Научном већу
Института за физику
Универзитета у Београду

Молба

Молим Научно веће Института за физику, Универзитета у Београду да покрене поступак мог избора у звање вишег научног сарадника.

Прилажем:

1. Мишљење руководиоца пројекта
2. Биографију
3. Опис научног рада
4. Квалитативне и квантитативне елементе за анализу
5. Списак радова
6. Копије радова
7. Цитираност

Љубица Давидовић

др Љубица Давидовић
научни сарадник

Naučnom veću Instituta za fiziku

Mišljenje rukovodioca projekta

Dr Ljubica Davidović je saradnik projekta ON171031 "Fizičke implikacije modifikovanog prostor-vremena". Zaposlena je u Institutu za fiziku Univerziteta u Beogradu od 2005. kao stipendista Ministarstva nauke, od 2006. kao istraživač pripravnik. Posle doktoriranja 2014. godine, 2015. je izabrana u zvanje naučni saradnik.

Dr Ljubica Davidović se u okviru projekta ON171031 bavi teorijom struna odnosno matematičkom fizikom, i iz te oblasti je u periodu od izbora za naučnog saradnika publikovala, u vodećim časopisima u oblasti (JHEP i EPJC), 5 radova, dok je još 1 rad na recenziji. U svom istraživačkom radu na projektu dr Davidović pokazuje visok nivo samostalnosti, što se izmedju ostalog vidi iz činjenice da je bila mentor master rada Ilije Ivaniševića a trenutno vodi njegovu doktorsku disertaciju. Osim tema iz teorije struna, dr Davidović se u okviru bilateralnog projekta SANU i Ruske akademije nauka "Fundamentalna i primenjena istraživanja u oblasti kvantne teorije informacija i kvantnog računanja" bavi temama iz kvantne mehanike, iz kojih je od prethodnog izbora objavila 4 rada; spisku publikovanih radova u ovom periodu treba dodati i 4 saopštenja sa medjunarodnih konferencija.

Osim uspešne medjunarodne saradnje na bilateralnom projektu, dr Ljubica Davidović je u više navrata bila član organizacionog komiteta medjunarodne Škole i konferencije moderne matematičke fizike MPHYS koju je ove godine Institut za fiziku organizovao deseti put. Dr Davidović predaje u odeljenju za talentovane učenike Zemunske gimnazije, a iz predmeta Elektromagnetizam i optika koji drži napisala je udžbenik.

Na osnovu gore navedenog, kao i na osnovu ličnog uvida u kvalitete kandidatkinje i značaj njenog učešća i rada na projektu ON171031, **predlažem Naučnom veću Instituta za fiziku da pokrene postupak za izbor dr Ljubice Davidović u zvanje viši naučni saradnik**. Za komisiju za izbor predlažem dr Branislava Cvetkovića (naučni savetnik, Institut za fiziku), dr Bojana Nikolića (viši naučni saradnik, Institut za fiziku), i prof. Voju Radovanovića (redovni profesor, Fizički fakultet).

Beograd, 29. oktobra 2019.

Sa poštovanjem,

Maja Burić

Prof. dr Maja Burić

rukovodilac projekta ON 171031 MPNTR Srbije

Биографија

Љубица Давидовић рођена је 17. децембра 1980. у Београду, где је завршила основну школу и Математичку гимназију, обе као као носилац Вукове дипломе. Завршила је Физички факултет Универзитета у Београду, смер теоријска физика, са просечном оценом **9,59**. Дипломирала је маја 2005., одбранивши дипломски рад под називом

„*Особине скаларног поља на некомутативном простору*"

са оценом **десет**. Рад је награђен наградом фонда „Проф. Др Љубомир Ћирковић", за најбољи дипломски рад одбрањен у школској 2004/2005. Награђена је стипендијом Норвешке владе (2002), као један од 500 најбољих студената у Србији.

Од августа 2005., прво као стипендиста Министарства науке а касније (април 2006.) као истраживач, ради на Институту за физику. Академски назив мастера стекла је 2007. године. Звање истраживач сарадник стекла је 2011. Просечна оцена на докторским студијама, смер Физика језгара, честица и поља јој је **десет**. Докторску дисертацију, рађену под руководством проф. др Бранислава Саздовића, под називом

„*Дирихлеова р-брана у слабо закривљеном простору*"

одбранила је марта 2014.. Звање научни сарадник стекла је 2015. године. Исте године Љубица Давидовић је добила националну стипендију L'Oreal- UNESCO за жене у науци. Љубица Давидовић је била рецензент за часописе Класична и квантна гравитација (Classical and Quantum Gravity) и Часопис за физику А:математичку и теоријску(Journal of Physics A: Mathematical and Theoretical).

Љубица Давидовић била је члан два национална пројекта:

1. од 2006. до 2009. пројекта под називом

   „ Моделовање и нумеричке симулације комплексних физичких система", ОИ 141035 и

2. од 2009. пројекта под називом

   „Физичке последице модификованог простор-времена", ОИ 171031.

Она је такође члан билатералног пројекта између Српске академије наука и уметности и Руске академије наука, који се реализује на Институту за физику и Институту за нуклеарне науке "Винча" и Грађевинском факултету у Србији и Институту за физику „Лебедев" у Русији. Назив пројекта је

„ Фундаментална и примењена истраживања у области квантне тероије информација и квантног рачунања".

Од 2009. Љубица Давидовић је члан локалног организационог комитета серија конференција под називом

,, Скуп математичке физике: школа и конференција о модерној математичкој физици".

Љубица Давидовић је била ментор за једну мастер тезу и тренутно је ментор за докторску дисертацију. Од 2016., Љубица Давидовић предаје "Електромагнетизам и оптику" у Земунској гимназији. Програм је намењен ученицима талентованим за физику. Она је аутор уџбеника за тај предмет који је тренутно на рецензији.

Преглед научне активности

У оквирима националног пројекта Министраства просвете, науке и технолошког развоја ОІ71031 под називом

„Физичке последице модификованог простор-времена”

испитивана је примена уопштене процедуре Т-дуализације која је реализована у докторској дисертацији Љубице Давидовић. Т-дуалност је симетрија која је први пут уочена у теорији струна и има значајну улогу у обједињењу наизглед различитих теорија. Уопштена процедура развијена је за бозонску струну која се креће у координатно зависним позадинским пољима, и то за слабо закривљено простор-време а омогућила је Т-дуализацију дуж произвољне координате независно од тога да ли се она појављује као аргумент позадинских поља или не. У оквирима докторске дисертацији је примењена на све координате простор-времена, чиме је добијена потпуно Т-дуализована теорија.

У даљем истраживању испитано је како су повезане теорије које се добијају парцијалном Т-дуализацијом, тј. применом Т-дуализационе процедуре на подскуп координата. Добијен је Т-дуализациони дијаграм који повезује почетну теорију са парцијално Т-дуализованом, добијеном Т-дуализацијом по произвољно изабраним координатама и потпуно Т-дуализованом теоријом. Простори на којима су ове теорије дефинисане су различити. Док је почетна теорија геометријска тј. дефинисана преко координата струне, све остале теорије су негеометријске и нелокалне. За њихов опис је поред координата струне потребно увести и њене дупле координате, а позадинска поља ових теорија зависе и од једних и од других. Добијени су Т-дуални закони трансформација координата, који повезују координате било које две теорије у дијаграму. Ови закони омогућили су испитивање облика Поасонових заграда у Т-дуалним просторима, са претпоставком да почетне координате задовољавају стандардне Поасонове заграде. Добијено је да је комутативна почетна теорија еквивалентна некомутативној потпуно Т-дуалној теорији. Сви параметри некомутативности су инфинитезимални и пропорционални јачини поља Калб-Рамоновог поља. Добијене су и алгебре Т-дуалних бројева намотаја и импулса преко истих бројева почетне теорије.

Истраживање је настављено разматрањем компликованијих позадинских поља. Разматрано је слабо закривљено простор-време другог реда у коме поред координатно зависног Калб-Рамоновог поља и метрика има координатно зависни члан квадратичан по јачини поља Калб-Рамоновог поља. У том случају Ричијев тензор је различит од нуле. Како позадинска поља немају глобалну симетрију помераја уопштена процедура Т-дуализације није примењива. Зато је та процедуре предефинисана тако да обухвати и случајве када се струна креће у просторима без глобалне симетрије.

Уопштена процедура Т-дуализације која је дефинисана за затворену бозонску струну искоришћена је за испитивање Т-дуалности отворене бозонске струне. Наиме, решавањем граничних услова отворене струне добија се ефективна теорија затворене струне. Стога се теорије почетне отворене струне и ефективне затворене струне могу сматрати еквивалентним. Проналажењем Т-дуалне ефективне теорије затворене струне по свим њеним координатама, могуће је одредити и теорију отворене струне чија је ефективна теорија управо добијена Т-дуална теорија. На тај начин добија се Т-дуална теорија почетне отворене струне.

Испитали смо и да ли су трансформације симетрије бозонске струне повезане Т-дуалношћу. Разматрали смо стандардну теорију затворене струне као и модификовану теорију отворене струне, модификовану тако да задржи исте симетрије које поседује затворена струна. Како је теорија струна конформно инваријантна теорија поља на светској површи, да би се одредиле трансформације коју очувавају законе физике, мора се наметнути изоморфизам између конформних теорија почетне и трансформисане конфигурације. Нашли смо трансформације симетрије које одговарају трансформацијама сличности тензора енергије-импулса, и показали да су њихови генератори Т-дуални. Посебно, показали смо да су опште координатне трансформације и локалне калибрационе трансформације Т-дуалне, стога смо закључили да Т-дуалност поред добро познатих замена замењује и симетрије теорија.

Разматрали смо и Т-дуализацију отворене струне у константним позадинским пољима која задовољава мешане граничне услове. Уколико одређена координата задовољава Дирихлеов гранични услов тада одговарајућа координата Т-дуалне теорије задовољава Нојманов услов. На граничне услове обе теорије применили смо Дираково процедуру и решили у њој добијене параметарски зависне везе. На решењима су добијене ефективне теорије за које је показано да су такође Т-дуалне.

У оквирима билатералног пројекта са Русијом под називом

"Фундаментална и примењена истраживања у области квантне теорије информација и квантног рачунања",

радови су обухватили следеће теме:

предложен је нови метод за одређивање Хусимијевих симбала, за операторе који су дати као производи произвољног броја оператора координата и импулса са произвољним поретком. За такав оператор се стандардно оператори координата и импулса представљају преко оператора креације и анихилације, па се израз антинормално уређује при чему се добија финални облик симбола. Нашом методом Хусимијев симбол се добија на много једноставнији начин, полазећи директно од експлицитне форме оператора без његове трансформације преко оператора креације и анихилације. Овом методом нашли смо средње вредности неких оператора. Показано је како се Хајзенбергова и Шредингер-Робертсонова релација неодређености, за координату и импулс, трансформишу при трансформацији скалирања $(q;p) \to (\lambda q; \lambda p)$. Дискутовано је о физичком смислу неких стања конструисаних помоћу ове трансформације.

Разматране су Хусимијеве функције $Q(q,p)$, које су квантне квазидистрибуције на фазном простору. Познато је да се при трансформацијама скалирања $(q;p) \to (\lambda q; \lambda p)$ Хусимијева функција било ког физичког стања трансформише у функцију која је такође Хусимијева функција неког физичког стања. Прецизније, доказано је да ако је $Q(q,p)$ Хусимијева функција тада је функција $\lambda^2 Q(\lambda q; \lambda p)$ такође Хусимијева. Стање коме одговара Хусимијева функција $\lambda^2 Q(\lambda q; \lambda p)$ се назива растегнуто стање. Разматрали смо особине растегнутих Фокових стања. Она се могу добити као резултат трансформације скалирања примењене на Фоково стање хармонијског осцилатора. Фокова стања хармонијског осцилатора су чиста стања док су растегнута Фокова стања мешана. Одредили смо експлицитан облик матрица густине растегнутих Фокових стања. Њихова структура се може описати помоћу негативне биномне расподеле. Графички смо приказали дистрибуције негативних биномних коефицијената за различита растегнута Фокова стања и фон Најманову ентропију најједноставнијег Фоковог стања.

Даље смо разматрали трансформације Хусимијевих функција $H(q,p)$ при трансформацијама скалирања, и то Хусимијевих функција произвољних суперпозиција N-фотонских стања хармонијског осцилатора. Развили смо метод који омогућава добијања растегнутих стања у која се ове суперпозиције трансформишу при скалирању. Испитивали смо особине тих растегнутих стања и израчунали експлицитан облик њихових матрица густине. Установили смо да структура матрице густине може бити описана коришћењем негативне биномне расподеле. Нашли смо изразе за енергију и ентропију растегнутих стања и израчунали средње вредности оператора броја стања. Одредили смо облик Хајзенбергове и Шредингер-Робертсонове релације неодређености за растегнута стања.

На крају смо размотрили и линеарни квантни појачавач који се састоји од N_A атома са два нивоа, и посматрали проблем појачања N-фотонског стања. N-фотонско стање одговара N-квантном стању хармонијског осцилатора. Показали смо да процес интеракције електромагнетног поља са атомима може бити повезан са одређеним трансформацијама фазног простора и функцијама дефинисаним на њему. Разматрали смо Хусимијеве функције $Q_N(q,p)$ N-квантног стања хармонијског осцилатора, које су дефинисане на фазном простору, изучавали трансформацију тих функција и нашли експлицитан облик матрице густине појачаног N-фотонског стања.

Елементи за квалитативну анализу

О квалитету научних резултата говори чињеница да је од 13 радова које је Љубица Давидовић објавила од стицања звања научног сарадника пет објављено у врхунским међународним часописима, два у истакнутим међународним часописима, два у међународним часописима. Треба напоменути да је један рад у врхунском међународном часопису Љубица Давидовић објавила самостално. Врхунски часописи у којима су радови објављени су Европски часопис за физику Ц:честице и поља (The European Physical Journal C:Particles and Fields), који је рангиран као шести (од 29) у области физика честица и поља док је Часопис физике високих енергија (Journal of High Energy Physics) рангиран као четврти и у истој области. Четири рада објављена у истакнутим и међународним часописима за физику су рађена у у оквиру билателарне сарадње са Физичким Институтом Лебедев у Москви (Русија).

Љубица Давидовић је редован предавач на међународној конференцији и школи из математичке физике која се одржава сваке две године у нашој земљи, у чијем је организационом одбору. Предавања из 2014.. и 2017. под називом
,,Парцијална Т-дуализација слабо закривљеног простора" ("Partial T-dualization in a weakly curved background") и ,,Симетрије бозонске струне" ("Symmetries of the bosonic string") су штампана у целини у Свескама физичких наука. Такође у Часопису за физику: серија конференције (Journal of Physics: Conference Series) штампано је у целини њено предавање са конференције Интеграбилни системи и квантне симетрије 2015. у Прагу под насловом ,, Т-дуализацица слабо закривљеног простор-времена"("T-dualization of a weakly curved background").

Љубица Давидовић је ментор за израду докторске дисертације Илије Иванишевића, који је положио све испите докторских студија на Физичком факултету на смеру физика језгара, честица и поља и тренутно као истраживач Института за физику ради на докторској дисертацији. Један рад у врхунском међународном часопису је близу публиковања док је други у припреми. Такође предстоји израда конференцијског рада са међународне конференције на којој је Илија Иванишевић презентовао најновије резултате истраживања.

Квантитативна анализа

Од 26. фебруара 2015. године када је стекла звање научног сарадника Љубица Давидовић је објавила тринаест радова који припадају следећим категоријама

Таблица 1.: *Радови по категоријама*

категорија	*број радова*	*вредност рада*	*укупно*
M21	*5*	*8*	*40*
M22	*2*	*5*	*10*
M23	*2*	*3*	*6*
M33	*4*	*1*	*4.*

Распоређени по правилнику вредновања научно истраживачког рада добија се следећи збирни број бодова

Таблица 2.: *Укупан број бодова*

диференцијални услов 50		**60**
обавезни 40	*M10+M20+M31+M32+M33+M41+M42+M90*	**60**
обавезни 30	*M11+M12+M21+M22+M23*	**56**

на основу чега се може видети да су услови за избор у звање вишег научног сарадника задовољени у све три категорије.

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www.mphys9.ipb.ac.rs/proceedings9/Davidovic.pdf ISBN: 978-86-82441-48-9.

Цитираност

Радови Љубице Давидовић су по бази SCOPUS цитирани 93 пута, искључујући самоцитате 46 пута.

Република Србија
МИНИСТАРСТВО ПРОСВЕТЕ,
НАУКЕ И ТЕХНОЛОШКОГ РАЗВОЈА
Комисија за стицање научних звања

Број:660-01-00042/409
26.02.2015. године
Б е о г р а д

На основу члана 22. става 2. члана 70. став 5. Закона о научноистраживачкој делатности ("Службени гласник Републике Србије", број 110/05 и 50/06 – исправка и 18/10), члана 2. става 1. и 2. тачке 1 – 4.(прилози) и члана 38. Правилника о поступку и начину вредновања и квантитативном исказивању научноистраживачких резултата истраживача ("Службени гласник Републике Србије", број 38/08) и захтева који је поднео

Институт за физику у Београду

Комисија за стицање научних звања на седници одржаној 26.02.2015. године, донела је

ОДЛУКУ
О СТИЦАЊУ НАУЧНОГ ЗВАЊА

Др Љубица Давидовић

стиче научно звање

Научни сарадник

у области природно-математичких наука - физика

О Б Р А З Л О Ж Е Њ Е

Институт за физику у Београду

утврдио је предлог број 949/1 од 15.07.2014. године на седници научног већа Института и поднео захтев Комисији за стицање научних звања број 986/1 од 23.07.2014. године за доношење одлуке о испуњености услова за стицање научног звања ***Научни сарадник***.

Комисија за стицање научних звања је по претходно прибављеном позитивном мишљењу Матичног научног одбора за физику на седници одржаној 26.02.2015. године разматрала захтев и утврдила да именована испуњава услове из члана 70. став 5. Закона о научноистраживачкој делатности ("Службени гласник Републике Србије", број 110/05 и 50/06 – исправка и 18/10), члана 2. става 1. и 2. тачке 1 – 4.(прилози) и члана 38. Правилника о поступку и начину вредновања и квантитативном исказивању научноистраживачких резултата истраживача ("Службени гласник Републике Србије", број 38/08) за стицање научног звања ***Научни сарадник***, па је одлучила као у изреци ове одлуке.

Доношењем ове одлуке именована стиче сва права која јој на основу ње по закону припадају.

Одлуку доставити подносиоцу захтева, именованој и архиви Министарства просвете, науке и технолошког развоја у Београду.

ПРЕДСЕДНИК КОМИСИЈЕ
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С. Стошић-Грујичић

ДРЖАВНИ СЕКРЕТАР
Др Александар Белић

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THE EUROPEAN PHYSICAL JOURNAL C

Regular Article - Theoretical Physics

T-duality diagram for a weakly curved background

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Abstract In one of our previous papers we generalized the Buscher T-dualization procedure. Here we will investigate the application of this procedure to the theory of a bosonic string moving in the weakly curved background. We obtain the complete T-dualization diagram, connecting the theories which are the result of the T-dualizations over all possible choices of the coordinates. We distinguish three forms of the T-dual theories: the initial theory, the theory obtained T-dualizing some of the coordinates of the initial theory and the theory obtained T-dualizing all of the initial coordinates. While the initial theory is geometric, all the other theories are non-geometric and additionally non-local. We find the T-dual coordinate transformation laws connecting these theories and show that the set of all T-dualizations forms an Abelian group.

1 Introduction

T-duality is a property of string theory that was not encountered in any point particle theory [1–4]. Its discovery was surprising, because it implies that there exist theories, defined for essentially different geometries of the compactified dimensions, which are physically equivalent. The origin of T-duality is seen in the possibility that, unlike a point particle, the string can wrap around compactified dimensions. But, no matter if one dimension is compactified on a circle of radius R or rather on a circle of radius l_s^2/R, where l_s is the fundamental string length scale, the theory will describe the string with the same physical properties. The investigation of T-duality does not cease to provide interesting new physical implications.

The prescription for obtaining the equivalent T-dual theories is given by the Buscher T-dualization procedure [5,6]. The procedure is applicable along the isometry directions, which allows the investigation of the backgrounds which do not depend on some coordinates. It is found that T-duality transforms geometric backgrounds to the non-geometric backgrounds with Q flux which are locally well defined, and these to different types of non-geometric backgrounds, backgrounds with R flux which are not well defined even locally [7,8]. A similar prescription can be used to obtain fermionic T-duality [9,10]. It is argued that the better understanding of T-duality should be sought for by doubling the coordinates, investigating the theories in which the background fields depend on both the usual space-time coordinates and their doubles [11–14], which would make the T-duality a manifest symmetry.

T-duality enables the investigation of the closed string non-commutativity. The coordinates of the closed string are commutative when the string moves in a constant background. In a 3-dimensional space with the Kalb–Ramond field depending on one of the coordinates, successive T-dualizations along isometry directions lead to a theory with Q flux and the non-commutative coordinates [15–17]. The novelty in the research is the generalized T-dualization procedure, realized in [18], addressing the bosonic string moving in the weakly curved background–constant gravitational field and coordinate dependent Kalb–Ramond field with an infinitesimal field strength. The non-commutativity characteristics of a closed string moving in the weakly curved background was considered in [19].

The generalized procedure is applicable to all the space-time coordinates on which the string backgrounds depend. In Ref. [18], it was first applied to all initial coordinates, which produces a T-dual theory; it was then applied to all the T-dual coordinates and the initial theory was obtained. In this paper, we will investigate the application of the generalized T-dualization procedure to an arbitrary set of coordinates. Let us denote the T-dualization along the direction x^μ by T^μ and

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the T-dualization along dual direction y_μ by T_μ. Choosing d arbitrary directions, we denote

$$\mathcal{T}^a = \circ_{n=1}^{d} T^{\mu_n}, \quad \mathcal{T}^i = \circ_{n=d+1}^{D} T^{\mu_n}, \quad \mathcal{T} = \circ_{n=1}^{D} T^{\mu_n}, \quad (1)$$

$$\mathcal{T}_a = \circ_{n=1}^{d} T_{\mu_n}, \quad \mathcal{T}_i = \circ_{n=d+1}^{D} T_{\mu_n}, \quad \widetilde{\mathcal{T}} = \circ_{n=1}^{D} T_{\mu_n}, \quad (2)$$

where $\mu_n \in (0, 1, \ldots, D-1)$, and $\circ$ denotes the composition of T-dualizations. We will apply T-dualizations (1) to the initial theory, and T-dualizations (2) to its completely T-dual theory (obtained in [18]). We will prove the following composition laws:

$$\mathcal{T}^i \circ \mathcal{T}^a = \mathcal{T}, \quad \mathcal{T}_i \circ \mathcal{T}_a = \widetilde{\mathcal{T}}, \quad \mathcal{T}_a \circ \mathcal{T}^a = 1, \quad (3)$$

where 1 denotes the identical transformation (T-dualization not performed). Therefore, the elements 1, $\mathcal{T}^a$ and $\mathcal{T}_a$, with $d = 1, \ldots, D$, form an Abelian group. We will find the explicit form of the resulting theories and the corresponding T-dual coordinate transformation laws. These results complete the T-dualization diagram connecting all the theories T-dual to the initial theory.

Throughout the whole article (except for Sect. 9) we assume that the Kalb–Ramond field depends on all coordinates. In that case all T-dual theories, except the initial theory, are non-geometric and non-local because they depend on the variable V^μ, which is a line integral of the derivatives of the dual coordinates. To all of these theories there corresponds a flux which is of the same type as the R flux unlike the non-geometric theories with Q flux, which have a local geometric description.

In Sects. 9.1 and 9.2, we present an example of the 3-dimensional torus, T^3 with H-flux, where Kalb–Ramond field depends only on coordinate x^3. Then T-dualizations along the isometry directions x^1 and x^2 lead to geometric background and the T-dualization along x^3 leads to non-geometric background. In Sect. 9.1 putting $D = 3$, $d = 1, 2$ with $B_{\mu\nu}$ depending on x^3 we reproduce the T-duality chain of Refs. [15–17].

In Sect. 9.2 we will compare the results of our paper with those of Ref. [8]. In our manuscript, the background fields' argument, the variable V^μ, incorporates all features of the non-geometric spaces. First, as pointed out in Ref. [8] it "eludes a geometric description even locally" because it is a line integral of the derivative. Second, we obtain non-associativity and breaking of Jacobi identity typical for the so called R-flux backgrounds. In Sect. 9.3 we present example of the 4-dimensional torus T^4 to generalize the case of Ref. [20] to critical surface.

The generalized T-dualization procedure originates from the Buscher T-dualization procedure. The first rule in the prescription is to replace the derivatives with the covariant derivatives. The new point in the prescription is the replacement of the coordinates in the background fields' argument with the invariant coordinates. The invariant coordinates are defined as the line integrals of the covariant derivatives of the original coordinates. Both covariant derivatives and invariant coordinates are defined using the gauge fields. These fields should be nonphysical, so one requires that their field strength should be zero. This is realized by adding the corresponding Lagrange multipliers' terms. As a consequence of the translational symmetry one can fix the coordinates along which the T-dualization is performed and obtain a gauge fixed action. An important cross-way in the T-dualization procedure is determined by the equations of motion of the gauge fixed action. Two equations of motion obtained varying this action are used to direct the procedure either back to the initial action or forward to the T-dual action. For the equation of motion obtained varying the action over the Lagrange multipliers, the gauge fixed action reduces to the initial action. For the equation of motion obtained varying the action over the gauge fields one obtains the T-dual theory. Comparing the solutions for the gauge fields in these two directions, one obtains the T-dual coordinate transformation laws.

2 T-duality in the weakly curved background

Let us consider the closed bosonic string propagating in the background with metric field $G_{\mu\nu}$, Kalb–Ramond field $B_{\mu\nu}$ and a dilaton field Φ, described by the action [3,4]

$$S[x] = \kappa \int_\Sigma d^2\xi \sqrt{-g}\Big[\Big(\frac{1}{2} g^{\alpha\beta} G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}} B_{\mu\nu}(x)\Big) \times \partial_\alpha x^\mu \partial_\beta x^\nu + \frac{1}{4\pi\kappa} \Phi(x) R^{(2)}\Big]. \quad (4)$$

The integration goes over a 2-dimensional world-sheet Σ parametrized by ξ^α ($\xi^0 = \tau$, $\xi^1 = \sigma$), $g_{\alpha\beta}$ is the intrinsic world-sheet metric, $R^{(2)}$ corresponding 2-dimensional scalar curvature, $x^\mu(\xi)$, $\mu = 0, 1, \ldots, D-1$ are the coordinates of the D-dimensional space-time, $\kappa = \frac{1}{2\pi\alpha'}$ with α' being the Regge slope parameter and $\varepsilon^{01} = -1$.

2.1 Weakly curved background

The requirement of the quantum conformal invariance of the world-sheet results in the space-time equations of motion for the background fields. In the lowest order in the slope parameter α' these equations are

$$\begin{aligned}
&R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu^{\ \rho\sigma} + 2 D_\mu \partial_\nu \Phi = 0, \\
&D_\rho B^\rho_{\ \mu\nu} - 2\partial_\rho \Phi B^\rho_{\ \mu\nu} = 0, \\
&4(\partial\Phi)^2 - 4 D_\mu \partial^\mu \Phi + \frac{1}{12} B_{\mu\nu\rho} B^{\mu\nu\rho} \\
&\quad + 4\pi\kappa(D-26)/3 - R = 0. \quad (5)
\end{aligned}$$

Here $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are the Ricci tensor and the covariant derivative with respect to the space-time metric. We will consider one of the simplest coordinate dependent solutions of (5), the weakly curved background. This background was considered in Refs. [21–23], where the influence of the boundary conditions on the non-commutativity of the open bosonic string has been investigated. The same approximation was considered in [16,19] in context of the closed string non-commutativity.

The weakly curved background is defined by

$$G_{\mu\nu}(x) = \text{const},$$
$$B_{\mu\nu}(x) = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho \equiv b_{\mu\nu} + h_{\mu\nu}(x),$$
$$\Phi(x) = \text{const}, \tag{6}$$

with $b_{\mu\nu}, B_{\mu\nu\rho} = \text{const}$. This background is the solution of the space-time equations of motion if the constant $B_{\mu\nu\rho}$ is taken to be infinitesimal and all the calculations are done in the first order in $B_{\mu\nu\rho}$, so that the curvature $R_{\mu\nu}$ can be neglected as the infinitesimal of the second order. Through the whole manuscript (with the exception of Sect. 9) we assume that the background has the topology of D-dimensional torus T^D, where the Kalb–Ramond field depends on all coordinates. In Sects. 9.1 and 9.2 we give an example of the 3-dimensional torus, T^3, with H-flux, where the Kalb–Ramond field depends only on the coordinate x^3, while in Sect. 9.3 we give an example of the 4-dimensional torus T^4 with constant background fields.

The assumption that $B_{\mu\nu\rho}$ is infinitesimal means that we consider the D-dimensional torus so large that for any choice of indices

$$\frac{B_{\mu\nu\rho}}{R_\mu R_\nu R_\rho} \ll 1 \tag{7}$$

holds [16], where R_μ are the radii of the torus. The H-flux background, considered in Refs. [8,16], is of the same type as the weakly curved background. However, this background depends just on x^3 and corresponds to the examples addressed in Sect. 9 of our paper. The background considered in the rest of the article depends on all coordinates.

In this paper we will investigate the T-dualization properties of the action (4) describing the closed string moving in the weakly curved background. Taking the conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$, the action (4) becomes

$$S[x] = \kappa \int_\Sigma d^2\xi\ \partial_+ x^\mu \Pi_{+\mu\nu}(x)\partial_- x^\nu, \tag{8}$$

with the background field composition equal to

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2}G_{\mu\nu}(x), \tag{9}$$

and the light-cone coordinates given by

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma), \qquad \partial_\pm = \partial_\tau \pm \partial_\sigma. \tag{10}$$

2.2 Complete T-dualization

The T-dualization of the closed string theory in the weakly curved background was presented in [18]. The procedure is related to a global symmetry of the theory

$$\delta x^\mu = \lambda^\mu. \tag{11}$$

The symmetry still exists in the presence of the nontrivial Kalb–Ramond field (6), but only in the case of the trivial mapping of the world-sheet into the space-time, because in that case the variation of the action (8)

$$\delta S = \frac{\kappa}{3}\varepsilon^{\alpha\beta}B_{\mu\nu\rho}\lambda^\rho \int d^2\xi\, \partial_\alpha x^\mu \partial_\beta x^\nu \tag{12}$$

after partial integration, using the identity $\varepsilon^{\alpha\beta}\partial_\alpha\partial_\beta = 0$, becomes

$$\delta S = \frac{\kappa}{3}B_{\mu\nu\rho}\lambda^\rho \epsilon^{\alpha\beta} \int d^2\xi\, \partial_\alpha(x^\mu \partial_\beta x^\nu), \tag{13}$$

which is equal to zero. This means that classically, directions which appear in the argument of Kalb–Ramond field are also Killing directions. However, the standard Buscher procedure cannot be applied to them, because background fields depend on the coordinates but not on their derivatives.

The T-dual picture of the theory, obtained on applying the T-dualization procedure to all the coordinates, is given by

$$S[y] = \kappa \int d^2\xi\ \partial_+ y_\mu\, {}^\star\Pi_+^{\mu\nu}\big(\Delta V(y)\big)\,\partial_- y_\nu$$
$$= \frac{\kappa^2}{2}\int d^2\xi\ \partial_+ y_\mu \Theta_-^{\mu\nu}\big(\Delta V(y)\big)\partial_- y_\nu, \tag{14}$$

where

$$\Theta_\pm^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}, \tag{15}$$

with

$$G_{E\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu},$$
$$\theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}, \tag{16}$$

being the effective metric and the non-commutativity parameter in Seiberg–Witten terminology of the open bosonic string theory [24]. The T-dual background fields are equal to

$${}^\star G^{\mu\nu}\big(\Delta V(y)\big) = (G_E^{-1})^{\mu\nu}\big(\Delta V(y)\big),$$
$${}^\star B^{\mu\nu}\big(\Delta V(y)\big) = \frac{\kappa}{2}\theta^{\mu\nu}\big(\Delta V(y)\big), \tag{17}$$

and their argument is given by

$$\Delta V^\mu(y) = -\frac{\kappa}{2}\left(\Theta^{\mu\nu}_{0-} + \Theta^{\mu\nu}_{0+}\right)\Delta y_\nu + \frac{\kappa}{2}\left(\Theta^{\mu\nu}_{0-} - \Theta^{\mu\nu}_{0+}\right)\Delta \tilde{y}_\nu = -\kappa\theta_0^{\mu\nu}\Delta y_\nu + (g^{-1})^{\mu\nu}\Delta\tilde{y}_\nu. \tag{18}$$

Here $\Theta^{\mu\nu}_{0\pm}$ is the zeroth order value of the field composition $\Theta^{\mu\nu}_\pm$ defined in (15) and $g_{\mu\nu} = G_{\mu\nu} - 4b^2_{\mu\nu}$ and $\theta_0^{\mu\nu} = -\frac{2}{\kappa}(g^{-1}bG^{-1})^{\mu\nu}$ are the zeroth order values of the effective fields (16). The variable $\Delta\tilde{y}_\mu$ is the double of the dual variable $\Delta y_\mu = y_\mu(\xi) - y_\mu(\xi_0)$, defined as the following line integral:

$$\Delta\tilde{y}_\mu = \int_P (d\tau y'_\mu + d\sigma \dot{y}_\mu) = \int_P d\xi^\alpha \varepsilon^\beta{}_\alpha \partial_\beta y_\mu, \tag{19}$$

taken along the path P, from the point $\xi_0^\alpha(\tau_0, \sigma_0)$ to the point $\xi^\alpha(\tau, \sigma)$.

The fact that we are working with the weakly curved background ensures that the T-dual background fields are the solution of the space-time equations (5). Because both dual metric ${}^\star G^{\mu\nu}$ and dual Kalb–Ramond field ${}^\star B^{\mu\nu}$ are linear in coordinates with infinitesimal coefficients, the dual Christoffel symbol ${}^\star\Gamma^{\nu\rho}_\mu$ and dual field strength ${}^\star B^{\mu\nu\rho}$ are constant and infinitesimal. In Eq. (114) of Sect. 8 we will show that T-dual dilaton field is ${}^\bullet\Phi = \Phi - \ln\det\sqrt{2\Pi_+}$, where Φ is constant and Π_+ is linear in coordinates with infinitesimal coefficients. So, ${}^\bullet\Phi$ is also linear in coordinates with infinitesimal coefficients, and $\partial_\mu{}^\bullet\Phi$ is constant and infinitesimal. Consequently, $D_\mu\partial_\nu{}^\bullet\Phi$, $\partial_\rho{}^\bullet\Phi B^\rho{}_{\mu\nu}$ and $(\partial_\mu{}^\bullet\Phi)^2$ are infinitesimals of the second order. So, all T-dual space-time equations, for the metric, for the Kalb–Ramond field and for dilaton field, are infinitesimals of the second order and as such are neglected.

The initial theory (8) and its completely T-dual theory (14) are connected by the T-dual coordinate transformation laws (eq. (42) of Ref. [18])

$$\partial_\pm x^\mu = -\kappa\Theta^{\mu\nu}(\Delta V)\partial_\pm y_\nu \mp 2\kappa\Theta^{\mu\nu}_{0\pm}\beta^\mp_\nu(V), \tag{20}$$

and its inverse (eq. (66) of Ref. [18])

$$\partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}(\Delta x)\partial_\pm x^\nu \mp 2\beta^\mp_\mu(x), \tag{21}$$

where $\beta^\pm_\mu(x) = \mp\frac{1}{2}h_{\mu\nu}(x)\partial_\mp x^\nu$. It is shown that

$$\mathcal{T}: S[x^\mu] \to S[y_\mu], \quad \tilde{\mathcal{T}}: S[y_\mu] \to S[x^\mu], \tag{22}$$

and therefore

$$\mathcal{T}\circ\tilde{\mathcal{T}} = 1. \tag{23}$$

3 T-dualization along arbitrary subset of coordinates $\mathcal{T}^a : S[x^\mu] \to S[x^i, y_a]$

In this section, we will learn what theory is obtained if one chooses to apply the T-dualization procedure to the action (8), along arbitrary d coordinates x^a, $\mathcal{T}^a : S[x^\mu] \to S[x^i, y_a]$, with $\mathcal{T}^a = \circ^d_{n=1}T^{\mu_n}$, $\mu_n \in (0, 1, \ldots, D-1)$.

The closed string action in the weakly curved background (6) has a global symmetry (11). One localizes the symmetry for the coordinates x^a, by introducing the gauge fields v^a_α and substituting the ordinary derivatives with the covariant derivatives

$$\partial_\alpha x^a \to D_\alpha x^a = \partial_\alpha x^a + v^a_\alpha. \tag{24}$$

The covariant derivatives are invariant under standard gauge transformations

$$\delta v^a_\alpha = -\partial_\alpha\lambda^a. \tag{25}$$

In the case of the weakly curved background, in order to obtain the gauge invariant action one should additionally substitute the coordinates x^a in the argument of the background fields with their invariant extension, defined by

$$\Delta x^a_{\text{inv}} \equiv \int_P d\xi^\alpha D_\alpha x^a = \int_P (d\xi^+ D_+ x^a + d\xi^- D_- x^a) = x^a - x^a(\xi_0) + \Delta V^a, \tag{26}$$

where

$$\Delta V^a \equiv \int_P d\xi^\alpha v^a_\alpha = \int_P (d\xi^+ v^a_+ + d\xi^- v^a_-). \tag{27}$$

To preserve the physical equivalence between the gauged and the original theory, one introduces the Lagrange multipliers y_a and adds term $\frac{1}{2}y_a F^a_{+-}$ to the Lagrangian, which will force the field strength $F^a_{+-} \equiv \partial_+ v^a_- - \partial_- v^a_+ = -2F^a_{01}$ to vanish. In this way, the gauge invariant action

$$\begin{aligned} S_{\text{inv}}[x^\mu, x^a_{\text{inv}}, y_a] &= \kappa\int d^2\xi \Big[\partial_+ x^i\Pi_{+ij}\big(x^i, \Delta x^a_{\text{inv}}\big)\partial_- x^j \\ &+ \partial_+ x^i\Pi_{+ia}\big(x^i, \Delta x^a_{\text{inv}}\big)D_- x^a \\ &+ D_+ x^a\Pi_{+ai}\big(x^i, \Delta x^a_{\text{inv}}\big)\partial_- x^i \\ &+ D_+ x^a\Pi_{+ab}\big(x^i, \Delta x^a_{\text{inv}}\big)D_- x^b \\ &+ \frac{1}{2}(v^a_+\partial_- y_a - v^a_-\partial_+ y_a)\Big] \end{aligned} \tag{28}$$

is obtained, where the last term is equal to $\frac{1}{2}y_a F^a_{+-}$ up to the total divergence. Now, we can fix the gauge taking $x^a(\xi) = x^a(\xi_0)$ and obtain the gauge fixed action

$S_{\text{fix}}[x^i, v^a_\pm, y_a]$

$$= \kappa \int d^2\xi \Big[\partial_+ x^i \Pi_{+ij}\big(x^i, \Delta V^a\big)\partial_- x^j + \partial_+ x^i \Pi_{+ia}\big(x^i, \Delta V^a\big) v_-^a + v_+^a \Pi_{+ai}\big(x^i, \Delta V^a\big)\partial_- x^i + v_+^a \Pi_{+ab}\big(x^i, \Delta V^a\big) v_-^b + \frac{1}{2}(v_+^a \partial_- y_a - v_-^a \partial_+ y_a)\Big]. \tag{29}$$

This action reduces to the initial one for the equations of motion obtained varying over the Lagrange multipliers. The T-dual action is obtained for the equations of motion for the gauge fields.

3.1 Regaining the initial action

Varying the gauge fixed action (29) over the Lagrange multipliers y_a one obtains the equations of motion

$$\partial_+ v_-^a - \partial_- v_+^a = 0, \tag{30}$$

which have the solution

$$v_\pm^a = \partial_\pm x^a. \tag{31}$$

On this solution the background fields' argument ΔV^a defined in (27) is path independent and reduces to

$$\Delta V^a(\xi) = x^a(\xi) - x^a(\xi_0). \tag{32}$$

The gauge fixed action (29) reduces to the initial action (8), but the background fields' argument is ΔV^a instead of x^i. However, the action (8) is invariant under the constant shift of coordinates, so shifting coordinates by $x^a(\xi_0)$ one obtains the exact form of the initial action.

3.2 The T-dual action

Using the equations of motion for the gauge fields, we eliminate them and obtain the T-dual action.

The equations of motion obtained varying the gauge fixed action (29) over the gauge fields $v_\pm^a$ are

$$\Pi_{\pm ai}\big(x^i, \Delta V^a\big)\partial_\mp x^i + \Pi_{\pm ab}\big(x^i, \Delta V^a\big) v_\mp^b + \frac{1}{2}\partial_\mp y_a = \pm \beta_a^\pm\big(x^i, V^a\big), \tag{33}$$

where

$$\beta_a^\pm\big(x^i, V^a\big) = \mp\frac{1}{2}\Big[h_{ai}(x^i)\partial_\mp x^i + h_{ab}(x^i)\partial_\mp V^b + h_{ai}\big(V^a\big)\partial_\mp x^i + h_{ab}\big(V^a\big)\partial_\mp V^b\Big] \tag{34}$$

is the contribution from the background fields' argument ΔV^a, defined in a same way as in Ref. [18], by $\delta_V S_{\rm fix} = -\kappa \int d^2\xi (\beta_a^+ \delta v_+^a + \beta_a^- \delta v_-^a)$. If the initial background $\Pi_{+\mu\nu}$ does not depend on the coordinates x^a, the corresponding beta functions are zero $\beta_a^\pm = 0$.

Multiplying Eq. (33) by $2\kappa \tilde\Theta_\mp^{ab}$, defined in (A.7), the inverse of the background fields composition $\Pi_{\pm ab}$, one obtains

$$v_\mp^a = -2\kappa \tilde\Theta_\mp^{ab}\big(x^i, \Delta V^a\big)\Big[\Pi_{\pm bi}\big(x^i, \Delta V^a\big)\partial_\mp x^i + \frac{1}{2}\partial_\mp y_b \mp \beta_b^\pm\big(x^i, V^a\big)\Big]. \tag{35}$$

Substituting (35) into the action (29), we obtain the T-dual action

$$\begin{aligned} S[x^i, y_a] &= \kappa \int d^2\xi \Big[\partial_+ x^i \overline{\Pi}_{+ij}\big(x^i, \Delta V^a(x^i, y_a)\big)\partial_- x^j \\ &\quad - \kappa\, \partial_+ x^i \Pi_{+ia}\big(x^i, \Delta V^a(x^i, y_a)\big) \times \tilde\Theta_-^{ab}\big(x^i, \Delta V^a(x^i, y_a)\big)\partial_- y_b \\ &\quad + \kappa\, \partial_+ y_a \tilde\Theta_-^{ab}\big(x^i, \Delta V^a(x^i, y_a)\big) \times \Pi_{+bi}\big(x^i, \Delta V^a(x^i, y_a)\big)\partial_- x^i \\ &\quad + \frac{\kappa}{2}\,\partial_+ y_a \tilde\Theta_-^{ab}\big(x^i, \Delta V^a(x^i, y_a)\big)\partial_- y_b \Big], \end{aligned} \tag{36}$$

where

$$\overline{\Pi}_{+ij} \equiv \Pi_{+ij} - 2\kappa \Pi_{+ia}\tilde\Theta_-^{ab}\Pi_{+bj}. \tag{37}$$

In order to find the explicit value of the background fields argument $\Delta V^a(x^i, y_a)$, it is enough to consider the zeroth order of the equations of motion for the gauge fields $v_\pm^a$ (35)

$$v_\pm^{(0)a} = -2\kappa \tilde\Theta_{0\pm}^{ab}\Big[\Pi_{0\mp bi}\partial_\pm x^{(0)i} + \frac{1}{2}\partial_\pm y_b^{(0)}\Big]. \tag{38}$$

Here $\tilde\Theta_{0\pm}^{ab}$ and $\Pi_{0\mp bi}$ stand for the zeroth order values of $\tilde\Theta_\pm^{ab}$ and $\Pi_{\mp bi}$, and they are defined in (A.11).

Substituting (38) into (27) we obtain

$$\begin{aligned} \Delta V^{(0)a}(x^i, y_a) &= -\kappa\Big[\tilde\Theta_{0+}^{ab}\Pi_{0-bi} + \tilde\Theta_{0-}^{ab}\Pi_{0+bi}\Big]\Delta x^{(0)i} \\ &\quad -\kappa\Big[\tilde\Theta_{0+}^{ab}\Pi_{0-bi} - \tilde\Theta_{0-}^{ab}\Pi_{0+bi}\Big]\Delta\tilde x^{(0)i} \\ &\quad -\frac{\kappa}{2}\Big[\tilde\Theta_{0+}^{ab} + \tilde\Theta_{0-}^{ab}\Big]\Delta y_b^{(0)} - \frac{\kappa}{2}\Big[\tilde\Theta_{0+}^{ab} - \tilde\Theta_{0-}^{ab}\Big]\Delta\tilde y_b^{(0)}. \end{aligned} \tag{39}$$

Here

$$\Delta\tilde y_a^{(0)} = \int_P (d\tau\, y_a^{(0)\prime} + d\sigma\, \dot y_a^{(0)}), \qquad \Delta\tilde x^{(0)i} = \int_P (d\tau\, x^{(0)\prime i} + d\sigma\, \dot x^{(0)i}), \tag{40}$$

are the variables T-dual to the coordinates y_a and x^i in the zeroth order in $B_{\mu\nu\rho}$, for $b_{\mu\nu} = 0$, which we call the double variables.

Thus, we obtain the explicit form of the T-dual action and conclude that it is given in terms of the original coordinates x^i and the dual coordinates y_a originating from the Lagrange multipliers. However, the background fields' argument depends not only on these variables but on their doubles as well. Because of this the theory is non-local as the double variables $\tilde{x}^i$ and $\tilde{y}_a$ are defined as line integrals.

The action (36) can be obtained from the initial action (8) under the following substitutions of the coordinate derivatives and the background fields:

$$\partial_\pm x^i \to \partial_\pm x^i, \quad \partial_\pm x^a \to \partial_\pm y_a, \tag{41}$$

$$\Pi_{+ij} \to {}^\bullet\Pi_{+ij}, \quad \Pi_{+ia} \to {}^\bullet\Pi_{+i}{}^{a},$$
$$\Pi_{+ai} \to {}^\bullet\Pi^a_{+i}, \quad \Pi_{+ab} \to {}^\bullet\Pi^{ab}_{+}, \tag{42}$$

where the dual background fields are

$${}^\bullet\Pi_{+ij} = \overline{\Pi}_{+ij}, \quad {}^\bullet\Pi_{+i}{}^{a} = -\kappa\Pi_{+ib}\tilde{\Theta}^{ba}_-,$$
$${}^\bullet\Pi^a_{+i} = \kappa\tilde{\Theta}^{ab}_-\Pi_{+bi}, \quad {}^\bullet\Pi^{ab}_{+} = \frac{\kappa}{2}\tilde{\Theta}^{ab}_-, \tag{43}$$

with $\overline{\Pi}_{+ij}$, $\Pi_{+\mu\nu}$, and $\tilde{\Theta}^{ab}_-$ defined in (37), (9), and (A.7). The argument of all T-dual background fields is $[x^i, V^a(x^i, y_a)]$. According to (27) and (39), it is non-local and consequently non-geometric. Calculating the symmetric and antisymmetric part of the T-dual field compositions (43), we find that the T-dual metric and Kalb–Ramond field are equal to

$$\begin{aligned}
{}^\bullet G_{ij} &= \overline{G}_{ij} = G_{ij} - G_{ia}(\tilde{G}^{-1}_E)^{ab}G_{bj}\\
&\quad -2\kappa\left(B_{ia}\tilde{\theta}^{ab}G_{bj} + G_{ia}\tilde{\theta}^{ab}B_{bj}\right) - 4B_{ia}(\tilde{G}^{-1}_E)^{ab}B_{bj},\\
{}^\bullet B_{ij} &= \overline{B}_{ij} = B_{ij} - \frac{\kappa}{2}G_{ia}\tilde{\theta}^{ab}G_{bj} - B_{ia}(\tilde{G}^{-1}_E)^{ab}G_{bj}\\
&\quad -G_{ia}(\tilde{G}^{-1}_E)^{ab}B_{bj} - 2\kappa B_{ia}\tilde{\theta}^{ab}B_{bj},\\
{}^\bullet G^{ab} &= (\tilde{G}^{-1}_E)^{ab},\\
{}^\bullet B^{ab} &= \frac{\kappa}{2}\tilde{\theta}^{ab},\\
{}^\bullet G^a{}_i &= \kappa\tilde{\theta}^{ab}G_{bi} + 2(\tilde{G}^{-1}_E)^{ab}B_{bi},\\
{}^\bullet B^a{}_i &= \kappa\tilde{\theta}^{ab}B_{bi} + \frac{1}{2}(\tilde{G}^{-1}_E)^{ab}G_{bi},
\end{aligned} \tag{44}$$

where $\tilde{G}_{Eab}$ and $\tilde{\theta}^{ab}$ are defined in (A.6) and (A.10). The T-dual background fields have the same form as in the flat background [1,5,25] but in the present case fields $B_{\mu\nu}$, $\tilde{G}^{-1ab}_E$ and $\tilde{\theta}^{ab}$ are coordinate dependent.

Comparing the solutions for the gauge fields (31) and (35), we obtain the T-dual coordinate transformation law

$$\begin{aligned}
\partial_\mp x^a \cong{}& -2\kappa\tilde{\Theta}^{ab}_\mp\big(x^i, \Delta V^a(x^i, y_a)\big)\\
&\times\Big[\Pi_{\pm bi}\big(x^i, \Delta V^a(x^i, y_a)\big)\partial_\mp x^i + \frac{1}{2}\partial_\mp y_b\\
&\mp\beta^\pm_b\big(x^i, V^a(x^i, y_a)\big)\Big].
\end{aligned} \tag{45}$$

4 Inverse T-dualization $\mathcal{T}_a : S[x^i, y_a] \to S[x^\mu]$

In this section we will show that T-dualization of the action $S[x^i, y_a]$, given by (36), along already treated directions y_a leads to the original action.

So, let us localize the global symmetry of the coordinates y_a

$$\delta y_a = \lambda_a, \tag{46}$$

of the action (36). Note that this is the symmetry, despite the coordinate dependence of the metric (44), due to the invariance of the background fields' argument [18]. Following the T-dualization procedure, we substitute the ordinary derivatives with the covariant ones

$$D_\pm y_a = \partial_\pm y_a + u_{\pm a}, \tag{47}$$

where $u_{\pm a}$ are gauge fields which transform as $\delta u_{\pm a} = -\partial_\pm\lambda_a$. We also substitute coordinates y_a in the background fields' argument with the invariant coordinates

$$\begin{aligned}
y^{\rm inv}_a &= \int_P(d\xi^+ D_+ y_a + d\xi^- D_- y_a)\\
&= y_a(\xi) - y_a(\xi_0) + \Delta U_a,
\end{aligned} \tag{48}$$

where

$$\Delta U_a = \int_P(d\xi^+ u_{+a} + d\xi^- u_{-a}). \tag{49}$$

In this way, adding the Lagrange multiplier term which makes the introduced gauge fields nonphysical, we obtain the gauge invariant action

$$\begin{aligned}
&S_{\rm inv}[x^i, y_a, y^{\rm inv}_a, z^a]\\
&= \kappa\int d^2\xi\Big[\partial_+ x^i\overline{\Pi}_{+ij}\big(x^i, \Delta V^a(x^i, y^{\rm inv}_a)\big)\partial_- x^j\\
&\quad -\kappa\,\partial_+ x^i\Pi_{+ia}\big(x^i, \Delta V^a(x^i, y^{\rm inv}_a)\big)\\
&\quad \times\tilde{\Theta}^{ab}_-\big(x^i, \Delta V^a(x^i, y^{\rm inv}_a)\big)D_- y_b\\
&\quad +\kappa\, D_+ y_a\tilde{\Theta}^{ab}_-\big(x^i, \Delta V^a(x^i, y^{\rm inv}_a)\big)\\
&\quad \times\Pi_{+bi}\big(x^i, \Delta V^a(x^i, y^{\rm inv}_a)\big)\partial_- x^i\\
&\quad +\frac{\kappa}{2}D_+ y_a\tilde{\Theta}^{ab}_-\big(x^i, \Delta V^a(x^i, y^{\rm inv}_a)\big)D_- y_b\\
&\quad +\frac{1}{2}(u_{+a}\partial_- z^a - u_{-a}\partial_+ z^a)\Big],
\end{aligned} \tag{50}$$

which after fixing the gauge by $y_a(\xi) = y_a(\xi_0)$ becomes

$$\begin{aligned}
&S_{\rm fix}[x^i, u_{\pm a}, z^a]\\
&= \kappa\int d^2\xi\Big[\partial_+ x^i\overline{\Pi}_{+ij}\big(x^i, \Delta V^a(x^i, \Delta U_a)\big)\partial_- x^j\\
&\quad -\kappa\,\partial_+ x^i\Pi_{+ia}\big(x^i, \Delta V^a(x^i, \Delta U_a)\big)\\
&\quad \times\tilde{\Theta}^{ab}_-\big(x^i, \Delta V^a(x^i, \Delta U_a)\big)u_{-b}
\end{aligned}$$

$$+\kappa\, u_{+a}\tilde{\Theta}_-^{ab}\big(x^i, \Delta V^a(x^i, \Delta U_a)\big) \times \Pi_{+bi}\big(x^i, \Delta V^a(x^i, \Delta U_a)\big)\partial_- x^i + \frac{\kappa}{2}\, u_{+a}\tilde{\Theta}_-^{ab}\big(x^i, \Delta V^a(x^i, \Delta U_a)\big)u_{-b} + \frac{1}{2}(u_{+a}\partial_- z^a - u_{-a}\partial_+ z^a)\Big], \quad (51)$$

where ΔV^a is defined in (39) and ΔU_a in (49).

