have local propagating degrees of freedom. In order to transform the BF action into such a theory, one adds an additional term to the action, commonly called the *simplicity constraint*. A very nice example is the Yang-Mills theory for the SU(N) group, which can be rewritten as a constrained BF theory in the following way:

$$S = \int B_I \wedge F^I + \lambda^I \wedge \left(B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b \right) + \zeta^{abI} \left(M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - g_{IJ} F^J \wedge \delta_a \wedge \delta_b \right). \tag{2.3}$$

Here $F \equiv \mathrm{d}A + A \wedge A$ is again the curvature 2-form for the connection $A \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{su}(N))$, and $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{su}(N))$ is the Lagrange multiplier 2-form. The Killing form $g_{IJ} \equiv$

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$\langle \tau_I, \tau_J \rangle_{\mathfrak{su}(N)} \propto f_{IK}{}^L f_{JL}{}^K$ is used to raise and lower the indices $I, J, \dots$ which count the generators of $\mathrm{SU}(N)$, where $f_{IJ}{}^K$ are the structure constants for the $\mathfrak{su}(N)$ algebra. In addition to the topological $B \wedge F$ term, we also have two simplicity constraint terms, featuring the Lagrange multiplier 2-form λ^I and the Lagrange multiplier 0-form ζ^{abI}. The 0-form M_{abI} is also a Lagrange multiplier, while g is the coupling constant for the Yang-Mills theory.

Finally, δ^a is a nondynamical 1-form, such that there exists a global coordinate frame in which its components are equal to the Kronecker symbol $\delta^a{}_\mu$ (hence the notation δ^a). The 1-form δ^a plays the role of a background field, and defines the global spacetime metric, via the equation

$$\eta_{\mu\nu} = \eta_{ab} \delta^a{}_\mu \delta^b{}_\nu \,, \tag{2.4}$$

where $\eta_{ab} \equiv \mathrm{diag}(-1, +1, +1, +1)$ is the Minkowski metric. Since the coordinate system is global, the spacetime manifold $\mathcal{M}_4$ is understood to be flat. The indices $a, b, \dots$ are local Lorentz indices, taking values $0, \dots, 3$. Note that the field δ^a has all the properties of the tetrad 1-form e^a in the flat Minkowski spacetime. Also note that the action (2.3) is manifestly diffeomorphism invariant and gauge invariant with respect to $\mathrm{SU}(N)$, but not background independent, due to the presence of δ^a.

The equations of motion are obtained by varying the action (2.3) with respect to the variables ζ^{abI}, M_{abI}, A^I, B_I, and λ^I, respectively (note that we do not take the variation of the action with respect to the background field δ^a):

$$M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - F_I \wedge \delta_a \wedge \delta_b = 0 \,, \tag{2.5}$$

$$-\frac{12}{g} \lambda^I \wedge \delta^a \wedge \delta^b + \zeta^{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f = 0 \,, \tag{2.6}$$

$$-\mathrm{d}B_I + f_{JI}{}^K B_K \wedge A^J + \mathrm{d}(\zeta^{ab}{}_I \delta_a \wedge \delta_b) - f_{JI}{}^K \zeta^{ab}{}_K \delta_a \wedge \delta_b \wedge A^J = 0 \,, \tag{2.7}$$

$$F_I + \lambda_I = 0 \,, \tag{2.8}$$

$$B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b = 0 \,, \tag{2.9}$$

From the algebraic equations (2.5), (2.6), (2.8) and (2.9) one obtains the multipliers as functions of the dynamical field A^I:

$$M_{abI} = \frac{1}{48} \varepsilon_{abcd} F_I{}^{cd} \,, \quad \zeta^{abI} = \frac{1}{4g} \varepsilon^{abcd} F^I{}_{cd} \,, \quad \lambda_{Iab} = F_{Iab} \,, \quad B_{Iab} = \frac{1}{2g} \varepsilon_{abcd} F_I{}^{cd} \,. \tag{2.10}$$

Here we used the notation $F_{Iab} = F_{I\mu\nu} \delta_a{}^\mu \delta_b{}^\nu$, where we used the fact that $\delta^a{}_\mu$ is invertible, and similarly for other variables. Using these equations and the differential equation (2.7) one obtains the equation of motion for gauge field A^I,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0 \,. \tag{2.11}$$

This is precisely the classical equation of motion for the free Yang-Mills theory. Note that in addition to the Yang-Mills theory, one can easily extend the action (2.3) in order to describe the massive vector field and obtain the Proca equation of motion. This is done by adding a mass term

$$-\frac{1}{4!} m^2 A_{I\mu} A^I{}_\nu \eta^{\mu\nu} \varepsilon_{abcd} \delta^a \wedge \delta^b \wedge \delta^c \wedge \delta^d \tag{2.12}$$

to the action (2.3). Of course, this term explicitly breaks the SU(N) gauge symmetry of the action.

Another example of the constrained BF theory is the Plebanski action for general relativity [15], see also [13] for a recent review. Starting from a gauge group SO(3, 1), one constructs a constrained BF action as

$$S = \int_{\mathcal{M}_4} B_{ab} \wedge R^{ab} + \phi_{abcd} B^{ab} \wedge B^{cd} \,. \tag{2.13}$$

Here R^{ab} is the curvature 2-form for the spin connection ω^{ab}, B_{ab} is the usual Lagrange multiplier 2-form, while ϕ_{abcd} is the Lagrange multiplier 0-form corresponding to the simplicity constraint term $B^{ab} \wedge B^{cd}$. It can be shown that the variation of this action with respect to B_{ab}, ω^{ab} and ϕ_{abcd} gives rise to equations of motion which are equivalent to vacuum general relativity. However, the tetrad fields appear in the model as a solution to the simplicity constraint equation of motion $B^{ab} \wedge B^{cd} = 0$. Thus, being intrinsically on-shell objects, they are not present in the action and cannot be quantized. This renders the Plebanski model unsuitable for coupling of matter fields to gravity [10, 12, 19]. Nevertheless, as a model for pure gravity, the Plebanski model has been successfully quantized in the context of spinfoam models, see [1, 2, 8, 9] for details and references.

2.2 2BF theory

In order to circumvent the issue of coupling of matter fields, a recent promising approach has been developed [12, 19–23] in the context of higher category theory [11]. In particular, one employs the higher category theory construction to generalize the BF action to the so-called 2BF action, by passing from the notion of a gauge group to the notion of a gauge 2-group. In order to introduce it, let us first give a short review of the 2-group formalism.

In the framework of category theory, the group as an algebraic structure can be understood as a specific type of category, namely a category with only one object and invertible morphisms [11]. The notion of a category can be generalized to the so-called *higher categories*, which have not only objects and morphisms, but also 2-morphisms (morphisms between morphisms), and so on. This process of generalization is called the *categorical ladder*. Similarly to the notion of a group, one can introduce a 2-group as a 2-category consisting of only one object, where all the morphisms and 2-morphisms are invertible. It has been shown that every strict 2-group is equivalent to a crossed module $(H \xrightarrow{\partial} G \,, \triangleright)$, see appendix A for definition. Here G and H are groups, δ is a homomorphism from H to G, while $\triangleright : G \times H \to H$ is an action of G on H.

An important example of this structure is a vector space V equipped with an isometry group O. Namely, V can be regarded as an Abelian Lie group with addition as a group operation, so that a representation of O on V is an action $\triangleright$ of O on the group V, giving rise to the crossed module $(V \xrightarrow{\partial} O \,, \triangleright)$, where the homomorphism ∂ is chosen to be trivial, i.e., it maps every element of V into a unit of O. We will make use of this example below to introduce the Poincaré 2-group.

Similarly to the case of an ordinary Lie group G which has a naturally associated notion of a connection α, giving rise to a BF theory, the 2-group structure has a naturally

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associated notion of a 2-connection $(\alpha\,,\beta)$, described by the usual $\mathfrak{g}$-valued 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$ and an $\mathfrak{h}$-valued 2-form $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$, where $\mathfrak{h}$ is a Lie algebra of the Lie group H. The 2-connection gives rise to the so-called *fake 2-curvature* $(\mathcal{F},\mathcal{G})$, given as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,, \qquad \mathcal{G} = d\beta + \alpha\wedge^{\triangleright}\beta\,. \tag{2.14}$$

Here $\alpha\wedge^{\triangleright}\beta$ means that α and β are multiplied as forms using $\wedge$, and simultaneously multiplied as algebra elements using $\triangleright$, see appendix A. The curvature pair $(\mathcal{F},\mathcal{G})$ is called fake because of the presence of the $\partial\beta$ term in the definition of $\mathcal{F}$, see [11] for details.

Using these variables, one can introduce a new action as a generalization of the BF action, such that it is gauge invariant with respect to both G and H groups. It is called the $2BF$ action and is defined in the following way [16, 17]:

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}} + \langle C\wedge\mathcal{G}\rangle_{\mathfrak{h}}\,, \tag{2.15}$$

where the 2-form $B \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{g})$ and the 1-form $C \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{h})$ are Lagrange multipliers. Also, $\langle\_\,,\_\rangle_{\mathfrak{g}}$ and $\langle\_\,,\_\rangle_{\mathfrak{h}}$ denote the G-invariant bilinear symmetric nondegenerate forms for the algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. As a consequence of the axiomatic structure of a crossed module (see appendix A), the bilinear form $\langle\_\,,\_\rangle_{\mathfrak{h}}$ is H-invariant as well. See [16, 17] for review and references.

Similarly to the BF action, the $2BF$ action is also topological, which can be seen from equations of motion. Varying with respect to B and C one obtains

$$\mathcal{F} = 0\,, \qquad \mathcal{G} = 0\,, \tag{2.16}$$

while varying with respect to α and β one obtains the equations for the multipliers,

$$\mathrm{d}B_\alpha - g_{\alpha\beta}{}^{\gamma} B_\gamma\wedge\alpha^\beta - \triangleright_{\alpha a}{}^{b} C_b\wedge\beta^a = 0\,, \tag{2.17}$$

$$\mathrm{d}C_a - \partial_a{}^{\alpha} B_\alpha + \triangleright_{\alpha a}{}^{b} C_b\wedge\alpha^\alpha = 0\,. \tag{2.18}$$

One can either show that these equations have only trivial solutions, or one can use the Hamiltonian analysis to show that there are no local propagating degrees of freedom (see for example [21, 22]), demostrating the topological nature of the theory.

An example of a 2-group relevant for physics is the Poincaré 2-group, which is constructed using the aforementioned example of a vector space equipped with an isometry group. One constructs a crossed module by choosing

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \tag{2.19}$$

while $\triangleright$ is a natural action of $\mathrm{SO}(3,1)$ on $\mathbb{R}^4$, and the map ∂ is trivial. The 2-connection (α,β) is given by the algebra-valued differential forms

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \tag{2.20}$$

where ω^{ab} is the spin connection, while M_{ab} and P_a are the generators of groups $\mathrm{SO}(3,1)$ and $\mathbb{R}^4$, respectively. The corresponding 2-curvature in this case is given by

$$\mathcal{F} = (\mathrm{d}\omega^{ab} + \omega^a{}_c\wedge\omega^{cb})M_{ab} \equiv R^{ab}M_{ab}\,, \quad \mathcal{G} = (\mathrm{d}\beta^a + \omega^a{}_b\wedge\beta^b)P_a \equiv \nabla\beta^a P_a \equiv G^a P_a\,, \tag{2.21}$$

where we have evaluated $\wedge^{\triangleright}$ using the equation $M_{ab} \triangleright P_c = \eta_{[bc}P_{a]}$. Note that, since ∂ is trivial, the fake curvature is the same as ordinary curvature. Using the bilinear forms

$$\langle M_{ab}, M_{cd} \rangle_{\mathfrak{g}} = \eta_{a[c}\eta_{bd]} \,, \qquad \langle P_a, P_b \rangle_{\mathfrak{h}} = \eta_{ab} \,, \tag{2.22}$$

one can show that 1-forms C^a transform in the same way as the tetrad 1-forms e^a under the Lorentz transformations and diffeomorphisms, so the fields C^a can be identified with the tetrads. Then one can rewrite the $2BF$ action (2.15) for the Poincaré 2-group as

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a \,. \tag{2.23}$$

In order to obtain general relativity, the topological action (2.23) can be modified by adding a convenient simplicity constraint, like it is done in the BF case:

$$S = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) . \tag{2.24}$$

Here λ_{ab} is a Lagrange multiplier 2-form associated to the simplicity constraint term, and l_p is the Planck length. Varying the action (2.24) with respect to B_{ab}, e_a, ω_{ab}, β_a and λ_{ab}, one obtains the following equations of motion:

$$R_{ab} - \lambda_{ab} = 0 \,, \tag{2.25}$$

$$\nabla \beta_a + \frac{1}{8\pi l_p^2} \varepsilon_{abcd} \lambda^{bc} \wedge e^d = 0 \,, \tag{2.26}$$

$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} = 0 \,, \tag{2.27}$$

$$\nabla e_a = 0 \,, \tag{2.28}$$

$$B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d = 0 \,. \tag{2.29}$$

The only dynamical fields are the tetrads e^a, while all other fields can be algebraically determined, as follows. From the equations (2.28) and (2.29) we obtain that $\nabla B^{ab} = 0$, from which it follows, using the equation (2.27), that $e_{[a} \wedge \beta_{b]} = 0$. Assuming that the tetrads are nondegenerate, $e \equiv \det(e^a{}_\mu) \neq 0$, it can be shown that this is equivalent to the condition $\beta^a = 0$ (for the proof see appendix in [12]). Therefore, from the equations (2.25), (2.27), (2.28) and (2.29) we obtain

$$\lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu} \,, \quad \beta^a{}_{\mu\nu} = 0 \,, \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu \,, \quad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu \,. \tag{2.30}$$

Here the Ricci rotation coefficients are defined as

$$\triangle^{ab}{}_\mu \equiv \frac{1}{2}(c^{abc} - c^{cab} + c^{bca}) e_{c\mu} \,, \tag{2.31}$$

where

$$c^{abc} = e^\mu{}_b e^\nu{}_c \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu \right) . \tag{2.32}$$

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Finally, the remaining equation (2.26) reduces to

$$\varepsilon_{abcd} R^{bc} \wedge e^d = 0\,, \tag{2.33}$$

which is nothing but the vacuum Einstein field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$. Therefore, the action (2.24) is classically equivalent to general relativity.

The main advantage of the action (2.24) over the Plebanski model and similar approaches lies in the fact that the tetrad fields are explicitly present in the topological sector of the theory. This allows one to couple matter fields in a straightforward way, as demonstrated in [12]. However, one can do even better, and couple gauge fields to gravity within a unified framework of 2-group formalism.

Let us demonstrate this on the example of the SU(N) Yang-Mills theory. Begin by modifying the Poincaré 2-group structure to include the SU(N) gauge group, as follows. We choose the two Lie groups as

$$G = \mathrm{SO}(3,1) \times \mathrm{SU}(N)\,, \qquad H = \mathbb{R}^4\,, \tag{2.34}$$

and we define the action $\triangleright$ of the group G in the following way. As in the case of the Poincaré 2-group, it acts on itself via conjugation. Next, it acts on H such that the SO(3,1) subgroup acts on $\mathbb{R}^4$ via the vector representation, while the action of SU(N) subgroup is trivial. The map ∂ also remains trivial, as before. The 2-connection (α, β) now obtains the form which reflects the structure of the group G,

$$\alpha = \omega^{ab} M_{ab} + A^I \tau_I\,, \qquad \beta = \beta^a P_a\,, \tag{2.35}$$

where A^I is the gauge connection 1-form, while τ_I are the SU(N) generators. The curvature for α is thus

$$\mathcal{F} = R^{ab} M_{ab} + F^I \tau_I\,, \qquad F^I \equiv \mathrm{d}A^I + f_{JK}{}^I A^J \wedge A^K\,. \tag{2.36}$$

The curvature for β remains the same as before, since the action $\triangleright$ of SU(N) on $\mathbb{R}^4$ is trivial, i.e., $\tau_I \triangleright P_a = 0$. Finally, the product structure of the group G implies that its Killing form $\langle \_,\_ \rangle_{\mathfrak{g}}$ reduces to the Killing forms for the SO(3,1) and SU(N), along with the identity $\langle M_{ab}, \tau_I \rangle_{\mathfrak{g}} = 0$.

Given a crossed module defined in this way, its corresponding topological $2BF$ action (2.15) becomes

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a\,, \tag{2.37}$$

where $B^I \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{su}(N))$ is the new Lagrange multiplier. In order to transform this topological action into action with nontrivial dynamics, we again introduce the appropriate simplicity constraints. The constraint giving rise to gravity is the same as in (2.24), while the constraint for the gauge fields is given as in the action (2.3) with the substitution $\delta^a \to e^a$:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda^I \wedge \left(B_I - \frac{12}{g} M_{abI} e^a \wedge e^b \right) + \zeta^{abI} \Big(M_{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - g_{IJ} F^J \wedge e_a \wedge e_b \Big)\,. \end{aligned} \tag{2.38}$$

It is crucial to note that the action (2.38) is a combination of the pure gravity action (2.24) and the Yang-Mills action (2.3), such that the nondynamical background field δ^a from (2.3) gets promoted to a dynamical field e^a. The relationship between these fields has already been hinted at in the equation (2.4), which describes the connection between δ^a and the flat spacetime metric $\eta_{\mu\nu}$. Once promoted to e^a, this field becomes dynamical, while the equation (2.4) becomes the usual relation between the tetrad and the metric,

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \,, \tag{2.39}$$

further confirming that the Lagrange multiplier C^a should be identified with the tetrad. Moreover, the total action (2.38) now becomes background independent, as expected in general relativity. All this is a consequence of the fact that the tetrad field is explicitly present in the topological sector of the action (2.24), establishing an improvement over the Plebanski model.

By varying the action (2.38) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, ζ^{abI}, M_{abI}, B_I, λ^I, A^I, and e^a, we obtain the following equations of motion, respectively:

$$R^{ab} - \lambda^{ab} = 0 \,, \tag{2.40}$$

$$\nabla B^{ab} - e^{[a} \wedge \beta^{b]} = 0 \,, \tag{2.41}$$

$$\nabla e^a = 0 \,, \tag{2.42}$$

$$B_{ab} - \frac{1}{16\pi l_p^2} \varepsilon_{abcd} e^c \wedge e^d = 0 \,, \tag{2.43}$$

$$M_{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F_I \wedge e_a \wedge e_b = 0 \,, \tag{2.44}$$

$$-\frac{12}{g} \lambda^I \wedge e^a \wedge e^b + \zeta^{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f = 0 \,, \tag{2.45}$$

$$F_I + \lambda_I = 0 \,, \tag{2.46}$$

$$B_I - \frac{12}{g} M_{abI} e^a \wedge e^b = 0 \,, \tag{2.47}$$

$$-\mathrm{d}B_I + B_K \wedge g_{JI}{}^K A^J + \mathrm{d}(\zeta_I^{ab} e_a \wedge e_b) - \zeta_K^{ab} e_a \wedge e_b \wedge g_{JI}{}^K A^J = 0 \,, \tag{2.48}$$

$$\begin{aligned} \nabla \beta_a + \frac{1}{8\pi l_p^2} \varepsilon_{abcd} \lambda^{bc} \wedge e^d - \frac{24}{g} M_{abI} \lambda^I \wedge e^b \\ + 4\zeta^{ef}{}^I M_{ef}{}_I \varepsilon_{abcd} e^b \wedge e^c \wedge e^d - 2\zeta_{ab}{}^I F_I \wedge e^b = 0 \,. \end{aligned} \tag{2.49}$$

In the above system of equations, we have two dynamical equations for e^a and A^I, while all other variables are algebraically determined from these. In particular, from equations (2.40)–(2.47), we have:

$$\lambda_{ab\mu\nu} = R_{ab\mu\nu} \,, \quad \beta_{a\mu\nu} = 0 \,, \quad \omega_{ab\mu} = \triangle_{ab\mu} \,, \quad \lambda_{abI} = F_{abI} \,, \quad B_{\mu\nu I} = -\frac{e}{2g} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}{}_I \,, \tag{2.50}$$

$$B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu \,, \quad M_{abI} = -\frac{1}{4eg} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma \,, \quad \zeta^{abI} = \frac{1}{4eg} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma \,.$$

Then, substituting all these into (2.48) and (2.49) we obtain the differential equation of motion for A^I,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + \Gamma^\rho{}_{\lambda\rho} F^{I\lambda\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0 \,, \tag{2.51}$$

where $\Gamma^{\lambda}{}_{\mu\nu}$ is the standard Levi-Civita connection, and a differential equation of motion for e^a,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \qquad T^{\mu\nu} \equiv -\frac{1}{4g}\left(F_{\rho\sigma}{}^{I}F^{\rho\sigma}{}_{I}g^{\mu\nu} + 4F^{\mu\rho}{}_{I}F_{\rho}{}^{\nu I}\right)\,. \tag{2.52}$$

The system of equations (2.50)–(2.52) is equivalent to the system (2.40)–(2.49). Note that we have again obtained that $\beta^a = 0$, as in the pure gravity case.

In this way, we see that both gravity and gauge fields can be represented within a unified framework of higher gauge theory based on a 2-group structure.

3 3*BF* models, scalar and fermion matter fields

While the structure of a 2-group can successfully accommodate both gravitational and gauge fields, unfortunately it cannot include other matter fields, such as scalars or fermions. In order to construct a unified description of all matter fields within the framework of higher gauge theory, we are led to make a further generalization, passing from the notion of a 2-group to the notion of a 3-group. As it turns out, the 3-group structure is a perfect fit for the description of all fields that are present in the Standard Model, coupled to gravity. Moreover, this structure gives rise to a new gauge group, which corresponds to the choice of the scalar and fermion fields present in the theory. This is a novel and unexpected result, which has the potential to open up a new avenue of research with the aim of explaining the structure of the matter sector of the Standard Model and beyond.

In order to demonstrate this in more detail, we first need to introduce the notion of a 3-group, which we will afterward use to construct constrained 3*BF* actions describing scalar and fermion fields on an equal footing with gravity and gauge fields.

3.1 3-groups and topological 3*BF* action

Similarly to the concepts of a group and a 2-group, one can introduce the notion of a 3-group in the framework of higher category theory, as a 3-category with only one object where all the morphisms, 2-morphisms and 3-morphisms are invertible. It has been proved that a strict 3-group is equivalent to a 2-*crossed module* [24], in the same way as a 2-group is equivalent to a crossed module.

A Lie 2-crossed module, denoted as $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$, is a algebraic structure specified by three Lie groups G, H and L, together with the homomorphisms δ and ∂, an action $\triangleright$ of the group G on all three groups, and a G-equivariant map

$$\{\_\,,\_\} : H \times H \to L\,.$$

called the Peiffer lifting. See appendix A for more details.

In complete analogy to the construction of *BF* and 2*BF* topological actions, one can define a gauge invariant topological 3*BF* action for the manifold $\mathcal{M}_4$ and 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$. Given $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ as Lie algebras corresponding to the groups G, H and L, one can introduce a 3-connection (α, β, γ) given by the algebra-valued differential forms $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$ and $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,,\mathfrak{l})$. The corresponding fake 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is then defined as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \qquad \mathcal{H} = \mathrm{d}\gamma + \alpha \wedge^{\triangleright} \gamma + \{\beta \wedge \beta\}\,. \tag{3.1}$$

see [24, 25] for details. Then, a $3BF$ action is defined as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F}\rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G}\rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H}\rangle_{\mathfrak{l}}\,, \tag{3.2}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$, $C \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ and $D \in \mathcal{A}^0(\mathcal{M}_4, \mathfrak{l})$ are Lagrange multipliers. The forms $\langle\_,\_\rangle_{\mathfrak{g}}$, $\langle\_,\_\rangle_{\mathfrak{h}}$ and $\langle\_,\_\rangle_{\mathfrak{l}}$ are G-invariant bilinear symmetric nondegenerate forms on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, respectively. Under certain conditions, the forms $\langle\_,\_\rangle_{\mathfrak{h}}$ and $\langle\_,\_\rangle_{\mathfrak{l}}$ are also H-invariant and L-invariant, see appendix B for details.

One can see that varying the action with respect to the variables B, C and D, one obtains the equations of motion

$$\mathcal{F} = 0\,, \qquad \mathcal{G} = 0\,, \qquad \mathcal{H} = 0\,, \tag{3.3}$$

while varying with respect to α, β, γ one obtains

$$\mathrm{d}B_\alpha - g_{\alpha\beta}{}^{\gamma} B_\gamma \wedge \alpha^\beta - \triangleright_{\alpha a}{}^{b} C_b \wedge \beta^a + \triangleright_{\alpha B}{}^{A} D_A \wedge \gamma^B = 0\,, \tag{3.4}$$

$$\mathrm{d}C_a - \partial_a{}^{\alpha} B_\alpha + \triangleright_{\alpha a}{}^{b} C_b \wedge \alpha^\alpha + 2X_{\{ab\}}{}^{A} D_A \wedge \beta^b = 0\,, \tag{3.5}$$

$$\mathrm{d}D_A - \triangleright_{\alpha A}{}^{B} D_B \wedge \alpha^\alpha + \delta_A{}^{a} C_a = 0\,. \tag{3.6}$$

Regarding the gauge transformations, the $3BF$ action is invariant with respect to three different types of transformations, generated by the groups G, H and L, respectively. Under the G-gauge transformations, the 3-connection transforms as

$$\alpha' = g^{-1}\alpha g + g^{-1}\mathrm{d}g\,, \qquad \beta' = g^{-1} \triangleright \beta\,, \qquad \gamma' = g^{-1} \triangleright \gamma\,, \tag{3.7}$$

where $g : \mathcal{M}_4 \to G$ is an element of the G-principal bundle over $\mathcal{M}_4$. Next, under the H-gauge transformations, generated by $\eta \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{h})$, the 3-connection transforms as

$$\alpha' = \alpha + \partial\eta\,, \qquad \beta' = \beta + \mathrm{d}\eta + \alpha' \wedge^{\triangleright} \eta - \eta \wedge \eta\,, \qquad \gamma' = \gamma - \{\beta' \wedge \eta\} - \{\eta \wedge \beta\}\,. \tag{3.8}$$

Finally, under the L-gauge transformations, generated by $\theta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{l})$, the 3-connection transforms as

$$\alpha' = \alpha\,, \qquad \beta' = \beta - \delta\theta\,, \qquad \gamma' = \gamma - \mathrm{d}\theta - \alpha \wedge \theta\,. \tag{3.9}$$

As a consequence of the definition (3.1) and the above transformation rules, the curvatures transform under the G-gauge transformations as

$$\mathcal{F} \to g^{-1}\mathcal{F}g\,, \qquad \mathcal{G} \to g^{-1} \triangleright \mathcal{G}\,, \qquad \mathcal{H} \to g^{-1} \triangleright \mathcal{H}\,, \tag{3.10}$$

under the H-gauge transformations as

$$\mathcal{F} \to \mathcal{F}\,, \qquad \mathcal{G} \to \mathcal{G} + \mathcal{F} \wedge^{\triangleright} \eta\,, \qquad \mathcal{H} \to \mathcal{H} - \{\mathcal{G}' \wedge \eta\} + \{\eta \wedge \mathcal{G}\}\,, \tag{3.11}$$

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and under the L-gauge transformations as

$$\mathcal{F} \to \mathcal{F} \,, \qquad \mathcal{G} \to \mathcal{G} \,, \qquad \mathcal{H} \to \mathcal{H} - \mathcal{F} \wedge^{\triangleright} \theta \,. \tag{3.12}$$

For more details, the reader is referred to [25].

In order to make the action (3.2) gauge invariant with respect to the transformations (3.7), (3.8) and (3.9), the Lagrange multipliers B, C and D must transform under the G-gauge transformations as

$$B \to g^{-1} B g \,, \qquad C \to g^{-1} \triangleright C \,, \qquad D \to g^{-1} \triangleright D \,, \tag{3.13}$$

under the H-gauge transformations as

$$B \to B + C' \wedge^{\mathcal{T}} \eta - \eta \wedge^{\mathcal{D}} \eta \wedge^{\mathcal{D}} D \,, \qquad C \to C + D \wedge^{\mathcal{X}_1} \eta + D \wedge^{\mathcal{X}_2} \eta \,, \qquad D \to D \,, \tag{3.14}$$

while under the L-gauge transformations they transform as

$$B \to B - D \wedge^{\mathcal{S}} \theta \,, \qquad C \to C \,, \qquad D \to D \,. \tag{3.15}$$

See appendix B for details, for the definition of the maps $\mathcal{T}$, $\mathcal{D}$, $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{S}$, and for the notation of the $\wedge^{\mathcal{T}}$, $\wedge^{\mathcal{D}}$, $\wedge^{\mathcal{X}_1}$, $\wedge^{\mathcal{X}_2}$, and $\wedge^{\mathcal{S}}$ products.

3.2 Constrained $3BF$ action for a real Klein-Gordon field

Once the topological $3BF$ action is specified, we can proceed with the construction of the constrained $3BF$ action, describing a realistic case of a scalar field coupled to gravity. In order to perform this construction, we have to define a specific 2-crossed module which gives rise to the topological sector of the action, and then we have to impose convenient simplicity constraints.

We begin by defining a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G \,, \triangleright \,, \{\_\,,\_\})$, as follows. The groups are given as

$$G = \mathrm{SO}(3,1) \,, \qquad H = \mathbb{R}^4 \,, \qquad L = \mathbb{R} \,. \tag{3.16}$$

The group G acts on itself via conjugation, on H via the vector representation, and on L via the trivial representation. This specifies the definition of the action $\triangleright$. The map ∂ is chosen to be trivial, as before. The map δ is also trivial, that is, every element of L is mapped to the identity element of H. Finally, the Peiffer lifting is trivial as well, mapping every ordered pair of elements in H to an identity element in L. This specifies one concrete 2-crossed module.

Given this choice of a 2-crossed module, the 3-connection $(\alpha \,, \beta \,, \gamma)$ takes the form

$$\alpha = \omega^{ab} M_{ab} \,, \qquad \beta = \beta^a P_a \,, \qquad \gamma = \gamma \mathbb{I} \,, \tag{3.17}$$

where $\mathbb{I}$ is the sole generator of the Lie group $\mathbb{R}$. From (3.1), the fake 3-curvature $(\mathcal{F} \,, \mathcal{G} \,, \mathcal{H})$ reduces to the ordinary 3-curvature,

$$\mathcal{F} = R^{ab} M_{ab} \,, \qquad \mathcal{G} = \nabla \beta^a P_a \,, \qquad \mathcal{H} = \mathrm{d}\gamma \,, \tag{3.18}$$

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where we used the fact that G acts trivially on L, that is, $M_{ab} \triangleright \mathbb{I} = 0$. The topological $3BF$ action (3.2) now becomes

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma \,, \tag{3.19}$$

where the bilinear form for L is $\langle \mathbb{I}, \mathbb{I} \rangle_{\mathfrak{l}} = 1$.

It is important to note that the Lagrange multiplier D in (3.2) is a 0-form and transforms trivially with respect to G, H and L gauge transformations for our choice of the 2-crossed module, as can be seen from (3.13), (3.14) and (3.15). Thus, D has all the *hallmark properties of a real scalar field*, allowing us to make identification between them, and conveniently relabel D into ϕ in (3.19). This is a crucial property of the 3-group structure in a 4-dimensional spacetime and is one of the main results of the paper. It follows the line of reasoning used in recognizing the Lagrange multiplier C^a in the $2BF$ action for the Poincaré 2-group as a tetrad field e^a. It is also important to stress that the choice of the third gauge group, L, dictates the number and the structure of the matter fields present in the action. In this case, $L = \mathbb{R}$ implies that we have only one real scalar field, corresponding to a single generator $\mathbb{I}$ of $\mathbb{R}$. The trivial nature of the action $\triangleright$ of $\mathrm{SO}(3,1)$ on $\mathbb{R}$ also implies that ϕ transforms as a scalar field. Finally, the scalar field appears as a degree of freedom in the topological sector of the action, making the quantization procedure feasible.

As in the case of BF and $2BF$ theories, in order to obtain nontrivial dynamics, we need to impose convenient simplicity constraints on the variables in the action (3.19). Since we are interested in obtaining the scalar field ϕ of mass m coupled to gravity in the standard way, we choose the action in the form:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda \wedge \left(\gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c \right) + \Lambda^{ab} \wedge \left(H_{abc} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b \right) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi^2 \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,. \end{aligned} \tag{3.20}$$

Note that the first row is the topological sector (3.19), the second row is the familiar simplicity constraint for gravity from the action (2.24), the third row contains the new simplicity constraints corresponding to the Lagrange multiplier 1-forms λ and Λ^{ab} and featuring the Lagrange multiplier 0-form H_{abc}, while the fourth row is the mass term for the scalar field.

Varying the total action (3.20) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, Λ_{ab}, γ, λ, H_{abc}, ϕ and e^a one obtains the equations of motion:

$$R^{ab} - \lambda^{ab} = 0 \,, \tag{3.21}$$

$$\nabla B^{ab} - e^{[a} \wedge \beta^{b]} = 0 \,, \tag{3.22}$$

$$\nabla e^a = 0 \,, \tag{3.23}$$

$$B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd}e^c \wedge e^d = 0\,, \tag{3.24}$$

$$H_{abc}\varepsilon^{cdef}e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b = 0\,, \tag{3.25}$$

$$\mathrm{d}\phi - \lambda = 0\,, \tag{3.26}$$

$$\gamma - \frac{1}{2}H_{abc}e^a \wedge e^b \wedge e^c = 0\,, \tag{3.27}$$

$$-\frac{1}{2}\lambda \wedge e^a \wedge e^b \wedge e^c + \varepsilon^{cdef}\Lambda^{ab} \wedge e_d \wedge e_e \wedge e_f = 0\,, \tag{3.28}$$

$$\mathrm{d}\gamma - \mathrm{d}(\Lambda^{ab} \wedge e_a \wedge e_b) - \frac{1}{4!}m^2\phi\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d = 0\,, \tag{3.29}$$

$$\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{3}{2}H_{abc}\lambda \wedge e^b \wedge e^c + 3H^{def}\varepsilon_{abcd}\Lambda_{ef} \wedge e^b \wedge e^c$$
$$-2\Lambda_{ab} \wedge \mathrm{d}\phi \wedge e^b - 2\frac{1}{4!}m^2\phi\varepsilon_{abcd}e^b \wedge e^c \wedge e^d = 0\,. \tag{3.30}$$

The dynamical degrees of freedom are e^a and ϕ, while the remaining variables are algebraically determined in terms of them. Specifically, the equations (3.21)–(3.28) give

$$\begin{aligned}
&\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, && \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,, \quad \gamma_{\mu\nu\rho} = -\frac{e}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\sigma\phi\,,\\
&\Lambda^{ab}{}_\mu = \frac{1}{12e}g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\partial_\nu\phi e^a{}_\rho e^b{}_\sigma\,, && \beta^a{}_{\mu\nu} = 0\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,,\\
&H^{abc} = \frac{1}{6e}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\phi e^a{}_\nu e^b{}_\rho e^c{}_\sigma\,, && \lambda_\mu = \partial_\mu\phi\,.
\end{aligned} \tag{3.31}$$

Note that from the equations (3.22), (3.23) and (3.24) it follows that $\beta^a = 0$, as in the pure gravity case. The equation of motion (3.29) reduces to the covariant Klein-Gordon equation for the scalar field,

$$\left(\nabla_\mu\nabla^\mu - m^2\right)\phi = 0\,. \tag{3.32}$$

Finally, the equation of motion (3.30) for e^a becomes:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \qquad T^{\mu\nu} \equiv \partial^\mu\phi\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\left(\partial_\rho\phi\partial^\rho\phi + m^2\phi^2\right)\,. \tag{3.33}$$

The system of equations (3.21)–(3.30) is equivalent to the system of equations (3.31)–(3.33). Note that in addition to the correct covariant form of the Klein-Gordon equation, we have also obtained the correct form of the stress-energy tensor for the scalar field.

3.3 Constrained 3BF action for the Dirac field

Now we pass to the more complicated case of the Dirac field. We first define a 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$ as follows. The groups are:

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^8(\mathbb{G})\,, \tag{3.34}$$

where $\mathbb{G}$ is the algebra of complex Grassmann numbers. The maps ∂, δ and the Peiffer lifting are trivial. The action of the group G on itself is given via conjugation, on H via vector representation, and on L via spinor representation, as follows. Denoting the

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8 generators of the Lie group $\mathbb{R}^8(\mathbb{G})$ as P_α and P^α, where the index α takes the values $1,\ldots,4$, the action of G on L is thus given explicitly as

$$M_{ab} \rhd P_\alpha = \frac{1}{2}(\sigma_{ab})^\beta{}_\alpha P_\beta \,, \qquad M_{ab} \rhd P^\alpha = -\frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta \,, \tag{3.35}$$

where $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, and γ_a are the usual Dirac matrices, satisfying the anticommutation rule $\{\gamma_a \,, \gamma_b\} = -2\eta_{ab}$.

As in the case of the scalar field, the choice of the group L dictates the matter content of the theory, while the action $\rhd$ of G on L specifies its transformation properties. To see this explicitly, let us construct the corresponding $3BF$ action. The 3-connection $(\alpha\,,\beta\,,\gamma)$ now takes the form

$$\alpha = \omega^{ab} M_{ab} \,, \qquad \beta = \beta^a P_a \,, \qquad \gamma = \gamma^\alpha P_\alpha + \bar\gamma_\alpha P^\alpha \,, \tag{3.36}$$

while the 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$, defined in (3.1), is given as

$$\mathcal{F} = R^{ab} M_{ab} \,, \mathcal{G} = \nabla \beta^a P_a \,, \tag{3.37}$$
$$\mathcal{H} = \left(\mathrm{d}\gamma^\alpha + \frac{1}{2}\omega^{ab}(\sigma_{ab})^\alpha{}_\beta \gamma^\beta \right) P_\alpha + \left(\mathrm{d}\bar\gamma_\alpha - \frac{1}{2}\omega^{ab}\bar\gamma_\beta(\sigma_{ab})^\beta{}_\alpha \right) P^\alpha \equiv (\overset{\rightarrow}{\nabla}\gamma)^\alpha P_\alpha + (\bar\gamma\overset{\leftarrow}{\nabla})_\alpha P^\alpha \,,$$

where we have used (3.35). The bilinear form $\langle \_,\_ \rangle_{\mathfrak{l}}$ is defined as

$$\langle P_\alpha, P_\beta\rangle_{\mathfrak{l}} = 0 \,, \qquad \langle P^\alpha, P^\beta\rangle_{\mathfrak{l}} = 0 \,, \qquad \langle P_\alpha, P^\beta\rangle_{\mathfrak{l}} = -\delta^\beta_\alpha \,, \qquad \langle P^\alpha, P_\beta\rangle_{\mathfrak{l}} = \delta^\alpha_\beta \,. \tag{3.38}$$

Note that, for general $A, B \in \mathfrak{l}$, we can write

$$\langle A, B\rangle_{\mathfrak{l}} = A^I B^J g_{IJ} \,, \qquad \langle B, A\rangle_{\mathfrak{l}} = B^J A^I g_{JI} \,. \tag{3.39}$$

Since we require the bilinear form to be symmetric, the two expressions must be equal. However, since the coefficients in $\mathfrak{l}$ are Grassmann numbers, we have $A^I B^J = -B^J A^I$, so it follows that $g_{IJ} = -g_{JI}$. Hence the antisymmetry of (3.38).

Now we use the properties of the group L and the action $\rhd$ of G on L to recognize the physical nature of the Lagrange multiplier D in (3.2). Indeed, the choice of the group L dictates that D contains 8 independent complex Grassmannian matter fields as its components. Moreover, due to the fact that D is a 0-form and that it transforms according to the spinorial representation of $\mathrm{SO}(3,1)$, we can identify its components with the Dirac bispinor fields, and write

$$D = \psi^\alpha P_\alpha + \bar\psi_\alpha P^\alpha \,, \tag{3.40}$$

where it is assumed that ψ and $\bar\psi$ are independent fields, as usual. This is again an illustration of the fact that information about the structure of the matter sector in the theory is specified by the choice of the group L in the 2-crossed module, and another main result of the paper.

Given all of the above, now we can finally write the $3BF$ action (3.2) corresponding to this choice of the 2-crossed module as

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + (\bar\gamma\overset{\leftarrow}{\nabla})_\alpha \psi^\alpha + \bar\psi_\alpha (\overset{\rightarrow}{\nabla}\gamma)^\alpha \,. \tag{3.41}$$

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In order to promote this action into a full theory of gravity coupled to Dirac fermions, we add the convenient constraint terms to the action, as follows:

$$
\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + (\bar{\gamma}\overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla}\gamma)^\alpha \\
& - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\
& - \lambda^\alpha \wedge \left(\bar{\gamma}_\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar{\psi}\gamma^d)_\alpha \right) + \bar{\lambda}_\alpha \wedge \left(\gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^\alpha \right) \\
& - \frac{1}{12} m \, \bar{\psi}\psi \, \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d + 2\pi i l_p^2 \, \bar{\psi}\gamma_5\gamma^a\psi \, \varepsilon_{abcd} e^b \wedge e^c \wedge \beta^d \, .
\end{aligned}
\tag{3.42}
$$

Here the first row is the topological sector, the second row is the gravitational simplicity constraint term from (2.24), while the third row contains the new simplicity constraints for the Dirac field corresponding to the Lagrange multiplier 1-forms λ^α and $\bar{\lambda}_\alpha$. The fourth row contains the mass term for the Dirac field, and a term which ensures the correct coupling between the torsion and the spin of the Dirac field, as specified by the Einstein-Cartan theory. Namely, we want to ensure that the torsion has the form

$$
T_a \equiv \nabla e_a = 2\pi l_p^2 s_a \, , \tag{3.43}
$$

where

$$
s_a = i\varepsilon_{abcd} e^b \wedge e^c \bar{\psi}\gamma_5\gamma^d\psi \tag{3.44}
$$

is the spin 2-form. Of course, other couplings should also be straightforward to implement, but we choose this particular coupling because we are interested in reproducing the standard Einstein-Cartan gravity coupled to the Dirac field.

Varying the action (3.42) with respect to B_{ab}, λ^{ab}, $\bar{\gamma}_\alpha$, γ^α, λ^α, $\bar{\lambda}_\alpha$, $\bar{\psi}_\alpha$, ψ^α, e^a, β^a and ω^{ab} one obtains the equations of motion:

$$
R^{ab} - \lambda^{ab} = 0 \, , \tag{3.45}
$$

$$
B_{ab} - \frac{1}{16\pi l_p^2} \varepsilon_{abcd} e^c \wedge e^d = 0 \, , \tag{3.46}
$$

$$
(\overrightarrow{\nabla}\psi)^\alpha - \lambda^\alpha = 0 \, , \tag{3.47}
$$

$$
(\bar{\psi}\overleftarrow{\nabla})_\alpha - \bar{\lambda}_\alpha = 0 \, , \tag{3.48}
$$

$$
\bar{\gamma}_\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar{\psi}\gamma^d)_\alpha = 0 \, , \tag{3.49}
$$

$$
\gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^\alpha = 0 \, , \tag{3.50}
$$

$$
\begin{aligned}
\mathrm{d}\gamma^\alpha + \omega^\alpha{}_\beta \wedge \gamma^\beta + \frac{i}{6} \lambda^\beta \wedge \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \gamma^{d\alpha}{}_\beta + \frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \psi^\alpha & \\
+ i2\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\gamma_5\gamma^d\psi)^\alpha = 0 \, , &
\end{aligned}
\tag{3.51}
$$

$$
\begin{aligned}
\mathrm{d}\bar{\gamma}_\alpha - \bar{\gamma}_\beta \wedge \omega^\beta{}_\alpha + \frac{i}{6} \bar{\lambda}_\beta \wedge \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \gamma^{d\beta}{}_\alpha - \frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \bar{\psi}_\alpha & \\
- i2\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\bar{\psi}\gamma_5\gamma^d)_\alpha = 0 \, , &
\end{aligned}
\tag{3.52}
$$

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$$\nabla\beta_a + 2\varepsilon_{abcd}\lambda^{bc}\wedge e^d - \frac{i}{2}\varepsilon_{abcd}\lambda^\alpha\wedge e^b\wedge e^c(\bar\psi\gamma^d)_\alpha + \frac{i}{2}\varepsilon_{abcd}\bar\lambda_\alpha\wedge e^b\wedge e^c(\gamma^d\psi)^\alpha$$
$$-\frac{1}{3}\varepsilon_{abcd}e^b\wedge e^c\wedge e^d m\bar\psi\psi - 4\pi l_p^2 i\varepsilon_{abcd}e^b\wedge\beta^c\bar\psi\gamma_5\gamma^d\psi = 0\,, \quad (3.53)$$
$$\nabla e_a - i2\pi l_p^2\varepsilon_{abcd}e^b\wedge e^c\bar\psi\gamma_5\gamma^d\psi = 0\,, \quad (3.54)$$
$$\nabla B_{ab} - e_{[a}\wedge\beta_{b]} + \bar\gamma\frac{1}{8}[\gamma_a,\gamma_b]\psi + \bar\psi\frac{1}{8}[\gamma_a,\gamma_b]\gamma = 0\,. \quad (3.55)$$

The dynamical degrees of freedom are e^a, ψ^α and $\bar\psi_\alpha$, while the remaining variables are determined in terms of the dynamical variables, and are given as:

$$\begin{aligned}
&B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, && \lambda^\alpha{}_\mu = (\overrightarrow{\nabla}_\mu\psi)^\alpha\,, && \bar\lambda_{\alpha\mu} = (\bar\psi\overleftarrow{\nabla}_\mu)_\alpha\,,\\
&\bar\gamma_{\alpha\mu\nu\rho} = i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\bar\psi\gamma^d)_\alpha\,, && \gamma^\alpha{}_{\mu\nu\rho} = -i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\gamma^d\psi)^\alpha\,, && \\
&\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, && \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu + K^{ab}{}_\mu\,. &&
\end{aligned} \quad (3.56)$$

Here $K^{ab}{}_\mu$ is the contorsion tensor, constructed in the standard way from the torsion tensor, whereas from (3.54) we have

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a\,, \quad (3.57)$$

which is precisely the desired equation (3.43). Further, from the equation (3.46) one obtains

$$\nabla B^{ab} = -\frac{1}{8\pi l_p^2}\varepsilon^{abcd}\left(e_c\wedge\nabla e_d\right)\,. \quad (3.58)$$

Substituting this expression in the equation (3.55) it follows that

$$2\varepsilon_{abcd}e^c\wedge\left(-\frac{1}{16\pi l_p^2}\nabla e^d + \frac{1}{8}s^d\right) - e_{[a}\wedge\beta_{b]} = 0\,. \quad (3.59)$$

The expression in the parentheses is equal to zero, according to the equation (3.54). From the remaining term $e_{[a}\wedge\beta_{b]} = 0$ it again follows that

$$\beta = 0\,. \quad (3.60)$$

Using this result, the equation of motion (3.51) for fermions becomes

$$\frac{i}{6}\varepsilon_{abcd}e^a\wedge e^b\wedge\left(2e^c\wedge\gamma^d\overrightarrow{\nabla} + \frac{im}{2}e^c\wedge e^d - 3(\nabla e^c)\gamma^d\right)\psi = 0\,. \quad (3.61)$$

Using equation (3.54), the last term in the parentheses vanishes, and the equation reduces to the covariant Dirac equation,

$$(i\gamma^a e^\mu{}_a\overrightarrow{\nabla}_\mu - m)\psi = 0\,, \quad (3.62)$$

where $e^\mu{}_a$ is the inverse tetrad. Similarly, the equation (3.52) gives the conjugated Dirac equation:

$$\bar\psi(i\overleftarrow{\nabla}_\mu e^\mu{}_a\gamma^a + m) = 0\,. \quad (3.63)$$

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Finally, the equation of motion (3.53) for tetrad field reduces to

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \qquad T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\gamma^{\nu}\overleftrightarrow{\nabla}^{a}e^{\mu}{}_{a}\psi - \frac{1}{2}g^{\mu\nu}\bar{\psi}\Big(i\gamma^{a}\overleftrightarrow{\nabla}_{\rho}e^{\rho}{}_{a} - 2m\Big)\psi\,, \quad (3.64)$$

Here, we used the notation $\overleftrightarrow{\nabla} = \overrightarrow{\nabla} - \overleftarrow{\nabla}$. The system of equations (3.45)–(3.55) is equivalent to the system of equations (3.56), (3.60), (3.62)–(3.64). As we expected, the equations of motion (3.57), (3.62), (3.63) and (3.64) are precisely the equations of motion of the Einstein-Cartan theory coupled to a Dirac field.

3.4 Constrained $3BF$ action for the Weyl and Majorana fields

A general solution of the Dirac equation is not an irreducible representation of the Lorentz group, and one can rewrite Dirac fermions as left-chiral and right-chiral fermion fields that both retain their chirality under Lorentz transformations, implying their irreducibility. Hence, it is useful to rewrite the action for left and right Weyl spinors as a constrained $3BF$ action. For simplicity, we will discuss only left-chiral spinor field, while the right-chiral field can be treated analogously. Both Weyl and Majorana fermions can be treated in the same way, the only difference being the presence of an additional mass term in the Majorana action.

We being by defining a 2-crossed module $(L \xrightarrow{\delta} H \xrightarrow{\partial} G\,, \triangleright\,, \{\_\,,\_\})$, as follows. The groups are:

$$G = \mathrm{SO}(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^4(\mathbb{G})\,. \tag{3.65}$$

The maps ∂, δ and the Peiffer lifting are trivial. The action $\triangleright$ of the group G on G, H and L is given in the same way as for the Dirac case, whereas the spinorial representation reduces to

$$M_{ab} \triangleright P^{\alpha} = \frac{1}{2}(\sigma_{ab})^{\alpha}{}_{\beta}P^{\beta}\,, \qquad M_{ab} \triangleright P_{\dot\alpha} = \frac{1}{2}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha}P_{\dot\beta}\,, \tag{3.66}$$

where $\sigma^{ab} = -\bar\sigma^{ab} = \frac{1}{4}(\sigma^a\bar\sigma^b - \sigma^b\bar\sigma^a)$, for $\sigma^a = (1,\vec\sigma)$ and $\bar\sigma^a = (1,-\vec\sigma)$, in which $\vec\sigma$ denotes the set of three Pauli matrices. The four generators of the group L are denoted as P^α and $P_{\dot\alpha}$, where the Weyl indices $\alpha, \dot\alpha$ take values $1, 2$.

The 3-connection $(\alpha\,,\beta\,,\gamma)$ now takes the form corresponding to this choice of Lie groups,

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma_\alpha P^\alpha + \bar\gamma^{\dot\alpha}P_{\dot\alpha}\,, \tag{3.67}$$

while the fake 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ defined in (3.1) is

$$\mathcal{F} = R^{ab}M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,, \tag{3.68}$$

$$\mathcal{H} = \left(\mathrm{d}\gamma_\alpha + \frac{1}{2}\omega^{ab}(\sigma^{ab})^{\beta}{}_{\alpha}\gamma_\beta\right)P^\alpha + \left(\mathrm{d}\bar\gamma^{\dot\alpha} + \frac{1}{2}\omega_{ab}(\bar\sigma^{ab})^{\dot\alpha}{}_{\dot\beta}\bar\gamma^{\dot\beta}\right)P_{\dot\alpha} \equiv (\overrightarrow{\nabla}\gamma)_\alpha P^\alpha + (\bar\gamma\overleftarrow{\nabla})^{\dot\alpha}P_{\dot\alpha}\,.$$

Introducing the spinor fields ψ_α and $\bar\psi^{\dot\alpha}$ via the Lagrange multiplier D as

$$D = \psi_\alpha P^\alpha + \bar\psi^{\dot\alpha}P_{\dot\alpha}\,, \tag{3.69}$$

and using the bilinear form $\langle\_,\_\rangle_{\mathfrak{l}}$ for the group L,

$$\langle P^\alpha, P^\beta\rangle_{\mathfrak{l}} = \varepsilon^{\alpha\beta}\,, \qquad \langle P_{\dot\alpha}, P_{\dot\beta}\rangle_{\mathfrak{l}} = \varepsilon_{\dot\alpha\dot\beta}\,, \qquad \langle P^\alpha, P_{\dot\beta}\rangle_{\mathfrak{l}} = 0\,, \qquad \langle P_{\dot\alpha}, P^\beta\rangle_{\mathfrak{l}} = 0\,, \tag{3.70}$$

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where $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$ are the usual two-dimensional antisymmetric Levi-Civita symbols, the topological $3BF$ action (3.2) for spinors coupled to gravity becomes

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \psi^\alpha \wedge (\vec{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}} \wedge (\bar{\gamma}\overleftarrow{\nabla})^{\dot{\alpha}} . \tag{3.71}$$

In order to obtain the suitable equations of motion for the Weyl spinors, we again introduce appropriate simplicity constraints, so that the action becomes:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \psi^\alpha \wedge (\vec{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot{\alpha}} \wedge (\bar{\gamma}\overleftarrow{\nabla})^{\dot{\alpha}} \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & - \lambda^\alpha \wedge \left(\gamma_\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} \right) - \bar{\lambda}_{\dot{\alpha}} \wedge \left(\bar{\gamma}^{\dot{\alpha}} + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta \right) \\ & - 4\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\bar{\psi}_{\dot{\alpha}} \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta) . \end{aligned} \tag{3.72}$$

The new simplicity constraints are in the third row, featuring the Lagrange multiplier 1-forms λ_α and $\bar{\lambda}^{\dot{\alpha}}$. Also, using the coupling between the Dirac field and torsion from Einstein-Cartan theory as a model, the term in the fourth row is chosen to ensure that the coupling between the Weyl spin tensor

$$s_a \equiv i\varepsilon_{abcd} e^b \wedge e^c \, \psi^\alpha \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} , \tag{3.73}$$

and torsion is given as:

$$T_a = 4\pi l_p^2 s_a . \tag{3.74}$$

The case of the Majorana field is introduced in exactly the same way, albeit with an additional mass term in the action, of the form:

$$-\frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d (\psi^\alpha \psi_\alpha + \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}) . \tag{3.75}$$

Varying the action (3.72) with respect to the variables B_{ab}, λ^{ab}, γ_α, $\bar{\gamma}^{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, e^a, β^a and ω^{ab} one again obtains the complete set of equations of motion, displayed in the appendix C. The only dynamical degrees of freedom are ψ_α, $\bar{\psi}^{\dot{\alpha}}$ and e^a, while the remaining variables are algebraically determined in terms of these as:

$$\begin{gathered} \lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu} , \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2} \varepsilon_{abcd} e^c{}_\mu e^d{}_\nu , \quad \lambda_{\alpha\mu} = \nabla_\mu \psi_\alpha , \quad \bar{\lambda}^{\dot{\alpha}}{}_\mu = \nabla_\mu \bar{\psi}^{\dot{\alpha}} , \\ \gamma_{\alpha\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \sigma^d{}_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}} , \quad \bar{\gamma}^{\dot{\alpha}}{}_{\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta , \quad \omega_{ab\mu} = \triangle_{ab\mu} + K_{ab\mu} . \end{gathered} \tag{3.76}$$

In addition, one also maintains the result $\beta = 0$ as before. Finally, the equations of motion for the dynamical fields are

$$\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta = 0 , \qquad \sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} = 0 , \tag{3.77}$$

and

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2 \, T^{\mu\nu} , \tag{3.78}$$

where

$$T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} - g^{\mu\nu}\frac{1}{2}\Big(i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi}\Big) . \qquad (3.79)$$

Here we have suppressed the spinor indices. In the case of the Majorana field, the equations of motion (3.76) remain the same, while the equations of motion for ψ_α and $\bar{\psi}^{\dot{\alpha}}$ take the form

$$i\sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} - m\psi_\alpha = 0\,, \qquad i\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta - m\bar{\psi}^{\dot{\alpha}} = 0\,, \qquad (3.80)$$

whereas the stress-energy tensor takes the form

$$\begin{aligned} T^{\mu\nu} \equiv\; & \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} \\ & - g^{\mu\nu}\frac{1}{2}\left[i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi} - \frac{1}{2}m\left(\psi\psi + \bar{\psi}\bar{\psi}\right)\right] . \end{aligned} \qquad (3.81)$$

4 The Standard Model

The Standard Model 3-group can be defined as:

$$G = \mathrm{SO}(3,1)\times\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)\,, \quad H = \mathbb{R}^4\,, \quad L = \mathbb{R}^4(\mathbb{C})\times\mathbb{R}^{64}(\mathbb{G})\times\mathbb{R}^{64}(\mathbb{G})\times\mathbb{R}^{64}(\mathbb{G})\,, \qquad (4.1)$$

where $\mathbb{C}$ denotes the field of complex numbers. The motivation for this choice of the group L is given in the table below.

1. lepton generation	red color 1. quark generation	green color 1. quark generation	blue color 1. quark generation
$\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L$	$\begin{pmatrix}u_r \\ d_r\end{pmatrix}_L$	$\begin{pmatrix}u_g \\ d_g\end{pmatrix}_L$	$\begin{pmatrix}u_b \\ d_b\end{pmatrix}_L$
$(\nu_e)_R$	$(u_r)_R$	$(u_g)_R$	$(u_b)_R$
$(e^-)_R$	$(d_r)_R$	$(d_g)_R$	$(d_b)_R$

We see that in order to introduce one generation of matter one needs to provide 16 spinors, or equivalently the group L has to be chosen as $L = \mathbb{R}^{64}(\mathbb{G})$. As there are three generations of matter, the part of the group L that corresponds to the fermion fields in the theory is chosen to be $L = \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G}) \times \mathbb{R}^{64}(\mathbb{G})$. To define the Higgs sector one needs two complex scalar fields $\begin{pmatrix}\phi^+ \\ \phi_0\end{pmatrix}$, or equivalently the scalar sector of the group L is given as $L = \mathbb{R}^4(\mathbb{C})$.

The maps ∂, δ and the Peiffer lifting are trivial. The action of the group G on itself is given via conjugation. The action of the $\mathrm{SO}(3,1)$ subgroup of G on H is via vector representation and the action of $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ subgroup on H is via trivial representation. The action of the $\mathrm{SO}(3,1)$ on L is via trivial representation for the generators corresponding to the scalar fields, i.e. the $\mathbb{R}^4(\mathbb{C})$ subgroup of L, and via spinor representation for the every quadruple of generators corresponding to the fermion fields, given as in the section 3. The information how spinors transform under the $SU(3) \times SU(2) \times U(1)$ group is encoded in the action of that subgroup of G on L, as specified in the table above. For simplicity, in the following, only one family of the lepton sector and only electroweak part of the gauge sector of the Standard model is considered.

The groups are chosen as:

$$G = SO(3,1) \times SU(2) \times U(1)\,, \qquad H = \mathbb{R}^4\,, \qquad L^{\text{leptons}} = \mathbb{R}^{16}(\mathbb{G}) \times \mathbb{R}^4(\mathbb{C})\,. \tag{4.2}$$

The 3-connection then takes the form

$$\begin{aligned} &\alpha = \omega^{ab} M_{ab} + W^I T_I + AY\,, \qquad\qquad \beta = \beta^a P_a\,, \\ &\gamma = \gamma_\alpha{}^{\tilde{L}} P^\alpha{}_{\tilde{L}} + \gamma^{\dot\alpha}{}_{\tilde{L}} P_{\dot\alpha}{}^{\tilde{L}} + \gamma_\alpha{}^{\tilde{R}} P^\alpha{}_{\tilde{R}} + \gamma^{\dot\alpha}{}_{\tilde{R}} P_{\dot\alpha}{}^{\tilde{R}} + \gamma^{\tilde{a}} P_{\tilde{a}}\,. \end{aligned} \tag{4.3}$$

Here the indices $I, J, \dots$ take the values $1,2,3$ and counts the Pauli matrices, generators of the group $SU(2)$, the indices $\tilde{L}, \tilde{L}', \dots$ take the values $1,2$ and count the components of left doublet, $\tilde{R}$ denotes the right singlet $(e^-)_R$ and right singlet $(\nu_e)_R$, and indices $\tilde{a}, \tilde{b}, \dots$ take values $1,2$ and count the components of the scalar doublet. It is also useful to define $\tilde{i} = (\tilde{L}, \tilde{R})$ which takes values $1, \dots, 4$.

The action of the group G on L is defined as:

$$\begin{aligned} &M_{ab} \rhd P^\alpha{}_i = \frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta{}_i\,, \qquad M_{ab} \rhd P_{\dot\alpha i} = \frac{1}{2}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha} P_{\dot\beta i}\,, \qquad M_{ab} \rhd P_{\tilde{a}} = 0\,, \\ &T_I \rhd P^\alpha{}_{\tilde{L}} = \frac{1}{2}(\sigma_I)^{\tilde{L}'}{}_{\tilde{L}} P^\alpha{}_{\tilde{L}'}\,, \qquad T_I \rhd P_{\dot\alpha \tilde{L}} = \frac{1}{2}(\sigma_I)^{\tilde{L}'}{}_{\tilde{L}} P_{\dot\alpha \tilde{L}'}\,, \\ &T_I \rhd P^\alpha{}_{\tilde{R}} = 0\,, \qquad T_I \rhd P_{\dot\alpha \tilde{R}} = 0\,, \qquad T_I \rhd P_{\tilde{a}} = \frac{1}{2}(\sigma_I)^{\tilde{b}}{}_{\tilde{a}} P_{\tilde{b}}\,, \\ &Y \rhd P^\alpha{}_{\tilde{L}} = -P^\alpha{}_{\tilde{L}}\,, \quad Y \rhd P^\alpha{}_{e_R} = -2P^\alpha{}_{e_R}\,, \quad Y \rhd P^\alpha{}_{\nu_R} = -2P^\alpha{}_{\nu_R}\,, \quad Y \rhd P_{\tilde{a}} = P_{\tilde{a}}\,, \\ &Y \rhd P_{\dot\alpha \tilde{L}} = -P_{\dot\alpha \tilde{L}}\,, \qquad Y \rhd P_{\dot\alpha e_R} = -2P_{\dot\alpha e_R}\,, \qquad Y \rhd P_{\dot\alpha \nu_R} = -2P_{\dot\alpha \nu_R}\,. \end{aligned} \tag{4.4}$$

The 3-curvatures are given as:

$$\begin{aligned} &\mathcal{F} = R^{ab} M_{ab} + F^I T_I + FY\,, \qquad \mathcal{G} = \nabla \beta^a P_a\,, \\ &\mathcal{H} = (\overset{\rightarrow}{\nabla} \gamma^{\tilde{L}})_\alpha P^\alpha{}_{\tilde{L}} + (\bar\gamma_{\tilde{L}} \overset{\leftarrow}{\nabla})^{\dot\alpha} P_{\dot\alpha}{}^{\tilde{L}} + (\overset{\rightarrow}{\nabla} \gamma^{\tilde{R}})_\alpha P^\alpha{}_{\tilde{R}} + (\bar\gamma_{\tilde{R}} \overset{\leftarrow}{\nabla})^{\dot\alpha} P_{\dot\alpha}{}^{\tilde{R}} + \mathrm{d}\gamma^{\tilde{a}} P_{\tilde{a}}\,. \end{aligned} \tag{4.5}$$

The topological $3BF$ action is defined as:

$$S = \int B_{ab} R^{ab} + B_I F^I + BF + e_a \nabla \beta^a + \psi^\alpha{}_{\tilde{i}} (\overset{\rightarrow}{\nabla} \gamma^{\tilde{i}})_\alpha + \bar\psi_{\dot\alpha}{}^{\tilde{i}} (\bar\gamma_{\tilde{i}} \overset{\leftarrow}{\nabla})^{\dot\alpha} + \phi^{\tilde{a}} \mathrm{d}\gamma_{\tilde{a}}\,. \tag{4.6}$$

At this point, it is useful to simplify the notation and denote all indices of the group G by $\hat\alpha$, of the group H by $\hat{a}$ and L by $\hat{A}$. In order to promote this action to a full theory of first lepton family coupled to electroweak gauge fields, Higgs field, and gravity, we again introduce the appropriate simplicity constraint, as follows

$$
\begin{aligned}
S = \int & B_{\hat\alpha} \wedge \mathcal{F}^{\hat\alpha} + e_{\hat a} \wedge \mathcal{G}^{\hat a} + D_{\hat A} \wedge \mathcal{H}^{\hat A} \\
& + \left(B_{\hat\alpha} - C_{\hat\alpha}{}^{\hat\beta} M_{cd\hat\beta} e^c \wedge e^d \right) \wedge \lambda^{\hat\alpha} - \left(\gamma_{\hat A} - e^a \wedge e^b \wedge e^c C_{\hat A}{}^{\hat B} M_{abc\hat B} \right) \wedge \lambda^{\hat A} \\
& + \zeta^{ab}{}_{\hat\alpha} \wedge \left(M_{ab}{}^{\hat\alpha} \varepsilon^{cdef} e_c \wedge e_d \wedge e_e \wedge e_f - F^{\hat\alpha} \wedge e_c \wedge e_d \right) \\
& + \zeta^{ab}{}_{\hat A} \wedge \left(M_{abc}{}^{\hat A} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - F^{\hat A} \wedge e_a \wedge e_b \right) \\
& - \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \, \left(Y_{\hat A\hat B\hat C} D^{\hat A} D^{\hat B} D^{\hat C} + M_{\hat A\hat B} D^{\hat A} D^{\hat B} + L_{\hat A\hat B\hat C\hat D} D^{\hat A} D^{\hat B} D^{\hat C} D^{\hat D} \right) \\
& - 4\pi i \, l_p^2 \, \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c D_{\hat A} T^{d\hat A}{}_{\hat B} D^{\hat B} \,,
\end{aligned}
\tag{4.7}
$$

where:

$$
\begin{aligned}
B_{\hat\alpha} &= \begin{bmatrix} B_{ab} & B_I & B \end{bmatrix} , \qquad \mathcal{F}^{\hat\alpha} = \begin{bmatrix} R_{ab} & F_I & F \end{bmatrix}^T , \qquad D_{\hat A} = \begin{bmatrix} \psi^\alpha{}_{\tilde L} & \bar\psi_{\dot\alpha\tilde L} & \psi^\alpha{}_R & \bar\psi_{\dot\alpha R} & \phi_{\tilde a} \end{bmatrix} , \\
\mathcal{H}^{\hat A} &= \begin{bmatrix} (\overrightarrow{\nabla}\gamma_{\tilde L})_\alpha & (\bar\gamma_{\tilde L}\overleftarrow{\nabla})^{\dot\alpha} & (\overrightarrow{\nabla}\gamma_{\tilde R})_\alpha & (\bar\gamma_{\tilde R}\overleftarrow{\nabla})^{\dot\alpha} & \mathrm{d}\gamma_{\tilde a} \end{bmatrix}^T , \quad \gamma_{\hat A} = \begin{bmatrix} \gamma^\alpha{}_{\tilde L} & \bar\gamma_{\dot\alpha\tilde L} & \gamma^\alpha{}_{\tilde R} & \bar\gamma_{\dot\alpha\tilde R} & \gamma_{\tilde a} \end{bmatrix} , \\
\lambda^{\hat\alpha} &= \begin{bmatrix} -\lambda^{ab} & \lambda^I & \lambda \end{bmatrix}^T , \qquad \zeta^{cd}{}_{\hat\alpha} = \begin{bmatrix} 0 & \zeta^{cd}{}_I & \zeta^{cd} \end{bmatrix} , \qquad \zeta^{ab}{}_{\hat A} = \begin{bmatrix} \zeta^{ab} & 0 & 0 \end{bmatrix} , \\
\lambda^{\hat A} &= \begin{bmatrix} \lambda_{\alpha L} & \bar\lambda^{\dot\alpha}{}_L & \lambda_{\alpha R} & \bar\lambda^{\dot\alpha}{}_R & \lambda^{\tilde a} \end{bmatrix}^T , \qquad M_{cd\hat\alpha} = \begin{bmatrix} \varepsilon_{abcd} & M_{cdI} & M_{cd} \end{bmatrix} , \\
M_{abc\hat A} &= \begin{bmatrix} \varepsilon_{abcd}\sigma^d{}_{\alpha\dot\beta}\bar\psi^{\dot\beta}{}_L & \varepsilon_{abcd}\bar\sigma^{d\dot\alpha\beta}\psi_{\beta L} & \varepsilon_{abcd}\sigma^d{}_{\alpha\dot\beta}\bar\psi^{\dot\beta}{}_R & \varepsilon_{abcd}\bar\sigma^{d\dot\alpha\beta}\psi_{\beta R} & M_{abc\tilde a} \end{bmatrix} .
\end{aligned}
$$

The matrices $C^{\hat\alpha}{}_{\hat\beta}$, $C^{\hat A}{}_{\hat B}$, $M_{\hat A\hat B}$, $Y_{\hat A\hat B\hat C}$, $L_{\hat A\hat B\hat C\hat D}$ and $T^{d\hat A}{}_{\hat B}$ are constant matrices, and carry the information about gauge coupling constants, mass of the Higgs field, Yukawa couplings and mixing angles, Higgs self-coupling constant and torsion coupling, respectively.