4.1 Regaining the T-dual action

The equations of motion obtained varying the gauge fixed action (51) over the Lagrange multipliers z^a

$$\partial_+ u_{-a} - \partial_- u_{+a} = 0, \quad (52)$$

have the solution

$$u_{\pm a} = \partial_\pm y_a. \quad (53)$$

On this solution the variable ΔU_a defined by (49) is path independent and reduces to

$$\Delta U_a(\xi) = y_a(\xi) - y_a(\xi_0), \quad (54)$$

and the gauge fixed action (51) reduces to the action (36).

4.2 Regaining the initial action

The equations of motion obtained varying the gauge fixed action (51) over the gauge fields $u_{\pm a}$ are

$$\kappa\tilde{\Theta}_\mp^{ab}\big(x^i, \Delta V^a(x^i, \Delta U_a)\big) \times\Big[\frac{1}{2}u_{\mp b} + \Pi_{\pm bi}\big(x^i, \Delta V^a(x^i, \Delta U_a)\big)\partial_\mp x^i\Big] + \frac{1}{2}\partial_\mp z^a = \pm\kappa\,\tilde{\Theta}_{0\mp}^{ab}\beta_b^\pm\big(x^i, V^a(x^i, U_a)\big), \quad (55)$$

where terms $\tilde{\Theta}_{0\mp}^{ab}\beta_b^\pm$ are the contribution from the variation over the background field argument

$$\delta_U S_{\text{fix}} = -\kappa^2\int d^2\xi\,\big(\delta u_{+a}\tilde{\Theta}_{0-}^{ab}\beta_b^+ + \delta u_{-a}\tilde{\Theta}_{0+}^{ab}\beta_b^-\big). \quad (56)$$

Here $\beta_a^\pm$ is of the same form as (34) and $\tilde{\Theta}_{0\mp}^{ab}$ is defined in (A.11).

Let us show that for the equations of motion (55), the gauge fixed action (51) will reduce to the initial action (8). Using the fact that $\tilde{\Theta}_\mp^{ab}$ is inverse to $2\kappa\Pi_{\pm ab}$, these equations of motion can be rewritten as

$$u_{\mp a} = -2\Pi_{\pm ai}\big(x^i, \Delta V^a(x^i, \Delta U_a)\big)\partial_\mp x^i -2\Pi_{\pm ab}\big(x^i, \Delta V^a(x^i, \Delta U_a)\big)\partial_\mp z^b \pm 2\beta_a^\pm\big(x^i, V^a(x^i, U_a)\big). \quad (57)$$

Substituting (57) into (51), using the definition (37) and the first relation in (A.22) one obtains

$$S[x^i, z^a] = \kappa\int_\Sigma d^2\xi\Big[\partial_+ x^i\Pi_{+ij}\partial_- x^j + \partial_+ x^i\Pi_{+ia}\partial_- z^a + \partial_+ z^a\Pi_{+ai}\partial_- x^i + \partial_+ z^a\Pi_{+ab}\partial_- z^b\Big]. \quad (58)$$

The explicit form of the argument of the background fields is obtained substituting the zeroth order of Eq. (57) into (49)

$$U_a^{(0)} = -2b_{ai}x^{(0)i} + G_{ai}\tilde{x}^{(0)i} - 2b_{ab}z^{(0)b} + G_{ab}\tilde{z}^{(0)b}. \quad (59)$$

Consequently, the argument of the background fields ΔV^a, defined in (39), is just

$$V^{(0)a}(x^i, U_a) = z^a. \quad (60)$$

So, the action (58) is equal to the initial action (8) with $x^\mu = (x^i, z^a)$.

Comparing the solutions for the gauge fields (53) and (57), we obtain the T-dual transformation law

$$\partial_\mp y_a \cong -2\Pi_{\pm ai}(x^i, z^a)\partial_\mp x^i - 2\Pi_{\pm ab}(x^i, z^a)\partial_\mp z^b \pm 2\beta_a^\pm\big(x^i, z^a\big). \quad (61)$$

Substituting $\partial_\mp y_a$ to (45) with the help of (60) one finds $\partial_\pm x^a = \partial_\pm z^a$. Therefore, (61) is the transformation inverse to (45), which confirms the relation $\mathcal{T}^a \circ \mathcal{T}_a = 1$.

5 T-dualization along all undualized coordinates $\mathcal{T}^i : S[x^i, y_a] \rightarrow S[y_\mu]$

In this section we will T-dualize the action (36), applying the T-dualization procedure to the undualized coordinates x^i. Substituting the ordinary derivatives $\partial_\pm x^i$ with the covariant derivatives

$$D_\pm x^i = \partial_\pm x^i + w_\pm^i, \quad (62)$$

where the gauge fields $w_\pm^i$ transform as $\delta w_\pm^i = -\partial_\pm\lambda^i$, substituting the coordinates x^i in the background field arguments with

$$\Delta x_{\text{inv}}^i = \int_P (d\xi^+ D_+ x^i + d\xi^- D_- x^i), \quad (63)$$

and adding the Lagrange multiplier term, we obtain the gauge invariant action

$$S_{\text{inv}}[x^i, x_{\text{inv}}^i, y] = \kappa\int d^2\xi\Big[D_+ x^i\overline{\Pi}_{+ij}\big(\Delta x_{\text{inv}}^i, \Delta V^a(\Delta x_{\text{inv}}^i, y_a)\big)D_- x^j - \kappa\, D_+ x^i\Pi_{+ia}\big(\Delta x_{\text{inv}}^i, \Delta V^a(\Delta x_{\text{inv}}^i, y_a)\big)$$

$$\times\,\tilde{\Theta}_-^{ab}\big(\Delta x^i_{\rm inv},\Delta V^a(\Delta x^i_{\rm inv},y_a)\big)\partial_- y_b$$
$$+\kappa\,\partial_+ y_a \tilde{\Theta}_-^{ab}\big(\Delta x^i_{\rm inv},\Delta V^a(\Delta x^i_{\rm inv},y_a)\big)$$
$$\times\,\Pi_{+bi}\big(\Delta x^i_{\rm inv},\Delta V^a(\Delta x^i_{\rm inv},y_a)\big)D_- x^i$$
$$+\frac{\kappa}{2}\,\partial_+ y_a \tilde{\Theta}_-^{ab}\big(\Delta x^i_{\rm inv},\Delta V^a(\Delta x^i_{\rm inv},y_a)\big)\partial_- y_b$$
$$+\frac{1}{2}(w^i_+\partial_- y_i - w^i_-\partial_+ y_i)\Big]. \tag{64}$$

Substituting the gauge fixing condition $x^i(\xi)=x^i(\xi_0)$ one obtains

$$\begin{aligned}
&S_{\rm fix}[x^i,w^i_\pm,y]\\
&=\kappa\int d^2\xi\Big[w^i_+\overline{\Pi}_{+ij}(\Delta W)w^j_-\\
&-\kappa\,w^i_+\Pi_{+ia}(\Delta W)\tilde{\Theta}_-^{ab}(\Delta W)\partial_- y_b\\
&+\kappa\,\partial_+ y_a\tilde{\Theta}_-^{ab}(\Delta W)\Pi_{+bi}(\Delta W)w^i_-\\
&+\frac{\kappa}{2}\,\partial_+ y_a\tilde{\Theta}_-^{ab}(\Delta W)\partial_- y_b\\
&+\frac{1}{2}(w^i_+\partial_- y_i - w^i_-\partial_+ y_i)\Big],
\end{aligned}\tag{65}$$

where $\Delta W^\mu=\big[\Delta W^i,\Delta V^a(\Delta W^i,y_a)\big]$ with ΔW^i defined by

$$\Delta W^i\equiv\int_P(d\xi^+ w^i_+ + d\xi^- w^i_-), \tag{66}$$

and $\Delta V^a=\Delta V^a(\Delta W^i,y_a)$ is defined in (39), where argument x^i is replaced by ΔW^i.

5.1 Regaining the T-dual action

The equations of motion for the Lagrange multipliers y_i are

$$\partial_+ w^i_- - \partial_- w^i_+ = 0, \tag{67}$$

and they have the solution

$$w^i_\pm=\partial_\pm x^i. \tag{68}$$

For this solution the background field argument ΔW^i defined in (66) reduces to

$$\Delta W^i(\xi)=x^i(\xi)-x^i(\xi_0), \tag{69}$$

so that the argument ΔV^a becomes

$$\Delta V^a(\Delta W^i,y^a)=\Delta V^a(x^i,y^a), \tag{70}$$

and therefore the gauge fixed action (65) reduces to the action (36).

5.2 From the gauge fixed action to the completely T-dual action

The equations of motion obtained varying the gauge fixed action (65) over $w^i_\pm$ are

$$\begin{aligned}
&\overline{\Pi}_{\pm ij}(\Delta W)w^j_\mp-\kappa\,\Pi_{\pm ia}(\Delta W)\tilde{\Theta}_\mp^{ab}(\Delta W)\partial_\mp y_b+\frac{1}{2}\partial_\mp y_i\\
&=\pm 2\kappa\,\overline{\Pi}_{\pm ij}\Theta_\mp^{j\mu}\beta^\pm_\mu(W),
\end{aligned}\tag{71}$$

where

$$\beta^\pm_\mu(V)=\mp\frac{1}{2}h_{\mu\nu}(V)\partial_\mp V^\nu. \tag{72}$$

Terms $\overline{\Pi}_{\pm ij}\Theta_\mp^{j\mu}\beta^\pm_\mu(W)$ are the contribution from the background fields' argument, defined by

$$\begin{aligned}
\delta_U S_{\rm fix}=-2\kappa^2\int d^2\xi\Big(&\delta w^i_+\overline{\Pi}_{+ij}\Theta_-^{j\mu}\beta^+_\mu\\
&+\delta w^i_-\overline{\Pi}_{-ij}\Theta_+^{j\mu}\beta^-_\mu\Big),
\end{aligned}\tag{73}$$

calculated using (A.15), (A.16), and (39).

Using the fact that the background field composition $\overline{\Pi}_{\pm ij}$ is invese to $2\kappa\Theta_\mp^{ij}$ defined by (A.22), we can rewrite the equation of motion (71) expressing the gauge fields as

$$\begin{aligned}
w^i_\mp=\;&2\kappa\Theta_\mp^{ij}(\Delta W)\Big[\kappa\,\Pi_{\pm ja}(\Delta W)\tilde{\Theta}_\mp^{ab}(\Delta W)\partial_\mp y_b\\
&-\frac{1}{2}\partial_\mp y_j\Big]\pm 2\kappa\,\Theta_{0\pm}^{i\mu}\beta^\pm_\mu(W).
\end{aligned}\tag{74}$$

Using the second relation in (A.23), we obtain

$$w^i_\mp=-\kappa\Theta_\mp^{i\mu}(\Delta W)\Big[\partial_\mp y_\mu\mp 2\beta^\pm_\mu(W)\Big]. \tag{75}$$

Substituting (75) into the gauge fixed action (65), we obtain

$$\begin{aligned}
&S[y]\\
&=\kappa\int d^2\xi\Big[\partial_+ y_i\Big(\kappa\Theta_-^{ij}-\kappa^2\Theta_-^{ik}\overline{\Pi}_{+kl}\Theta_-^{lj}\Big)\partial_- y_j\\
&+\Big(-\kappa^2\Theta_-^{ij}\overline{\Pi}_{+jk}\Theta_-^{ka}+\frac{\kappa}{2}\Theta_-^{ia}-\kappa^2\Theta_-^{ij}\Pi_{+jb}\tilde{\Theta}_-^{ba}\Big)\\
&\times\partial_+ y_i\,\partial_- y_a\\
&+\Big(-\kappa^2\Theta_-^{aj}\overline{\Pi}_{+jk}\Theta_-^{ki}+\frac{\kappa}{2}\Theta_-^{ai}-\kappa^2\tilde{\Theta}_-^{ab}\Pi_{+bj}\Theta_-^{ji}\Big)\\
&\times\partial_+ y_a\,\partial_- y_i+\partial_+ y_a\Big(\frac{\kappa}{2}\tilde{\Theta}_-^{ab}-\kappa^2\Theta_-^{ai}\overline{\Pi}_{+ij}\Theta_-^{jb}\\
&-\kappa^2\Theta_-^{ai}\Pi_{+ic}\tilde{\Theta}_-^{cb}-\kappa^2\tilde{\Theta}_-^{ac}\Pi_{+ci}\Theta_-^{ib}\Big)\partial_- y_b\Big].
\end{aligned}\tag{76}$$

Using (A.22), (A.27), and (A.29) one can rewrite this action as

$$S[y]=\frac{\kappa^2}{2}\int d^2\xi\,\partial_+ y_\mu\Theta_-^{\mu\nu}(\Delta W)\partial_- y_\nu. \tag{77}$$

In order to find the background fields' argument ΔW^i, we consider the zeroth order of Eqs. (75), and we conclude that

$$\Delta W^i = -\kappa\theta_0^{i\mu}\Delta y_\mu + (g^{-1})^{i\mu}\Delta\tilde{y}_\mu. \tag{78}$$

Using (A.28) and (A.23), we find that $\Delta V^a(\Delta W^i, y^a)$ defined in (39) equals

$$\Delta V^a(\Delta W^i, y_a) = -\kappa\theta_0^{a\mu}\Delta y_\mu + (g^{-1})^{a\mu}\Delta\tilde{y}_\mu. \tag{79}$$

Therefore, we conclude that the background fields' argument is equal to (18), so that the action (77) is the completely T-dual action (14), which is in agreement with Ref. [18]. Comparing the solutions for the gauge fields (68) and (75), we obtain the T-dual transformation law

$$\partial_\mp x^i \cong -\kappa\Theta_\mp^{i\mu}\big(\Delta V(y)\big)\Big[\partial_\mp y_\mu \mp 2\beta_\mu^\pm\big(V(y)\big)\Big]. \tag{80}$$

One can verify that two successive T-duality transformations (45) and (80) correspond to the total T-duality transformation (20). Indeed, the relation (80) is just the ith component of this transformation. Substituting $\partial_\pm x^i$ from (80) into (45), using (A.25) and (A.29), we obtain

$$\partial_\pm x^a = -\kappa\Theta_\pm^{a\mu}(\Delta V)\Big[\partial_\pm y_\mu \pm 2\beta_\mu^\mp(V)\Big],$$

which is just the ath component of the complete T-duality transformation. So, we confirm that $\mathcal{T}^a \circ \mathcal{T}^i = \mathcal{T}$.

6 Inverse T-dualization along arbitrary subset of the dual coordinates $\mathcal{T}_i : S[y_\mu] \to S[x^i, y_a]$

Finally, in this section we will show that the T-dualization of the completely T-dual action (14), along arbitrary subset of the dual coordinates y_i leads to T-dual action (36). So, let us start with the T-dual action

$$S[y] = \frac{\kappa^2}{2}\int d^2\xi\ \partial_+ y_\mu \Theta_-^{\mu\nu}\big(\Delta V(y)\big)\partial_- y_\nu, \tag{81}$$

which is globally invariant to the constant shift of coordinates y_μ

$$\delta y_\mu = \lambda_\mu. \tag{82}$$

We localize this symmetry for the coordinates y_i and obtain the locally invariant action

$$\begin{aligned} &S_{\text{inv}}[y, y_i^{\text{inv}}, z^i] \\ &= \frac{\kappa^2}{2}\int d^2\xi\Big[D_+ y_i \Theta_-^{ij}\big(\Delta V(y_i^{\text{inv}}, y_a)\big)D_- y_j \\ &\quad + D_+ y_i \Theta_-^{ia}\big(\Delta V(y_i^{\text{inv}}, y_a)\big)\partial_- y_a \\ &\quad + \partial_+ y_a \Theta_-^{ai}\big(\Delta V(y_i^{\text{inv}}, y_a)\big)D_- y_i \\ &\quad + \partial_+ y_a \Theta_-^{ab}\big(\Delta V(y_i^{\text{inv}}, y_a)\big)\partial_- y_b \\ &\quad + \frac{1}{\kappa}(u_{+i}\partial_- z^i - u_{-i}\partial_+ z^i)\Big], \end{aligned} \tag{83}$$

where $D_\pm y_i = \partial_\pm y_i + u_{\pm i}$ are the covariant derivatives. The gauge fields $u_{\pm i}$ transform as $\delta u_{\pm i} = -\partial_\pm \lambda_i$ and the invariant coordinates are defined by $y_i^{\text{inv}} = \int_P (d\xi^+ D_+ y_i + d\xi^- D_- y_i)$. After fixing the gauge by $y_i(\xi) = y_i(\xi_0)$, the action becomes

$$\begin{aligned} &S_{\text{fix}}[y_a, u_{\pm i}, z^i] \\ &= \frac{\kappa^2}{2}\int d^2\xi\Big[u_{+i}\Theta_-^{ij}\big(\Delta V(\Delta U_i, y_a)\big)u_{-j} \\ &\quad + u_{+i}\Theta_-^{ia}\big(\Delta V(\Delta U_i, y_a)\big)\partial_- y_a \\ &\quad + \partial_+ y_a \Theta_-^{ai}\big(\Delta V(\Delta U_i, y_a)\big)u_{-i} \\ &\quad + \partial_+ y_a \Theta_-^{ab}\big(\Delta V(\Delta U_i, y_a)\big)\partial_- y_b \\ &\quad + \frac{1}{\kappa}(u_{+i}\partial_- z^i - u_{-i}\partial_+ z^i)\Big], \end{aligned} \tag{84}$$

where $\Delta U_i = \int_P (d\xi^+ u_{+i} + d\xi^- u_{-i})$.

6.1 Regaining the T-dual action

The equations of motion obtained varying the gauge fixed action (84) over the Lagrange multipliers

$$\partial_+ u_{-i} - \partial_- u_{+i} = 0, \tag{85}$$

have the solution

$$u_{\pm i} = \partial_\pm y_i. \tag{86}$$

On this solution the variable ΔU_i reduces to

$$\Delta U_i(\xi) = y_i(\xi) - y_i(\xi_0), \tag{87}$$

and therefore

$$\Delta V^\mu(\Delta U_i, y_a) = \Delta V^\mu(y). \tag{88}$$

So, the action (84) becomes the action (81).

6.2 Obtaining the T-dual action

The equations of motion obtained varying the action (84) over $u_{\pm i}$ are

$$\begin{aligned} &\kappa\Theta_\mp^{ij}\big(\Delta V(\Delta U_i, y_a)\big)u_{\mp j} + \kappa\Theta_\mp^{ia}\big(\Delta V(\Delta U_i, y_a)\big)\partial_\mp y_a \\ &\quad + \partial_\mp z^i = \pm 2\kappa\Theta_{0\mp}^{i\mu}\beta_\mu^\pm\big(V(U_i, y_a)\big), \end{aligned} \tag{89}$$

where $\beta_\mu^\pm$ are given by (72). The terms with beta function come from the variation over the argument U_i

$$\delta_U S_{\text{fix}} = -\kappa^2\int d^2\xi\big(\delta u_{+i}\Theta_{0-}^{i\mu}\beta_\mu^+ + \delta u_{-i}\Theta_{0+}^{i\mu}\beta_\mu^-\big), \tag{90}$$

and are calculated using (A.15) and (18). Using the fact that $2\kappa\overline{\Pi}_{\mp ij}$ is the inverse of $\Theta^{ij}_{\pm}$, the equation (89) can be rewritten as

$$u_{\mp i} = -2\overline{\Pi}_{\pm ij}\big(\Delta V(\Delta U_i, y_a)\big)\Big[\kappa\Theta^{ja}_{\mp}\big(\Delta V(\Delta U_i, y_a)\big) \times\partial_{\mp}y_a + \partial_{\mp}z^j \mp 2\kappa\Theta^{j\mu}_{0\mp}\beta^{\pm}_{\mu}\big(V(U_i, y_a)\big)\Big]. \quad (91)$$

Substituting (91) into the gauge fixed action (84), using (A.25) we obtain

$$S[z^i, y_a] = \frac{\kappa^2}{2}\int d^2\xi\Big[\frac{2}{\kappa}\partial_+ z^i\overline{\Pi}_{+ij}\partial_- z^j + 2\partial_+ z^i\overline{\Pi}_{+ij}\Theta^{jb}_{-}\partial_- y_b - 2\partial_+ y_a\Theta^{ai}_{-}\overline{\Pi}_{+ij}\partial_- z^j + \partial_+ y_a\tilde{\Theta}^{ab}_{-}\partial_- y_b\Big], \quad (92)$$

which with the help of (A.29) becomes

$$S[z^i, y_a] = \frac{\kappa^2}{2}\int d^2\xi\Big[\frac{2}{\kappa}\partial_+ z^i\overline{\Pi}_{+ij}\partial_- z^j -2\partial_+ z^i\Pi_{+ia}\tilde{\Theta}^{ab}_{-}\partial_- y_b + 2\partial_+ y_a\tilde{\Theta}^{ab}_{-}\Pi_{+bj}\partial_- z^j +\partial_+ y_a\tilde{\Theta}^{ab}_{-}\partial_- y_b\Big]. \quad (93)$$

In order to find the argument of the background fields $\Delta V(\Delta U_i, y_a)$, one considers the zeroth order of Eqs. (91) and obtains

$$\Delta U^{(0)}_i = -\Big[\overline{\Pi}_{0+ij} + \overline{\Pi}_{0-ij}\Big]\Delta z^{(0)j} +\Big[\overline{\Pi}_{0+ij} - \overline{\Pi}_{0-ij}\Big]\Delta\tilde{z}^{(0)j} -\kappa\Big[\overline{\Pi}_{0+ij}\Theta^{ja}_{0-} + \overline{\Pi}_{0-ij}\Theta^{ja}_{0+}\Big]\Delta y^{(0)}_a +\kappa\Big[\overline{\Pi}_{0+ij}\Theta^{ja}_{0-} - \overline{\Pi}_{0-ij}\Theta^{ja}_{0+}\Big]\Delta\tilde{y}^{(0)}_a, \quad (94)$$

where the double variables are defined in analogy with (40). Substituting (94) into (18), we obtain

$$\Delta V^i(\Delta U_i, y_a) = \Delta z^i, \quad (95)$$

and

$$\Delta V^a(\Delta U_i, y_a) = -\kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} + \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta z^{(0)i} -\kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} - \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta\tilde{z}^{(0)i} -\frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} + \tilde{\Theta}^{ab}_{0-}\Big]\Delta y^{(0)}_b -\frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} - \tilde{\Theta}^{ab}_{0-}\Big]\Delta\tilde{y}^{(0)}_b, \quad (96)$$

which is exactly (39) with $z^i = x^i$. So, we can conclude that the action (93) is equal to the T-dual action (36).

Comparing the solutions for the gauge fields (86) and (91), we obtain the T-dual transformation law

$$\partial_{\mp}y_i \cong -2\overline{\Pi}_{\pm ij}\big(\Delta z^i, \Delta V^a(\Delta U_i(z^i, y_a), y_a)\big) \times\Big[\kappa\Theta^{ja}_{\mp}\big(\Delta z^i, \Delta V^a(\Delta U_i(z^i, y_a), y_a)\big)\partial_{\mp}y_a + \partial_{\mp}z^j \mp 2\kappa\Theta^{j\mu}_{0\mp}\beta^{\pm}_{\mu}\big(z^i, V^a(U_i(z^i, y_a), y_a)\big)\Big]. \quad (97)$$

These transformations are inverse to (80), so that $\mathcal{T}^i\circ\mathcal{T}_i = 1$. Successively applying (97) and (61), using (A.29) and (A.25), we obtain the ith component of the inverse law of the total T-dualization (21). Its ath component is (61), so we confirm that $\mathcal{T}_a\circ\mathcal{T}_i = \tilde{\mathcal{T}}$.

7 Group of the T-dual transformation laws

In this section we will recapitulate the coordinate transformation laws between the theories considered. In Sect. 3, we performed the T-dualization procedure along the coordinates x^a

$$\mathcal{T}^a : S[x^\mu] \to S[x^i, y_a], \quad (98)$$

and obtained the following coordinate transformation law: (45)

$$\partial_{\mp}x^a \cong -2\kappa\tilde{\Theta}^{ab}_{\mp}\big(x^i, \Delta V^a(x^i, y_a)\big) \times\Big[\Pi_{\pm bi}\big(x^i, \Delta V^a(x^i, y_a)\big)\partial_{\mp}x^i + \frac{1}{2}\partial_{\mp}y_b \mp\beta^{\pm}_b\big(x^i, V^a(x^i, y_a)\big)\Big] \quad (99)$$

where V^a and $\beta^{\pm}_a$ are given by (39) and (34). In the zeroth oder this law implies

$$x^{(0)a} \cong V^{(0)a}(x^i, y_a). \quad (100)$$

In Sect. 4, starting from the action $S[x^i, y_a]$ we performed the T-dualization procedure along the coordinates y_a

$$\mathcal{T}_a : S[x^i, y_a] \to S[x^\mu], \quad (101)$$

and obtained the transformation law (61)

$$\partial_{\mp}y_a \cong -2\Pi_{\pm a\mu}(x)\partial_{\mp}x^\mu \pm 2\beta^{\pm}_a(x), \quad (102)$$

which is the law inverse to (99) and in the zeroth order it implies

$$y^{(0)}_a \cong U^{(0)}_a(x). \quad (103)$$

Multiplying the transformation law (99) from the left side by $\Pi_{\pm ca}(x) \cong \Pi_{\pm ca}\big(x^i, \Delta V^a(x^i, y_a)\big)$, using (100), we obtain the transformation law (102). So, we confirm that $\mathcal{T}^a\circ\mathcal{T}_a = 1$.

In Sect. 5, starting once again from the action $S[x^i, y_a]$, we performed the T-dualization procedure along the undualized coordinates x^i

$$\mathcal{T}^i : S[x^i, y_a] \to S[y_\mu], \tag{104}$$

and obtained the coordinate transformation law (80)

$$\partial_\mp x^i \cong -\kappa \Theta^{i\mu}_\mp\big(\Delta V(y)\big)\Big[\partial_\mp y_\mu \mp 2\beta^\pm_\mu\big(V(y)\big)\Big], \tag{105}$$

where V^μ and $\beta^\pm_\mu$ are given by (18) and (72). In the zeroth order it gives

$$x^{(0)i} \cong V^{(0)i}(y). \tag{106}$$

The two successive T-duality transformations (99) and (105) give the complete transformation (20), so that $\mathcal{T}^a \circ \mathcal{T}^i = \mathcal{T}$.

In Sect. 6, starting from the completely T-dual action $S[y]$, we performed the T-dualization procedure along the coordinates y_i

$$\mathcal{T}_i : S[y_\mu] \to S[x^i, y_a], \tag{107}$$

and obtained (97)

$$\begin{aligned}\partial_\mp y_i \cong& -2\overline{\Pi}_{\pm ij}\big(\Delta x^i, \Delta V^a(\Delta U_i(x^i, y_a), y_a)\big)\\ &\times\Big[\kappa\Theta^{ja}_\mp\big(\Delta x^i, \Delta V^a(\Delta U_i(x^i, y_a), y_a)\big)\partial_\mp y_a + \partial_\mp x^j\\ &\mp 2\kappa\Theta^{j\mu}_{0\mp}\beta^\pm_\mu\big(x^i, V^a(U_i(x^i, y_a), y_a)\big)\Big],\end{aligned} \tag{108}$$

with V^a, U_i, and $\beta^\pm_\mu$ given by (79), (94), and (72). In the zeroth order this law implies

$$y_i^{(0)} \cong U_i^{(0)}(x^i, y_a). \tag{109}$$

Multiplying (108) from the left by

$\Theta^{ki}_\mp\big(\Delta x^i, \Delta V^a(y)\big) \cong \Theta^{ki}_\mp\big(\Delta x^i, \Delta V^a(\Delta U_i(x^i, y_a), y_a)\big)$,

using (106), we obtain the transformation law (105), so that $\mathcal{T}^i \circ \mathcal{T}_i = 1$. Successively applying (108) and (102), using (A.29) and (A.25), we obtain the ith component of the inverse law of the complete T-dualization (21). Its ath component is (102), so we confirm that $\mathcal{T}_a \circ \mathcal{T}_i = \tilde{\mathcal{T}}$.

We can conclude that the elements $1, \mathcal{T}^a$ and $\mathcal{T}_a$, with $d = 1, \ldots, D$, form an Abelian group. The element $\mathcal{T}^a$ is the inverse of the element $\mathcal{T}_a$.

8 Dilaton field in the weakly curved background

The T-duality transformation of the dilaton field in the weakly curved background was considered in Ref. [26]. For completeness and further use, we give here a brief recapitulation of some basic steps of the treatment.

It is well known that a dilaton transformation has a quantum origin. So, let us start with the path integral for the gauge fixed action

$$\mathcal{Z} = \int dv^\mu_+ dv^\mu_- dy_\mu e^{iS_{\rm fix}(v_\pm, \partial_\pm y)}, \tag{110}$$

where

$$S_{\rm fix}(v_\pm, \partial_\pm y) = S_0 + S_1\,, \tag{111}$$

with S_1 being the infinitesimal part of the action

$$\begin{aligned}S_0 &= \kappa\int d^2\xi\,[v^\mu_+\Pi_{0+\mu\nu}v^\nu_- + \frac{1}{2}(v^\mu_+\partial_- y_\mu - v^\mu_-\partial_+ y_\mu)]\,,\\ S_1 &= \kappa\int d^2\xi\; v^\mu_+ h_{\mu\nu}(V)v^\nu_-\,.\end{aligned} \tag{112}$$

For a constant background ($S_1 = 0$) the path integral is Gaussian and it equals $(\det \Pi_{0+\mu\nu})^{-1}$. In our case the background is coordinate dependent and thus the integral is not Gaussian. The fact that we work with an infinitesimal parameter enables us to show that the final result is formally the same as in the flat case [26],

$$\mathcal{Z} = \int dy_\mu \frac{1}{\det(\Pi_{+\mu\nu}(V))} e^{i\,{}^\star S(y)}, \tag{113}$$

where ${}^\star S(y) = \frac{\kappa^2}{2}\int d^2\xi\; \partial_+ y_\mu\, \Theta^{\mu\nu}_-(V)\partial_- y_\nu$ is the complete T-dual action and $\Pi_{+\mu\nu}(V) = B_{\mu\nu}(V) + \frac{1}{2}G_{\mu\nu}$. Consequently, although for the weakly curved background the functional integration over $v_\pm$ is of the third degree, it produces formally the same result as in the flat space (where the action is Gaussian),

$${}^\bullet\Phi = \Phi - \ln\det\sqrt{2\Pi_{+ab}}. \tag{114}$$

Using the expressions for T-dual fields (43) we can find the relations between the determinants

$$\begin{aligned}\det(2\Pi_{\pm ab}) &= \frac{1}{\det(2\,{}^\bullet\Pi^{ab}_\pm)} = \sqrt{\frac{\det G_{ab}}{\det {}^\bullet G^{ab}}}\\ &= \sqrt{\frac{\det G_{\mu\nu}}{\det {}^\bullet G_{\mu\nu}}},\end{aligned} \tag{115}$$

where because of the relation $\Pi_{\pm ab} = B_{ab} \pm \frac{1}{2}G_{ab}$ we put in the factor 2 for convenience. The symbol ${}^\bullet G_{\mu\nu}$ denotes metric in the whole space-time after partial T-dualization along x^a directions. With the help of last relation we can show that the change of space-time measure in the path integral is correct

$$\begin{aligned}\sqrt{\det G_{\mu\nu}}\,dx^i dx^a &\to \sqrt{\det G_{\mu\nu}}\,dx^i \frac{1}{\det(2\,\Pi_{+ab})}\,dy_a\\ &= \sqrt{\det {}^\bullet G_{\mu\nu}}\,dx^i dy_a,\end{aligned} \tag{116}$$

when we performed T-dualization T^a along x^a directions.

9 Comparison with the existing facts

9.1 T-dualization chain for the background with H flux

In this section we will compare our results with the T-dualization chain of Ref. [16]. The coordinates of the $D = 3$-dimensional torus will be denoted by x^1, x^2, x^3. Because of the different notation, the background fields considered in this paper and those considered in [16], which will be denoted $\mathcal{G}$ and $\mathcal{B}$, are related by

$$\mathcal{B}_{\mu\nu} = -2B_{\mu\nu}, \quad \mathcal{G}_{\mu\nu} = G_{\mu\nu}, \quad \mu, \nu = 1, 2, 3. \tag{117}$$

Nontrivial components of the background considered in Ref. [16] are

$$\mathcal{G}_{\mu\nu} = \delta_{\mu\nu}, \quad \mathcal{B}_{12} = Hx^3, \tag{118}$$

which in our notation corresponds to the background fields

$$G_{\mu\nu} = \delta_{\mu\nu}, \quad B_{12} = -\frac{1}{2}Hx^3. \tag{119}$$

Let us first compare the results in the case $d = 1$, corresponding to the transition

T^1 : torus with H-flux → twisted torus.

To do so, let us perform T-dualization along the direction x^1, $T^1 : S[x] \to S[y_1, x^2, x^3]$, for the string moving in the background (119). The indices take the values $a, b \in \{1\}$ and $i, j \in \{2, 3\}$. Because the only nontrivial component of the Kalb–Ramond field is $B_{ai} = -\frac{1}{2}Hx^3\delta_{i2}$, the effective fields are just $\tilde{G}^E_{\mu\nu} = \delta_{\mu\nu}$ and $\tilde{\theta}^{ab} = 0$. So, the T-dual background fields (44), in the linear order in H, are

$$\begin{aligned} &{}^\bullet G_{ij} = \delta_{ij}, \quad {}^\bullet B_{ij} = 0,\\ &{}^\bullet G^{ab} = \delta^{ab}, \quad {}^\bullet B^{ab} = 0,\\ &{}^\bullet G^a{}_i = -Hx^3\delta_{i2}, \quad {}^\bullet B^a{}_i = 0. \end{aligned} \tag{120}$$

Therefore

$${}^\bullet G_{\mu\nu} = \begin{pmatrix} 1 & -Hx^3 & 0\\ -Hx^3 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} = {}^\bullet\mathcal{G}_{\mu\nu}, \tag{121}$$

and

$${}^\bullet B_{\mu\nu} = 0 = {}^\bullet\mathcal{B}_{\mu\nu}, \tag{122}$$

so our result is in agreement with that of Ref. [16].

Now, let us make the comparison in the case $d = 2$, which corresponds to the transition

$T^1 \circ T^2$: torus with H-flux → Q-flux non-geometry.

Instead to perform T^2 dualization, from twisted torus to Q-flux non-geometry as in [16], we will start from the initial background with H-flux and perform T-dualizations along x^1 and x^2, $T^1 \circ T^2 : S[x] \to S[y_1, y_2, x^3]$. The indices take the values $a, b \in \{1, 2\}$ and $i, j \in \{3\}$. Because the only nontrivial contribution to the Kalb–Ramond field B_{ab} is $B_{12} = -\frac{1}{2}Hx^3$, the effective background fields are $\tilde{G}^E_{ab} = \delta_{ab}$, $\bar{G}^E_{ij} = \delta_{ij}$, and the only nonzero component of $\tilde{\theta}^{ab}$ is $\tilde{\theta}^{12} = \frac{1}{\kappa}Hx^3$. The T-dual background fields linear in H are therefore

$${}^\bullet G_{ij} = \delta_{ij}, \quad {}^\bullet G^{ab} = \delta^{ab}, \quad {}^\bullet G^a i = 0, \tag{123}$$

and

$${}^\bullet B_{ij} = 0, \quad {}^\bullet B^{12} = \frac{1}{2}Hx^3, \quad {}^\bullet B^a{}_i = 0. \tag{124}$$

Consequently

$${}^\bullet G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix} = {}^\bullet\mathcal{G}_{\mu\nu}, \tag{125}$$

$${}^\bullet\mathcal{B}_{\mu\nu} = -2\,{}^\bullet B_{\mu\nu} = \begin{pmatrix} 0 & -Hx^3 & 0\\ Hx^3 & 0 & 0\\ 0 & 0 & 0 \end{pmatrix}, \tag{126}$$

so the results of this paper and [16] in this case coincide.

9.2 Non-associativity of R-flux background and breaking of Jacobi identity

In Refs. [18,19] we obtained T-dual transformation laws connecting T-dual coordinates y_μ with the initial coordinates x^μ. Here we will reduce our case to the 3-dimensional torus with H-flux considered in [8]. Then, the full T-dualization along all coordinates corresponds to the so-called R-flux. So, we are going to calculate its characteristic features: non-associativity relation and breaking of Jacobi identity.

We will work in the background of Sect. 9.1 consisting of euclidean flat metric $G_{\mu\nu}$ and Kalb–Ramond field with one nontrivial component $B_{12} = -\frac{1}{2}Hx^3$. T-dual transformation laws for coordinates y_μ ($\mu = 1, 2, 3$) are of the form

$$y_1' \cong \frac{1}{\kappa}\pi_1 + \frac{1}{2}Hx^3x'^2, \tag{127}$$

$$y_2' \cong \frac{1}{\kappa}\pi_2 - \frac{1}{2}Hx^3x'^1, \tag{128}$$

$$y_3' \cong \frac{1}{\kappa}\pi_3, \tag{129}$$

where π_1, π_2, π_3 are canonically conjugated momenta for coordinates x^1, x^2, x^3, respectively. The initial space is a geometric one, so, the standard Poisson algebra is satisfied,

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu{}_\nu \delta(\sigma - \bar\sigma),$$
$$\{x^\mu, x^\nu\} = \{\pi_\mu, \pi_\nu\} = 0. \tag{130}$$

From (127)–(129) we obtain

$$\{y'_\mu(\sigma), y'_\nu(\bar\sigma)\} = -\frac{1}{2\kappa} H \varepsilon_{\mu\nu\rho} x'^\rho \delta(\sigma - \bar\sigma), \tag{131}$$

which, after two partial integrations, produces

$$\{y_\mu(\sigma), y_\nu(\bar\sigma)\} = \frac{1}{2\kappa} H \varepsilon_{\mu\nu\rho} \left[x^\rho(\sigma) - x^\rho(\bar\sigma)\right] \theta(\sigma - \bar\sigma), \tag{132}$$

where $\varepsilon_{\mu\nu\rho}$ is the 3-dimensional Levi-Civita tensor ($\varepsilon_{123} = 1$) and the function $\theta(\sigma)$ is defined as

$$\theta(\sigma) \equiv \begin{cases} 0 & \text{if} \quad \sigma = 0 \\ 1/2 & \text{if} \quad 0 < \sigma < 2\pi, \quad \sigma \in [0, 2\pi]. \\ 1 & \text{if} \quad \sigma = 2\pi \end{cases} \tag{133}$$

Using the standard Poisson algebra (130) and transformation laws (127)–(129), after one partial integration, we get

$$\begin{aligned} &\{\{y_\mu(\sigma_1), y_\nu(\sigma_2)\}, y_\rho(\sigma_3)\} \\ &\quad = \frac{1}{2\kappa^2} H \varepsilon_{\mu\nu\rho} [\theta(\sigma_2 - \sigma_1)\theta(\sigma_1 - \sigma_3) \\ &\quad\quad + \theta(\sigma_1 - \sigma_2)\theta(\sigma_2 - \sigma_3)], \end{aligned} \tag{134}$$

Now we have all ingredients to calculate the non-associativity relation

$$\begin{aligned} &\{\{y_\mu(\sigma_1), y_\nu(\sigma_2)\}, y_\rho(\sigma_3)\} - \{y_\mu(\sigma_1), \{y_\nu(\sigma_2), y_\rho(\sigma_3)\}\} \\ &\quad = \frac{1}{2\kappa^2} H \varepsilon_{\mu\nu\rho} [2\theta(\sigma_3 - \sigma_2)\theta(\sigma_2 - \sigma_1) \\ &\quad\quad + \theta(\sigma_1 - \sigma_3)\theta(\sigma_3 - \sigma_2) + \theta(\sigma_3 - \sigma_1)\theta(\sigma_1 - \sigma_2)] \end{aligned} \tag{135}$$

and breaking of Jacobi identity

$$\begin{aligned} &\{y_\mu(\sigma_1), y_\nu(\sigma_2), y_\rho(\sigma_3)\} \\ &\quad \equiv \{\{y_\mu(\sigma_1), y_\nu(\sigma_2)\}, y_\rho(\sigma_3)\} + \{\{y_\nu(\sigma_2), y_\rho(\sigma_3)\}, y_\mu(\sigma_1)\} \\ &\quad\quad + \{\{y_\rho(\sigma_3), y_\mu(\sigma_1)\}, y_\nu(\sigma_2)\} \\ &\quad = \frac{1}{\kappa^2} H \varepsilon_{\mu\nu\rho} [\theta(\sigma_1 - \sigma_2)\theta(\sigma_2 - \sigma_3) \\ &\quad\quad + \theta(\sigma_3 - \sigma_1)\theta(\sigma_1 - \sigma_2) + \theta(\sigma_2 - \sigma_3)\theta(\sigma_3 - \sigma_1)]. \end{aligned} \tag{136}$$

For example, for $\sigma_1 = 2\pi + \sigma$ and $\sigma_2 = \sigma_3 = \sigma$ one has

$$\{y_\mu(2\pi + \sigma), y_\nu(\sigma), y_\rho(\sigma)\} = -\frac{1}{\kappa^2} H \varepsilon_{\mu\nu\rho}. \tag{137}$$

In the approach of this article, the background of the T-dual theory depends on the non-local variable V^μ, which incorporates the main features of the non-geometric spaces. Reducing our procedure to three dimensions and using the backgrounds of Refs. [8,16,27], we showed that our structure of arguments of background fields proves the proposal of Refs. [8,27] that non-associativity and breaking of Jacobi identity are features of R-flux background.

9.3 Critical surface

Let us generalize the discussion of Ref. [20] where the critical surface, which separates equivalent sections of background fields, generalizes the critical radius. Using the dilaton field analysis, namely the relation (115), we can conclude that T-duality maps the theories with a given

$$\det(2\Pi_{\pm ab})$$

into the theories with

$$1/\det(2\Pi_{\pm ab}),$$

so that all different theories are in the region

$$\det(2\Pi_{\pm ab}) \le 0.$$

The theories which background fields satisfy the condition $\det(2\Pi_{\pm ab}) = 1$, are mapped into each other under T-duality. This is a generalization of the critical radius and can be consider as a critical surface. So, relation (115) implies $\sqrt{\det G_{ab}} = \sqrt{\det {}^\bullet G^{ab}}$, which means that a dual volume is equal to the initial one. At the critical surface the extended symmetry should be expected.

Let us, following [20], give an example of the relation between the original and T-dual background fields. We will consider the initial background in the 4-dimensional torus T^4 given by

$$G_{\mu\nu} = g\delta_{\mu\nu}, \quad B_{\mu\nu} = b^i E^i_{\mu\nu}, \tag{138}$$

where

$$E^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad E^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$
$$E^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \tag{139}$$

satisfies

$$E^i E^j = -\delta^{ij} I + \varepsilon^{ijk} E^k, \qquad \varepsilon^{123} = 1. \tag{140}$$

The zero modes of the T-dual metric and T-dual Kalb–Ramond field (17) for the initial fields (138) are

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu} = \frac{g}{g^2+b^2}\, I \tag{141}$$

and

$$^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu} = -\frac{1}{2}\frac{b^i}{g^2+b^2}\, E^i, \tag{142}$$

with $b^2 = b^i b^i$. They have the same form as the initial fields (138)

$$^\star G_{\mu\nu} = {}^\star g\delta_{\mu\nu}, \quad {}^\star B_{\mu\nu} = {}^\star b^i E^i_{\mu\nu}, \tag{143}$$

with

$$^\star g = \frac{g}{g^2+b^2}, \quad {}^\star b = -\frac{b^i}{g^2+b^2}. \tag{144}$$

One easily shows

$$^\star g^2 + {}^\star b^2 = \frac{1}{g^2+b^2}. \tag{145}$$

In spheric coordinates one has

$$(g, b^1, b^2, b^3) = (r\cos\theta, r\sin\theta\cos\varphi, \\ r\sin\theta\sin\varphi\cos\varphi_1, r\sin\theta\sin\varphi\sin\varphi_1), \tag{146}$$

so $g^2 + b^2 = r^2$, and using (144) one obtains

$$({}^\star g, {}^\star b^1, {}^\star b^2, {}^\star b^3) \\ = \left(\frac{1}{r}\cos\theta, -\frac{1}{r}\sin\theta\cos\varphi, -\frac{1}{r}\sin\theta\sin\varphi\cos\varphi_1, \right. \\ \left. -\frac{1}{r}\sin\theta\sin\varphi\sin\varphi_1\right). \tag{147}$$

Therefore, T-duality transforms $(r, \theta, \varphi, \varphi_1)$ to

$$({}^\star r, {}^\star\theta, {}^\star\varphi, {}^\star\varphi_1) = \left(\frac{1}{r}, -\theta, \varphi, \varphi_1\right). \tag{148}$$

From the relation $\Pi_\pm G^{-1}\Pi_\mp = -\frac{1}{4}G_E$ we find

$$\det(2\Pi_{\pm\mu\nu}) = \frac{g^2}{{}^\star g^2} = (g^2+b^2)^2 = r^4. \tag{149}$$

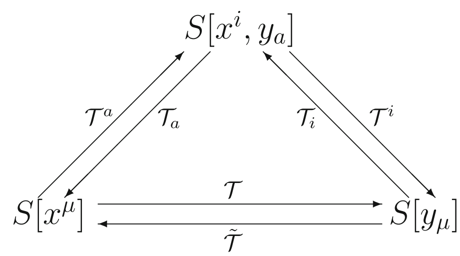
Backgrounds corresponding to $r = 1$ are mapped into themselves. The subset of this is the fixed surface with the condition

$$\det(2\Pi_{\pm\mu\nu}) = r^4 = 1, \theta = 0$$

or $g = 1$, $b^i = 0$.

10 Conclusion

In this paper, we considered the closed string propagating in the weakly curved background (6), composed of a constant metric $G_{\mu\nu}$ and a linearly coordinate dependent Kalb–Ramond field $B_{\mu\nu}$, with infinitesimal field strength. We investigated the application of the generalized T-dualization procedure on the arbitrary set of coordinates and obtained the following T-duality diagram:



Let us stress that generalized T-dualization procedure enables the T-dualization along arbitrary direction, even if the background fields depend on these directions. The consequence of this procedure is that the arguments of the background fields, such as ΔV^a, are non-local. They are non-local by definition, as they are the line integrals of the gauge fields. Once the explicit form is obtained the non-locality is seen in a fact that they depend on double coordinates $\tilde{x}$ and $\tilde{y}$, which are the line integrals of the τ and σ derivatives of the original coordinates. To all the theories considered, except the initial theory, there corresponds the non-geometric, non-local flux.

The generalized T-dualization procedure was first applied along arbitrary d ($d = 1, \ldots, D-1$) coordinates $x^a = \{x^{\mu_1}, \ldots, x^{\mu_d}\}$. We obtained the T-dual action $S[x^i, y_a]$, given by Eq. (36) with the dual background fields equal to

$$^\bullet\Pi_{+ij} = \overline{\Pi}_{+ij}, \quad {}^\bullet\Pi_{+i}{}^a = -\kappa\Pi_{+ib}\tilde{\Theta}_-^{ba}, \\ ^\bullet\Pi_{+i}^a = \kappa\tilde{\Theta}_-^{ab}\Pi_{+bi}, \quad {}^\bullet\Pi_+^{ab} = \frac{\kappa}{2}\tilde{\Theta}_-^{ab}. \tag{150}$$

The argument of all background fields, $[x^i, V^a(x^i, y_a)]$, depends nonlinearly on coordinates x^i, y_a through their doubles $\tilde{x}^i$, $\tilde{y}_a$ [see (39) and (40)]. All actions $S[x^i, y_a]$ are physically equivalent, but they are described with coordinates $x^i = \{x^{\mu_{d+1}}, \ldots, x^{\mu_D}\}$, for the untreated directions and dual coordinates $y_a = \{y_{\mu_1}, \ldots, y_{\mu_d}\}$, for the dualized directions. The case $d = D$ corresponds to the completely T-dual action with the T-dual fields $\frac{\kappa}{2}\Theta_-^{\mu\nu}\big(V(y)\big)$ and the case $d = 0$ to the initial action with the background $\Pi_{+\mu\nu}(x)$.

Applying the procedure to the T-dual action along dual directions $y_a = \{y_{\mu_1}, \ldots, y_{\mu_d}\}$ we obtained the initial theory, and applying it to the untreated directions $x^i = \{x^{\mu_{d+1}}, \ldots, x^{\mu_D}\}$ we obtained the completely T-dual theory. All these derivations confirmed that the set of all T-dualizations forms an Abelian group. The neutral element of the group is the unexecuted T-dualization, while the T-dualizations along some subset of original directions $\mathcal{T}^a$ is inverse to the T-dualizations along the set of the corresponding dual directions $\mathcal{T}_a$.

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Appendix A: The background field compositions

The background field compositions $\Pi_{\pm\mu\nu}$ of the initial theory are

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}, \tag{A.1}$$

where $G_{\mu\nu}$ and $B_{\mu\nu}$ are the initial metric and the initial Kalb–Ramond field. The background field compositions $\Theta^{\mu\nu}_{\pm}$ of the T-dual theory are

$$\Theta^{\mu\nu}_{\pm} \equiv -\frac{2}{\kappa}(G_E^{-1}\Pi_{\pm}G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}, \tag{A.2}$$

with $G_{E\mu\nu}$ being the effective metric

$$G_{E\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \tag{A.3}$$

and $\theta^{\mu\nu}$ being the parameter of non-commutativity

$$\theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}. \tag{A.4}$$

These background field compositions satisfy

$$\Pi_{\pm\mu\nu}\Theta^{\nu\rho}_{\mp} = \Theta^{\rho\nu}_{\pm}\Pi_{\mp\nu\mu} = \frac{1}{2\kappa}\delta^{\rho}_{\mu}. \tag{A.5}$$

Let us define the analogs of $\Theta^{\mu\nu}_{\pm}$ in the d- and $D-d$-dimensional subspaces determined by coordinates $x^a = \{x^{\mu_1}, \ldots, x^{\mu_d}\}$ and $x^i = \{x^{\mu_{d+1}}, \ldots, x^{\mu_D}\}$, where $d = 1, 2, \ldots, D-1$. The effective metrics in these subspaces are defined by

$$\begin{aligned}\tilde{G}_{Eab} &\equiv G_{ab} - 4B_{ac}(\tilde{G}^{-1})^{cd}B_{db},\\ \bar{G}_{Eij} &\equiv G_{ij} - 4B_{ik}(\bar{G}^{-1})^{kl}B_{lj},\end{aligned} \tag{A.6}$$

where $\tilde{G}_{ab} \equiv G_{ab}$ and $\bar{G}_{ij} \equiv G_{ij}$. Using these we define the following field compositions:

$$\begin{aligned}\tilde{\Theta}^{ab}_{\pm} &\equiv -\frac{2}{\kappa}(\tilde{G}_E^{-1})^{ac}\Pi_{\pm cd}(\tilde{G}^{-1})^{db},\\ \bar{\Theta}^{ij}_{\pm} &\equiv -\frac{2}{\kappa}(\bar{G}_E^{-1})^{ik}\Pi_{\pm kl}(\bar{G}^{-1})^{lj},\end{aligned} \tag{A.7}$$

which are in fact the inverses of $2\kappa\Pi_{\mp ab}$ and $2\kappa\Pi_{\mp ij}$

$$\begin{aligned}\tilde{\Theta}^{ab}_{\pm}\Pi_{\mp bc} &= \Pi_{\mp cb}\tilde{\Theta}^{ba}_{\pm} = \frac{1}{2\kappa}\delta^a_c,\\ \bar{\Theta}^{ij}_{\pm}\Pi_{\mp jk} &= \Pi_{\mp kj}\bar{\Theta}^{ji}_{\pm} = \frac{1}{2\kappa}\delta^i_k.\end{aligned} \tag{A.8}$$

Analogously as the fields theta $\Theta^{\mu\nu}_{\pm}$ defined in the whole space by (A.2), the theta fields defined in the subspaces can be separated into antisymmetric and symmetric parts as

$$\begin{aligned}\tilde{\Theta}^{ab}_{\pm} &= \tilde{\theta}^{ab} \mp \frac{1}{\kappa}(\tilde{G}_E^{-1})^{ab},\\ \bar{\Theta}^{ij}_{\pm} &= \bar{\theta}^{ij} \mp \frac{1}{\kappa}(\bar{G}_E^{-1})^{ij},\end{aligned} \tag{A.9}$$

where

$$\begin{aligned}\tilde{\theta}^{ab} &\equiv -\frac{2}{\kappa}(\tilde{G}_E^{-1})^{ac}B_{cd}(\tilde{G}^{-1})^{db},\\ \bar{\theta}^{ij} &\equiv -\frac{2}{\kappa}(\bar{G}_E^{-1})^{ik}B_{kl}(\bar{G}^{-1})^{lj}.\end{aligned} \tag{A.10}$$

In the zeroth order the quantities $\Pi_{\pm\mu\nu}$, $\Theta^{\mu\nu}_{\pm}$, $\tilde{\Theta}^{ab}_{\pm}$, and $\bar{\Theta}^{ij}_{\pm}$ reduce to

$$\begin{aligned}\Pi_{0\pm\mu\nu} &= b_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu},\\ \Theta^{\mu\nu}_{0\pm} &= -\frac{2}{\kappa}(g^{-1})^{\mu\rho}\Pi_{0\pm\rho\sigma}(G^{-1})^{\sigma\nu} = \theta^{\mu\nu}_0 \mp \frac{1}{\kappa}(g^{-1})^{\mu\nu},\\ \tilde{\Theta}^{ab}_{0\pm} &= -\frac{2}{\kappa}(\tilde{g}^{-1})^{ac}\,\Pi_{0\pm cd}(\tilde{G}^{-1})^{db} = \tilde{\theta}^{ab}_0 \mp \frac{1}{\kappa}(\tilde{g}^{-1})^{ab},\\ \bar{\Theta}^{ij}_{0\pm} &= -\frac{2}{\kappa}(\bar{g}^{-1})^{ik}\,\Pi_{0\pm kl}(\bar{G}^{-1})^{lj} = \bar{\theta}^{ij}_0 \mp \frac{1}{\kappa}(\bar{g}^{-1})^{ij},\end{aligned} \tag{A.11}$$

where the zeroth order effective metrics are

$$\begin{aligned}g_{\mu\nu} &= G_{\mu\nu} - 4b_{\mu\rho}(G^{-1})^{\rho\sigma}b_{\sigma\nu},\\ \tilde{g}_{ab} &= G_{ab} - 4b_{ac}(\tilde{G}^{-1})^{cd}b_{db},\\ \bar{g}_{ij} &= G_{ij} - 4b_{ik}(\bar{G}^{-1})^{kl}b_{lj},\end{aligned} \tag{A.12}$$

and the zeroth order non-commutativity parameters are

$$\begin{aligned}\theta^{\mu\nu}_0 &= -\frac{2}{\kappa}(g^{-1})^{\mu\rho}b_{\rho\sigma}(G^{-1})^{\sigma\nu},\\ \tilde{\theta}^{ab}_0 &= -\frac{2}{\kappa}(\tilde{g}^{-1})^{ac}\,b_{cd}(\tilde{G}^{-1})^{db}\\ \bar{\theta}^{ij}_0 &= -\frac{2}{\kappa}(\bar{g}^{-1})^{ik}\,b_{kl}(\bar{G}^{-1})^{lj}.\end{aligned} \tag{A.13}$$

Quantities $\Pi_{0\pm\mu\nu}$, $\Theta^{\mu\nu}_{0\pm}$, $\tilde{\Theta}^{ab}_{0\pm}$, and $\bar{\Theta}^{ij}_{0\pm}$ satisfy

$$\Pi_{0\pm\mu\nu}\Theta^{\nu\rho}_{0\mp} = \Theta^{\rho\nu}_{0\pm}\Pi_{0\mp\nu\mu} = \frac{1}{2\kappa}\delta^{\rho}_{\mu},$$
$$\Pi_{0\pm ab}\tilde{\Theta}^{bc}_{0\mp} = \tilde{\Theta}^{cb}_{0\pm}\Pi_{0\mp ba} = \frac{1}{2\kappa}\delta^{c}_{a},$$
$$\Pi_{0\pm ij}\bar{\Theta}^{jk}_{0\mp} = \bar{\Theta}^{kj}_{0\pm}\Pi_{0\mp ji} = \frac{1}{2\kappa}\delta^{k}_{i}. \tag{A.14}$$

The non-commutativity parameters theta $\Theta^{\mu\nu}_{\pm}$, $\tilde{\Theta}^{ab}_{\pm}$, and $\bar{\Theta}^{ij}_{\pm}$ can be expressed as

$$\Theta^{\mu\nu}_{\pm} = \Theta^{\mu\nu}_{0\pm} - 2\kappa\Theta^{\mu\rho}_{0\pm}h_{\rho\sigma}\Theta^{\sigma\nu}_{0\pm},$$
$$\tilde{\Theta}^{ab}_{\pm} = \tilde{\Theta}^{ab}_{0\pm} - 2\kappa\tilde{\Theta}^{ac}_{0\pm}h_{cd}\tilde{\Theta}^{db}_{0\pm},$$
$$\bar{\Theta}^{ij}_{\pm} = \bar{\Theta}^{ij}_{0\pm} - 2\kappa\bar{\Theta}^{ik}_{0\pm}h_{kl}\bar{\Theta}^{lj}_{0\pm}. \tag{A.15}$$

Appendix A.1: Relations between field compositions

In Sect. 3.2 we introduced the background field composition

$$\overline{\Pi}_{\pm ij} \equiv \Pi_{\pm ij} - 2\kappa\Pi_{\pm ia}\tilde{\Theta}^{ab}_{\mp}\Pi_{\pm bj}, \tag{A.16}$$

and analogously we define

$$\widetilde{\Pi}_{\pm ab} \equiv \Pi_{\pm ab} - 2\kappa\Pi_{\pm ai}\bar{\Theta}^{ij}_{\mp}\Pi_{\pm jb}. \tag{A.17}$$

Here we will show that these quantities are the inverses of the ordinary non-commutativity parameters theta, projected to the i- and a-subspaces [see (A.22)].

Let us express the tensors $\Pi_{\pm\mu\nu}$ and $\Theta^{\mu\nu}_{\pm}$, which satisfy (A.5), in a block-wise form as

$$\Pi_{\pm\mu\nu} = \begin{pmatrix} \Pi_{\pm ij} & \Pi_{\pm ib} \\ \Pi_{\pm aj} & \Pi_{\pm ab} \end{pmatrix}, \quad \Theta^{\mu\nu}_{\pm} = \begin{pmatrix} \Theta^{ij}_{\pm} & \Theta^{ib}_{\pm} \\ \Theta^{aj}_{\pm} & \Theta^{ab}_{\pm} \end{pmatrix}. \tag{A.18}$$

We will use the definition of block-wise inversion, which states that the inverse of the matrix of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A.19}$$

equals

$$M^{-1} = \begin{pmatrix} (A-BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -D^{-1}C(A-BD^{-1}C)^{-1} & (D-CA^{-1}B)^{-1} \end{pmatrix}. \tag{A.20}$$

Applying (A.20) to the first matrix in (A.18), Eq. (A.5) implies

$$2\kappa\Theta^{ij}_{\mp} = \left(\Pi_{\pm ij} - 2\kappa\Pi_{\pm ia}\tilde{\Theta}^{ab}_{\mp}\Pi_{\pm bj}\right)^{-1},$$
$$2\kappa\Theta^{ib}_{\mp} = -2\kappa\bar{\Theta}^{ij}_{\mp}\Pi_{\pm ja}(\Pi_{\pm ab} - 2\kappa\Pi_{\pm ak}\bar{\Theta}^{kl}_{\mp}\Pi_{\pm lb})^{-1},$$
$$2\kappa\Theta^{aj}_{\mp} = -2\kappa\tilde{\Theta}^{ab}_{\mp}\Pi_{\pm bi}(\Pi_{\pm ij} - 2\kappa\Pi_{\pm ic}\tilde{\Theta}^{cd}_{\mp}\Pi_{\pm dj})^{-1},$$
$$2\kappa\Theta^{ab}_{\mp} = (\Pi_{\pm ab} - 2\kappa\Pi_{\pm ai}\bar{\Theta}^{ij}_{\mp}\Pi_{\pm jb})^{-1}, \tag{A.21}$$

and we can conclude that (A.16) and (A.17) are the inverses of $2\kappa\Theta^{ij}_{\mp}$ and $2\kappa\Theta^{ab}_{\mp}$, respectively. So, we can write

$$\overline{\Pi}_{\pm ij}\Theta^{jk}_{\mp} = \Theta^{kj}_{\mp}\overline{\Pi}_{\pm ji} = \frac{1}{2\kappa}\delta^{k}_{i},$$
$$\widetilde{\Pi}_{\pm ab}\Theta^{bc}_{\mp} = \Theta^{cb}_{\mp}\widetilde{\Pi}_{\pm ba} = \frac{1}{2\kappa}\delta^{c}_{a}, \tag{A.22}$$

and

$$\Theta^{ib}_{\mp} = -2\kappa\bar{\Theta}^{ij}_{\mp}\Pi_{\pm ja}\Theta^{ab}_{\mp},$$
$$\Theta^{aj}_{\mp} = -2\kappa\tilde{\Theta}^{ab}_{\mp}\Pi_{\pm bi}\Theta^{ij}_{\mp}. \tag{A.23}$$

Applying (A.20) to the second matrix in (A.18), Eq. (A.5) implies

$$2\kappa\Pi_{\mp ij} = (\Theta^{ij}_{\pm} - 2\kappa\Theta^{ia}_{\pm}\widetilde{\Pi}_{ab\mp}\Theta^{bj}_{\pm})^{-1},$$
$$2\kappa\Pi_{\mp ib} = -2\kappa\overline{\Pi}_{\mp ij}\Theta^{ja}_{\pm}(\Theta^{ab}_{\pm} - 2\kappa\Theta^{ak}_{\pm}\overline{\Pi}_{\mp kl}\Theta^{lb}_{\pm})^{-1},$$
$$2\kappa\Pi_{\mp aj} = -2\kappa\widetilde{\Pi}_{ab\mp}\Theta^{bi}_{\pm}(\Theta^{ij}_{\pm} - 2\kappa\Theta^{ic}_{\pm}\widetilde{\Pi}_{\mp cd}\Theta^{dj}_{\pm})^{-1},$$
$$2\kappa\Pi_{\mp ab} = (\Theta^{ab}_{\pm} - 2\kappa\Theta^{ai}_{\pm}\overline{\Pi}_{\mp ij}\Theta^{jb}_{\pm})^{-1}, \tag{A.24}$$

so using (A.8) we conclude that

$$\bar{\Theta}^{ij}_{\pm} = \Theta^{ij}_{\pm} - 2\kappa\Theta^{ia}_{\pm}\widetilde{\Pi}_{ab\mp}\Theta^{bj}_{\pm},$$
$$\tilde{\Theta}^{ab}_{\pm} = \Theta^{ab}_{\pm} - 2\kappa\Theta^{ai}_{\pm}\overline{\Pi}_{\mp ij}\Theta^{jb}_{\pm}, \tag{A.25}$$

and

$$\Pi_{\mp ib} = -2\kappa\overline{\Pi}_{\mp ij}\Theta^{ja}_{\pm}\Pi_{\mp ab},$$
$$\Pi_{\mp aj} = -2\kappa\widetilde{\Pi}_{\mp ab}\Theta^{bi}_{\pm}\Pi_{\mp ij}. \tag{A.26}$$

Let us derive some useful relations between these quantities. Equation (A.5), for $\mu = a$, $\nu = i$ and $\mu = i$, $\nu = a$, becomes

$$\Pi_{\pm ab}\Theta^{bi}_{\mp} = -\Pi_{\pm aj}\Theta^{ji}_{\mp},$$
$$\Pi_{\pm ij}\Theta^{ja}_{\mp} = -\Pi_{\pm ib}\Theta^{ba}_{\mp}, \tag{A.27}$$

while taking $\mu = a$, $\nu = b$ and $\mu = i$, $\nu = j$ we obtain

$$\Pi_{\pm ac}\Theta^{cb}_{\mp} + \Pi_{\pm ai}\Theta^{ib}_{\mp} = \frac{1}{2\kappa}\delta^{b}_{a},$$
$$\Pi_{\pm ia}\Theta^{aj}_{\mp} + \Pi_{\pm ik}\Theta^{kj}_{\mp} = \frac{1}{2\kappa}\delta^{j}_{i}. \tag{A.28}$$

Multiplying Eq. (A.27) from the left with $\tilde{\Theta}^{ca}_{\mp}$ and from the right with $\bar{\Pi}_{\mp ik}$ we get the relation

$$\Theta^{ci}_{\mp}\overline{\Pi}_{\pm ik} = -\tilde{\Theta}^{ca}_{\mp}\Pi_{\pm ak}, \tag{A.29}$$

while multiplying Eq. (A.28) from the right with $\bar{\Theta}^{ki}_{\mp}$ and from the left with $\tilde{\Pi}_{\pm ac}$, we obtain

$$\Theta^{ka}_{\mp}\widetilde{\Pi}_{\pm ac} = -\bar{\Theta}^{ki}_{\mp}\Pi_{\pm ic}. \tag{A.30}$$

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T-dualization in a curved background in absence of a global symmetry[1]

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Abstract: We investigate T-duality of a closed string moving in a weakly curved background of the second order. A previously discussed weakly curved background consisted of a flat metric and a linearly coordinate dependent Kalb-Ramond field with an infinitesimal strength. The background here considered differs from the above in a coordinate dependent metric of the second order. Consequently, the corresponding Ricci tensor is nonzero. As this background does not posses the global shift symmetry the generalized Buscher T-dualization procedure is not applicable to it. We redefine it and make it applicable to backgrounds without the global symmetry.

Keywords: Bosonic Strings, String Duality

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Contents

1 Introduction

T-duality [1–5] was first introduced to represent the fact that the toroidal compactifications [6–9] of a closed string to a radius R and a radius $1/R$ are equivalent. Although the compactified theories are defined for different target spaces, their spectrum is the same. The observed symmetry lead to investigation for discovering the connections between the theories having the same spectrum, what resulted in T-dualization procedures. These prescriptions consist of rules which transform the given theory to its T-dual theory. The T-dual theories describing strings moving in a geometrically different backgrounds, having the same predictions, were found for some particular backgrounds. The general procedure, applicable to an arbitrary background is still to be determined.

T-duality was found to be connected with the isometries of a sigma model. This discovery was included as an inevitable condition for T-dualization, as the isometry was built in the T-dualization procedure. The first procedure determining the T-duals of the constant background fields was the Buscher procedure [10–13]. This procedure, called the standard T-dualization procedure, is founded in localizing the global isometry by introducing the gauge fields, whose field strength is set to zero by a Lagrange multiplier term. The gauged theory reduces to the initial theory for the equations of motion for the Lagrange multiplier and to the T-dual theory for the equations of motion for the gauge fields. This procedure enabled investigation of the coordinate dependent backgrounds as well, when the T-dualizations are performed along the directions which do not appear as the background fields arguments.

The generalized T-dualization proposed in paper [14], addressing the non-abelian isometries lead to observation that the application of T-dualization procedures can lead to theories without an isometry. These theories are obviously T-dual to the initial theories, but, the initial theories can not be obtained as a result of the same T-dualization procedure starting with the theory without isometry. This observation implied that T-duality in general must be understood from some other perspective.

The investigation of the relations between non geometric backgrounds lead to a new generalized T-dualization in string field theory [15]. It was proposed that particular non geometric backgrounds should be understood as string backgrounds which are the result of generalized T-dualization applied along nonisometry directions. In paper [16], the conditions for a background to have a geometric or non geometric T-dual were sought for. It was concluded that the large class of sigma models that cannot be gauged can be T-dualized.

In paper [17], one considers similarity transformations of the stress energy tensor of a conformal field theory which do not change the Virasoro alebra. There exists the transformations of the background fields which produce the same change of the stress tensor as the change generated by some similarity transformations. A particular generator of a similarity transformation produces symmetry transformation, such as general coordinate transformations and gauge transformations of a Kalb-Ramond field. However, some particular forms of these generators produce T-duality transformations at critical radius. Investigation leads to T-dualization techniques directly applicable to an arbitrary string backgrounds.

In paper [18], a generalization of a Buscher T-dualization procedure was given. The generalized procedure is applicable to backgrounds depending on all the space-time coordinates, along arbitrary background fields argument. The procedure was realized for a weakly curved background which consists of a constant metric and a coordinate dependent Kalb-Ramond field with an infinitesimal strength. The difference between the generalized and standard Buscher procedure is in an invariant coordinate. Standardly, one substitutes the derivatives with the covariant derivatives to obtain the gauge invariant action, but in the generalized procedure one additionally substitutes the argument of the background field with the invariant argument. It was confirmed in [18] that the generalized T-dualization procedure does not harm the interchange of equations of motion and Bianchi identities [19]. However, it strongly changes the geometry of a target space. The geometric space is transformed to a double non-geometric space. The commutative space is transformed to a non-commutative space, as shown in [20]. The closed string non-commutativity was previously investigated in [21–25]. The application of a procedure to an arbitrary set of coordinates was considered in [26]. It was concluded that the geometric background again transforms to a double space, with double coordinates present for both T-dualized and undualized directions.

In the present paper, we consider the weakly curved background of the second order. We take a metric which consists of a constant and quadratic in coordinate term and linearly coordinate dependent Kalb-Ramond field. This background does not posses the global shift symmetry. In comparison to the previously considered backgrounds there is an additional difference. The Ricci tensor of the metric here considered is nonzero. The background has to be the solution of the space-time equations of motion, obtained from the demand of the conformal invariance of the quantum theory. To satisfy these equations one takes the coordinate dependent parts to be infinitesimal.

The original form of the generalized Buscher T-dualization procedure [18] is not applicable to a weakly curved background of the second order. Here, we search for the procedure which will be applicable and will preserve the general features of the previous procedure. We find the appropriate formulation and investigate the properties and the consequences of the new generalization. We apply the procedure along all space-time coordinates and obtain the T-dual theory. We obtain a geometrical structure that differs from the double non geometrical space. The dual background field arguments do not depend only on the dual coordinate and its double. However the application of the procedure to all dual coordinates leads again to the initial theory. We obtain T-dual coordinate transformation laws and confirm that T-duality interchanges equations of motion and Bianchi identities.

2 Bosonic string action and choice of background

Let us consider the closed bosonic string propagating in the background fields: a metric $G_{\mu\nu}$ and a Kalb-Ramond antisymmetric tensor field $B_{\mu\nu}$, described by the action

$$S[x] = \kappa \int_\Sigma d^2\xi\, \partial_\alpha x^\mu \left[\frac{1}{2}\eta^{\alpha\beta} G_{\mu\nu}(x) + \epsilon^{\alpha\beta} B_{\mu\nu}(x)\right] \partial_\beta x^\nu, \quad \varepsilon^{01} = -1. \tag{2.1}$$

The integration goes over two-dimensional world-sheet Σ parametrized by $\xi^\alpha, \alpha = 0, 1$ ($\xi^0 = \tau$, $\xi^1 = \sigma$). The coordinates of the D-dimensional space-time are marked by $x^\mu(\xi)$, $\mu = 0, 1, \ldots, D-1$. From the action principle one obtains the equations of motion

$$\ddot{x}^\mu - x''^\mu + 2B^\mu{}_{\nu\rho} \dot{x}^\nu x'^\rho + \Gamma^\mu_{\nu\rho}(\dot{x}^\nu \dot{x}^\rho - x'^\nu x'^\rho) = 0, \tag{2.2}$$

where $B^\mu{}_{\nu\rho} = (G^{-1})^{\mu\sigma} B_{\sigma\nu\rho}$ and $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$ and $\Gamma^\mu_{\nu\rho} = \frac{1}{2}(G^{-1})^{\mu\sigma}(\partial_\nu G_{\rho\sigma} + \partial_\rho G_{\sigma\nu} - \partial_\sigma G_{\nu\rho})$ is a Christoffel symbol.

Introducing the light-cone coordinates and their derivatives

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma), \qquad \partial_\pm = \partial_\tau \pm \partial_\sigma, \tag{2.3}$$

the action (2.1) can be rewritten as

$$S[x] = \kappa \int_\Sigma d^2\xi\ \partial_+ x^\mu \Pi_{+\mu\nu}(x) \partial_- x^\nu, \tag{2.4}$$

where $\Pi_{\pm\mu\nu}$ is the combination of background fields, defined by

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2}G_{\mu\nu}(x). \tag{2.5}$$

The equation of motion (2.2) can be rewritten as

$$\partial_+\partial_- x^\mu + \Big(\Gamma^\mu_{\nu\rho} - B^\mu{}_{\nu\rho}\Big)\partial_+ x^\nu \partial_- x^\rho = 0. \tag{2.6}$$

In order to obtain a conformally invariant quantum theory, the background fields must obey the space-time equations of motion, which for the constant dilaton field have the following form

$$R_{\mu\nu} - B_{\mu\rho\sigma}B_\nu{}^{\rho\sigma} = 0\,, \tag{2.7}$$

$$D_\rho B^\rho{}_{\mu\nu} = 0, \tag{2.8}$$

where $R_{\mu\nu}$ is Ricci tensor defined by

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}, \quad R^\rho{}_{\mu\sigma\nu} = \Gamma^\rho_{\mu\nu,\sigma} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\tau_{\mu\nu}\Gamma^\rho_{\tau\sigma} - \Gamma^\tau_{\mu\sigma}\Gamma^\rho_{\tau\nu}, \tag{2.9}$$

and D_μ is a covariant derivative

$$D_\rho B^\sigma{}_{\mu\nu} = \partial_\rho B^\sigma{}_{\mu\nu} + \Gamma^\sigma_{\varepsilon\rho}B^\varepsilon{}_{\mu\nu} - \Gamma^\varepsilon_{\mu\rho}B^\sigma{}_{\varepsilon\nu} - \Gamma^\varepsilon_{\nu\rho}B^\sigma{}_{\mu\varepsilon}. \tag{2.10}$$

We will consider the following solution of the space-time equations of motion (2.7), (2.8)

$$G_{\mu\nu}(x) = g_{\mu\nu} + 3h^2_{\mu\nu}(x), \quad B_{\mu\nu}(x) = b_{\mu\nu} + h_{\mu\nu}(x), \tag{2.11}$$

with $g_{\mu\nu}, b_{\mu\nu} = const$ and $h_{\mu\nu} \equiv \frac{1}{3}B_{\mu\nu\rho}x^\rho$, $\quad h^2_{\mu\nu} \equiv (hg^{-1}h)_{\mu\nu}$, where $B_{\mu\nu\rho}$ is constant and infinitesimal. Throughout the paper the calculation will be done up to the second order in $B_{\mu\nu\rho}$. We will refer to this solution as the weakly curved background of the second order. More general discussion about the solutions of the space-time equations of motion, up to the second order terms, can be found in ref. [27].

Let us demonstrate that (2.11) satisfies (2.7) and (2.8). The inverse metric and the Cristoffel symbol are

$$(G^{-1})^{\mu\nu} = (g^{-1})^{\mu\nu} - 3(h^2)^{\mu\nu}, \tag{2.12}$$

$$\Gamma^\mu_{\nu\rho} = -B^\mu{}_{\nu\sigma}h^\sigma{}_\rho - B^\mu{}_{\rho\sigma}h^\sigma{}_\nu, \tag{2.13}$$

with $(h^2)^{\mu\nu} = (g^{-1}hg^{-1}hg^{-1})^{\mu\nu}$ and $B^\mu{}_{\nu\sigma}h^\sigma{}_\rho = (g^{-1})^{\mu\varepsilon}B_{\varepsilon\nu\sigma}(g^{-1})^{\sigma\tau}h_{\tau\rho}$. Therefore, the Ricci tensor equals

$$R_{\mu\nu} = B_{\mu\rho\sigma}B^{\rho\sigma}{}_\nu, \tag{2.14}$$

which is just the eq. (2.7). The equation (2.8) is satisfied, because the term corresponding to the first term in (2.10) is zero and the others can be neglected as the third order terms in $B_{\mu\nu\rho}$.

Let us notice, using (2.9) and (2.13), that the coefficient in the second order term of the metric (2.11) is in fact the Riemann curvature tensor

$$G_{\mu\nu} = g_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\sigma}x^\rho x^\sigma. \tag{2.15}$$

The solutions which where previously investigated in this context where the constant background and the weakly curved background of the first order. In both cases the Ricci tensor $R_{\mu\nu}$ is absent, in the first case because it equals zero and in the second because it is neglected as the second order term. Here, the Ricci tensor is of the second order and its contribution becomes nontrivial because we work up to the second order in $B_{\mu\nu\rho}$.

3 T-dualization procedure

In the majority of papers addressing T-dualization of a bosonic string theory, one performs T-dualizations along directions on which the background does not depend. The first procedure, applicable to coordinates on which the background fields depend, the generalization of the Buscher T-dualization procedure, was presented in [18]. It was applied to a bosonic string moving in the weakly curved background, composed of a constant metric and a linearly coordinate dependent Kalb-Ramond field with an infinitesimal strength. This theory has a global shift symmetry. This fact is used in the T-dualization prescription, which relies on gauging the global symmetry. The locally invariant action was built substituting the ordinary derivatives with the covariant ones and substituting the coordinate in the argument of the background fields with the invariant coordinate (a line integral of a covariant derivatives of the original coordinates). The physical equivalence was achieved by introduction of the Lagrange multiplier term, which makes the gauge fields nonphysical.

3.1 Auxiliary action

Here, we will consider the weakly curved background of the second order. As in the first order weakly curved background, the Kalb-Ramond field is linear in coordinate and has an infinitesimal field strength. The metric however, beside of a constant term has a quadratic in coordinate part which is an infinitesimal of the second order. Such a metric has an infinitesimal but nonzero Ricci tensor $R_{\mu\nu} \neq 0$. The bosonic string theory in this background does not possess the shift symmetry. However, defining of the new T-dualization rules on the grounds of the existing procedure is still possible. The main object in the conventional procedure, is the gauge fixed action which reduces to the initial action for the equations of motion for the Lagrange multipliers and becomes T-dual action for the equations of motion for the gauge fields. Here we will define its substitution, which inherits these two features. We postulate the auxiliary action by

$$S_{\text{aux}}[y, v_\pm] = \kappa \int d^2\xi \left[v_+^\mu \Pi_{+\mu\nu}(\Delta V) v_-^\nu + \frac{1}{2}(v_+^\mu \partial_- y_\mu - v_-^\mu \partial_+ y_\mu) \right]. \tag{3.1}$$

It can be obtained from the initial action (2.4), by making the following substitutions

$$\partial_\pm x^\mu \to v_\pm^\mu, \quad x^\mu \to \Delta V^\mu \tag{3.2}$$

and adding the Lagrange multiplier y_μ term. This action is of the same form as the gauge fixed action, however, $v_\pm^\mu$ are here some auxiliary fields, which take over the role of the gauge fields. Similarly as in [18], the argument of the background fields is the line integral of the auxiliary fields taken along a path P (from ξ_0 to ξ)

$$\Delta V^\mu[v_+, v_-] \equiv \int_P d\xi^\alpha v^\mu_\alpha = \int_P (d\xi^+ v^\mu_+ + d\xi^- v^\mu_-). \tag{3.3}$$

Note that as well as in ref. [18], the equation of motion with respect to y_μ forces the "field strength" to vanish $\partial_+ v^\mu_- - \partial_- v^\mu_+ = 0$, which is just the condition for the path independence of ΔV^μ. In the resulting theories, the argument reduces to $\Delta V^\mu = V^\mu(\xi) - V^\mu(\xi_0)$ and we will chose the value of $V^\mu(\xi_0)$ to be zero.

3.2 From the auxiliary to the initial and T-dual theory

Let us confirm that the auxiliary action (3.1) becomes the initial action (2.4) for the equations of motion obtained varying over the Lagrange multiplier y_μ

$$\partial_+ v^\mu_- - \partial_- v^\mu_+ = 0. \tag{3.4}$$

Using their solution

$$v^\mu_\pm = \partial_\pm x^\mu, \tag{3.5}$$

one obtains $V^\mu(\xi) = x^\mu(\xi)$, and therefore taking $x^\mu(\xi_0) = 0$ the auxiliary action reduces to the initial action (2.4).

The equations of motion for the auxiliary fields $v^\mu_\pm$ are

$$\Pi_{\mp\mu\nu}(V) v^\nu_\pm + \frac{1}{2}\partial_\pm y_\mu = \mp\beta^\mp_\mu(V). \tag{3.6}$$

Here the functions $\beta^\pm_\mu$ are defined by

$$\delta_V S_{\text{aux}} = -\kappa \int d\xi^2 \beta^\alpha_\mu \delta v^\mu_\alpha, \tag{3.7}$$

where $\delta_V S_{\text{aux}}$ stands for the variation of the action (3.1) over the background field argument V^μ.

Let us introduce the following background fields: an effective metric and a non-commutativity parameter, defined by

$$G^E_{\mu\nu} \equiv \left(G - 4BG^{-1}B\right)_{\mu\nu}, \quad \theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G^{-1}_E BG^{-1})^{\mu\nu}, \tag{3.8}$$

and their combinations

$$\Theta^{\mu\nu}_\pm = -\frac{2}{\kappa}(G^{-1}_E \Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G^{-1}_E)^{\mu\nu}, \tag{3.9}$$

which are the inverses of the background field compositions $2\kappa\Pi_{\mp\mu\nu}$. Now, one can rewrite the equations of motion (3.6) as

$$v^\mu_\pm(y) = -\kappa\, \Theta^{\mu\nu}_\pm\big(V(y)\big)\Big[\partial_\pm y_\nu \pm 2\beta^\mp_\nu\big(V(y)\big)\Big]. \tag{3.10}$$

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The equation (3.10) is not the solution of (3.6), because $v^\mu_\pm$ appears within the argument V^μ of both $\Theta^{\mu\nu}_\pm$ and $\beta^\mp_\mu$. We will solve this equation iteratively.

The T-dual theory is obtained, by inserting the equation of motion (3.10) into the action (3.1)

$$^\star S[y,v_\pm]=\frac{\kappa^2}{2}\int d^2\xi\,\Big[\partial_+ y_\mu\Theta^{\mu\nu}_-\big(V(y)\big)\partial_- y_\nu+4\beta^-_\mu\big(V(y)\big)\Theta^{\mu\nu}_-\big(V(y)\big)\beta^+_\nu\big(V(y)\big)\Big]. \tag{3.11}$$

In order to obtain the explicit form of the T-dual action one has to calculate the beta functions $\beta^\pm_\mu$ for a concrete background, solve (3.10) to find the explicit y-dependence of the auxiliary fields $v^\mu_\pm=v^\mu_\pm(y)$, and therefrom determine the argument of the dual background fields $V^\mu(y)$.

4 T-dual action in a weakly curved background

Let us find the explicit expression for the T-dual action (3.11), in the weakly curved background of the second order. The main task is to obtain the $\beta^\pm_\mu$ functions (B.6), which are calculated in appendix B. Because they are infinitesimal, it is enough to consider their first order value and to determine the zeroth order value of V^μ, in order to calculate the last term in the action

$$4\beta^-_{1\mu}\big(V_0(y)\big)\Theta^{\mu\nu}_{0-}\beta^+_{1\nu}\big(V_0(y)\big)=\partial_+V^\mu_0 h_{\mu\nu}\big(V_0(y)\big)\Theta^{\nu\rho}_{0-}h_{\rho\sigma}\big(V_0(y)\big)\partial_-V^\sigma_0. \tag{4.1}$$

Let us find the explicit form of the dual background fields argument. We will solve the equations (3.10) iteratively and find the zeroth and the first order in $B_{\mu\nu\rho}$ values of the auxiliary fields $v^\mu_\pm(y)$. In the zeroth order one has

$$v^\mu_{0\pm}(y)=-\kappa\,\Theta^{\mu\nu}_{0\pm}\,\partial_\pm y_\nu, \tag{4.2}$$

consequently the zeroth order value of V^μ defined in (3.3) is

$$\begin{aligned}V^\mu_0 &= -\frac{\kappa}{2}\big(\Theta^{\mu\nu}_{0+}+\Theta^{\mu\nu}_{0-}\big)y^{(0)}_\nu-\frac{\kappa}{2}\big(\Theta^{\mu\nu}_{0+}-\Theta^{\mu\nu}_{0-}\big)\tilde y^{(0)}_\nu\\ &= -\kappa\theta^{\mu\nu}_0 y^{(0)}_\nu+(g^{-1}_E)^{\mu\nu}\tilde y^{(0)}_\nu,\end{aligned} \tag{4.3}$$

where $\tilde y_\mu$ is a double coordinate defined by

$$\tilde y_\mu\equiv\int_P d\xi^\alpha\varepsilon^\beta{}_\alpha\partial_\beta y_\mu. \tag{4.4}$$

Now, using (4.3) and (A.8), the last term (4.1) in the T-dual action (3.11), becomes

$$\begin{aligned}4\beta^-_{1\mu}\big(V_0(y)\big)\Theta^{\mu\nu}_{0-}\beta^+_{1\nu}\big(V_0(y)\big) &= -\frac{\kappa}{2}\partial_+y_\mu\Big[\Theta_{1-}\big(V_0(y)\big)\Pi_{0+}\Theta_{1-}\big(V_0(y)\big)\Big]^{\mu\nu}\partial_-y_\nu\\ &\equiv -\,\partial_+y_\mu\,\Delta^{\mu\nu}_-\big(V_0(y)\big)\partial_-y_\nu,\end{aligned} \tag{4.5}$$

with $\Delta^{\mu\nu}_\pm$ explicitly given by (A.9).

The first order value of the auxiliary field $v^\mu_\pm$, defined by (3.10), is obtained using the first order value of $\beta^\pm_\mu$, (B.6), and the expressions for $\Theta^{\mu\nu}_{1\pm}$ given by (A.8)

$$v^\mu_{1\pm}(y) = -\kappa\, \Theta^{\mu\nu}_{0\pm}\partial_\pm y^{(1)}_\nu - \frac{3\kappa}{2}\Theta^{\mu\nu}_{1\pm}(V_0(y))\partial_\pm y^{(0)}_\nu. \tag{4.6}$$

Let us note that, because of (4.2) and (4.6), the complete first order value of the auxiliary field can be written as

$$v^{(1)\mu}_\pm(y) = -\kappa\, {}^\diamond\Theta^{(1)\mu\nu}_\pm(V(y))\partial_\pm y_\nu, \tag{4.7}$$

where ${}^\diamond\Theta^{\mu\nu}_\pm$ is defined in (A.14).

Substituting (4.6) into (3.3), we obtain the first order value of V^μ

$$\begin{aligned} V^\mu_1(y) = & -\frac{\kappa}{2}(\Theta^{\mu\nu}_{0+} + \Theta^{\mu\nu}_{0-})y^{(1)}_\nu - \frac{\kappa}{2}(\Theta^{\mu\nu}_{0+} - \Theta^{\mu\nu}_{0-})\tilde{y}^{(1)}_\nu \\ & -\frac{\kappa^3}{2}(\Theta^{\mu\nu}_{0+}+\Theta^{\mu\nu}_{0-})B_{\nu\rho\sigma}\Big(\Theta^{\sigma\tau}_{0+}\Theta^{\rho\varepsilon}_{0+}M_{+\tau\varepsilon}(y)+\Theta^{\sigma\tau}_{0-}\Theta^{\rho\varepsilon}_{0-}M_{-\tau\varepsilon}(y)+\Theta^{\sigma\tau}_{0-}\Theta^{\rho\varepsilon}_{0+}\widetilde{(y^{(0)}_{-\tau}y^{(0)}_{+\varepsilon})}\Big) \\ & -\frac{\kappa^3}{2}(\Theta^{\mu\nu}_{0+}-\Theta^{\mu\nu}_{0-})B_{\nu\rho\sigma}\Big(\Theta^{\sigma\tau}_{0+}\Theta^{\rho\varepsilon}_{0+}M_{+\tau\varepsilon}(y)-\Theta^{\sigma\tau}_{0-}\Theta^{\rho\varepsilon}_{0-}M_{-\tau\varepsilon}(y)+\Theta^{\sigma\tau}_{0-}\Theta^{\rho\varepsilon}_{0+}\, y^{(0)}_{-\tau}y^{(0)}_{+\varepsilon}\Big), \end{aligned} \tag{4.8}$$

where

$$M_{\pm\mu\nu}(y) \equiv \frac{1}{2}\int d\xi^\pm \left(y^{(0)}_{\pm\mu}\partial_\pm y^{(0)}_{\pm\nu} - y^{(0)}_{\pm\nu}\partial_\pm y^{(0)}_{\pm\mu}\right), \tag{4.9}$$

and $\widetilde{(y^{(0)}_{-\tau}y^{(0)}_{+\varepsilon})}$ is a double (defined by (4.4)) of the quantity $y^{(0)}_{-\tau}y^{(0)}_{+\varepsilon}$.

Once the argument of the background fields is calculated, we can write the explicit form of the T-dual action

$$\begin{aligned} {}^\star S[y] &= \frac{\kappa^2}{2}\int d^2\xi\ \partial_+ y_\mu\, {}^\dagger\Theta^{\mu\nu}_-(V(y))\partial_- y_\nu \\ &\equiv \frac{\kappa^2}{2}\int d^2\xi\ \partial_+ y_\mu\Big[\Theta^{\mu\nu}_-(V(y)) - \Delta^{\mu\nu}_-(V_0(y))\Big]\partial_- y_\nu, \end{aligned} \tag{4.10}$$

with $V^\mu = V^{(1)\mu} = V^\mu_0 + V^\mu_1$ given by (4.3) and (4.8). The second term in the dual background fields composition $\Delta^{\mu\nu}_-$ is the contribution from the term quadratic in β, and has a form (A.9).

Comparing the initial action (2.4) with the T-dual action (4.10), one can conclude that they are equal under the following transformations

$$\begin{aligned} \partial_\pm x^\mu &\to \partial_\pm y_\mu, \\ \Pi_{+\mu\nu}(x) \to {}^\star\Pi^{\mu\nu}_+(y) &= \frac{\kappa}{2}\Big(\Theta^{\mu\nu}_-(V(y)) - \Delta^{\mu\nu}_-(V_0(y))\Big) \\ &\equiv \frac{\kappa}{2}\,{}^\dagger\Theta^{\mu\nu}_-(V(y)). \end{aligned} \tag{4.11}$$

The T-dual metric (the symmetric part of the T-dual background fields composition) and the T-dual Kalb-Ramond field (the antisymmetric part) are

$${}^\star G^{\mu\nu} = (G^{-1}_E)^{\mu\nu} - \kappa^2(\theta_0 h\theta_0 h g^{-1}_E + \theta_0 h g^{-1}_E h\theta_0 + g^{-1}_E h\theta_0 h\theta_0)^{\mu\nu} - (g^{-1}_E h g^{-1}_E h g^{-1}_E)^{\mu\nu}, \tag{4.12}$$

and

$$^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu} - \frac{\kappa^3}{2}(\theta_0 h\theta_0 h\theta_0)^{\mu\nu} - \frac{\kappa}{2}(\theta_0 h g_E^{-1} h g_E^{-1} + g_E^{-1} h\theta_0 h g_E^{-1} + g_E^{-1} h g_E^{-1} h\theta_0)^{\mu\nu}. \quad (4.13)$$

The characteristics of the dual geometry, are considered in section 7.

5 T-dual of T-dual

Let us now follow the prescription of section 3, and show that the T-dual of a T-dual theory is the original theory. To obtain the auxiliary action of the T-dual action (4.10), let us substitute the dual coordinate derivatives $\partial_\pm y_\mu$ with some auxiliary fields $u_{\pm\mu}$, substitute the coordinate in the argument of background fields with $\Delta U_\mu = \int(d\xi^+ u_{+\mu} + d\xi^- u_{-\mu})$ and require the "flatness" of $u_{\pm\mu}$ by introducing the Lagrange multiplier terms

$$^\star S_{\text{aux}}[z, u_\pm] = \frac{\kappa}{2}\int d^2\xi \Big[\kappa u_{+\mu}\, {}^\dagger\Theta_-^{\mu\nu}\big(V(\Delta U)\big)\, u_{-\nu} + u_{+\mu}\partial_- z^\mu - u_{-\mu}\partial_+ z^\mu\Big]. \quad (5.1)$$

For the solution $u_{\pm\mu} = \partial_\pm y_\mu$ of the equations of motion $\partial_- u_{+\mu} + \partial_+ u_{-\mu} = 0$, which are obtained varying the auxiliary action over the Lagrange multiplier z^μ, the variable U_μ reduces to y_μ ($y_\mu(\xi_0) = 0$). So, the auxiliary action reduces to the original one.

The original theory should be obtained for the equations of motion for the auxiliary fields $u_{\pm\mu}$

$$\kappa\, {}^\dagger\Theta_\pm^{\mu\nu}\big(V(U)\big) u_{\pm\nu} + \partial_\pm z^\mu = \mp\kappa\, {}^\star\beta^{\mp\mu}\big(V(U)\big), \quad (5.2)$$

with ${}^\star\beta^{\mp\mu}$ defined by (C.2), being the contribution from the variation over the background fields argument. Multiplying the equations by ${}^\dagger\Pi_{\mp\mu\nu}$ (the inverse of the background field composition ${}^\dagger\Theta_\pm^{\mu\nu}$, defined in (A.11)) one obtains

$$u_{\pm\mu} = -2\kappa\, {}^\dagger\Pi_{\mp\mu\nu}\big(V(U)\big)\Big[\frac{1}{\kappa}\partial_\pm z^\nu \pm {}^\star\beta^{\mp\nu}\big(V(U)\big)\Big]. \quad (5.3)$$

Using the last equation and the first order value of the dual beta function ${}^\star\beta^{\mp\nu}$ given by (C.15), we can determine the value of the variable U_μ, up to the first order

$$U_\mu^{(1)}(z) = -2b_{\mu\nu}z^\nu + g_{\mu\nu}\tilde{z}^\nu - B_{\mu\nu\rho}\Big[M_+^{\rho\nu}(z^{(0)}) + M_-^{\rho\nu}(z^{(0)}) - \big(\widetilde{z_+^{(0)\rho} z_-^{(0)\nu}}\big)\Big], \quad (5.4)$$

with $M_\pm^{\mu\nu}(z)$ defined in (4.9). Substituting (5.4) to (4.3) and (4.8) we confirm that $V^\mu(U) = z^\mu$.

So, substituting (5.3) to the action (5.1), we obtain

$$^{\star\star}S_{\text{aux}}[z] = \kappa\int d^2\xi \Big[\partial_+ z^\mu\, {}^\dagger\Pi_{+\mu\nu}(z)\partial_- z^\nu + \kappa^2\, {}^\star\beta^{-\mu}(z)\, {}^\dagger\Pi_{+\mu\nu}(z)\, {}^\star\beta^{+\nu}(z)\Big]. \quad (5.5)$$

Using the first order value of ${}^\star\beta^{\pm\mu}$, given by (C.15), the second term of the action becomes $-2\kappa\partial_+ z^\mu(\Pi_{0+}\Delta_-(z)\Pi_{0+})_{\mu\nu}\partial_- z^\nu$ and therefore the action (5.5) is just the initial action (2.4)

$$^{\star\star}S_{\text{aux}}[z] = \kappa\int d^2\xi\, \partial_+ z^\mu\, \Pi_{+\mu\nu}(z)\partial_- z^\nu. \quad (5.6)$$

6 Features of T-duality

In the previous sections, we showed how the original and its T-dual theory can be transformed one to the other. In both directions, both theories follow from the auxiliary action and are obtained for a concrete form of the auxiliary fields. Comparing these auxiliary fields one obtains the T-dual coordinate transformation laws.

In section 3, we showed how the original theory can be transformed into its T-dual theory. So, comparing the expressions for the auxiliary fields (3.5) and (3.10), one obtains the T-dual coordinate transformation law

$$\partial_\pm x^\mu \cong -\kappa\, \Theta^{\mu\nu}_\pm\big(V(y)\big)\Big[\partial_\pm y_\nu \pm 2\beta^\mp_\nu\big(V(y)\big)\Big]. \tag{6.1}$$

In the first order this law implies

$$x^{(1)\mu} \cong V^{(1)\mu}(y). \tag{6.2}$$

Substituting the beta functions (B.6) into the transformation law (6.1), we obtain

$$\partial_\pm x^\mu \cong -\kappa\left[{}^\diamond\Theta^{\mu\nu}_\pm\big(V(y)\big) + \Delta^{\mu\nu}_\pm\big(V(y)\big)\right]\partial_\pm y_\nu, \tag{6.3}$$

with ${}^\diamond\Theta^{\mu\nu}_\pm$ are $\Delta^{\mu\nu}_\pm$ given by (A.14) and (A.9). Using these laws one can show that the equation of motion of the original theory transform to an identity (Bianchi identity) in the T-dual theory, and vice verse. From (6.1) and (6.3), using (6.2) one obtains

$$\beta^\pm_\mu(x) = \mp\big({}^\diamond\Pi_{\pm\mu\nu}(x) - \Pi_{\pm\mu\nu}(x)\big)\partial_\mp x^\nu. \tag{6.4}$$

In section 5, the T-dual theory was transformed to the original theory. Comparing the solutions for the auxiliary fields we obtain the following T-dual coordinate transformation law

$$\kappa\partial_\pm y_\mu \cong -2\kappa\, {}^\dagger\Pi_{\mp\mu\nu}(z)\Big(\partial_\pm z^\nu \pm \kappa\, {}^\star\beta^{\mp\nu}(z)\Big). \tag{6.5}$$

In the first order this law implies

$$y^{(1)}_\mu \cong U^{(1)}_\mu(z). \tag{6.6}$$

Substituting the explicit value of the dual beta function (C.15), we obtain

$$\partial_\pm y_\mu \cong -2\, {}^\diamond\Pi_{\mp\mu\nu}(z)\partial_\pm z^\nu, \tag{6.7}$$

with ${}^\diamond\Pi_{\mp\mu\nu}$ defined in (A.15). Eliminating $\partial_\pm z^\mu$ from (6.5) and (6.7), using (6.2) one obtains

$${}^\star\beta^{\pm\mu}\big(V(y)\big) = \mp\Big({}^\diamond\Theta^{\mu\nu}_\mp - \Theta^{\mu\nu}_\mp + 2\Delta^{\mu\nu}_\mp\Big)\partial_\pm y_\nu. \tag{6.8}$$

Let us show that the T-dual coordinate transformation laws (6.1) and (6.5) are inverse to each other. Multiplying (6.5) by ${}^\dagger\Theta_\pm(V(y)) \cong {}^\dagger\Theta_\pm(z)$ we obtain

$$\kappa\, {}^\dagger\Theta^{\mu\nu}_\pm(V(y))\partial_\pm y_\nu \cong -\partial_\pm z^\mu \mp \kappa\, {}^\star\beta^{\mp\mu}(z). \tag{6.9}$$

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Using (C.15), (A.10) and (A.14) it becomes

$$\begin{aligned}\partial_\pm z^\mu &\cong -\kappa\, {}^\dagger\Theta_\pm^{\mu\nu}(V(y))\partial_\pm y_\nu \mp 2\kappa\, {}^\diamond\Theta_\pm^{(1)\mu\nu}(V(y))\beta_\nu^\mp(V(y)) \\ &= -\kappa\, \Theta_\pm^{\mu\nu}(V(y))\Big[\partial_\pm y_\nu \pm 2\beta_\nu^\mp(V(y))\Big] \\ &\quad +\kappa\Delta_\pm^{\mu\nu}(V(y))\partial_\pm y_\nu \mp \kappa\, \Theta_{1\pm}^{\mu\nu}(V(y))\beta_\nu^\mp(V(y)).\end{aligned} \tag{6.10}$$

Recalling the definitions (A.9), (A.8) and (B.6) the last two terms cancel out and one obtains

$$\partial_\pm z^\mu \cong -\kappa\, \Theta_\pm^{\mu\nu}(V(y))\Big[\partial_\pm y_\nu \pm 2\beta_\nu^\mp(V(y))\Big], \tag{6.11}$$

which is just (6.1). The equivalent conclusion that (6.3) and (6.7) are inverse to each other, follows from (A.16).

Let us finally show, that using the T-dual coordinate transformation laws one can confirm that the equations of motion and the Bianchi identities of the original and T-dual theory interchange. Applying the transformation law (6.7) to the identity

$$\partial_+\partial_- y_\mu - \partial_-\partial_+ y_\mu = 0, \tag{6.12}$$

we obtain

$$\partial_+\partial_- z^\mu + ({}^\diamond\Pi_+ - {}^\diamond\Pi_-)^{-1\mu\nu}\Big(\partial_\rho{}^\diamond\Pi_{+\nu\sigma} - \partial_\sigma{}^\diamond\Pi_{-\nu\rho}\Big)\partial_+ z^\rho\partial_- z^\sigma = 0. \tag{6.13}$$

Using the explicit form of the composition ${}^\diamond\Pi_{\pm\mu\nu}$, given by (A.15), expression (A.19) and the value of the Christoffel symbol (2.13), we obtain

$$({}^\diamond\Pi_+ - {}^\diamond\Pi_-)^{-1\mu\nu}\Big(\partial_\rho{}^\diamond\Pi_{+\nu\sigma} - \partial_\sigma{}^\diamond\Pi_{-\nu\rho}\Big) = -B^\mu{}_{\rho\sigma} + \Gamma^\mu_{\rho\sigma}. \tag{6.14}$$

So, (6.13) is the initial equation of motion (2.6).

The equation of motion of the T-dual theory (4.10) is

$$\partial_+\Big[({}^\diamond\Theta_- + \Delta_-)^{\mu\nu}(V(y))\partial_- y_\nu\Big] - \partial_-\Big[({}^\diamond\Theta_+ + \Delta_+)^{\mu\nu}(V(y))\partial_+ y_\nu\Big] = 0. \tag{6.15}$$

Using the T-dual coordinate transformation law (6.7) (with $z^\mu = x^\mu$) and (6.2), we obtain

$$\partial_+\Big[({}^\diamond\Theta_- + \Delta_-)^{\mu\nu}(x){}^\diamond\Pi_{+\nu\rho}(x)\partial_- x^\rho\Big] - \partial_-\Big[({}^\diamond\Theta_+ + \Delta_+)^{\mu\nu}(x){}^\diamond\Pi_{-\nu\rho}(x)\partial_+ x^\rho\Big] = 0, \tag{6.16}$$

which with a help of (A.16) is just the identity

$$\partial_+\partial_- x^\mu - \partial_-\partial_+ x^\mu = 0. \tag{6.17}$$

7 Original and dual geometries

Let us discuss what the geometry of the T-dual theory looks like and compare it with the geometry of the original theory. To simplify discussion we will put the constant part of the original Kalb-Ramond field, $b_{\mu\nu}$, to zero. In fact it appears in front of a topological term and can contribute only in the quantum theory.

Let us first note the substantial difference between our T-dual theories and the standard σ-formulations of string theories. In our approach argument of the background fields is expression V^μ, which is a line integral of the T-dual coordinate derivatives. For $b_{\mu\nu} = 0$ it essentially depends on a double coordinate $\tilde{y}_\mu \equiv \int_P d\xi^\alpha \varepsilon^\beta{}_\alpha \partial_\beta y_\mu$, eq. (4.4), which makes the T-dual theory non-geometric. In some particular examples such theories are known as the theories with R-flux. For these theories, the equations of motion are not necessarily equal to the standardly derived space-time equations of motion for the background fields depending just on the coordinate x^μ or y_μ. Although we do not expect that all relations between background fields of the original and T-dual theories in our approach will coincide with those in the literature, we are going to compare them.

Let us first make a simple qualitative analysis. In the approximation of the first order it is easy to see, without calculation, that the T-dual space-time equations of motion are satisfied. In fact both dual metric ${}^\star G^{\mu\nu}$ and dual Kalb-Ramond field ${}^\star B^{\mu\nu}$ are linear in coordinates with infinitesimal coefficients. Consequently, the dual Christoffel ${}^\star\Gamma_\mu^{\nu\rho}$ and the dual field strength ${}^\star B^{\mu\nu\rho}$ are constant and infinitesimal. So, both dual space-time equations, for the metric and for the Kalb-Ramond field, are equal to the second order infinitesimals which are neglected, meaning they are satisfied. In our case of the second order approximation, a similar analysis shows that both Riemann tensor and square of the Kalb-Ramond field are constant and the second order infinitesimals in both initial and T-dual theories.

In this section we are going to discuss the following issues: the geometries and the space-time equations of motion of the initial and T-dual theories as well as the integrability conditions of ref. [28].