5 Conclusions

Let us summarize the results of the paper. In section 2 we have given a short reminder of the BF theory and described how one can use it to construct the action for general relativity (the well known Plebanski model), and the action for the Yang-Mills theory in flat spacetime, in a novel way. Passing on to higher gauge theory, we have reviewed the formalism of 2-groups and the corresponding $2BF$ theory, using it again to construct the action for general relativity (a model first described in [12]), and the unified action of general relativity and Yang-Mills theory, both naturally described using the 2-group formalism. With this background material in hand, in section 3 we have used the idea of a categorical ladder yet again, generalizing the $2BF$ theory to $3BF$ theory, with the underlying structure of a 3-group instead of a 2-group. This has led us to the main insight that the *scalar and fermion fields can be specified using a gauge group*, namely the third gauge group, denoted L, present in the 2-crossed module corresponding to a given 3-group. This has allowed us to single out specific gauge groups corresponding to the Klein-Gordon, Dirac, Weyl and Majorana fields, and to construct the relevant constrained $3BF$ actions that describe all these fields coupled to gravity in the standard way.

The obtained results represent the fundamental building blocks for the construction of the complete Standard Model of elementary particles coupled to Einstein-Cartan gravity as a $3BF$ action with suitable simplicity constraints, as demonstrated in section 4. In this way, we can complete the first step of the spinfoam quantization programme for the complete theory of gravity and all matter fields, as specified in the Introduction. This is a clear improvement over the ordinary spinfoam models based on an ordinary constrained BF theory.

In addition to this, the gauge group which determines the matter spectrum of the theory is a completely novel structure, not present in the Standard Model. This new gauge group stems from the 3-group structure of the theory, so it is not surprising that it is invisible in the ordinary formulation of the Standard Model, since the latter does not use any 3-group structure in an explicit way. In this paper, we have discussed the choices of this group which give rise to all relevant matter fields, and these can simply be directly multiplied to give the group corresponding to the full Standard Model, encoding the quark and lepton families and all other structure of the matter spectrum. However, the true potential of the matter gauge group lies in a possibility of nontrivial unification of matter fields, by choosing it to be something other than the ordinary product of its component groups. For example, instead of choosing $\mathbb{R}^8(\mathbb{G})$ for the Dirac field, one can try a noncommutative SU(3) group, which also contains 8 generators, but its noncommutativity requires that the maps δ and $\{\_\,,\_\}$ be nontrivial, in order to satisfy the axioms of a 2-crossed module. This, in turn, leads to a distinction between 3-curvature and fake 3-curvature, which can have consequences for the dynamics of the theory. In this way, by studying nontrivial choices of a 3-group, one can construct various different 3-group-unified models of gravity and matter fields, within the context of higher gauge theory. This idea resembles the ordinary grand unification programme within the framework of the standard gauge theory, where one constructs various different models of vector fields by making various choices for the Yang-Mills gauge group. The detailed discussion of these 3-group unified models is left for future work.

As far as the spinfoam quantization programme is concerned, having completed the step 1 (as outlined in the Introduction), there is a clear possibility to complete the steps 2 and 3 as well. First, the fact that the full action is written completely in terms of differential forms of various degrees, allows us to adapt it to a triangulated spacetime manifold, in the sense of Regge calculus. In particular, all fields and their field strengths present in the $3BF$ action can be naturally associated to the appropriate d-dimensional simplices of a 4-dimensional triangulation, by matching 0-forms to vertices, 1-forms to edges, etc. This leads us to the following table:

d	triangulation	dual triangulation	form	fields	field strengths
0	vertex	4-polytope	0-form	ϕ, $\psi_{\tilde{\alpha}}$, $\bar{\psi}^{\tilde{\alpha}}$	
1	edge	3-polyhedron	1-form	ω^{ab}, A^I, e^a	
2	triangle	face	2-form	β^a, B^{ab}	R^{ab}, F^I, T^a
3	tetrahedron	edge	3-form	γ, $\gamma_{\tilde{\alpha}}$, $\bar{\gamma}^{\tilde{\alpha}}$	$\mathcal{G}^a$
4	4-simplex	vertex	4-form		$\mathcal{H}$, $\mathcal{H}_{\tilde{\alpha}}$, $\bar{\mathcal{H}}^{\tilde{\alpha}}$

Once the classical Regge-discretized topological $3BF$ action is constructed, one can attempt to construct a state sum Z which defines the path integral for the theory. The topological nature of the pure $3BF$ action, together with the underlying structure of the 3-group, should ensure that such a state sum Z is a topological invariant, in the sense that it is triangulation independent. Unfortunately, in order to perform this step precisely, one needs a generalization of the Peter-Weyl and Plancharel theorems to 2-groups and 3-groups, a mathematical result that is presently still missing. The purpose of the Peter-Weyl theorem is to provide a decomposition of a function on a group into a sum over the corresponding irreducible representations, which ultimately specifies the appropriate spectrum of labels for the d-simplices in the triangulation, fixing the domain of values for the fields living on those d-simplices. In the case of 2-groups and especially 3-groups, the representation theory has not been developed well enough to allow for such a construction, with a consequence of the missing Peter-Weyl theorem for 2-groups and 3-groups. However, until the theorem is proved, we can still try to *guess* the appropriate structure of the irreducible representations of the 2- and 3-groups, as was done for example in [12], leading to the so-called *spincube model* of quantum gravity.

Finally, if we remember that for the purpose of physics we are not really interested in a topological theory, but instead in one which contains local propagating degrees of freedom, we are therefore not really engaged in constructing a topological invariant Z, but rather a state sum which describes nontrivial dynamics. In particular, we need to impose the simplicity constraints onto the state sum Z, which is the step 3 of the spinfoam quantization programme. In light of that, one of the main motivations and also main results of our paper was to rewrite the action for gravity and matter in a way that explicitly distinguishes the topological sector from the simplicity constraints. Imposing the constraints is therefore straightforward in the context of a 3-group gauge theory, and completing this step would ultimately lead us to a state sum corresponding to a tentative theory of quantum gravity with matter. This is also a topic for future work.

In the end, let us also mention that aside from the unification and quantization programmes, there is also a plethora of additional studies one can perform with the constrained $3BF$ action, such as the analysis of the Hamiltonian structure of the theory (suitable for a potential canonical quantization programme), the idea of imposing the simplicity constraints using a spontaneous symmetry breaking mechanism, and finally a detailed study of the mathematical structure and properties of the simplicity constraints. This list is of course not conclusive, and there may be many more interesting related topics to study in both physics and mathematics.

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A Category theory, 2-groups and 3-groups

Definition 1 (Pre-crossed module and crossed module) *A pre-crossed module* $(H \stackrel{\partial}{\rightarrow} G\,, \triangleright)$ *of groups G and H, is given by a group map* $\partial : H \rightarrow G$*, together with a left action* $\triangleright$ *of G on H, by automorphisms, such that for each* $h_1\,, h_2 \in H$ *and* $g \in G$ *the following identity hold:*

$$g\partial h g^{-1} = \partial(g \triangleright h)\,.$$

In a pre-crossed module the **Peiffer commutator** *is defined as:*

$$\langle h_1\,, h_2\rangle_{\mathrm{p}} = h_1 h_2 h_1^{-1}\partial(h_1) \triangleright h_2^{-1}\,.$$

A pre-crossed module is said to be a **crossed module** *if all of its Peiffer commutators are trivial, which is to say that*

$$(\partial h) \triangleright h' = h h' h^{-1}\,,$$

i.e. the **Peiffer identity** *is satisfied.*

Definition 2 (2-crossed module) *A 2-crossed module* $(L \stackrel{\delta}{\rightarrow} H \stackrel{\partial}{\rightarrow} G, \triangleright, \{-,-\})$ *is given by three groups G, H and L, together with maps* ∂ *and* δ *such that:*

$$L \stackrel{\delta}{\rightarrow} H \stackrel{\partial}{\rightarrow} G\,,$$

where $\partial\delta = 1$*, an action* $\triangleright$ *of the group G on all three groups, and an G-equivariant map called the* **Peiffer lifting***:*

$$\{-\,,-\} : H \times H \rightarrow L\,.$$

The following identities are satisfied:

1. *The maps* ∂ *and* δ *are G-equivariant, i.e. for each* $g \in G$ *and* $h \in H$*:*

$$g \triangleright \partial(h) = \partial(g \triangleright h)\,, \qquad g \triangleright \delta(l) = \delta(g \triangleright l)\,,$$

   the action of the group G on the groups H and L is a smooth left action by automorphisms, i.e. for each $g, g_1, g_2 \in G$, $h_1, h_2 \in H$, $l_1, l_2 \in L$ *and* $e \in H, L$*:*

$$g_1 \triangleright (g_2 \triangleright e) = (g_1 g_2) \triangleright e\,, \quad g \triangleright (h_1 h_2) = (g \triangleright h_1)(g \triangleright h_2)\,, \quad g \triangleright (l_1 l_2) = (g \triangleright l_1)(g \triangleright l_2)\,,$$

   and the Peiffer lifting is G-equivariant, i.e. for each h_1, $h_2 \in H$ *and* $g \in G$*:*

$$g \triangleright \{h_1\,, h_2\} = \{g \triangleright h_1,\, g \triangleright h_2\}\,;$$

2. *the action of the group G on itself is via conjugation, i.e. for each* $g\,, g_0 \in G$*:*

$$g \triangleright g_0 = g\, g_0\, g^{-1}\,;$$

3. *In a* 2*-crossed module the structure* $(L \xrightarrow{\delta} H, \triangleright')$ *is a crossed module, with action of the group* H *on the group* L *is defined for each* $h \in H$ *and* $l \in L$ *as:*

$$h \triangleright' l = l\,\{\delta(l)^{-1},\, h\}\,,$$

but $(H \xrightarrow{\partial} G\,, \triangleright)$ *may not be one, and the Peiffer identity does not necessary hold. However, when* ∂ *is chosen to be trivial and group* H *Abelian, the Peiffer identity is satisfied, i.e. for each* $h,\, h' \in H$*:*

$$\delta(h) \triangleright h' = h\, h'\, h^{-1}\,;$$

4. $\delta(\{h_1, h_2\}) = \langle h_1\,, h_2\rangle_{\mathrm{p}}, \qquad \forall h_1, h_2 \in H,$
5. $[l_1, l_2] = \{\delta(l_1)\,, \delta(l_2)\}, \qquad \forall l_1\,, l_2 \in L.$ *Here, the notation* $[l, k] = lkl^{-1}k^{-1}$ *is used;*
6. $\{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\}\partial(h_1) \triangleright \{h_2, h_3\}, \qquad \forall h_1, h_2, h_3 \in H;$
7. $\{h_1, h_2 h_3\} = \{h_1, h_2\}\{h_1, h_3\}\{\langle h_1, h_3\rangle_{\mathrm{p}}^{-1}, \partial(h_1) \triangleright h_2\}, \qquad \forall h_1, h_2, h_3 \in H;$
8. $\{\delta(l), h\}\{h, \delta(l)\} = l(\partial(h) \triangleright l^{-1}), \qquad \forall h \in H\,, \quad \forall l \in L.$

Definition 3 (Differential pre-crossed module, differential crossed module) *A differential pre-crossed module* $(\mathfrak{h} \xrightarrow{\partial} \mathfrak{g}\,, \triangleright)$ *of algebras* $\mathfrak{g}$ *and* $\mathfrak{h}$ *is given by a Lie algebra map* $\partial : \mathfrak{h} \to \mathfrak{g}$ *together with an action* $\triangleright$ *of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *such that for each* $\underline{h} \in \mathfrak{h}$ *and* $\underline{g} \in \mathfrak{g}$*:*

$$\partial(\underline{g} \triangleright \underline{h}) = [\underline{g}, \partial(\underline{h})]\,.$$

The action $\triangleright$ *of* $\mathfrak{g}$ *on* $\mathfrak{h}$ *is on left by derivations, i.e. for each* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ *and each* $\underline{g} \in \mathfrak{g}$*:*

$$\underline{g} \triangleright [\underline{h}_1,\, \underline{h}_2] = [\underline{g} \triangleright \underline{h}_1,\, \underline{h}_2] + [\underline{h}_1,\, \underline{g} \triangleright \underline{h}_2]\,.$$

In a differential pre-crossed module, the Peiffer commutators are defined for each $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ *as:*

$$\langle \underline{h}_1,\, \underline{h}_2\rangle_{\mathrm{p}} = [\underline{h}_1, \underline{h}_2] - \partial(\underline{h}_1) \triangleright \underline{h}_2\,.$$

The map $(\underline{h}_1,\, \underline{h}_2) \in \mathfrak{h} \times \mathfrak{h} \to \langle \underline{h}_1,\, \underline{h}_2\rangle_{\mathrm{p}} \in \mathfrak{h}$ *is bilinear* $\mathfrak{g}$*-equivariant map called the* **Peiffer paring***, i.e. all* $\underline{h}_1\,, \underline{h}_2 \in \mathfrak{h}$ *and* $\underline{g} \in \mathfrak{g}$ *satisfy the following identity:*

$$\underline{g} \triangleright \langle \underline{h}_1\,, \underline{h}_2\rangle_{\mathrm{p}} = \langle \underline{g} \triangleright \underline{h}_1\,, \underline{h}_2\rangle + \langle \underline{h}_1\,, \underline{g} \triangleright \underline{h}_2\rangle_{\mathrm{p}}\,.$$

A differential pre-crossed module is said to be a **differential crossed module** *if all of its Peiffer commutators vanish, which is to say that for each* $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$*:*

$$\partial(\underline{h}_1) \triangleright \underline{h}_2 = [\underline{h}_1,\, \underline{h}_2]\,.$$

Definition 4 (Differential 2**-crossed module)** *A differential* 2*-crossed module is given by a complex of Lie algebras:*

$$\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}\,,$$

together with left action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{h}$, $\mathfrak{l}$, by derivations, and on itself via adjoint representation, and a $\mathfrak{g}$-equivariant bilinear map called the **Peiffer lifting***:*

$$\{-\,,\,-\}:\mathfrak{h}\times\mathfrak{h}\to\mathfrak{l}$$

Fixing the basis in algebra $T_A\in\mathfrak{l}$, $t_a\in\mathfrak{h}$ and $\tau_\alpha\in\mathfrak{g}$:

$$[T_A,T_B]=f_{AB}{}^C\,T_C\,,\qquad [t_a,t_b]=f_{ab}{}^c\,t_c\,,\qquad [\tau_\alpha,\tau_\beta]=f_{\alpha\beta}{}^\gamma\,\tau_\gamma\,,$$

one defines the maps ∂ and δ as:

$$\partial(t_a)=\partial_a{}^\alpha\,\tau_\alpha\,,\qquad \delta(T_A)=\delta_A{}^a\,t_a\,,$$

and action of $\mathfrak{g}$ on the generators of $\mathfrak{l}$, $\mathfrak{h}$ and $\mathfrak{g}$ is, respectively:

$$\tau_\alpha\triangleright T_A=\triangleright_{\alpha A}{}^B\,T_B\,,\qquad \tau_\alpha\triangleright t_a=\triangleright_{\alpha a}{}^b\,t_b\,,\qquad \tau_\alpha\triangleright\tau_\beta=\triangleright_{\alpha\beta}{}^\gamma\,\tau_\gamma\,.$$

Note that when η is $\mathfrak{g}$-valued differential form and ω is $\mathfrak{l}$, $\mathfrak{h}$ or $\mathfrak{g}$ valued differential form the previous action is defined as:

$$\eta\triangleright\omega=\eta^\alpha\wedge\omega^A\,\triangleright_{\alpha A}{}^B\,T_B\,,\qquad \eta\triangleright\omega=\eta^\alpha\wedge\omega^a\,\triangleright_{\alpha a}{}^b\,t_b\,,\qquad \eta\triangleright\omega=\eta^\alpha\wedge\omega^\beta f_{\alpha\beta}{}^\gamma\,\tau_\gamma\,.$$

The coefficients $X_{ab}{}^A$ are introduced as:

$$\{t_a,\,t_b\}=X_{ab}{}^A T_A\,.$$

The following identities are satisfied:

1. *In the differential crossed module $(L\xrightarrow{\delta}H\,,\triangleright')$ the action $\triangleright'$ of $\mathfrak{h}$ on $\mathfrak{l}$ is defined for each $\underline{h}\in\mathfrak{h}$ and $\underline{l}\in\mathfrak{l}$ as:*

   $$\underline{h}\triangleright'\underline{l}=-\{\delta(\underline{l}),\,\underline{h}\}\,,$$

   or written in the basis where $t_a\triangleright' T_A=\triangleright'{}_{aA}{}^B T_B$ the previous identity becomes:

   $$\triangleright'{}_{aA}{}^B=-\delta_A{}^b X_{ba}{}^B\,;$$

2. *The action of $\mathfrak{g}$ on itself is via adjoint representation:*

   $$\triangleright_{\alpha\beta}{}^\gamma=f_{\alpha\beta}{}^\gamma\,;$$

3. *The action of $\mathfrak{g}$ on $\mathfrak{h}$ and $\mathfrak{l}$ is equivariant, i.e. the following identities are satisfied:*

   $$\partial_a{}^\beta f_{\alpha\beta}{}^\gamma=\triangleright_{\alpha a}{}^b\partial_b{}^\gamma\,,\qquad \delta_A{}^a\,\triangleright_{\alpha a}{}^b=\triangleright_{\alpha A}{}^B\delta_B{}^b\,;$$

4. *The Peiffer lifting is $\mathfrak{g}$-equivariant, i.e. for each $\underline{h}_1,\underline{h}_2\in\mathfrak{h}$ and $\underline{g}\in\mathfrak{g}$:*

   $$\underline{g}\triangleright\{\underline{h}_1,\underline{h}_2\}=\{\underline{g}\triangleright\underline{h}_1,\underline{h}_2\}+\{\underline{h}_1,\,\underline{g}\triangleright\underline{h}_2\}\,,$$

   or written in the basis:

   $$X_{ab}{}^B\triangleright_{\alpha B}{}^A=\triangleright_{\alpha a}{}^c X_{cb}{}^A+\triangleright_{\alpha b}{}^c X_{ac}{}^A\,;$$

5. $\delta(\{\underline{h}_1, \underline{h}_2\}) = \langle \underline{h}_1, \underline{h}_2 \rangle_{\rm p}$, $\quad \forall \underline{h}_1, \underline{h}_2 \in \mathfrak{h}$, *i.e.*

$$X_{ab}{}^{A}\delta_A{}^c = f_{ab}{}^c - \partial_a{}^\alpha \rhd_{\alpha b}{}^c ;$$

6. $[\underline{l}_1, \underline{l}_2] = \{\delta(\underline{l}_1), \delta(\underline{l}_2)\}$, $\quad \forall \underline{l}_1, \underline{l}_2 \in \mathfrak{l}$, *i.e.*

$$f_{AB}{}^C = \delta_A{}^a \delta_B{}^b X_{ab}{}^C ;$$

7. $\{[\underline{h}_1, \underline{h}_2], \underline{h}_3\} = \partial(\underline{h}_1) \rhd \{\underline{h}_2, \underline{h}_3\} + \{\underline{h}_1, [\underline{h}_2, \underline{h}_3]\} - \partial(\underline{h}_2) \rhd \{\underline{h}_1, \underline{h}_3\} - \{\underline{h}_2, [\underline{h}_1, \underline{h}_3]\}$, $\forall \underline{h}_1, \underline{h}_2, \underline{h}_3 \in \mathfrak{h}$, *i.e.*

$$\{[\underline{h}_1,\underline{h}_2],\underline{h}_3\} = \{\partial(\underline{h}_1)\rhd\underline{h}_2,\underline{h}_3\} - \{\partial(\underline{h}_2)\rhd\underline{h}_1,\underline{h}_3\} - \{\underline{h}_1,\delta\{\underline{h}_2,\underline{h}_3\}\} + \{\underline{h}_2,\delta\{\underline{h}_1,q,\underline{h}_3\}\},$$

$$f_{ab}{}^d X_{dc}{}^B = \partial_a{}^\alpha X_{bc}{}^A \rhd_{\alpha A}{}^B + X_{ad}{}^B f_{bc}{}^d - \partial_b{}^\alpha \rhd_{\alpha A}{}^B X_{ac}{}^A - X_{bd}{}^B f_{ac}{}^d ;$$

8. $\{\underline{h}_1, [\underline{h}_2, \underline{h}_3]\} = \{\delta\{\underline{h}_1, \underline{h}_2\}, \underline{h}_3\} - \{\delta\{\underline{h}_1, \underline{h}_3\}, \underline{h}_2\}$, $\quad \forall \underline{h}_1, \underline{h}_2, \underline{h}_3 \in \mathfrak{h}$, *i.e.*

$$X_{ad}{}^A f_{bc}{}^d = X_{ab}{}^B \delta_B{}^d X_{dc}{}^A - X_{ac}{}^B \delta_B{}^d X_{db}{}^A ;$$

9. $\{\delta(\underline{l}), \underline{h}\} + \{\underline{h}, \delta(\underline{l})\} = -\partial(\underline{h}) \rhd \underline{l}$, $\quad \forall \underline{l} \in \mathfrak{l}, \quad \forall \underline{h} \in \mathfrak{h}$, *i.e.*

$$\delta_A{}^a X_{ab}{}^B + \delta_A{}^a X_{ba}{}^B = -\partial_b{}^\alpha \rhd_{\alpha A}{}^B .$$

Note that the property 6. implies that either trivial map δ or the trivial Peiffer lifting imply that L is an Abelian group. Conversely, if L is Abelian, property 6. implies that either the map δ or the Peiffer lifting is trivial, or both.

In the case of an Abelian group H and trivial map ∂, among the aforementioned properties the only non-trivial remaining are:

1. $\delta\{\underline{h}_1, \underline{h}_2\} = 0$, $\quad \forall \underline{h}_1, \underline{h}_2 \in \mathfrak{h}$;
2. $[\underline{l}_1, \underline{l}_2] = \{\delta(\underline{l}_1), \delta(\underline{l}_2)\}$, $\quad \forall \underline{l}_1, \underline{l}_2 \in \mathfrak{l}$;
3. $\{\delta(\underline{l}), \underline{h}\} = -\{\underline{h}, \delta(\underline{l})\}$, $\quad \forall \underline{h} \in \mathfrak{h}, \quad \forall \underline{l} \in \mathfrak{l}$.

A reader intrested in more details about 3-groups is referred to [25].

B The construction of gauge-invariant actions for 3*BF* theory

Symmetric bilinear invariant nondegenerate forms are defined as:

$$\langle T_A, T_B \rangle_{\mathfrak{l}} = g_{AB} , \qquad \langle t_a, t_b \rangle_{\mathfrak{h}} = g_{ab} , \qquad \langle \tau_\alpha, \tau_\beta \rangle_{\mathfrak{g}} = g_{\alpha\beta} .$$

They satisfy the following properties:

- $\langle \_ , \_ \rangle_{\mathfrak{g}}$ is G-invariant:

$$\langle g\tau_\alpha g^{-1}, g\tau_\beta g^{-1} \rangle_{\mathfrak{g}} = \langle \tau_\alpha, \tau_\beta \rangle_{\mathfrak{g}} , \quad \forall g \in G ;$$

- $\langle \_ , \_ \rangle_{\mathfrak{h}}$ is G-invariant:

$$\langle g \rhd t_a \,, g \rhd t_b \rangle_{\mathfrak{h}} = \langle t_a \,, t_b \rangle_{\mathfrak{h}} \,, \quad \forall g \in G \,,$$

  and, when $(H \xrightarrow{\partial} G \,, \rhd)$ is a crossed module, consequently H-invariant:

$$\langle h t_a h^{-1} \,, h t_b h^{-1} \rangle_{\mathfrak{h}} = \langle \partial(h) \rhd t_a \,, \partial(h) \rhd t_b \rangle_{\mathfrak{h}} = \langle t_a \,, t_b \rangle_{\mathfrak{h}} \,, \quad \forall h \in H \,;$$

- $\langle \_ , \_ \rangle_{\mathfrak{l}}$ is G-invariant:

$$\langle g \rhd T_A \,, g \rhd T_B \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall g \in G \,,$$

  and in the case when the Peiffer lifting or the map δ is trivial consequently H-invariant:

$$\langle h \rhd' T_A \,, h \rhd' T_B \rangle_{\mathfrak{l}} = \langle T_A - \{\delta(T_A), h\} \,, T_B - \{\delta(T_B), h\} \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall h \in H \,.$$

  From the H-invariance of $\langle \_ , \_ \rangle_{\mathfrak{l}}$ and properties of a crossed module $(L \xrightarrow{\delta} H \,, \rhd')$ follows L-invariance:

$$\langle l T_A l^{-1} \,, l T_B l^{-1} \rangle_{\mathfrak{l}} = \langle \delta(l) \rhd' T_A \,, \delta(l) \rhd' T_B \rangle_{\mathfrak{l}} = \langle T_A \,, T_B \rangle_{\mathfrak{l}} \,, \quad \forall l \in L \,.$$

From the invariance of the bilinear forms follows the existence of gauge-invariant topological $3BF$ action of the form:

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle \mathcal{C} \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle \mathcal{D} \wedge \mathcal{H} \rangle_{\mathfrak{l}} \,, \tag{B.1}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4 \,, \mathfrak{g})$, $C \in \mathcal{A}^1(\mathcal{M}_4 \,, \mathfrak{h})$ and $D \in \mathcal{A}^0(\mathcal{M}_4 \,, \mathfrak{l})$ are Lagrange multipliers, and $\mathcal{F} \in \mathcal{A}^2(\mathcal{M}_4 \,, \mathfrak{g})$, $\mathcal{G} \in \mathcal{A}^3(\mathcal{M}_4 \,, \mathfrak{h})$ and $\mathcal{H} \in \mathcal{A}^4(\mathcal{M}_4 \,, \mathfrak{l})$ are curvatures defined as in (3.1). Written in the basis:

$$\begin{aligned} \mathcal{F} &= \frac{1}{2} \mathcal{F}^{\alpha}{}_{\mu\nu} \tau_{\alpha} \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \,, \qquad \mathcal{G} = \frac{1}{3!} \mathcal{G}^{a}{}_{\mu\nu\rho} t_a \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\rho} \,, \\ \mathcal{H} &= \frac{1}{4!} \mathcal{H}^{A}{}_{\mu\nu\rho\sigma} T_A \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\rho} \wedge \mathrm{d}x^{\sigma} \,, \end{aligned}$$

the coefficients are:

$$\begin{aligned} \mathcal{F}^{\alpha}{}_{\mu\nu} &= \partial_{\mu} \alpha^{\alpha}{}_{\nu} - \partial_{\nu} \alpha^{\alpha}{}_{\mu} + f_{\beta\gamma}{}^{\alpha} \alpha^{\beta}{}_{\mu} \alpha^{\gamma}{}_{\nu} - \beta^{a}{}_{\mu\nu} \partial_{a}{}^{\alpha} \,, \\ \mathcal{G}^{a}{}_{\mu\nu\rho} &= \partial_{\mu} \beta^{a}{}_{\nu\rho} + \partial_{\nu} \beta^{a}{}_{\rho\mu} + \partial_{\rho} \beta^{a}{}_{\mu\nu} \\ &\quad + \alpha^{\alpha}{}_{\mu} \beta^{b}{}_{\nu\rho} \rhd_{\alpha b}{}^{a} + \alpha^{\alpha}{}_{\nu} \beta^{b}{}_{\rho\mu} \rhd_{\alpha b}{}^{a} + \alpha^{\alpha}{}_{\rho} \beta^{b}{}_{\mu\nu} \rhd_{\alpha b}{}^{a} - \gamma^{A}{}_{\mu\nu\rho} \delta_{A}{}^{a} \,, \\ \mathcal{H}^{A}{}_{\mu\nu\rho\sigma} &= \partial_{\mu} \gamma^{A}{}_{\nu\rho\sigma} - \partial_{\nu} \gamma^{A}{}_{\rho\sigma\mu} + \partial_{\rho} \gamma^{A}{}_{\sigma\mu\nu} - \partial_{\sigma} \gamma^{A}{}_{\mu\nu\rho} \\ &\quad + 2\beta^{a}{}_{\mu\nu} \beta^{b}{}_{\rho\sigma} X_{\{ab\}}{}^{A} - 2\beta^{a}{}_{\mu\rho} \beta^{b}{}_{\nu\sigma} X_{\{ab\}}{}^{A} + 2\beta^{a}{}_{\mu\sigma} \beta^{b}{}_{\nu\rho} X_{\{ab\}}{}^{A} \\ &\quad + \alpha^{\alpha}{}_{\mu} \gamma^{B}{}_{\nu\rho\sigma} \rhd_{\alpha B}{}^{A} - \alpha^{\alpha}{}_{\nu} \gamma^{B}{}_{\rho\sigma\mu} \rhd_{\alpha B}{}^{A} + \alpha^{\alpha}{}_{\rho} \gamma^{B}{}_{\sigma\mu\nu} \rhd_{\alpha B}{}^{A} - \alpha^{\alpha}{}_{\sigma} \gamma^{B}{}_{\mu\nu\rho} \rhd_{\alpha B}{}^{A} \,. \end{aligned}$$

Note that the wedge product $A \wedge B$ when A is a 0-form and B is a p-form is defined as $A \wedge B = \frac{1}{p!} A B_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge x^{\mu_p}$.

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Given G-invariant symmetric non-degenerate bilinear forms in $\mathfrak{g}$ and $\mathfrak{h}$, one can define a bilinear antisymmetric map $\mathcal{T} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}$ by the rule:

$$\langle \mathcal{T}(\underline{h}_1, \underline{h}_2) , \underline{g} \rangle_{\mathfrak{g}} = -\langle \underline{h}_1, \underline{g} \rhd \underline{h}_2 \rangle_{\mathfrak{h}}, \qquad \forall \underline{h}_1, \underline{h}_2 \in \mathfrak{h} , \quad \forall \underline{g} \in \mathfrak{g} .$$

See [17] for more properties and the construction of $2BF$ invariant topological action using this map. To define $3BF$ invariant topological action one has to first define a bilinear antisymmetric map $\mathcal{S} : \mathfrak{l} \times \mathfrak{l} \to \mathfrak{g}$ by the rule:

$$\langle \mathcal{S}(\underline{l}_1, \underline{l}_2), \underline{g} \rangle_{\mathfrak{g}} = -\langle \underline{l}_1, \underline{g} \rhd \underline{l}_2 \rangle_{\mathfrak{l}} , \qquad \forall \underline{l}_1, \forall \underline{l}_2 \in \mathfrak{l} , \quad \forall \underline{g} \in \mathfrak{g} .$$

Note that $\langle \_ , \_ \rangle_{\mathfrak{g}}$ is non-degenerate and

$$\langle \underline{l}_1, \underline{g} \rhd \underline{l}_2 \rangle_{\mathfrak{l}} = -\langle \underline{g} \rhd \underline{l}_1, \underline{l}_2 \rangle_{\mathfrak{l}} = -\langle \underline{l}_2, \underline{g} \rhd \underline{l}_1 \rangle_{\mathfrak{l}} , \qquad \forall \underline{g} \in \mathfrak{g}, \quad \forall \underline{l}_1, \underline{l}_2 \in \mathfrak{l} .$$

Morever, given $g \in G$ and $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$ one has:

$$\mathcal{S}(g \rhd \underline{l}_1, g \rhd \underline{l}_2) = g \, \mathcal{S}(\underline{l}_1, \underline{l}_2) \, g^{-1} ,$$

since for each $\underline{g} \in \mathfrak{g}$ and $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$:

$$\begin{aligned} \langle \underline{g}, g^{-1} \mathcal{S}(g \rhd \underline{l}_1 , g \rhd \underline{l}_2) g \rangle_{\mathfrak{g}} &= \langle g \underline{g} g^{-1}, \mathcal{S}(g \rhd \underline{l}_1, g \rhd \underline{l}_2) \rangle_{\mathfrak{g}} \\ &= -\langle (g \, \underline{g} \, g^{-1}) \rhd g \rhd \underline{l}_1, g \rhd \underline{l}_2 \rangle_{\mathfrak{l}} \\ &= -\langle \underline{g} \rhd \underline{l}_1 , \underline{l}_2 \rangle_{\mathfrak{l}} = \langle \underline{g} , \mathcal{S}(\underline{l}_1, \underline{l}_2) \rangle_{\mathfrak{g}} , \end{aligned}$$

where the following mixed relation has been used:

$$g \rhd (\underline{g} \rhd \underline{l}) = (g \, \underline{g} \, g^{-1}) \rhd g \rhd \underline{l} . \tag{B.2}$$

We thus have the following identity:

$$\mathcal{S}(\underline{g} \rhd \underline{l}_1, \underline{l}_2) + \mathcal{S}(\underline{l}_1, \underline{g} \rhd \underline{l}_2) = [\underline{g}, \mathcal{S}(\underline{l}_1, \underline{l}_2)] .$$

As far as the bilinear antisymmetric map $\mathcal{S} : l \times l \to g$, one can write it in the basis:

$$\mathcal{S}(T_A, T_B) = \mathcal{S}_{AB}{}^{\alpha} \tau_{\alpha} ,$$

so that the defining relation for $\mathcal{S}$ becomes the relation:

$$\mathcal{S}_{AB}{}^{\alpha} g_{\alpha\beta} = - \rhd_{\alpha[B}{}^{C} g_{A]C} .$$

Given two $\mathfrak{l}$-valued forms η and ω, one can define a $\mathfrak{g}$-valued form:

$$\omega \wedge^{\mathcal{S}} \eta = \omega^A \wedge \eta^B \mathcal{S}_{AB}{}^{\alpha} \tau_{\alpha} .$$

Now one can define the transformations of the Lagrange multipliers under L-gauge transformations (3.15).

Further, to define the transformations of the Lagrange multipliers under H-gauge transformations one needs to define the bilinear map $\mathcal{X}_1 : \mathfrak{l} \times \mathfrak{h} \to \mathfrak{h}$ by the rule:

$$\langle \mathcal{X}_1(\underline{l},\, \underline{h}_1),\, \underline{h}_2 \rangle_{\mathfrak{h}} = -\langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\} \rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}\,,$$

and bilinear map $\mathcal{X}_2 : \mathfrak{l} \times \mathfrak{h} \to \mathfrak{h}$ by the rule:

$$\langle \mathcal{X}_2(\underline{l},\, \underline{h}_2),\, \underline{h}_1 \rangle_{\mathfrak{h}} = -\langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\} \rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}\,.$$

As far as the bilinear maps $\mathcal{X}_1$ and $\mathcal{X}_2$ one can define the coefficients in the basis as:

$$\mathcal{X}_1(T_A, t_a) = \mathcal{X}_{1Aa}{}^b\, t_b\,, \qquad \mathcal{X}_2(T_A, t_a) = \mathcal{X}_{2Aa}{}^b\, t_b\,.$$

When written in the basis the defining relations for the maps $\mathcal{X}_1$ and $\mathcal{X}_2$ become:

$$\mathcal{X}_{1Ab}{}^c g_{ac} = -X_{ba}{}^B g_{AB}\,, \qquad \mathcal{X}_{2Ab}{}^c g_{ac} = -X_{ab}{}^B g_{AB}\,.$$

Given $\mathfrak{l}$-valued differential form ω and $\mathfrak{h}$-valued differential form η, one defines a $\mathfrak{h}$-valued form as:

$$\omega \wedge^{\mathcal{X}_1} \eta = \omega^A \wedge \eta^a \mathcal{X}_{1Aa}{}^b t_b\,, \qquad \omega \wedge^{\mathcal{X}_2} \eta = \omega^A \wedge \eta^a \mathcal{X}_{2Aa}{}^b t_b\,.$$

Given any $g \in G$, $\underline{l} \in \mathfrak{l}$ and $\underline{h} \in \mathfrak{h}$ one has:

$$\mathcal{X}_1(g \rhd \underline{l},\, g^{-1} \rhd \underline{h}) = g \rhd \mathcal{X}_1(\underline{l},\, \underline{h})\,, \qquad \mathcal{X}_2(g \rhd \underline{l},\, g \rhd \underline{h}) = g^{-1} \rhd \mathcal{X}_2(\underline{l},\, \underline{h})\,,$$

since for each $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ and $\underline{l} \in \mathfrak{l}$:

$$\begin{aligned}\langle \underline{h}_2,\, g^{-1} \rhd \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{h}_1) \rangle_{\mathfrak{h}} &= \langle g \rhd \underline{h}_2,\, \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{h}_1) \rangle_{\mathfrak{h}} = \langle g \rhd \underline{l},\, \{g \rhd \underline{h}_1,\, g \rhd \underline{h}_2\} \rangle_{\mathfrak{l}} \\ \langle g \rhd \underline{l},\, g \rhd \{\underline{h}_1,\, \underline{h}_2\} \rangle_{\mathfrak{l}} &= \langle \underline{l},\, \{\underline{h}_1,\, \underline{h}_2\} \rangle_{\mathfrak{l}} = \langle \underline{h}_2,\, \mathcal{X}_1(l,\, \underline{h}_1) \rangle_{\mathfrak{h}}\,,\end{aligned}$$

and similarly for $\mathcal{X}_2$. Finaly, one needs to define a trilinear map $\mathcal{D} : \mathfrak{h} \times \mathfrak{h} \times \mathfrak{l} \to \mathfrak{g}$ by the rule:

$$\langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l}),\, \underline{g} \rangle_{\mathfrak{g}} = -\langle \underline{l},\, \{\underline{g} \rhd \underline{h}_1,\, \underline{h}_2\} \rangle_{\mathfrak{l}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}, \quad \forall \underline{g} \in \mathfrak{g}\,,$$

One can define the coefficients of the trilinear map as:

$$\mathcal{D}(t_a,\, t_b,\, T_A) = \mathcal{D}_{abA}{}^\alpha \tau_\alpha\,,$$

and the defining relation for the map $\mathcal{D}$ expressed in terms of coefficients becomes:

$$\mathcal{D}_{abA}{}^\beta g_{\alpha\beta} = -\rhd_{\alpha a}{}^c X_{cb}{}^B g_{AB}\,.$$

Given two $\mathfrak{h}$-valued forms ω and η, and $\mathfrak{l}$-valued form ξ, the g-valued form is given by the formula:

$$\omega \wedge^{\mathcal{D}} \eta \wedge^{\mathcal{D}} \xi = \omega^a \wedge \eta^b \wedge \xi^A \mathcal{D}_{abA}{}^\beta \tau_\beta\,.$$

The following compatibility relation between the maps $\mathcal{X}_1$ and $\mathcal{D}$ hold:

$$\langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l}),\, \underline{g} \rangle_{\mathfrak{g}} = \langle \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, \underline{h}_2 \rangle_{\mathfrak{h}}\,, \qquad \forall \underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}\,, \quad \forall \underline{l} \in \mathfrak{l}, \quad \forall \underline{g} \in \mathfrak{g}\,, \tag{B.3}$$

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which one can prove valid from the defining relations in terms of the coefficients. One can demonstrate that for each $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$ and $g \in G$:

$$\mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l}) = g\, \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})\, g^{-1}\,,$$

since for each $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$, $\underline{g} \in \mathfrak{g}$ and $g \in G$:

$$\begin{aligned}
\langle g^{-1}\mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l})g,\, \underline{g}\rangle_{\mathfrak{g}} &= \langle \mathcal{D}(g \rhd \underline{h}_1,\, g \rhd \underline{h}_2,\, g \rhd \underline{l}),\, g\underline{g}g^{-1}\rangle_{\mathfrak{g}} \\
&= \langle \mathcal{X}_1(g \rhd \underline{l},\, g\underline{g}g^{-1} \rhd g \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{X}_1(g \rhd \underline{l},\, g \rhd \underline{g} \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle g \rhd \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, g \rhd \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{X}_1(\underline{l},\, \underline{g} \rhd \underline{h}_1),\, \underline{h}_2\rangle_{\mathfrak{h}} \\
&= \langle \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})\,,\underline{g}\rangle_{\mathfrak{g}}\,,
\end{aligned}$$

where the relation (B.2) and the compatibility relation (B.3) were used. We thus have for each $\underline{h}_1,\, \underline{h}_2 \in \mathfrak{h}$, $\underline{l} \in \mathfrak{l}$ and $\underline{g} \in \mathfrak{g}$ the following identity:

$$\mathcal{D}(\underline{g} \rhd \underline{h}_1,\, \underline{h}_2,\, \underline{l}) + \mathcal{D}(\underline{h}_1,\, \underline{g} \rhd \underline{h}_2,\, \underline{l}) + \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{g} \rhd \underline{l}) = [\underline{g},\, \mathcal{D}(\underline{h}_1,\, \underline{h}_2,\, \underline{l})]\,.$$

Now one can define the transformations of the Lagrange multipliers under H-gauge transformations as in (3.14).

C The equations of motion for the Weyl and Majorana fields

The action for the Weyl spinor field coupled to gravity is given by (3.72). The variation of this action with respect to the variables B_{ab}, λ^{ab}, γ_α, $\bar{\gamma}^{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, e^a, β^a and ω^{ab} one obtains the complete set of equations of motion, as follows:

$$\begin{aligned}
R^{ab} - \lambda^{ab} &= 0\,, \\
B_{ab} - \frac{1}{16\pi l_p^2}\varepsilon_{abcd}e^c \wedge e^d &= 0\,, \\
\nabla\psi_\alpha + \lambda_\alpha &= 0\,, \\
\nabla\bar{\psi}^{\dot{\alpha}} + \bar{\lambda}^{\dot{\alpha}} &= 0\,, \\
-\gamma_\alpha + \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}} &= 0\,, \\
-\bar{\gamma}^{\dot{\alpha}} + \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta &= 0\,, \\
\nabla\gamma_\alpha - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}} &= 0\,, \\
\nabla\bar{\gamma}^{\dot{\alpha}} - \frac{i}{6}\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \bar{\sigma}^{d\dot{\alpha}\beta}\lambda_\beta &= 0\,, \\
\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{i}{2}\varepsilon_{abcd}e^b \wedge e^c \wedge (\bar{\lambda}_{\dot{\alpha}}\bar{\sigma}^{d\dot{\alpha}\beta}\psi_\beta + \lambda^\alpha\sigma^d{}_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}) & \\
-8\pi i l_p^2\varepsilon_{abcd}e^b\beta^c\big(\psi^\alpha(\sigma^d)_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}\big) &= 0\,,
\end{aligned}$$

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$$\nabla e_a - 4\pi l_p^2 \varepsilon_{abcd} e^b \wedge e^c \wedge (\bar{\psi}_{\dot{\alpha}} \bar{\sigma}^{d\dot{\alpha}\beta} \psi_\beta) = 0\,,$$
$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} - \frac{1}{2} \gamma \sigma^{ab}{}_\alpha{}^\beta \psi_\beta - \frac{1}{2} \bar{\gamma}_{\dot{\alpha}} \bar{\sigma}^{ab\dot{\alpha}}{}_{\dot{\beta}} \bar{\psi}^{\dot{\beta}} = 0\,.$$

In the case of the Majorana field, one adds the mass term (3.75) to the action (3.72). Then, the variation of the action with respect to B_{ab}, ψ^{ab}, γ^α, $\bar{\gamma}_{\dot{\alpha}}$, λ_α, $\bar{\lambda}^{\dot{\alpha}}$, ψ_α, $\bar{\psi}_I^{\dot{\alpha}}$, e^a, β^a and ω_{ab} gives the equations of motion for the Majorana case, as follows:

$$\begin{aligned}
R^{ab} - \lambda^{ab} &= 0\,,\\
B_{ab} - \frac{1}{16\pi l_p^2} \varepsilon_{abcd} e^c \wedge e^d &= 0\,,\\
-\nabla \psi_\alpha + \lambda_\alpha &= 0\,,\\
-\nabla \bar{\psi}^{\dot{\alpha}} + \lambda^{\dot{\alpha}} &= 0\,,\\
\gamma^\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \bar{\psi}_{\dot{\beta}} (\bar{\sigma}^d)^{\dot{\beta}\alpha} &= 0\,,\\
\bar{\gamma}_{\dot{\alpha}} - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \psi^\beta (\sigma^d)_{\beta\dot{\alpha}} &= 0\,,\\
\nabla \gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} \lambda^{\dot{\beta}} \wedge e^a \wedge e^b \wedge e^c (\sigma^d)^\alpha{}_{\dot{\beta}} - \frac{1}{6}\, m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \psi^\alpha &\\
-4i\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c \bar{\psi}_{\dot{\beta}} (\bar{\sigma}^d)^{\dot{\beta}\alpha} &= 0\,,\\
\nabla \bar{\gamma}_{\dot{\alpha}} + \frac{i}{6} \varepsilon_{abcd} \lambda_\beta \wedge e^a \wedge e^b \wedge e^c (\bar{\sigma}^d)_{\dot{\alpha}}{}^\beta - \frac{1}{6} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \psi_{\dot{\alpha}} &\\
-4i\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c \psi^\beta (\sigma^d)_{\beta\dot{\alpha}} &= 0\,,\\
\nabla \beta^a + \frac{1}{8\pi l_p^2} \varepsilon_{abcd} \lambda^{bc} \wedge e^d + \frac{i}{2} \varepsilon_{abcd} \lambda_\alpha \wedge e^b \wedge e^c \bar{\psi}_{\dot{\beta}} (\bar{\sigma}^d)^{\dot{\beta}\alpha} + \frac{i}{2} \varepsilon_{abcd} \lambda^{\dot{\alpha}} \wedge e^b \wedge e^c \psi^\beta (\sigma^d)_{\beta\dot{\alpha}} &\\
-\frac{1}{3} m \varepsilon_{abcd} e^b \wedge e^c \wedge e^d (\psi^\alpha \psi_\alpha + \bar{\psi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}) - 8\pi i l_p^2 \varepsilon_{abcd} e^b \beta^c \big(\psi^\alpha (\sigma^d)_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}}\big) &= 0\,,\\
\nabla e_a - 4i\pi l_p^2 \varepsilon_{abcd} e^b \wedge e^c \big(\psi^\alpha (\sigma^d)_{\alpha\dot{\beta}} \bar{\psi}^{\dot{\beta}}\big) &= 0\,,\\
\nabla B_{ab} - e_{[a} \wedge \beta_{b]} - \frac{1}{2} \psi^\alpha (\sigma^{ab})_\alpha{}^\beta \gamma_\beta - \frac{1}{2} \bar{\psi}_{\dot{\alpha}} (\bar{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\gamma}^{\dot{\beta}} &= 0\,.
\end{aligned}$$

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Hamiltonian Analysis for the Scalar Electrodynamics as 3BF Theory

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Abstract: The higher category theory can be employed to generalize the BF action to the so-called 3BF action, by passing from the notion of a gauge group to the notion of a gauge 3-group. The theory of scalar electrodynamics coupled to Einstein–Cartan gravity can be formulated as a constrained 3BF theory for a specific choice of the gauge 3-group. The complete Hamiltonian analysis of the 3BF action for the choice of a Lie 3-group corresponding to scalar electrodynamics is performed. This analysis is the first step towards a canonical quantization of a 3BF theory, an important stepping stone for the quantization of the complete scalar electrodynamics coupled to Einstein–Cartan gravity formulated as a 3BF action with suitable simplicity constraints. It is shown that the resulting dynamic constraints eliminate all propagating degrees of freedom, i.e., the 3BF theory for this choice of a 3-group is a topological field theory, as expected.

Keywords: Hamiltonian analysis; higher gauge theory; BF theory; topological theory; scalar electrodynamics

1. Introduction

The vast majority of physics community agrees that the quantum theory of gravity is necessary, even if they disagree on the quantization approach. The theory of loop quantum gravity is one of the well-formulated possible candidates for the desired theory of quantum gravity [1–3]. There are two approaches within the theory—the canonical and the covariant quantization method. The covariant quantization method is focused on obtaining a generating functional, by considering a triangulated spacetime manifold and defining the functional as a state sum over all configurations of a field living on simplices of the triangulation [2].

One of the key tools in the covariant quantization approach is the so-called BF theory. Given a Lie group G and its corresponding Lie algebra $\mathfrak{g}$, one considers a $\mathfrak{g}$-valued connection 1-form A, and its corresponding field strength 2-form $F \equiv \mathrm{d}A + A \wedge A$. Multiplying F with a $\mathfrak{g}$-valued Lagrange multiplier 2-form B and integrating over a four-dimensional spacetime manifold $\mathcal{M}$, one obtains the action of the BF theory,

$$S_{BF}[A,B] = \int_{\mathcal{M}} \langle B \wedge F \rangle_{\mathfrak{g}} \,,$$

where $\langle \_ , \_ \rangle_{\mathfrak{g}}$ is a G-invariant non-degenerate symmetric bilinear form. The BF theory derives its name from the symbols B and F for the Lagrange multiplier and the field strength present in the action. As it is defined, the BF theory is topological, containing no local propagating degrees of freedom. Therefore, for the purpose of building physically relevant actions, attention usually focuses not on the pure BF theory, but rather on the theory with constraints. The constrained BF models are based on deformations of the BF theory [4], by adding constraints to the topological BF action that promote some of the gauge degrees of freedom into physical ones. The well known example is the Plebanski model for general relativity [5]. Constrained BF models represent a starting point in the spinfoam approach to the construction of quantum gravity models [2].

The main shortcoming of building a quantum gravity model using a BF theory is the fact that it is very hard, if not impossible, to write the action for matter fields (specifically scalar and fermion fields) in the form of a constrained BF theory. Thus, the spinfoam quantization method is limited to pure gravity, and the problem of consistently coupling matter fields to gravity in this framework becomes highly nontrivial. One of the proposed ways to circumvent this issue is to generalize the notion of a BF theory using the mathematical apparatus of higher category theory.

The higher category theory [6] can be employed to generalize the BF action to the so-called nBF action, by passing from the notion of a gauge group to the notion of a gauge n-group (for a comprehensive review of n-groups see for example [7], and also Appendix C). Specifically, the notion of a 3-group in the framework of higher category theory is introduced as a 3-category with only one object where all the morphisms, 2-morphisms and 3-morphisms are invertible. Based on this generalization, recently a constrained $3BF$ action has been introduced, which describes the full Standard Model coupled to Einstein–Cartan gravity [8].

As a first step to the study of the Hamiltonian structure of such theories, in this work, we discuss the simplest nontrivial toy example, namely the theory of scalar electrodynamics coupled to gravity. The standard way to define scalar electrodynamics coupled to gravity is by the action:

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[-\frac{1}{16\pi l_p^2} R - \frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + g^{\mu\nu} \nabla_\mu \phi^* \nabla_\nu \phi - m^2 \phi^* \phi \right] . \tag{1}$$

Here, $g_{\mu\nu}$ is the spacetime metric, $g \equiv \det(g_{\mu\nu})$ is its determinant, R is the corresponding curvature scalar, and l_p is the Planck length, its square being equal to the Newton's gravitational constant, $l_p^2 = G$, in the natural system of units $\hbar = c = 1$. The total covariant derivative ∇_μ of the complex scalar field ϕ is defined as $\nabla_\mu \phi = (\partial_\mu + iqA_\mu)\phi$, and thus coupled to the electromagnetic potential A_μ via the coupling constant q (the electric charge of the field ϕ). See Appendix A for more detailed notation. In the next section, we will reformulate this model as a classically equivalent constrained $3BF$ theory for a specific choice of the gauge 3-group. Moreover, for reasons of simplicity, in the Hamiltonian analysis, we will focus only on the topological sector, disregarding the simplicity constraints. The Hamiltonian structure of the theory is important for various reasons, primarily for the canonical quantization program.

The layout of the paper is as follows. In Section 2, we introduce the 3-group structure corresponding to the theory of scalar electrodynamics coupled to Einstein–Cartan gravity and the corresponding constrained $3BF$ action. Section 3 contains the Hamiltonian analysis for the topological, $3BF$ sector of the action, with the resulting first-class and second-class constraints present in the theory, and their mutual Poisson brackets. In Section 4, we analyze the Bianchi identities that the first-class constraints satisfy, which enforce restrictions in the sense of Hamiltonian analysis, and reduce the number of independent first-class constraints present in the theory. Section 5 focuses on the counting of the dynamical degrees of freedom present in the theory, based on the results from Sections 3 and 4. Encouraged by these results, in Section 6, we construct the generator of the gauge symmetries for the topological theory and we find the form variations of all variables and their canonical momenta. Finally, Section 7 is devoted to the discussion of the results and the possible future lines of research. The Appendices contain various technical details.

The notation and conventions are as follows. The local Lorentz indices are denoted by the Latin letters $a, b, c, \dots$, take values $0, 1, 2, 3$, and are raised and lowered using the Minkowski metric η_{ab} with signature $(-, +, +, +)$. Spacetime indices are denoted by the Greek letters $\mu, \nu, \dots$, and are raised and lowered by the spacetime metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ are the tetrad fields. The inverse tetrad is denoted as $e^\mu{}_a$, so that the standard orthogonality conditions hold: $e^a{}_\mu e^\mu{}_b = \delta^a_b$ and $e^a{}_\mu e^\nu{}_a = \delta^\nu_\mu$. When needed, spacetime indices will be split into time and space indices, denoted with a 0 and lowcase Latin indices $i, j, \ldots$, respectively. All other indices that appear in the paper are dependent on the context, and their usage is explicitly defined in the text where they appear. The antisymmetrization over two indices is introduced with the factor one half that is $A_{[a_1|a_2 \ldots a_{n-1}|a_n]} = \frac{1}{2}\left(A_{a_1 a_2 \ldots a_{n-1} a_n} - A_{a_n a_2 \ldots a_{n-1} a_1}\right)$, and the total antisymmetrization is introduced as $A_{[a_1 \ldots a_n]} = \frac{1}{n!}\sum_{\sigma \in S_n}(-1)^{\mathrm{sign}(\sigma)} A_{a_{\sigma(1)} \ldots a_{\sigma(n)}}$.

2. Scalar Electrodynamics as a Constrained 3BF Action

Let us begin by providing a short introduction into the construction and structure of a $3BF$ theory, after which we will impose appropriate simplicity constraints, in order to obtain the equations of motion for scalar electrodynamics coupled to gravity.

As was discussed in detail in [8], one formulates a topological $3BF$ action by specifying a particular gauge Lie 3-group. It has been proved that any strict 3-group is equivalent to a 2-crossed module [9,10]. A gauge theory for the manifold $\mathcal{M}_4$ and 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$ can be constructed for the following choice of the three Lie groups as:

$$G = SO(3,1) \times U(1)\,, \quad H = \mathbb{R}^4\,, \quad L = \mathbb{R}^2\,.$$

The maps ∂ and δ are chosen to be trivial. The action of the algebra $\mathfrak{g}$ on $\mathfrak{h}$ and $\mathfrak{l}$ is chosen as:

$$\begin{aligned} &M_{ab} \triangleright P_c = \triangleright_{ab,c}{}^{d}\, P_d = \delta_{[a|}{}^{d} \eta_{|b]c}\, P_d = \eta_{[b|c}\, P_{|a]}\,, \qquad && T \triangleright P_a = 0\,, \\ &M_{ab} \triangleright P_A = 0\,, && T \triangleright P_A = \triangleright_A{}^B\, P_B \end{aligned} \tag{2}$$

where M_{ab} denote the six generators of $\mathfrak{so}(3,1)$, T is the sole generator of $\mathfrak{u}(1)$, P_a are the four generators of $\mathbb{R}^4$ and P_A are the two generators of $\mathbb{R}^2$. In the previous expression, the action of the algebra $\mathfrak{u}(1)$ on the algebra $\mathbb{R}^2$ is defined via

$$\triangleright_A{}^B = iq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The action of the algebra $\mathfrak{g}$ on itself is by definition given via the adjoint representation and, for the choice $\mathfrak{g} = \mathfrak{so}(3,1) \times \mathfrak{u}(1)$, one obtains

$$\begin{gathered} M_{ab} \triangleright M_{cd} = \triangleright_{ab\,,cd}{}^{ef}\, M_{ef} = f_{ab\,,cd}{}^{ef}\, M_{ef} = \eta_{ad} M_{bc} + \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}\,, \\ M_{ab} \triangleright T = 0\,, \quad T \triangleright M_{ab} = 0\,, \quad T \triangleright T = 0\,, \end{gathered} \tag{3}$$

as the consequence of the direct product structure and the Abelian nature of the subgroup $U(1)$. The Peiffer lifting

$$\{\_\,,\_\} : H \times H \to L$$

is also trivial, i.e., all the coefficients $X_{ab}{}^A$ are equal to zero:

$$\{P_a\,, P_b\} \equiv X_{ab}{}^A T_A = 0\,. \tag{4}$$

Given Lie algebras $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$, one can introduce a 3-connection (α, β, γ) given by the algebra-valued differential forms $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{h})$ and $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,, \mathfrak{l})$. The corresponding fake 3-curvature $(\mathcal{F}\,, \mathcal{G}\,, \mathcal{H})$ is then defined as:

$$\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha \wedge^{\triangleright} \beta - \delta\gamma\,, \qquad \mathcal{H} = \mathrm{d}\gamma + \alpha \wedge^{\triangleright} \gamma + \{\beta \wedge \beta\}\,, \tag{5}$$

see [9,10] for details. For this specific choice of a 3-group, where $\alpha = \omega + A$, given by the algebra-valued differential forms $\omega \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{so}(3,1))$, $A \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{u}(1))$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathbb{R}^4)$ and $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,, \mathbb{R}^2)$, the corresponding 3-curvature $(\mathcal{F}\,, \mathcal{G}\,, \mathcal{H})$ is defined as

$$\begin{aligned} \mathcal{F} &= R^{ab}M_{ab} + FT &&= \left(\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}\right)M_{ab} + \mathrm{d}A\; T\,, \\ \mathcal{G} &= \mathcal{G}^a P_a &&= \left(\mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b\right)P_a\,, \\ \mathcal{H} &= \mathcal{H}^A P_A &&= \left(\mathrm{d}\gamma^A + \rhd_B{}^A A \wedge \gamma^B\right)P_A\,. \end{aligned} \tag{6}$$

Note that the connection ω^{ab} is not present in the last expression, as follows from the definition of the action $\rhd$ and the Peiffer lifting $\{\_\,,\_\}$, see Equations (2) and (4):

$$\begin{aligned} \mathcal{H} &= \mathrm{d}\gamma + \alpha \wedge^{\rhd} \gamma + \{\beta \wedge \beta\} \\ &= \mathrm{d}\gamma^A P_A + (\omega^{ab}M_{ab} + AT) \wedge^{\rhd} (\gamma^A P_A) \\ &= \mathrm{d}\gamma^A P_A + \omega^{ab} \wedge \gamma^A M_{ab} \rhd P_A + A \wedge \gamma^A T \rhd P_A \\ &= \mathrm{d}\gamma^A P_A + A \wedge \gamma^A \rhd_A{}^B P_B \\ &= (\mathrm{d}\gamma^A + \rhd_B{}^A A \wedge \gamma^B) P_A\,. \end{aligned} \tag{7}$$

The coefficients of the differential 2-forms F and R^{ab}, 3-form $\mathcal{G}$, and 4-form $\mathcal{H}$ are:

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\,, \\ R^{ab}{}_{\mu\nu} &= \partial_\mu \omega^{ab}{}_\nu - \partial_\nu \omega^{ab}{}_\mu + \omega^a{}_{c\mu}\omega^{cb}{}_\nu - \omega^a{}_{c\nu}\omega^{cb}{}_\mu\,, \\ \mathcal{G}^a{}_{\mu\nu\rho} &= \partial_\mu \beta^a{}_{\nu\rho} + \partial_\nu \beta^a{}_{\rho\mu} + \partial_\rho \beta^a{}_{\mu\nu} + \omega^a{}_{b\mu}\,\beta^b{}_{\nu\rho} + \omega^a{}_{b\nu}\,\beta^b{}_{\rho\mu} + \omega^a{}_{b\rho}\,\beta^b{}_{\mu\nu}\,, \\ \mathcal{H}^A{}_{\mu\nu\rho\sigma} &= \partial_\mu \gamma^A{}_{\nu\rho\sigma} - \partial_\nu \gamma^A{}_{\rho\sigma\mu} + \partial_\rho \gamma^A{}_{\sigma\mu\nu} - \partial_\sigma \gamma^A{}_{\mu\nu\rho} \\ &\quad + \rhd_B{}^A A_\mu \gamma^B{}_{\nu\rho\sigma} - \rhd_B{}^A A_\nu \gamma^B{}_{\rho\sigma\mu} + \rhd_B{}^A A_\rho \gamma^B{}_{\sigma\mu\nu} - \rhd_B{}^A A_\sigma \gamma^B{}_{\mu\nu\rho}\,. \end{aligned} \tag{8}$$

Now, one can define a gauge invariant $3BF$ action as:

$$S_{3BF} = \int_{\mathcal{M}_4} \Big(\langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}} + \langle D \wedge \mathcal{H} \rangle_{\mathfrak{l}} \Big)\,, \tag{9}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{so}(3,1))$, $C \in \mathcal{A}^1(\mathcal{M}_4\,, \mathbb{R}^4)$ and $D \in \mathcal{A}^0(\mathcal{M}_4\,, \mathbb{R}^2)$ are Lagrange multipliers. The forms $\langle\_\,,\_\rangle_{\mathfrak{g}}$, $\langle\_\,,\_\rangle_{\mathfrak{h}}$ and $\langle\_\,,\_\rangle_{\mathfrak{l}}$ are G-invariant bilinear symmetric nondegenerate forms on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, respectively, defined as

$$\langle M_{ab}\,, M_{cd}\rangle_{\mathfrak{g}} = g_{ab,cd}\,, \quad \langle T\,, T\rangle_{\mathfrak{g}} = 1\,, \quad \langle M_{ab}\,, T\rangle_{\mathfrak{g}} = 0\,, \quad \langle P_a\,, P_b\rangle_{\mathfrak{h}} = g_{ab}\,, \quad \langle P_A\,, P_B\rangle_{\mathfrak{l}} = g_{AB}\,,$$

where

$$g_{ab,cd} = \eta_{a[c|}\eta_{b|d]}\,, \qquad g_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad g_{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Identifying the Lagrange multiplier C^a as the tetrad field e^a, and the Lagrange multiplier D^A as the doublet of scalar fields ϕ^A,

$$\phi = \phi^A P_A = \phi P_1 + \phi^* P_2\,,$$

based on their transformation properties as discussed in [8,11], the Lagrangian of the action (9) obtains the form:

$$S_{3BF} = \int_{\mathcal{M}_4} \mathrm{d}^4x\, \epsilon^{\mu\nu\rho\sigma} \big(\frac{1}{4} B^{ab}{}_{\mu\nu} R^{cd}{}_{\rho\sigma}\, g_{ab,cd} + \frac{1}{4} B_{\mu\nu} F_{\rho\sigma} + \frac{1}{3!} e^a{}_\mu \mathcal{G}^b{}_{\nu\rho\sigma}\, g_{ab} + \frac{1}{4!} \phi^A \mathcal{H}^B{}_{\mu\nu\rho\sigma}\, g_{AB} \big) . \tag{10}$$

Varying the action with respect to all the variables, one obtains the equations of motion:

varied variable	equation of motion	varied variable	equation of motion
δB^{ab}	$R_{ab} = 0$	δB	$F = 0$
$\delta\omega^{ab}$	$\nabla B_{ab} - e_{[a|} \wedge \beta_{|b]} = 0$	δA	$\mathrm{d}B + \phi_A \rhd_B{}^A \gamma^B = 0$
δe^a	$\mathcal{G}_a = 0$	$\delta\beta^a$	$\nabla e_a = 0$
$\delta\phi^A$	$\nabla\gamma_A = 0$	$\delta\gamma^A$	$\nabla\phi_A = 0$

(11)

Since one is interested in the doublet of scalar fields ϕ^A of mass m and charge q minimally coupled to gravity and electromagnetic field, we impose additional simplicity constraint terms to the topological action (9), in order to obtain the appropriate equations of motion equivalent to the equations of motion for the action (1):

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B \wedge F + e_a \wedge \nabla\beta^a + \phi_A \nabla\gamma^A \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda^A \wedge \left(\gamma_A - \frac{1}{2} H_{abcA} e^a \wedge e^b \wedge e^c \right) + \Lambda^{abA} \wedge \left(H_{abcA} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \nabla\phi_A \wedge e_a \wedge e_b \right) \\ & + \lambda \wedge \left(B - \frac{12}{q} M_{ab} e^a \wedge e^b \right) + \zeta^{ab} \left(M_{ab} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b \right) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi_A \phi^A \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d . \end{aligned} \tag{12}$$

For the notation used here and the equations of motion obtained by varying the action (12), see Appendix A.

The dynamical degrees of freedom are the tetrad fields e^a, the scalar doublet ϕ^A, and the electromagnetic potential A, while the remaining variables are algebraically determined in terms of them, as shown in Appendix A. The equation of motion for the field ϕ^A reduces to the covariant Klein-Gordon equation for the scalar field,

$$\left(\nabla_\mu \nabla^\mu - m^2 \right) \phi_A = 0 . \tag{13}$$

The differential equation of motion for the field A is:

$$\nabla_\mu F^{\mu\nu} = j^\nu , \qquad j^\mu \equiv \frac{1}{2} \left(\nabla^\nu \phi^A \rhd^B{}_A \phi_B - \phi_A \rhd_B{}^A \nabla^\nu \phi^B \right) = iq \left(\nabla\phi^* \phi - \phi^* \nabla\phi \right) . \tag{14}$$

Finally, the equation of motion for e^a becomes:

$$\begin{aligned} R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi l_p^2\, T^{\mu\nu} , \\ T^{\mu\nu} \equiv \nabla^\mu \phi_A \nabla^\nu \phi^A - \frac{1}{2} g^{\mu\nu} \left(\nabla_\rho \phi_A \nabla^\rho \phi^A + m^2 \phi_A \phi^A \right) - \frac{1}{4q} \left(F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} + 4 F^{\mu\rho} F_\rho{}^\nu \right) . \end{aligned} \tag{15}$$

3. The Hamiltonian Analysis

The Hamiltonian analysis of the constrained $3BF$ action (12) for scalar electrodynamics is exceedingly complicated to study. A testament to this is the level of complexity of the constrained $2BF$ formulation of general relativity [12], which is merely one sector in the action (12). Therefore, in this paper, we will limit ourselves to the topological sector of the theory, namely the unconstrained $3BF$ theory (9), which consists of the terms in the first row of Equation (12), and is written in full detail in Equation (10). One should be aware that this restriction changes various properties of the theory. Namely, the simplicity constraints (everything but the first row in Equation (12)) substantially modify the dynamics of the theory—they increase the number of local propagating degrees of freedom of the theory, a property that was known since the original Plebanski model [5]. On the other hand, the unconstrained $3BF$ theory (9) is important even in its own right, and the Hamiltonian analysis may give important insight into the structure of both the unconstrained and the constrained theory.

In what follows, the complete Hamiltonian analysis for the action (9) is presented, see [13] for an overview and a comprehensive introduction of the Hamiltonian analysis. The Hamiltonian analysis for a $2BF$ action is performed in [12,14–16].