7.1 Geometry of the original theory

For the original theory we take

$$G_{\mu\nu}(x) = g_{\mu\nu} + 3h^2_{\mu\nu}(x)\,, \qquad B_{\mu\nu}(x) = h_{\mu\nu}(x)\,, \tag{7.1}$$

so that the corresponding Christoffel symbols are linear in coordinate and infinitesimals of the second order while the Kalb-Ramond field strength is a constant infinitesimal of the first order

$$\Gamma^\mu_{\nu\rho} = -B^\mu{}_{\nu\sigma} h^\sigma{}_\rho - B^\mu{}_{\rho\sigma} h^\sigma{}_\nu\,, \qquad H_{\mu\nu\rho} = B_{\mu\nu\rho}\,. \tag{7.2}$$

Therefore, the Riemann tensor is a constant infinitesimal of the second order

$$R^\rho{}_{\sigma\mu\nu} = \frac{2}{3} B_{\mu\nu}{}^\varepsilon B_\varepsilon{}^\rho{}_\sigma + \frac{1}{3}(B^\rho{}_{\mu\varepsilon} B^\varepsilon{}_{\sigma\nu} - B_{\sigma\mu}{}^\varepsilon B_\varepsilon{}^\rho{}_\nu), \tag{7.3}$$

which produces the constant second order infinitesimal Ricci tensor

$$R_{\mu\nu} = B_{\mu\rho\sigma} B^{\rho\sigma}{}_\nu. \tag{7.4}$$

Note that the covariant derivative of the field strength is equal to the ordinary derivative $D_\mu H^\mu{}_{\rho\sigma} = \partial_\mu H^\mu{}_{\rho\sigma}$. This is the consequence of the fact that the Christoffel symbols are infinitesimals of the second order and the terms ΓH are infinitesimals of the third order which should be neglect in our case. When $H^\mu{}_{\rho\sigma}$ is constant one has $D_\mu H^\mu{}_{\rho\sigma} = 0$.

Therefore, the space-time equations of motion can be written in the form

$$S_{\mu\nu} = 0\,, \qquad D_\mu H^\mu{}_{\rho\sigma} = 0\,, \tag{7.5}$$

where for future benefits, following ref. [28], we introduced the tensors

$$S^\rho{}_{\sigma\mu\nu} = R^\rho{}_{\sigma\mu\nu} - \frac{2}{3}B_{\mu\nu}{}^\varepsilon B_\varepsilon{}^\rho{}_\sigma - \frac{1}{3}(B^\rho{}_{\mu\varepsilon}B^\varepsilon{}_{\sigma\nu} - B_{\sigma\mu}{}^\varepsilon B_\varepsilon{}^\rho{}_\nu)\,, \tag{7.6}$$

and

$$S_{\mu\nu} = S^\rho{}_{\mu\rho\nu}\,. \tag{7.7}$$

Note that the coefficients in front of the squares of field strength differ from that in ref. [28], because of a different notation. In both articles they are adjusted in such a way that $S_{\mu\nu} = 0$ is the space-time equation of motion.

7.2 Geometry of the T-dual theory

The background fields of the T-dual theory are

$$^\star G^{\mu\nu} = (g^{-1})^{\mu\nu}\,, \qquad ^\star B^{\mu\nu}(V) = -(g^{-1}hg^{-1})^{\mu\nu}(V)\,, \tag{7.8}$$

where

$$V^\mu = (g^{-1})^{\mu\nu}\tilde{y}_\nu. \tag{7.9}$$

In comparison to the original theory, the term h^2 of the metric tensor is missing and the background fields depend on V^μ instead of x^μ.

The dual metric is constant and therefore the dual Christoffel symbol, dual Riemann and Ricci tensors are zero. But, the dual Kalb-Ramond field strength is constant infinitesimal of the first order $^\star H_{\mu\nu\rho} = -B_{\mu\nu\rho}$. As we explained in the beginning of this section the T-dual fields do not satisfy the standard space-time equations of motion because the space is nongeometric and the background fields depend on the dual coordinate $\tilde{y}$. The T-dual space-time equations of motion are

$$^\star R^{\mu\nu} = 0\,, \qquad ^\star D^{\mu\star} H_\mu{}^{\rho\sigma} = 0\,, \tag{7.10}$$

where again dual covariant derivative is equal to the ordinary derivative. It is interesting to note that although the initial theory is curved, the corresponding T-dual is flat. It seems that at least in the second order the T-duality acts as a parallelizable transformation. This assumption should be checked in the higher orders of approximations.

7.3 Relation with ref. [28]

Although, as we explained, we should not expect for our background to satisfy the pseudo-duality conditions of ref. [28], we are going to discuss the relation with this article. Let us

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first note that $\beta^{\pm}_\mu$ functions, introduced in the generalized T-dualization procedure, originate from the fact that T-dual background fields do not depend on the coordinate y_μ but on its dual $V^\mu = (g^{-1})^{\mu\nu}\tilde{y}_\nu$. So, in order to compare our relations with the conditions derived in the literature we will omit the term $\beta^-_\mu \Theta^{\mu\nu}_- \beta^+_\nu$. Then, the T-dual metric tensor acquires the quadratic term and the Kalb-Ramond field is unchanged

$$ {}^\star G^{\mu\nu} = (g^{-1})^{\mu\nu} + (h^2)^{\mu\nu}(V)\,, \qquad {}^\star B^{\mu\nu}(V) = -(g^{-1}hg^{-1})^{\mu\nu}(V)\,. \tag{7.11} $$

Secondly, one can note that the pseudoduality relation in ref. [28], which is the starting point in that paper corresponds to the relation (6.7) in this paper. Taking $b_{\mu\nu} = 0$ the relation (6.7) reduces to

$$ \partial_\pm y_\mu \cong \mp(g \mp 3h + 6h^2)_{\mu\nu}(x)\partial_\pm x^\nu \equiv \pm T_\pm \partial_\pm x^\nu\,, \tag{7.12} $$

where $T_\pm$ is the notation from the article [28] where only the case $T_+ = T_-$ was treated. Obviously, the T-dual coordinate transformation laws differ at least by the term $3h$.

The Christoffel symbol and the Kalb-Ramond field strength for the background fields (7.11) are

$$ {}^\star\Gamma^{\nu\rho}_\mu = -\frac{1}{3}\Big(B_\mu{}^\nu{}_\sigma h^{\sigma\rho} + B_\mu{}^\rho{}_\sigma h^{\sigma\nu}\Big), \quad {}^\star H^{\mu\nu\rho} = -B^{\mu\nu\rho}. \tag{7.13} $$

The Riemann tensor is

$$ {}^\star R^{\mu\sigma\nu}_\rho = -\frac{2}{9}B_\rho{}^\mu{}_\varepsilon B^{\varepsilon\nu\sigma} - \frac{1}{9}B_\rho{}^\nu{}_\varepsilon B^{\varepsilon\mu\sigma} + \frac{1}{9}B_\rho{}^\sigma{}_\varepsilon B^{\varepsilon\mu\nu}, \tag{7.14} $$

and the Ricci tensor equals

$$ {}^\star R^{\mu\nu} = \frac{1}{3}B^{\mu\varepsilon\rho}B_{\varepsilon\rho}{}^\nu. \tag{7.15} $$

Note that the Christoffel symbols, Riemann and Ricci tensors are one third of the corresponding variables of the original theory. The same as in the original theory, the Christoffel symbols are infinitesimals of the second order and the covariant derivative of the field strength is equal to the ordinary derivative ${}^\star D^{\mu\star}H^{\mu\rho\sigma} = \partial^{\mu\star}H^{\mu\rho\sigma}$. Because the tensor ${}^\star H^{\mu\rho\sigma}$ is constant the right hand side is zero.

The dual space-time equations of motion can be written as

$$ {}^\star S^{\mu\nu} = 0\,, \qquad {}^\star D^{\mu\star}H_\mu{}^{\rho\sigma} = 0\,, \tag{7.16} $$

where we define dual tensors ${}^\star S_\rho{}^{\sigma\mu\nu}$ with additional coefficient $\frac{1}{3}$ in the last three terms in comparison to the tensor $S^\rho{}_{\sigma\mu\nu}$

$$ {}^\star S_\rho{}^{\sigma\mu\nu} = {}^\star R_\rho{}^{\sigma\mu\nu} - \frac{2}{9}B^{\mu\nu}{}_\varepsilon B^\varepsilon{}_\rho{}^\sigma - \frac{1}{9}(B_\rho{}^{\mu\varepsilon}B_\varepsilon{}^{\sigma\nu} - B^{\sigma\mu}{}_\varepsilon B^\varepsilon{}_\rho{}^\nu)\,, \tag{7.17} $$

and as usual

$$ {}^\star S^{\mu\nu} = {}^\star S_\rho{}^{\mu\rho\nu}\,. \tag{7.18} $$

The pseudoduality conditions of the ref. [28], in our notation read as follows

$$^\star S^\rho{}_{\sigma\mu\nu} = -S^\rho{}_{\sigma\mu\nu}\,, \qquad ^\star D_\mu {}^\star H^\mu{}_{\rho\sigma} = -D_\mu H^\mu{}_{\rho\sigma}\,. \tag{7.19}$$

Note that because both equations are infinitesimal we can raise and lower indices with the constant part of the metric. Both pseudoduality conditions are fulfilled because all terms are separately equal to zero. The second relation is valid without derivatives as well $^\star H^\mu{}_{\rho\sigma} = -H^\mu{}_{\rho\sigma}$.

8 Conclusion

In this paper, we presented the T-dualization procedure applicable to string backgrounds with nontrivial Ricci tensor and without isometries. The procedure is the generalization of the one given in paper [18], for a weakly curved background. It was applied to a string moving in the weakly curved background of the second order, composed of a linearly coordinate dependent Kalb-Ramond field with an infinitesimal strength and a metric with an infinitesimal of the second order quadratic in coordinate term.

The generalized Buscher procedure was not applicable to the second order weakly curved background, because the action does not possess a global symmetry. If there is no global symmetry, there is no corresponding gauge symmetry, which is the crucial ingredient of the T-dualization procedure. However, it is possible to construct an auxiliary action, which plays the role of the gauge fixed action. The auxiliary action is constructed from the initial action, substituting the derivatives of the coordinates by some auxiliary fields, and the background fields argument by a line integral of these auxiliary fields. This action reduces to the initial action and to the T-daul action on its equations of motion. So, there is a full analogy between the T-dualization procedures for backgrounds with and without a global symmetry. The only difference is in fact the interpretation of the auxiliary fields which are understood as the gauge fields in the case of a background with symmetry.

The realization of the new generalized Buscher procedure is more complicated. The main problem is to solve the equations of motion, obtained varying the auxiliary action with respect to the auxiliary fields, in terms of the Lagrange multipliers. To solve them, one should iteratively calculate the argument of the background fields and the beta functions defined in appendices B and C. It turns out that the argument of the dual background fields has a more complicated form then in the weakly curved background of the first order. This argument represents the complicated structure of the dual geometry. In the first order the argument is given in terms of the dual coordinate y_μ and its double $\tilde{y}_\mu$. In the second order the argument is given in terms of the dual coordinate, its double and an additional form $M_{\pm\mu\nu}(y) \equiv \frac{1}{2}\int d\xi^\pm \left(y^{(0)}_{\pm\mu}\partial_\pm y^{(0)}_{\pm\nu} - y^{(0)}_{\pm\nu}\partial_\pm y^{(0)}_{\pm\mu}\right)$, which can be interpreted as the left and the right "angular momentum".

Applying the T-dualization procedure to all the coordinates of the initial theory, the theory transforms to a T-dual theory. The initial background which is curved and geometric transforms to a non geometric curved background

$$\Pi_{+\mu\nu}(x) \to {}^\star\Pi^{\mu\nu}_+(y) = \frac{\kappa}{2}\,{}^\dagger\Theta^{\mu\nu}_-\big(V(y)\big),$$

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where $\Theta_-^{\mu\nu}$ is defined in (4.11). Consequently the T-dual theory can not be directly compared with the standard theories, where the background fields depend on the ordinary coordinates. The T-dual Riemann tensor is zero, which means that T-dual background is flat. It would be interesting to check whether T-duality in the higher orders can make the target space parallelizable. Although the T-dual background is flat, certain adjustments can be made in order for our approximation to satisfy the relations analogue to the general relations of ref. [28].

Applying the procedure to all the dual coordinates, T-dual theory transforms to the initial theory. Comparing the solutions for the auxiliary fields we obtain the T-dual coordinate transformation laws, connecting initial and dual coordinates, which are inverse to each other. Using these laws one confirms that the equations of motion and the Bianchi identity of one theory transform to the Bianchi identity and the equations of motion of the other theory. Because these laws have obtained the second order correction, they will enable further investigation of the non-commutativity properties of the spaces connected by T-duality. Furthermore, the laws are the basis for a double formulation [29–32] where T-duality is interpreted as an exchange of the initial and dual coordinates.

So, we showed that the T-dualization of the theory with a non-trivial Ricci tensor and without global symmetry is possible, and that it does not break the standard features of T-duality. The non geometric structure of T-dual theory is much richer than in the cases previously analyzed and may be a subject of further investigations.

A The expansion of the background fields

All the expressions will be divided into its zeroth, first and second order values, for example $G_{\mu\nu} = G_{0\mu\nu} + G_{1\mu\nu} + G_{2\mu\nu}$. By

$$G^{(1)}_{\mu\nu} = G_{0\mu\nu} + G_{1\mu\nu}, \tag{A.1}$$

we mark the value up to the first order. The inverse of $G_{\mu\nu}$ is given by

$$(G^{-1})^{\mu\nu} = (G^{-1})_0^{\mu\nu} - \Big[(G^{-1})_0\Big(G_1 + G_2 - G_1(G^{-1})_0G_1\Big)(G^{-1})_0\Big]^{\mu\nu} \tag{A.2}$$

- Original background fields

$$\begin{aligned} &G_{0\mu\nu} = g_{\mu\nu}, &&B_{0\mu\nu} = b_{\mu\nu}, &&\Pi_{0\pm\mu\nu} = b_{\mu\nu} \pm \frac{1}{2}g_{\mu\nu},\\ &G_{1\mu\nu} = 0, &&B_{1\mu\nu} = h_{\mu\nu}, &&\Pi_{1\pm\mu\nu} = h_{\mu\nu},\\ &G_{2\mu\nu} = 3h^2_{\mu\nu}, &&B_{2\mu\nu} = 0, &&\Pi_{2\pm\mu\nu} = \pm\frac{3}{2}h^2_{\mu\nu}. \end{aligned} \tag{A.3}$$

- Inverse of a metric $(G^{-1})^{\mu\nu}$

$$\begin{aligned} (G^{-1})_0^{\mu\nu} &= (g^{-1})^{\mu\nu},\\ (G^{-1})_1^{\mu\nu} &= 0,\\ (G^{-1})_2^{\mu\nu} &= -3(g^{-1}h^2g^{-1})^{\mu\nu}. \end{aligned} \tag{A.4}$$

- Effective metric $(G_E)_{\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$

$$\begin{aligned}
(G_{E0})_{\mu\nu} &= g_{\mu\nu} - 4b^2_{\mu\nu} = (g_E)_{\mu\nu}, \\
(G_{E1})_{\mu\nu} &= -4(bh + hb)_{\mu\nu}, \\
(G_{E2})_{\mu\nu} &= -h^2_{\mu\nu} + 12(bh^2b)_{\mu\nu}.
\end{aligned} \tag{A.5}$$

- Effective metric inverse

$$\begin{aligned}
(G_E^{-1})_0^{\mu\nu} &= (g_E^{-1})^{\mu\nu}, \\
(G_E^{-1})_1^{\mu\nu} &= 4\Big[g_E^{-1}\Big(bh + hb\Big)g_E^{-1}\Big]^{\mu\nu} = -2\kappa(\theta_0 h g_E^{-1} + g_E^{-1} h\theta_0)^{\mu\nu}, \\
(G_E^{-1})_2^{\mu\nu} &= -\Big[g_E^{-1}\Big(-h^2 + 12bh^2b - 16(bh+hb)g_E^{-1}(bh+hb)\Big)g_E^{-1}\Big]^{\mu\nu} \\
&= -3(g_E^{-1}h^2g_E^{-1})^{\mu\nu} + 6\kappa^2(\theta_0 h^2\theta_0)^{\mu\nu} + 4(g_E^{-1}hg_E^{-1}hg_E^{-1})^{\mu\nu} \\
&\quad +4\kappa^2(\theta_0 h g_E^{-1} h\theta_0 + \theta_0 h g_E^{-1} h\theta_0 + g_E^{-1}h\theta_0 h\theta_0)^{\mu\nu}.
\end{aligned} \tag{A.6}$$

- Parameter of noncommutativity $\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$

$$\begin{aligned}
\theta_0^{\mu\nu} &= -\frac{2}{\kappa}(g_E^{-1}bg^{-1})^{\mu\nu}, \\
\theta_1^{\mu\nu} &= -\frac{2}{\kappa}(g_E^{-1}(h + 4bhb)g_E^{-1})^{\mu\nu} = -\frac{2}{\kappa}(g_E^{-1}hg_E^{-1} + \kappa^2\theta_0 h\theta_0)^{\mu\nu}, \\
\theta_2^{\mu\nu} &= -3(\theta_0 h^2 g_E^{-1} + g_E^{-1}h^2\theta_0)^{\mu\nu} + 4\kappa^2(\theta_0 h\theta_0 h\theta_0)^{\mu\nu} \\
&\quad +4(\theta_0 h g_E^{-1} h g_E^{-1} + g_E^{-1} h\theta_0 h g_E^{-1} + g_E^{-1} h g_E^{-1} h\theta_0)^{\mu\nu}.
\end{aligned} \tag{A.7}$$

- Theta function $\Theta_\pm^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}$

$$\begin{aligned}
\Theta_{0\pm}^{\mu\nu} &= \theta_0^{\mu\nu} \mp \frac{1}{\kappa}(g_E^{-1})^{\mu\nu}, \\
\Theta_{1\pm}^{\mu\nu} &= -2\kappa\Big[\Theta_{0\pm}h\Theta_{0\pm}\Big]^{\mu\nu}, \\
\Theta_{2\pm}^{\mu\nu} &= \Theta_{0\pm}^{\mu\rho}\Big[\pm 3\kappa h^2 + 4\kappa^2 h\Theta_{0\pm}h\Big]_{\rho\sigma}\Theta_{0\pm}^{\sigma\nu} \\
&= \pm 3\kappa\Theta_{0\pm}^{\mu\rho}h^2_{\rho\sigma}\Theta_{0\pm}^{\sigma\nu} + 4\Delta_\pm^{\mu\nu}.
\end{aligned} \tag{A.8}$$

- The second order contribution $\partial_+ y_\mu\, \Delta_-^{\mu\nu}\big(V_0(y)\big)\partial_- y_\nu \equiv -4\beta^-_{1\mu}\big(V_0(y)\big)\Theta_{0-}^{\mu\nu}\beta^+_{1\nu}\big(V_0(y)\big)$

$$\Delta_\pm^{\mu\nu} = \frac{\kappa}{2}\Theta_{1\pm}^{\mu\rho}\Pi_{0\mp\rho\sigma}\Theta_{1\pm}^{\sigma\nu} = \kappa^2\Big(\Theta_{0\pm}h\Theta_{0\pm}h\Theta_{0\pm}\Big)^{\mu\nu}. \tag{A.9}$$

- Dual background fields composition ${}^\dagger\Theta_\pm^{\mu\nu} = \Theta_\pm^{\mu\nu} - \Delta_\pm^{\mu\nu}$

$$\begin{aligned}
{}^\dagger\Theta_{0\pm}^{\mu\nu} &= \Theta_{0\pm}^{\mu\nu}, \\
{}^\dagger\Theta_{1\pm}^{\mu\nu} &= \Theta_{1\pm}^{\mu\nu}, \\
{}^\dagger\Theta_{2\pm}^{\mu\nu} &= 3\kappa^2\Big[\Theta_{0\pm}h\Big(\theta_0 \pm \frac{1}{\kappa}(g^{-1} - g_E^{-1})\Big)h\Theta_{0\pm}\Big]^{\mu\nu}.
\end{aligned} \tag{A.10}$$

- Dual background fields composition ${}^{\dagger}\Pi_{\pm\mu\nu} = \Pi_{\pm\mu\nu} + 2\kappa\Pi_{\pm\mu\rho}\Delta^{\rho\sigma}_{\mp}\Pi_{\pm\sigma\nu}$

$$\begin{aligned}
{}^{\dagger}\Pi_{0\pm\mu\nu} &= \Pi_{0\pm\mu\nu}, \\
{}^{\dagger}\Pi_{1\pm\mu\nu} &= \Pi_{1\pm\mu\nu}, \\
{}^{\dagger}\Pi_{2\pm\mu\nu} &= \Pi_{2\pm\mu\nu} + \frac{\kappa}{2}h_{\mu\rho}\Theta^{\rho\sigma}_{0\mp}h_{\sigma\nu}.
\end{aligned} \tag{A.11}$$

The last two compositions are inverse to each other

$$ {}^{\dagger}\Pi_{\pm\mu\nu}{}^{\dagger}\Theta^{\nu\rho}_{\mp} = \frac{1}{2\kappa}\delta^{\rho}_{\mu}. \tag{A.12}$$

- Functions ${}^{\diamond}F_{\pm}$

$$ {}^{\diamond}F_{\pm} \equiv \sum_{n=0}^{2}\left(1+\frac{n}{2}\right)F_{n\pm}, \tag{A.13}$$

$$ {}^{\diamond}\Theta^{\mu\nu}_{\pm} \equiv \Theta^{\mu\nu}_{0\pm} + \frac{3}{2}\Theta^{\mu\nu}_{1\pm} + 2\Theta^{\mu\nu}_{2\pm}, \tag{A.14}$$

$$ {}^{\diamond}\Pi_{\pm\mu\nu} \equiv \Pi_{0\pm\mu\nu} + \frac{3}{2}\Pi_{1\pm\mu\nu} + 2\Pi_{2\pm\mu\nu}. \tag{A.15}$$

- Inverses of functions ${}^{\diamond}\Theta^{\mu\nu}_{\pm}$ and ${}^{\diamond}\Pi_{\pm\mu\nu}$

$$\begin{aligned}
({}^{\diamond}\Pi_{\pm\mu\nu} + 2\kappa\Pi_{0\pm\mu\rho}\Delta^{\rho\sigma}_{\mp}\Pi_{0\pm\sigma\nu}){}^{\diamond}\Theta^{\nu\rho}_{\mp} &= \frac{1}{2\kappa}\delta^{\rho}_{\mu}, \\
{}^{\diamond}\Pi_{\pm\mu\nu}({}^{\diamond}\Theta^{\nu\rho}_{\mp} + \Delta^{\nu\rho}_{\mp}) &= \frac{1}{2\kappa}\delta^{\rho}_{\mu}.
\end{aligned} \tag{A.16}$$

In the first order one has

$$ {}^{\diamond}\Pi^{(1)}_{\pm\mu\nu}{}^{\diamond}\Theta^{(1)\nu\rho}_{\mp} = \frac{1}{2\kappa}\delta^{\rho}_{\mu}. \tag{A.17}$$

- Difference

$$ {}^{\diamond}\Pi_{+\mu\nu} - {}^{\diamond}\Pi_{-\mu\nu} = g_{\mu\nu} + 6h^{2}_{\mu\nu}, \tag{A.18}$$

and its inverse

$$ ({}^{\diamond}\Pi_{+\mu\nu} - {}^{\diamond}\Pi_{-\mu\nu})^{-1} = (g^{-1})^{\mu\nu} - 6(h^2)^{\mu\nu}. \tag{A.19}$$

B Beta function $\beta^{\pm}_{\mu}$

Let us calculate the beta functions defined in (3.7), for the action (3.1). The variation of the action over the background fields argument is

$$ \delta_V S_{\text{aux}} = \kappa\int d^2\xi\left[\varepsilon^{\alpha\beta}\partial_\rho B_{\mu\nu} + \frac{1}{2}\eta^{\alpha\beta}\partial_\rho G_{\mu\nu}\right]\partial_\alpha V^\mu\partial_\beta V^\nu\delta V^\rho. \tag{B.1}$$

Partially integrating, using the zeroth order equation of motion $\eta^{\alpha\beta}\partial_\alpha\partial_\beta V^\mu = 0$ in a quadratic in $B_{\mu\nu\rho}$ terms, we obtain

$$\begin{aligned}
\delta_V S_{\text{aux}} = -\kappa\int d^2\xi\Bigg[&\left(\varepsilon^{\alpha\beta}\partial_\rho B_{\mu\nu} + \frac{1}{2}\eta^{\alpha\beta}\partial_\rho G_{\mu\nu}\right)V^\mu\partial_\beta V^\nu\delta v^\rho_\alpha \\
&+\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\rho G_{\mu\nu}V^\mu\partial_\beta V^\nu\delta V^\rho\Bigg].
\end{aligned} \tag{B.2}$$

Using the explicit value of the initial metric the second term can be rewritten as

$$\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\partial_\rho G_{\mu\nu}V^\mu\partial_\beta V^\nu\delta V^\rho = \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha G_{2\mu\rho}(V)\partial_\beta V^\mu\delta V^\rho = \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\Big(G_{2\mu\rho}(V)\partial_\beta V^\mu\Big)\delta V^\rho, \tag{B.3}$$

and therefore

$$\delta_V S_{\rm aux} = -\kappa\int d^2\xi\Big[\Big(\varepsilon^{\alpha\beta}\partial_\rho B_{\mu\nu} + \frac{1}{2}\eta^{\alpha\beta}\partial_\rho G_{\mu\nu}\Big)V^\mu\partial_\beta V^\nu\delta v^\rho_\alpha - \frac{1}{2}\eta^{\alpha\beta}G_{2\mu\rho}(V)\partial_\beta V^\mu\delta v^\rho_\alpha\Big]. \tag{B.4}$$

So, the beta functions, defined by (3.7), are

$$\begin{aligned}\beta^\alpha_\rho &= \varepsilon^{\alpha\beta}\partial_\rho B_{\mu\nu}V^\mu\partial_\beta V^\nu + \frac{1}{2}\eta^{\alpha\beta}\Big(\partial_\rho G_{\mu\nu}(V)V^\mu - G_{2\nu\rho}(V)\Big)\partial_\beta V^\nu\\ &= -\Big(\epsilon^{\alpha\beta}h_{\mu\nu}(V) + 3\eta^{\alpha\beta}h^2_{\mu\nu}(V)\Big)\partial_\beta V^\nu,\end{aligned} \tag{B.5}$$

and in the light-cone coordinates they become

$$\beta^\pm_\rho(V) = \frac{1}{2}\Big(\beta^0_\rho \pm \beta^1_\rho\Big) = \frac{1}{2}\Big(\mp h_{\rho\nu}(V) - 3h^2_{\rho\nu}(V)\Big)\partial_\mp V^\nu. \tag{B.6}$$

C Dual beta function ${}^\star\beta^{\pm\mu}$

In this section we will find the beta functions for the dual theory auxiliary action (5.1)

$${}^\star S_{\rm aux} = \frac{\kappa^2}{2}\int d^2\xi\Big[u_{+\mu}\,{}^\dagger\Theta^{\mu\nu}_-\big(V(U)\big)\,u_{-\nu} + \frac{1}{\kappa}(u_{+\mu}\partial_- z^\mu - u_{-\mu}\partial_+ z^\mu)\Big]. \tag{C.1}$$

We define them as usual by

$$\begin{aligned}\delta_V{}^\star S_{\rm aux} &= \frac{\kappa^2}{2}\int d^2\xi\, u_{+\mu}\,\partial_\rho{}^\dagger\Theta^{\mu\nu}_-\big(V(U)\big)\,u_{-\nu}\,\delta V^\rho(U)\\ &= -\frac{\kappa^2}{2}\int d^2\xi\Big[{}^\star\beta^{+\mu}\delta u_{+\mu} + {}^\star\beta^{-\mu}\delta u_{-\mu}\Big].\end{aligned} \tag{C.2}$$

Multiplying the equation (4.7) by ${}^\diamond\Pi^{(1)}_{\mp\mu\nu}$, defined by (A.15) (the inverse of ${}^\diamond\Theta^{(1)\mu\nu}_\pm$, see (A.17)), one obtains the auxiliary fields $u_{\pm\mu}$ in terms of the auxiliary fields $v^\mu_\pm$

$$u^{(1)}_{\pm\mu} = -2{}^\diamond\Pi^{(1)}_{\mp\mu\nu}(V(U))v^{(1)\mu}_\pm(U). \tag{C.3}$$

Substituting these expressions to the first expression in (C.2) we obtain

$$\delta_V{}^\star S_{\rm aux} = -2\kappa^2\int d^2\xi\,\partial_+V^\mu F_{\mu\nu,\rho}(V(U))\partial_- V^\nu\delta V^\rho(U), \tag{C.4}$$

with the background field composition $F_{\mu\nu,\rho}$ defined by

$$F_{\mu\nu,\rho} \equiv \Big({}^\diamond\Pi_+\,\partial_\rho{}^\dagger\Theta_-{}^\diamond\Pi_+\Big)_{\mu\nu} = -\frac{1}{2\kappa}\partial_\rho\Big[h + \frac{3}{2}h\Big(\kappa\Theta_{0-} + g^{-1}\Big)h\Big]_{\mu\nu}, \tag{C.5}$$

where the second expression is obtained using (A.15) and (A.10). Partially integrating in (C.4) we obtain

$$\delta_{V^\star} S_{\text{aux}} = \kappa^2 \int d^2\xi \Big\{ \Big[V^\mu \partial_+ F_{\mu\nu,\rho}(V) \partial_- V^\nu + \partial_+ V^\mu \partial_- F_{\mu\nu,\rho}(V) V^\nu \Big] \delta V^\rho(U) + V^\mu F_{\mu\nu,\rho}(V) \partial_- V^\nu \delta v_+^\rho(U) + \partial_+ V^\mu F_{\mu\nu,\rho}(V) V^\nu \delta v_-^\rho(U) \Big\}. \tag{C.6}$$

The term $F_{\mu\nu,\rho}(V)\big(V^\mu\partial_+\partial_- V^\nu + \partial_+\partial_- V^\mu V^\nu\big)\delta V^\rho$ is absent, because, the antisymmetric in first two indices part of $F_{\mu\nu,\rho}$ gives zero, while its symmetric part is of the second order and therefore the whole expression vanishes on the zeroth order equation of motion $\partial_+\partial_- V^\mu = 0$.

Taking the variation of (4.7), using (A.14), one obtains

$$\delta v_\pm^\mu(U) = -\kappa^\diamond \Theta_\pm^{(1)\mu\nu}(V(U))\delta u_{\pm\nu} + \delta v_\pm^{\star\mu}(U), \tag{C.7}$$

$$\delta v_\pm^{\star\mu}(U) = -\frac{3\kappa}{2}\partial_\rho \Theta_{1\pm}^{\mu\nu} u_{0\pm\nu} \delta V_0^\rho(U_0). \tag{C.8}$$

Using (A.8) and (4.2) one observes that

$$\delta v_\pm^{\star\mu}(U) = -3\kappa \Theta_{0\pm}^{\mu\nu} \partial_\rho h_{\nu\sigma} \partial_\pm V^\sigma \delta V^\rho. \tag{C.9}$$

Let us calculate the contribution from $\delta v_\pm^{\star\mu}(U)$ terms. Because these terms are of the first order it is enough to use the first order value of $F_{\mu\nu,\rho}$, defined by (C.5). One obtains

$$\begin{aligned} \delta_{V^\star} S_{\text{aux}} &= \kappa^2 \int d^2\xi \Big(V^\mu F_{\mu\nu,\rho}(V) \partial_- V^\nu \delta v_+^{\star\rho}(U) + \partial_+ V^\mu F_{\mu\nu,\rho}(V) V^\nu \delta v_-^{\star\rho}(U) \Big) \\ &= -\frac{3\kappa^2}{2} \int d^2\xi\, \partial_+ V^\mu \partial_\rho (h\Theta_{0-}h)_{\mu\nu}(V) \partial_- V^\nu \delta V^\rho. \end{aligned} \tag{C.10}$$

Partially integrating we obtain

$$\begin{aligned} \delta_{V^\star} S_{\text{aux}} = \frac{3\kappa^2}{4} \int d^2\xi \Big[\Big(& V^\mu \partial_+ \partial_\rho (h\Theta_{0-}h)_{\mu\nu}(V) \partial_- V^\nu \\ & + \partial_+ V^\mu \partial_- \partial_\rho (h\Theta_{0-}h)_{\mu\nu}(V) V^\nu \Big) \delta V^\rho \\ & + V^\mu \partial_\rho (h\Theta_{0-}h)_{\mu\nu}(V) \partial_- V^\nu \delta v_+^\rho + \partial_+ V^\mu \partial_\rho (h\Theta_{0-}h)_{\mu\nu}(V) V^\nu \delta v_-^\rho \Big], \end{aligned} \tag{C.11}$$

where we again used the zeroth order equation of motion $\partial_+\partial_- V^\mu = 0$. Finally, substituting (C.7) and (C.5) into (C.6), using (C.11), and noting that $\partial_\pm \partial_\rho h_{\mu\nu} = 0$, the variation of the auxiliary action becomes

$$\begin{aligned} \delta_{V^\star} S_{\text{aux}} = \kappa^2 \int d^2\xi \Big\{ & \Big[-\frac{3}{4\kappa} V^\mu \partial_+ \partial_\rho h^2_{\mu\nu}(V) \partial_- V^\nu - \frac{3}{4\kappa} \partial_+ V^\mu \partial_- \partial_\rho h^2_{\mu\nu}(V) V^\nu \Big] \delta V^\rho(U) \\ & - \frac{1}{2\kappa} V^\mu \partial_\rho \Big(h + \frac{3}{2} h^2 \Big)_{\mu\nu} (V) \partial_- V^\nu (-\kappa)^\diamond \Theta_+^{(1)\rho\sigma}(V) \delta u_{+\sigma} \\ & - \frac{1}{2\kappa} \partial_+ V^\mu \partial_\rho \Big(h + \frac{3}{2} h^2 \Big)_{\mu\nu} (V) V^\nu (-\kappa)^\diamond \Theta_-^{(1)\rho\sigma}(V) \delta u_{-\sigma} \Big\}. \end{aligned} \tag{C.12}$$

The first two terms can be rewritten as

$$V^\mu\partial_+\partial_\rho h^2_{\mu\nu}\partial_-V^\nu + \partial_+V^\mu\partial_-\partial_\rho h^2_{\mu\nu}V^\nu = \partial_+V^\mu\partial_-h^2_{\mu\rho} + \partial_-V^\mu\partial_+h^2_{\mu\rho}$$
$$= \partial_-\left(\partial_+V^\mu h^2_{\mu\rho}\right) + \partial_+\left(\partial_-V^\mu h^2_{\mu\rho}\right). \quad \text{(C.13)}$$

Partially integrating, using (C.7), we obtain

$$\delta_V{}^\star S_{\text{aux}} = -\frac{\kappa^2}{2}\int d^2\xi \left\{\delta u_{-\mu}{}^\diamond\Theta_+^{(1)\mu\nu}(V)(h-3h^2)_{\nu\rho}(V)\partial_+V^\rho \right.$$
$$\left. +\delta u_{+\mu}{}^\diamond\Theta_-^{(1)\mu\nu}(V)(-h-3h^2)_{\nu\rho}(V)\partial_-V^\rho\right\}. \quad \text{(C.14)}$$

Finally, recalling (B.6) we obtain the dual beta functions

$$^\star\beta^{\pm\mu}(V(U)) = 2\,{}^\diamond\Theta_\mp^{(1)\mu\nu}(V(U))\beta_\nu^\pm(V(U)). \quad \text{(C.15)}$$

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Open string T-duality in a weakly curved background

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Abstract We consider a theory of an open string moving in a weakly curved background, composed of a constant metric and a linearly coordinate dependent Kalb–Ramond field with an infinitesimal field strength. We find its T-dual using the generalized Buscher procedure developed for the closed string moving in a weakly curved background, and the fact that solving the boundary conditions, the open string theory transforms to the effective closed string theory. So, T-dualizing the effective theory along all effective directions we obtain its T-dual theory and resume the open string theory which has such an effective theory. In this way we obtain the open string theory T-dual.

1 Introduction

T-duality (reviewed in [1,2]) is a symmetry of a string spectrum, exchanging the momentum and the winding numbers, a symmetry which was not encountered in any point particle theory [1–4], a symmetry which is therefore naturally assumed to be connected with the fact that the strings unlike the point particles can wrap around compactified dimensions [5–7]. T-duality is a symmetry which connects string theories which are physically equivalent but describe strings which move in space-times with different geometries of the compactified dimensions. The theory with one dimension compactified on a circle of radius R is by T-duality physically equivalent to the theory with one dimension compactified on a circle of radius l_s^2/R, where l_s is the fundamental string length scale.

The existence of T-duality initiated the search for the T-dualization procedure, the set of rules which give a prescription for determining all string theories T-dual to the given theory. The problem was solved for some particular backgrounds. The first investigations addressing the problem were done for a string sigma model describing a string moving in a background containing a metric $G_{\mu\nu}$, an antisymmetric field $B_{\mu\nu}$ and a dilaton field Φ, and it was required that the metric admits at least one abelian isometry which leaves the action invariant. The investigation resulted in the Buscher procedure [8–10]. This procedure offers the prescription, founded in localizing the global isometry. The main object of the procedure is a gauge invariant action, which carries the information on both the initial theory and the T-dual theory, so that both of these theories can be obtained from the qauge invariant action for one of the two of its equations of motion. The procedure is applicable to the constant background and the coordinate dependent backgrounds, but only along directions which do not appear as the background field's arguments.

It was found that in a T-dualization of a background along the isometry direction one mixes the background fields $G_{\mu\nu}$ and $B_{\mu\nu}$, which explains how T-duality relates backgrounds with different geometrical and topological properties. The search for the equivalent string backgrounds goes beyond the Buscher procedure. If one considers a string sigma model as a theory of maps from two-dimensional space to a manifold M, the geometrical string background are obtained if the transition functions in the overlapping coordinate patches are diffeomorphisms and gauge transformations. If T-duality is included into transition functions one obtains non-geometrical string backgrounds [11]. Even more non-geometrical backgrounds are known to exist [12], which in some cases arise from the consistent string backgrounds performing generalized T-dualities in non-isometry directions.

T-dualization procedures were developed for the closed strings and then extended to the open string case. T-duality of an open string differs from the closed string T-duality, due to a simple fact that the open string does not have a winding number. Its ends move on the Dirichlet branes and can simply unwind [13]. It was shown in [14] that Dirichlet branes have a significant role in string duality. The open string T-duality along the isometry directions was investigated using the standard Buscher procedure [15], canonical transformations [15,16], the functional integral approach

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[16], and using new formalisms for nonabelian isometry [16–18]. T-duality relates open strings to Dirichlet branes, which is a limiting case of duality between pairs of Dirichlet branes [19]. Poisson–Lie T-duality enabled the investigation of backgrounds without isometries.

References [20–22] offered the generalized Buscher T-dualization procedure, which is applicable along an arbitrary coordinate regardless of whether this coordinate appears as an argument of the background fields or not. The main difference between the standard and the generalized procedure is that, in the generalized procedure, in addition to the introduction of the gauge fields and the covariant derivatives one should also introduce an invariant argument which is defined as a line integral of the covariant derivatives of the original argument. The generalized procedure was realized for the closed string moving in a curved background with an infinitesimal coordinate dependence, named the weakly curved background [20]. A more general procedure was defined in [21] for a weakly curved background of the second order, which does not possess the global isometry. In this paper, we will use the first version of the generalized T-dualization procedure to address the T-duality of an open string moving in the weakly curved background.

The investigation of an open bosonic string moving in a weakly curved background, presented in Refs. [23–25], dedicated to solving the boundary conditions, showed that the effective theory obtained for the solution of the boundary conditions is in fact a closed string theory. The boundary conditions were treated by Dirac method in [23,24]. Reference [25] included a close examination of the parity of the boundary conditions, equations of motion and the consistency conditions. The two approaches lead to equivalent solutions. The effective theory, obtained for the solution, is defined on the doubled target space, which consists of a symmetric and an antisymmetric variables $(q^\mu, \tilde{q}^\mu)$, with the first variable being the even part of the initial coordinate and the second variable being the double of the first variable. The effective metric depends on the effective coordinate q^μ, while the effective Kalb–Ramond field depends on a double variable $\tilde{q}^\mu$.

In this paper, we will apply the generalized Buscher procedure [20–22], developed for the closed string moving in the weakly curved background, to the effective closed string theory, obtained for the solution of the open string boundary conditions, along all effective coordinates. We will obtain its T-dual theory. We will show that T-dualization of the obtained theory along all dual coordinates leads to the initial theory. Our main goal is to find the open string T-dual. So, we shall assume that the obtained T-dual theory is an effective theory of some open string theory. We search for the explicit form of this theory. Demanding that the effective theory of the unknown theory is exactly the T-dual theory, we obtain a T-dual of the initial open string theory. The obtained theory is defined in the geometrical space, on the contrary to the closed string case, where the T-dualization led to a T-dual theory defined on the non-geometrical double space. The relations between the initial background and its T-dual differ from those in the closed string case.

2 Open string theory in a weakly curved background

Let us consider an open bosonic string moving in a background defined by a metric tensor $G_{\mu\nu}$, an antisymmetric Kalb–Ramond field $B_{\mu\nu}$, and a dilaton field Φ, described by the action [3,4]

$$S[x] = \kappa \int_\Sigma d^2\xi \sqrt{-g}\left[\left(\frac{1}{2}g^{\alpha\beta}G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}}B_{\mu\nu}(x)\right) \times \partial_\alpha x^\mu \partial_\beta x^\nu + \frac{1}{4\pi\kappa}\Phi(x)R^{(2)}\right], \tag{1}$$

where the integration goes over a two-dimensional world-sheet Σ parametrized by ξ^α ($\xi^0 = \tau,\ \xi^1 = \sigma$), $g_{\alpha\beta}$ is the intrinsic world-sheet metric, $R^{(2)}$ corresponding 2-dimensional scalar curvature, $x^\mu(\xi)$, $\mu = 0, 1, \ldots, D-1$ are the coordinates of the D-dimensional space-time, $\kappa = \frac{1}{2\pi\alpha'}$ with α' being the Regge slope parameter and $\varepsilon^{01} = -1$. We will also use the notation $\dot{x} = \frac{\partial x}{\partial \tau}$, $x' = \frac{\partial x}{\partial \sigma}$. In order to have a conformal invariance on a quantum level the background fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ have to obey the following space-time equations of motion (given in the lowest order in the slope parameter α'):

$$\begin{aligned}
&R_{\mu\nu} - \frac{1}{4}B_{\mu\rho\sigma}B_\nu^{\ \rho\sigma} + 2D_\mu\partial_\nu\Phi = 0,\\
&D_\rho B^\rho_{\ \mu\nu} - 2\partial_\rho\Phi B^\rho_{\ \mu\nu} = 0,\\
&4(\partial\Phi)^2 - 4D_\mu\partial^\mu\Phi + \frac{B_{\mu\nu\rho}B^{\mu\nu\rho}}{12} - R + 4\pi\kappa\frac{D-26}{3} = 0,
\end{aligned} \tag{2}$$

where $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are the Ricci tensor and the covariant derivative with respect to the space-time metric. We will consider the open string moving in the weakly curved background [26–29], defined by

$$\begin{aligned}
&G_{\mu\nu} = \text{const},\\
&B_{\mu\nu}(x) = b_{\mu\nu} + h_{\mu\nu}(x), \quad h_{\mu\nu}(x) = \frac{1}{3}B_{\mu\nu\rho}x^\rho,\\
&\Phi = \text{const},
\end{aligned} \tag{3}$$

where the quantities $b_{\mu\nu}$ and $B_{\mu\nu\rho}$ are constant. For such a background the equations of motion (2) reduce to

$$R_{\mu\nu} - \frac{1}{4}B_{\mu\rho\sigma}B_\nu^{\ \rho\sigma} = 0\,,$$
$$D_\rho B^\rho_{\ \mu\nu} = 0, \tag{4}$$
$$\frac{1}{12}B_{\mu\nu\rho}B^{\mu\nu\rho} - R + 4\pi\kappa\frac{D-26}{3} = 0.$$

The weakly curved background is the solution of these equations for $D = 26$, if the quadratic terms in $B_{\mu\nu\rho}$ can be neglected. So, we choose $B_{\mu\nu\rho}$ infinitesimal and work in the linear order in $B_{\mu\nu\rho}$.

The action (1) for the string moving in the weakly curved background (3) can be written as

$$S[x] = \kappa\int_\Sigma d^2\xi\ \partial_+ x^\mu \Pi_{+\mu\nu}(x)\partial_- x^\nu, \tag{5}$$

where the light-cone coordinates and their derivatives are $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\partial_\pm = \partial_\tau \pm \partial_\sigma$, and $\Pi_{\pm\mu\nu}$ is a combination of the background fields, defined by

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2}G_{\mu\nu}. \tag{6}$$

There is a direction of research in differential geometry in which the above theory is interpreted as a theory of maps x^μ from the two-dimensional space Σ to a manifold M with metric G and a closed 3-form H [30]. For a geometric string background, the background fields in overlaps of the coordinate patches are related by diffeomorphisms and gauge transformations. The local choices fit together to form a space-time manifold.

The minimal action principle for the open string, described by (5), gives the equation of motion

$$\partial_+\partial_- x^\mu - B^\mu_{\ \nu\rho}\partial_+ x^\nu \partial_- x^\rho = 0 \tag{7}$$

and the boundary conditions on the string endpoints

$$\gamma^0_\mu\Big|_{\sigma=0,\pi} = 0, \tag{8}$$

where

$$\gamma^0_\mu \equiv \frac{\delta\mathcal{L}}{\delta x'^\mu} = G_{\mu\nu}x'^\nu - 2B_{\mu\nu}\dot{x}^\nu$$
$$= \Pi_{+\mu\nu}\partial_- x^\nu + \Pi_{-\mu\nu}\partial_+ x^\nu. \tag{9}$$

Solving of these boundary conditions was a subject of investigation of Refs. [23–25]. In the first two papers, the boundary conditions were treated as constraints and we applied the Dirac consistency procedure. We obtained an infinite number of constraints, gathered them into two parameter dependent constraints, which were solved. We obtained the form of the initial coordinates satisfying the boundary conditions. In the paper [25], we obtained the analogous result by examining the parity of the equations of motion, boundary conditions and the consistency conditions. The solution of the boundary condition at $\sigma = 0$ is

$$\partial_\pm x^\mu = (G^{-1})^{\mu\rho}[G_{\rho\nu} - A_{\rho\nu}(\tilde{q}) \pm 2B_{\rho\nu}(q)]\partial_\pm q^\nu, \tag{10}$$

where

$$A_{\rho\nu}(\tilde{q}) = [h(\tilde{q}) - 12bh(\tilde{q})b - 12h(b\tilde{q})b + 12bh(b\tilde{q})]_{\rho\nu},$$

q^μ is the even part of the initial coordinate,

$$q^\mu(\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^{2n}}{(2n)!}x^{(2n)\mu}\Big|_{\sigma=0}, \tag{11}$$

and $\tilde{q}$ is its double, which satisfies

$$\dot{\tilde{q}}^\mu = q'^\mu, \qquad \tilde{q}'^\mu = \dot{q}^\mu. \tag{12}$$

Changing the domain to $\sigma \in [-\pi, \pi]$, one enables the solution (10) to satisfy the boundary condition at the other string endpoint as well. For details see [25]. Substituting the solution (10) to the initial action we obtain the effective theory, given in terms of effective variables $(q^\mu, \tilde{q}^\mu)$

$$S^{\text{eff}} = \kappa\int d\tau \int_{-\pi}^{\pi} d\sigma\ \partial_+ q^\mu\ \Pi^{\text{eff}}_{+\mu\nu}(q, 2b\tilde{q})\ \partial_- q^\nu, \tag{13}$$

with an effective background

$$\Pi^{\text{eff}}_{\pm\mu\nu}(q, 2b\tilde{q}) \equiv B^{\text{eff}}_{\mu\nu}(2b\tilde{q}) \pm \frac{1}{2}G^{\text{eff}}_{\mu\nu}(q). \tag{14}$$

The effective theory is defined on the doubled space $(q^\mu, \tilde{q}^\mu)$, with a double coordinate appearing in a solution of the boundary conditions. Usually, the doubled geometry is introduced in an investigation of T-duality. A coordinate and its double give an origin to the winding and the momentum number, the exchange of which by T-duality does not change the physics. The doubled description [31,32] makes T-duality a manifest symmetry.

The effective metric and the Kalb–Ramond field in (14) are explicitly given by

$$G^{\text{eff}}_{\mu\nu}(q) = G^E_{\mu\nu}(q) := (G - 4B^2(q))_{\mu\nu},$$
$$B^{\text{eff}}_{\mu\nu}(2b\tilde{q}) = -\frac{\kappa}{2}(g_E\Delta\theta(2b\tilde{q})g_E)_{\mu\nu}, \tag{15}$$

where $\Delta\theta^{\mu\nu}$ is the infinitesimal part of the non-commutativity parameter

$$\theta^{\mu\nu} = -\frac{2}{\kappa}[G_E^{-1}BG^{-1}]^{\mu\nu}$$
$$= \theta_0^{\mu\nu} - \frac{2}{\kappa}[g_E^{-1}(h + 4bhb)g_E^{-1}]^{\mu\nu}, \tag{16}$$

defined in analogy with the flat space-time non-commutativity parameter introduced in [33]. One can show that $\theta^{\mu\nu}$ is indeed the non-commutativity parameter by considering the phase spaces of the initial and the effective theory, as in [23]. The initial phase space consists of the initial coordinates x^μ and the momenta π_μ. Solving the boundary conditions one obtains expressions for x^μ and π_μ given in terms of the effective coordinates q^μ and the effective momenta p_μ (the even part of the initial momenta). So, the initial phase space transforms to the effective phase space, whose variables satisfy the star brackets

$$^\star\{q^\mu(\sigma), p_\nu(\bar\sigma)\} = 2\delta^\mu_\nu \delta_S(\sigma, \bar\sigma), \tag{17}$$

defined in Appendix B of Ref. [23], which are the analogs of the Dirac brackets of the initial space. As the initial coordinates which solve the boundary conditions depend on both effective coordinates and the effective momenta, they are non-commutative. Taking

$$X^\mu(\sigma) = x^\mu(\sigma) - x^\mu_{\text{cm}}, \quad x^\mu_{\text{cm}} = \frac{1}{\pi}\int_0^\pi d\sigma x^\mu(\sigma), \tag{18}$$

one finds that the non-commutativity parameter is exactly $\theta^{\mu\nu}$, given by (16),

$$^\star\{X^\mu(\sigma), X^\nu(\bar\sigma)\} = \theta^{\mu\nu}[X(\sigma)] \begin{cases} -1 & \sigma, \bar\sigma = 0, \\ 1 & \sigma, \bar\sigma = \pi, \\ 0 & \text{otherwise}. \end{cases} \tag{19}$$

3 T-dualization of the effective theory

In this section we will apply the generalized Buscher T-dualization procedure, developed in [20–22], to the effective theory (13) and find its T-dual. In general, the procedure could be applied along arbitrary set of the effective coordinates. In this paper, we will investigate only the application along all effective coordinates. The effective theory is given in terms of the effective coordinate q^μ and its double $\tilde q^\mu$, so the overall procedure will look like the T-dualization of the T-dual of the closed string theory in the weakly curved background, because the weakly curved background T-dual is also defined in doubled space, composed of a dual coordinate (Lagrange multiplier) and its double.