Under the standard assumption that the spacetime manifold is globally hyperbolic, $\mathcal{M}_4 = \mathbb{R} \times \Sigma_3$, the Lagrangian of the action (9) has the form:

$$L_{3BF} = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\, \epsilon^{\mu\nu\rho\sigma} \Big(\frac{1}{4} B^{ab}{}_{\mu\nu}\, R^{cd}{}_{\rho\sigma}\, g_{ab,cd} + \frac{1}{4} B_{\mu\nu} F_{\rho\sigma} + \frac{1}{3!}\, e^a{}_\mu\, \mathcal{G}^b{}_{\nu\rho\sigma}\, g_{ab} + \frac{1}{4!}\, \phi^A \mathcal{H}^B{}_{\mu\nu\rho\sigma}\, g_{AB} \Big) . \quad (16)$$

The canonical momentum $\pi(q)$ corresponding for the canonical coordinate q from the set of all variables in the theory, $q \in \{B^{ab}{}_{\mu\nu}, \omega^{ab}{}_\mu, B_{\mu\nu}, A_\mu, e^a{}_\mu, \beta^a{}_{\mu\nu}, \phi^A, \gamma^A{}_{\mu\nu\rho}\}$, is obtained as a derivative of the Lagrangian with respect to the appropriate velocity,

$$\pi(q) \equiv \frac{\delta L}{\delta \partial_0 q} ,$$

giving:

$$\begin{array}{rclrcl} \pi(B)_{ab}{}^{\mu\nu} &=& 0\,, & \pi(\omega)_{ab}{}^{\mu} &=& \epsilon^{0\mu\nu\rho} B_{ab\nu\rho}\,, \\ \pi(B)^{\mu\nu} &=& 0\,, & \pi(A)^{\mu} &=& \frac{1}{2}\epsilon^{0\mu\nu\rho} B_{\nu\rho}\,, \\ \pi(e)_a{}^{\mu} &=& 0\,, & \pi(\beta)_a{}^{\mu\nu} &=& -\epsilon^{0\mu\nu\rho} e_{a\rho}\,, \\ \pi(\phi)_A &=& 0\,, & \pi(\gamma)_A{}^{\mu\nu\rho} &=& \epsilon^{0\mu\nu\rho} \phi_A\,. \end{array} \quad (17)$$

Since these momenta cannot be inverted for the time derivatives of the variables, they all give rise to primary constraints:

$$\begin{array}{rclrcl} P(B)_{ab}{}^{\mu\nu} &\equiv& \pi(B)_{ab}{}^{\mu\nu} \approx 0\,, & P(\omega)_{ab}{}^{\mu} &\equiv& \pi(\omega)_{ab}{}^{\mu} - \epsilon^{0\mu\nu\rho} B_{ab\nu\rho} \approx 0\,, \\ P(B)^{\mu\nu} &\equiv& \pi(B)^{\mu\nu} \approx 0\,, & P(A)^{\mu} &\equiv& \pi(A)^{\mu} - \frac{1}{2}\epsilon^{0\mu\nu\rho} B_{\nu\rho} \approx 0\,, \\ P(e)_a{}^{\mu} &\equiv& \pi(e)_a{}^{\mu} \approx 0\,, & P(\beta)_a{}^{\mu\nu} &\equiv& \pi(\beta)_a{}^{\mu\nu} + \epsilon^{0\mu\nu\rho} e_{a\rho} \approx 0\,, \\ P(\phi)_A &\equiv& \pi(\phi)_A \approx 0\,, & P(\gamma)_A{}^{\mu\nu\rho} &\equiv& \pi(\gamma)_A{}^{\mu\nu\rho} - \epsilon^{0\mu\nu\rho} \phi_A \approx 0\,. \end{array} \quad (18)$$

Here, the symbol "≈" denotes the so-called "weak" equality, i.e., the equality that holds on a subspace of the phase space determined by the constraints, while the equality that holds for any point of the phase space is referred to as the "strong" equality and it is denoted by the symbol "=". The expressions "on-shell" and "off-shell" are used for weak and strong equalities, respectively, and henceforth will be used in this paper.

The fundamental Poisson brackets are defined as:

$$\begin{aligned}
\{\, B^{ab}{}_{\mu\nu}(x)\,,\, \pi(B)_{cd}{}^{\rho\sigma}(y)\,\} &= 4\delta^a{}_{[c}\delta^b{}_{d]}\delta^\rho{}_{[\mu}\delta^\sigma{}_{\nu]}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \omega^{ab}{}_{\mu}(x)\,,\, \pi(\omega)_{cd}{}^{\nu}(y)\,\} &= 2\delta^a{}_{[c}\delta^b{}_{d]}\delta^\nu{}_{\mu}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, B_{\mu\nu}(x)\,,\, \pi(B)^{\rho\sigma}(y)\,\} &= 2\delta^\rho{}_{[\mu}\delta^\sigma{}_{\nu]}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, A_{\mu}(x)\,,\, \pi(A)^{\nu}(y)\,\} &= \delta^\nu{}_{\mu}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, e^{a}{}_{\mu}(x)\,,\, \pi(e)_b{}^{\nu}(y)\,\} &= \delta^a{}_b\delta^\nu{}_{\mu}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \beta^{a}{}_{\mu\nu}(x)\,,\, \pi(\beta)_b{}^{\rho\sigma}(y)\,\} &= 2\delta^a{}_b\,\delta^\rho{}_{[\mu}\delta^\sigma{}_{\nu]}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \phi^{A}(x)\,,\, \pi(\phi)_B(y)\,\} &= \delta^A{}_B\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, \gamma^{A}{}_{\mu\nu\rho}(x)\,,\, \pi(\gamma)_B{}^{\alpha\beta\gamma}(y)\,\} &= 3!\delta^A{}_B\,\delta^\alpha{}_{[\mu}\delta^\beta{}_{\nu}\delta^\gamma{}_{\rho]}\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned} \tag{19}$$

Using these relations, one can calculate the algebra between the primary constraints,

$$\begin{aligned}
\{\, P(B)^{abjk}(x)\,,\, P(\omega)_{cd}{}^{i}(y)\,\} &= 4\epsilon^{0ijk}\,\delta^a{}_{[c}\delta^b{}_{d]}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, P(B)^{jk}(x)\,,\, P(A)^{i}(y)\,\} &= \epsilon^{0ijk}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, P(e)^{ak}\,,\, P(\beta)_b{}^{ij}(y)\,\} &= -\epsilon^{0ijk}\,\delta^a{}_b(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\, P(\phi)^{A}(x)\,,\, P(\gamma)_B{}^{ijk}(y)\,\} &= \epsilon^{0ijk}\,\delta^A{}_B\,\delta^{(3)}(\vec{x}-\vec{y})\,,
\end{aligned} \tag{20}$$

while all other Poisson brackets vanish. The canonical on-shell Hamiltonian is defined by

$$\begin{aligned}
H_c = \int_{\Sigma_3} \mathrm{d}^3\vec{x}\,\Big[&\frac{1}{4}\pi(B)_{ab}{}^{\mu\nu}\,\partial_0 B^{ab}{}_{\mu\nu} + \frac{1}{2}\pi(\omega)_{ab}{}^{\mu}\,\partial_0\omega^{ab}{}_{\mu} + \frac{1}{2}\pi(B)^{\mu\nu}\,\partial_0 B_{\mu\nu} + \pi(A)^{\mu}\,\partial_0 A_{\mu}\\
&+ \pi(e)_a{}^{\mu}\,\partial_0 e^a{}_{\mu} + \frac{1}{2}\pi(\beta)_a{}^{\mu\nu}\,\partial_0\beta^a{}_{\mu\nu} + \pi(\phi)_A\,\partial_0 D^A + \frac{1}{3!}\pi(\gamma)_A{}^{\mu\nu\rho}\,\partial_0\gamma^A{}_{\mu\nu\rho}\Big] - L\,.
\end{aligned} \tag{21}$$

Rewriting the Hamiltonian (21) such that all the velocities are multiplied by the first class constraints and therefore in an on-shell quantity they drop out, one obtains:

$$\begin{aligned}
H_c = -\int_{\Sigma_3} \mathrm{d}^3\vec{x}\,\epsilon^{0ijk}\Big[&\frac{1}{2}B_{ab0i}\,R^{ab}{}_{jk} + \frac{1}{2}B_{0i}F_{jk} + \frac{1}{6}e_{a0}\,\mathcal{G}^a{}_{ijk} + \beta^a{}_{0i}\nabla_j e_{ak}\\
&+ \frac{1}{2}\omega^{ab}{}_0\Big(\nabla_i B_{ab\,jk} - e_{[a|i}\,\beta_{|b]jk}\Big) + \frac{1}{2}A_0\Big(\partial_i B_{jk} + \frac{1}{3}\,\phi_A\ \rhd_B{}^A\,\gamma^B{}_{ijk}\Big) + \frac{1}{2}\gamma^A{}_{0ij}\nabla_k\phi_A\Big]\,.
\end{aligned} \tag{22}$$

This expression does not depend on any of the canonical momenta and it contains only the fields and their spatial derivatives. By adding a Lagrange multiplier λ for each of the primary constraints we can build the off-shell Hamiltonian, which is given by:

$$\begin{aligned}
H_T = H_c + \int_{\Sigma_3} \mathrm{d}^3\vec{x}\Big[&\frac{1}{4}\lambda(B)^{ab}{}_{\mu\nu}P(B)_{ab}{}^{\mu\nu} + \frac{1}{2}\lambda(\omega)^{ab}{}_{\mu}P(\omega)_{ab}{}^{\mu} + \frac{1}{2}\lambda(B)_{\mu\nu}P(B)^{\mu\nu} + \lambda(A)_{\mu}P(A)^{\mu}\\
&+\lambda(e)^a{}_{\mu}P(e)_a{}^{\mu} + \frac{1}{2}\lambda(\beta)^a{}_{\mu\nu}P(\beta)_a{}^{\mu\nu} + \lambda(\phi)^A P(\phi)_A + \frac{1}{3!}\lambda(\gamma)^A{}_{\mu\nu\rho}P(\gamma)_A{}^{\mu\nu\rho}\Big]\,.
\end{aligned} \tag{23}$$

Since the primary constraints must be preserved in time, one must impose the following requirement:

$$\dot{P} \equiv \{\, P\,,\, H_T\,\} \approx 0\,, \tag{24}$$

for each primary constraint P. By using the consistency condition (24) for the primary constraints $P(B)_{ab}{}^{0i}$, $P(\omega)_{ab}{}^{0}$, $P(B)^{0i}$, $P(A)^{0}$, $P(e)_a{}^{0}$, $P(\beta)_a{}^{0i}$, and $P(\gamma)_A{}^{0ij}$,

$$
\begin{aligned}
&\dot{P}(B)_{ab}{}^{0i} \approx 0\,, \quad \dot{P}(\omega)_{ab}{}^{0} \approx 0\,, \quad \dot{P}(B)^{0i} \approx 0\,, \quad \dot{P}(A)^{0} \approx 0\,, \\
&\dot{P}(e)_a{}^{0} \approx 0\,, \quad \dot{P}(\beta)_a{}^{0i} \approx 0\,, \quad \dot{P}(\gamma)_A{}^{0ij} \approx 0\,,
\end{aligned}
\tag{25}
$$

one obtains the secondary constraints $\mathcal{S}$,

$$
\begin{array}{llllll}
\mathcal{S}(R)_{ab}{}^{i} & \equiv & \epsilon^{0ijk} R_{ab\,jk} \approx 0\,, & \mathcal{S}(\nabla B)_{ab} & \equiv & \epsilon^{0ijk}\left(\nabla_i B_{ab\,jk} - e_{[a|i}\,\beta_{|b]\,jk}\right) \approx 0\,, \\
\mathcal{S}(F)^{i} & \equiv & \frac{1}{2}\epsilon^{0ijk} F_{jk} \approx 0\,, & \mathcal{S}(\nabla B) & \equiv & \frac{1}{2}\epsilon^{0ijk}\left(\partial_i B_{jk} + \frac{1}{3}\,\phi_A \rhd_B{}^{A}\,\gamma^{B}{}_{ijk}\right) \approx 0\,, \\
\mathcal{S}(\mathcal{G})_a & \equiv & \frac{1}{6}\epsilon^{0ijk}\mathcal{G}_{aijk} \approx 0\,, & \mathcal{S}(\nabla e)_a{}^{i} & \equiv & \epsilon^{0ijk}\nabla_j e_{ak} \approx 0\,, \\
\mathcal{S}(\nabla\phi)_A{}^{ij} & \equiv & \epsilon^{0ijk}\nabla_k\phi_A \approx 0\,, & & &
\end{array}
\tag{26}
$$

while in the case of $P(B)_{ab}{}^{jk}$, $P(\omega)_{ab}{}^{k}$, $P(B)^{jk}$, $P(A)^{k}$, $P(e)_a{}^{k}$, $P(\beta)_a{}^{jk}$, $P(\phi)_A$ and $P(\gamma)_A{}^{ijk}$ the consistency conditions

$$
\begin{aligned}
&\dot{P}(B)_{ab}{}^{jk} \approx 0\,, \quad \dot{P}(\omega)_{ab}{}^{k} \approx 0\,, \quad \dot{P}(B)^{jk} \approx 0\,, \quad \dot{P}(A)^{k} \approx 0\,, \\
&\dot{P}(e)_a{}^{k} \approx 0\,, \quad \dot{P}(\beta)_a{}^{jk} \approx 0\,, \quad \dot{P}(\phi)_A \approx 0\,, \quad \dot{P}(\gamma)_A{}^{ijk} \approx 0\,,
\end{aligned}
\tag{27}
$$

determine the following Lagrange multipliers:

$$
\begin{array}{llllll}
\lambda(\omega)_{ab}{}^{i} & \approx & \nabla^i\,\omega_{ab\,0}\,, & \lambda(B)^{ij} & \approx & 2\partial^{[i|}\,B^{0|j]} + \gamma_A{}^{0ij} \rhd_B{}^{A}\,\phi^B\,, \\
\lambda(A)^{i} & \approx & \partial^i A_0\,, & \lambda(\beta)_a{}^{ij} & \approx & 2\nabla^{[i|}\,\beta_a{}^{0|j]} - \omega_{ab}{}^{0}\,\beta^{b\,ij}\,, \\
\lambda(\phi)^{A} & \approx & A^0 \rhd^{A}{}_{B}\,\phi^B\,, & \lambda(e)_a{}^{i} & \approx & \nabla^i\,e_a{}^{0} - \omega_a{}^{b\,0}\,e_b{}^{i}\,, \\
\lambda(B)_{ab}{}^{ij} & \approx & \multicolumn{4}{l}{2\nabla^{[i|}B_{ab}{}^{0|j]} + e_{[a|\,0}\beta_{|b]}{}^{ij} - 2e_{[a|}{}^{[i|}\beta_{|b]}{}^{0|j]} + 2\omega_{[a|}{}^{c}B_{|b]}{}^{c\,ij}\,,} \\
\lambda(\gamma)_A{}^{ijk} & \approx & \multicolumn{4}{l}{-A^0 \rhd_A{}^{B}\,\gamma_B{}^{ijk} + \nabla^i\gamma_A{}^{0jk} - \nabla^j\gamma_A{}^{0ik} + \nabla^k\gamma_A{}^{0ij}\,.}
\end{array}
\tag{28}
$$

Note that the consistency conditions leave the Lagrange multipliers

$$
\lambda(B)^{ab}{}_{0i}\,, \qquad \lambda(\omega)^{ab}{}_{0}\,, \qquad \lambda(B)_{0i}\,, \qquad \lambda(A)_0\,, \qquad \lambda(e)^a{}_0\,, \qquad \lambda(\beta)^a{}_{0i}\,, \qquad \lambda(\gamma)^A{}_{0ij} \tag{29}
$$

undetermined. The consistency conditions of the secondary constraints do not produce new constraints, since one can show that

$$
\begin{array}{lllll}
\dot{\mathcal{S}}(R)^{abi} & = & \{\mathcal{S}(R)^{abi}\,,\,H_T\} & = & \omega^{[a|}{}_{c0}\,\mathcal{S}(R)^{c|b]i}\,, \\
\dot{\mathcal{S}}(\nabla B) & = & \{\mathcal{S}(\nabla B),\,H_T\} & = & -\rhd_B{}^{A}\,\gamma^B{}_{0ij}\,\mathcal{S}(\nabla\phi)_A{}^{ij}\,, \\
\dot{\mathcal{S}}(\mathcal{G})^{a} & = & \{\mathcal{S}(\mathcal{G})^{a}\,,\,H_T\} & = & \beta_{b0k}\,\mathcal{S}(R)^{abk} - \omega^{ab}{}_{0}\,\mathcal{S}(\mathcal{G})_b\,, \\
\dot{\mathcal{S}}(\nabla e)_a{}^{i} & = & \{\mathcal{S}(\nabla e)_a{}^{i}\,,\,H_T\} & = & e^b{}_0\,\mathcal{S}(R)_{ab}{}^{i} - \omega_a{}^{b}{}_0\,\mathcal{S}(\nabla e)_b{}^{i}\,, \\
\dot{\mathcal{S}}(\nabla\phi)_A{}^{ij} & = & \{\mathcal{S}(\nabla\phi)_A{}^{ij}\,,\,H_T\} & = & A_0 \rhd_A{}^{B}\mathcal{S}(\nabla\phi)_B{}^{ij}\,, \\
\dot{\mathcal{S}}(F)^{i} & = & \{\mathcal{S}(F)^{i}\,,\,H_T\} & = & 0\,, \\
\dot{\mathcal{S}}(\nabla B)_{ab} & = & \{\mathcal{S}(\nabla B)_{ab}\,,\,H_T\} & = & \mathcal{S}(R)_{[a|c}{}^{k}\,B^c{}_{|b]0k} + \omega_{[a|}{}^{c}{}_{0}\mathcal{S}(\nabla B)_{|b]c} \\
& & & & -\beta_{[a|0k}\,\mathcal{S}(\nabla e)_{|b]}{}^{k} + e_{[a|0}\,\mathcal{S}(\mathcal{G})_{|b]}\,.
\end{array}
\tag{30}
$$

Then, the total Hamiltonian can be written as

$$\begin{aligned} H_T \;=\; \int_{\Sigma_3} \mathrm{d}^3\vec{x} \; \Big[& \frac{1}{2}\lambda(B)_{ab}{}^{0i}\,\Phi(B)^{ab}{}_i + \frac{1}{2}\lambda(\omega)_{ab}{}^{0}\,\Phi(\omega)^{ab} + \lambda(B)^{0i}\,\Phi(B)_i + \lambda(A)^0\,\Phi(A) \\ & + \lambda(e)_a{}^0\,\Phi(e)^a + \lambda(\beta)_a{}^{0i}\,\Phi(\beta)^a{}_i + \frac{1}{2}\lambda(\gamma)_A{}^{0ij}\Phi(\gamma)^A{}_{ij} \\ & - \frac{1}{2}B_{ab0i}\,\Phi(R)^{abi} - \frac{1}{2}\omega_{ab0}\,\Phi(\nabla B)^{ab} - B_{0i}\,\Phi(F)^i - A_0\,\Phi(\nabla B) \\ & - e_{a0}\,\Phi(\mathcal{G})^a - \beta_{a0i}\,\Phi(\nabla e)^{ai} - \frac{1}{2}\gamma_{A0ij}\,\Phi(\nabla\phi)^{Aij} \Big] , \end{aligned} \tag{31}$$

where

$$\begin{aligned} \Phi(B)^{ab}{}_i &= P(B)^{ab}{}_{0i}\,, & \Phi(\gamma)^A{}_{ij} &= P(\gamma)^A{}_{0ij}\,, \\ \Phi(\omega)^{ab} &= P(\omega)^{ab}{}_0\,, & \Phi(F)^i &= \mathcal{S}(F)^i - \partial_j P(B)^{ij}\,, \\ \Phi(B)_i &= P(B)_{0i}\,, & \Phi(R)^{abi} &= \mathcal{S}(R)^{abi} - \nabla_j P(B)^{ab\,ij}\,, \\ \Phi(A) &= P(A)_0\,, & \Phi(\mathcal{G})^a &= \mathcal{S}(\mathcal{G})^a + \nabla_i P(e)^{a\,i} - \tfrac{1}{4}\,\beta_{b\,ij}\,P(B)^{ab\,ij}\,, \\ \Phi(e)^a &= P(e)^a{}_0\,, & \Phi(\nabla e)^{a\,i} &= \mathcal{S}(\nabla e)^{a\,i} - \nabla_j P(\beta)^{a\,ij} + \tfrac{1}{2}\,e_{b\,j}\,P(B)^{ab\,ij}\,, \\ \Phi(\beta)^a{}_i &= P(\beta)^a{}_{0i}\,, & \Phi(\nabla\phi)^{A\,ij} &= \mathcal{S}(\nabla\phi)^{A\,ij} + \nabla_k P(\gamma)^{A\,ijk} - \rhd_B{}^A\,\phi^B\,P(B)^{ij}\,, \\ \Phi(\nabla B) &= \mathcal{S}(\nabla B) + \partial_i P(A)^i + \frac{1}{3!}\,\gamma^A{}_{ijk}\,\rhd_A{}^B\,P(\gamma)_B{}^{ijk} - \phi_A\,\rhd_B{}^A\,P(\phi)^B\,, \\ \Phi(\nabla B)^{ab} &= \mathcal{S}(\nabla B)^{ab} + \nabla_i P(\omega)^{abi} + B^{[a|}{}_{c\,ij}\,P(B)^{c|b]\,ij} - 2e^{[a|}{}_i\,P(e)^{|b]\,i} - \beta^{[a|}{}_{ij}\,P(\beta)^{|b]\,ij}\,, \end{aligned} \tag{32}$$

are the first-class constraints, while

$$\begin{aligned} &\chi(B)_{ab}{}^{jk} = P(B)_{ab}{}^{jk}\,, \quad \chi(B)^{jk} = P(B)^{jk}\,, \quad \chi(e)_a{}^i = P(e)_a{}^i\,, \quad \chi(\phi)_A = P(\phi)_A\,, \\ &\chi(\omega)_{ab}{}^i = P(\omega)_{ab}{}^i\,, \quad \chi(A)^i = P(A)^i\,, \quad \chi(\beta)_a{}^{ij} = P(\beta)_a{}^{ij}\,, \quad \chi(\gamma)_A{}^{ijk} = P(\gamma)_A{}^{ijk}\,, \end{aligned} \tag{33}$$

are the second-class constraints.

The PB algebra of the first-class constraints is given by:

$$\begin{aligned} \{\,\Phi(\mathcal{G})^a(x)\,,\,\Phi(\nabla e)_b{}^i(y)\,\} &= -\,\Phi(R)^a{}_b{}^i(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\,\Phi(\mathcal{G})^a(x)\,,\,\Phi(\nabla B)_{bc}(y)\,\} &= 2\delta^a{}_{[b|}\,\Phi(\mathcal{G})_{|c]}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\,\Phi(\nabla e)^a{}_i(x)\,,\,\Phi(\nabla B)_{bc}(y)\,\} &= 2\delta^a{}_{[b|}\Phi(\nabla e)_{|c]i}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\,\Phi(R)^{abi}(x)\,,\,\Phi(\nabla B)_{cd}(y)\,\} &= -4\delta^{[a|}{}_{[c}\,\Phi(R)^{|b]}{}_{d]}{}^i(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\,\Phi(\nabla B)^{ab}(x)\,,\,\Phi(\nabla B)_{cd}(y)\,\} &= -4\delta^{[a|}{}_{[c|}\,\Phi(\nabla B)^{|b]}{}_{|d]}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,, \\ \{\,\Phi(\nabla B)(x)\,,\,\Phi(\nabla\phi)_A{}^{ij}(y)\,\} &= -2\rhd^B{}_A\;\Phi(\nabla\phi)_B{}^{ij}(x)\delta^{(3)}(\vec{x}-\vec{y})\,. \end{aligned} \tag{34}$$

The PB algebra between the first and the second-class constraints is given by:

$$
\begin{aligned}
\{\,\Phi(R)^{abi}(x)\,,\,\chi(\omega)_{cd}{}^{j}(y)\,\} &= 4\,\delta^{[a|}{}_{[c|}\,\chi(B)^{|b]}{}_{|d]}{}^{ij}(x)\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\mathcal{G})^{a}(x)\,,\,\chi(\omega)_{cd}{}^{i}(y)\,\} &= 2\,\delta^{a}{}_{[c|}\,\chi(e)_{|d]}{}^{i}(x)\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\mathcal{G})^{a}(x)\,,\,\chi(\beta)_{c}{}^{jk}(y)\,\} &= -\frac{1}{2}\,\chi(B)^{a}{}_{c}{}^{jk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla e)^{ai}(x)\,,\,\chi(\omega)_{cd}{}^{j}(y)\,\} &= -2\,\delta^{a}{}_{[c|}\,\chi(\beta)_{|d]}{}^{ij}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla e)^{ai}(x)\,,\,\chi(e)_{b}{}^{j}(y)\,\} &= \frac{1}{2}\,\chi(B)^{a}{}_{b}{}^{ij}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\chi(\omega)_{cd}{}^{i}(y)\,\} &= 4\,\delta^{[a|}{}_{[c|}\,\chi(\omega)_{|d]}{}^{|b]i}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)(x)\,,\,\chi(A)^{i}(y)\,\} &= 2\,\chi(A)^{i}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\chi(\beta)_{c}{}^{jk}(y)\,\} &= -2\delta^{[a|}{}_{c}\,\chi(\beta)^{|b]jk}\,\delta^{(3)}(x-y)\,,\\
\{\,\Phi(\nabla B)(x)\,,\,\chi(\gamma)_{A}{}^{ijk}(y)\,\} &= \rhd_{A}{}^{B}\;\chi(\gamma)_{B}{}^{ijk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\chi(B)_{cd}{}^{jk}(y)\,\} &= 4\,\delta^{[a|}{}_{[c}\,\chi(B)_{d]}{}^{|b]jk}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)^{ab}(x)\,,\,\chi(e)_{a}{}^{i}(y)\,\} &= -2\delta^{[a|}{}_{c}\,\chi(e)^{|b]i}\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla B)(x)\,,\,\chi(\phi)_{A}(y)\,\} &= -\rhd^{B}{}_{A}\,\chi(\phi)_{B}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla\phi)^{Aij}(x)\,,\,\chi(A)^{k}(y)\,\} &= -\rhd_{B}{}^{A}\,\chi(\gamma)^{Bijk}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,,\\
\{\,\Phi(\nabla\phi)^{Aij}(x)\,,\,\chi(\phi)_{B}(y)\,\} &= -\rhd_{B}{}^{A}\,\chi(B)^{ij}(x)\,\delta^{(3)}(\vec{x}-\vec{y})\,.
\end{aligned}
\tag{35}
$$

The PB algebra between the second-class constraints has already been calculated, and is given in Equations (20).

4. The Bianchi Identities

In order to calculate the number of degrees of freedom in the theory, one needs to make use of the *Bianchi identities* (BI), as well as additional, *generalized Bianchi identities* (GBI) that are an analogue of the ordinary BI for the additional fields present in the theory.

One uses BI associated with the 1-form fields ω^{ab} and e^{a}, as well as the GBI for the 1-form A. Namely, the corresponding 2-form curvatures

$$
R^{ab} = \mathrm{d}\omega^{ab} + \omega^{a}{}_{c}\wedge\omega^{cb}\,, \qquad T^{a} = \mathrm{d}e^{a} + \omega^{a}{}_{b}\wedge e^{b}\,, \qquad F = \mathrm{d}A\,, \tag{36}
$$

satisfy the following identities:

$$
\epsilon^{\lambda\mu\nu\rho}\,\nabla_{\mu}R^{ab}{}_{\nu\rho} = 0\,, \tag{37}
$$

$$
\epsilon^{\lambda\mu\nu\rho}\left(\nabla_{\mu}T^{a}{}_{\nu\rho} - R^{ab}{}_{\mu\nu}\,e_{b\rho}\right) = 0\,, \tag{38}
$$

$$
\epsilon^{\lambda\mu\nu\rho}\,\nabla_{\mu}F_{\nu\rho} = 0\,. \tag{39}
$$

Choosing the free index to be time coordinate $\lambda = 0$, these indentities, as the time-independent parts of the Bianchi identities, become the off-shell restrictions in the sense of the Hamiltonian analysis. On the other hand, choosing the free index to be a spatial coordinate, one obtains time-dependent pieces of the Bianchi identities, which do not enforce any restrictions, but can instead be derived as a consequence of the Hamiltonian equations of motion.

There are also GBI associated with the 2-form fields B^{ab}, B and β^a. The corresponding 3-form curvatures are given by

$$S^{ab} = \mathrm{d}B^{ab} + 2\omega^{[a|}{}_c \wedge B^{c\,|b]}\,, \qquad P = \mathrm{d}B\,, \qquad G^a = \mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b\,. \tag{40}$$

Differentiating these expressions, one obtains the following GBI:

$$\epsilon^{\lambda\mu\nu\rho}\left(\frac{1}{3}\nabla_\lambda\, S^{ab}{}_{\mu\nu\rho} - R^{[a|\,c}{}_{\lambda\mu}\, B_c{}^{|b]}{}_{\nu\rho}\right) = 0\,, \tag{41}$$

$$\epsilon^{\lambda\mu\nu\rho}\partial_\lambda\, P_{\mu\nu\rho} = 0\,, \tag{42}$$

$$\epsilon^{\lambda\mu\nu\rho}\left(\frac{2}{3}\nabla_\lambda\, G^a{}_{\mu\nu\rho} - R^{ab}{}_{\lambda\mu}\,\beta_{b\,\nu\rho}\right) = 0\,. \tag{43}$$

However, in four-dimensional spacetime, these identities will be single-component equations, with no free spacetime indices, and therefore necessarily feature time derivatives of the fields. Thus, they do not impose any off-shell restictions on the canonical variables.

Finally, there is also GBI associated with the 0-form ϕ. The corresponding 1-form curvature is:

$$Q^A = \mathrm{d}\phi^A + \rhd_B{}^A\, A \wedge \phi^B\,, \tag{44}$$

so that the GBI associated with this curvature is:

$$\epsilon^{\lambda\mu\nu\rho}\left(\nabla_\nu Q^A{}_\rho - \frac{1}{2}\rhd_B{}^A\, F_{\nu\rho}\phi^B\right) = 0. \tag{45}$$

This GBI consists of 12 component equations, corresponding to six possible choices of the free antisymmetrized spacetime indices $\lambda\mu$, and the 2 possible choices of the free group index A. However, not all of these 12 identities are independent. This can be seen by taking the derivative of the Equation (45) and obtaining eight identities of the form

$$\rhd_B{}^A\,\epsilon^{\lambda\mu\nu\rho}\,\partial_\mu\, F_{\nu\rho}\,\phi^B = 0\,, \tag{46}$$

which are automatically satisfied because of the GBI (39). One concludes there are only four independent identities (45). Now, fixing the value $\lambda = 0$, one obtains the time-independent components of both Equations (45) and (46),

$$\epsilon^{0ijk}\left(\nabla_j Q^A{}_k - \frac{1}{2}\rhd_B{}^A\, F_{jk}\phi^B\right) = 0\,, \tag{47}$$

and

$$\rhd_B{}^A\,\epsilon^{0ijk}\,\partial_i\, F_{jk}\,\phi^B = 0\,. \tag{48}$$

Of these, there are six components in Equation (47), but, because of the two components of Equation (48), there are overall only four independent GBI relevant for the Hamiltonian analysis.

5. Number of Degrees of Freedom

Let us now show that the structure of the constraints implies that there are no local degrees of freedom (DoF) in a $3BF$ theory. In the general case, if there are N initial fields in the theory and there are F independent first-class constraints per space point and S independent second-class constraints per space point, then the number of local DoF, i.e., the number of independent field components, is given by

$$n = N - F - \frac{S}{2}\,. \tag{49}$$

Equation (49) is a consequence of the fact that S second-class constraints are equivalent to vanishing of $S/2$ canonical coordinates and $S/2$ of their momenta. The F first-class constraints are equivalent to vanishing of F canonical coordinates, and since the first-class constraints generate the gauge symmetries, we can impose F gauge-fixing conditions for the corresponding F canonical momenta. Consequently, there are $2N-2F-S$ independent canonical coordinates and momenta and therefore $2n = 2N-2F-S$, giving rise to Equation (49).

In our case, N can be determined from the Table 1, giving rise to a total of $N = 120$ canonical coordinates. Similarly, the number of independent components for the second class constraints is determined by the Table 2, so that $S = 70$.

Table 1. The number of components for all fields present in the theory.

$\omega^{ab}{}_\mu$	A_μ	$\beta^a{}_{\mu\nu}$	$\gamma^A{}_{\mu\nu\rho}$	$B^{ab}{}_{\mu\nu}$	$B_{\mu\nu}$	$e^a{}_\mu$	ϕ^A
24	4	24	8	36	6	16	2

Table 2. The number of components for the second class constraints present in the theory.