So, let us apply the generalized Buscher T-dualization procedure to the effective theory (13) along all effective directions q^μ. The first task is to localize the global symmetry $\delta q^\mu = \lambda^\mu = \text{const}$ and find the gauge invariant action. Following the procedure, we substitute the light-cone derivatives $\partial_\pm q^\mu$ of the effective theory (13) with the covariant derivatives $D_\pm q^\mu$, defined by

$$D_\pm q^\mu = \partial_\pm q^\mu + v^\mu_\pm, \tag{20}$$

where $v^\mu_\pm$ are the gauge fields, which transform as $\delta v^\mu_\pm = -\partial_\pm \lambda^\mu$. We also substitute the argument of the background fields, with an invariant argument, which is obtained substituting the effective coordinate q^μ and its double $\tilde q^\mu$ with an invariant effective coordinate and its double, defined as a line integrals of the covariant derivatives of the effective coordinate and its double, with the line integrals taken along the path P, from the initial point ξ^α_0 to the point ξ^α. The last step in forming the gauge invariant action is adding the Lagrange multiplier term $\frac{1}{2}(v^\mu_+\partial_-\varrho_\mu - v^\mu_-\partial_+\varrho_\mu)$, which makes the introduced gauge fields nonphysical. Fixing the gauge by $q^\mu(\xi) = q^\mu(\xi_0)$, we obtain the gauge fixed action, which reads

$$\begin{aligned} \mathcal{S}_{\text{fix}} = \kappa \int d^2\xi [& v^\mu_+ \Pi^{\text{eff}}_{+\mu\nu}(\Delta V, 2b\Delta\tilde V) v^\nu_- \\ & + \frac{1}{2}(v^\mu_+\partial_-\varrho_\mu - v^\mu_-\partial_+\varrho_\mu)], \end{aligned} \tag{21}$$

where ΔV^μ and $\Delta\tilde V^\mu$ are the following line integrals of the gauge fields:

$$\begin{aligned} \Delta V^\mu &= \int_P (d\xi^+ v^\mu_+ + d\xi^- v^\mu_-), \\ \Delta\tilde V^\mu &= \int_P (d\xi^+ v^\mu_+ - d\xi^- v^\mu_-). \end{aligned} \tag{22}$$

The next step in the T-dualization procedure is finding the equations of motion of the gauge fixed action. For the equation of motion for the Lagrange multiplier the action will reduce to the initial action, while for the equation for the gauge fields the action will become the T-dual action. Varying the action (21), one obtains the following equations of motion:

$$\partial_- v^\mu_+ - \partial_+ v^\mu_- = 0 \tag{23}$$

and

$$\Pi^{eff}_{\pm\mu\nu}(\Delta V, 2b\Delta\tilde V) v^\nu_\mp + \frac{1}{2}\partial_\mp \varrho_\mu = \pm\beta^\pm_\mu(V), \tag{24}$$

where the beta functions $\beta^\pm_\mu(V)$ come from the variation over the background field's arguments, and they are defined by

$$\begin{aligned} \delta_A \mathcal{S}_{\text{fix}} = \kappa \int d\xi^2 \Big[& \varepsilon^{\alpha\beta}\partial_\rho B^{\text{eff}}_{\mu\nu}(2b\delta\tilde V)^\rho \\ & + \frac{1}{2}\eta^{\alpha\beta}\partial_\rho G^{\text{eff}}_{\mu\nu}\delta V^\rho \Big] \partial_\alpha V^\mu \partial_\beta V^\nu \\ = -\kappa \int & d^2\xi (\beta^+_\mu \delta v^\mu_+ + \beta^-_\mu \delta v^\mu_-). \end{aligned} \tag{25}$$

Partially integrating, using the fact that $\partial_\alpha \delta\tilde V^\mu = \varepsilon^\beta{}_\alpha \delta v^\mu_\beta$, one obtains the explicit form of the beta functions

$$\beta^\pm_\mu(V) = \Big[- b_\mu{}^\sigma \partial_\sigma B^{\text{eff}}_{\nu\rho} + \frac{1}{4}\partial_\mu G^{\text{eff}}_{\nu\rho} \Big] V^\nu \partial_\mp V^\rho. \tag{26}$$

Substituting the explicit values of the effective background fields (15), (A.2), and (16) one obtains

$$\beta^{\pm}_{\mu}(V) = -\frac{1}{3}[b_{\mu}{}^{\sigma} B_{\sigma\nu\rho} + b_{\nu}{}^{\sigma} B_{\sigma\rho\mu} + b_{\rho}{}^{\sigma} B_{\sigma\nu\mu} - 4b_{\mu}{}^{\sigma} b_{\nu}{}^{\tau} b_{\rho}{}^{\varepsilon} B_{\sigma\tau\varepsilon}]V^{\nu}\partial_{\mp}V^{\rho}. \quad (27)$$

3.1 Regaining the effective theory

Solving Eq. (23), by

$$v^{\mu}_{\pm} = \partial_{\pm}q^{\mu}, \quad (28)$$

one obtains $V^{\mu} = q^{\mu}$ and $\tilde{V}^{\mu} = \tilde{q}^{\mu}$. Substituting these relations to the gauge fixed action (21), one confirms that it reduces to the effective action (13).

3.2 T-dual theory

To obtain the T-dual action, one should substitute the expressions for the gauge fields to the gauge fixed action. The gauge fields are obtained, multiplying the equation of motion (24) by the inverse of $\Pi^{eff}_{\pm\mu\nu}$, $(\Theta^{eff}_{\mp})^{\mu\nu}$ defined in (A.9),

$$v^{\mu}_{\pm} = -\kappa(\Theta^{\text{eff}}_{\pm})^{\mu\nu}(\Delta V, 2b\Delta\tilde{V})[\partial_{\pm}\varrho_{\nu} \pm 2\beta^{\mp}_{\nu}(V)]. \quad (29)$$

The beta functions will not contribute to the T-dual action because they are infinitesimal and appear within the quadratic term. So, the T-dual action reads

$$^{\star}\mathcal{S} = \kappa\int d^2\xi\, \partial_{+}\varrho_{\mu}\frac{\kappa}{2}(\Theta^{\text{eff}}_{-})^{\mu\nu}(\Delta V(\varrho), 2b\Delta\tilde{V}(\varrho))\partial_{-}\varrho_{\nu}. \quad (30)$$

Let us calculate the argument of the background fields. Using the zeroth order of the equations of motion (24) and (A.5),

$$\Pi^{\text{eff}}_{0\pm\mu\nu}v^{\nu}_{0\mp} + \frac{1}{2}\partial_{\mp}\varrho_{\mu} = \pm\frac{1}{2}g^{\text{eff}}_{\mu\nu}v^{\nu}_{0\mp} + \frac{1}{2}\partial_{\mp}\varrho_{\mu} = 0, \quad (31)$$

and the fact that the open string effective metric (A.2) is the same in both initial and the effective background (A.4), one obtains the explicit value of the arguments (22),

$$V^{\mu}_{0}(\varrho) = (g^{-1}_{E})^{\mu\nu}(G^{\text{eff}}, B^{\text{eff}})\tilde{\varrho}_{\nu} = (g^{-1}_{E})^{\mu\nu}\tilde{\varrho}_{\nu},$$
$$\tilde{V}^{\mu}_{0}(\varrho) = (g^{-1}_{E})^{\mu\nu}(G^{\text{eff}}, B^{\text{eff}})\varrho_{\nu} = (g^{-1}_{E})^{\mu\nu}\varrho_{\nu}. \quad (32)$$

Comparing the forms of the effective action (13) and the T-dual action (30), we see that they are equal under the following transformations:

$$\partial_{\pm}q^{\mu} \to \partial_{\pm}\varrho_{\mu}, \qquad \Pi^{eff}_{+\mu\nu} \to {}^{\star}\Pi^{\mu\nu}_{+}, \quad (33)$$

where the T-dual background is

$$^{\star}\Pi^{\mu\nu}_{+}(\Delta V, 2b\Delta\tilde{V}) := \frac{\kappa}{2}(\Theta^{eff}_{-})^{\mu\nu}(\Delta V, 2b\Delta\tilde{V}). \quad (34)$$

Using (A.9), we obtain the T-dual metric ${}^{\star}G^{\mu\nu}$, which depends on the first variable ΔV^{μ} and the T-dual Kalb–Ramond field ${}^{\star}B^{\mu\nu}$, which depends on the second variable $2b^{\mu}_{\nu}\Delta\tilde{V}^{\nu}$

$$^{\star}G^{\mu\nu} = (G^{-1}_{E})^{\mu\nu}(\Delta V),$$
$$^{\star}B^{\mu\nu} = \frac{\kappa}{2}(\theta^{eff})^{\mu\nu}(2b\Delta\tilde{V}) = \frac{\kappa}{2}\Delta\theta^{\mu\nu}(2b\Delta\tilde{V}). \quad (35)$$

We see that the effective metric has transformed to its inverse and that the Kalb–Ramond field has transformed to the infinitesimal part of the non-commutativity parameter.

Comparing the actions one could conclude that the relation between the variables of the effective theory and its T-dual is simple as in (33). However, the real connection is given by the T-dual coordinate transformation law, which is obtained comparing the expressions for the gauge fields (28) and (29), and it reads

$$\partial_{\pm}q^{\mu} \cong -\kappa(\Theta^{\text{eff}}_{\pm})^{\mu\nu}(\Delta V(\varrho), 2b\Delta\tilde{V}(\varrho)) \times[\partial_{\pm}\varrho_{\nu} \pm 2\beta^{\mp}_{\nu}(V(\varrho))]. \quad (36)$$

In the zeroth order this law gives

$$q^{(0)\mu} \cong V^{(0)\mu}(\varrho), \quad (37)$$

which will be useful later on.

4 T-dualization of T-dual theory

Let us now show that the T-dualization of the T-dual theory (30) leads to the initial effective theory (13). Following the T-dualization procedure, we localize the global symmetry $\delta\varrho_{\mu} = \lambda_{\mu} = \text{const}$. We introduce the gauge fields $u_{\pm\mu}$, which transform as $\delta u_{\pm\mu} = -\partial_{\pm}\lambda_{\mu}$, substitute the ordinary derivatives $\partial_{\pm}\varrho_{\mu}$ in the T-dual action (30) by the covariant derivatives $D_{\pm}\varrho_{\mu} = \partial_{\pm}\varrho_{\mu} + u_{\pm\mu}$, substitute the coordinate ϱ_{μ} and its double $\tilde{\varrho}_{\mu}$ in the background field's argument by an invariant coordinate $\varrho^{\text{inv}}_{\mu} = \varrho_{\mu}(\xi) - \varrho_{\mu}(\xi_0) + \Delta U_{\mu}$ and its double $\tilde{\varrho}^{\text{inv}}_{\mu} = \tilde{\varrho}_{\mu}(\xi) - \tilde{\varrho}_{\mu}(\xi_0) + \Delta\widetilde{U}_{\mu}$ where $\Delta U_{\mu} = \int_P(d\xi^{+}u_{+\mu} + d\xi^{-}u_{-\mu})$, and $\Delta\widetilde{U}_{\mu} = \int_P(d\xi^{+}u_{+\mu} - d\xi^{-}u_{-\mu})$, add the Lagrange multiplier ς^{μ} term, and fix the gauge by $\varrho_{\mu}(\xi) = \varrho_{\mu}(\xi_0)$. In this way, we obtain the gauge fixed action for the T-dual action (30), which reads

$$^{\star}\mathcal{S}_{fix} = \kappa\int d^2\xi\left[\frac{\kappa}{2}(\Theta^{\text{eff}}_{-})^{\mu\nu}(\Delta V(\Delta U), 2b\Delta\tilde{V}(\Delta U)) \times u_{+\mu}u_{-\nu} + \frac{1}{2}(u_{+\mu}\partial_{-}\varsigma^{\mu} - u_{-\mu}\partial_{+}\varsigma^{\nu})\right]. \quad (38)$$

Varying it over ς^{μ} and $u_{\pm\mu}$, one obtains the following equations of motion:

$$\partial_{+}u_{-\mu} - \partial_{-}u_{+\mu} = 0 \quad (39)$$

and

$$\frac{\kappa}{2}(\Theta^{\text{eff}}_{\pm})^{\mu\nu}(\Delta V(\Delta U), 2b\Delta\tilde{V}(\Delta U))u_{\pm\nu} + \frac{1}{2}\partial_{\pm}\varsigma^{\mu}$$
$$= \mp\kappa(\Theta^{\text{eff}}_{0\pm})^{\mu\nu}\beta^{\mp}_{\nu}(V(U)), \qquad (40)$$

where the right hand side is the contribution from the variation over the background field's arguments

$$\delta_{A^\star}\mathcal{S}_{fix} = \frac{\kappa^2}{2}\int d^2\xi\Big[\varepsilon^{\alpha\beta}\partial_\rho(\theta^{\text{eff}})^{\mu\nu}(2b\delta\tilde{V}(U))^{\rho}$$
$$+\frac{1}{2\kappa}\eta^{\alpha\beta}\partial_\rho((G^{\text{eff}}_E)^{-1})^{\mu\nu}\delta V^{\rho}(U)\Big]\partial_\alpha U_\mu\partial_\beta U_\nu$$
$$= -\kappa\int d^2\xi[\delta u_{+\mu}\kappa(\Theta^{\text{eff}}_{0-})^{\mu\nu}\beta^{+}_{\nu}(V(U))$$
$$+\delta u_{-\mu}\kappa(\Theta^{\text{eff}}_{0+})^{\mu\nu}\beta^{-}_{\nu}(V(U))], \qquad (41)$$

with the beta functions $\beta^{\pm}_{\mu}$ given by (26). Multiplying Eq. (40) by the inverse of $(\Theta^{\text{eff}}_{\pm})^{\mu\nu}$, $\Pi^{\text{eff}}_{\mp\mu\nu}$, defined in (14), we obtain the gauge fields,

$$u_{\pm\mu} = -2\Pi^{\text{eff}}_{\mp\mu\nu}(\Delta V(\Delta U), 2b\Delta\tilde{V}(\Delta U))\partial_{\pm}\varsigma^{\nu}$$
$$\mp 2\beta^{\mp}_{\mu}(V(U)). \qquad (42)$$

The gauge fixed action (38) reduces to its initial theory (30) for the equation of motion for the Lagrange multiplier (39) and to the effective theory for the equation of motion for the gauge fields (42). One can verify that the solution of Eq. (39) is

$$u_{\pm\mu} = \partial_{\pm}\varrho_{\mu}, \qquad (43)$$

which implies $V^{\mu}(U) = V^{\mu}(\varrho)$, and $\tilde{V}^{\mu}(U) = \tilde{V}^{\mu}(\varrho)$ transforms the gauge fixed action (38) to the T-dual action (30). On the other hand, substituting the gauge fields (42) to the gauge fixed action (38), using the zeroth order value of the gauge fields,

$$u_{0\pm\mu} = -2\Pi^{\text{eff}}_{0\mp\mu\nu}\partial_{\pm}\varsigma^{\nu} = \pm g^{E}_{\mu\nu}\partial_{\pm}\varsigma^{\nu}, \qquad (44)$$

which implies

$$U^{(0)}_{\mu} = g^{E}_{\mu\nu}\tilde{\varsigma}^{\nu}, \qquad (45)$$

while $V^{(0)\mu}(U^{(0)}) = \varsigma^{\mu}$ and $\tilde{V}^{(0)\mu}(U^{(0)}) = \tilde{\varsigma}^{\mu}$, one obtains the effective theory (13), with $q^{\mu} = \varsigma^{\mu}$. The results show that T-dual of the T-dual is the initial theory.

Comparing the expressions for the gauge fields (43) and (42), we obtain the T-dual coordinate transformation law

$$\partial_{\pm}\varrho_{\mu} \cong -2\Pi^{\text{eff}}_{\mp\mu\nu}(\Delta V(\Delta U), 2b\Delta\tilde{V}(\Delta U))\partial_{\pm}\varsigma^{\nu}$$
$$\mp 2\beta^{\mp}_{\mu}(V(U)), \qquad (46)$$

which is the inverse of the law (36). In the zeroth order this law gives

$$\varrho^{(0)}_{\mu} \cong U^{(0)}_{\mu}(q). \qquad (47)$$

5 Open string T-dual

We started with the open string described by coordinates x^{μ}, solved the boundary conditions and obtained the effective string described by the even part of the initial coordinates q^{μ}, then we T-dualized the effective theory and obtained the T-dual string described by coordinates ϱ_{μ} (which were originally the Lagrange multipliers). Now, our goal is to find an open string theory such that its effective theory, obtained for the solution of the boundary conditions is exactly the T-dual theory (30). So, obviously the coordinates of the open T-dual string should have a lower index y_{μ}, in order for their even part to be the T-dual coordinate ϱ_{μ}, ones its boundary conditions are solved. Consequently the open T-dual background should have upper indices $\widetilde{G}^{\mu\nu}$, $\widetilde{B}^{\mu\nu}$. What are the relations between the open string background and its T-dual, and the relations between their coordinates will become evident once the open string T-dual is found, i.e. once the connection between the effective theory of the theory we search for and the T-dual theory (30) is made.

So, let us find the open string theory

$$\widetilde{S}[y] = \kappa\int_{\Sigma} d^2\xi\ \partial_{+}y_{\mu}\widetilde{\Pi}^{\mu\nu}_{+}(y)\partial_{-}y_{\nu}, \qquad (48)$$

where $\widetilde{\Pi}^{\mu\nu}_{+} = \widetilde{B}^{\mu\nu} \pm \frac{1}{2}\widetilde{G}^{\mu\nu}$ and $\widetilde{B}^{\mu\nu} = \tilde{b}^{\mu\nu} + \frac{1}{3}\widetilde{B}^{\mu\nu\rho}y_{\rho}$, such that its effective theory (13),

$$\widetilde{S}^{\text{eff}} = \kappa\int d\tau\int_{-\pi}^{\pi} d\sigma$$
$$\times\partial_{+}q_{\mu}(y)\widetilde{\Pi}^{\mu\nu}_{+\text{eff}}(q(y), 2\tilde{b}\tilde{q}(y))\partial_{-}q_{\nu}(y), \qquad (49)$$

is the T-dual theory (30). The effective background is composed of the metric $(\widetilde{G}^{\text{eff}})^{\mu\nu} = \widetilde{G}^{\mu\nu}_{E} = (\widetilde{G} - 4\widetilde{B}^2)^{\mu\nu}$ and the Kalb–Ramond field $(\widetilde{B}^{\text{eff}})^{\mu\nu} = -\frac{\kappa}{2}(\tilde{g}_E\Delta\tilde{\theta}\tilde{g}_E)^{\mu\nu}$. The effective variable is $q_{\mu}(y)$, which is the even part of the variable y_{μ} and $\tilde{q}_{\mu}(y)$ is its double. Let us first make a connection between the variables of these two theories. We suppose that

$$q_{\mu}(y) = C_{\mu\nu}(g^{-1}_{E})^{\nu\rho}\tilde{\varrho}_{\rho},$$
$$\bar{q}_{\mu}(y) = D_{\mu\nu}2(G^{-1}bg^{-1}_{E})^{\nu\rho}\varrho_{\rho}. \qquad (50)$$

Then $\partial_{\pm}q_{\mu}(y) = \pm C_{\mu\nu}(g^{-1}_{E})^{\nu\rho}\partial_{\pm}\varrho_{\rho}$, so equating the actions (49) and (30) one obtains the condition for the background fields

$$g^{-1}_{E}C^{T}\widetilde{\Pi}_{+\text{eff}}(q(y), 2\tilde{b}\tilde{q}(y))Cg^{-1}_{E}$$
$$= -\frac{\kappa}{2}\Theta^{\text{eff}}_{-}(g^{-1}_{E}\tilde{\varrho}, 2G^{-1}bg^{-1}_{E}\varrho). \qquad (51)$$

In the zeroth order this condition becomes

$$g^{-1}_{E}C^{T}\tilde{g}_{E}Cg^{-1}_{E} = -g^{-1}_{E}, \qquad (52)$$

which implies

$$\tilde{G} = -(C^T)^{-1} G C^{-1}, \quad \tilde{b} = \pm (C^T)^{-1} b C^{-1}. \tag{53}$$

Let us denote the variables of the T-dual theory by

$$z^\mu = (g_E^{-1})^{\mu\nu} \tilde{\varrho}_\nu, \quad t^\mu = 2(G^{-1} b g_E^{-1})^{\mu\nu} \varrho_\nu. \tag{54}$$

Using (53) and (50) we can note that

$$\bar{q}_\mu(y) = 2(\tilde{G}^{-1}\tilde{b})_\mu{}^\nu \tilde{q}_\nu(y) = \mp 2(CG^{-1}bC^{-1})_\mu{}^\nu \tilde{q}_\nu(y) = \mp C_{\mu\nu} t^\nu, \tag{55}$$

on the other hand from (50) it follows that $\bar{q}_\mu(y) = D_{\mu\nu} t^\nu$, so we conclude that $D = \mp C$. Therefore, the coordinate that solves the boundary conditions of the unknown theory (48) is just

$$y_\mu = q_\mu(y) + \bar{q}_\mu(y) = C_{\mu\nu}(z^\nu \mp t^\nu) = C_{\mu\nu}\big((g_E^{-1})^{\nu\rho}\tilde{\varrho}_\rho \pm \kappa \theta_0^{\nu\rho} \varrho_\rho\big). \tag{56}$$

Now, we can write the first order part of the condition (51) as

$$g_E^{-1} C^T \tilde{G}_{E1}(Cz) C g_E^{-1} = -(G_E^{-1})_1(z), \quad g_E^{-1} C^T \tilde{g}_E \Delta\tilde{\theta}(\mp Ct) \tilde{g}_E C g_E^{-1} = \Delta\theta(t). \tag{57}$$

Using the explicit values of the effective open string metric, its inverse and the non-commutativity parameter (A.2), (A.6), and (A.8), we obtain

$$\mp\big[bC^T \tilde{h}(Cz) C + C^T \tilde{h}(Cz) C b\big] = bh(z) + h(z)b, \quad \mp\big[C^T \tilde{h}(Ct) C + 4bC^T \tilde{h}(Ct) C b\big] = h(t) + 4bh(t)b, \tag{58}$$

with the following solution:

$$\tilde{h}(Ct) = \mp (C^T)^{-1} h(t) C^{-1}. \tag{59}$$

Finally, we determine the open string T-dual background. It reads

$$\widetilde{G} = -(C^T)^{-1} G C^{-1}, \quad \widetilde{B}(y) = \pm (C^T)^{-1}(b - h(C^{-1}y)) C^{-1}. \tag{60}$$

The effective theory (which gives the space-time equations of motion (2)) for the T-dual background (60) remains the same. In a constant background, the dual background would be just (53), choosing the upper solution for $\tilde{b}$ one has

$$\widetilde{G} \pm \tilde{b} = -(C^T)^{-1}(G \mp b) C^{-1}, \tag{61}$$

which choosing $C = G \mp b$ becomes

$$\widetilde{G} \pm \tilde{b} = -(G \pm b)^{-1}, \tag{62}$$

which is in agreement with the standard T-duality relation (13) of Ref. [34]. The other solution would lead to an equivalent conclusion.

Comparing the initial open string theory (5) and its T-dual (48) with the background (60), we see that they are equal under the following transformations:

$$\partial_\pm x^\mu \to \partial_\pm y_\mu, \quad G \to \widetilde{G} = -(C^T)^{-1} G C^{-1}, \quad B(x) = b + h(x) \to \widetilde{B}(y) = \pm (C^T)^{-1}(b - h(C^{-1}y)) C^{-1}. \tag{63}$$

Choosing

$$C_{\mu\nu} = \widetilde{G}^{-1}_{\mu\nu}, \tag{64}$$

which is by (60) just $C_{\mu\nu} = -G_{\mu\nu}$, the T-dual of the open string in the weakly curved background becomes

$$\widetilde{G}^{\mu\nu} = -(G^{-1})^{\mu\nu}, \quad \widetilde{B}^{\mu\nu}(y) = \tilde{b}^{\mu\nu} + \frac{1}{3}\widetilde{B}^{\mu\nu\rho} y_\rho, \tag{65}$$

with the constant part of the Kalb–Ramond field equal to

$$\tilde{b}^{\mu\nu} = \pm\big(G^{-1} b G^{-1}\big)^{\mu\nu} \tag{66}$$

and the field strength of the T-dual Kalb–Ramond field equal to

$$\widetilde{B}^{\mu\nu\rho} = \pm (G^{-1})^{\mu\sigma}(G^{-1})^{\nu\tau}(G^{-1})^{\rho\varepsilon} B_{\sigma\tau\varepsilon}, \tag{67}$$

and therefore $\widetilde{B}^{\mu\nu}(y) = \pm(G^{-1}B(G^{-1}y)G^{-1})^{\mu\nu}$. In this case, the transformation of the background fields (63) is just

$$\Pi_{\pm\mu\nu}(x) \to \widetilde{\Pi}_\pm^{\mu\nu}(y) = \begin{cases} \big(G^{-1}\big)^{\mu\rho} \Pi_{\mp\rho\sigma}(G^{-1}y)\big(G^{-1}\big)^{\sigma\nu}, \\ -\big(G^{-1}\big)^{\mu\rho} \Pi_{\pm\rho\sigma}(G^{-1}y)\big(G^{-1}\big)^{\sigma\nu}. \end{cases} \tag{68}$$

6 Conclusion

In this paper we were looking for a T-dual of an open string moving in a weakly curved background. The starting theory, S, was a subject of investigation in Refs. [23–25], where it was shown that, solving the boundary conditions at the open string endpoints, one obtains the effective closed string described by the effective closed string theory S^{eff}, defined on the doubled space $(q^\mu, \tilde{q}^\mu)$. The T-duals of such a theory can be determined using the generalized Buscher T-dualization procedure, which we developed earlier in Refs. [20–22]. In this paper we applied the T-dualization procedure to the effective theory S^{eff} along all effective directions q^μ. We obtained the T-dual theory ${}^\star S^{\text{eff}}$. Applying the procedure to the obtained theory along all dual directions ϱ_μ, we returned to the effective theory. So, we proved $S^{\text{eff}} \overset{T}{\longleftrightarrow} {}^\star S^{\text{eff}}$. Finally, we searched for the open string theory $\tilde{S}$ such that its effective theory is ${}^\star S^{\text{eff}}$ exactly. We found the explicit form of $\tilde{S}$.

The relations between the theories investigated are represented in the following diagram:

$$
\begin{array}{ccc}
S=\kappa\int_\Sigma d^2\xi\,\partial_+x^\mu\Pi_{+\mu\nu}(x)\partial_-x^\nu & \xrightarrow{\text{solving BC}} & S^{\text{eff}}=\kappa\int_{\Sigma^\star} d^2\xi\,\partial_+q^\mu\,\Pi^{\text{eff}}_{+\mu\nu}(q,2b\tilde q)\partial_-q^\nu \\
\big\downarrow T & & \big\downarrow T \\
\tilde S=\kappa\int_\Sigma d^2\xi\,\partial_+y_\mu\widetilde\Pi_+^{\mu\nu}(y)\partial_-y_\nu & \xrightarrow{\text{solving BC}} & {}^\star S^{\text{eff}}=\frac{\kappa^2}{2}\int_{\Sigma^\star} d^2\xi\,\partial_+\varrho_\mu(\Theta^{\text{eff}}_-)^{\mu\nu}(g_E^{-1}\tilde\varrho,2bg_E^{-1}\varrho)\partial_-\varrho_\nu.
\end{array}
$$

On the left hand side are the open string theories, the original theory depending on the original coordinate x^μ and its T-dual depending on the dual coordinate y_μ. On the right hand side are the effective theories of the open string theories, obtained for the solution of the boundary conditions. These theories are defined on the doubled spaces, which consist of the effective variables which are the even parts of the coordinates of the theories they originate from and their doubles. The effective theories are T-dual to each other, and their variables are connected by the T-dual coordinate transformation laws (36) and (46). The obtained T-dual coordinate transformation laws, could be used for investigation of the relations between the geometrical properties of the corresponding spaces. Let us notice that the T-dualization of a closed string theory [20–22] led to a T-dual theory with the target space which significantly differs from the initial space. While the initial theory was defined in the ordinary space the T-dual was defined in the doubled space. In the open string case investigated here, the T-dualization does not cause such a transition. Both the initial theory and its T-dual are defined on the geometrical space.

The metrics of the above theories are

$$
\begin{array}{ccc}
G_{\mu\nu}=\text{const} & \xrightarrow{\text{solving BC}} & G^{\text{eff}}_{\mu\nu}=G^E_{\mu\nu}(q) \\
\big\downarrow T & & \big\downarrow T \\
\widetilde G^{\mu\nu}=\widetilde{\text{const}} & \xrightarrow{\text{solving BC}} & \widetilde G^{\mu\nu}_{\text{eff}}=(G_E^{-1})^{\mu\nu}(g_E^{-1}\tilde\varrho),
\end{array}
$$

and the Kalb–Ramond fields are

$$
\begin{array}{ccc}
B_{\mu\nu}(x)=b_{\mu\nu}+\frac{1}{3}B_{\mu\nu\rho}x^\rho & \xrightarrow{\text{solving BC}} & B^{\text{eff}}_{\mu\nu}=-\frac{\kappa}{2}(g_E\Delta\theta(2b\tilde q)g_E)_{\mu\nu} \\
\big\downarrow T & & \big\downarrow T \\
\widetilde B^{\mu\nu}(y)=\tilde b^{\mu\nu}+\frac{1}{3}\widetilde B^{\mu\nu\rho}y_\rho & \xrightarrow{\text{solving BC}} & \widetilde B^{\mu\nu}_{\text{eff}}=\frac{\kappa}{2}\theta^{\mu\nu}_{\text{eff}}(2bg_E^{-1}\varrho),
\end{array}
$$

with $\widetilde G^{\mu\nu}$ and $\widetilde B^{\mu\nu}(y)$ given explicitly by (60).

One can notice that the relation between the initial background and its T-dual in the open string case differs from that in the closed string case, as it should be expected. In the closed string case, the T-duality transforms the constant metric of the weakly curved background to a coordinate dependent effective metric inverse, while the linearly coordinate dependent Kalb–Ramond field is transformed into a coordinate dependent non-commutativity parameter. In the open string case, the important role in the relation between the T-dual backgrounds plays a matrix C, which is introduced to define the connection between the variables of the open string theory T-dual and the effective theory T-dual. It is found that T-duality transforms the constant metric of the weakly curved background to a constant T-dual metric, while the coordinate dependent Kalb–Ramond field transforms again to the coordinate dependent field, which is in this case of the same structure as the initial field.

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Appendix A: Background fields

The background fields used in the paper will be separated into their constant and coordinate dependent values.

- The weakly curved background

  $G_{0\mu\nu}=\text{const},$

  $G_{1\mu\nu}=0,$

$$B_{0\mu\nu} = b_{\mu\nu} = \text{const},$$
$$B_{1\mu\nu}(x) = h_{\mu\nu}(x) = \frac{1}{3}B_{\mu\nu\rho}x^{\rho}, \quad B_{\mu\nu\rho} = \text{const}. \tag{A.1}$$

- Effective metric $G^{\text{eff}}_{\mu\nu}(G, B) = (G_E)_{\mu\nu}(G, B) = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}$

$$G^{\text{eff}}_{0\mu\nu} = (G_{E0})_{\mu\nu} = g_{\mu\nu} - 4b^2_{\mu\nu} = (g_E)_{\mu\nu},$$
$$G^{\text{eff}}_{1\mu\nu} = (G_{E1})_{\mu\nu} = -4(bh + hb)_{\mu\nu}. \tag{A.2}$$

- Effective Kalb–Ramond field $B^{\text{eff}}_{\mu\nu}$

$$B^{\text{eff}}_{0\mu\nu} = 0,$$
$$B^{\text{eff}}_{1\mu\nu} = -\frac{\kappa}{2}(g_E\Delta\theta g_E)_{\mu\nu} = (h + 4bhb)_{\mu\nu}. \tag{A.3}$$

Note that, because the effective Kalb–Ramond field is infinitesimal,

$$G_E(G^{\text{eff}}(x), B^{\text{eff}}(y)) = G^{\text{eff}}(x) = G_E(x). \tag{A.4}$$

- Background field combination $\Pi^{\text{eff}}_{\pm\mu\nu} = B^{\text{eff}}_{\mu\nu}(y) \pm \frac{1}{2}G^{\text{eff}}_{\mu\nu}(x)$

$$\Pi^{\text{eff}}_{0\pm\mu\nu} = \pm\frac{1}{2}(g_E)_{\mu\nu},$$
$$\Pi^{\text{eff}}_{1\pm\mu\nu} = (h(y) + 4bh(y)b)_{\mu\nu} \mp 2(bh(x) + h(x)b)_{\mu\nu}. \tag{A.5}$$

- Effective metric inverse

$$(G_E^{-1})^{\mu\nu} = (G_{E0}^{-1} - G_{E0}^{-1}G_{E1}G_{E0}^{-1})^{\mu\nu},$$

$$(G_E^{-1})_0^{\mu\nu} = (g_E^{-1})^{\mu\nu},$$
$$(G_E^{-1})_1^{\mu\nu} = 4(g_E^{-1}(bh + hb)g_E^{-1})^{\mu\nu}. \tag{A.6}$$

- Non-commutativity parameter $\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$

$$\theta_0^{\mu\nu} = -\frac{2}{\kappa}(g_E^{-1}bG^{-1})^{\mu\nu},$$
$$\theta_1^{\mu\nu} = \Delta\theta^{\mu\nu} = -\frac{2}{\kappa}[g_E^{-1}(h + 4bhb)g_E^{-1}]^{\mu\nu}. \tag{A.7}$$

- Effective non-commutativity parameter

$$\theta^{\mu\nu}_{\text{eff}} := \theta^{\mu\nu}(G_{\text{eff}}(x), B_{\text{eff}}(y))$$
$$= -\frac{2}{\kappa}(G_E^{-1}(G_{\text{eff}}(x), B_{\text{eff}}(y))B_{\text{eff}}(y)G^{-1}_{\text{eff}}(x))^{\mu\nu}$$
$$\theta^{\mu\nu}_{0\text{eff}} = 0,$$
$$\theta^{\mu\nu}_{1\text{eff}}(x, y) = \Delta\theta^{\mu\nu}(y)$$
$$= -\frac{2}{\kappa}(g_E^{-1}(h(y) + 4bh(y)b)g_E^{-1})^{\mu\nu}. \tag{A.8}$$

- Tensor $\Theta^{\mu\nu}_{\pm} = -\frac{2}{\kappa}(G_E^{-1}\Pi_{\pm}G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}$.
- Effective tensor

$$(\Theta^{\text{eff}}_{\pm})^{\mu\nu}(x, y) \equiv \Theta^{\mu\nu}_{\pm}(G_{\text{eff}}(x), B_{\text{eff}}(y))$$
$$= \theta^{\mu\nu}_{\text{eff}}(y) \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}(x),$$

$$(\Theta^{\text{eff}}_{0\pm})^{\mu\nu}(x, y) = \mp\frac{1}{\kappa}(g_E^{-1})^{\mu\nu},$$
$$(\Theta^{\text{eff}}_{1\pm})^{\mu\nu}(x, y) = \theta^{\mu\nu}_{1\text{eff}}(y) \mp \frac{1}{\kappa}(G_E^{-1})_1^{\mu\nu}(x). \tag{A.9}$$

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The T-dual symmetries of a bosonic string

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Abstract We investigate whether the symmetry transformations of a bosonic string are connected by T-duality. We start with a standard closed string theory. We continue with a modified open string theory, modified to preserve the symmetry transformations possessed by the closed string theory. Because the string theory is conformally invariant world-sheet field theory, in order to find the transformations which preserve the physics, one has to demand the isomorphism between the conformal field theories corresponding to the initial and the transformed field configurations. We find the symmetry transformations corresponding to the similarity transformation of the energy-momentum tensor, and find that their generators are T-dual. Particularly, we find that the general coordinate and local gauge transformations are T-dual, so we conclude that T-duality in addition to the well-known exchanges, transforms symmetries of the initial and its T-dual theory into each other.

1 Introduction

One of the most important notions in theoretical physics is a symmetry. What is a symmetry of the string theory is not yet clear mainly because the theory itself is not yet formulated in a background independent way, which would enlighten its deeper principles. However, it is believed that the symmetry does exist, and that it will lead to finding the physically indistinguishable solutions to the string equations of motion and choosing the correct vacuum [1,2].

String theory revealed a T-duality, a symmetry which is a consequence of the string's extended nature. T-duality connects seemingly different string theories by exchanging, for example, the characteristics of the strings momentum and winding [3–6]. So, it should exchange the symmetries of string theories as well. If that holds, one can say that symmetries of string theories always appear in pairs.

In this paper, we will investigate symmetry of the space-time in which the bosonic string moves, using a world-sheet formulation [1]. The formalism differs from the usual, where the symmetry is a transformation of the space-time fields which leave the classical action invariant. This concept of a symmetry does not apply now, because only the world-sheet values of the space-time fields appear in the string action. Still, a symmetry should be a change in the space-time fields which does not change the physics. Thus, suppose one considers a string theory with some chosen space-time field configuration, and a string theory with a modified space-time configuration. How does one determine if these two descriptions are physically equivalent? The string theory is a conformally invariant world-sheet field theory. The physics is determined by the conformal field theory, corresponding to the field configuration in question. The transformation on the fields will be a symmetry if the corresponding conformal field theories are isomorphic [1,2].

If one is given a conformal field theory, one will obtain a physically identical conformal field theory by performing a similarity transformation on the operators of the initial conformal field theory

$$\hat{O} \to e^{-i\hat{\Gamma}}\hat{O}e^{i\hat{\Gamma}}.$$

This transformation does not change the algebraic properties, so the new theory will be physically the same as the initial theory. However, the transformation will in general make changes to the world-sheet energy-momentum tensor. If these changes can be interpreted as changes in the space-time fields, then the latter are the symmetry transformations of the target space.

This idea was introduced in [1,7], where this automorphism of the operator algebra was seen as an analog of the change of variables in a partition function. The problem of finding symmetries was reduced to the problem of finding the operator generating the symmetry transformation. The first

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investigations were treating the string massless fields, but the method was then generalized to treat conformal deformations of conformal field theory [8,9].

The classical analog of a similarity transformation is a transformation of a variable of interest by a Poisson bracket between a generator and a variable. In this paper, we will apply these transformations to the string sigma model energy-momentum tensor. We will start by considering a small modification of the space-time background $\Pi_{\pm\mu\nu} \to \Pi_{\pm\mu\nu} + \delta\Pi_{\pm\mu\nu}$. It will induce the transformation of the energy-momentum tensor. We will demand that the new energy-momentum tensor satisfies the Virasoro algebra as well. This means that the transformed theory is physically equivalent to the initial theory, or that these field transformations are the symmetry of space-time theory. In this way we will find a transformation of space-time fields corresponding to a similarity transformation and the generator of this symmetry.

We will consider both closed and open string theory. For the open string we will consider a modified action which different from the closed string action has an additional surface term which enables the invariance of the complete action to the general coordinate transformations and the gauge transformations, which are the symmetries of the closed string. For the open string theory, the boundary conditions can be satisfied by choosing either the Neumann or the Dirichlet boundary condition for every coordinate direction. If the choice is made, the modified action surface term is given in terms of the corresponding Neumann and the Dirichlet gauge fields. The closed string symmetries remain the symmetries of the open string theory taking the appropriate transformation of these gauge fields.

If one includes T-duality into the consideration, one can conclude that the general coordinate transformations and the local gauge transformations are not independent. Comparing their generators, using the T-dual coordinate transformation laws, one concludes that they are T-dual. Therefore, the symmetries are T-dual and the complete generator of symmetries is self-dual.

2 The bosonic string essentials

The quantization of the bosonic string theory, describing the string moving in a background consisting of a space-time metric $G_{\mu\nu}$, a Kalb–Ramond field $B_{\mu\nu}$ and a dilaton field Φ, leads to the conclusion that in order to have a conformal invariance on the quantum level the energy-momentum tensor components $\hat{T}_\pm(\varphi)$, with $\varphi = (G_{\mu\nu}, B_{\mu\nu}, \Phi)$, have to obey the Virasoro algebras [2,10,11],

$$\left[\hat{T}_\pm(\varphi(\sigma)), \hat{T}_\pm(\varphi(\bar{\sigma}))\right] = i\hbar\left[\hat{T}_\pm(\varphi(\sigma)) + \hat{T}_\pm(\varphi(\bar{\sigma}))\right]\delta'(\sigma-\bar{\sigma}),$$
$$\left[\hat{T}_\pm(\varphi(\sigma)), \hat{T}_\mp(\varphi(\bar{\sigma}))\right] = 0. \tag{1}$$

From these conditions follow the space-time equations of motion which space-time fields $G_{\mu\nu}$, $B_{\mu\nu}$, Φ have to obey. In order to obtain the symmetries of the space-time equations of motion, one does not need to find their explicit form. It is sufficient to consider the transformations which do not change the above relations.

2.1 The conformal gauge

The action which was quantized [12] for a constant dilaton field reads

$$S[x] = \kappa \int_\Sigma d^2\xi \sqrt{-g}\Big[\frac{1}{2}g^{\alpha\beta}G_{\mu\nu}(x) + \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}}B_{\mu\nu}(x)\Big]\partial_\alpha x^\mu \partial_\beta x^\nu, \quad (\varepsilon^{01} = -1), \tag{2}$$

where the integration goes over a two-dimensional world sheet Σ with coordinates ξ^α ($\xi^0 = \tau$, $\xi^1 = \sigma$). $g_{\alpha\beta}$ is the intrinsic world-sheet metric and $x^\mu(\xi)$, $\mu = 0, 1, \ldots, D-1$ are the coordinates of the D-dimensional space-time and $\kappa = \frac{1}{2\pi\alpha'}$.

Taking a conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$, the action becomes

$$S[x] = \kappa \int_\Sigma d^2\xi\ \partial_+ x^\mu \Pi_{+\mu\nu}(x)\partial_- x^\nu, \tag{3}$$

with

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2}G_{\mu\nu}(x), \tag{4}$$

given in terms of the light-cone coordinates $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, $\partial_\pm = \partial_0 \pm \partial_1$.

The momentum corresponding to x^μ is

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \kappa G_{\mu\nu}(x)\dot{x}^\nu - 2\kappa B_{\mu\nu}(x)x'^\nu, \tag{5}$$

and therefore the canonical Hamiltonian for the theory (3) equals

$$\mathcal{H}_c = \frac{1}{4\kappa}(G^{-1})^{\mu\nu}\Big[j_{+\mu}j_{+\nu} + j_{-\mu}j_{-\nu}\Big], \tag{6}$$

where the currents $j_{\pm\mu}$ are given by

$$j_{\pm\mu} = \pi_\mu + 2\kappa\Pi_{\pm\mu\nu}(x)x'^\nu. \tag{7}$$

One can rewrite the Hamiltonian in terms of the energy-momentum tensor components

$$T_\pm = \mp\frac{1}{4\kappa}(G^{-1})^{\mu\nu}j_{\pm\mu}j_{\pm\nu}, \tag{8}$$

as

$$\mathcal{H}_C = T_- - T_+. \tag{9}$$

2.2 The gauge invariant approach

Let us in this subsection consider a string theory without a gauge fixing. If one takes the following parametrization of the world-sheet metric tensor $g_{\alpha\beta}$ [13]:

$$g_{\alpha\beta} = e^{2F}\hat{g}_{\alpha\beta} = \frac{1}{2}e^{2F}\begin{pmatrix} -2h^-h^+ & h^- + h^+ \\ h^- + h^+ & -2 \end{pmatrix}, \tag{10}$$

with $h^- > h^+$, the action (2) becomes

$$S = 2\kappa \int_\Sigma d^2\xi \sqrt{-\hat{g}}\ \hat{\partial}_+ x^\mu \Pi_{+\mu\nu}\hat{\partial}_- x^\nu, \tag{11}$$

where $\Pi_{\pm\mu\nu}$ is defined by (4) and the partial derivative is given by

$$\hat{\partial}_\pm = \frac{\sqrt{2}}{h^- - h^+}(\partial_0 + h^\mp \partial_1). \tag{12}$$

Varying the action over x^μ, one obtains the equations of motion

$$\hat{\nabla}_\pm \hat{\partial}_\mp x^\mu + \Gamma^\mu_{\mp\nu\rho}\partial_\pm x^\nu \partial_\mp x^\rho = 0, \tag{13}$$

with $\hat{\nabla}_\pm$ being the covariant derivative [13], defined by

$$\hat{\nabla}_\pm x_n = (\hat{\partial}_\pm + n\hat{\omega}_\pm)x_n, \quad \omega_\pm = \mp\frac{\sqrt{2}h^{\mp\prime}}{h^- - h^+}, \tag{14}$$

where x_n is a scalar, vector or tensor and n is the sum of its world-sheet indices, taking 1 for plus and -1 for minus. The generalized connection is defined by

$$\Gamma^\mu_{\pm\nu\rho} = \Gamma^\mu_{\nu\rho} \pm B^\mu_{\nu\rho}, \tag{15}$$

given in terms of the Christoffel symbol by

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}(G^{-1})^{\mu\sigma}(\partial_\nu G_{\rho\sigma} + \partial_\rho G_{\sigma\nu} - \partial_\sigma G_{\nu\rho}),$$

and the field strength of the field $B_{\mu\nu}$,

$$B^\mu_{\nu\rho} = (G^{-1})^{\mu\sigma}B_{\sigma\nu\rho} = (G^{-1})^{\mu\sigma}(\partial_\sigma B_{\nu\rho} + \partial_\nu B_{\rho\sigma} + \partial_\rho B_{\sigma\nu}).$$

The momentum corresponding to x^μ is

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = \frac{\kappa G_{\mu\nu}(x)}{h^- - h^+}\Big(2\dot{x}^\nu + (h^+ + h^-)x'^\nu\Big) -2\kappa B_{\mu\nu}(x)x'^\nu. \tag{16}$$

One can extract $\dot{x}^\mu$ from the last equation, to obtain

$$\dot{x}^\mu = \frac{(G^{-1})^{\mu\nu}}{2\kappa}\Big(h^-(\pi_\nu + 2\kappa\Pi_{-\nu\rho}x'^\rho) - h^+(\pi_\nu + 2\kappa\Pi_{+\nu\rho}x'^\rho)\Big). \tag{17}$$

Using the currents (7), the coordinate derivatives over world-sheet parameters become

$$\dot{x}^\mu = \frac{(G^{-1})^{\mu\nu}}{2\kappa}\Big(h^- j_{-\nu} - h^+ j_{+\nu}\Big), \tag{18}$$

$$x'^\mu = \frac{(G^{-1})^{\mu\nu}}{2\kappa}\Big(j_{+\nu} - j_{-\nu}\Big). \tag{19}$$

The canonical Hamiltonian density $\mathcal{H}_c = \pi_\mu\dot{x}^\mu - \mathcal{L}$ is

$$\mathcal{H}_c = -h^- T_+ - h^+ T_-, \tag{20}$$

with $T_\pm$ defined in (8). For $h^\pm = \mp 1$, one returns to the conformal gauge.

3 The symmetries of space-time

In this section, we will search for the symmetries of the space-time in which the closed and the open strings propagate, and for their generators. We will investigate the change of the world-sheet energy-momentum tensor caused by the change in the space-time fields. We will demand that the transformed energy-momentum tensor $T_\pm + \delta T_\pm$ still obeys the classical analog of the Virasoro algebra (1).

The energy-momentum tensor components $T_\pm$ satisfy two independent copies of the Virasoro algebra. To find a symmetry of the equations of motion, one should conclude what kind of transformation of fields $\varphi \to \varphi + \delta\varphi$, and consequently of the energy-momentum tensor

$$\hat{T}_\pm(\varphi + \delta\varphi) = \hat{T}_\pm(\varphi) + \delta\hat{T}_\pm(\varphi),$$
$$\delta\hat{T}_\pm(\varphi) = {}_G\hat{T}^{\mu\nu}_\pm \delta G_{\mu\nu} + {}_B\hat{T}^{\mu\nu}_\pm \delta B_{\mu\nu} + {}_\Phi\hat{T}_\pm \delta\Phi, \tag{21}$$

conserves the Virasoro algebra. One does not need to know the explicit form of the space-time equations of motion to find its symmetry transformations. In order to have a conserved Virasoro algebra, one should find transformations for which the following conditions are fulfilled:

$$\Big[\hat{T}_\pm(\varphi(\sigma)), \delta\hat{T}_\pm(\varphi(\bar{\sigma}))\Big] + \Big[\delta\hat{T}_\pm(\varphi(\sigma)), \hat{T}_\pm(\varphi(\bar{\sigma}))\Big] = i\hbar\Big[\delta\hat{T}_\pm(\varphi(\sigma)) + \delta\hat{T}_\pm(\varphi(\bar{\sigma}))\Big]\delta'(\sigma - \bar{\sigma}),$$
$$\Big[\delta\hat{T}_\pm(\varphi(\sigma)), \hat{T}_\mp(\varphi(\bar{\sigma}))\Big] + \Big[\hat{T}_\pm(\varphi(\sigma)), \delta\hat{T}_\mp(\varphi(\bar{\sigma}))\Big] = 0. \tag{22}$$

It is known [14] that a similarity transformation applied to $\hat{T}_\pm$,

$$\hat{T}_\pm \to e^{-i\hat{\Gamma}}\hat{T}_\pm e^{i\hat{\Gamma}},$$

ensures the physical equivalence of the corresponding theories, and it makes the change in $\hat{T}_\pm$, which corresponds to a change in the space-time fields, without changing the physics. This kind of change in the space-time fields is therefore a symmetry transformation. The similarity transformation implies that the change of $\hat{T}_\pm$ is just

$$\delta \hat{T}_\pm(\varphi) = -i\Big[\hat{\Gamma}, \hat{T}(\varphi)\Big]. \tag{23}$$

One can confirm that the last relation solves the conditions for the Virasoro algebra conservation (22).

In the subsequent sections, we will be interested in finding the change in the space-time fields, which transform $T_\pm$ in a way which preserves the classical version of the Virasoro algebra. We will search for a generator Γ_Λ (where Λ is some parameter) such that its Poisson bracket with energy-momentum components $T_\pm$ produces the variation $\delta T_\pm = \{\Gamma, T_\pm\}$, equal to the change of energy-momentum tensor caused by the variation of fields $\varphi \to \varphi + \delta\varphi$. If such a generator exists, then the previous variation is a symmetry transformation of the space-time.

3.1 T-duality of the closed string symmetry generators

Our goal in this and the subsequent sections is to find the generators of the general symmetry transformations corresponding to the similarity transformation. Thus, let us suppose the background fields undergo a small change in value $\Pi_{\pm\mu\nu} \to \Pi_{\pm\mu\nu} + \delta\Pi_{\pm\mu\nu}$. Let us find the generators of the symmetries Γ, for this transformation of the background fields. The currents change by

$$\delta j_{\pm\mu} = 2\kappa\,\delta\Pi_{\pm\mu\nu}(x)x'^\nu, \tag{24}$$

and therefore

$$\delta T_\pm = \frac{1}{2\kappa}\delta\Pi_{\pm\mu\nu}\, j^\mu_\pm j^\nu_\mp. \tag{25}$$

Let us determine the algebra of the currents (7). Using the standard Poisson brackets between the coordinates and the momenta

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu\,\delta(\sigma-\bar\sigma), \tag{26}$$

one obtains

$$\begin{aligned}\{j_{\pm\mu}(\sigma), j_{\pm\nu}(\bar\sigma)\} =\;& \\ &\pm 2\kappa\,\Gamma_{\mp\mu,\nu\rho}\, x'^\rho(\sigma)\delta(\sigma-\bar\sigma)\\ &\pm 2\kappa\, G_{\mu\nu}(x(\sigma))\delta'(\sigma-\bar\sigma),\\ \{j_{\pm\mu}(\sigma), j_{\mp\nu}(\bar\sigma)\} =\;& \\ &\pm 2\kappa\,\Gamma_{\mp\rho,\mu\nu}\, x'^\rho(\sigma)\delta(\sigma-\bar\sigma),\end{aligned} \tag{27}$$

where the generalized connection is defined by (15). Consequently, the Poisson brackets between $T_\pm$, defined by (8), and currents are

$$\begin{aligned}&\{T_\pm(\sigma), j_{\pm\mu}(\bar\sigma)\}\\ &\quad = \pm\frac{1}{2\kappa}\Gamma_{\mp\nu,\mu\rho}\, j^\nu_\pm j^\rho_\mp\,\delta(\sigma-\bar\sigma) - j_{\pm\mu}(\sigma)\delta'(\sigma-\bar\sigma),\\ &\{T_\pm(\sigma), j_{\mp\mu}(\bar\sigma)\} = \pm\frac{1}{2\kappa}\Gamma_{\mp\rho,\nu\mu}\, j^\nu_\pm j^\rho_\mp\,\delta(\sigma-\bar\sigma).\end{aligned} \tag{28}$$

Finally, we obtain the Virasoro algebra

$$\begin{aligned}&\{T_\pm(\sigma), T_\pm(\bar\sigma)\} = -\Big[T_\pm(\sigma) + T_\pm(\bar\sigma)\Big]\delta'(\sigma-\bar\sigma),\\ &\{T_\pm(\sigma), T_\mp(\bar\sigma)\} = 0,\end{aligned} \tag{29}$$

in agreement with the condition (1).