$\chi(B)_{ab}{}^{jk}$	$\chi(B)^{jk}$	$\chi(e)_a{}^i$	$\chi(\phi)_A$	$\chi(\omega)_{ab}{}^i$	$\chi(A)^i$	$\chi(\beta)_a{}^{ij}$	$\chi(\gamma)_A{}^{ijk}$
18	3	12	2	18	3	12	2

The first-class constraints are not all independent because of BI and GBI. To see that, take the derivative of $\Phi(R)^{abi}$ to obtain

$$\nabla_i \Phi(R)^{abi} = \varepsilon^{0ijk} \nabla_i R^{ab}{}_{jk} + \frac{1}{2} R^{c[a|}{}_{ij} P(B)_c{}^{|b]ij} \,. \tag{50}$$

The first term on the right-hand side is zero off-shell because $\epsilon^{ijk}\,\nabla_i R^{ab}{}_{jk} = 0$, which is a $\lambda = 0$ component of the BI (37). The second term on the right-hand side is also zero off-shell, since it is a product of two constraints,

$$R^{c[a|}{}_{ij}\, P(B)_c{}^{|b]ij} \equiv \frac{1}{2} \epsilon_{0ijk} \mathcal{S}(R)^{c[a|k}\, P(B)_c{}^{|b]ij} = 0 \,. \tag{51}$$

Therefore, we have the off-shell identity

$$\nabla_i \Phi(R)^{abi} = 0 \,, \tag{52}$$

which means that six components of $\Phi(R)^{abi}$ are not independent of the others. In an analogous fashion, taking the derivative of $\Phi(F)^i$, one obtains

$$\partial_i \Phi(F)^i = \varepsilon^{0ijk}\, \partial_i F_{jk} + \frac{1}{2}\, F_{ij}\, P(B)^{ij} \,. \tag{53}$$

The first term on the right-hand side is zero off-shell because $\epsilon^{ijk}\,\partial_i F_{jk} = 0$, which is a $\lambda = 0$ component of the GBI (37). The second term on the right-hand side is also zero off-shell, since it is a product of two constraints,

$$F_{ij}\, P(B)^{ij} \equiv \frac{1}{2} \epsilon_{0ijk}\, \mathcal{S}(F)^k\, P(B)^{ij} = 0 \,. \tag{54}$$

Therefore, we have the off-shell identity

$$\partial_i \Phi(F)^i = 0 \,, \tag{55}$$

which means that one component of $\Phi(F)^i$ is not independent of the others. Similarly, one can demonstrate that

$$\nabla_i\Phi(\nabla e)_a{}^i - \frac{1}{2}\,\Phi(R)_{ab}{}^i\, e^b{}_i + \frac{1}{4}\epsilon^{0ijk}\mathcal{S}(R)_{abk}\, P(\beta)^b{}_{ij} = \frac{1}{2}\epsilon^{0ijk}\left(\nabla_i T_{ajk} - R_{ab\,ij}\, e^b{}_k\right)\,. \tag{56}$$

The right-hand side of the Equation (56) is the $\lambda = 0$ component of the BI (38), so that Equation (56) gives the relation:

$$\nabla_i\Phi(\nabla e)_a{}^i - \frac{1}{2}\,\Phi(R)_{ab}{}^i\, e^b{}_i = 0\,, \tag{57}$$

where we have omitted the term that is the product of two constraints. This relation means that four components of the constraints $\Phi(\nabla e)_a{}^i$ and $\Phi(R)_{ab}{}^i$ can be expressed in terms of the rest. Finally, one can also demonstrate that

$$\begin{aligned}\nabla_i\Phi(\nabla\phi)_A{}^{ij} &- \frac{1}{2}\epsilon_{0ikl} \rhd_A \mathcal{S}(F)^l\,\chi(\gamma)_B{}^{ijk} + \rhd^B{}_A\,\phi_B\,\Phi(F)^j \\ &+ \frac{1}{2}\epsilon_{0ilm} \rhd^B{}_A\, P(B)^{ij}\,\mathcal{S}(\nabla\phi)_B{}^{lm} = \epsilon^{0ijk}\left(\nabla_i Q_{Ak} + \frac{1}{2}\rhd^B{}_A\, F_{ik}\,\phi_B\right)\,,\end{aligned} \tag{58}$$

which gives

$$\nabla_i\Phi(\nabla\phi)_A{}^{ij} + \frac{1}{2}\rhd^B{}_A\,\phi_B\,\Phi(F)^j = 0\,, \tag{59}$$

for $\lambda = 0$ component of the GBI (45), where we have again used that the product of two contraints is zero off-shell. This relation suggests that six components of two first-class constraints, $\Phi(\nabla\phi)_A{}^{ij}$ and $\Phi(F)^j$, are not independent of the others. However, in the previous section, we have discussed that only four of these six identities are mutually independent, which means that we have only four independent identities (59). A rigorous proof of this statement entails the evaluation of the corresponding Wronskian, and is left for future work.

Taking into account all of the above indentites (52), (55), (57), and (59), we can finally evaluate the total number of independent first-class constraints. From the Table 3, one can see that the total number of components of the first-class constraints is given by $F^* = 100$. However, the number of independent components of the first-class constraints is $F = 85$, obtained by subtracting the six relations (52), one relation (55), four relations (57) and four relations (59).

Table 3. The number of components for the first class constraints present in the theory. The identities (52), (55), (57), and (59) reduce the number of components which are independent. This reduction is explicitly denoted in the table.

$\Phi(B)_{ab}{}^i$	$\Phi(B)^i$	$\Phi(e)_a$	$\Phi(\omega)_{ab}$	$\Phi(A)$	$\Phi(\beta)_a{}^i$	$\Phi(\gamma)_A{}^{ij}$	$\Phi(R)_{ab}{}^i$	$\Phi(F)^i$	$\Phi(\mathcal{G})_a$	$\Phi(\nabla e)_a{}^i$	$\Phi(\nabla B)_{ab}$	$\Phi(\nabla B)$	$\Phi(\nabla\phi)_A{}^{ij}$
18	3	4	6	1	12	6	18 − 6	3 − 1	4	12 − 4	6	1	6 − 4

Therefore, substituting all the obtained results into Equation (49), one gets

$$n = 120 - 85 - \frac{70}{2} = 0, \tag{60}$$

which means that there are no propagating DoF in a $3BF$ theory described by the action (10).

6. Generator of the Gauge Symmetry

Based on the results of the Hamiltonian analysis of the action (10), it can also be interesting to calculate the generator of the complete gauge symmetry of the action. The gauge generator of the theory is obtained by using the Castellani's procedure (see Chapter V in [13] for details of the procedure), and one gets the following result (see Appendix B for details of the calculation):

$$
\begin{aligned}
G \;=\; \int_{\Sigma_3} \mathrm{d}^3\vec{x} \Bigg(& \frac{1}{2}(\nabla_0 \epsilon^{ab}{}_i)\Phi(B)_{ab}{}^i - \frac{1}{2}\epsilon^{ab}{}_i \Phi(R)_{ab}{}^i + \frac{1}{2}(\nabla_0 \epsilon^{ab})\Phi(\omega)_{ab} - \frac{1}{2}\epsilon^{ab}\Phi(\nabla B)_{ab} \\
& + (\partial_0 \epsilon_i)\Phi(B)^i - \epsilon_i \Phi(F)^i + (\partial_0 \epsilon)\Phi(A) - \epsilon \Phi(\nabla B) \\
& + (\nabla_0 \epsilon^a)\Phi(e)_a - \epsilon^a \Phi(\mathcal{G})_a + (\nabla_0 \epsilon^a{}_i)\Phi(\beta)_a{}^i - \epsilon^a{}_i \Phi(\nabla e)_a{}^i \\
& + \frac{1}{2}(\nabla_0 \epsilon^A{}_{ij})\Phi(\gamma)_A{}^{ij} - \frac{1}{2}\epsilon^A{}_{ij}\Phi(\nabla\phi)_A{}^{ij} \\
& + \epsilon^{ab}\left(\beta_{[a|0i}P(\beta)_{|b]}{}^i + e_{[a|0}P(e)_{|b]} + B_{[a|c0i}P(B)^c{}_{|b]}{}^i \right) - \epsilon\, \gamma_{A0ij} \rhd_B{}^A P(\gamma)^{Bij} \\
& + \epsilon^a \beta_{b0i} P(B)^{abi} + \epsilon^a{}_i\, e_{b0} P(B)_a{}^{bi} \Bigg) .
\end{aligned} \tag{61}
$$

Here, $\epsilon^{ab}{}_i$, ϵ^{ab}, ϵ_i, ϵ, ϵ^a, $\epsilon^a{}_i$ and $\epsilon^A{}_{ij}$ are the independent parameters of the gauge transformations.

Furthermore, one can employ the gauge generator to calculate the form-variations for all canonical coordinates and their corresponding momenta, by computing the Poisson bracket of the chosen variable $A(t,\vec{x})$ and the generator (61):

$$
\delta_0 A(t,\vec{x}) = \{ A(t,\vec{x})\,, G \} . \tag{62}
$$

The results are given as follows:

$$
\begin{aligned}
\delta_0 \omega^{ab}{}_0 &= \nabla_0 \epsilon^{ab} , & \delta_0 \pi(\omega)_{ab}{}^0 &= -2\epsilon_{[a|}{}^c{}_i \pi(B)_{c|b]}{}^{0i} - 2\epsilon_{[a|}{}^c \pi(\omega)_{c|b]}{}^0 , \\
& & & \quad +2\epsilon_{[a|}\pi(e)_{|b]}{}^0 + 2\epsilon_{[a|i}\pi(\beta)_{|b]}{}^{0i} , \\
\delta_0 \omega^{ab}{}_i &= \nabla_i \epsilon^{ab} , & \delta_0 \pi(\omega)_{ab}{}^i &= -2\epsilon_{[a|}{}^c{}_j\, \pi(B)_{c|b]}{}^{ij} - 2\epsilon_{[a|}{}^c{}_i\, \pi(\omega)_{|b]c}{}^i \\
& & & \quad +2\epsilon_{[a|}\, \pi(e)_{|b]i} + 2\epsilon_{[a|j}\pi(\beta)_{|b]}{}^{ij} \\
& & & \quad +2\epsilon^{0ijk}\, \nabla_{[j|}\epsilon_{ab|k]} + \epsilon^{0ijk}\epsilon_{[a|}\beta_{|b]jk} , \\
\delta_0 B^{ab}{}_{0i} &= \nabla_0 \epsilon^{ab}{}_i + \epsilon^{[a|}{}_i e^{|b]}{}_0 & \delta_0 \pi(B)_{ab}{}^{0i} &= 2\epsilon_{[a|c}\, \pi(B)_{|b]}{}^{ci} , \\
&\quad +2\epsilon^{[a|c}B^{|b]}{}_{c0i} + \epsilon^{[a|}\beta^{|b]}{}_{0i} , & & \\
\delta_0 B^{ab}{}_{ij} &= 2\nabla_{[i|}\epsilon^{ab}{}_{|j]} + 2\epsilon^{[a|c}B^{|b]}{}_{cij} & \delta_0 \pi(B)_{ab}{}^{ij} &= 2\epsilon_{[a|c}\, \pi(B)_{|b]}{}^{cij} , \\
&\quad +2\epsilon^{[a|}{}_{[i}e^{|b]}{}_{j]} + \epsilon^{[a|}\beta^{|b]}{}_{ij} , & & \\
\delta_0 A_0 &= \partial_0 \epsilon , & \delta_0 \pi(A)^0 &= -\tfrac{1}{2}\epsilon^A{}_{ij} \rhd^B{}_A\, \pi(\gamma)_B{}^{0ij} , \\
\delta_0 A_i &= \partial_i \epsilon , & \delta_0 \pi(A)^i &= \epsilon^{0ijk}\partial_j \epsilon_k - \tfrac{1}{2}\epsilon^A{}_{jk} \rhd^B{}_A\, \pi(\gamma)_B{}^{ijk} , \\
\delta_0 B_{0i} &= \partial_0 \epsilon_i , & \delta_0 \pi(B)^{0i} &= 0 , \\
\delta_0 B_{ij} &= 2\,\partial_{[i|}\epsilon_{|j]} + \epsilon^A{}_{ij} \rhd^B{}_A\, \phi_B , & \delta_0 \pi(B)^{ij} &= -\epsilon^{0ijk}\partial_k \epsilon , \\
\delta_0 \beta^a{}_{0i} &= \nabla_0 \epsilon^a{}_i - \epsilon^{ab}\beta_{b0i} , & \delta_0 \pi(\beta)_a{}^{0i} &= -\epsilon_{ab}\pi(\beta)^{b0i} + \tfrac{1}{2}\epsilon^b \pi(B)_{ab}{}^{0i} , \\
\delta_0 \beta^a{}_{ij} &= 2\nabla_{[i|}\epsilon^a{}_{|j]} - \epsilon^{ab}\,\beta_{bij} , & \delta_0 \pi(\beta)_a{}^{ij} &= -\epsilon_{ab}\, \pi(\beta)^{bij} + \tfrac{1}{2}\epsilon^b\, \pi(B)_{ab}{}^{ij} \\
& & & \quad -\epsilon^{0ijk}\, \nabla_k \epsilon^a , \\
\delta_0 e^a{}_0 &= \nabla_0 \epsilon^a - \epsilon^{ab}\, e_{b0} , & \delta_0 \pi(e)_a{}^0 &= -\epsilon_{ab}\, \pi(e)^{b0} + \tfrac{1}{2}\epsilon^b{}_i\, \pi(B)_{ab}{}^{0i} , \\
\delta_0 e^a{}_i &= \nabla_i \epsilon^a - \epsilon^{ab}\, e_{bi} , & \delta_0 \pi(e)_a{}^i &= -\epsilon_{ab}\, \pi(e)^{bi} + \epsilon^{0ijk}\Big(\nabla_{[j|}\epsilon_{a|k]} + \epsilon_{ab}\beta^{bjk} \Big) \\
& & & \quad +\tfrac{1}{2}\epsilon^b{}_j\, \pi(B)_{ab}{}^{ij} ,
\end{aligned}
$$

$$
\begin{aligned}
\delta_0\gamma^A{}_{0ij} &= \nabla_0\epsilon^A{}_{ij} - \epsilon\,\gamma^B{}_{0ij} \rhd^A{}_B\,, & \delta_0\pi(\gamma)_A{}^{0ij} &= \epsilon \rhd^B{}_A\,\pi(\gamma)_B{}^{0ij}\,,\\
\delta_0\gamma^A{}_{ijk} &= -\,\epsilon\,\gamma^B{}_{ijk} \rhd_B{}^A + \nabla_i\epsilon^A{}_{jk} & \delta_0\pi(\gamma)_A{}^{ijk} &= \epsilon \rhd_A{}^B\left(\pi(\gamma)_B{}^{ijk} + \epsilon^{0ijk}\phi_B\right),\\
&\quad -\nabla_j\epsilon^A{}_{ik} + \nabla_k\epsilon^A{}_{ij}\,, & & \qquad (63)\\
\delta_0\phi^A &= \epsilon\,\phi^B \rhd^A{}_B\,, & \delta_0\pi(\phi)_A &= -\epsilon \rhd^B{}_A\,\pi(\phi)_B + \frac{1}{3!}\,\epsilon\,\epsilon^{0ijk} \rhd^B{}_A\,\gamma_{Bijk}\\
& & &\quad -\frac{1}{2}\rhd_{A\,B}\,\epsilon^B{}_{ij}\,\pi(B)^{ij} - \frac{1}{2}\epsilon^{0ijk}\,\nabla_i\epsilon^A{}_{jk}\,,
\end{aligned}
$$

These transformations are an extension of the form-variations in the case of the Poincaré 2-group obtained in [17].

7. Conclusions

Let us summarize the results of the paper. In Section 2, we have demonstated in detail how to use the idea of a categorical ladder to introduce the 3-group structure corresponding to the theory of scalar electrodynamics coupled to Einstein–Cartan gravity. We have introduced the topological $3BF$ action corresponding to this choice of a 3-group, as well as the constrained $3BF$ action which gives rise to the standard equations of motion for the scalar electrodynamics. In order to perform the canonical quantization of this theory, the complete Hamiltonian analysis of the full theory with constraints has to be performed, but the important step towards this goal is the Hamiltonian analysis of the topological $3BF$ action. This has been done in Section 3. Here, the first-class and second-class constraints of the theory, as well as their Poisson brackets, have been obtained. In Section 4, we have discussed the Bianchi identities and also the generalized Bianchi identities, since they enforce restrictions in the sense of Hamiltonian analysis, and reduce the number of independent first-class constraints present in the theory. With this background material in hand, in Section 5, the counting of the dynamical degrees of freedom present in the theory has been performed and it was established that the considered $3BF$ action is a topological theory, i.e., the diffeomorphism invariant theory without any propagating degrees of freedom. In Section 6, we have constructed the generator of the gauge symmetries for the theory, and we found the form-variations for all the variables and their canonical momenta.

The results obtained in this paper represent the straightforward generalization of Hamiltonian analysis done in [15] for the Poincaré 2-group, and a first example of the Hamiltonian analysis of a $3BF$ action. The fact that the theory was found to be topological is nontrivial, since it relies on the existence of the generalized Bianchi identities, which have been identified for the first time. In addition to that, it was demonstrated that the algebra of constraint closes, which is an important consistency check for the theory. There is another very interesting aspect of the constraint algebra. Namely, one can recognize, looking at the structure of Equations (34) that the subalgebra generated by the first-class constraint $\Phi(\nabla\phi)_A{}^{ij}$ is in fact an *ideal* of the constraint algebra because the Poisson bracket between this constraint and all other constraints is again proportional to that constraint. It is curious that precisely the constraint $\Phi(\nabla\phi)_A{}^{ij}$ is the only one related to the Lie group L from the 3-group, according to its index structure, and also that the structure constant of the ideal is determined by the action $\rhd$ of the group G on L. Let us also note that the action $\rhd$ appears as well in the structure constants of the algebra between the first-class and second-class constraints.

The results of this work open several avenues for future research. From the point of view of mathematics, the relationship between the algebraic structures mentioned above should be understood in more detail. More generally, one should understand the correspondence between the gauge group generated by the generator (61) and the 3-group structure used to define the theory. This is not viable in the special case of the 3-group discussed in this work, but instead needs to be done in the case of a generic 3-group, where homomorphisms δ and ∂ and the Peiffer lifting $\{\_,\_\}$ are nontrivial. From the point of view of physics, the obtained results represent the fundamental building blocks for the construction of the quantum theory of scalar electrodynamics coupled to gravity, as well as a convenient model to discuss before proceeding to the Hamiltonian analysis and canonical quantization of the full Standard Model coupled to gravity, formulated as a $3BF$ action with suitable constraints [8]. Both the Hamiltonian analysis of constrained $3BF$ models and the corresponding canonical quantization programme need to be further developed in order to achieve these goals. Our work is a first step in this direction.

Finally, let us note in the end that the above list of topics for future research is by no means complete, and there are potentially many other interesting topics that can be studied in this context.

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Abbreviations

The following abbreviations are used in this manuscript:

LQG	Loop Quantum Gravity
BI	Bianchi Identities
GBI	Generalized Bianchi Identities
DoF	Degrees of Freedom
PB	Poisson Bracket

Appendix A. The Equations of Motion for the Scalar Electrodynamics

The action of scalar electrodynamics coupled to Einstein–Cartan gravity is given in the form (12):

$$
\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B \wedge F + e_a \wedge \nabla \beta^a + \phi_A \nabla \gamma^A \\
& - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\
& + \lambda^A \wedge \left(\gamma_A - \frac{1}{2} H_{abcA} e^a \wedge e^b \wedge e^c \right) + \Lambda^{abA} \wedge \left(H_{abcA} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \nabla \phi_A \wedge e_a \wedge e_b \right) \\
& + \lambda \wedge \left(B - \frac{12}{q} M_{ab} e^a \wedge e^b \right) + \zeta^{ab} \left(M_{ab} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b \right) \\
& - \frac{1}{2 \cdot 4!} m^2 \phi_A \phi^A \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,.
\end{aligned}
\tag{A1}
$$

Varying the total action (12) with respect to the variables B_{ab}, B, ω_{ab}, β_a, λ_{ab}, Λ^{abA}, γ^A, λ^A, H_{abcA}, ζ^{ab}, M_{ab}, λ, A, ϕ^A and e^a, one obtains the equations of motion:

$$R^{ab} - \lambda^{ab} = 0 \,, \tag{A2}$$

$$F + \lambda = 0 \,, \tag{A3}$$

$$\nabla B^{ab} - e^{[a|} \wedge \beta^{|b]} = 0 \,, \tag{A4}$$

$$\nabla e^a = 0 \,, \tag{A5}$$

$$B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d = 0 \,, \tag{A6}$$

$$H_{abcA}\varepsilon^{cdef}e_d \wedge e_e \wedge e_f - \nabla\phi_A \wedge e_a \wedge e_b = 0\,, \tag{A7}$$

$$\nabla\phi_A - \lambda_A = 0\,, \tag{A8}$$

$$\gamma_A - \frac{1}{2}H_{abcA}e^a \wedge e^b \wedge e^c = 0\,, \tag{A9}$$

$$-\frac{1}{2}\lambda^A \wedge e^a \wedge e^b \wedge e^c + \varepsilon^{cdef}\Lambda^{abA} \wedge e_d \wedge e_e \wedge e_f = 0\,, \tag{A10}$$

$$M_{ab}\varepsilon_{cdef}e^c \wedge e^d \wedge e^e \wedge e^f - F \wedge e_a \wedge e_b = 0\,, \tag{A11}$$

$$-\frac{12}{q}\lambda \wedge e^a \wedge e^b + \zeta^{ab}\varepsilon_{cdef}e^c \wedge e^d \wedge e^e \wedge e^f = 0\,, \tag{A12}$$

$$B - \frac{12}{g}M_{ab}e^a \wedge e^b = 0\,, \tag{A13}$$

$$-\,\mathrm{d}B + \mathrm{d}(\zeta^{ab}e_a \wedge e_b) - \phi_A \rhd_B{}^A\gamma^B - \Lambda^{abA} \rhd^B{}_A\,\phi_B \wedge e_a \wedge e_b = 0\,, \tag{A14}$$

$$\nabla\gamma_A - \nabla(\Lambda^{ab}{}_A \wedge e_a \wedge e_b) - \frac{1}{4!}m^2\,\phi_A\varepsilon_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d = 0\,, \tag{A15}$$

$$\begin{aligned}
&\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d + \frac{3}{2}H_{abcA}\lambda^A \wedge e^b \wedge e^c + 3H^{defA}\varepsilon_{abcd}\Lambda_{efA} \wedge e^b \wedge e^c \\
&- 2\Lambda_{abA} \wedge \nabla\phi^A \wedge e^b - 2\frac{1}{4!}m^2\phi_A\,\phi^A\varepsilon_{abcd}e^b \wedge e^c \wedge e^d \\
&- \frac{24}{q}M_{ab}\lambda \wedge e^b + 4\zeta^{ef}M_{ef}\varepsilon_{abcd}e^b \wedge e^c \wedge e^d - 2\zeta_{ab}F \wedge e^b = 0\,.
\end{aligned} \tag{A16}$$

The dynamical degrees of freedom are the tetrad fields e^a, the scalar field ϕ^A, and the electromagnetic potential A, while the remaining variables are algebraically determined in terms of them. Specifically, Equations (A2)–(A13) give

$$\begin{gathered}
\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,, \qquad \gamma^A{}_{\mu\nu\rho} = -\frac{1}{2e}\varepsilon_{\mu\nu\rho\sigma}\,\nabla^\sigma\phi^A\,, \\
\Lambda^{abA}{}_\mu = \frac{1}{12e}g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\nabla_\nu\phi^A\,e^a{}_\rho e^b{}_\sigma\,, \qquad \beta^a{}_{\mu\nu} = 0\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \\
H^{abcA} = \frac{1}{6e}\varepsilon^{\mu\nu\rho\sigma}\,\nabla_\mu\phi^A\,e^a{}_\nu e^b{}_\rho e^c{}_\sigma\,, \qquad \lambda^A{}_\mu = \nabla_\mu\phi^A\,, \\
\lambda_{\mu\nu} = F_{\mu\nu}\,, \quad B_{\mu\nu} = -\frac{1}{2eq}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\,, \\
M^{ab} = -\frac{1}{4e}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,e^a{}_\rho e^b{}_\sigma\,, \quad \zeta^{ab} = \frac{1}{4eq}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}\,e^a{}_\rho e^b{}_\sigma\,.
\end{gathered} \tag{A17}$$

Note that from the Equations (A4)–(A6) it follows that $\beta^a = 0$, as in the pure gravity case. The equation of motion (A15) reduces to the covariant Klein–Gordon equation for the scalar field coupled to the electromagnetic potential A,

$$\left(\nabla_\mu\nabla^\mu - m^2\right)\phi_A = 0\,. \tag{A18}$$

From Equation (A14), we obtain the differential equation of motion for the field A:

$$\nabla_\mu F^{\mu\nu} = j^\nu\,, \qquad j^\mu \equiv \frac{1}{2}\left(\nabla^\nu\phi^A \rhd^B{}_A\phi_B - \phi_A \rhd_B{}^A\nabla^\nu\phi^B\right) = iq\left(\nabla\phi^*\,\phi - \phi^*\nabla\phi\right). \tag{A19}$$

Finally, the equation of motion (A16) for e^a becomes:

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,,$$
$$T^{\mu\nu} \equiv \nabla^\mu\phi_A\,\nabla^\nu\phi^A - \frac{1}{2}g^{\mu\nu}\left(\nabla_\rho\phi_A\,\nabla^\rho\phi^A + m^2\phi_A\,\phi^A\right) - \frac{1}{4q}\left(F_{\rho\sigma}F^{\rho\sigma}g^{\mu\nu} + 4F^{\mu\rho}F_\rho{}^\nu\right)\,. \tag{A20}$$

The system of Equations (A2)–(A16) is equivalent to the system of Equations (A17)–(A20).

Appendix B. The Calculation of the Gauge Generator

The gauge generator of the theory is obtained by the standard Castellani procedure (see [13] for an introduction). One starts from the generic form for the generator,

$$\begin{aligned} G \;=\; \int_{\Sigma_3} \partial^3\vec{x}\Big(&\frac{1}{2}(\partial_0\epsilon^{ab}{}_i)G_{1ab}{}^i + \frac{1}{2}\epsilon^{ab}{}_iG_{0ab}{}^i + \frac{1}{2}(\partial_0\epsilon^{ab})G_{1ab} + \frac{1}{2}\epsilon^{ab}G_{0ab} \\ &+ (\partial_0\epsilon_i)G_1{}^i + \epsilon_iG_0{}^i + (\partial_0\epsilon)G_1 + \epsilon G_0 \\ &+ (\partial_0\epsilon^a)G_{1a} + \epsilon^aG_{0a} + (\partial_0\epsilon^a{}_i)G_{1a}{}^i + \epsilon^a{}_iG_{0a}{}^i \\ &+ \frac{1}{2}(\partial_0\epsilon^A{}_{ij})G_{1A}{}^{ij} + \frac{1}{2}\epsilon^A{}_{ij}G_{0A}{}^{ij}\Big)\,, \end{aligned} \tag{A21}$$

where the generators G_0 and G_1 are obtained by the standard prescription [13]:

$$\begin{aligned} G_1 &= C_{PFC}\,, \\ G_0 + \{\,G_1\,,\,H_T\,\} &= C_{PFC}\,, \\ \{\,G_0\,,\,H_T\,\} &= C_{PFC}\,, \end{aligned} \tag{A22}$$

where C_{PFC} is a primary first-class constraint. For example, one choses $G_{1ab}{}^i = \Phi(B)_{ab}{}^i$. From the conditions

$$\begin{aligned} G_{0ab}{}^i + \{\,\Phi(B)_{ab}{}^i\,,\,H_T\,\} = G_{0ab}{}^i + \Phi(R)_{ab}{}^i &= C_{PFC}\,, \\ \{\,G_{0ab}{}^i\,,\,H_T\,\} = C_{PFC}{}^* &= \{\,C_{PFC} - \Phi(R)_{ab}{}^i\,,\,H_T\,\}\,, \end{aligned} \tag{A23}$$

we solve for $G_{0ab}{}^i$ by determining C_{PFC} from the second equation. Evaluating one PB, one can reexpress the second equation in the form:

$$\{\,C_{PFC}\,,\,H_T\,\} = C_{PFC}{}^* + 2\omega_{[a|}{}^d{}_0\Phi(R)_{|b]d}{}^i = \{\,2\omega_{[a|}{}^d{}_0P(B)_{|b]d}{}^i\,,\,H_T\,\}\,. \tag{A24}$$

From the second equality, we recognize that

$$C_{PFC} = 2\omega_{[a|}{}^d{}_0P(B)_{|b]d}{}^i\,, \tag{A25}$$

which can then be substituted into the first condition above, giving

$$G_{0ab}{}^i = 2\omega_{[a|}{}^d{}_0\Phi(B)_{|b]d}{}^i - \Phi(R)_{ab}{}^i\,. \tag{A26}$$

One thus obtains

$$\frac{1}{2}(\partial_0\epsilon^{ab}{}_i)(G_1)_{ab}{}^i + \frac{1}{2}\epsilon^{ab}{}_iG_{0ab}{}^i = \frac{1}{2}\nabla_0\epsilon^{ab}{}_i\Phi(B)_{ab}{}^i - \frac{1}{2}\epsilon^{ab}{}_i\Phi(R)_{ab}{}^i\,.$$

The other G_0 and G_1 terms are obtained in a similar way, and the generator (61) is derived.

Appendix C. Introduction to 3-Groups

The notion of a 3-*group* is usually introduced in the framework of higher category theory [6]. In category theory, every *group* can be understood as a *category* which has only one element, and morphisms which are all invertible. The group elements are then individual morphisms that map the category element to itself, while the group operation is the categorical composition of the morphisms. In such a case, the axioms of the category guarantee the validity of all axioms of a group. This kind of construction can be generalized to 2-*groups*, 3-*groups* and, in general, *n-groups*. Namely, a 2-group is by definition a 2-category which has only one element, and whose morphisms and 2-morhisms (i.e., morphisms between morphisms) are invertible. Similarly, a 3-group is by definition a 3-category which has only one element, while its morphisms, 2-morphisms, and 3-morphisms are invertible.

The above definition of a 3-group is very abstract, and while theoretically very important, in itself not very useful for practical calculations and applications in physics. Fortunately, there is a theorem of equivalence between 3-groups and the so-called 2-*crossed modules*, which are algebraic structures with more familiar properties [9,10]. For the applications in physics, attention focuses on the so-called *strict Lie* 3-*groups*, and their corresponding differential (Lie algebra) structure, which corresponds to the *differential Lie* 2-*crossed module*. Let us therefore give a brief overview of the latter.

A *differential Lie* 2-*crossed module* $(\mathfrak{l} \xrightarrow{\delta} \mathfrak{h} \xrightarrow{\partial} \mathfrak{g}, \triangleright, \{\_\,,\_\})$ is given by three Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, maps $\delta : \mathfrak{l} \to \mathfrak{h}$ and ∂: $\mathfrak{h} \to \mathfrak{g}$, together with a map called the Peiffer lifting,

$$\{\_\,,\_\} : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{l}\,, \tag{A27}$$

and an action $\triangleright$ of the algebra $\mathfrak{g}$ on all three algebras.

Let us introduce the bases in the three algebras, $\tau_\alpha \in \mathfrak{g}$, $t_a \in \mathfrak{h}$ and $T_A \in \mathfrak{l}$, and structure constants in those bases, as follows:

$$[\tau_\alpha\,,\tau_\beta] = f_{\alpha\beta}{}^{\gamma}\tau_\gamma\,, \quad [t_a\,,t_b] = f_{ab}{}^{c}t_c\,, \quad [T_A\,T_B] = f_{AB}{}^{C}T_C\,. \tag{A28}$$

Now, the maps ∂ and δ can be written as

$$\partial(t_a) = \partial_a{}^{\alpha}\,\tau_\alpha\,, \qquad \delta(T_A) = \delta_A{}^{a}\,t_a\,, \tag{A29}$$

and the action of the algebra $\mathfrak{g}$ on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ as:

$$\tau_\alpha \triangleright \tau_\beta = \triangleright_{\alpha\beta}{}^{\gamma}\,\tau_\gamma\,, \quad \tau_\alpha \triangleright t_a = \triangleright_{\alpha a}{}^{b}\,t_b\,, \quad \tau_\alpha \triangleright T_A = \triangleright_{\alpha A}{}^{B}\,T_B\,. \tag{A30}$$

Finally, the Peiffer lifting can be encoded into coefficients $X_{ab}{}^{A}$ as:

$$\{t_a,\,t_b\} = X_{ab}{}^{A}\,T_A\,. \tag{A31}$$

A differential Lie 2-crossed module has the following properties (we write all equations in the abstract and their corresponding component forms, side by side):

1. The action of the algebra $\mathfrak{g}$ on itself is via the adjoint representation, i.e., $\forall g, g_1 \in \mathfrak{g}$:

$$g \triangleright g_1 = [g, g_1]\,, \qquad \triangleright_{\alpha\beta}{}^{\gamma} = f_{\alpha\beta}{}^{\gamma}\,. \tag{A32}$$

2. The action of the algebra $\mathfrak{g}$ on algebras $\mathfrak{h}$ and $\mathfrak{l}$ is $\mathfrak{g}$-equivariant, i.e., $\forall g \in \mathfrak{g}$, $h \in \mathfrak{h}$, $l \in \mathfrak{l}$:

$$\partial(g \triangleright h) = g \triangleright \partial(h)\,, \qquad \partial_a{}^{\beta}\,f_{\alpha\beta}{}^{\gamma} = \triangleright_{\alpha a}{}^{b}\,\partial_b{}^{\gamma}\,, \tag{A33}$$

$$\delta(g \triangleright l) = g \triangleright \delta(l)\,, \qquad \delta_A{}^{a}\,\triangleright_{\alpha a}{}^{b} = \triangleright_{\alpha A}{}^{B}\,\delta_B{}^{b}\,. \tag{A34}$$

3. The Peiffer lifting is a $\mathfrak{g}$-equivariant map, i.e., for every $g \in \mathfrak{g}$ and $h_1, h_2 \in \mathfrak{h}$:

$$g \rhd \{h_1, h_2\} = \{g \rhd h_1, h_2\} + \{h_1, g \rhd h_2\}, \qquad X_{ab}{}^B \rhd_{\alpha B}{}^A = \rhd_{\alpha a}{}^c X_{cb}{}^A + \rhd_{\alpha b}{}^c X_{ac}{}^A . \tag{A35}$$

4. For every $h_1,\ h_2 \in \mathfrak{h}$, the following identity holds:

$$\delta(\{h_1,\ h_2\}) = [h_1\,, h_2] - \partial(h_1) \rhd h_2\,, \qquad X_{ab}{}^A\, \delta_A{}^c = f_{ab}{}^c - \partial_a{}^\alpha\ \rhd_{\alpha b}{}^c . \tag{A36}$$

5. For all $l_1,\ l_2 \in \mathfrak{l}$, the following identity holds:

$$[l_1,\ l_2] = \{\delta(l_1),\ \delta(l_2)\}\ , \qquad f_{AB}{}^C = \delta_A{}^a\, \delta_B{}^b\, X_{ab}{}^C . \tag{A37}$$

6. For all $h_1, h_2, h_3 \in \mathfrak{h}$:

$$\begin{aligned} \{[h_1,\ h_2],\ h_3\} = &\ \partial(h_1) \rhd \{h_2,\ h_3\} + \{h_1,\ [h_2,\ h_3]\} - \partial(h_2) \rhd \{h_1,\ h_3\} - \{h_2,\ [h_1,\ h_3]\}\ , \\ f_{ab}{}^d\, X_{dc}{}^B = &\ \partial_a{}^\alpha\, X_{bc}{}^A \rhd_{\alpha A}{}^B + X_{ad}{}^B\, f_{bc}{}^d - \partial_b{}^\alpha \rhd_{\alpha A}{}^B\, X_{ac}{}^A - X_{bd}{}^B\, f_{ac}{}^d . \end{aligned} \tag{A38}$$

7. For all $h_1, h_2, h_3 \in \mathfrak{h}$:

$$\begin{aligned} \{h_1,\ [h_2,\ h_3]\} = &\ \{\delta\,\{h_1,\ h_2\}\,, h_3\} - \{\delta\,\{h_1,\ h_3\}\,,\ h_2\}\ , \\ X_{ad}{}^A\, f_{bc}{}^d = &\ X_{ab}{}^B\, \delta_B{}^d\, X_{dc}{}^A - X_{ac}{}^B\, \delta_B{}^d X_{db}{}^A . \end{aligned} \tag{A39}$$

8. For all $l \in \mathfrak{l}$ and $\forall h \in \mathfrak{h}$:

$$\{\delta(l),\ h\} + \{h,\ \delta(l)\} = -\partial(h) \rhd l\,, \qquad 2\,\delta_A{}^a\, X_{\{ab\}}{}^B = -\partial_b{}^\alpha \rhd_{\alpha A}{}^B . \tag{A40}$$

Finally, when dealing with various algebra valued differential forms, one multiplies them as differential forms using the ordinary wedge product $\wedge$, and simultaneously as algebra elements using one of maps defined above. For example, the product with an action $\wedge^{\rhd}$ of the $\mathfrak{g}$-valued n-form ρ on the $\mathfrak{h}$-valued m-form η is defined as:

$$\begin{aligned} \rho \wedge^{\rhd} \eta = &\ \frac{1}{n!m!}\, \rho^\alpha{}_{\mu_1\dots\mu_m}\, \eta^a{}_{\nu_1\dots\nu_n}\, \tau_\alpha \rhd t_a\, \mathrm{d}x^{\mu_1} \wedge \dots \mathrm{d}x^{\mu_m} \wedge \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_n} \\ = &\ \frac{1}{n!m!}\, \rho^\alpha{}_{\mu_1\dots\mu_m}\, \eta^a{}_{\nu_1\dots\nu_n}\, \rhd_{\alpha a}{}^b t_b\, \mathrm{d}x^{\mu_1} \wedge \dots \mathrm{d}x^{\mu_m} \wedge \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_n} . \end{aligned} \tag{A41}$$

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Construction and examples of higher gauge theories*

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Abstract

We provide several examples of higher gauge theories, constructed as generalizations of a BF model to $2BF$ and $3BF$ models with constraints. Using the framework of higher category theory, we introduce appropriate 2-groups and 3-groups, and construct the actions for the corresponding constrained $2BF$ and $3BF$ theories. In this way, we can construct actions which describe the correct dynamics of Yang-Mills, Klein-Gordon, Dirac, Weyl, and Majorana fields coupled to Einstein-Cartan gravity. Each action is naturally split into a topological sector and a sector with simplicity constraints. The properties of the higher gauge group structure opens up a possibility of a nontrivial unification of all fields.