We will suppose the generator of the symmetries in the following form:

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-, \quad \mathcal{G}_\pm = \int d\sigma\, \Lambda^\mu_\pm\big(x(\sigma)\big) j_{\pm\mu}(\sigma). \tag{30}$$

Using (28), one obtains the Poisson brackets between $T_\pm$ and the generators,

$$\begin{aligned}&\{T_\pm(\sigma), \mathcal{G}_\pm(\bar\sigma)\} = \pm\frac{1}{2\kappa}\Big[\Gamma^\mu_{\mp\rho\nu}\Lambda^\rho_\pm + \partial_\nu\Lambda^\mu_\pm\Big] j_{\pm\mu} j^\nu_\mp,\\ &\{T_\pm(\sigma), \mathcal{G}_\mp(\bar\sigma)\} = \pm\frac{1}{2\kappa}\Big[\Gamma^\nu_{\mp\mu\rho}\Lambda^\rho_\mp + \partial_\mu\Lambda^\nu_\mp\Big] j^\mu_\pm j_{\mp\nu}.\end{aligned} \tag{31}$$

If one defines the generalized covariant derivatives by

$$D_{\pm\mu}\Lambda^\nu = \partial_\mu\Lambda^\nu + \Gamma^\nu_{\pm\rho\mu}\Lambda^\rho = D_\mu\Lambda^\nu \pm B^\nu{}_{\rho\mu}\Lambda^\rho, \tag{32}$$

one rewrites (31) as

$$\begin{aligned}&\{T_\pm(\sigma), \mathcal{G}_\pm(\bar\sigma)\} = \pm\frac{1}{2\kappa}\Big(D_{\mp\nu}\Lambda^\mu_\pm\Big) j_{\pm\mu} j^\nu_\mp,\\ &\{T_\pm(\sigma), \mathcal{G}_\mp(\bar\sigma)\} = \pm\frac{1}{2\kappa}\Big(D_{\pm\mu}\Lambda^\nu_\mp\Big) j^\mu_\pm j_{\mp\nu}.\end{aligned} \tag{33}$$

We know that $T_\pm$ transforms as (25), therefore we search for a generator $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ such that

$$\delta T_\pm = \{\mathcal{G}, T_\pm\} = \frac{1}{2\kappa}\delta\Pi_{\pm\mu\nu}\, j^\mu_\pm j^\nu_\mp, \tag{34}$$

which implies

$$\delta\Pi_{\pm\mu\nu} = \mp\Big(D_{\mp\nu}\Lambda_{\pm\mu} + D_{\pm\mu}\Lambda_{\mp\nu}\Big). \tag{35}$$

Taking

$$\Lambda_{\pm\mu} = \xi_\mu \pm \Lambda_\mu, \tag{36}$$

one obtains

$$\begin{aligned}&\delta G_{\mu\nu} = -2(D_\mu\xi_\nu + D_\nu\xi_\mu),\\ &\delta B_{\mu\nu} = D_\mu\Lambda_\nu - D_\nu\Lambda_\mu - 2B_{\mu\nu}{}^\rho\xi_\rho.\end{aligned} \tag{37}$$

Using the currents (7) and the gauge parameter (36), we rewrite the generator $\mathcal{G}$ as

$$\mathcal{G} = \int d\sigma \Big[2\xi^\mu \pi_\mu + 2\kappa (2\xi^\mu B_{\mu\nu} + \Lambda^\mu G_{\mu\nu}) x'^\nu \Big]. \quad (38)$$

To simplify the last expression, one can define another gauge parameter

$$\widetilde{\Lambda}_\nu = 2\xi^\mu B_{\mu\nu} + \Lambda^\mu G_{\mu\nu} = \Lambda_\nu - 2B_{\nu\mu}\xi^\mu, \quad (39)$$

so that

$$\mathcal{G} = 2\int d\sigma \Big[\xi^\mu \pi_\mu + \widetilde{\Lambda}_\mu \kappa x'^\mu \Big]. \quad (40)$$

In terms of the new parameter the Kalb–Ramond field transforms as

$$\delta B_{\mu\nu} = D_\mu \widetilde{\Lambda}_\nu - D_\nu \widetilde{\Lambda}_\mu + 2\Big[D_\mu \big(B_{\nu\rho}\xi^\rho \big) - D_\nu \big(B_{\mu\rho}\xi^\rho \big) - \xi^\rho B_{\rho\mu\nu} \Big], \quad (41)$$

and the generator (40) is rewritten as

$$\mathcal{G} = \mathcal{G}_\xi + \mathcal{G}_{\widetilde{\Lambda}}. \quad (42)$$

Therefore, the closed string described by (11) is invariant under the general coordinate transformations

$$\delta_\xi G_{\mu\nu} = -2(D_\mu \xi_\nu + D_\nu \xi_\mu),$$
$$\delta_\xi B_{\mu\nu} = -2\xi^\rho B_{\rho\mu\nu} + 2(\partial_\mu b_\nu - \partial_\nu b_\mu), \quad b_\mu = B_{\mu\nu}\xi^\nu, \quad (43)$$

with $D_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma^\rho_{\mu\nu}\xi_\rho$, and the local gauge transformations

$$\delta_\Lambda G_{\mu\nu} = 0,$$
$$\delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad (44)$$

where we omit the tilde on Λ_μ.

If one keeps in mind T-duality, the present form of the generator offers interesting conclusions. T-duality connects physically equivalent string sigma models, and the connection between T-dual string backgrounds and their variables is simplest in the constant background case. In that case, the well-known T-duality relation [15,16]

$$\pi_\mu \cong \kappa x'^\mu$$

stands. So, T-duality interchanges the sigma derivative of the coordinates with the momenta. The consequence of the above relation for the generator of symmetry (42) is that its constituents turn out to be T-dual as well,

$$\mathcal{G}_\xi \cong \mathcal{G}_{\widetilde{\Lambda}},$$

which makes the complete generator $\mathcal{G}$ self-dual. Because of this, the local gauge transformations and the general coordinate transformations are T-dual also. The same relation, however, does not hold in more complicated backgrounds, when T-duality is performed along the nonisometry directions or in backgrounds without the global shift symmetry. These backgrounds were discussed in [17–19] where the generalized T-dualization procedure, applicable along an arbitrary space-time direction was presented and elaborated.

4 The open string and its symmetries

The open string described by the same action as the closed string is not invariant to the above symmetries. The change in the action caused by the general coordinate transformations is

$$\delta_\xi S = 2\sqrt{2}\kappa \int d\tau \xi^\nu \Big[-h^- \Pi_{+\nu\mu}\hat{\partial}_- x^\mu + h^+ \Pi_{-\nu\mu}\hat{\partial}_+ x^\mu \Big] \Big|_{\sigma=0}^{\sigma=\pi}, \quad (45)$$

for the equation of motion (13) and the change by the local gauge transformations is

$$\delta_\Lambda S = 2\kappa \int d\tau \Lambda_\mu \dot{x}^\mu \Big|_{\sigma=0}^{\sigma=\pi}. \quad (46)$$

The first expression can be rewritten as

$$\delta S = -2\int d\tau \xi^\mu \gamma^{(0)}_\mu \Big|_{\sigma=0}^{\sigma=\pi},$$

where

$$\gamma^{(0)}_\mu = -\sqrt{2}\kappa \Big[-h^- \Pi_{+\nu\mu}\hat{\partial}_- x^\mu + h^+ \Pi_{-\nu\mu}\hat{\partial}_+ x^\mu \Big] = \frac{\kappa G_{\mu\nu}(x)}{h^- - h^+} \Big[(h^- + h^+)\dot{x}^\nu + 2h^- h^+ x'^\nu \Big] + 2\kappa B_{\mu\nu}(x)\dot{x}^\nu. \quad (47)$$

The boundary conditions of the open string are given in terms of this variable,

$$\gamma^{(0)}_\mu \delta x^\mu \Big|_{\sigma=0}^{\sigma=\pi} = 0. \quad (48)$$

In Ref. [20] the way to gain invariance to the transformation (44) was shown, and in Refs. [21,22] the open string action invariant under both (43) and (44) was presented, which different from the standard action has an additional surface term,

$$S_{bon} = 2\int d\tau \Big[\kappa A_\mu(x)\dot{x}^\mu - \bar{A}_\mu(x)(G^{-1})^{\mu\nu}\gamma^{(0)}_\nu \Big] \Big|_{\sigma=0}^{\sigma=\pi}. \quad (49)$$

This term makes the open string theory invariant under both general coordinate and local gauge transformations, if the introduced vector fields A_μ and $\bar{A}_\mu$ transform as

$$\delta_\Lambda A_\mu = -\Lambda_\mu,$$
$$\delta_\xi \bar{A}_\mu = -\xi_\mu. \tag{50}$$

For each of the coordinates one can fulfill the boundary conditions (48), by choosing either the Neumann or the Dirichlet boundary condition. If we mark the coordinates with the Neumann condition by x^a, $a = 0, 1, \ldots, p$ and the coordinates with the Dirichlet condition by x^i, $i = p+1, \ldots, D-1$, the surface term (49) reduces to

$$S_{\text{bon}} = 2\int d\tau \Big[\kappa A_a^N(x)\dot{x}^a - A_i^D(x)(G^{-1})^{ij}\gamma_j^{(0)}\Big]\Big|_{\sigma=0}^{\sigma=\pi}, \tag{51}$$

where A_a^N and A_i^D are $(p+1)$- and $(D-p-1)$-dimensional vector gauge fields, the former living on the Dp-brane and the latter orthogonal to the Dp-brane. The Neumann vector field is as usual coupled to the coordinate time parameter derivative and the Dirichlet vector field is coupled to the variable $\gamma_\mu^{(0)}$ related to the boundary condition, depending on both world-sheet parameter derivatives of the coordinates.

4.1 Field strengths

It is well known that, in the bosonic string action, the surface term can be rewritten in the form of the Kalb–Ramond term. If all the boundary conditions are Neumann, then the action on the boundary,

$$S_{\text{bon}} = 2\kappa \int d\tau A_\mu^N(x)\dot{x}^\mu \Big|_{\sigma=0}^{\sigma=\pi}, \tag{52}$$

can be rewritten as

$$S_{\text{bon}} = \kappa \int d^2\xi\, \mathcal{F}_{\mu\nu}^N \varepsilon^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu, \tag{53}$$

with

$$\mathcal{F}_{\mu\nu}^N = \partial_\mu A_\nu^N(x) - \partial_\nu A_\mu^N(x). \tag{54}$$

For arbitrary choice of the boundary conditions the action on the boundary is given by (51). Let us restrict our investigation to the following metric:

$$G_{\mu\nu} = \begin{bmatrix} G_{ab} & 0 \\ 0 & G_{ij} \end{bmatrix}. \tag{55}$$

In that case, (51) can be rewritten using (48) as

$$S_{\text{bon}} = 2\kappa \int d\tau \Big[\mathcal{A}_\mu^{(0)}(x)\dot{x}^\mu + \mathcal{A}_\mu^{(1)} x'^\mu\Big]\Big|_{\sigma=0}^{\sigma=\pi}, \tag{56}$$

with

$$\mathcal{A}_a^{(0)} = A_a^N,$$
$$\mathcal{A}_i^{(0)} = -\frac{h^- + h^+}{h^- - h^+} A_i^D + 2(BG^{-1})_i{}^j A_j^D,$$
$$\mathcal{A}_a^{(1)} = 0,$$
$$\mathcal{A}_i^{(1)} = -2\frac{h^- h^+}{h^- - h^+} A_i^D. \tag{57}$$

In terms of the field strengths, the surface term becomes

$$S_{\text{bon}} = 2\kappa \int_\Sigma d^2\xi \sqrt{-\hat{g}}\, \hat{\partial}_+ x^\mu \mathcal{F}_{\mu\nu} \hat{\partial}_- x^\nu, \tag{58}$$
$$\mathcal{F}_{\mu\nu} = \mathcal{F}_{\mu\nu}^{(a)} + \frac{1}{2}\mathcal{F}_{\mu\nu}^{(s)},$$

with

$$\mathcal{F}_{ab}^{(a)} = \partial_a A_b^N - \partial_b A_a^N,$$
$$\mathcal{F}_{ij}^{(a)} = -\frac{h^- + h^+}{h^- - h^+}(\partial_i A_j^D - \partial_j A_i^D) + 2\partial_i((BG^{-1})_j{}^k A_k^D)$$
$$-2\partial_j((BG^{-1})_i{}^l A_l^D),$$
$$\mathcal{F}_{ab}^{(s)} = 0,$$
$$\mathcal{F}_{ij}^{(s)} = -\frac{2}{h^-}\partial_i \mathcal{A}_j^{(1)} + \frac{2}{h^+}\partial_j \mathcal{A}_i^{(1)}. \tag{59}$$

The above calculations are done for an open string moving in constant background fields. If a coordinate dependent background is assumed, then the field strengths $\mathcal{F}_{ij}^{(s)}$ and $\mathcal{F}_{ij}^{(a)}$ will have additional terms, coming from $\frac{h^+ - h^-}{h^+ h^-}\mathcal{A}_i^{(1)}\Gamma^i_{\mp\nu\rho}$.

Comparing the boundary actions (58) with the action (11), we conclude that the addition of the surface term has changed the background fields by

$$G_{\mu\nu} \to G_{\mu\nu} + \mathcal{F}_{\mu\nu}^{(s)} \equiv \mathcal{G}_{\mu\nu},$$
$$B_{\mu\nu} \to B_{\mu\nu} + \mathcal{F}_{\mu\nu}^{(a)} \equiv \mathcal{B}_{\mu\nu}. \tag{60}$$

4.2 The symmetry generators for the open string

The metric and the Kalb–Ramond field are changed in comparison with the closed string case to $G_{\mu\nu} \to \mathcal{G}_{\mu\nu} = G_{\mu\nu} + \mathcal{F}_{\mu\nu}^{(s)}$ and $B_{\mu\nu} \to \mathcal{B}_{\mu\nu} = B_{\mu\nu} + \mathcal{F}_{\mu\nu}^{(a)}$. Instead of the transformations (37), the open string symmetry transformations, for a theory with mixed boundary conditions, are

$$\delta\mathcal{G}_{\mu\nu} = -2(\mathcal{D}_\mu \xi_\nu + \mathcal{D}_\nu \xi_\mu),$$
$$\delta\mathcal{B}_{\mu\nu} = \mathcal{D}_\mu \Lambda_\nu - \mathcal{D}_\nu \Lambda_\mu - 2\mathcal{B}_{\mu\nu}{}^\rho \xi_\rho, \tag{61}$$

where

$$\mathcal{D}_\mu \Lambda^\nu = \partial_\mu \Lambda^\nu + \Gamma^\nu_{\rho\mu}(\mathcal{G})\Lambda^\rho$$

is a covariant derivative corresponding to the metric $\mathcal{G}_{\mu\nu}$. By $\mathcal{B}_\mu{}^\rho{}_\nu$ we denoted the field strength of the Kalb–Ramond field. The field itself is changed; however, the field strength of the additional part is zero and therefore $\mathcal{B}_{\mu\nu\rho} = B_{\mu\nu\rho}$.

The generator of the transformation is

$$\mathcal{G} = \int d\sigma \Big[2\xi^\mu \pi_\mu + 2\kappa (2\xi^\mu \mathcal{B}_{\mu\nu} + \Lambda^\mu \mathcal{G}_{\mu\nu}) x'^\nu \Big], \tag{62}$$

with

$$\pi_\mu = \kappa \mathcal{G}_{\mu\nu}(x)\dot{x}^\nu - 2\kappa \mathcal{B}_{\mu\nu}(x) x'^\nu. \tag{63}$$

Using the explicit form of the transformed metric, we obtain the connection

$$\Gamma_{\mu,\nu\rho}(\mathcal{G}) = \Gamma_{\mu,\nu\rho} - 2\partial_\nu \partial_\rho A^D_\mu. \tag{64}$$

If one choses only Neumann boundary conditions, the metric remains the same as $\mathcal{F}^{(s)}_{\mu\nu} = 0$, so that only the Kalb–Ramond field changes, but not its field strength.

5 Conclusion

We considered the general coordinate and the local gauge transformations of the bosonic string, and we showed that they are T-dual to each other. We started with the bosonic string theory in a conformal gauge and in a gauge invariant form. One of the purposes of the latter was to find the separation of the Hamiltonian into two energy-momentum tensors satisfying two copies of the Virasoro algebras. These tensors are represented as products of currents, which are used for defining the generators of symmetries that were investigated. The generators were defined as the integrals over the spatial world-sheet parameter of the weighted currents.

Given the form of the generator, we investigated how it affects the variables, whose transformation is defined by a Poisson bracket between the generator of symmetry and the corresponding variable. We were interested in finding the explicit form of the change in the energy-momentum tensor caused by such a transformation. This form of transformation is used as a classical analog of the quantum transformation, known to preserve the Virasoro algebra between the energy-momentum tensor components. So, both the initial and the transformed energy-momentum tensor describe the same physics. Because of that, we were interested in finding the small variations of background fields such that the change in energy-momentum tensor they cause is exactly the considered transformation. In fact, if there exists a generator such that the described equality is possible, then the obtained transformations of space-time fields represent the symmetry of the theory.

We considered the standard closed string theory, and a modified open string theory which in comparison to the standard open string theory has an additional surface term. This term was chosen in such a way as to cancel the obstacle for the closed string symmetries to be the open string symmetries as well. Introduction of the Neumann vector fields on the boundary is equivalent to a change of Kalb–Ramond field. In the case considered here, however, the vector fields are not coupled only to the time derivative of the coordinates but also to the functions which define the boundary conditions on the string endpoints. Therefore, both metric and Kalb–Ramond field are changed by the surface term.

We found the transformations of the background fields, which transform the energy-momentum tensor in such a way that the Virasoro algebra is preserved. It turned out that the generators of the symmetries (40) can be separated in such a way that one part of the generator, linear in x'^μ, represents the generator of the well-known local gauge transformation of the Kalb–Ramond field and the other part, proportional to π_μ, generates the general coordinate transformations. Since these quantities are T-dual $\kappa x'^\mu \cong \pi_\mu$, we conclude that these symmetries are T-dual. Thus, we showed that T-duality has an additional feature: it interchanges the symmetries of a theory. The generator of general coordinate transformations and local gauge transformations of the initial theory is T-dual to the generator of local gauge transformations and the general coordinate transformations of the T-dual theory.

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Effective theories of two T-dual theories are also T-dual

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Abstract We investigate how T-duality and solving the boundary conditions of the open bosonic string are related. We start by considering the T-dualization of the open string moving in the constant background. We take that the coordinates of the initial theory satisfy either Neumann or Dirichlet boundary conditions. It follows that the coordinates of T-dual theory satisfy exactly the opposite set of boundary conditions. We treat the boundary conditions of both theories as constraints, and apply the Dirac procedure to them, which results in forming σ-dependent constraints. We solve these constraints and obtain the effective theories for the solution. We show that the effective closed string theories are also T-dual.

1 Introduction

T-duality [1–3], first observed in string theory, interchanges the string momenta and winding numbers, leaving the spectrum unchanged. Its description on the string sigma model level was first given by Buscher [4,5]. The Buscher procedure [6] covers the T-dualization of the coordinates on which the background fields do not depend. The generalized Buscher procedure, applicable to the arbitrary coordinate of the coordinate dependent background was proposed in [7]. The T-dual theory obtained by this prescription is nongeometric, described in terms of the dual coordinates and their double. The double field theories are investigated in [8,9].

The nongeometricity appears naturally when considering the open bosonic string moving in a weakly curved background with all coordinates satisfying the Neumann boundary conditions. The problem of solving these boundary conditions was considered in [10–12]. In the first two papers the conditions were treated as constraints in a Dirac procedure. In the third, the solution of boundary conditions was presupposed in a form expressing the odd coordinate and momenta parts in terms of their even parts. Both treatments lead to effective theories, obtained for the solution of boundary condition, defined in nongeometric space given in terms of even parts of coordinates and of their doubles.

In this paper we consider the open string moving in the constant background fields: metric $G_{\mu\nu}$ and antisymmetric Kalb–Ramond field $B_{\mu\nu}$. It is well known that the constant Kalb–Ramond field does not affect the dynamics in the world-sheet interior but it contributes to its boundary and causes the noncommutativity of the string coordinates. Also, we consider the T-dual theory, obtained applying the T-dualization procedure to the above theory. The T-dual theory has a standard action describing the T-dual string moving in the background with a T-dual metric ${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}$, which is an inverse of the effective metric and a dual Kalb–Ramond field ${}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}$ which is the noncommutativity parameter, in Seiberg–Witten terminology of the open bosonic string theory [13].

We consider the mixed boundary conditions, for both initial and T-dual coordinates and solve them using techniques developed in [10–12]. We chose the Neumann boundary conditions for coordinate directions x^a and Dirichlet conditions for the rest of the coordinates x^i of the initial theory. As usual, using the T-dual coordinate transformation laws one shows that the chosen boundary conditions of the initial theory transform to the boundary conditions of the T-dual theory, so that y_a satisfy the Dirichlet and y_i the Neumann boundary conditions.

We treat all boundary conditions as constraints and follow the Dirac procedure. The new constraints are found, first as a Poisson bracket between the hamiltonian and the boundary conditions, and every subsequent as a Poisson bracket between the hamiltonian and the previous constraint. Using the Taylor expansion, we represent this infinite set of constraints we obtain, by only two σ-dependent constraints [14–

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17], one for each endpoint. Imposing 2π-periodicity, to the variables building the constraints, one observes that the constraints at $\sigma = \pi$ can be expressed in terms of that at $\sigma = 0$, and that in fact solving one pair of constraints one solves the other pair as well.

We can separate the constraints into even and odd parts under world-sheet parity transformation ($\Omega : \sigma \to -\sigma$), separating the variables building the constraints into even and odd parts. Solving the σ-dependent constraints, one reduces the phase space by half. Halves of the original canonical variables are treated as effective variables: the independent variables and their canonical conjugates. For the solution of the constraints we obtain the effective theories, defined in terms of the effective variables. We examine their characteristics and confirm that the effective theories of two T-dual theories are also T-dual.

The paper is organized as follows: In Sect. 2 we consider the standard open bosonic string action and we choose the boundary conditions for every coordinate. Then, we find the T-dual theory, and show that T-dual coordinates satisfy exactly the opposite boundary condition for a given direction of the T-dual space-time, than for the corresponding direction of the original space-time. In Sect. 3, we rewrite the boundary conditions in the canonical form and find the new constraints following the Dirac procedure. We gather the constraints into σ-dependent constraints, separate the canonical variables into their even and odd parts, and solve the constraints. In Sect. 4 we find the noncommutativity relations for coordinates and momenta of both initial and T-dual theories. In Sect. 5 we calculate the effective theories, which will be obtained from the initial theories for the solution of the constraints. We show that the effective theories of the initial and T-dual theory remain T-dual, and find the effective T-duality coordinate transformation laws.

2 The open bosonic string and its T-dual

The bosonic string sigma model, describes the bosonic string moving in a curved background associated with the massless bosonic fields: a metric field $G_{\mu\nu}$, a Kalb–Ramond field $B_{\mu\nu}$ and a dilaton field Φ. The dynamics is described by the action [18–20]

$$S[x] = \kappa \int_\Sigma d^2\xi \sqrt{-g}\left[\left(\frac{1}{2}g^{\alpha\beta}G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}}B_{\mu\nu}(x)\right)\partial_\alpha x^\mu \partial_\beta x^\nu + \Phi(x)R^{(2)}\right]. \quad (1)$$

The integration goes over two-dimensional world-sheet Σ parametrized by ξ^α ($\xi^0 = \tau,\ \xi^1 = \sigma$), $g_{\alpha\beta}$ is the intrinsic world-sheet metric, $R^{(2)}$ corresponding 2-dimensional scalar curvature, $x^\mu(\xi),\ \mu = 0, 1, ..., D-1$ are the coordinates of the D-dimensional space-time, $\kappa = \frac{1}{2\pi\alpha'}$ with α' being the Regge slope parameter and $\varepsilon^{01} = -1$. The space-time fields in which the string moves have to obey the space-time equations of motion, in order to have a conformal invariance on the quantum level. If the dilaton field is taken to be zero, and the conformal gauge is considered $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$, the action can be rewritten as

$$S = \kappa \int d\xi^2 \partial_+ x^\mu \Pi_{+\mu\nu}\partial_- x^\nu, \quad (2)$$

with the background field composition

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2}G_{\mu\nu}(x), \quad (3)$$

and the light-cone coordinates given by

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma), \quad \partial_\pm = \partial_\tau \pm \partial_\sigma. \quad (4)$$

From the minimal action principle one obtains the equations of motion and the boundary conditions

$$\gamma^{(0)}_\mu \delta x^\mu \Big|_0^\pi = 0, \quad (5)$$

where

$$\gamma^{(0)}_\mu = \kappa\left(\Pi_{+\mu\nu}\partial_- x^\nu + \Pi_{-\mu\nu}\partial_+ x^\nu\right). \quad (6)$$

For the closed string the boundary conditions are fulfilled because of the periodicity of its coordinates. In the open string case, for each of the space-time coordinates one can fulfill the boundary conditions (5) by choosing either the Neumann or the Dirichlet boundary condition. Let us choose the Neumann condition for coordinates $x^a,\ a = 0, 1, \ldots, p$ and the Dirichlet condition for coordinates $x^i,\ i = p+1, \ldots, D-1$, which read

$$\begin{aligned} &\mathit{Neumann}: \quad \gamma^{(0)}_a\Big|_{\partial\Sigma} = 0, \\ &{}_N\gamma^0_a \equiv \gamma^{(0)}_a = \kappa\left(\Pi_{+ab}\partial_- x^b + \Pi_{-ab}\partial_+ x^b\right), \\ &\mathit{Dirichlet}: \quad \kappa\dot{x}^i\Big|_{\partial\Sigma} = 0, \quad {}_D\gamma^i_0 \equiv \kappa\dot{x}^i. \end{aligned} \quad (7)$$

We consider the block diagonal constant metric and Kalb–Ramond field $G_{\mu\nu} = const$, $B_{\mu\nu} = const$

$$G_{\mu\nu} = \begin{pmatrix} G_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} B_{ab} & 0 \\ 0 & B_{ij} \end{pmatrix}. \quad (8)$$

2.1 Open string theory T-dual

Let us find a T-dual of the open string theory described by the action (2). In order to find the T-dual action, one substitutes the ordinary derivatives with the covariant derivatives $D_\pm x^\mu = \partial_\pm x^\mu + v^\mu_\pm$, defined in terms of the gauge fields $v^\mu_\pm$. One adds the Lagrange multiplier term to make the introduced gauge fields nonphysical. The gauge is fixed taking $x^\mu(\xi) = 0$. Next, one finds the equations of motion varying the obtained gauge fixed action over the gauge fields $v^\mu_\pm$. The

T-dual action is obtained by substituting the expressions for the gauge fields obtained from these equations of motion, into the gauge fixed action. The T-dual action reads [10]

$$^\star S = \frac{\kappa^2}{2}\int d\xi^2 \partial_+ y_\mu \Theta_-^{\mu\nu}\partial_- y_\nu, \tag{9}$$

The dual background field composition equals

$$^\star\Pi_\pm^{\mu\nu} = \frac{\kappa}{2}\Theta_\mp^{\mu\nu} = -\left(G_E^{-1}\Pi_\mp G^{-1}\right)^{\mu\nu},$$
$$(G_E)_{\mu\nu} = (G - 4BG^{-1}B)_{\mu\nu}, \tag{10}$$

where G_E is the effective metric. The T-dual metric is its inverse

$$^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \tag{11}$$

and a T-dual Kalb–Ramond field is

$$^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}, \tag{12}$$

where $\theta^{\mu\nu} = -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$ is the noncommutativity parameter.

Because of the choice (8), the composition of the T-dual background fields is also block diagonal

$$\Theta_\pm^{\mu\nu} = \begin{pmatrix} \Theta_\pm^{ab} & 0 \\ 0 & \Theta_\pm^{ij} \end{pmatrix}, \tag{13}$$

given in terms of the inverse of the initial metric and the effective metric

$$(G^{-1})^{\mu\nu} = \begin{pmatrix} (G^{-1})^{ab} & 0 \\ 0 & (G^{-1})^{ij} \end{pmatrix},$$
$$(G_E^{-1})^{\mu\nu} = \begin{pmatrix} (G_E^{-1})^{ab} & 0 \\ 0 & (G_E^{-1})^{ij} \end{pmatrix}, \tag{14}$$

by

$$\Theta_\pm^{ab} = -\frac{2}{\kappa}\left(G_E^{-1}\right)^{ac}\Pi_{\pm cd}(G^{-1})^{db} = \theta^{ab} \mp \frac{1}{\kappa}\left(G_E^{-1}\right)^{ab},$$
$$\Theta_\pm^{ij} = -\frac{2}{\kappa}\left(G_E^{-1}\right)^{ik}\Pi_{\pm kl}(G^{-1})^{lj} = \theta^{ij} \mp \frac{1}{\kappa}\left(G_E^{-1}\right)^{ij}, \tag{15}$$

where $(G_E)_{ab} = G_{ab} - 4B_{ac}(G^{-1})^{cd}B_{db}$ and $(G_E)_{ij} = G_{ij} - 4B_{ik}(G^{-1})^{kl}B_{lj}$. The components of the non-commutativity parameter are

$$\theta^{ab} = -\frac{2}{\kappa}\left(G_E^{-1}\right)^{ac} B_{cd}(G^{-1})^{db},$$
$$\theta^{ij} = -\frac{2}{\kappa}\left(G_E^{-1}\right)^{ik} B_{kl}(G^{-1})^{lj}. \tag{16}$$

The coordinates of the initial and the T-dual theory are connected by T-duality coordinate transformation laws, which read

$$\partial_\pm x^\mu \cong -\kappa\Theta_\pm^{\mu\nu}\partial_\pm y_\nu,$$
$$\partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu. \tag{17}$$

The T-dual boundary conditions are

$$^\star\gamma^{(0)\mu}\,\delta y_\mu\Big|_0^\pi = 0, \tag{18}$$

where

$$^\star\gamma^{(0)\mu} = \frac{\kappa^2}{2}\Big[\Theta_-^{\mu\nu}\partial_- y_\nu + \Theta_+^{\mu\nu}\partial_+ y_\nu\Big]. \tag{19}$$

The T-dual theory (9) is equivalent to an open string theory (2) with chosen boundary conditions (7), if the T-dual boundary conditions are fulfilled in a Neumann way for coordinates y_i and in a Dirichlet way for y_a

$$Neumann\!: \quad ^\star\gamma^{(0)i}\Big|_{\partial\Sigma} = 0,$$
$$^\star_N\gamma_0^i \equiv {}^\star\gamma^{(0)i} = \frac{\kappa^2}{2}\Big[\Theta_-^{ij}\partial_- y_j + \Theta_+^{ij}\partial_+ y_j\Big],$$
$$Dirichlet\!: \quad \kappa\dot{y}_a\Big|_{\partial\Sigma} = 0, \qquad ^\star_D\gamma_a^0 \equiv \kappa\dot{y}_a. \tag{20}$$

This is because of the T-duality transformation law (17), which gives

$$-\kappa\dot{x}^\mu \cong {}^\star\gamma^{(0)\mu}(y), \quad \gamma_\mu^{(0)}(x) \cong -\kappa\dot{y}_\mu, \tag{21}$$

and consequently

$$_D\gamma_0^i \equiv \kappa\dot{x}^i \cong -{}^\star\gamma^{(0)i} \equiv -{}^\star_N\gamma_0^i,$$
$$_N\gamma_a^0 \equiv \gamma_a^{(0)} \cong -\kappa\dot{y}_a = -{}^\star_D\gamma_a^0. \tag{22}$$

So, performing T-dualization one changes the type of the boundary conditions which the coordinates in i and a directions satisfy.

3 Dirac consistency procedure applied to the boundary conditions

The coordinates of the initial and T-dual open string satisfy the appropriate set of the boundary conditions (7) and (20), obtained from the actions (2) and (9). In this section, we are going to treat them as constraints and we will apply the Dirac consistency procedure. In order to implement the procedure, let us find the canonical form of the boundary conditions, and express them in terms of the currents building the energy-momentum tensors, and consequently the hamiltonians.

The momenta conjugated to the coordinates of the initial and T-dual theories (2) and (9) are

$$\pi_\mu = -2\kappa B_{\mu\nu}x'^\nu + \kappa G_{\mu\nu}\dot{x}^\nu,$$
$$^\star\pi^\mu = -\kappa^2\theta^{\mu\nu}y'_\nu + \kappa\left(G_E^{-1}\right)^{\mu\nu}\dot{y}_\nu$$
$$= -2\kappa\,{}^\star B^{\mu\nu}y'_\nu + \kappa\,{}^\star G^{\mu\nu}\dot{y}_\nu. \tag{23}$$

The energy-momentum tensor components for the initial theory can be expressed in terms of currents

$$j_{\pm\mu} = \pi_\mu + 2\kappa\Pi_{\pm\mu\nu}x'^\nu, \tag{24}$$

as

$$T_\pm = \mp \frac{1}{4\kappa}(G^{-1})^{\mu\nu} j_{\pm\mu} j_{\pm\nu}. \tag{25}$$

Using the first relation in (23), the currents can be rewritten in terms of coordinates as

$$j_{\pm\mu} = \kappa G_{\mu\nu}\partial_\pm x^\nu. \tag{26}$$

The canonical hamiltonian density is

$$\mathcal{H}_c = T_- - T_+ = \frac{1}{4\kappa}(G^{-1})^{\mu\nu}\Big[j_{+\mu}j_{+\nu} + j_{-\mu}j_{-\nu}\Big]. \tag{27}$$

The hamiltonian density and the energy-momentum tensor of the T-dual theory

$$\begin{aligned} {}^\star T_\pm &= \mp \frac{1}{4\kappa}({}^\star G^{-1})^{\mu\nu}\, {}^\star j_{\pm\mu}\, {}^\star j_{\pm\nu}, \\ {}^\star\mathcal{H}_c &= {}^\star T_- - {}^\star T_+ \\ &= \frac{1}{4\kappa}({}^\star G^{-1})^{\mu\nu}\Big[{}^\star j_{+\mu}\,{}^\star j_{+\nu} + {}^\star j_{-\mu}\,{}^\star j_{-\nu}\Big] \end{aligned} \tag{28}$$

are expressed in terms of the dual currents given by

$${}^\star j_\pm^\mu = {}^\star\pi^\mu + 2\kappa\, {}^\star\Pi_\pm^{\mu\nu} y'_\nu, \tag{29}$$

where ${}^\star\Pi_\pm^{\mu\nu}$ is defined in (10). Using the second relation in (23) one obtains

$${}^\star j_\pm^\mu = \kappa\left(G_E^{-1}\right)^{\mu\nu}\partial_\pm y_\nu. \tag{30}$$

3.1 The Dirac procedure applied to the initial theory

Let us treat the Neumann and Dirichlet boundary conditions (7) of the initial theory as canonical constraints and apply the Dirac consistency procedure to them, following [10,11]. The simplest way to obtain the explicit form of these constraints is using the currents defined in (24). Because the hamiltonian is already expressed in terms of these currents, all that remains is to find their algebra.

The algebra of currents [21] in a constant background is given by

$$\begin{aligned} \big\{j_{\pm\mu}(\sigma), j_{\pm\nu}(\bar\sigma)\big\} &= \pm 2\kappa\, G_{\mu\nu}\,\delta'(\sigma-\bar\sigma), \\ \big\{j_{\pm\mu}(\sigma), j_{\mp\nu}(\bar\sigma)\big\} &= 0. \end{aligned} \tag{31}$$

Using the expressions for momenta (23), one can rewrite the Neumann (N) and Dirichlet (D) boundary conditions (7) in a canonical form

$$\begin{aligned} {}_N\gamma_a^0 &= \Pi_{+ab}(G^{-1})^{bc} j_{-c} + \Pi_{-ab}(G^{-1})^{bc} j_{+c}, \\ {}_D\gamma_0^i &= \kappa\dot x^i = \frac{1}{2}(G^{-1})^{ij}(j_{+j} + j_{-j}). \end{aligned} \tag{32}$$

Following the Dirac procedure, one can impose consistency to these constraints. The additional constraints are defined for every $n \geq 1$ by

$${}_N\gamma_a^n = \Big\{H_c, {}_N\gamma_a^{n-1}\Big\}, \quad {}_D\gamma_n^i = \Big\{H_c, {}_D\gamma_{n-1}^i\Big\}, \tag{33}$$

with $H_c = \int d\sigma\mathcal{H}_c$ being the canonical hamiltonian.

All these constraints can be gathered into only two constraints, which depend on the space parameter of the world-sheet. We will multiply every constraint by an appropriate degree of the world-sheet space parameter σ and add the terms together, forming two sigma dependent constraints

$$\Gamma_a^N(\sigma) = \sum_{n\geq 0}\frac{\sigma^n}{n!}\,{}_N\gamma_a^n\Big|_{\sigma=0}, \quad \Gamma_D^i(\sigma) = \sum_{n\geq 0}\frac{\sigma^n}{n!}\,{}_D\gamma_n^i\Big|_{\sigma=0}. \tag{34}$$

One could in principle consider another reparametrization of the world-sheet given by $\eta = f(\sigma)$ so that the parameters of string endpoints remain 0 and π i.e. one demands $f(0) = 0$ and $f(\pi) = \pi$, and on the interval $(0, \pi)$ f should be an increasing function $f' > 0$. Then, one would consider a Taylor expansion

$$\Gamma_a^N(\eta) = \sum_{n\geq 0}\frac{\eta^n}{n!}\,{}_N\gamma_a^n\Big|_{\eta=0}, \quad \Gamma_D^i(\eta) = \sum_{n\geq 0}\frac{\eta^n}{n!}\,{}_D\gamma_n^i\Big|_{\eta=0}.$$

Solving these constraints would be exactly the same, only the argument of variables would change to $f(\sigma)$, as if one has performed the reparametrization of all effective variables $q^\mu, \bar q^\mu, p_\mu, \bar p_\mu$, defined later in this section. At the end, one would obtain the noncommutativity relations of the same form.

Because the background fields are constant, the Poisson bracket between the hamiltonian and the currents will produce the first σ-derivative of the currents

$$\Big\{H_c, j_{\pm\mu}(\sigma)\Big\} = \mp j'_{\pm\mu}(\sigma). \tag{35}$$

Consequently, the n-th constraints will be given in terms of the n-th derivative of the currents

$$\begin{aligned} {}_N\gamma_a^n &= \Pi_{+ab}(G^{-1})^{bc} j_{-c}^{(n)} + \Pi_{-ab}(G^{-1})^{bc}(-1)^n j_{+c}^{(n)}, \\ {}_D\gamma_n^i &= \frac{(G^{-1})^{ij}}{2}\Big[(-1)^n j_{+j}^{(n)} + j_{-j}^{(n)}\Big]. \end{aligned} \tag{36}$$

So, the constraints read

$$\begin{aligned} \Gamma_a^N(\sigma) &= \sum_{n\geq 0}\frac{\sigma^n}{n!}\Big[\Pi_{+ab}(G^{-1})^{bc} j_{-c}^{(n)} \\ &\quad + \Pi_{-ab}(G^{-1})^{bc}(-1)^n j_{+c}^{(n)}\Big]\Big|_{\sigma=0}, \\ \Gamma_D^i(\sigma) &= \frac{1}{2}\sum_{n\geq 0}\frac{\sigma^n}{n!}(G^{-1})^{ij}\Big[(-1)^n j_{+j}^{(n)} + j_{-j}^{(n)}\Big]\Big|_{\sigma=0}, \end{aligned} \tag{37}$$

where (n) marks the n-th partial derivative over σ. Summing, we obtain the explicit form of the sigma dependent constraints

$$\begin{aligned} \Gamma_a^N(\sigma) &= \Pi_{+ab}(G^{-1})^{bc} j_{-c}(\sigma) + \Pi_{-ab}(G^{-1})^{bc} j_{+c}(-\sigma), \\ \Gamma_D^i(\sigma) &= \frac{1}{2}(G^{-1})^{ij}\Big[j_{+j}(-\sigma) + j_{-j}(\sigma)\Big]. \end{aligned} \tag{38}$$

The Poisson brackets of σ-dependent constraints are

$$\left\{\Gamma_a^N(\sigma), \Gamma_b^N(\bar\sigma)\right\} = -\kappa (G_E)_{ab}\,\delta'(\sigma-\bar\sigma),$$
$$\left\{\Gamma_D^i(\sigma), \Gamma_D^j(\bar\sigma)\right\} = -2\kappa (G^{-1})^{ij}\delta'(\sigma-\bar\sigma). \tag{39}$$

Therefore, they are of the second class and one can solve them.

Obviously, the parameter dependent constraints are given in terms of currents depending on either σ or $-\sigma$. Therefore, in order to obtain the constraints in terms of the independent canonical variables, it is useful to divide the latter into their even and odd parts, with respect to σ. For the initial coordinates one has

$$x^\mu = q^\mu + \bar q^\mu,$$
$$q^\mu = \sum_{n\geq 0} \frac{\sigma^{2n}}{(2n)!} x^{(2n)\mu}\Big|_{\sigma=0},$$
$$\bar q^\mu = \sum_{n\geq 0} \frac{\sigma^{2n+1}}{(2n+1)!} x^{(2n+1)\mu}\Big|_{\sigma=0}, \tag{40}$$

and for the momenta one has

$$\pi_\mu = p_\mu + \bar p_\mu,$$
$$p_\mu = \sum_{n\geq 0} \frac{\sigma^{2n}}{(2n)!} \pi_\mu^{(2n)}\Big|_{\sigma=0},$$
$$\bar p_\mu = \sum_{n\geq 0} \frac{\sigma^{2n+1}}{(2n+1)!} \pi_\mu^{(2n+1)}\Big|_{\sigma=0}. \tag{41}$$

It is well known that this separation, leads to a solvable form of the constraints which now read

$$\Gamma_a^N(\sigma) = 2(BG^{-1})_a{}^b\, p_b + \bar p_a - \kappa (G_E)_{ab}\bar q'^b,$$
$$\Gamma_D^i(\sigma) = (G^{-1})^{ij}\Big[p_j - \kappa G_{jk} q'^k + 2\kappa B_{jk}\bar q'^k\Big]. \tag{42}$$

Using the above expressions for the constraints of the initial theory

$$\Gamma_a^N(\sigma) = 0, \quad \Gamma_D^i(\sigma) = 0, \tag{43}$$

one obtains the solution

$$\bar p_a = 0, \quad \bar q'^a = -\theta^{ab} p_b,$$
$$q'^i = 0, \quad p_i = -2\kappa B_{ij}\bar q'^j. \tag{44}$$

3.1.1 The constraints at $\sigma=\pi$

In order to derive constraints at the other string end-point $\sigma=\pi$, we will multiply every constraint with the appropriate power of $\sigma-\pi$ and sum the products to obtain two sigma dependent constraints

$${}_\pi\Gamma_a^N(\sigma) = \sum_{n\geq 0} \frac{(\sigma-\pi)^n}{n!}\, {}_N\gamma_a^n\Big|_{\sigma=\pi},$$
$${}_\pi\Gamma_D^i(\sigma) = \sum_{n\geq 0} \frac{(\sigma-\pi)^n}{n!}\, {}_D\gamma_n^i\Big|_{\sigma=\pi}. \tag{45}$$

Substituting the canonical form of the constraints (36), we obtain

$${}_\pi\Gamma_a^N(\sigma) = \sum_{n\geq 0} \frac{(\sigma-\pi)^n}{n!}\Big[\Pi_{+ab}(G^{-1})^{bc} j_{-c}^{(n)} + \Pi_{-ab}(G^{-1})^{bc}(-1)^n j_{+c}^{(n)}\Big]\Big|_{\sigma=\pi},$$
$${}_\pi\Gamma_D^i(\sigma) = \frac{1}{2}\sum_{n\geq 0} \frac{(\sigma-\pi)^n}{n!}(G^{-1})^{ij}\Big[(-1)^n j_{+j}^{(n)} + j_{-j}^{(n)}\Big]\Big|_{\sigma=\pi}. \tag{46}$$

Summing, we obtain the explicit form of the sigma dependent constraints

$${}_\pi\Gamma_a^N(\sigma) = \Pi_{+ab}(G^{-1})^{bc} j_{-c}(\sigma) + \Pi_{-ab}(G^{-1})^{bc} j_{+c}(2\pi-\sigma)$$
$${}_\pi\Gamma_D^i(\sigma) = \frac{1}{2}(G^{-1})^{ij}\Big[j_{+j}(2\pi-\sigma) + j_{-j}(\sigma)\Big]. \tag{47}$$

Comparing the constraints (47) and (38), one observes that they are equal if

$$j_{+a}(2\pi-\sigma) = j_{+a}(-\sigma),$$
$$j_{+i}(2\pi-\sigma) = j_{+i}(-\sigma). \tag{48}$$

It follows that if we extend the domain [10] of the variables building the currents, i.e. original coordinates and momenta and demand their 2π-periodicity

$$x^\mu(\sigma+2\pi) = x^\mu(\sigma),$$
$$\pi_\mu(\sigma+2\pi) = \pi_\mu(\sigma), \tag{49}$$

then the σ-dependent constraints for $\sigma=0$ and $\sigma=\pi$ are equal, and their solution is given by (44).

3.2 T-dual theory constraints

The canonical form of the T-dual boundary conditions (20) is obtained using the expression for the T-dual momenta, given by the second relation of (23) and the dual currents (29). The conditions are rewritten as

$${}^\star_N\gamma_0^i = \frac{\kappa}{2}\Big[\Theta_-^{ij} G_{jk}^E\, {}^\star j_-^k + \Theta_+^{ij} G_{jk}^E\, {}^\star j_+^k\Big], \tag{50}$$

and

$${}^\star_D\gamma_a^0 = \frac{1}{2} G_{ab}^E\Big({}^\star j_+^b + {}^\star j_-^b\Big). \tag{51}$$

Although the boundary conditions of the initial and the T-dual theory are related by (22), so that the Neumann and the Dirichlet conditions of the initial theory transform to the Dirichlet and the Neumann conditions in the T-dual theory, one can observe that the form of Neumann and Dirichlet conditions has not changed. Rewriting (50) and (51) as

$$
\begin{aligned}
{}^\star_N\gamma_0^i &= {}^\star\Pi_{+ij}({}^\star G^{-1})^{jk}\,{}^\star j_{-k} + {}^\star\Pi_{-ij}({}^\star G^{-1})^{jk}\,{}^\star j_{+k},\\
{}^\star_D\gamma_a^0 &= \frac{1}{2}({}^\star G^{-1})_{ab}\Big({}^\star j_+^b + {}^\star j_-^b\Big). \qquad (52)
\end{aligned}
$$

using ${}^\star\Pi_\pm^{ij} = \frac{\kappa}{2}\Theta_\mp^{ij}$ and ${}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}$, we see that they are of the same form as (32) keeping in mind the T-duality relations

$$
\Pi_{\pm\mu\nu} \to {}^\star\Pi_\pm^{\mu\nu}, \quad G_{\mu\nu} \to {}^\star G^{\mu\nu}, \quad j_{\pm\mu} \to {}^\star j_\pm^\mu. \qquad (53)
$$

Using the Dirac procedure, analogue to that for the initial theory, the following σ-dependent constraints are obtained

$$
\begin{aligned}
{}^\star\Gamma_N^i(\sigma) &= \frac{\kappa}{2}\Big[\Theta_-^{ij}G_{jk}^E\,{}^\star j_-^k(\sigma) + \Theta_+^{ij}G_{jk}^E\,{}^\star j_+^k(-\sigma)\Big],\\
{}^\star\Gamma_a^D(\sigma) &= \frac{1}{2}G_{ab}^E\big[{}^\star j_+^b(-\sigma) + {}^\star j_-^b(\sigma)\big]. \qquad (54)
\end{aligned}
$$

The constraints at $\sigma = \pi$ produce the same result if we demand the 2π-periodicity for the T-dual canonical variables $y_\mu(\sigma + 2\pi) = y_\mu(\sigma)$ and ${}^\star\pi^\mu(\sigma + 2\pi) = {}^\star\pi^\mu(\sigma)$.

Separating the dual variables into the odd and even parts with respect to $\sigma = 0$, in a same way as in (40) and (41)

$$
\begin{aligned}
y_\mu &= k_\mu + \bar{k}_\mu,\\
{}^\star\pi^\mu &= {}^\star p^\mu + {}^\star\bar{p}^\mu, \qquad (55)
\end{aligned}
$$

one obtains the sigma dependent constraints of the following form

$$
\begin{aligned}
{}^\star\Gamma_N^i(\sigma) &= -2(G^{-1}B)^i{}_j\,{}^\star p^j + {}^\star\bar{p}^i - \kappa(G^{-1})^{ij}\bar{k}_j',\\
{}^\star\Gamma_a^D(\sigma) &= G_{ab}^E\Big[{}^\star p^b + \kappa^2\theta^{bc}\bar{k}_c' - \kappa\left(G_E^{-1}\right)^{bc}k_c'\Big]. \qquad (56)
\end{aligned}
$$

The T-dual constraints are also of the second class. So, we can solve them

$$
{}^\star\Gamma_a^D(\sigma) = 0, \quad {}^\star\Gamma_N^i(\sigma) = 0, \qquad (57)
$$

by

$$
\begin{aligned}
{}^\star\bar{p}^i &= 0, \quad \bar{k}_i' = -\frac{2}{\kappa}B_{ij}\,{}^\star p^j,\\
{}^\star p^a &= -\kappa^2\theta^{ab}\bar{k}_b', \qquad k_a' = 0. \qquad (58)
\end{aligned}
$$

4 Noncommutativity of the effective variables

Solving the constraints (43), has reduced the phase space by half. The σ-derivative of coordinates and the momenta for the solution (44) are

$$
x'^\mu = \begin{cases} q'^a - \theta^{ab}p_b, & \mu=a,\\ \bar{q}'^i, & \mu=i, \end{cases} \qquad (59)
$$

and

$$
\pi_\mu = \begin{cases} p_a, & \mu=a,\\ \bar{p}_i - 2\kappa B_{ij}\bar{q}'^j, & \mu=i. \end{cases} \qquad (60)
$$

By solving the constraints, one has eliminated parts of initial coordinates $\bar{q}^a$, q^i and momenta $\bar{p}_a$, p_i, and one is left with variables q^a, $\bar{q}^i$, p_a, $\bar{p}_i$ which are considered as fundamental variables. Note that in N-sector the new fundamental variables are even q^a, p_a, while in D-sector the new fundamental variables are odd $\bar{q}^i$, $\bar{p}_i$.

For an arbitrary function $F(x, \pi)$ defined on the initial phase space, one introduces its restriction on the reduced phase space by $f = F(x,\pi)\Big|_{\Gamma_\mu=0}$. The Poisson brackets in the effective phase space are the Dirac brackets [21] of the initial phase space associated with the second class constraints $\Gamma_\mu = 0$. The new brackets are denoted by star

$$
{}^\star\{f, g\} = \{F, G\}_{Dirac}\Big|_{\Gamma_\mu=0}. \qquad (61)
$$

The Poisson brackets of the effective variables are considered in app. “Appendix A” (for details see [10]).

So, by looking at the solution of the constraints, we can observe that in the Neumann-subspace, the coordinates depend on both effective coordinates and momenta, while in the Dirichlet-subspace the momenta depend on both effective coordinates and momenta. Therefore, using (A.6) we can conclude that in N-subspace, the coordinates do not commute

$$
{}^\star\{x^a(\sigma), x^b(\bar{\sigma})\} = 2\theta^{ab}\theta(\sigma + \bar{\sigma}), \qquad (62)
$$

while in the D-subspace the momenta do not commute

$$
{}^\star\{\pi_i(\sigma), \pi_j(\bar{\sigma})\} = 4\kappa B_{ij}\delta'(\sigma + \bar{\sigma}). \qquad (63)
$$

The dual coordinate σ-derivative and the dual momenta for the solution (58) of the dual constraints (57) are

$$
y_\mu' = \begin{cases} \bar{k}_a', & \mu=a,\\ k_i' - \frac{2}{\kappa}B_{ij}\,{}^\star p^j, & \mu=i, \end{cases} \qquad (64)
$$

and

$$
{}^\star\pi^\mu = \begin{cases} {}^\star\bar{p}^a - \kappa^2\theta^{ab}\bar{k}_b', & \mu=a,\\ {}^\star p^i, & \mu=i. \end{cases} \qquad (65)
$$

By solving the T-dual constraints one has eliminated the variables $k_a, \bar{k}_i, {}^\star p^a, {}^\star\bar{p}^i$. Therefore, the new fundamental variables are $\bar{k}_a, {}^\star\bar{p}^a$ (odd) in the D-sector and $k_i, {}^\star p^i$ (even) in the N-sector.

In this description, we see that the coordinates in the D-subspace commute while in the N-subspace they are not commutative

$$^\star\{y_i(\sigma), y_j(\bar\sigma)\} = \frac{4}{\kappa} B_{ij}\theta(\sigma+\bar\sigma) = 2\,^\star\theta_{ij}\theta(\sigma+\bar\sigma). \quad (66)$$

The momenta are commutative in the N-subspace, while in the D-subspace they are noncommutative

$$^\star\{^\star\pi^a(\sigma), ^\star\pi^b(\bar\sigma)\} = 2\kappa^2\theta^{ab}\delta'(\sigma+\bar\sigma) = 4\kappa\,^\star B^{ab}\delta'(\sigma+\bar\sigma). \quad (67)$$

So, N and D-sectors of the initial and T-dual theories replace their characteristics. Note that in all cases the Kalb–Ramond field is the source of noncommutativity.

5 Effective theories

By extending the domain of the initial coordinates and momenta, the solution of the constraint in one string endpoint solves the constraint in the other string endpoint as well, as in [10]. If we substitute the solution of the constraints into the canonical hamiltonians, we will obtain the effective hamiltonians. Using the equations of motion for momenta, we will find the corresponding effective lagrangians. Effective theories describe the closed effective string.

When choosing the Neumann boundary conditions for all directions the new basic canonical variables, the effective variables, are the even coordinate and momenta parts $q^\mu(\sigma)$ and $p_\mu(\sigma)$. Choosing the mixed boundary conditions both odd and even parts of initial coordinates and momenta remain the basic canonical variables for some directions. So, the effective hamiltonian for the initial theory will be given in terms of odd $\bar q^i$, $\bar p_i$ in Dirichlet directions and even q^a, p_a in Neumann directions and the effective T-dual hamiltonian in terms of odd $\bar k_a$, $^\star\bar p^a$ in Dirichlet directions and even k_i, $^\star p^i$ in Neumann directions. The corresponding effective lagrangians will consequently depend on both even and odd coordinate parts q^a, $\bar q^i$ and k_i, $\bar k_a$.

5.1 Effective energy-momentum tensors

For the solution (59), (60) of the boundary conditions, the a-th and the i-th component of the initial currents $j_{\pm\mu}$, reduce to

$$j_{\pm a} = \mp\kappa G_{ab}\Theta^{bc}_\pm\, j^N_{\pm c}, \quad j^N_{\pm c} \equiv p_c \pm \kappa G^E_{cd} q'^d,$$
$$j_{\pm i} = \bar p_i \pm \kappa G_{ij}\bar q'^j \equiv j^D_{\pm i}. \quad (68)$$

The energy-momentum tensor components (25) in a background (8) read

$$T_\pm = \mp\frac{1}{4\kappa}\Big[(G^{-1})^{ab} j_{\pm a} j_{\pm b} + (G^{-1})^{ij} j_{\pm i} j_{\pm j}\Big], \quad (69)$$

and reduce to

$$T^{eff}_\pm = \mp\frac{1}{4\kappa}\Big[\left(G_E^{-1}\right)^{ab}\; j^N_{\pm a} j^N_{\pm b} + (G^{-1})^{ij} j^D_{\pm i} j^D_{\pm j}\Big] \equiv T^N_\pm + T^D_\pm, \quad (70)$$

for the solution of constraints.