1. Introduction

The quantization of the gravitational field is one of the fundamental open problems in modern physics. There are various approaches to this problem, some of which have developed into vast research frameworks. One of such frameworks is the Loop Quantum Gravity approach, which aims to establish a nonperturbative quantization of gravity, both canonically and covariantly [1, 2, 3]. The covariant approach is slightly more general, and

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focuses on providing a possible rigorous definition of the path integral for the gravitational field,

$$Z = \int \mathcal{D}g \, e^{iS[g]} \,. \tag{1}$$

This is done by considering a triangulation of a spacetime manifold, and defining the path integral as a discrete state sum of the gravitational field configurations living on the simplices in the triangulation. This quantization technique is known as the *spinfoam* quantization method, and is performed via the following three steps:

(1) one writes the classical action $S[g]$ as a constrained BF action;

(2) one uses the Lie group structure, underlying the topological sector of the action, to define a triangulation-independent state sum Z;

(3) one imposes the simplicity constraints on the state sum, promoting it into a triangulation-dependent state sum, which serves as a definition for the path integral (1).

So far, this quantization prescription has been implemented for various choices of the gravitational action, of the Lie group, and of the spacetime dimension. For example, in 3 dimensions, historically the first spinfoam model is known as the Ponzano-Regge model [4]. In 4 dimensions there are multiple models, depending on the choice of the Lie group and the way one imposes the simplicity constraints [5, 6, 7, 8, 9]. While these models do give a definition for the gravitational path integral, none of them are able to consistently include matter fields. Including the matter fields has so far had limited success [10], mainly due to the absence of the tetrad fields from the topological sector of the theory.

In order to resolve this issue, a new approach has been developed, using the framework of *higher gauge theory* (see [11] for a review). In particular, one uses the idea of a *categorical ladder* to generalize the BF action (based on a Lie group) into a $2BF$ action (based on the so-called 2-group structure). A suitable choice of the *Poincaré 2-group* introduces the needed tetrad fields into the topological sector of the action [12]. While this result opened up a possibility to couple matter fields to gravity, the matter fields could not be naturally expressed using the underlying algebraic structure of a 2-group, rendering the spinfoam quantization method inapplicable. Namely, the matter sector could indeed be added to the classical action, but could not be expressed itself as a constrained $2BF$ theory, which means that the steps 1–3 above could not be performed for the matter sector of the action, but only for gravity.

This final issue has recently been resolved in [13], by passing from the 2-group structure to the 3-group structure, generalizing the action one step further in the categorical ladder. This generalization naturally gives rise to the so-called $3BF$ action, which turns out to be suitable for a unified description of both gravity and matter fields. The steps of the categorical ladder and their corresponding structures are summarized as follows:

categorical structure	algebraic structure	linear structure	topological action	degrees of freedom
Lie group	Lie group	Lie algebra	BF theory	gauge fields
Lie 2-group	Lie crossed module	differential Lie crossed module	$2BF$ theory	tetrad fields
Lie 3-group	Lie 2-crossed module	differential Lie 2-crossed module	$3BF$ theory	scalar and fermion fields

The purpose of this paper is to give a systematic overview of the constructions of classical BF, $2BF$ and $3BF$ actions, both pure and constrained, in order to demonstrate the categorical ladder procedure and the construction of higher gauge theories. In other words, we focus on the step 1 of the spinfoam quantization programme.

The layout of the paper is as follows. Section 2 deals with models based on a BF theory. First we discuss the pure, topological BF theory, and then pass on to the the physically more interesting Yang-Mills theory in Minkowski spacetime and the Plebanski formulation of general relativity. In Section 3 we study the first step in the categorical ladder, namely models based on the $2BF$ theory. After introducing the pure $2BF$ theory, we study the relevant formulation of general relativity [12], and then the coupled Einstein-Yang-Mills theory. Then, in Section 4 we perform the second step in the categorical ladder, passing on to models based on the $3BF$ theory. After the introduction of the pure $3BF$ model, we construct constrained $3BF$ actions for the cases of Klein-Gordon, Dirac, Weyl and Majorana fields, all coupled to the Einstein-Cartan gravity in the standard way. As we shall see, the scalar and fermion fields will be *naturally associated to a new gauge group*, generalizing the purpose of a gauge group in the Yang-Mills theory, which opens up a possibility of an algebraic classification of matter fields. Finally, Section 5 contains a discussion and conclusions.

The notation and conventions are as follows. The local Lorentz indices are denoted by the Latin letters $a, b, c, \dots$, take values $0, 1, 2, 3$, and are raised and lowered using the Minkowski metric η_{ab} with signature $(-,+,+,+)$. Spacetime indices are denoted by the Greek letters $\mu, \nu, \dots$, and are raised and lowered by the spacetime metric $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$, where $e^a{}_\mu$ are the tetrad fields. The inverse tetrad is denoted as $e^\mu{}_a$. All other indices that appear in the paper are dependent on the context, and their usage is explicitly defined in the text where they appear. We work in the natural system of units where $c = \hbar = 1$, and $G = l_p^2$, where l_p is the Planck length.

2. BF theory

We begin with a short review of BF theories. See [14, 15, 16] for additional information.

2.1. Pure BF theory

Given a Lie group G, and denoting its corresponding Lie algebra as $\mathfrak{g}$, one introduces the pure BF action as follows (we limit ourselves to the physically relevant case of 4-dimensional spacetime manifolds $\mathcal{M}_4$):

$$S_{BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} \,. \tag{2}$$

Here, $\mathcal{F} \equiv \mathrm{d}\alpha + \alpha \wedge \alpha$ is the curvature 2-form for the algebra-valued connection 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$, and $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ is a Lagrange multiplier 2-form, while $\langle \_\,, \_ \rangle_{\mathfrak{g}}$ denotes a G-invariant bilinear symmetric nondegenerate form.

One can see from (2) that the action is diffeomorphism invariant, and it is also gauge invariant with respect to G, provided that B transforms as a scalar with respect to G.

Varying the action (2) with respect to B^β and α^β, where the index β is the group G index (which counts the generators of $\mathfrak{g}$), one obtains the following equations of motion,

$$\mathcal{F}^\beta = 0\,, \qquad \nabla B^\beta \equiv \mathrm{d}B^\beta + f_{\gamma\delta}{}^\beta \alpha^\gamma \wedge B^\delta = 0\,, \tag{3}$$

where $f_{\gamma\delta}{}^\beta$ are the structure constants of the Lie group G. From the first equation of motion, one immediately sees that α is a flat connection, meaning that $\alpha = 0$ up to gauge transformations. Given this, the second equation of motion implies that B is constant. Therefore, there are no local propagating degrees of freedom, and the theory is called *topological*.

2.2. Yang-Mills theory

In physics one is usually interested in theories which are not topological, i.e., which have local propagating degrees of freedom. As a rule of thumb, one recognizes that the theory does have local propagating degrees of freedom if one of the equations of motion is a second-order partial differential equation, usually featuring a D'Alambertian operator $\Box$ in some form. In order to transform the pure BF action into such a theory, one adds an additional term to the action, commonly called the *simplicity constraint*. The resulting action is called a *constrained BF theory*. A nice example is the Yang-Mills theory for the $SU(N)$ group in Minkowski spacetime, which can be rewritten as a constrained BF theory in the following way:

$$\begin{aligned} S \;=\; & \int B_I \wedge F^I + \lambda^I \wedge \left(B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b \right) \\ & \quad + \zeta^{abI} \left(M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - g_{IJ} F^J \wedge \delta_a \wedge \delta_b \right). \end{aligned} \tag{4}$$

Here $F \equiv \mathrm{d}A + A \wedge A$ is again the curvature 2-form for the connection $A \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{su}(N))$, and $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{su}(N))$ is the Lagrange multiplier

2-form. The Killing form $g_{IJ} \equiv \langle \tau_I \,, \tau_J \rangle_{\mathfrak{su}(N)} \propto f_{IK}{}^L f_{JL}{}^K$ is used to raise and lower the indices $I, J, \dots$ which count the generators of $SU(N)$, while $f_{IJ}{}^K$ are the structure constants for the $\mathfrak{su}(N)$ algebra. In addition to the topological $B \wedge F$ term, there are also two simplicity constraint terms present, featuring two Lagrange multipliers, a 2-form λ^I and a 0-form ζ^{abI}. The 0-form M_{abI} is also a Lagrange multiplier, while g is the coupling constant for the Yang-Mills theory.

Finally, δ^a is a nondynamical 1-form, such that there exists a global coordinate frame in which its components are equal to the Kronecker symbol $\delta^a{}_\mu$ (hence the notation δ^a). The 1-form δ^a plays the role of a background field, and defines the global spacetime metric, via the equation

$$\eta_{\mu\nu} = \eta_{ab} \delta^a{}_\mu \delta^b{}_\nu \,, \tag{5}$$

where $\eta_{ab} \equiv \mathrm{diag}(-1, +1, +1, +1)$ is the Minkowski metric. Since the coordinate system is global, the spacetime manifold $\mathcal{M}_4$ is understood to be flat. The indices $a, b, \dots$ are local Lorentz indices, taking values $0, \dots, 3$. Note that the field δ^a has all the properties of the tetrad 1-form e^a in the flat Minkowski spacetime. Also note that the action (4) is manifestly diffeomorphism invariant and gauge invariant with respect to $SU(N)$, but not background independent, due to the presence of δ^a.

Varying the action (4) with respect to the variables ζ^{abI}, M_{abI}, A^I, B_I, and λ^I, respectively (but not with respect to the background field δ^a), we obtain the equations of motion:

$$M_{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f - F_I \wedge \delta_a \wedge \delta_b = 0 \,, \tag{6}$$

$$-\frac{12}{g} \lambda^I \wedge \delta^a \wedge \delta^b + \zeta^{abI} \varepsilon_{cdef} \delta^c \wedge \delta^d \wedge \delta^e \wedge \delta^f = 0 \,, \tag{7}$$

$$-\mathrm{d}B_I + f_{JI}{}^K B_K \wedge A^J + \mathrm{d}(\zeta^{ab}{}_I \delta_a \wedge \delta_b) - f_{JI}{}^K \zeta^{ab}{}_K \delta_a \wedge \delta_b \wedge A^J = 0 \,, \tag{8}$$

$$F_I + \lambda_I = 0 \,, \tag{9}$$

$$B_I - \frac{12}{g} M_{abI} \delta^a \wedge \delta^b = 0 \,, \tag{10}$$

From the equations (6), (7), (9) and (10) one obtains the multipliers as algebraic functions of the field strength $F^I{}_{\mu\nu}$ for the dynamical field A^I:

$$\begin{aligned} M_{abI} &= \frac{1}{48} \varepsilon_{abcd} F_I{}^{cd} \,, & \zeta^{abI} &= \frac{1}{4g} \varepsilon^{abcd} F^I{}_{cd} \,, \\ \lambda_{Iab} &= F_{Iab} \,, & B_{Iab} &= \frac{1}{2g} \varepsilon_{abcd} F_I{}^{cd} \,. \end{aligned} \tag{11}$$

Here we used the notation $F_{Iab} = F_{I\mu\nu}\delta_a{}^\mu\delta_b{}^\nu$, and similarly for other variables, where we exploited the fact that $\delta^a{}_\mu$ is invertible. Using these equations and the differential equation (8) one obtains the equation of motion for gauge field $A^I{}_\mu$,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0\,. \tag{12}$$

This is precisely the classical equation of motion for the free Yang-Mills theory. Note that this is a second-order partial differential equation for the field $A^I{}_\mu$, and moreover contains the $\Box$ operator in the first term.

In addition to the Yang-Mills theory, one can easily extend the action (4) in order to describe the massive vector field and obtain the Proca equation of motion. This is done by adding a mass term

$$-\frac{1}{4!}m^2 A_{I\mu}A^I{}_\nu \eta^{\mu\nu}\varepsilon_{abcd}\delta^a \wedge \delta^b \wedge \delta^c \wedge \delta^d \tag{13}$$

to the action (4). Of course, this term explicitly breaks the $SU(N)$ gauge symmetry of the action.

2.3. Plebanski general relativity

The second example of the constrained BF theory is the Plebanski action for general relativity [16, 14]. Using the Lorentz group $SO(3,1)$ as a gauge group, one constructs a constrained BF action as

$$S = \int_{\mathcal{M}_4} B_{ab} \wedge R^{ab} + \phi_{abcd} B^{ab} \wedge B^{cd}\,. \tag{14}$$

Here R^{ab} is the curvature 2-form for the spin connection ω^{ab}, B_{ab} is the usual Lagrange multiplier 2-form, while ϕ_{abcd} is the additional Lagrange multiplier 0-form multiplying the term $B^{ab} \wedge B^{cd}$ to form a simplicity constraint. It can be shown that the variation of this action with respect to B_{ab}, ω^{ab} and ϕ_{abcd} gives rise to the equations of motion of vacuum general relativity. However, in this model the tetrad fields appear only as a solution of the simplicity constraint equation of motion $B^{ab} \wedge B^{cd} = 0$. Therefore, being intrinsically on-shell objects, the tetrad fields are not present in the action itself and cannot be quantized. This renders the Plebanski model unsuitable for coupling of matter fields to gravity [10, 12, 20]. Nevertheless, regarded as a model for pure gravity, the Plebanski model has been successfully quantized in the context of spinfoam models [8, 9, 1, 2].

3. $2BF$ theory

In this section we perform the first step of the *categorical ladder*, generalizing the algebraic notion of a group to the notion of a 2-group. This leads to the generalization of the BF theory to the $2BF$ theory, also sometimes called $BFCG$ theory [11, 17, 18, 19].

3.1. Pure $2BF$ theory

In order to circumvent the issue of tetrad fields not being present in the Plebanski action, in the context of higher category theory [11] a recent promising approach has been developed [12, 21, 22, 23, 20, 24]. As an essential ingredient, let us first give a short review of the 2-group formalism.

Within the framework of category theory, the group as an algebraic structure can be understood as a category with only one object and invertible morphisms [11]. Additionally, the notion of a category can be generalized to the so-called *higher categories*, which have not only objects and morphisms, but also 2-morphisms (morphisms between morphisms), and so on. This process of generalization is called the *categorical ladder*. Using this process, one can introduce the notion of a 2-*group* as a 2-category consisting of only one object, where all the morphisms and all 2-morphisms are invertible. It has been shown that every strict 2-group is equivalent to a *crossed module* $(H \stackrel{\partial}{\to} G\,, \rhd)$, see [13] for detailed definitions. Here G and H are groups, ∂ is a homomorphism from H to G, while $\rhd : G \times H \to H$ is an action of G on H.

Similarly to the case of an ordinary Lie group G which has a naturally associated notion of a connection α, giving rise to a BF theory, the 2-group structure has a naturally associated notion of a 2-connection $(\alpha\,, \beta)$, described by the usual $\mathfrak{g}$-valued 1-form $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{g})$ and an $\mathfrak{h}$-valued 2-form $\beta \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{h})$, where $\mathfrak{h}$ is a Lie algebra of the Lie group H. The 2-connection gives rise to the so-called *fake* 2-*curvature* $(\mathcal{F}, \mathcal{G})$, given as

$$\mathcal{F} = \mathrm{d}\alpha + \alpha \wedge \alpha - \partial\beta\,, \qquad \mathcal{G} = d\beta + \alpha \wedge^{\rhd} \beta\,. \tag{15}$$

Here $\alpha \wedge^{\rhd} \beta$ means that α and β are multiplied as forms using $\wedge$, and simultaneously multiplied as algebra elements using $\rhd$, see [13]. The curvature pair $(\mathcal{F}, \mathcal{G})$ is called "fake" because of the presence of the additional term $\partial\beta$ in the definition of $\mathcal{F}$ [11].

Using the structure of a 2-group, or equivalently the crossed module, one can generalize the BF action to the so-called $2BF$ action, defined as follows [17, 18]:

$$S_{2BF} = \int_{\mathcal{M}_4} \langle B \wedge \mathcal{F} \rangle_{\mathfrak{g}} + \langle C \wedge \mathcal{G} \rangle_{\mathfrak{h}}\,. \tag{16}$$

Here the 2-form $B \in \mathcal{A}^2(\mathcal{M}_4\,, \mathfrak{g})$ and the 1-form $C \in \mathcal{A}^1(\mathcal{M}_4\,, \mathfrak{h})$ are Lagrange multipliers. Also, $\langle \_\,, \_ \rangle_{\mathfrak{g}}$ and $\langle \_\,, \_ \rangle_{\mathfrak{h}}$ denote the G-invariant bilinear symmetric nondegenerate forms for the algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. As a consequence of the axiomatic structure of a crossed module (see [13]), the bilinear form $\langle \_\,, \_ \rangle_{\mathfrak{h}}$ is H-invariant as well. See [17, 18] for review and references.

Similarly to the BF action, the $2BF$ action is also topological, which can be seen from equations of motion. Varying with respect to B^{α} and C^a one obtains

$$\mathcal{F}^{\alpha} = 0\,, \qquad \mathcal{G}^a = 0\,, \tag{17}$$

where indices a count the generators of the group H. Varying with respect to α^α and β^a one obtains the equations for the multipliers,

$$\mathrm{d}B_\alpha - g_{\alpha\beta}{}^\gamma B_\gamma \wedge \alpha^\beta - \rhd_{\alpha a}{}^b C_b \wedge \beta^a = 0\,, \tag{18}$$

$$\mathrm{d}C_a - \partial_a{}^\alpha B_\alpha + \rhd_{\alpha a}{}^b C_b \wedge \alpha^\alpha = 0\,. \tag{19}$$

We can again see that the equations of motion are only first-order and have only very simple solutions (note that this is not a sufficient argument for the absence of local propagating degrees of freedom — a counterexample is the Dirac equation, being a first-order partial differential equation which *does* have propagating degrees of freedom). One can additionally use the Hamiltonian analysis to rigorously demonstrate that there are no local propagating degrees of freedom [22, 23]. Thus the $2BF$ theory is also topological.

3.2. General relativity

An important example of a crossed module structure is a vector space V equipped with an isometry group O. Namely, V can be regarded as an Abelian Lie group with addition as a group operation, so that a representation of O on V is an action $\rhd$ of O on the group V, giving rise to the crossed module $(V \stackrel{\partial}{\to} O\,, \rhd)$, where the homomorphism ∂ is chosen to be trivial (it maps every element of V into a unit of O).

We can employ this construction to introduce the *Poincaré 2-group*. One constructs a crossed module by choosing

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,. \tag{20}$$

The map ∂ is trivial, while $\rhd$ is a natural action of $SO(3,1)$ on $\mathbb{R}^4$, defined by the equation

$$M_{ab} \rhd P_c = \eta_{[bc} P_{a]}\,, \tag{21}$$

where M_{ab} and P_a are the generators of groups $SO(3,1)$ and $\mathbb{R}^4$, respectively. The action $\rhd$ of $SO(3,1)$ on itself is given via conjugation. At the level of the algebra, conjugation reduces to the action via the adjoint representation, so that

$$M_{ab} \rhd M_{cd} = [\, M_{ab}\,,\, M_{cd}\,] \equiv \eta_{ad} M_{bc} - \eta_{ac} M_{bd} + \eta_{bc} M_{ad} - \eta_{bd} M_{ac}\,. \tag{22}$$

The 2-connection (α, β) is given by the algebra-valued differential forms

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \tag{23}$$

where ω^{ab} is called the spin connection. The corresponding 2-curvature in this case is given by

$$\begin{aligned} \mathcal{F} &= (\mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}) M_{ab} &&\equiv R^{ab} M_{ab}\,, \\ \mathcal{G} &= (\mathrm{d}\beta^a + \omega^a{}_b \wedge \beta^b) P_a &&\equiv \nabla \beta^a P_a \equiv G^a P_a\,, \end{aligned} \tag{24}$$

Note that, since ∂ is trivial, the fake curvature is the same as ordinary curvature. Introducing the bilinear forms

$$\langle M_{ab}\,, M_{cd}\rangle_{\mathfrak{g}} = \eta_{a[c}\eta_{bd]}\,, \qquad \langle P_a\,, P_b\rangle_{\mathfrak{h}} = \eta_{ab}\,, \tag{25}$$

one can show that 1-forms C^a transform in the same way as the tetrad 1-forms e^a under the Lorentz transformations and diffeomorphisms, so the fields C^a can be identified with the tetrads. Then one can rewrite the pure $2BF$ action (16) for the Poincaré 2-group as

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a\,. \tag{26}$$

Note that the above step of recognizing that $C^a \equiv e^a$ was crucial, since we now see that the tetrad fields are explicitly present in the $2BF$ action for the Poincaré 2-group.

In order to promote (26) to an action for general relativity, we add a convenient simplicity constraint term:

$$S = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a - \lambda_{ab} \wedge \Big(B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c \wedge e_d\Big)\,. \tag{27}$$

Here λ_{ab} is a Lagrange multiplier 2-form associated to the simplicity constraint term, and l_p is the Planck length. Note that the term "simplicity constraint" derives its name from the fact that the constraint imposes the property of *simplicity* on B^{ab} — a 2-form is said to be *simple* if it can be written as an exterior product of two 1-forms.

Varying the action (27) with respect to B_{ab}, e_a, ω_{ab}, β_a and λ_{ab}, we obtain the following equations of motion:

$$R_{ab} - \lambda_{ab} = 0\,, \tag{28}$$

$$\nabla\beta_a + \frac{1}{8\pi l_p^2}\varepsilon_{abcd}\lambda^{bc} \wedge e^d = 0\,, \tag{29}$$

$$\nabla B_{ab} - e_{[a} \wedge \beta_{b]} = 0\,, \tag{30}$$

$$\nabla e_a = 0\,, \tag{31}$$

$$B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c \wedge e_d = 0\,. \tag{32}$$

Given this system of equations, all fields can be algebraically determined in terms of the tetrads ${e^a}_\mu$, as follows. From the equations (31) and (32) we obtain that $\nabla B^{ab} = 0$, from which it follows, using the equation (30), that

$e_{[a} \wedge \beta_{b]} = 0$. Assuming that the tetrads are nondegenerate, $e \equiv \det(e^a{}_\mu) \neq 0$, it can be shown that this is equivalent to $\beta^a = 0$ [12]. Therefore, from the equations (28), (30), (31) and (32) we obtain

$$\lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu}\,, \quad \beta^a{}_{\mu\nu} = 0\,, \quad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \quad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu\,. \tag{33}$$

Here the Ricci rotation coefficients are defined as

$$\triangle^{ab}{}_\mu \equiv \frac{1}{2}(c^{abc} - c^{cab} + c^{bca})e_{c\mu}\,, \tag{34}$$

where

$$c^{abc} = e^\mu{}_b e^\nu{}_c \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu\right)\,. \tag{35}$$

The last equation establishes that the spin connection 1-form ω^{ab} is expressed as a function of the tetrads, which then implies the same for the curvature 2-form R^{ab}. Finally, the remaining equation (29) then reduces to

$$\varepsilon_{abcd}R^{bc} \wedge e^d = 0\,, \tag{36}$$

which is nothing but the vacuum Einstein field equation,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\,.$$

Therefore, the action (27) is classically equivalent to general relativity.

3.3. Einstein-Yang-Mills theory

As we have already mentioned above, the main advantage of the action (27) over the Plebanski model lies in the fact that the tetrad fields are explicitly present in the topological sector of the action. This allows one to couple matter fields in a straightforward way [12]. However, one can do even more [13], and couple the $SU(N)$ Yang-Mills fields to gravity within a unified framework of 2-group formalism.

Namely, we can modify the Poincaré 2-group structure to include the $SU(N)$ gauge group, as follows. We choose the two Lie groups as

$$G = SO(3,1) \times SU(N)\,, \qquad H = \mathbb{R}^4\,, \tag{37}$$

and we define the action $\triangleright$ of the group G in the following fashion. As in the case of the Poincaré 2-group, it acts on itself via conjugation. Next, it acts on H such that the $SO(3,1)$ subgroup acts on $\mathbb{R}^4$ via the vector representation (21), while the action of the $SU(N)$ subgroup is trivial,

$$\tau_I \triangleright P_a = 0\,, \tag{38}$$

where τ_I are the $SU(N)$ generators. The map ∂ also remains trivial, as before. The form of the 2-connection (α, β) now reflects the structure of the group G,

$$\alpha = \omega^{ab} M_{ab} + A^I \tau_I \,, \qquad \beta = \beta^a P_a \,, \tag{39}$$

where A^I is the gauge connection 1-form. Next, the curvature for α then becomes

$$\mathcal{F} = R^{ab} M_{ab} + F^I \tau_I \,, \qquad F^I \equiv \mathrm{d}A^I + f_{JK}{}^I A^J \wedge A^K \,. \tag{40}$$

The curvature for β remains the same as before, because of (38). Finally, the product structure of the group G implies that its Killing form $\langle \_ , \_ \rangle_{\mathfrak{g}}$ reduces to the Killing forms for the $SO(3,1)$ and $SU(N)$, along with the identity $\langle M_{ab} \,, \tau_I \rangle_{\mathfrak{g}} = 0$.

Given a crossed module defined in this way, its corresponding pure $2BF$ action (16) becomes

$$S_{2BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a \,, \tag{41}$$

where $B^I \in \mathcal{A}^2(\mathcal{M}_4 \,, \mathfrak{su}(N))$ is the new Lagrange multiplier. The action (41) is topological, and again we add appropriate simplicity constraint terms, in order to transform it into action with nontrivial dynamics. The constraint giving rise to gravity is the same as in (27), while the constraint for the gauge fields is given as in the action (4) with the substitution $\delta^a \to e^a$. Putting everything together, we obtain:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + B^I \wedge F_I + e_a \wedge \nabla \beta^a \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) + \lambda^I \wedge \left(B_I - \frac{12}{g} M_{abI} e^a \wedge e^b \right) \\ & + \zeta^{abI} \left(M_{abI} \varepsilon_{cdef} e^c \wedge e^d \wedge e^e \wedge e^f - g_{IJ} F^J \wedge e_a \wedge e_b \right) . \end{aligned} \tag{42}$$

It is crucial to note that the Yang-Mills simplicity constraints in (42) are obtained from the Yang-Mills action (4) by substituting the nondynamical background field δ^a from (4) with a dynamical field e^a. The relationship between these fields has already been hinted at in the equation (5), which describes the connection between δ^a and the flat spacetime metric $\eta_{\mu\nu}$. Once promoted to e^a, this field becomes dynamical due to the presence of gravitational terms, while the equation (5) becomes the usual relation between the tetrad and the metric,

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu \,, \tag{43}$$

further confirming the identification $C^a = e^a$. Moreover, the total action (42) now becomes background independent, as expected in general relativity. All this is a consequence of the fact that the tetrad field is explicitly present in the topological sector of the action (27), and represents a clear improvement over the Plebanski model.

Taking the variations of the action (42) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, ζ^{abI}, M_{abI}, B_I, λ^I, A^I, and e^a, we obtain equations of motion. Similarly as before, all variables can be algebraically expressed as functions of A^I and e^a and their derivatives:

$$\begin{gathered}
\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \beta_{a\mu\nu} = 0\,, \qquad \omega_{ab\mu} = \triangle_{ab\mu}\,, \qquad \lambda_{abI} = F_{abI}\,, \\
B_{\mu\nu I} = -\frac{e}{2g}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}{}_I\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu\,, \\
M_{abI} = -\frac{1}{4eg}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma\,, \qquad \zeta^{abI} = \frac{1}{4eg}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^I e^a{}_\rho e^b{}_\sigma\,.
\end{gathered} \tag{44}$$

In addition, we obtain two differential equations — An equation for A^I,

$$\nabla_\rho F^{I\rho\mu} \equiv \partial_\rho F^{I\rho\mu} + \Gamma^\rho{}_{\lambda\rho}F^{I\lambda\mu} + f_{JK}{}^I A^J{}_\rho F^{K\rho\mu} = 0\,, \tag{45}$$

where $\Gamma^\lambda{}_{\mu\nu}$ is the standard Levi-Civita connection, and an equation for e^a,

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \tag{46}$$

where

$$T^{\mu\nu} \equiv -\frac{1}{4g}\left(F_{\rho\sigma}{}^I F^{\rho\sigma}{}_I g^{\mu\nu} + 4F^{\mu\rho}{}_I F_\rho{}^{\nu I}\right)\,. \tag{47}$$

In this way, we see that both gravity and gauge fields can be successfully represented within a unified framework of higher gauge theory, based on a 2-group structure. A generalization from $SU(N)$ Yang-Mills case to more complicated cases such as $SU(3)\times SU(2)\times U(1)$ is completely straightforward.

4. $3BF$ theory

While the structure of a 2-group can successfully describe both gravitational and gauge fields, unfortunately it cannot accommodate other matter fields, such as scalars or fermions. In order to remedy this drawback, we make one further step in the categorical ladder, passing from the notion of a 2-group to the notion of a 3-group. As it turns out, the 3-group structure is excellent for the description of all fields that are present in the Standard Model, coupled to gravity. Moreover, a 3-group contains one more gauge group, which is novel and corresponds to the choice of the scalar and fermion fields present in the theory. This is an unexpected and beautiful result, not present in ordinary gauge theory.

As before, we will begin by introducing the notion of a 3-group, and constructing the corresponding $3BF$ action. Afterwards, we will modify this action by adding appropriate simplicity constraints, giving rise to theories with expected nontrivial dynamics. Along the way, we shall see that scalar and fermion fields are being treated pretty much on an equal footing with gravity and gauge fields.

4.1. Pure $3BF$ theory

Similarly to the concepts of a group and a 2-group, one can introduce the notion of a 3-group in the framework of higher category theory, as a 3-category with only one object where all the morphisms, 2-morphisms and 3-morphisms are invertible. Also, in the same way as a 2-group is equivalent to a crossed module, it was proved that a strict 3-group is equivalent to a 2-*crossed module* [25].

A Lie 2-crossed module, denoted as $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$, is an algebraic structure specified by three Lie groups G, H and L, together with the homomorphisms δ and ∂, an action $\triangleright$ of the group G on all three groups, and a G-equivariant map

$$\{\_\,,\_\} : H \times H \to L\,.$$

called the Peiffer lifting. The maps ∂, δ, $\triangleright$ and the Peiffer lifting satisfy certain axioms, so that the resulting structure is equivalent to a 3-group [13].

Like in the cases of BF and $2BF$ actions, we can introduce a gauge invariant topological $3BF$ action over the manifold $\mathcal{M}_4$ for a given 2-crossed module $(L \overset{\delta}{\to} H \overset{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$. Denoting $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$ as Lie algebras corresponding to the groups G, H and L, respectively, one can introduce a 3-connection (α,β,γ) given by the algebra-valued differential forms $\alpha \in \mathcal{A}^1(\mathcal{M}_4\,,\mathfrak{g})$, $\beta \in \mathcal{A}^2(\mathcal{M}_4\,,\mathfrak{h})$ and $\gamma \in \mathcal{A}^3(\mathcal{M}_4\,,\mathfrak{l})$. The corresponding fake 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is then defined as

$$\begin{gathered} \mathcal{F} = \mathrm{d}\alpha + \alpha\wedge\alpha - \partial\beta\,, \qquad \mathcal{G} = \mathrm{d}\beta + \alpha\wedge^{\triangleright}\beta - \delta\gamma\,, \\ \mathcal{H} = \mathrm{d}\gamma + \alpha\wedge^{\triangleright}\gamma + \{\beta\wedge\beta\}\,, \end{gathered} \tag{48}$$

see [25, 26] for details. Note that γ is a 3-form, while its corresponding field strength $\mathcal{H}$ is a 4-form, necessitating that the spacetime manifold be at least 4-dimensional. Then, a $3BF$ action is defined as

$$S_{3BF} = \int_{\mathcal{M}_4} \langle B\wedge\mathcal{F}\rangle_{\mathfrak{g}} + \langle C\wedge\mathcal{G}\rangle_{\mathfrak{h}} + \langle D\wedge\mathcal{H}\rangle_{\mathfrak{l}}\,, \tag{49}$$

where $B \in \mathcal{A}^2(\mathcal{M}_4, \mathfrak{g})$, $C \in \mathcal{A}^1(\mathcal{M}_4, \mathfrak{h})$ and $D \in \mathcal{A}^0(\mathcal{M}_4, \mathfrak{l})$ are Lagrange multipliers. Note that in precisely 4 spacetime dimensions the Lagrange multiplier D corresponding to $\mathcal{H}$ is a 0-form, i.e. a scalar function. The functionals $\langle\_\,,\_\rangle_{\mathfrak{g}}$, $\langle\_\,,\_\rangle_{\mathfrak{h}}$ and $\langle\_\,,\_\rangle_{\mathfrak{l}}$ are G-invariant bilinear symmetric nondegenerate forms on $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{l}$, respectively. Under certain conditions, the forms $\langle\_\,,\_\rangle_{\mathfrak{h}}$ and $\langle\_\,,\_\rangle_{\mathfrak{l}}$ are also H-invariant and L-invariant.