The dual energy-momentum tensor components are

$$^\star T_\pm = \mp\frac{1}{4\kappa}\Big[(^\star G^{-1})_{ab}\,^\star j^a_\pm\,^\star j^b_\pm + (^\star G^{-1})_{ij}\,^\star j^i_\pm\,^\star j^j_\pm\Big]. \quad (71)$$

The dual currents reduce for the solution (64) and (65) to

$$^\star j^a_\pm = {}^\star\bar p^a \pm \kappa\left(G_E^{-1}\right)^{ab}\bar k'_b \equiv {}^\star j^a_{D\pm},$$
$$^\star j^i_\pm = \pm\kappa\Theta^{ij}_\mp G_{jk}\,^\star j^k_{N\pm}, \quad ^\star j^k_{N\pm} = {}^\star p^k \pm \kappa(G^{-1})^{kl}k'_l, \quad (72)$$

and therefore the energy-momentum tensor components become

$$^\star T^{eff}_\pm = \mp\frac{1}{4\kappa}\Big[(G_E)_{ab}\,^\star j^a_{D\pm}\,^\star j^b_{D\pm} + G_{ij}\,^\star j^i_{N\pm}\,^\star j^j_{N\pm}\Big] \equiv {}^\star T^D_\pm + {}^\star T^N_\pm. \quad (73)$$

Note that in opposite to the initial currents, the T-dual currents with index a are Dirichlet's, while the currents with index i are Neumann's.

5.2 Effective hamiltonians

The effective canonical hamiltonian for theory (2) is

$$\mathcal{H}^{eff}_c = T^{eff}_- - T^{eff}_+, \quad (74)$$

and the effective T-dual canonical hamiltonian for (9) is

$$^\star\mathcal{H}^{eff}_c = {}^\star T^{eff}_- - {}^\star T^{eff}_+. \quad (75)$$

The effective hamiltonian (74), expressed in terms of effective variables with the help of (68), reads

$$\mathcal{H}^{eff}_c = \mathcal{H}^{eff}_N(q^a, p_a) + \mathcal{H}^{eff}_D(\bar q^i, \bar p_i), \quad (76)$$

where

$$\mathcal{H}^{eff}_N(q^a, p_a) = \frac{\kappa}{2} q'^a G^E_{ab} q'^b + \frac{1}{2\kappa} p_a (G_E^{-1})^{ab} p_b,$$
$$\mathcal{H}^{eff}_D(\bar q^i, \bar p_i) = \frac{\kappa}{2}\bar q'^i G_{ij}\bar q'^j + \frac{1}{2\kappa}\bar p_i (G^{-1})^{ij}\bar p_j. \quad (77)$$

The effective T-dual hamiltonian (75), expressed in terms of effective variables with the help of (72), reads

$$^\star\mathcal{H}^{eff}_c = {}^\star\mathcal{H}^{eff}_D(\bar k_a, {}^\star\bar p^a) + {}^\star\mathcal{H}^{eff}_N(k_i, {}^\star p^i), \quad (78)$$

where

$$^\star\mathcal{H}^{eff}_D(\bar k_a, {}^\star\bar p^a) = \frac{\kappa}{2}\bar k'_a\left(G_E^{-1}\right)^{ab}\bar k'_b + \frac{1}{2\kappa}{}^\star\bar p^a (G_E)_{ab}{}^\star\bar p^b,$$

$$
{}^\star\mathcal{H}^{eff}_N(k_i, {}^\star p^i) = \frac{\kappa}{2}\, k'_i (G^{-1})^{ij} k'_j + \frac{1}{2\kappa} {}^\star p^i G_{ij} {}^\star p^j. \tag{79}
$$

5.3 Effective Lagrangians

The lagrangians of the effective theories (76) and (78) are given by

$$
\mathcal{L}^{eff} = \Big[\pi_\mu \dot{x}^\mu - \mathcal{H}_c(x,\pi)\Big]\Big|_{\Gamma_\mu=0}, \tag{80}
$$

$$
{}^\star\mathcal{L}^{eff} = \Big[{}^\star\pi^\mu \dot{y}_\mu - {}^\star\mathcal{H}_c(y, {}^\star\pi)\Big]\Big|_{{}^\star\Gamma_\mu=0}. \tag{81}
$$

The effective lagrangians can be separated into

$$
\begin{aligned}
\mathcal{L}^{eff} &= \mathcal{L}_N(q,p) + \mathcal{L}_D(\bar{q},\bar{p}),\\
{}^\star\mathcal{L}^{eff} &= {}^\star\mathcal{L}_D(\bar{k}, {}^\star\bar{p}) + {}^\star\mathcal{L}_N(k, {}^\star p),
\end{aligned} \tag{82}
$$

with

$$
\begin{aligned}
\mathcal{L}_N(q,p) &= p_a \dot{q}^a - \mathcal{H}^{eff}_N(q^a, p_a),\\
\mathcal{L}_D(\bar{q},\bar{p}) &= \bar{p}_i \dot{\bar{q}}^i - \mathcal{H}^{eff}_D(\bar{q}^i, \bar{p}_i),\\
{}^\star\mathcal{L}_D(\bar{k}, {}^\star\bar{p}) &= {}^\star\bar{p}^a \dot{\bar{k}}_a - {}^\star\mathcal{H}^{eff}_D(\bar{k}_a, {}^\star\bar{p}^a),\\
{}^\star\mathcal{L}_N(k, {}^\star p) &= {}^\star p_i \dot{k}_i - {}^\star\mathcal{H}^{eff}_N(k_i, {}^\star p^i).
\end{aligned} \tag{83}
$$

The explicit forms of the effective lagrangians are found by eliminating the momenta from (83), using the equations of motion for them

$$
p_a = \kappa G^E_{ab} \dot{q}^b, \quad \bar{p}_i = \kappa G_{ij} \dot{\bar{q}}^j, \tag{84}
$$

and

$$
{}^\star\bar{p}^a = \kappa \left(G_E^{-1}\right)^{ab} \dot{\bar{k}}_b, \quad {}^\star p^i = \kappa (G^{-1})^{ij} \dot{k}_j. \tag{85}
$$

For these equations the σ-derivatives of the initial and T-dual coordinates, given by (59) and (64), become

$$
x'^\mu = \begin{cases} q'^a + 2(G^{-1}B)^a{}_b\, \dot{q}^b, & \mu=a,\\ \bar{q}'^i, & \mu=i, \end{cases} \tag{86}
$$

and

$$
y'_\mu = \begin{cases} \bar{k}'_a, & \mu=a,\\ k'_i - 2(BG^{-1})_i{}^j \dot{k}_j, & \mu=i. \end{cases} \tag{87}
$$

In order to find the expression for the initial and the T-dual coordinate we need to introduce a double coordinate $\tilde{q}^a$ of the even part of the initial coordinate q^a

$$
\dot{\tilde{q}}^a = q'^a, \quad \tilde{q}'^a = \dot{q}^a, \tag{88}
$$

and a double coordinate $\tilde{k}_i$ of the even part of the T-dual coordinate k_i

$$
\dot{\tilde{k}}_i = k'_i, \quad \tilde{k}'_i = \dot{k}_i. \tag{89}
$$

The coordinates become

$$
x^\mu = \begin{cases} q^a + 2(G^{-1}B)^a{}_b\, \tilde{q}^b, & \mu=a,\\ \bar{q}^i, & \mu=i, \end{cases} \tag{90}
$$

and

$$
y_\mu = \begin{cases} \bar{k}_a, & \mu=a,\\ k_i - 2(BG^{-1})_i{}^j \tilde{k}_j, & \mu=i. \end{cases} \tag{91}
$$

For the equations (84) and (85), the currents (68) and (72) reduce to

$$
j^N_{\pm a} = \kappa (G_E)_{ab}\, \partial_\pm q^b, \quad j^D_{\pm i} = \kappa G_{ij}\, \partial_\pm \bar{q}^j, \tag{92}
$$

and

$$
{}^\star j^a_{D\pm} = \kappa \left(G_E^{-1}\right)^{ab} \partial_\pm \bar{k}_b, \quad {}^\star j^i_{N\pm} = \kappa (G^{-1})^{ij}\, \partial_\pm k_j. \tag{93}
$$

So, after elimination of the momenta the effective lagrangians become

$$
\begin{aligned}
\mathcal{L}^{eff} &= \mathcal{L}_N(q) + \mathcal{L}_D(\bar{q}),\\
{}^\star\mathcal{L}^{eff} &= {}^\star\mathcal{L}_D(\bar{k}) + {}^\star\mathcal{L}_N(k),
\end{aligned} \tag{94}
$$

where the lagrangians (83) reduced to

$$
\begin{aligned}
\mathcal{L}_N(q) &= \frac{\kappa}{2} G^E_{ab}\, \eta^{\alpha\beta} \partial_\alpha q^a \partial_\beta q^b,\\
\mathcal{L}_D(\bar{q}) &= \frac{\kappa}{2} G_{ij}\, \eta^{\alpha\beta} \partial_\alpha \bar{q}^i \partial_\beta \bar{q}^j,\\
{}^\star\mathcal{L}_D(\bar{k}) &= \frac{\kappa}{2} (G_E^{-1})^{ab}\, \eta^{\alpha\beta} \partial_\alpha \bar{k}_a \partial_\beta \bar{k}_b,\\
{}^\star\mathcal{L}_N(k) &= \frac{\kappa}{2} (G^{-1})^{ij}\, \eta^{\alpha\beta} \partial_\alpha k_i \partial_\beta k_j.
\end{aligned} \tag{95}
$$

5.4 T-duality between effective theories

Let us now introduce coordinates

$$
Q^\mu = \begin{bmatrix} q^a \\ \bar{q}^i \end{bmatrix}, \quad K_\mu = \begin{bmatrix} \bar{k}_a \\ k_i \end{bmatrix}, \tag{96}
$$

and the corresponding canonically conjugated momenta

$$
P_\mu = \begin{bmatrix} p_a \\ \bar{p}_i \end{bmatrix}, \quad {}^\star P^\mu = \begin{bmatrix} {}^\star\bar{p}^a \\ {}^\star p^i \end{bmatrix}. \tag{97}
$$

The currents $j^N_{\pm a}$ and $j^D_{\pm i}$ defined in (68) and the currents ${}^\star j^a_{D\pm}$ and ${}^\star j^i_{n\pm}$ defined in (72), can be gathered into currents

$$
\hat{j}_{\pm\mu} = \begin{bmatrix} j^N_{\pm a} \\ j^D_{\pm i} \end{bmatrix}, \quad {}^\star\hat{j}^\mu_\pm = \begin{bmatrix} {}^\star j^a_{D\pm} \\ {}^\star j^i_{N\pm} \end{bmatrix}. \tag{98}
$$

They satisfy

$$
\hat{j}_{\pm\mu} = P_\mu \pm \kappa G^{eff}_{\mu\nu} Q'^\nu,
$$

$$ {}^\star \hat{j}^\mu_\pm = {}^\star P^\mu \pm \kappa\, {}^\star G^{\mu\nu}_{eff} K'_\nu, \tag{99}$$

where

$$G^{eff}_{\mu\nu} = \begin{pmatrix} G^E_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix},$$
$$ {}^\star G^{\mu\nu}_{eff} = \begin{pmatrix} \left(G_E^{-1}\right)^{ab} & 0 \\ 0 & (G^{-1})^{ij} \end{pmatrix}. \tag{100}$$

The effective energy-momentum components (70) and (73) can be rewritten as

$$T^{eff}_\pm = \mp \frac{1}{4\kappa}\left(G^{-1}_{eff}\right)^{\mu\nu} \hat{j}_{\pm\mu}\, \hat{j}_{\pm\nu},$$
$$ {}^\star T^{eff}_\pm = \mp \frac{1}{4\kappa}\left({}^\star G^{-1}_{eff}\right)_{\mu\nu} {}^\star \hat{j}^\mu_\pm\, {}^\star \hat{j}^\nu_\pm, \tag{101}$$

and the effective hamiltonians (76) and (78) are therefore

$$\mathcal{H}^{eff} = \frac{\kappa}{2} Q'^\mu G^{eff}_{\mu\nu} Q'^\nu + \frac{1}{2\kappa} P_\mu \left(G^{-1}_{eff}\right)^{\mu\nu} P_\nu,$$
$$ {}^\star\mathcal{H}^{eff} = \frac{\kappa}{2} K'_\mu {}^\star G^{\mu\nu}_{eff} K'_\nu + \frac{1}{2\kappa} {}^\star P^\mu \left({}^\star G^{-1}_{eff}\right)_{\mu\nu} {}^\star P^\nu. \tag{102}$$

Using the T-duality relations

$$\kappa x'^\mu \cong {}^\star\pi^\mu, \quad \kappa y'_\mu \cong \pi_\mu, \tag{103}$$

and (59), (65), (64), (60) one obtains

$$\kappa q'^a - \kappa\theta^{ab} p_b \cong {}^\star\bar{p}^a - \kappa^2\theta^{ab}\bar{k}'_b, \kappa\bar{q}'^i \cong {}^\star p^i, \tag{104}$$

and

$$\kappa\bar{k}'_a \cong p_a, \kappa k'_i - 2B_{ij}{}^\star p^j \cong \bar{p}_i - 2\kappa B_{ij}\bar{q}'^j. \tag{105}$$

Separating the odd and even parts one obtains

$$\kappa q'^a \cong {}^\star\bar{p}^a, \quad \kappa\bar{k}'_a \cong p_a,$$
$$\kappa\bar{q}'^i \cong {}^\star p^i, \quad \kappa k'_i \cong \bar{p}_i, \tag{106}$$

which gives

$$\kappa Q'^\mu \cong {}^\star P^\mu, \quad \kappa K'_\mu \cong P_\mu. \tag{107}$$

Comparing the background fields (100), we see that they are T-dual to each other as expected, because by T-duality the metric should transform to the inverse of the effective metric. In our case, in absence of the effective Kalb–Ramond field this means the T-dual metric should be inverse to the initial metric, what is just the case

$$(G^{eff}_{\mu\nu})^{-1} = \begin{pmatrix} G^E_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix}^{-1} = \begin{pmatrix} (G_E^{-1})^{ab} & 0 \\ 0 & (G^{-1})^{ij} \end{pmatrix} = {}^\star G^{\mu\nu}_{eff}. \tag{108}$$

Using (107) and (108) we can conclude that the effective hamiltonians (102) are T-dual to each other.

The corresponding lagrangians (94) are given by

$$\mathcal{L}^{eff} = \dot{Q}^\mu P_\mu - \mathcal{H}^{eff}(Q, P),$$
$$ {}^\star\mathcal{L}^{eff} = \dot{K}_\mu {}^\star P^\mu - {}^\star\mathcal{H}^{eff}(K, {}^\star P), \tag{109}$$

which for the equations of motion for momenta (84) and (85)

$$P_\mu = \kappa G^{eff}_{\mu\nu} \dot{Q}^\nu, \quad {}^\star P_\mu = \kappa\, {}^\star G^{eff}_{\mu\nu} \dot{K}_\nu, \tag{110}$$

become

$$\mathcal{L}^{eff} = \frac{\kappa}{2}\eta^{\alpha\beta}\,\partial_\alpha Q^\mu\, G^{eff}_{\mu\nu}\,\partial_\beta Q^\nu,$$
$$ {}^\star\mathcal{L}^{eff} = \frac{\kappa}{2}\eta^{\alpha\beta}\,\partial_\alpha K_\mu\, {}^\star G^{\mu\nu}_{eff}\,\partial_\beta K_\nu. \tag{111}$$

Combining (107) with (110) one obtains

$$Q'^\mu \cong {}^\star G^{\mu\nu}_{eff} \dot{K}_\nu, \quad K'_\mu \cong G^{eff}_{\mu\nu} \dot{Q}^\nu. \tag{112}$$

Therefore, the effective and T-dual effective variables Q^μ and K_μ are connected by

$$\partial_\pm K_\mu \cong \pm G^{eff}_{\mu\nu}\,\partial_\pm Q^\nu. \tag{113}$$

This is the T-dual effective coordinate transformation law. Using it together with (108), one can conclude that the effective lagrangians (111) are T-dual. This law is in agreement with the T-dual coordinate transformation law (17), for $B_{\mu\nu} = 0$

$$\partial_\pm y_\mu \cong \pm G_{\mu\nu}\partial_\pm x^\nu, \tag{114}$$

keeping in mind that the metric is replaced by the effective metric $G_{\mu\nu} \to G^{eff}_{\mu\nu}$.

6 Conclusion

In the present paper we show that solving the constraints obtained applying the Dirac consistency procedure to mixed boundary conditions of the open bosonic string, which leads to the effective theory and the T-dualization of the bosonic string theory can be performed in an arbitrary order. We started considering the string described by the open string sigma model. The string is moving in the constant metric $G_{\mu\nu}$ and a constant Kalb–Ramond field $B_{\mu\nu}$. We chose the Neumann boundary conditions for some directions x^a and the Dirichlet boundary conditions for all other directions x^i.

We treated the boundary conditions as constraints, and applied the Dirac procedure. The boundary conditions where given in terms of coordinates and momenta, which we rewrote in terms of currents building the energy-momentum tensor components. By Dirac procedure the new constraints are found commuting the hamiltonian with the known constraints. The canonical form of constraints allowed us a simple calculation of the exact form of the infinitely many constraints. From these constraints we formed two σ-dependent constraints, for every string endpoint, by multiplying every obtained constraint with the appropriate power of σ for constraints in $\sigma = 0$ and $\pi - \sigma$ for constraints in $\sigma = \pi$, and adding these terms together into Taylor expansions. The constraints at $\sigma = 0$ and $\sigma = \pi$ were found to be equivalent by imposing 2π-periodicity condition for the canonical variables x^μ and π_μ.

The σ-dependent constraints are of the second class. To solve them we introduced even and odd parts of the initial canonical variables. We found the solution and expressed the σ-derivative of the initial coordinate x'^μ and the initial momentum π_μ in terms of even parts q^a, p_a of x^μ, π_μ in Neumann directions and of their odd parts $\bar{q}^i$, $\bar{p}_i$ in Dirichlet directions, see (59) and (60). For the solution of constraints, the theory reduced to the effective theory. We obtained the effective energy-momentum tensors (70) and the effective hamiltonian (76). For the equations of motion for momenta, we obtained the corresponding effective lagrangian (94).

We also found the T-dual of the initial theory. We applied the Dirac procedure to the mixed boundary conditions of the T-dual theory. The constraints where solved, which reduced the phase space to $\bar{k}_a, {}^\star\bar{p}^a$ in D-sector and $k_i, {}^\star p^i$ in N-sector. For the solution of T-dual constraints we obtained the T-dual effective energy-momentum tensors (73) and the T-dual effective hamiltonian (78), as well as the corresponding T-dual effective lagrangian (94).

The canonically conjugated effective variables are now pairs q^a, p_a and $\bar{q}^i$, $\bar{p}_i$ for the initial and $k_i, {}^\star p^i$ and $\bar{k}_a, {}^\star\bar{p}^a$ for the T-dual effective theory. The effective variables in both effective theories satisfy the modified Poisson brackets considered in "Appendix A". Therefore, if the variable of the initial theory depends on both effective coordinates and effective momenta of any pair, it will be noncommutative. One observes that in N-sector coordinates do not commute (62) and also the momenta of the D-sector of the initial theory (63). In T-dual theory the roles are exchanged so that in D-sector coordinates do not commute (66) and also the momenta of the N-sector (67).

This is different, in comparison to the choice of the Neumann boundary conditions for all directions [10–12]. In that case, solving the constraints leads to full elimination of odd variables. Also, when considering a weakly curved background, with a coordinate dependent Kalb–Ramond field with an infinitesimal field strength, the effective theory turned out to be non-geometric. It is defined in the effective space-time composed of the even coordinate and its double $x^\mu \to q^\mu$, $\tilde{q}^\mu$. This fact lead to appearance of nontrivial effective Kalb–Ramond field, depending on the double effective coordinate $B_{\mu\nu}(x) \to B^{eff}_{\mu\nu}(2b\tilde{q})$. It would be interesting to find the corresponding field in the mixed boundary conditions case. For constant initial background fields, considered in this paper the effective fields are constant. But, the nongeometricity can still be seen. It appears in a fact that coordinates of the initial and T-dual theories, can not be expressed without an introduction of double coordinates, see (90) and (91).

The obtained effective theories, defined in terms of the effective variables, where compared using the T-dualization procedure. It was confirmed that the corresponding background fields (the effective metrics $G^{eff}_{\mu\nu}$ and ${}^\star G^{\mu\nu}_{eff}$ (100)) are T-dual to each other. Also, the effective variables of the initial effective theory are confirmed to be T-dual to the T-dual effective variables of the T-dual effective theory, by obtaining the T-duality law connecting them. This law was an appropriate reduction of the standard T-duality coordinate transformation law. Therefore, we showed the T-duality of the reduced bosonic string theories. Consequently, all the theories on the following diagram are equivalent

$$
\begin{array}{ccc}
\kappa \int d\xi^2 \partial_+ x^\mu \Pi_{+\mu\nu} \partial_- x^\nu & \overset{T}{\longleftrightarrow} & \frac{\kappa^2}{2} \int d\xi^2 \partial_+ y_\mu \Theta_-^{\mu\nu} \partial_- y_\nu \\
\Big\downarrow {\scriptstyle \Gamma=0} & & \Big\downarrow {\scriptstyle {}^\star\Gamma=0} \\
\frac{\kappa}{2} \int d\xi^2 \partial_+ Q^\mu G^{eff}_{\mu\nu} \partial_- Q^\nu & \overset{T}{\longleftrightarrow} & \frac{\kappa}{2} \int d\xi^2 \partial_+ K_\mu {}^\star G^{\mu\nu}_{eff} \partial_- K_\nu
\end{array}
$$

So, we confirmed that two procedures, the T-dualization procedure and the solving of the mixed boundary conditions, treated as constraints in the Dirac consistency procedure, do commute.

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Appendix A: Brackets between effective variables

The effective theory is given in terms of the odd and even parts of the initial coordinates and momenta. These parts do not satisfy the ordinary Poisson brackets because they are not the arbitrary functions, but contain only even or odd powers of σ. Additionally their domain is changed in order to solve the boundary conditions in both string endpoints. The new fundamental variables satisfy the modified Poisson brackets, defined with the appropriate delta functions.

The standard Poisson brackets between the initial coordinates and the momenta

$$\{x^\mu(\sigma), \pi_\nu(\bar{\sigma})\} = \delta^\mu_\nu \delta(\sigma - \bar{\sigma}), \tag{A.1}$$

give

$$\{q^\mu(\sigma), p_\nu(\bar{\sigma})\} = \delta^\mu{}_\nu \delta_S(\sigma, \bar{\sigma}),$$

$$\{\bar{q}^\mu(\sigma), \bar{p}_\nu(\bar{\sigma})\} = \delta^\mu_\nu \delta_A(\sigma, \bar{\sigma}), \tag{A.2}$$

where δ_S and δ_A are defined by

$$\delta_S(\sigma, \bar{\sigma}) = \frac{1}{2}[\delta(\sigma - \bar{\sigma}) + \delta(\sigma + \bar{\sigma})],$$
$$\delta_A(\sigma, \bar{\sigma}) = \frac{1}{2}[\delta(\sigma - \bar{\sigma}) - \delta(\sigma + \bar{\sigma})], \tag{A.3}$$

and the domain is $[-\pi, \pi]$. The even and odd coordinate parts satisfy

$$\int_{-\pi}^{\pi} d\bar{\sigma} q^\mu(\bar{\sigma}) \delta_S(\bar{\sigma}, \sigma) = q^\mu(\sigma),$$
$$\int_{-\pi}^{\pi} d\bar{\sigma} \bar{q}^\mu(\bar{\sigma}) \delta_A(\bar{\sigma}, \sigma) = \bar{q}^\mu(\sigma). \tag{A.4}$$

Separating integration domain in two parts, from $-\pi$ to 0 and from 0 to π, and changing the integration variable in the first part $\bar{\sigma} \to -\bar{\sigma}$, we obtain

$$2\int_{0}^{\pi} d\bar{\sigma} q^\mu(\bar{\sigma}) \delta_S(\bar{\sigma}, \sigma) = q^\mu(\sigma),$$
$$2\int_{0}^{\pi} d\bar{\sigma} \bar{q}^\mu(\bar{\sigma}) \delta_A(\bar{\sigma}, \sigma) = \bar{q}^\mu(\sigma). \tag{A.5}$$

So, the unit functions on the interval $[0, \pi]$ for functions with only an even or odd power in σ are $2\delta_S(\bar{\sigma}, \sigma)$ and $2\delta_A(\bar{\sigma}, \sigma)$, respectively. Therefore, the brackets which we use are

$$^\star\{q^\mu(\sigma), p_\nu(\bar{\sigma})\} = 2\delta^\mu_\nu \delta_S(\sigma, \bar{\sigma}),$$
$$^\star\{\bar{q}^\mu(\sigma), \bar{p}_\nu(\bar{\sigma})\} = 2\delta^\mu_\nu \delta_A(\sigma, \bar{\sigma}), \quad \sigma, \bar{\sigma} \in [0, \pi]. \tag{A.6}$$

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Derivation of the Husimi symbols without antinormal ordering, scale transformation and uncertainty relations

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Abstract

We propose a new method for the derivation of Husimi symbols, for operators that are given in the form of products of an arbitrary number of coordinates, and momentum operators, in an arbitrary order. For such an operator, in the standard approach, one expresses coordinate and momentum operators as a linear combination of the creation and annihilation operators, and then uses the antinormal ordering to obtain the final form of the symbol. In our method, one obtains the Husimi symbol in a much more straightforward fashion, departing directly from operator explicit form without transforming it through creation and annihilation operators. With this method the mean values of some operators are found. It is shown how the Heisenberg and the Schrödinger–Robertson uncertainty relations, for position and momentum, are transformed under scale transformation $(q;\, p) \rightarrow (\lambda q;\, \lambda p)$. The physical sense of some states which can be constructed with this transformation is also discussed.

Keywords: average values, Husimi function, polynomials, operators, scaling transform, uncertainty relation

1. Introduction

In classical statistical mechanics, in order to find the mean value of any function $F(q, p)$ defined on the phase space, one has to integrate that function over the phase space, weighted with an appropriate probability density function, i.e.

$$\langle F \rangle = \int F(q, p)\rho(q, p)\mathrm{d}q\mathrm{d}p. \tag{1}$$

Here, $\rho(q, p)$ is a probability density, which means that the integral of this function over a certain region of phase space gives the probability of having the system in that region of the phase space. In quantum mechanics, to each observable $F(q, p)$ one assigns an operator $\hat{A}$. However, if one wants to keep the resemblance to classical statistical mechanics, i.e. to still compute the mean value of an operator by some formula similar to (1), some additional steps are needed. First, an analogue of the classical distribution function has to be chosen, and this analogue comes in the form of the quasidistribution function $D(q, p)$. The mean value is now calculated similarly to (1) as:

$$\langle \hat{A} \rangle = \int A_D(q, p)D_\rho(q, p)\mathrm{d}q\mathrm{d}p. \tag{2}$$

A few notes about the function $A_D(q, p)$ are in order. This function should be assigned to each operator $\hat{A}$. The process of assignment is far from trivial, and is one the main themes of this work. The function is defined on the whole phase space, and must fulfil the condition that the mean value $\langle \hat{A} \rangle$ of an operator $\hat{A}$ at a given state, described by the

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quasidistribution $D(q, p)$, is given by (2). The function $A_D(q, p)$ is called the symbol of the operator $\hat{A}$, corresponding to the quasidistribution $D(q, p)$. In quantum mechanics a number of quasidistribution functions are used, hence, different symbols are assigned to the same operator, $\hat{A}$, depending on the quasidistribution [1–5]. So, a problem arises —how do we find the symbol of the given operator for the concrete quasidistribution?

In a number of cases the answer to this question is known. Concretely, the symbols are known for the following quasidistributions: Wigner function $W(q, p)$ [8], Husimi-Kano $Q(q, p)$ [6, 7] and Glauber-Sudarsan $P(q, p)$ [9, 10]. If the operator $\hat{A}$ comes in the form of creation and annihilation operators' bivariate polynomial, then its W, Q, and P-symbols are obtained using operations of symmetrization, antinormal and normal ordering [11–13].

The sequence of the actions is as follows: first, in a given operator which is the function of the coordinate and momentum operators, one expresses the mentioned operators as corresponding linear combinations of creation and annihilation operators. To this form of the operator the procedure of symmetrization or antisymmetrization, depending on the quasiprobability used, is applied. For this ordered form the scalar function can be directly obtained. This procedure, in principle, solves the problem of the determination of the symbols for the polynomial operators, but if the polynomial is complicated, the procedure can be very tedious.

In this paper, we propose a new, simpler, way of determining the Q-symbols. In our approach, we simply replace operators $\hat{q}$ and $\hat{p}$ at the places where they stand in the original operator, with differential operators which we will define below. This is done explicitly, without using any operations such as symmetrization or operator ordering. The obtained differential operator acts on the Husimi function giving the appropriate Q- symbol of the original operator. The crucial fact is that for obtaining the Q- symbol of an operator, the explicit form of the Husimi function for a given state is not needed. One uses only its general structure which is the same for all concrete Husimi functions. In section 2 we give the Husimi function in a form that is used below. Our novel method is presented in section 3.

In section 4, with the help of this method, we evaluate the mean values of some operators and analyze the behavior of the uncertainty relations under scale transformation $(q; p) \to (\lambda q; \lambda p)$. We show that as a result of the transformation, the right-hand side of the inequalities increases. This result can be used to explain some of the tunnelling phenomena.

In section 5 we investigate the properties of 'stretched Fock states'. These states can be achieved by applying of the scale transformation to Fock states of the harmonic oscillator.

2. Husimi function and the mean value problem

The Husimi functions are determined by the density operator and the set of coherent states of a harmonic oscillator [6, 7]. Let us consider some state, described by the density operator $\hat{\rho}$, and $\langle x|\alpha\rangle$ is a coherent state. Then, the Husimi function of the state $\hat{\rho}$ is defined by

$$Q\left(\alpha, \alpha^*\right) = \frac{1}{\pi}\int \langle\alpha \mid x\rangle \rho(x, y)\langle y \mid \alpha\rangle \mathrm{d}x\mathrm{d}y. \tag{3}$$

Here, $\alpha = \alpha_r + \mathrm{i}\alpha_i$ is an arbitrary complex number, and $\rho(x, y)$ is the kernel of the density operator in the coordinate representation. For the complex number α, which determines the coherent state, we will use expression $\alpha = (q + \mathrm{i}p)/\sqrt{2}$ and will regard the Husimi function Q as a function of p and q

$$Q(q, p) = \frac{1}{2\pi\hbar}\langle q, p\,|\hat{\rho}|\, q, p\rangle. \tag{4}$$

Here $|q, p\rangle$ is the coherent state, given in terms of the variables q and p:

$$\langle x \mid q, p\rangle = \left(\frac{1}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}(x - q)^2 + \mathrm{i}px - \frac{\mathrm{i}}{2}qp\right]. \tag{5}$$

The Husimi function of the state, given by the wave function $\psi(x)$, has the following form

$$\begin{aligned} Q(q, p) &= \frac{1}{2}\left(\frac{1}{\pi}\right)^{3/2} \int \exp\left[-\frac{1}{2}(y - q)^2 + \mathrm{i}py\right]\psi^*(y) \\ &\quad\times \psi(x) \exp\left[-\frac{1}{2}(x - q)^2 - \mathrm{i}px\right]\mathrm{d}x\mathrm{d}y \\ &= C\int F(q, p; x. y)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{6}$$

Using the Husimi function one can also determine the mean value of the operator, by applying the formula (2).

The Q-symbol of the operator, $\hat{A}$, in the standard approach, is determined with the help of the antinormal ordering of the creation and annihilation operators in the expression for the operator, $\hat{A}$. If the operator, $\hat{A}$, is a low order polynomial of the creation and annihilation operators, or of coordinates and momenta, then one can find the explicit form of its symbols by performing the antinormal ordering. In a number of cases, however, the procedure could be tedious. In this paper we propose a new procedure for determining the Q- symbols of an operator of arbitrary form, without using any operator ordering.

3. Deriving the Husimi symbols without antinormal ordering

Let us introduce the operator

$$\hat{X} = q + \frac{1}{2}\frac{\partial}{\partial q} + \frac{\mathrm{i}}{2}\frac{\partial}{\partial p}. \tag{7}$$

Using the explicit form of the function (6), one can prove the following formula

$$\begin{aligned} \int \hat{X}Q(q, p)\mathrm{d}q\mathrm{d}p &= \int qQ(q, p)\mathrm{d}q\mathrm{d}p \\ &= \int \psi(x)x\psi^*(x)\mathrm{d}x. \end{aligned} \tag{8}$$

The right-hand side of (8) is, by definition, the mean value of the coordinate operator. So, this equation shows that the mean value of the coordinate operator, when the state is described by the Husimi function, may be represented by the left-hand side of (8). So, in order to find the mean value of the coordinate operator $\hat{x}$, when the quantum state is described by the Husimi function, one should apply the operator $\hat{X}$ (7) on the Husimi function of the corresponding state and integrate the obtained result over the whole phase space (q, p). The equation (8) presents two alternative expressions for the mean value of the coordinate operator, one using the wave function for the description of the states and the other using the Husimi function.

Let us generalize the obtained results to the case of arbitrary exponent of the coordinate operator. It can be easily seen, that the mean value of some operator $K(\hat{X})$, where K is a polynomial of one variable, can be calculated by the following formula

$$\left\langle K\left(\hat{X}\right)\right\rangle = C\int \tilde{K}(q)F(q, p; x, y)\mathrm{d}x\mathrm{d}y\mathrm{d}q\mathrm{d}p. \tag{9}$$

In the case where $K(\hat{X})$ is a monomial $K(\hat{X}) = (\hat{X})^n$ and with the help of mathematical induction one can prove that the functions $\tilde{K}_n(q)$ are related by the recurrent relation

$$\tilde{K}_{n+1}(q) = q\tilde{K}_n(q) - \frac{1}{2}\frac{\partial}{\partial q}\tilde{K}_n(q), \quad \tilde{K}_0 = 1. \tag{10}$$

$$\tilde{K}_n(q) = \sum_{0\leqslant s\leqslant[n/2]} \frac{(-1)^s n!}{s!(n-2s)!4^s} q^{n-2s}. \tag{11}$$

Here $[n/2]$ is the integer part of a number $n/2$.

So, in order to derive the mean value of the operator $K(\hat{X})$, one has to apply this operator to the Husimi function and integrate over the parameter space (q, p), which determines the coherent state (5). The result is a polynomial of the variable q.

Almost the same procedure can be used in order to obtain the mean values of the momenta operators. To this end, in addition to the operator $\hat{X}$ defined in (7), consider the operator $\hat{P}$ defined as:

$$\hat{P} = p - \frac{\mathrm{i}}{2}\frac{\partial}{\partial q} + \frac{1}{2}\frac{\partial}{\partial p}. \tag{12}$$

Operators (7), (12) satisfy the commutation relation

$$[\hat{P}, \hat{X}] = -i. \tag{13}$$

For the operator $\hat{P}$ the following relation holds

$$\hat{P}Q = C\int\left(p - \frac{\mathrm{i}}{2}\frac{\partial}{\partial q} + \frac{1}{2}\frac{\partial}{\partial p}\right)F(q, p; x, y)\mathrm{d}x\mathrm{d}y$$
$$= C\int (p + \mathrm{i}q - \mathrm{i}x)F(q, p; x, y)\mathrm{d}x\mathrm{d}y. \tag{14}$$

Integrating this expression over the phase space parameters (q, p), we obtain

$$\int \hat{P}Q(q, p)\mathrm{d}q\mathrm{d}p = C\int pF(q, p; x, y)\mathrm{d}x\mathrm{d}y\mathrm{d}q\mathrm{d}p. \tag{15}$$

By induction one can prove that

$$\left\langle \hat{p}^n\right\rangle = \int \hat{P}^n Q(q, p)\mathrm{d}q\mathrm{d}p$$
$$= C\int L_n(p)F(q, p; x, y)\mathrm{d}x\mathrm{d}y\mathrm{d}q\mathrm{d}p. \tag{16}$$

The polynomials $L_n(p)$ are related by the following recurrence relation, which is the analogue of (10)

$$L_{n+1}(p) = pL_n(p) - \frac{1}{2}\frac{\partial}{\partial p}L_n(p), \quad L_0 = 1;$$
$$L_n(p) = \sum_{0\leqslant s\leqslant[n/2]} \frac{(-1)^s n!}{s!(n-2s)!4^s} p^{n-2s}. \tag{17}$$

Here $[n/2]$ is the integer part of a number $n/2$.

Let us now consider the following more general problem. Consider the operator which is the monomial of the coordinate and momenta operators. In the coordinate representation it can be represented as the polynomial of operators x and $\frac{\partial}{\partial x}$. More precisely, it will be the monomial expression

$$\hat{A}\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right) = \left(-\mathrm{i}\frac{\partial}{\partial x}\right)x \ldots x\left(-\mathrm{i}\frac{\partial}{\partial x}\right). \tag{18}$$

The mean value of this operator in the state $\psi(x)$ is determined by the formula

$$\langle\hat{A}\rangle = C\int \psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2 + \mathrm{i}py\right]$$
$$\cdot\exp\left[-\frac{1}{2}(x-q)^2 - \mathrm{i}px\right]$$
$$\times A\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q. \tag{19}$$

One can find the expression for this mean value with the help of mathematical induction. Supposing that the mean value of the operator (19) is determined by

$$\left\langle\hat{A}\right\rangle = \int \hat{P}\hat{X}\ldots\hat{X}\hat{P}Q(q, p)\mathrm{d}q\mathrm{d}p, \tag{20}$$

i.e. supposing that the following equality holds

$$\int \hat{P}\hat{X}\ldots\hat{X}\hat{P}Q(q, p)\mathrm{d}q\mathrm{d}p$$
$$= C\int \psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2 + \mathrm{i}py\right]$$
$$\cdot\exp\left[-\frac{1}{2}(x-q)^2 - \mathrm{i}px\right]$$
$$\times A\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q. \tag{21}$$

Let us now consider the following operator

$$x\hat{A}\left(x, -\mathrm{i}\frac{\partial}{\partial x}\right) = x\left(-\mathrm{i}\frac{\partial}{\partial x}\right)x \ldots x\left(-\mathrm{i}\frac{\partial}{\partial x}\right). \tag{22}$$

Its mean value has a form

$$\left\langle x\hat{A}\right\rangle = C\int \psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2+\mathrm{i}py\right] \cdot \exp\left[-\frac{1}{2}(x-q)^2-\mathrm{i}px\right] \times xA\left(x,-\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q$$
$$= C\int \mathrm{d}p\mathrm{d}q\hat{X}\int \mathrm{d}x\mathrm{d}y\psi^*(x) \times \exp\left[-\frac{1}{2}(y-q)^2+\mathrm{i}py\right] \times \exp\left[-\frac{1}{2}(x-q)^2-\mathrm{i}px\right] \cdot A\left(x,-\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)$$
$$= \int \hat{X}\hat{P}\hat{X}\ \dots\ \hat{X}\hat{P}Q(q,p)\mathrm{d}p\mathrm{d}q. \tag{23}$$

This result shows that in order to obtain the mean value of the operator $x\hat{A}$, one has to apply the operator $\hat{X}\hat{P}\hat{X}\ \dots\ \hat{X}\hat{P}$ to the Husimi function $Q(q,p)$ and integrate the obtained result over dpdq.

Analogously, one can consider the operator

$$\left(-i\frac{\partial}{\partial x}\right)\hat{A}\left(x,-\mathrm{i}\frac{\partial}{\partial x}\right) = \left(-\mathrm{i}\frac{\partial}{\partial x}\right)\left(-\mathrm{i}\frac{\partial}{\partial x}\right)x\ \dots\ x\left(-\mathrm{i}\frac{\partial}{\partial x}\right). \tag{24}$$

Its mean value has the form

$$\left\langle\left(-\mathrm{i}\frac{\partial}{\partial x}\right)\hat{A}\right\rangle = C\int \psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2+\mathrm{i}py\right] \cdot \exp\left[-\frac{1}{2}(x-q)^2-\mathrm{i}px\right] \times\left(-\mathrm{i}\frac{\partial}{\partial x}\right)A\left(x,-\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q. \tag{25}$$

Integrating by parts one obtains

$$C\int \psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2+\mathrm{i}py-\frac{1}{2}(x-q)^2-\mathrm{i}px\right] \cdot(p-\mathrm{i}x+\mathrm{i}q)A\left(x,-\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x)\mathrm{d}x\mathrm{d}y\mathrm{d}p\mathrm{d}q$$
$$= C\int \mathrm{d}p\mathrm{d}q\hat{P}\int \mathrm{d}x\mathrm{d}y\psi^*(x)\exp\left[-\frac{1}{2}(y-q)^2+\mathrm{i}py\right] \times\exp\left[-\frac{1}{2}(x-q)^2-\mathrm{i}px\right]$$
$$A\left(x,-\mathrm{i}\frac{\partial}{\partial x}\right)\psi(x) = \int \hat{P}\hat{P}\hat{X}\ \dots\ \hat{X}\hat{P}Q(q,p)\mathrm{d}p\mathrm{d}q. \tag{26}$$

From formulas (23), (26) it can be seen that, in order to determine the mean value of the operator (18), one has to apply the operators $\hat{X}$, $\hat{P}$ to the Husimi function $Q(q,p)$ in the order in which the operators x, $-\mathrm{i}\frac{\partial}{\partial x}$ appear in the operator $A\left(x,-\mathrm{i}\frac{\partial}{\partial x}\right)$, and integrate this result over (q,p).

4. Evaluation of mean values and uncertainty relations

Let us consider a state with a Husimi function $Q(q,p)$. It was shown in [14] that if Q(q, p) is a Husimi function of a quantum state and $\lambda^2<1$, then the transformed function $\lambda^2Q(\lambda q,\lambda p)$ is a Husimi function of some quantum state too. A state with the Husimi function $\lambda^2Q(\lambda q,\lambda p)$ we call a stretched state.

In this section we consider the problem of evaluation of average values of some operators. Let us consider a Hamiltonian operator

$$\bar{H} = \frac{\hbar\omega}{2}\left(\hat{q}^2+\hat{p}^2\right). \tag{27}$$

The average value of the energy $\bar{E}$ of a state with a Husimi function $Q(q,p)$ reads

$$\bar{E} = \int\frac{\hbar\omega}{2}\left(q^2+p^2-1\right)Q(q,p)\mathrm{d}q\mathrm{d}p = \int\frac{\hbar\omega}{2}\left(q^2+p^2\right)Q(q,p)\mathrm{d}q\mathrm{d}p-\frac{\hbar\omega}{2}. \tag{28}$$

The average value of the energy $\bar{E}_\lambda$ of a stretched state with a Husimi function $\lambda^2Q(\lambda q,\lambda p)$ reads

$$\bar{E}_\lambda = \int\frac{\hbar\omega}{2}\left(q^2+p^2-1\right)\lambda^2Q(\lambda q,\lambda p)\mathrm{d}q\mathrm{d}p = \int\frac{1}{\lambda^2}\frac{\hbar\omega}{2}\left((\lambda q)^2+(\lambda p)^2\right)Q(\lambda q,\lambda p)\mathrm{d}(\lambda q)\mathrm{d}(\lambda p)-\frac{\hbar\omega}{2} = \frac{1}{\lambda^2}\bar{E}+\frac{1-\lambda^2}{\lambda^2}\frac{\hbar\omega}{2}. \tag{29}$$

One can see from the expression (29) that energy of a state increases after the transform $(q;p)\to(\lambda q;\lambda p)$.

Let us consider now the Heisenberg uncertainty relation

$$\sigma_{qq}\sigma_{pp}\geqslant\frac{1}{4}\hbar^2,\quad \sigma_{qq}=\left\langle\hat{q}^2\right\rangle-\left\langle\hat{q}\right\rangle^2,\quad \sigma_{pp}=\left\langle\hat{p}^2\right\rangle-\left\langle\hat{p}\right\rangle^2, \tag{30}$$

and the Schrödinger–Robertson uncertainty relation

$$\sigma_{qq}\sigma_{pp}-\sigma_{qp}^2\geqslant\frac{1}{4}\hbar^2;\quad \sigma_{qp}=\frac{1}{2}\left\langle\hat{p}\hat{q}+\hat{q}\hat{p}\right\rangle-\left\langle\hat{q}\right\rangle\left\langle\hat{p}\right\rangle. \tag{31}$$

The dispersions σ_{qq} and σ_{pp} can be evaluated with the help of the Husimi functions

$$\sigma_{qq}=\int\left(q^2-\frac{1}{2}\right)Q(q,p)\mathrm{d}q\mathrm{d}p-\left(\int qQ(q,p)\mathrm{d}q\mathrm{d}p\right)^2,$$
$$\sigma_{pp}=\int\left(p^2-\frac{1}{2}\right)Q(q,p)\mathrm{d}q\mathrm{d}p-\left(\int pQ(q,p)\mathrm{d}q\mathrm{d}p\right)^2. \tag{32}$$

Also

$$\sigma_{qp} = \int qpQ(q, p)\mathrm{d}q\mathrm{d}p - \int qQ(q, p)\mathrm{d}q\mathrm{d}p \int pQ(q, p)\mathrm{d}q\mathrm{d}p. \quad (33)$$

For the stretched states the formulas (32), (33) take the form

$$\sigma_{qq\lambda} = \int \left(q^2 - \frac{1}{2}\right)\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p - \left(\int q\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p\right)^2,$$
$$\sigma_{pp\lambda} = \int \left(p^2 - \frac{1}{2}\right)\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p - \left(\int p\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p\right)^2,$$
$$\sigma_{qp\lambda} = \int qp\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p - \int q\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p \int p\lambda^2 Q(\lambda q, \lambda p)\mathrm{d}q\mathrm{d}p. \quad (34)$$

From the formulas (34) one can find values of dispersions σ_{qq} and σ_{pp} for stretched states.

$$\sigma_{qq\lambda} = \frac{1}{\lambda^2}\sigma_{qq} + \frac{1 - \lambda^2}{2\lambda^2},$$
$$\sigma_{pp\lambda} = \frac{1}{\lambda^2}\sigma_{pp} + \frac{1 - \lambda^2}{2\lambda^2}, \quad \sigma_{qp\lambda} = \frac{1}{\lambda^2}\sigma_{qp}. \quad (35)$$

We see that for stretched states the Heisenberg uncertainty relation reads

$$\sigma_{qq\lambda}\sigma_{pp\lambda} = \frac{1}{\lambda^4}\left(\sigma_{qq}\sigma_{pp} + \frac{1}{2}\left(1 - \lambda^2\right)\left(\sigma_{qq} + \sigma_{pp}\right) + \frac{1}{4}\left(1 - \lambda^2\right)^2\right) \geqslant \frac{1}{4\lambda^4}\hbar^2, \quad (36)$$

and the Schrödinger–Robertson uncertainty relation for stretched states reads

$$\sigma_{qq\lambda}\sigma_{pp\lambda} - \sigma_{qp\lambda}^2 = \frac{1}{\lambda^4}\left(\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 + \frac{1}{2}\left(1 - \lambda^2\right)\left(\sigma_{qq} + \sigma_{pp}\right) + \frac{1}{4}\left(1 - \lambda^2\right)^2\right) \geqslant \frac{1}{4\lambda^4}\hbar^2. \quad (37)$$

One can interpret the inequalities (36) and (37) in the sense that the scaling transform $(q; p) \to (\lambda q; \lambda p)$ provides an 'effective Planck's constant' value $\hbar_{\mathrm{eff}} = \hbar/\lambda^2$. For $\lambda^2 \ll 1$ the effective Planck's constant satisfies the inequality

$$\hbar_{\mathrm{eff}} \gg \hbar. \quad (38)$$

A similar situation appeared in the case of correlated states [15]. For these states the value of $\hbar_{\mathrm{eff}} = \hbar/\sqrt{1 - r^2}$ depends on the correlation coefficient $r = \sigma_{xp}/\sqrt{\sigma_x \sigma_p}$ between the coordinate and momentum.

The value of Planck's constant $\hbar$ is responsible for purely quantum phenomena such as quantum tunnelling [16]. The well-known quasiclassical formula for the transmission probability through the potential barrier $U(x)$ reads

$$D \approx \exp\left(-\frac{2}{\hbar}\int_a^b \sqrt{2m(U(x) - E)}\,\mathrm{d}x\right). \quad (39)$$

Here m is the mass of particle and E is its energy.

The above formula together with the inequality (38) shows that for the larger constant $\hbar_{zrmeff}$ the quantum tunnelling effect is enhanced. In [17], it was advocated that the transmission probability for the correlated wave packets with the nonzero correlation coefficient r between the coordinate and the momentum can be higher than for uncorrelated packets, and that the increase of this probability can be described by replacing the true Planck's constant $\hbar$ with the effective constant $\hbar_{\mathrm{eff}} = \hbar/\sqrt{1 - r^2}$. This remark has been done in [15] and developed in [18–22].

We believe that, as correlated states, the stretched states can be used to explain some physical phenomena.

The scaling transformation arises in a natural way in a number of physical problems and especially in the problem of the most quiet phase insensitive amplification of a quantum state [23]. In this case, the parameter λ is equal to the inverse value of the coefficient of amplification $G = 1/\lambda$.

5. Stretched fock states

We will now, as an example, apply the general obtained results to the case of the harmonic oscillator.

It was shown in [24] that a Fock state of the harmonic oscillator is transformed under the scale transformation in the mixed state, which is described by the density matrix

$$\hat{\rho}_N = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\frac{(N + k)!}{k!}\left(1 - \lambda^2\right)^k |N + k\rangle\langle N + k|, \quad \lambda^2 < 1. \quad (40)$$

These Fock stretched states consist of pure states $|N + k\rangle$, $k = 0, 1, 2, \ldots\infty$. Every one of these pure states $|N + k\rangle$ is present in the mixed state with the probability

$$c_k^N = \frac{\lambda^{2N+2}(N + k)!}{N!k!}\left(1 - \lambda^2\right)^k. \quad (41)$$

The distribution of pure states is described by a negative binomial distribution [25]

$$f(k, r, p) = \binom{r + k - 1}{k}p^r q^k; \; p + q = 1; \quad k = 0, 1, 2, \ldots \quad (42)$$

Using the properties of this distribution, it is possible to find the average photon number in a stretched state (40)

$$\langle n\rangle = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}(N+k)\frac{(N+k)!}{k!}\left(1-\lambda^2\right)^k = \frac{N+1}{\lambda^2} - 1. \quad (43)$$

And the dispersion of the photon number

$$\sigma_n = \langle n^2\rangle - (\langle n\rangle)^2 = \frac{(N+1)\left(1-\lambda^2\right)}{\lambda^4}. \quad (44)$$

The dispersions σ_{qq}, σ_{pp} and σ_{qp} for the stretched Fock states (40) can be found directly with the help of the expression (40).

$$\langle n|\, q^2\, |n\rangle = n + \frac{1}{2}. \quad (45)$$

$$\sigma_{qq\lambda} = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\left(N+k+\frac{1}{2}\right)\frac{(N+k)!}{k!}\left(1-\lambda^2\right)^k = \left(N+\frac{1}{2}\right) + \frac{1}{\lambda^2}(N+1)\left(1-\lambda^2\right) = \frac{\sigma_{qq}}{\lambda^2} + \frac{1-\lambda^2}{2\lambda^2}. \quad (46)$$

We see that the expression (46) coincides with the expression (35). The same is true for the dispersions σ_{pp} and σ_{qp}.

6. Conclusion

A new method that allows the Q-symbols of operators to be obtained without resorting to the anti-normal-ordering operation has been developed. In order to achieve this aim, an explicit form of coherent states, in terms of which the Husimi function is used, has been constructed. The proposed formalism is based on using the operators $\hat{X}$ and $\hat{P}$, which constitute a Heisenberg algebra, and are a certain generalization of the standard coordinate and momentum operators. We hope that this approach can also be used to construct and analyze other quasiprobability distributions. With the help of this formalism the mean values of some operators are calculated. It is shown how the uncertainty relations are transformed under scale transformations. The right-hand side of these relations can greatly increase. This fact can be used to study the tunnelling effect. We have also found, in an explicit form, the density matrix for the scaling-transformed Husimi functions of Fock states for a harmonic oscillator. These stretched states can be used in the study of the quantum tunnelling phenomenon.