One can see that varying the action with respect to the variables B^α, C^a and D^A (where indices A count the generators of the group L), one obtains the equations of motion

$$\mathcal{F}^\alpha = 0\,, \qquad \mathcal{G}^a = 0\,, \qquad \mathcal{H}^A = 0\,, \tag{50}$$

while varying with respect to α^α, β^a, γ^A one obtains

$$\mathrm{d}B_\alpha - g_{\alpha\beta}{}^\gamma B_\gamma \wedge \alpha^\beta - \rhd_{\alpha a}{}^b C_b \wedge \beta^a + \rhd_{\alpha B}{}^A D_A \wedge \gamma^B = 0\,, \tag{51}$$

$$\mathrm{d}C_a - \partial_a{}^\alpha B_\alpha + \rhd_{\alpha a}{}^b C_b \wedge \alpha^\alpha + 2X_{\{ab\}}{}^A D_A \wedge \beta^b = 0\,, \tag{52}$$

$$\mathrm{d}D_A - \rhd_{\alpha A}{}^B D_B \wedge \alpha^\alpha + \delta_A{}^a C_a = 0\,. \tag{53}$$

4.2. Klein-Gordon theory

Now we proceed to demonstrate that one can use the 3-group structure and the corresponding $3BF$ theory to describe the Klein-Gordon field coupled to general relativity. We begin by specifying a 2-crossed module, which is used to construct the topological $3BF$ theory, and then we impose appropriate simplicity constraints to obtain the desired equations of motion.

We specify a 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \rhd\,, \{\_\,,\_\})$, as follows. The groups are given as

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}\,. \tag{54}$$

The group G acts on itself via conjugation, on H via the vector representation, and on L via the trivial representation. This specifies the definition of the action $\rhd$. The map ∂ is chosen to be trivial, as before. The map δ is also trivial, that is, every element of L is mapped to the identity element of H. Finally, the Peiffer lifting is trivial as well, mapping every ordered pair of elements in H to an identity element in L. This specifies one concrete 2-crossed module which, as we shall see below, corresponds to gravity and one real scalar field.

Given this choice of a 2-crossed module, the 3-connection $(\alpha\,,\beta\,,\gamma)$ takes the form

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma \mathbb{I}\,, \tag{55}$$

where $\mathbb{I}$ is the sole generator of the Lie group $\mathbb{R}$. Since the homomorphisms ∂ and δ are trivial, as well as the Peiffer lifting, the fake 3-curvature (48) reduces to the ordinary 3-curvature,

$$\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla \beta^a P_a\,, \qquad \mathcal{H} = \mathrm{d}\gamma\,, \tag{56}$$

where we used the fact that G acts trivially on L, that is, $M_{ab} \rhd \mathbb{I} = 0$. This means that the 3-form γ transforms as a scalar with respect to Lorentz symmetry. Consequently, its Lagrange multiplier D also transforms as a scalar, since it also belongs to the algebra $\mathfrak{l}$. Since D is also a 0-form, it transforms as a scalar with respect to diffeomorphisms as well. In other words, D completely behaves as a real scalar field, so we relabel it into more traditional notation, $D \equiv \phi$, and write the pure $3BF$ action (49) as:

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma \,, \tag{57}$$

where the bilinear form for L is $\langle \mathbb{I} \,, \mathbb{I} \rangle_{\mathfrak{l}} = 1$.

The existence of a scalar field in the $3BF$ action is a crucial property of a 3-group in a 4-dimensional spacetime, just like identifying the Lagrange multiplier C^a with a tetrad field e^a was a crucial property of the $2BF$ action and the Poincaré 2-group. We can also see that the choice of the third gauge group, L, dictates the number and the structure of the matter fields present in the action. In this case, $L = \mathbb{R}$ implies that we have only one real scalar field, corresponding to a single generator $\mathbb{I}$ of $\mathbb{R}$. The trivial nature of the action $\rhd$ of $SO(3,1)$ on $\mathbb{R}$ implies that ϕ transforms as a scalar field. Finally, the scalar field appears in the topological sector of the action, making the quantization procedure feasible.

As in the case of BF and $2BF$ theories, we need to add appropriate simplicity constraints to the action (57). In order to obtain the Klein-Gordon field ϕ of mass m coupled to gravity in the standard way, the action takes the form:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + \phi \, \mathrm{d}\gamma \\ & - \lambda_{ab} \wedge \left(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \right) \\ & + \lambda \wedge \left(\gamma - \frac{1}{2} H_{abc} e^a \wedge e^b \wedge e^c \right) \\ & + \Lambda^{ab} \wedge \left(H_{abc} \varepsilon^{cdef} e_d \wedge e_e \wedge e_f - \mathrm{d}\phi \wedge e_a \wedge e_b \right) \\ & - \frac{1}{2 \cdot 4!} m^2 \phi^2 \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d \,. \end{aligned} \tag{58}$$

The first row is the topological sector (57), the second row is the familiar simplicity constraint for gravity from the action (27), the third and fourth rows contain the new simplicity constraints featuring the Lagrange multiplier 1-forms λ and Λ^{ab} and the 0-form H_{abc}, while the fifth row is the mass term for the scalar field.

The variation of (58) with respect to the variables B_{ab}, ω_{ab}, β_a, λ_{ab}, Λ_{ab}, γ, λ, H_{abc}, ϕ and e^a gives us the equations of motion. As before, all variables can be algebraically expressed in terms of the tetrads e^a and the scalar field ϕ:

$$
\begin{gathered}
\lambda_{ab\mu\nu} = R_{ab\mu\nu}\,, \qquad \omega^{ab}{}_{\mu} = \triangle^{ab}{}_{\mu}\,, \qquad \gamma_{\mu\nu\rho} = -\frac{e}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^{\sigma}\phi\,, \\
\beta^{a}{}_{\mu\nu} = 0\,, \qquad \Lambda^{ab}{}_{\mu} = \frac{1}{12e}g_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\partial_{\nu}\phi e^{a}{}_{\rho}e^{b}{}_{\sigma}\,, \qquad \lambda_{\mu} = \partial_{\mu}\phi\,, \\
H^{abc} = \frac{1}{6e}\varepsilon^{\mu\nu\rho\sigma}\partial_{\mu}\phi e^{a}{}_{\nu}e^{b}{}_{\rho}e^{c}{}_{\sigma}\,, \qquad B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^{c}{}_{\mu}e^{d}{}_{\nu}\,.
\end{gathered} \tag{59}
$$

The equations of motion for e^a and ϕ, however, are differential equations. The equation for the scalar field becomes the covariant Klein-Gordon equation,

$$\left(\nabla_{\mu}\nabla^{\mu} - m^2\right)\phi = 0\,, \tag{60}$$

while the equation for the tetrads is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2\, T^{\mu\nu}\,, \tag{61}$$

where

$$T^{\mu\nu} \equiv \partial^{\mu}\phi\partial^{\nu}\phi - \frac{1}{2}g^{\mu\nu}\left(\partial_{\rho}\phi\partial^{\rho}\phi + m^2\phi^2\right) \tag{62}$$

is the stress-energy tensor for a single real scalar field.

4.3. Einstein-Cartan-Dirac theory

In order to describe the Dirac field coupled to Einstein-Cartan gravity, we follow the same procedure as for the case of the scalar field, but now we choose the 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \triangleright\,, \{\_\,,\_\})$ in a different way, as follows. The groups are:

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^8(\mathbb{G})\,, \tag{63}$$

where $\mathbb{G}$ is the algebra of complex Grassmann numbers. The maps ∂, δ and the Peiffer lifting are trivial, as before. The action of the group G on itself is given via conjugation, on H via vector representation, and on L via spinor representation, in the following way. Denoting the 8 generators of the Lie group $\mathbb{R}^8(\mathbb{G})$ as P_α and P^α, where the index α takes the values $1,\dots,4$, the action $\triangleright$ of G on L is thus given explicitly as

$$M_{ab} \triangleright P_{\alpha} = \frac{1}{2}(\sigma_{ab})^{\beta}{}_{\alpha}P_{\beta}\,, \qquad M_{ab} \triangleright P^{\alpha} = -\frac{1}{2}(\sigma_{ab})^{\alpha}{}_{\beta}P^{\beta}\,, \tag{64}$$

where $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, and γ_a are the usual Dirac matrices, satisfying the anticommutation rule $\{\gamma_a\,, \gamma_b\} = -2\eta_{ab}$.

As in the case of the scalar field, the choice of the group L dictates the matter content of the theory, while the action $\triangleright$ of G on L specifies its transformation properties.

Let us now proceed to construct the $3BF$ action. The 3-connection $(\alpha\,,\beta\,,\gamma)$ takes the form

$$\alpha = \omega^{ab} M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma^\alpha P_\alpha + \bar{\gamma}_\alpha P^\alpha\,, \tag{65}$$

while the 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is given as

$$\begin{aligned} &\mathcal{F} = R^{ab} M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,, \\ \mathcal{H} &= \Big(\mathrm{d}\gamma^\alpha + \frac{1}{2}\omega^{ab}(\sigma_{ab})^\alpha{}_\beta\gamma^\beta\Big)P_\alpha + \Big(\mathrm{d}\bar{\gamma}_\alpha - \frac{1}{2}\omega^{ab}\bar{\gamma}_\beta(\sigma_{ab})^\beta{}_\alpha\Big)P^\alpha \\ &\equiv (\overset{\rightarrow}{\nabla}\gamma)^\alpha P_\alpha + (\bar{\gamma}\overset{\leftarrow}{\nabla})_\alpha P^\alpha\,, \end{aligned} \tag{66}$$

where we have used (64). The bilinear form $\langle\_\,,\_\rangle_{\mathfrak{l}}$ is defined via its action on the generators:

$$\begin{aligned} \langle P_\alpha\,, P_\beta\rangle_{\mathfrak{l}} = 0\,, &\qquad \langle P^\alpha\,, P^\beta\rangle_{\mathfrak{l}} = 0\,, \\ \langle P_\alpha\,, P^\beta\rangle_{\mathfrak{l}} = -\delta_\alpha^\beta\,, &\qquad \langle P^\alpha\,, P_\beta\rangle_{\mathfrak{l}} = \delta^\alpha_\beta\,. \end{aligned} \tag{67}$$

Note that the bilinear form defined in this way is antisymmetric, rather than symmetric, when it acts on the generators. The reason for this is the following. For general $A, B \in \mathfrak{l}$, we want the bilinear form to be symmetric. Expanding A and B into components, we can write

$$\langle A\,, B\rangle_{\mathfrak{l}} = A^I B^J g_{IJ}\,, \qquad \langle B\,, A\rangle_{\mathfrak{l}} = B^J A^I g_{JI}\,. \tag{68}$$

Since we require the bilinear form to be symmetric, the two expressions must be equal. However, since the coefficients in $\mathfrak{l}$ are Grassmann numbers, we have $A^I B^J = -B^J A^I$, so it follows that $g_{IJ} = -g_{JI}$. Hence the antisymmetry of (67) — it compensates for the anticommutativity property of the Grassman coefficients, making the bilinear form symmetric for general algebra elements $A, B \in \mathfrak{l}$.

Now we employ the action $\triangleright$ of G on L to determine the transformation properties of the Lagrange multiplier D in (49). Indeed, the choice of the group L dictates that D contains 8 independent complex Grassmannian matter fields as its components. Moreover, due to the fact that D is a 0-form and that it transforms according to the spinorial representation of $SO(3,1)$, we can identify its components with the Dirac bispinor fields, and write

$$D = \psi^\alpha P_\alpha + \bar{\psi}_\alpha P^\alpha\,. \tag{69}$$

This is again an illustration of the fact that information about the structure of the matter sector in the theory is specified by the choice of the group L

in the 2-crossed module, and its transformation properties with respect to the Lorentz group are fixed by the action $\triangleright$.

Given all of the above, we write the corresponding pure $3BF$ action as:

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + (\bar{\gamma} \overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla} \gamma)^\alpha \,. \tag{70}$$

In order to obtain the action that gives us the dynamics of Einstein-Cartan theory of gravity coupled to a Dirac field, we add the following simplicity constraints:

$$\begin{aligned} S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla \beta^a + (\bar{\gamma} \overleftarrow{\nabla})_\alpha \psi^\alpha + \bar{\psi}_\alpha (\overrightarrow{\nabla} \gamma)^\alpha \\ & - \lambda_{ab} \wedge \Big(B^{ab} - \frac{1}{16\pi l_p^2} \varepsilon^{abcd} e_c \wedge e_d \Big) \\ & - \lambda^\alpha \wedge \Big(\bar{\gamma}_\alpha - \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\bar{\psi} \gamma^d)_\alpha \Big) \\ & + \bar{\lambda}_\alpha \wedge \Big(\gamma^\alpha + \frac{i}{6} \varepsilon_{abcd} e^a \wedge e^b \wedge e^c (\gamma^d \psi)^\alpha \Big) \\ & - \frac{1}{12} m \, \bar{\psi} \psi \, \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d + 2\pi i l_p^2 \, \bar{\psi} \gamma_5 \gamma^a \psi \, \varepsilon_{abcd} e^b \wedge e^c \wedge \beta^d . \end{aligned} \tag{71}$$

Similarly to the previous case of the scalar field, we recognize the topological sector in the first row, the gravitational simplicity constraint in the second row, while the third and fourth rows contain the new simplicity constraints for the Dirac field, featuring the Lagrange multiplier 1-forms λ^α and $\bar{\lambda}_\alpha$. The fifth row contains the mass term for the Dirac field, and a term which ensures the correct coupling between the torsion and the spin of the Dirac field. In particular, we want to obtain

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a \,, \tag{72}$$

as one of the equations of motion, where

$$s_a = i \varepsilon_{abcd} e^b \wedge e^c \bar{\psi} \gamma_5 \gamma^d \psi \tag{73}$$

is the Dirac spin 2-form. Of course, other alternative coupling choices are possible, but we choose this one since this is the traditional coupling most often discussed in textbooks.

The variation of the action (71) with respect to B_{ab}, λ^{ab}, $\bar{\gamma}_\alpha$, γ^α, λ^α, $\bar{\lambda}_\alpha$, $\bar{\psi}_\alpha$, ψ^α, e^a, β^a and ω^{ab}, again gives us equations of motion, which can be algebraically solved for all fields as functions of e^a, ψ and $\bar{\psi}$:

$$B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd}e^c{}_\mu e^d{}_\nu \,, \qquad \lambda^\alpha{}_\mu = (\overrightarrow{\nabla}_\mu\psi)^\alpha \,, \qquad \bar{\lambda}_{\alpha\mu} = (\bar{\psi}\overleftarrow{\nabla}_\mu)_\alpha \,,$$
$$\bar{\gamma}_{\alpha\mu\nu\rho} = i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\bar{\psi}\gamma^d)_\alpha \,, \qquad \gamma^\alpha{}_{\mu\nu\rho} = -i\varepsilon_{abcd}e^a{}_\mu e^b{}_\nu e^c{}_\rho(\gamma^d\psi)^\alpha \,,$$
$$\beta^a{}_{\mu\nu} = 0 \,, \qquad \lambda_{ab\mu\nu} = R_{ab\mu\nu} \,, \qquad \omega^{ab}{}_\mu = \triangle^{ab}{}_\mu + K^{ab}{}_\mu \,. \tag{74}$$

Here $K^{ab}{}_\mu$ is the contorsion tensor, constructed in the standard way from the torsion tensor. In addition, we also obtain

$$T_a \equiv \nabla e_a = 2\pi l_p^2 s_a \,, \tag{75}$$

which is precisely the desired equation (72) for the torsion. Finally, the differential equations of motion for ψ and $\bar{\psi}$ are the standard covariant Dirac equation,

$$(i\gamma^a e^\mu{}_a \overrightarrow{\nabla}_\mu - m)\psi = 0 \,, \tag{76}$$

and its conjugate,

$$\bar{\psi}(i\overleftarrow{\nabla}_\mu e^\mu{}_a \gamma^a + m) = 0 \,, \tag{77}$$

where $e^\mu{}_a$ is the inverse tetrad. The differential equation of motion for e^a is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2 \, T^{\mu\nu} \,, \tag{78}$$

where

$$T^{\mu\nu} \equiv \frac{i}{2}\bar{\psi}\gamma^a\overleftrightarrow{\nabla}^\nu e^\mu{}_a\psi - \frac{1}{2}g^{\mu\nu}\bar{\psi}\Big(i\gamma^a\overleftrightarrow{\nabla}_\rho e^\rho{}_a - 2m\Big)\psi \,, \tag{79}$$

Here, we used the notation $\overleftrightarrow{\nabla} = \overrightarrow{\nabla} - \overleftarrow{\nabla}$. As expected, the equations of motion (75), (76), (77) and (78) are precisely the equations of motion of the Einstein-Cartan-Dirac theory.

4.4. Weyl and Majorana fields coupled to Einstein-Cartan gravity

As is well known, the Dirac fermions are not an irreducible representation of the Lorentz group, and one can rewrite them as left-chiral and right-chiral irreducible Weyl fermion fields. Hence, it is useful to construct the 2-crossed module and a constrained $3BF$ action for left and right Weyl spinors. For simplicity, we will discuss only the left-chiral spinor field (the right-chiral can be studied analogously). Additionally, we can also describe Majorana fermions using the same formalism, the only difference being the presence of an additional mass term in the Majorana action.

We soecify a 2-crossed module $(L \stackrel{\delta}{\to} H \stackrel{\partial}{\to} G\,, \rhd\,, \{\_\,,\_\})$, in a way similar to the Dirac case, as follows. The groups are:

$$G = SO(3,1)\,, \qquad H = \mathbb{R}^4\,, \qquad L = \mathbb{R}^4(\mathbb{G})\,. \tag{80}$$

The maps ∂, δ and the Peiffer lifting are trivial. The action $\rhd$ of the group G on G, H and L is given in the same way as for the Dirac case, whereas the spinorial representation reduces to

$$M_{ab} \rhd P^\alpha = \frac{1}{2}(\sigma_{ab})^\alpha{}_\beta P^\beta\,, \qquad M_{ab} \rhd P_{\dot\alpha} = \frac{1}{2}(\bar\sigma_{ab})^{\dot\beta}{}_{\dot\alpha} P_{\dot\beta}\,, \tag{81}$$

where $\sigma^{ab} = -\bar\sigma^{ab} = \frac{1}{4}(\sigma^a\bar\sigma^b - \sigma^b\bar\sigma^a)$, for $\sigma^a = (1,\vec\sigma)$ and $\bar\sigma^a = (1,-\vec\sigma)$, in which $\vec\sigma$ denotes the set of three Pauli matrices. The four generators of the group L are denoted as P^α and $P_{\dot\alpha}$, where the Weyl indices $\alpha, \dot\alpha$ take values $1,2$.

The 3-connection $(\alpha\,,\beta\,,\gamma)$ takes the form

$$\alpha = \omega^{ab}M_{ab}\,, \qquad \beta = \beta^a P_a\,, \qquad \gamma = \gamma_\alpha P^\alpha + \bar\gamma^{\dot\alpha} P_{\dot\alpha}\,, \tag{82}$$

while the 3-curvature $(\mathcal{F}\,,\mathcal{G}\,,\mathcal{H})$ is

$$\begin{aligned} &\mathcal{F} = R^{ab}M_{ab}\,, \qquad \mathcal{G} = \nabla\beta^a P_a\,, \\ \mathcal{H} &= \big(\mathrm{d}\gamma_\alpha + \frac{1}{2}\omega^{ab}(\sigma^{ab})^\beta{}_\alpha\gamma_\beta\big)P^\alpha + \big(\mathrm{d}\bar\gamma^{\dot\alpha} + \frac{1}{2}\omega_{ab}(\bar\sigma^{ab})^{\dot\alpha}{}_{\dot\beta}\bar\gamma^{\dot\beta}\big)P_{\dot\alpha} \\ &\equiv (\overset{\rightarrow}{\nabla}\gamma)_\alpha P^\alpha + (\bar\gamma\overset{\leftarrow}{\nabla})^{\dot\alpha}P_{\dot\alpha}\,. \end{aligned} \tag{83}$$

The Lagrange multiplier D now contains as coefficients the spinor fields ψ_α and $\bar\psi^{\dot\alpha}$,

$$D = \psi_\alpha P^\alpha + \bar\psi^{\dot\alpha}P_{\dot\alpha}\,, \tag{84}$$

and the bilinear form $\langle\_\,,\_\rangle_{\mathfrak{l}}$ for the group L is

$$\begin{aligned} \langle P^\alpha\,, P^\beta\rangle_{\mathfrak{l}} = \varepsilon^{\alpha\beta}\,, \qquad \langle P_{\dot\alpha}\,, P_{\dot\beta}\rangle_{\mathfrak{l}} = \varepsilon_{\dot\alpha\dot\beta}\,, \\ \langle P^\alpha\,, P_{\dot\beta}\rangle_{\mathfrak{l}} = 0\,, \qquad \langle P_{\dot\alpha}\,, P^\beta\rangle_{\mathfrak{l}} = 0\,, \end{aligned} \tag{85}$$

where $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot\alpha\dot\beta}$ are the usual two-dimensional antisymmetric Levi-Civita symbols.

The pure $3BF$ action (49) now becomes

$$S_{3BF} = \int_{\mathcal{M}_4} B^{ab}\wedge R_{ab} + e_a \wedge \nabla\beta^a + \psi^\alpha \wedge (\overset{\rightarrow}{\nabla}\gamma)_\alpha + \bar\psi_{\dot\alpha}\wedge(\bar\gamma\overset{\leftarrow}{\nabla})^{\dot\alpha}\,. \tag{86}$$

In order to obtain the suitable equations of motion for the Weyl spinors, we again introduce appropriate simplicity constraints, to obtain:

$$
\begin{aligned}
S = \int_{\mathcal{M}_4} & B^{ab} \wedge R_{ab} + e_a \wedge \nabla\beta^a + \psi^\alpha \wedge (\overrightarrow{\nabla}\gamma)_\alpha + \bar{\psi}_{\dot\alpha} \wedge (\bar\gamma\overleftarrow{\nabla})^{\dot\alpha} \\
& - \lambda_{ab} \wedge (B^{ab} - \frac{1}{16\pi l_p^2}\varepsilon^{abcd} e_c \wedge e_d) \\
& - \lambda^\alpha \wedge (\gamma_\alpha + \frac{i}{6}\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \sigma^d{}_{\alpha\dot\beta}\bar\psi^{\dot\beta}) \\
& - \bar\lambda_{\dot\alpha} \wedge (\bar\gamma^{\dot\alpha} + \frac{i}{6}\varepsilon_{abcd} e^a \wedge e^b \wedge e^c \bar\sigma^{d\dot\alpha\beta}\psi_\beta) \\
& - 4\pi l_p^2 \varepsilon_{abcd} e^a \wedge e^b \wedge \beta^c (\bar\psi_{\dot\alpha}\bar\sigma^{d\dot\alpha\beta}\psi_\beta)\,.
\end{aligned}
\tag{87}
$$

The new simplicity constraints, in the third and fourth rows, feature the Lagrange multiplier 1-forms λ_α and $\bar\lambda^{\dot\alpha}$. Also, in analogy to the coupling between the spin and the torsion in Einstein-Cartan-Dirac theory, the term in the fifth row is chosen to ensure that the coupling between the Weyl spin tensor

$$
s_a \equiv i\varepsilon_{abcd} e^b \wedge e^c \; \psi^\alpha \sigma^d{}_{\alpha\dot\beta}\bar\psi^{\dot\beta} \tag{88}
$$

and torsion is given as:

$$
T_a = 4\pi l_p^2 s_a\,. \tag{89}
$$

The action for the Majorana field is precisely the same, but for an additional mass term in the action:

$$
-\frac{1}{12} m \varepsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d (\psi^\alpha\psi_\alpha + \bar\psi_{\dot\alpha}\bar\psi^{\dot\alpha})\,. \tag{90}
$$

The variation of the action (87) with respect to the variables B_{ab}, λ^{ab}, γ_α, $\bar\gamma^{\dot\alpha}$, λ_α, $\bar\lambda^{\dot\alpha}$, ψ_α, $\bar\psi^{\dot\alpha}$, e^a, β^a and ω^{ab} gives us the equations of motion, which can be algebraically solved for all variables as functions of ψ_α, $\bar\psi^{\dot\alpha}$ and e^a:

$$
\begin{gathered}
\beta^a{}_{\mu\nu} = 0\,, \quad \lambda^{ab}{}_{\mu\nu} = R^{ab}{}_{\mu\nu}\,, \quad \lambda_{\alpha\mu} = \nabla_\mu\psi_\alpha\,, \quad \bar\lambda^{\dot\alpha}{}_\mu = \nabla_\mu\bar\psi^{\dot\alpha}\,, \\
B_{ab\mu\nu} = \frac{1}{8\pi l_p^2}\varepsilon_{abcd} e^c{}_\mu e^d{}_\nu\,, \quad \omega_{ab\mu} = \triangle_{ab\mu} + K_{ab\mu}\,, \\
\gamma_{\alpha\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \sigma^d{}_{\alpha\dot\beta}\bar\psi^{\dot\beta}\,, \quad \bar\gamma^{\dot\alpha}{}_{\mu\nu\rho} = i\varepsilon_{abcd} e^a{}_\mu e^b{}_\nu e^c{}_\rho \bar\sigma^{d\dot\alpha\beta}\psi_\beta\,.
\end{gathered}
\tag{91}
$$

In addition, one also obtains (89). Finally, the differential equations of motion for the spinor and tetrad fields are

$$
\bar\sigma^{a\dot\alpha\beta} e^\mu{}_a \nabla_\mu\psi_\beta = 0\,, \qquad \sigma^a{}_{\alpha\dot\beta} e^\mu{}_a \nabla_\mu\bar\psi^{\dot\beta} = 0\,, \tag{92}
$$

and

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = 8\pi l_p^2 \, T^{\mu\nu} \,, \tag{93}$$

where

$$\begin{aligned} T^{\mu\nu} \;\; \equiv \;\; & \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} \\ & -\frac{1}{2}g^{\mu\nu}\Big(i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi} \Big) \,. \end{aligned} \tag{94}$$

Here we have suppressed the spinor indices, for simplicity. In the case of the Majorana field, the equations of motion (91) remain the same. The equations of motion for ψ_α and $\bar{\psi}^{\dot{\alpha}}$ obtain the additional mass term,

$$i\sigma^a{}_{\alpha\dot{\beta}} e^\mu{}_a \nabla_\mu \bar{\psi}^{\dot{\beta}} - m\psi_\alpha = 0 \,, \qquad i\bar{\sigma}^{a\dot{\alpha}\beta} e^\mu{}_a \nabla_\mu \psi_\beta - m\bar{\psi}^{\dot{\alpha}} = 0 \,, \tag{95}$$

while the stress-energy tensor becomes

$$\begin{aligned} T^{\mu\nu} \;\; \equiv \;\; & \frac{i}{2}\bar{\psi}\bar{\sigma}^b e^\nu{}_b \nabla^\mu \psi + \frac{i}{2}\psi\sigma^b e^\nu{}_b \nabla^\mu \bar{\psi} \\ & -g^{\mu\nu}\frac{1}{2}\left[i\bar{\psi}\bar{\sigma}^a e^\lambda{}_a \nabla_\lambda \psi + i\psi\sigma^a e^\lambda{}_a \nabla_\lambda \bar{\psi} - \frac{1}{2}m\left(\psi\psi + \bar{\psi}\bar{\psi}\right) \right] \,. \end{aligned} \tag{96}$$

5. Conclusions

Let us summarize the results of the paper. In Section 2 we have introduced the BF theory and discussed models based on constrained BF action, in particular the Yang-Mills theory in Minkowski spacetime and the Plebanski formulation of general relativity. Section 3 was devoted to the first step in the categorical ladder and the $2BF$ theory. After introducing the notions of a 2-group, a crossed module, and the corresponding $2BF$ theory, we have studied the $2BF$ formulation of general relativity and the Einstein-Yang-Mills theory. Then, in Section 4 we have performed one more step in the categorical ladder, and introduced the notions of a 3-group, 2-crossed module, and the $3BF$ theory. This structure was employed to construct the constrained $3BF$ actions for the cases of Klein-Gordon, Dirac, Weyl and Majorana fields, each coupled to the Einstein-Cartan gravity in the standard way. In those descriptions, it turned out that the scalar and fermion fields are associated to a *new gauge group*, similar to the gauge fields being associated to a gauge group in the Yang-Mills theory. This opens up a possibility of a classification of matter fields based on an algebraic structure of a 3-group.

All the obtained results serve to complete the first step of the spinfoam quantization programme, as outlined in the Introduction. This paves the way to the study of steps 2 and 3 of the programme. Namely, the full action for gravity, gauge fields and matter is written completely in the langulage of differential forms, which can be easily adapted to a triangulated spacetime manifold, in the sense of Regge calculus. This can be seen in the following table:

d	triangulation	dual triangulation	form	fields	field strengths
0	vertex	4-polytope	0-form	ϕ, $\psi_{\tilde{\alpha}}$, $\bar{\psi}^{\tilde{\alpha}}$	
1	edge	3-polyhedron	1-form	ω^{ab}, A^I, e^a	
2	triangle	face	2-form	β^a, B^{ab}	R^{ab}, F^I, T^a
3	tetrahedron	edge	3-form	γ, $\gamma_{\tilde{\alpha}}$, $\bar{\gamma}^{\tilde{\alpha}}$	$\mathcal{G}^a$
4	4-simplex	vertex	4-form		$\mathcal{H}$, $\mathcal{H}_{\tilde{\alpha}}$, $\bar{\mathcal{H}}^{\tilde{\alpha}}$

This data can be utilized to construct a Regge-discretized topological $3BF$ action, and from that a state sum Z, giving rise to a rigorous definition of the path integral

$$Z = \int \mathcal{D}g \int \mathcal{D}\phi \; e^{iS[g,\phi]} \,, \tag{97}$$

which is a generalization of (1) in the sense that it adds matter fields (including the gauge boson sector) to gravity at the quantum level. Being a topological theory, and given the underlying structure of the 3-group, a pure $3BF$ action ought to ensure the topological invariance of the state sum Z, i.e., Z should be triangulation independent. This step, however, requires the generalizations of the Peter-Weyl and Plancharel theorems to 2-groups and 3-groups, which are unfortunately still missing (though there are some attempts to circumvent them at least in the 2-group case [27, 28]). Namely, the purpose of the Peter-Weyl and Plancharel theorems is to provide a decomposition of a function on a group into a sum over the corresponding irreducible representations, which then specifies the spectrum of labels for the simplices in the triangulation, and fixes the domain of values for the fields living on those simplices. In the absence of the two theorems, one can still try to *guess* the irreducible representations of the 2- and 3-groups, as was done for example in the *spincube model* of quantum gravity [12], or to try to construct the state sum using other techniques, as was done in [27, 28]).

Of course, when building a realistic theory, we are not interested in a topological theory, but instead in one which contains local propagating degrees of freedom. Thus the state sum Z need not be a topological invariant. This is obtained via the step 3 of the spinfoam quantization programme, by imposing the simplicity constraints on Z. The classical actions discussed in this paper manifestly distinguish the topological sector from the simplicity constraints, which have been explicitly determined. Imposing them should thus be a straightforward procedure for a given Z. Completing this programme would ultimately lead us to a tentative state sum describing both gravity and matter at a quantum level, which is a topic for future research.

In addition to the construction of a full quantum theory of gravity, there are also many additional possible studies of the classical constrained $3BF$ action. For example, a Hamiltonian analysis of the theory could be interesting for the canonical quantization programme, and some work has begun in this area [29]. Also, it is worth looking into the idea of imposing the simplicity constraints using a spontaneous symmetry breaking mechanism. Finally, one can also study in more depth the mathematical structure and properties of the simplicity constraints. The list is not conclusive, and there may be many other interesting topics to study.

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Република Србија
Универзитет у Београду
Физички факултет
Д.Бр.2017/8009
Датум: 25.10.2017. године

На основу члана 161 Закона о општем управном поступку и службене евиденције издаје се

УВЕРЕЊЕ

Раденковић (Саша) Тијана, бр. индекса 2017/8009, рођена 21.03.1992. године, Београд, Београд-Савски Венац, Република Србија, уписана школске 2017/2018. године, у статусу: финансирање из буџета; тип студија: докторске академске студије; студијски програм: Физика.

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Д.Бр.2017/8009
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Диплома

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Диплома

Тијана, Саша, Раденковић

рођена 21. марта 1992. године, Београд, Република Србија, уписана школске 2016/2017. године, а дана 27. септембра 2017. године завршила је мастер академске студије, другог степена, на студијском програму Теоријска и експериментална физика, обима 60 (шездесет) бодова ЕСПБ са просечном оценом 9,33 (девет и 33/100).

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Проф. др Иван Белча

Ректор
Проф. др Иванка Поповић

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ДОКТОРСКЕ СТУДИЈЕ

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Школска година 2019/2020

Подаци о студенту

Име: TIJANA

Презиме: RADENKOVIĆ

Број индекса: 8009/2017

Научна област дисертације: KVANTNA POLJA, ČESTICE I GRAVITACIJA

Подаци о ментору докторске дисертације

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Научна област: KVANTNA GRAVITACIJA

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Институција: INSTITUT ZA FIZIKU

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Наслов

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Потпис ментора: Марко Војиновић

Датум	27.1.2020.	Потпис студента	Тијана Раденковић

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Датум 01.07.2020.

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[signature]