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SCALE TRANSFORMATIONS IN PHASE SPACE AND STRETCHED STATES OF A HARMONIC OSCILLATOR

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We consider scale transformations $(q,p) \to (\lambda q, \lambda p)$ in phase space. They induce transformations of the Husimi functions $H(q,p)$ defined in this space. We consider the Husimi functions for states that are arbitrary superpositions of n-particle states of a harmonic oscillator. We develop a method that allows finding so-called stretched states to which these superpositions transform under such a scale transformation. We study the properties of the stretched states and calculate their density matrices in explicit form. We establish that the density matrix structure can be described using negative binomial distributions. We find expressions for the energy and entropy of stretched states and calculate the means of the number-of-states operator. We give the form of the Heisenberg and Robertson–Schrödinger uncertainty relations for stretched states.

Keywords: phase space, Husimi function, scale transformation, harmonic oscillator, stretched state, uncertainty relation

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1. Introduction

We propose a new method for constructing the quantum states arising in certain physical processes. This method is based on using quasiprobability distributions to describe quantum states. These distributions are defined on the phase space, and physical processes are associated with some transformations of this space. Phase-space transformations induce transformations of the functions defined on them. Determining the physical states corresponding to the transformed quasiprobability distributions, we can find the result

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of the action of a physical process on the initial quantum state. This is the general scheme of our approach. Here, we use Husimi functions as quasiprobability distributions and consider the amplification of quantum states as a physical process. This process can be associated with a scale transformation in the phase space.

It is known that quantum mechanics admits several mutually equivalent formulations. Moreover, one of them can be more convenient for a particular problem, which justifies their joint existence and study. The most popular is the formulation where a quantum state is associated with a vector in a Hilbert space and observable quantities are associated with operators acting in this space. Another also very popular formulation is based on using so-called quasiprobability distributions in phase space. Historically, this approach is related to attempts to use the similarity between the statistical nature of quantum phenomena and classical statistical processes.

In classical statistical mechanics, a physical state is associated with some distribution function $\rho(q,p)$ defined on the phase space of a system. This function is the probability density for the system to be in the state characterized by the parameters q and p. The arguments are the coordinates and momenta, where q is exactly the coordinate of the point in space where the system is at a given instant and p is exactly the momentum which the system has at this instant. Moreover, we assume that the coordinates and momenta can be measured simultaneously. Knowing the distribution function for a system, we can calculate its various characteristics, for instance, the means of certain quantities. For this, a physical quantity is also associated with a corresponding function $F(q,p)$ defined on the phase space, and to find its mean in the state with a distribution function $\rho(q,p)$, we must calculate the integral

$$\langle F\rangle = \int F(q,p)\rho(q,p)\,dq\,dp. \tag{1}$$

From the very beginning of quantum mechanics, there were attempts to regard it as a statistical theory and develop a formalism similar to classical statistical theory, namely, to associate a quantum state with some function $D(q,p)$ that is defined on the phase space and uniquely characterizes the state. Each operator $\hat{A}$ related to an observable quantity can be associated with the function $A_D(q,p)$, which is also defined over the whole phase space such that the mean $\langle\hat{A}\rangle$ of the operator $\hat{A}$ in the state defined by a quasiprobability distribution $D(q,p)$ can be calculated as the integral

$$\langle\hat{A}\rangle = \int A_D(q,p)D_\rho(q,p)\,dq\,dp. \tag{2}$$

The function $D(q,p)$ is called the quasiprobability distribution associated with a given quantum state, and the function $A_D(q,p)$ is the $\hat{A}$-operator symbol, constructed according to the given quasiprobability distribution $D(q,p)$.

There is a set of quasiprobability states in quantum mechanics, and each operator $\hat{A}$ therefore has several symbols. The Wigner $W(q,p)$ [1], Husimi–Kano $Q(q,p)$ [2], [3], and Glauber–Sudarshan $P(q,p)$ [4], [5] functions are the most famous of them. General properties of quasiprobability distributions were studied in [6]–[9], where methods for constructing operator symbols associated with these quasiprobability distributions were also developed. Problems of applying them in quantum optics were considered in [10]–[14].

In addition to various quasiprobability distributions, there is a true probability distribution in quantum mechanics defined on the phase space and completely determining the quantum state. It is called the symplectic tomogram of a quantum state [15]. Properties of tomograms were discussed in [16].

Here, we deal with the Husimi–Kano function $Q(q,p)$, which we call the Husimi function for brevity. Its definition and general properties are presented in Sec. 2. The principal idea of our approach is as follows. There is a set of quantum states for which the exact analytic form of the Husimi functions is known. We can consider a transformation of the phase space (q,p) that induces a transformation of the Husimi functions. As a result, new functions arise that depend on the transformation parameters. We can try to represent these new functions as a sum of already known Husimi functions with coefficients depending on the parameters of the phase-space transformation. Using this sum of Husimi functions, we can then find the density matrix of the transformed state. In several cases, the phase-space transformation can be related to a particular physical process.

We consider a scale transformation in the phase space of the form

$$(q,p) \to (\lambda q, \lambda p), \quad |\lambda|^2 \leq 1. \tag{3}$$

It was proved in [17] that if $Q(q,p)$ is a Husimi function of a quantum state and $\lambda < 1$, then

$$Q_\lambda(q,p) = \lambda^2 Q(\lambda q, \lambda p) \tag{4}$$

is also a Husimi function for some quantum state.

In Sec. 3, we show that transformation (4) of the Husimi function naturally arises in quantum optics problems. This stimulates our interest in this problem. We study what happens with the states of a harmonic oscillator under such a transformation.

In Sec. 4, we consider the pure state that is an arbitrary superposition of n-particle states. We show that a pure state becomes a mixed state as a result of this transformation, and we find its density matrix. This mixed λ state contains an infinite set of pure states, and the probabilities with which these pure states are included into a mixed state form a negative binomial distribution.

In the case where we deal with an initial single N-particle state, the transformed λ state contains all M-particle states with $M \geq N$. Moreover, the less the parameter λ^2 is, the smoother the distribution of the given pure states in the mixed state. Roughly speaking, we can assume that under λ transformation (3), states with $M > N$ arise from the state $|N\rangle$. We call such mixed states stretched states. In Sec. 5, we find the means of the particle number operator for stretched states. In Sec. 6, we calculate the entropy of the von Neumann stretched states. In Sec. 7, we find the forms of the Heisenberg and Robertson–Schrödinger uncertainty relations for stretched states. For such states, we show that the factor λ^{-4} appears in the right-hand side of the uncertainty relations, i.e., the uncertainty of states increases under λ transformation (3). We briefly discuss the possible physical effects of this fact.

Everywhere in this paper, we assume that $|\lambda| < 1$. But the question of what happens and what states can appear for $|\lambda| > 1$ arises. In Sec. 8, we formally apply the argumentation scheme used for $|\lambda| < 1$ to the case $|\lambda| > 1$. We show that in this case, the Husimi functions of n-particle states of the harmonic oscillator are transformed into quantities that are not the Husimi functions of any quantum states.

2. Husimi functions of harmonic-oscillator states

We consider the one-dimensional harmonic oscillator. Its Hamiltonian $\widehat{H}$ is defined as

$$\widehat{H} = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + \frac{\omega^2}{2}x^2 = \frac{1}{2}(\hat{p}^2 + \omega^2\hat{q}^2) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{5}$$

where the coordinate and momentum operators $\hat{q}$ and $\hat{p}$ and also the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ are

$$\begin{aligned} &\hat{p} = -i\hbar\frac{d}{dx}, \qquad \hat{q} = x, \\ &\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\hat{p} + i\omega\hat{q}), \qquad \hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\hat{p} - i\omega\hat{q}). \end{aligned} \tag{6}$$

For the Hamiltonian H, the n-particle states $|n\rangle$ are its eigenfunctions:

$$\widehat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle. \tag{7}$$

The coherent states of the harmonic oscillator are

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle. \tag{8}$$

Here, α is an arbitrary complex number.

Let there be a quantum state defined by the density operator $\hat{\rho}$. Using coherent states (8), we can then construct its Husimi function:

$$Q(\alpha, \alpha^*) = \frac{1}{\pi}\int \langle\alpha|x\rangle\rho(x,y)\langle y|\alpha\rangle\,dx\,dy. \tag{9}$$

If the quantum state is pure and is described by the wave function $|\psi\rangle$, then its Husimi function is

$$Q(q,p) = \langle\alpha|\psi\rangle\langle\psi|\alpha\rangle. \tag{10}$$

The Husimi function is defined on the phase space with the coordinates (q,p). The scale transformation $(q,p) \to (\lambda q, \lambda p)$ in this space was considered in [17]. It was shown that if $Q(q,p)$ is a Husimi function of a quantum state, then $\lambda^2 Q(\lambda q, \lambda p)$ is also a Husimi function of some quantum state if $|\lambda|^2 \leq 1$.

Below, we show that transformation (3) can be associated with certain physical processes, for instance, with the state of an electromagnetic field passing through a quantum amplifier [18], [19]. Therefore, the problem of constructing an explicit form of such transformations for a particular state and studying their properties is relevant. In several cases, this problem can be solved exactly.

Here, we consider a harmonic oscillator and find the density matrices of those states into which the superpositions of its n-particle states are mapped under scale transformation (3). For this, we use a special method based on some properties of Husimi functions of such states.

3. Connection of scale transformations with quantum optics problems

Before proceeding to a systematic development of the formalism, we show the relation of this approach to quantum optics problems and explain how scale transformation (3) appears in such a formulation of the problem. The general idea can be understood using an example of a simple linear light amplifier consisting of partially inverted two-level atoms. The resonance Hamiltonian for the interaction between the field and the atoms is

$$\widehat{H} = \hbar k \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{pmatrix}. \tag{11}$$

This is an interaction Hamiltonian in the Jaynes–Cummings model. It has several interesting properties, in particular, supersymmetry [20]. For this Hamiltonian, the equation for the density matrix $\hat{\rho}$ of an electromagnetic field can be written in the first approximation as [13]

$$\frac{\partial\hat{\rho}}{\partial t} = -kN_1(\hat{a}\hat{a}^\dagger\hat{\rho} - 2\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger) - kN_2(\hat{a}^\dagger\hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}). \tag{12}$$

Here, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the electromagnetic field, N_1 and N_2 are the populations of the upper and lower levels of a two-level atoms, and k is the amplification coefficient.

Using relation (9) between the density matrix and the Husimi function, we can pass from operator equation (12) to an ordinary differential equation for the Husimi function. Using this equation, the expression for the Husimi function for the state at the exit from a quantum amplifier was obtained in [18]:

$$Q_{\rm out}(\alpha,t)=\frac{1}{G^2}Q_{\rm in}\left(\frac{\alpha}{G}\right)=\left\langle\frac{\alpha}{G}\middle|\hat\rho_{\rm in}\middle|\frac{\alpha}{G}\right\rangle, \tag{13}$$

where

$$G(t)=e^{2(N_1-N_2)kt}. \tag{14}$$

We see that expression (13) coincides with (4) at $\lambda=G^{-1}$. Hence, scale transformation (3) in the phase space turns out to be related to the action of a quantum amplifier, and the form of this transformation is defined by the structure of Hamiltonian (11). Accordingly, the action of the amplifier on an arbitrary quantum state can be described using the scale transformation in the phase space.

Formally, we here deal with only the states of a harmonic oscillator, but keeping in mind that the method developed here is assumed to be applicable to quantum optics problems, we sometimes call these states n-photon or n-particle Fock states.

4. Density matrices of stretched states

We first consider the N-particle state $|N\rangle$. Its Husimi function is

$$Q_N(q,p)=\langle\alpha|N\rangle\langle N|\alpha\rangle=e^{-|\alpha|^2}\frac{|\alpha|^{2N}}{N!}. \tag{15}$$

After scale transformation (3), we have

$$Q_N^\lambda(q,p)=\lambda^2e^{-\lambda^2|\alpha|^2}\frac{\lambda^{2N}|\alpha|^{2N}}{N!}. \tag{16}$$

We want to represent expression (16) as a sum of Husimi functions $Q_j(q,p)$ with different j. For this, we write it as

$$Q_N^\lambda(q,p)=e^{-\lambda^2|\alpha|^2}\frac{\lambda^{2N+2}|\alpha|^{2N}}{N!}=e^{-|\alpha|^2}e^{(1-\lambda^2)|\alpha|^2}\frac{\lambda^{2N+2}|\alpha|^{2N}}{N!}. \tag{17}$$

Expanding the exponential $e^{(1-\lambda^2)|\alpha|^2}$ in a series, we now obtain

$$\begin{aligned}Q_N^\lambda(q,p)&=e^{-|\alpha|^2}\sum_{j=0}^\infty\frac{(1-\lambda^2)^j|\alpha|^{2j}}{j!}\frac{\lambda^{2N+2}|\alpha|^{2N}}{N!}=\\&=\sum_{j=0}^\infty\lambda^{2N+2}\frac{(1-\lambda^2)^j(N+j)!}{j!\,N!}e^{-|\alpha|^2}\frac{|\alpha|^{2(N+j)}}{(N+j)!}=\\&=\sum_{j=0}^\infty\lambda^{2N+2}\frac{(1-\lambda^2)^j(N+j)!}{j!\,N!}Q_{N+j}(q,p).\end{aligned} \tag{18}$$

Passing from the Husimi function to the density matrix, we find that state (16) corresponds to the density matrix

$$\hat\rho_N^\lambda=\sum_{j=0}^\infty\lambda^{2N+2}\frac{(1-\lambda^2)^j(N+j)!}{j!\,N!}|N+j\rangle\langle N+j|. \tag{19}$$

This density matrix has a diagonal form with the first N diagonal elements (with the labels $0, 1, \ldots, N-1$) equal to zero and the other main diagonal elements

$$F_{N+j}^{N} = \frac{(1-\lambda^2)^j (N+j)!}{j!\, N!}\lambda^{2N+2}, \quad j = 0, 1, \ldots. \tag{20}$$

Quantities (20) form a negative binomial distribution. The elements of this distribution are given by [21]

$$f(k,r,p) = \binom{r+k-1}{k} p^r q^k = \frac{(r+k-1)!}{(r-1)!\, k!} p^r (1-p)^k, \quad k = 0, 1, 2, \ldots. \tag{21}$$

They are determined by two parameters r and p, $q = 1-p$, and k is the element label in the distribution. In our case, $r = N+1$, $k = j$, and $p = \lambda^2$. We therefore have

$$F_{N+j}^{N} = f(j, N+1, \lambda^2). \tag{22}$$

We now consider a state that is a superposition of two k-particle states $|M\rangle$ and $|N\rangle$. Its wave function and density matrix are

$$\begin{aligned} &\psi_{M,N} = c_M|M\rangle + c_N|N\rangle, \qquad |c_M|^2 + |c_N|^2 = 1, \\ &\hat{\rho}_{M,N} = (c_M|M\rangle + c_N|N\rangle)(c_M^*\langle M| + c_N^*\langle N|). \end{aligned} \tag{23}$$

We find the density matrix of the stretched state $\hat{\rho}_{M,N}^{\lambda}$. The Husimi function of state (23) can be written as

$$\begin{aligned} Q_{M,N}(\alpha) &= \langle\alpha|\rho_{M,N}|\alpha\rangle = \\ &= e^{-|\alpha|^2/2}\left(c_M \frac{(\alpha^*)^M}{\sqrt{M!}} + c_N \frac{(\alpha^*)^N}{\sqrt{N!}}\right) e^{-|\alpha|^2/2}\left(c_M^* \frac{\alpha^M}{\sqrt{M!}} + c_N^* \frac{\alpha^N}{\sqrt{N!}}\right). \end{aligned} \tag{24}$$

After transformation (3), we have

$$\begin{aligned} Q_{M,N}^{\lambda}(\lambda q, \lambda p) &= \lambda^2 e^{-\lambda^2|\alpha|^2}\left(c_M \frac{\lambda^M(\alpha^*)^M}{\sqrt{M!}} + c_N \frac{\lambda^N(\alpha^*)^N}{\sqrt{N!}}\right)\left(c_M^* \frac{\lambda^M\alpha^M}{\sqrt{M!}} + c_N^* \frac{\lambda^N\alpha^N}{\sqrt{N!}}\right) = \\ &= e^{-|\alpha|^2}\lambda^2 e^{(1-\lambda^2)|\alpha|^2}\left(c_M \frac{\lambda^M(\alpha^*)^M}{\sqrt{M!}} + c_N \frac{\lambda^N(\alpha^*)^N}{\sqrt{N!}}\right)\left(c_M^* \frac{\lambda^M\alpha^M}{\sqrt{M!}} + c_N^* \frac{\lambda^N\alpha^N}{\sqrt{N!}}\right). \end{aligned} \tag{25}$$

As in the previous case, we expand the exponential $e^{(1-\lambda^2)|\alpha|^2}$ in a series, and Husimi function (25) then becomes

$$Q_{M,N}^{\lambda}(\lambda q, \lambda p) = e^{-|\alpha|^2}\lambda^2 \sum_{j=0}^{\infty} \frac{(1-\lambda^2)^j|\alpha|^{2j}}{j!}\left(c_M \frac{\lambda^M(\alpha^*)^M}{\sqrt{M!}} + c_N \frac{\lambda^N(\alpha^*)^N}{\sqrt{N!}}\right)\left(c_M^* \frac{\lambda^M\alpha^M}{\sqrt{M!}} + c_N^* \frac{\lambda^N\alpha^N}{\sqrt{N!}}\right).$$

We now take the relations

$$e^{-|\alpha|^2/2}\frac{\alpha^{s+j}}{\sqrt{(s+j)!}} = \langle s+j|\alpha\rangle, \qquad e^{-|\alpha|^2/2}\frac{(\alpha^*)^{k+j}}{\sqrt{(k+j)!}} = \langle\alpha|k+j\rangle \tag{26}$$

into account. Using them, we obtain the expression for Husimi function (25):

$$\begin{aligned} Q_{N,M}^{\lambda}(\lambda q, \lambda p) = \sum_{j=0}^{\infty} \frac{\lambda^2(1-\lambda^2)^j}{j!}\langle\alpha|\left(\sqrt{\frac{(M+j)!}{M!}}\lambda^M c_M|M+j\rangle + \sqrt{\frac{(N+j)!}{N!}}\lambda^N c_N|N+j\rangle\right) \times \\ \times\left(\sqrt{\frac{(M+j)!}{M!}}\lambda^M c_M^*\langle M+j| + \sqrt{\frac{(N+j)!}{N!}}\lambda^N c_N^*\langle N+j|\right)|\alpha\rangle. \end{aligned} \tag{27}$$

It hence follows that the density matrix of the stretched state $\hat{\rho}^{\lambda}_{M,N}$ is

$$\hat{\rho}^{\lambda}_{N,M} = \sum_{j=0}^{\infty} \frac{\lambda^2(1-\lambda^2)^j}{j!} \left(\sqrt{\frac{(M+j)!}{M!}} \lambda^M c_M |M+j\rangle + \sqrt{\frac{(N+j)!}{N!}} \lambda^N c_N |N+j\rangle \right) \times$$
$$\times \left(\sqrt{\frac{(M+j)!}{M!}} \lambda^M c^*_M \langle M+j| + \sqrt{\frac{(N+j)!}{N!}} \lambda^N c^*_N \langle N+j| \right). \qquad (28)$$

We consider the structure of this matrix.

Density matrix (28) has three nonzero diagonals. Let $M = N + k$. Then the first N elements with the labels $0, 1, \ldots, N-1$ on the main diagonal are equal to zero. The next k diagonal elements with the labels $N, \ldots, N+k-1$ coincide with the first k nonzero diagonal elements of density matrix (19),

$$D_{N+j,N+j} = \frac{(1-\lambda^2)^j (N+j)!}{j!\, N!} \lambda^{2+2N} |c_N|^2, \quad j = 0, 1, \ldots, k-1. \qquad (29)$$

The remaining elements on the main diagonal of density matrix (28) have the form of the sum of the diagonal elements of the density matrices ρ^{λ}_N and ρ^{λ}_M, which are given by formula (19):

$$D_{N+k+j,N+k+j} = \frac{(1-\lambda^2)^{k+j} (N+k+j)!}{(k+j)! N!} \lambda^{2+2N} |c_N|^2 +$$
$$+ \frac{(1-\lambda^2)^j (M+j)!}{j! M!} \lambda^{2+2M} |c_M|^2, \quad j = 0, 1, \ldots. \qquad (30)$$

Therefore, the main diagonal of the density matrix $\hat{\rho}^{\lambda}_{N,M}$ given by (28) has the form of the sum of the main diagonals of the density matrices $\hat{\rho}^{\lambda}_N$ and $\hat{\rho}^{\lambda}_M$ multiplied by the coefficients $|c_N|^2$ and $|c_M|^2$.

In addition to the main diagonal, density matrix (28) has nonzero elements on two more diagonals, above and below the main diagonal at a distance of k steps. These nonzero elements are located with the coordinates $(N+i, N+k+i)$ and $(N+k+i, N+i)$, $i = 0, 1, \ldots$. The elements of the density matrix with the coordinates $(N+j, M+j)$ and $(M+j, N+j)$ have the forms

$$\begin{aligned} D_{N+j,M+j} &= \frac{(1-\lambda^2)^j}{j!} \lambda^{N+M+2} \sqrt{\frac{(N+j)!\,(M+j)!}{N!\,M!}} c_N c^*_M, \\ D_{M+j,N+j} &= \frac{(1-\lambda^2)^j}{j!} \lambda^{N+M+2} \sqrt{\frac{(N+j)!\,(M+j)!}{N!\,M!}} c^*_N c_M, \end{aligned} \qquad j = 0, 1, \ldots. \qquad (31)$$

We can now consider a general case of an arbitrary state of the harmonic oscillator. The strategy remains the same as in the case of state (19), which is a superposition of two Fock states.

We consider a pure state that is an arbitrary superposition of n-particle states of a harmonic oscillator:

$$|\psi\rangle_{\Sigma} = \sum_{k=0}^{\infty} c_k |k\rangle, \qquad \sum_{k=0}^{\infty} |c_k|^2 = 1. \qquad (32)$$

Its Husimi function is

$$Q_{\Sigma}(q,p) = \langle\alpha|\psi\rangle\langle\psi|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{k=0}^{\infty} \frac{(\alpha^*)^k}{\sqrt{k!}} c_k e^{-|\alpha|^2/2} \sum_{s=0}^{\infty} \frac{\alpha^s}{\sqrt{s!}} c^*_s. \qquad (33)$$

After a λ-scale transformation, Husimi function (33) becomes

$$Q_\Sigma^\lambda(q,p) = \lambda^2 Q(\lambda q, \lambda p) = \lambda^2 e^{-\lambda^2|\alpha|^2} \sum_{k=0}^{\infty} \frac{\lambda^k(\alpha^*)^k}{\sqrt{k!}} c_k \sum_{s=0}^{\infty} \frac{\lambda^s \alpha^s}{\sqrt{s!}} c_s^* =$$
$$= e^{-|\alpha|^2} \lambda^2 e^{(1-\lambda^2)|\alpha|^2} \sum_{k=0}^{\infty} \frac{\lambda^k(\alpha^*)^k}{\sqrt{k!}} c_k \sum_{s=0}^{\infty} \frac{\lambda^s \alpha^s}{\sqrt{s!}} c_s^*. \tag{34}$$

We expand the exponential $e^{(1-\lambda^2)|\alpha|^2}$ in a series, and expression (34) then becomes

$$\lambda^2 Q_\Sigma(\lambda q, \lambda p) = e^{-|\alpha|^2} \lambda^2 \sum_{j=0}^{\infty} \frac{(1-\lambda^2)^j |\alpha|^{2j}}{j!} \sum_{k=0}^{\infty} \frac{\lambda^k(\alpha^*)^k}{\sqrt{k!}} c_k \sum_{s=0}^{\infty} \frac{\lambda^s \alpha^s}{\sqrt{s!}} c_s^*. \tag{35}$$

It can be written as

$$\lambda^2 Q_\Sigma(\lambda q, \lambda p) = e^{-|\alpha|^2} \sum_{j=0}^{\infty} \lambda^2 \frac{(1-\lambda^2)^j}{j!} \sum_{k=0}^{\infty} \frac{\lambda^k(\alpha^*)^{k+j}}{\sqrt{k!}} c_k \sum_{s=0}^{\infty} \frac{\lambda^s \alpha^{s+j}}{\sqrt{s!}} c_s^*. \tag{36}$$

As before, we must now take relations (26) into account. Using them, we write equality (36) in the form

$$\lambda^2 Q_\Sigma(\lambda q, \lambda p) = \sum_{j=0}^{\infty} \lambda^2 \frac{(1-\lambda^2)^j}{j!} \sum_{k=0}^{\infty} \sqrt{\frac{(k+j)!}{k!}} \lambda^k c_k \langle \alpha | k+j \rangle \sum_{s=0}^{\infty} \sqrt{\frac{(s+j)!}{s!}} \lambda^s c_s^* \langle s+j | \alpha \rangle. \tag{37}$$

We see that expression (37) is the Husimi function of a state with the density matrix

$$\hat\rho_\Sigma^\lambda = \sum_{j=0}^{\infty} \frac{\lambda^2(1-\lambda^2)^j}{j!} \left(\sum_{k=0}^{\infty} \sqrt{\frac{(k+j)!}{k!}} \lambda^k c_k |k+j\rangle \right) \left(\sum_{s=0}^{\infty} \sqrt{\frac{(s+j)!}{s!}} \lambda^s c_s^* \langle s+j| \right). \tag{38}$$

We find that after λ-scale transformation (3), pure state (32) becomes a mixed state described by density matrix (38).

We now consider the structure of density matrix (38). For this, as an example, we use density matrix (28) corresponding to the stretched state obtained from a state that is a superposition of two Fock states. For such a state, the main diagonal of the density matrix has the form of the sum of the main diagonals of the density matrices corresponding to the single Fock states, and the elements of each of these matrices are multiplied by the square of the absolute value of the coefficient with which this single state is included in the superposition.

If state (32) is a superposition of more than two n-particle states, then the main diagonal of density matrix (38) of the corresponding stretched state has a similar structure. Namely, it has the form of a sum of the main diagonals of the density matrices corresponding to the single n-particle states, and elements of each of them are multiplied by $|c_k|^2$, the square of the absolute value of the coefficient with which this single state $|k\rangle$ is included in superposition (32).

We introduce the notation

$$\begin{aligned} d_{N,j} &= \lambda^{N+1} \sqrt{\frac{(1-\lambda^2)^j (N+j)!}{N!\, j!}} c_N, \\ d_{N,j}^* &= \lambda^{N+1} \sqrt{\frac{(1-\lambda^2)^j (N+j)!}{N!\, j!}} c_N^*, \end{aligned} \qquad j = 0, 1, \ldots. \tag{39}$$

With this notation, the elements on the main diagonal of density matrix (28) become

$$D_{n,n} = \sum_{i=0}^{n} |d_{i,n-i}|^2, \quad i = 0, 1, \ldots, n, \quad n = 0, 1, \ldots. \tag{40}$$

In addition to the main diagonal, there are nonzero elements of density matrix (38) on other diagonals parallel to the main one. Their structure is analogous to the structure of the secondary diagonals of density matrix (28). We describe elements on these diagonals. For this, we use notation (31). In fact, these elements are sums of quantities of type (31).

We first consider the diagonal nearest to the main diagonal of matrix (28) and located above it. Its elements have the coordinates $(n, n+1)$, $n = 0, 1, 2, \ldots$ and are expressed as

$$D_{n,n+1} = \sum_{j=0}^{n} d_{j,n-j} d^*_{j+1,n-j}. \tag{41}$$

The summation limits in this formula are defined by the k-particle states included in superposition state (32). If state (32) contains a finite number of k-particle states and the highest of them is the state $|K\rangle$, then formula (41) becomes

$$D_{n,n+1} = \begin{cases} \displaystyle\sum_{j=0}^{n} d_{j,n-j} d^*_{j+1,n-j}, & n < K, \\ \displaystyle\sum_{j=0}^{K-1} d_{j,n-j} d^*_{j+1,n-j}, & n \geq K. \end{cases} \tag{42}$$

The values of the elements on other diagonals above the main diagonal are defined as

$$D_{n,n+k} = \begin{cases} \displaystyle\sum_{j=0}^{n} d_{j,n-j} d^*_{j+k,n-j}, & n \leq K-k, \\ \displaystyle\sum_{j=0}^{K-k} d_{j,n-j} d^*_{j+k,n-j}, & n \geq K-k. \end{cases} \tag{43}$$

If some terms in state (32) are absent, i.e., some coefficients $c_k = 0$, then the corresponding quantities $d_{k,j} = 0$, $j = 0, 1, \ldots$, and the corresponding terms in sum (39) are absent. The values of the elements of the density matrix below the main diagonal are defined by the Hermitian property:

$$D_{n+k,n} = D^*_{n,n+k}. \tag{44}$$

This is the structure of density matrix (38) of the stretched state corresponding to state (32). We study some of its properties below, but we first note the following fact. The elements on the main diagonal of matrix (38) are linear combinations of terms from negative binomial distributions (21). In our case, these distributions differ one from another by the parameter N, while the parameter λ is the same in all of them. The elements on the diagonals above and below the main diagonal are linear combinations of the distribution terms of the form

$$F(N, M; \lambda)_j = \frac{(1-\lambda^2)^j}{j!} \lambda^{N+M+2} \sqrt{\frac{(N+j)!(M+j)!}{N!\,M!}}, \quad j = 0, 1, \ldots. \tag{45}$$

Distributions (45) are characterized by the three parameters N, M, and λ, where N and M are integers and $\lambda^2 \leq 1$. For $N = M$, they become the usual negative binomial distributions. We call distributions (45) double negative binomial distributions. Their properties will be studied in another paper.

5. Means of the state number operator

We now study some properties of stretched states. First, we find the means of the particle number of stretched state (38). We do this for state (21) first.

The mean of particles in the state given by the density matrix ρ is defined by the expression

$$\langle \hat{n} \rangle = \mathrm{Tr}(\hat{N}\hat{\rho}), \tag{46}$$

where $\hat{N} = \hat{a}^+\hat{a}$ is the particle number operator.

For state (21), we have

$$\langle \hat{n}_N \rangle = \sum_{j=0}^{\infty} \frac{\lambda^2(1-\lambda^2)^j}{j!}\left(\frac{(N+j)!}{N!}\lambda^{2N}(N+j)\right). \tag{47}$$

It is known that we have the relation

$$S_N^0(x) = \sum_{j=0}^{\infty} \frac{(N+j)!}{j!}x^j = \frac{N!}{(1-x)^{N+1}}, \quad |x| < 1. \tag{48}$$

Substituting $x = 1-\lambda^2$ in it, we obtain

$$\sum_{j=0}^{\infty} \lambda^{2N+2}\frac{(N+j)!}{N!\,j!}(1-\lambda^2)^j = 1. \tag{49}$$

We now calculate the quantity

$$S_N^1(x) = \sum_{j=0}^{\infty} j\frac{(N+j)!}{j!}x^j = x\frac{d}{dx}S_N^0(x) = \frac{x(N+1)!}{(1-x)^{N+2}}. \tag{50}$$

Using relations (49) and (50), we obtain the expression for the mean of the particle number in stretched state (21)

$$\langle \hat{n}_N \rangle = N + (S_N^0)^{-1}S_N^1 = N + \frac{(N+1)(1-\lambda^2)}{\lambda^2} = \frac{N+1}{\lambda^2} - 1. \tag{51}$$

We now find an expression for the dispersion of the particle number in state (21), which is defined as

$$\sigma_n = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2. \tag{52}$$

To calculate its value, we need the mean $\langle \hat{n}_N^2 \rangle$. It can be found:

$$\langle \hat{n}_N^2 \rangle = \sum_{j=0}^{\infty} \lambda^{2N+2}\frac{(1-\lambda^2)^j}{j!}\frac{(N+j)!}{N!}(N+j)^2 =$$
$$= N^2 + 2N\langle n_N \rangle + \sum_{j=0}^{\infty} \lambda^{2N+2}\frac{(1-\lambda^2)^j}{j!}\frac{(N+j)!}{N!}j^2. \tag{53}$$

We next have

$$S_N^2(x) = \sum_{i=0}^{\infty} i^2\frac{(N+i)!}{i!}x^i =$$
$$= x\frac{d}{dx}S_1 = \left(x\frac{d}{dx}\right)^2 S_0 = \frac{x(N+1)}{1-x} + \frac{x^2(N+1)(N+2)}{(1-x)^2}. \tag{54}$$

Hence, using relations (51), we obtain the expression for the dispersion of the particle number in state (21):

$$\sigma_n = \langle \hat{n}_N^2 \rangle - \langle \hat{n}_N \rangle^2 = \frac{x(N+1)}{(1-x)^2} = \frac{(N+1)(1-\lambda^2)}{\lambda^4}. \tag{55}$$

We have found the quantities $\langle \hat{n}_N \rangle$ and $\langle \hat{n}_N^2 \rangle$. We now obtain $\langle \hat{n}_N^l \rangle$ for state (21). In this case, we have

$$\langle \hat{n}_N^l \rangle = \sum_{j=0}^{\infty} \lambda^{2N+2} \frac{(1-\lambda^2)^j}{j!} \frac{(N+j)!}{N!} (N+j)^l =$$
$$= \sum_{j=0}^{\infty} F_{N+j}^N \sum_{k=0}^{l} \binom{l}{k} j^k N^{l-k} = \sum_{k=0}^{l} G_N^k(\lambda) \binom{l}{k} N^{l-k}. \tag{56}$$

Here, $\binom{l}{k} = \frac{l!}{(l-k)!\,k!}$ are the usual binomial coefficients, and the quantities $G_N^k(\lambda)$ are

$$G_N^k(\lambda) = \sum_{j=0}^{\infty} \lambda^{2N+2} \frac{(1-\lambda^2)^j}{j!} \frac{(N+j)!}{N!} j^k. \tag{57}$$

We calculate these sums. As shown above, equality (48) holds. Using it, we obtain

$$\begin{aligned} S_N^1(x) &= \sum_{i=0}^{\infty} i \frac{(N+i)!}{i!} x^i = x \frac{d}{dx} S_0, \\ S_N^2(x) &= \sum_{i=0}^{\infty} i^2 \frac{(N+i)!}{i!} x^i = x \frac{d}{dx} S_1 = \left(x \frac{d}{dx} \right)^2 S_0, \\ &\vdots \\ S_N^k(x) &= \sum_{i=0}^{\infty} i^k \frac{(N+i)!}{i!} x^i = \left(x \frac{d}{dx} \right)^k S_0, \\ &\vdots \end{aligned} \tag{58}$$

Using these relations, we obtain expressions for the $G_N^k(\lambda)$:

$$G_N^k(\lambda) = (S_N^0(1-\lambda^2))^{-1} S_N^k(1-\lambda^2). \tag{59}$$

Hence, we have the formula

$$\langle \hat{n}^l \rangle = \sum_{k=0}^{l} (S_N^0(1-\lambda^2))^{-1} S_N^k(1-\lambda^2) \binom{l}{k} N^{l-k}. \tag{60}$$

We note that both positive and negative binomial coefficients are used to construct the $\langle \hat{n}_N^l \rangle$.

We now calculate the values of quantities (51) and (60) for the stretched states with density matrix (38). We first do this for state (28). In this case, the mean particle number $\langle \hat{n}_{N,V} \rangle$ is

$$\langle \hat{n}_{N,V} \rangle = \sum_{j=0}^{\infty} \frac{\lambda^2(1-\lambda^2)^j}{j!} \left(\frac{(N+j)!}{N!} \lambda^{2N} |c_N|^2 (N+j) + \frac{(M+j)!}{M!} \lambda^{2M} |c_M|^2 (M+j) \right) =$$
$$= \frac{1}{\lambda^2} \big(|c_N|^2 N + |c_M|^2 M \big) + \frac{1}{\lambda^2} - 1, \qquad |c_N|^2 + |c_M|^2 = 1. \tag{61}$$

The mean of an arbitrary power of the particle number operator $\langle(\hat{n}_{N,V})^l\rangle$ for state (28) is

$$\langle(\hat{n}_{N,V})^l\rangle = \sum_{j=0}^{\infty} \frac{\lambda^2(1-\lambda^2)^j}{j!}\left(\frac{(N+j)!}{N!}\lambda^{2N}|c_N|^2(N+j)^l + \frac{(M+j)!}{M!}\lambda^{2M}|c_M|^2(M+j)^l\right) =$$

$$= \frac{1}{\lambda^2}\big(|c_N|^2\langle\hat{n}_N^l\rangle + |c_M|^2\langle\hat{n}_M^l\rangle\big) + \frac{1}{\lambda^2} - 1, \qquad |c_N|^2 + |c_M|^2 = 1. \tag{62}$$

In the case of state (38) of general form, the quantities $\langle\hat{n}_\Sigma\rangle$ and $\langle(\hat{n}_\Sigma)^l\rangle$ can be written as

$$\begin{aligned} \langle\hat{n}_\Sigma\rangle &= \frac{1}{\lambda^2}\sum_{k=0}^{\infty}|c_k|^2 k + \frac{1}{\lambda^2} - 1, \\ \langle(\hat{n}_\Sigma)^l\rangle &= \frac{1}{\lambda^2}\sum_{k=0}^{\infty}|c_k|^2\langle\hat{n}_k^l\rangle + \frac{1}{\lambda^2} - 1, \end{aligned} \qquad \sum_{k=0}^{\infty}|c_k|^2 = 1. \tag{63}$$

We see that formulas (63) for the means of powers of the particle number operator for state (38) are the sums of similar expressions for state (21). This can be explained because only the diagonal elements of density matrix (38) are included in formulas (63) and these elements equal the sum of diagonal elements of density matrices (21) for the stretched states $|k\rangle$ multiplied by the squares of the absolute values of the coefficients c_k with which they are included in state (32).

6. Entropy of stretched states

We now consider the von Neumann entropy H of stretched states. Let there be a state defined by a density matrix $\hat{\rho}$. Then the quantity H is

$$H = -\sum_k \lambda_k \log_2 \lambda_k. \tag{64}$$

Here, λ_k is an eigenvalue of the density matrix ρ. Moreover, we assume that $0\cdot\log_2 0 = 0$. It follows from this definition that the entropy of a pure state is equal to zero because the idempotency of the density matrix of the pure state $\hat{\rho}^2 = \hat{\rho}$ implies that some of its eigenvalues are unity and the others are zero.

If we have a mixed state, then its entropy is defined by the distribution of the λ_k. The value λ_k defines the probability of finding the system in the state associated with the given eigenvalue of the density matrix. It is easy to understand that the more uniformly the probabilities λ_k are distributed, the greater entropy (64) is, and it reaches its maximum when these probabilities are equal to each other. In the case where sum (64) has N terms, the maximum of the von Neumann entropy H is at $\lambda_k = 1/N$ and is $H = \log_2 N$.

We now consider the stretched states corresponding to the N-particle state of the harmonic oscillator. Its density matrix is

$$\hat{\rho}_N = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\frac{(N+k)!}{k!}(1-\lambda^2)^k|N+k\rangle\langle N+k|, \quad \lambda^2 < 1. \tag{65}$$

Because this matrix is diagonal, its elements

$$c_k^N = \frac{\lambda^{2N+2}(N+k)!}{N!\,k!}(1-\lambda^2)^k \tag{66}$$

are its eigenvalues, and they can be directly substituted in expression (64).

Therefore, to find the von Neumann entropy of state (65), we must calculate the quantity

$$H_N = -\sum_{k=0}^{\infty} c_k^N \log_2 c_k^N = \\ = -\sum_{k=0}^{\infty} \frac{\lambda^{2N+2}(N+k)!}{N!\,k!}(1-\lambda^2)^k \log_2\left(\frac{\lambda^{2N+2}(N+k)!}{N!\,k!}(1-\lambda^2)^k\right). \tag{67}$$

If $N=0$, then we have

$$H_0 = -\lambda^2 \sum_{k=0}^{\infty} (1-\lambda^2)^k \log_2(\lambda^2(1-\lambda^2)^k) = -\log_2 \lambda^2 - \log_2(1-\lambda^2)\frac{1-\lambda^2}{\lambda^2}. \tag{68}$$

If $\lambda \to 1$, then $H_0 \to 0$, and this corresponds to the fact that the entropy of a pure N-particle state is zero. If $\lambda \to 0$, then $H_0 \to \infty$. This means that the uncertainty in the probability of detecting an arbitrary n-particle state increases.

If there is a stretched state corresponding to superposition (32) of n-particle states of a harmonic oscillator, then its density matrix (38) is not diagonal, and we must find its eigenvalues to calculate the entropy of such a state.

7. Uncertainty relation for stretched states

In the preceding section, we calculated the means of the particle number operator and its powers for the stretched state obtained from an arbitrary superposition of k-particle Fock states of a harmonic oscillator. In this section, we consider some other operators and present the form of the Heisenberg and Robertson–Schrödinger uncertainty relations for stretched states. In [22], a new method for constructing the Husimi symbols was proposed. It is especially effective for an operator that is polynomial in the coordinate and momentum operators $\hat{q}$ and $\hat{p}$. These are the operators we treat here.

We first consider the harmonic oscillator Hamiltonian

$$\widehat{H} = \frac{\hbar\omega}{2}\big(\hat{q}^2 + \hat{p}^2\big). \tag{69}$$

Its Husimi symbol is

$$K_{\mathrm{H}}(q,p) = q^2 + p^2 - 1. \tag{70}$$

The mean energy $\bar{E}$ of the state characterized by the Husimi function $Q(q,p)$ is

$$\bar{E} = \int \frac{\hbar\omega}{2}(q^2+p^2-1)Q(q,p)\,dq\,dp = \int \frac{\hbar\omega}{2}(q^2+p^2)Q(q,p)\,dq\,dp - \frac{\hbar\omega}{2}. \tag{71}$$

We now find the mean energy of the stretched state corresponding to the state with the Husimi function $Q(q,p)$. The Husimi function of the stretched state can be written as $Q_\lambda(q,p) = \lambda^2 Q(\lambda q, \lambda p)$, and the mean energy $\bar{E}_\lambda$ of the state with such a Husimi function is defined by

$$\bar{E}_\lambda = \int K_{\mathrm{H}}(q,p) Q_\lambda(q,p)\,dq\,dp = \\ = \int \frac{1}{\lambda^2}\frac{\hbar\omega}{2}\big((\lambda q)^2 + (\lambda p)^2 - 1\big)Q(\lambda q, \lambda p)\,d(\lambda q)\,d(\lambda p) = \frac{1}{\lambda^2}\bar{E} + \frac{1-\lambda^2}{\lambda^2}\frac{\hbar\omega}{2}. \tag{72}$$

This expression holds for all stretched states of a harmonic oscillator. Because $|\lambda|^2 < 1$, we can recognize that the energy of a stretched state increases under a scale transformation $(q,p) \to (\lambda q, \lambda p)$. For the Fock states, the mean energy of the corresponding stretched states can be found explicitly. Hence, in the case of superposition (32), we have

$$\bar{E}_{\Sigma\lambda} = \frac{1}{\lambda^2}\frac{\hbar\omega}{2}\sum_{k=0}^{\infty}|c_k|^2 k + \frac{1-\lambda^2}{\lambda^2}\frac{\hbar\omega}{2}. \tag{73}$$

Using the expression for the means of operators in terms of the Husimi functions of states, we can establish the form of the uncertainty relations for the stretched states. We consider the Heisenberg and Robertson–Schrödinger uncertainty relations here. They can be written in the general forms

$$\sigma_{qq}\sigma_{pp} \geq \frac{1}{4}\hbar^2, \qquad \sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 \geq \frac{1}{4}\hbar^2, \tag{74}$$

and they hold for any quantum state. We now determine what happens with these relations when we pass to the stretched states. For this, we must calculate the quantities

$$\sigma_{qq} = \langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2, \qquad \sigma_{pp} = \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2, \qquad \sigma_{qp} = \frac{1}{2}\langle\hat{p}\hat{q} + \hat{q}\hat{p}\rangle - \langle\hat{q}\rangle\langle\hat{p}\rangle.$$

Using the Husimi function, we can write the dispersions σ_{qq} and σ_{pp} and the quantity σ_{qp} as

$$\begin{aligned}
\sigma_{qq} &= \int\biggl(q^2 - \frac{1}{2}\biggr)Q(q,p)\,dq\,dp - \biggl(\int qQ(q,p)\,dq\,dp\biggr)^2,\\
\sigma_{pp} &= \int\biggl(p^2 - \frac{1}{2}\biggr)Q(q,p)\,dq\,dp - \biggl(\int pQ(q,p)\,dq\,dp\biggr)^2,\\
\sigma_{qp} &= \int qpQ(q,p)\,dq\,dp - \int qQ(q,p)\,dq\,dp\int qQ(q,p)\,dq\,dp.
\end{aligned} \tag{75}$$

For stretched states. these formulas become

$$\begin{aligned}
\sigma_{qq\lambda} &= \int\biggl(q^2 - \frac{1}{2}\biggr)\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp - \biggl(\int q\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp\biggr)^2,\\
\sigma_{pp\lambda} &= \int\biggl(p^2 - \frac{1}{2}\biggr)\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp - \biggl(\int p\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp\biggr)^2,\\
\sigma_{qp\lambda} &= \int qp\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp - \int q\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp\int q\lambda^2 Q(\lambda q,\lambda p)\,dq\,dp.
\end{aligned} \tag{76}$$

These expressions yield the values of σ_{qq}, σ_{pp}, and σ_{qp} for stretched states:

$$\sigma_{qq\lambda} = \frac{1}{\lambda^2}\sigma_{qq} + \frac{1-\lambda^2}{\lambda^2}, \qquad \sigma_{pp\lambda} = \frac{1}{\lambda^2}\sigma_{pp} + \frac{1-\lambda^2}{\lambda^2}, \qquad \sigma_{qp\lambda} = \frac{1}{\lambda^2}\sigma_{qp}. \tag{77}$$

Using the obtained expressions, we find the modification of the Heisenberg and Robertson–Schrödinger uncertainty relations in passing to the stretched states,

$$\sigma_{qq\lambda}\sigma_{pp\lambda} = \frac{1}{\lambda^4}\biggl(\sigma_{qq}\sigma_{pp} + \frac{1}{2}(1-\lambda^2)(\sigma_{qq}+\sigma_{pp}) + \frac{1}{4}(1-\lambda^2)^2\biggr) \geq \frac{1}{4\lambda^4}\hbar^2,$$

$$\sigma_{qq\lambda}\sigma_{pp\lambda} - \sigma_{qp\lambda}^2 = \frac{1}{\lambda^4}\biggl(\sigma_{qq}\sigma_{pp} - \sigma_{qp}^2 + \frac{1}{2}(1-\lambda^2)(\sigma_{qq}+\sigma_{pp}) + \frac{1}{4}(1-\lambda^2)^2\biggr) \geq \frac{1}{4\lambda^4}\hbar^2.$$

We see that the right-hand sides of these uncertainty relations contain the factor λ^{-4}. For $|\lambda| < 1$, their values therefore increase and, in general, can become arbitrarily large. Some other states have the same property, for instance, the so-called correlated states that arise as a generalization of coherent states [23]. These states have a large coordinate and momentum uncertainty. In [24]–[26], it was proposed to use this property of such states to describe some phenomena in which the observed probability of transmission through the potential barrier is larger than usual. The increase in the tunneling probability can be formally associated with an "increase" of the Planck constant $\hbar$, i.e., we can assume that the scale transformation $(q,p) \to (\lambda q, \lambda p)$ generates an "effective Planck constant" $\hbar_{\text{eff}} = \hbar/\lambda^2$. The effective Planck constant for $\lambda^2 \ll 1$ satisfies the inequality $\hbar_{\text{eff}} \gg \hbar$.

8. The case of large λ

Up to now in our considerations, we assumed that $\lambda < 1$. Moreover, we relied on a theorem proved in [17]. According to this theorem, if $Q(q,p)$ is the Husimi function of a quantum state and $\lambda < 1$, then $\lambda^2 Q(\lambda q, \lambda p)$ is also a Husimi function of some quantum state. There is no known statements of this type for $\lambda > 1$. To clarify what kind of problems can arise in this case, we try to implement the construction used for $\lambda < 1$ to the case $\lambda > 1$.

We first consider the case $1 < \lambda < 2$ and choose the N-particle Fock state $|N\rangle$. Its Husimi function is

$$Q^N(q,p) = e^{-|\alpha|^2} \frac{|\alpha|^{2N}}{N!}. \tag{78}$$

After the transformation $(q,p) \to (\lambda q, \lambda p)$, it becomes

$$\lambda^2 Q^N(\lambda q, \lambda p) = \lambda^{2+2N} e^{-\lambda^2|\alpha|^2} \frac{|\alpha|^{2N}}{N!} = e^{-4|\alpha|^2} \lambda^{2+2N} e^{(4-\lambda^2)|\alpha|^2} \frac{|\alpha|^{2N}}{N!}. \tag{79}$$

As before, we want to represent this function as a sum of quantities of form (78). For this, we expand the exponential $e^{(4-\lambda^2)|\alpha|^2}$ in a series:

$$e^{(4-\lambda^2)|\alpha|^2} = \sum_{j=0}^{\infty} \frac{(4-\lambda^2)^j}{j!} |\alpha|^{2j}. \tag{80}$$

Substituting this expansion in (79), for $1 < \lambda < 2$, we obtain

$$\begin{aligned} \lambda^2 Q^N(\lambda q, \lambda p) &= \sum_{j=0}^{\infty} \frac{(4-\lambda^2)^j}{N!\,j!} \lambda^{2+2N} e^{-4|\alpha|^2} |\alpha|^{2N+2j} = \\ &= \sum_{j=0}^{\infty} \frac{(4-\lambda^2)^j}{N!\,j!} \lambda^{2+2N} e^{-4|\alpha|^2} \frac{(4|\alpha|^2)^{N+j}}{4^{N+j}} = \\ &= \sum_{j=0}^{\infty} \frac{(4-\lambda^2)^j (N+j)!}{N!\,j!} \lambda^{2+2N} \frac{1}{4^{N+j}} \left(e^{-4|\alpha|^2} \frac{(4|\alpha|^2)^{N+j}}{(N+j)!} \right) = \\ &= 4 \sum_{j=0}^{\infty} \frac{(N+j)!}{N!\,j!} \left(1 - \frac{\lambda^2}{4}\right)^j \left(\frac{\lambda^2}{4}\right)^{1+N} Q^{N+j}(2q, 2p). \end{aligned} \tag{81}$$

In the general case where $s-1 < \lambda < s$, we have

$$\lambda^2 Q^N(\lambda q, \lambda p) = s^2 \sum_{j=0}^{\infty} \frac{(N+j)!}{N!\,j!} \left(1 - \frac{\lambda^2}{s^2}\right)^j \left(\frac{\lambda^2}{s^2}\right)^{1+N} Q^{N+j}(sq, sp). \tag{82}$$

As before, the coefficients of the quantities $Q^{N+j}(sq, sp)$ in sums (81) and (82) are the elements of a negative binomial distribution. Their sum is equal to unity, and series (81) and (82) therefore converge. Hence, the only open problem is to interpret the quantities

$$Q^N(sq, sp) = s^{2N} e^{-s^2|\alpha|^2} \frac{|\alpha|^{2N}}{N!} = e^{-s^2q^2 - s^2p^2} \frac{(s^2q^2 + s^2p^2)^N}{N!}. \tag{83}$$

In [27], the problem of the conditions that must be satisfied by a function $F(q, p)$ (defined on the phase space) for this function to be the Husimi function of a quantum state was studied. It was shown that the Gauss function

$$F(q, p) = Ce^{-\lambda^2q^2 - \lambda^2p^2} \tag{84}$$

is the Husimi function of a quantum state if and only if $\lambda \leq 1$. Functions (83) do not satisfy this criterion; consequently, no quantum state is associated with them, i.e., they are not Husimi functions. Therefore, scale transformation (3) for $\lambda^2 > 1$, in general, does not transform a Husimi function into another Husimi function.

9. Conclusion

We have proposed a new method for constructing the states arising as a result of quantum processes. This method is based on using quasiprobability distributions, more specifically, Husimi functions. The chain of arguments is as follows. Let some process be described by a Hamiltonian H. Using this Hamiltonian, we can find an operator equation for the density matrix ρ. Using the relation between the Husimi function Q and the density matrix ρ, we can pass from this operator equation to an ordinary differential equation for the Husimi function.

The principle underlying our method is that solutions of a given differential equation are associated with transformations of a phase space. Such a transformation induces a transformation of the Husimi functions defined in it. Expressing such transformed functions $\widetilde{Q}$ in terms of the already known Husimi functions, we can perform the inverse transformation and find the density matrix $\tilde{\rho}$ of the transformed state. Here, we implemented this program for scale transformation (3) in the phase space, which models the operation of a quantum amplifier [18], [19]. We constructed the density matrix for a stretched state arising as a result of the action of this amplifier on n-particle states of a harmonic oscillator and on their superpositions. We plan to consider other states and other transformations of the phase space in the future, in particular, states leading to weakening of Fock states rather than their amplification.

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SCALING TRANSFORM AND STRETCHED STATES IN QUANTUM MECHANICS

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Abstract

We consider the Husimi $Q(q,p)$-functions which are quantum quasiprobability distributions on the phase space. It is known that, under a scaling transform $(q;p) \to (\lambda q; \lambda p)$, the Husimi function of any physical state is converted into a function which is also the Husimi function of some physical state. More precisely, it has been proved that, if $Q(q,p)$ is the Husimi function, the function $\lambda^2 Q(\lambda q; \lambda p)$ is also the Husimi function. We call a state with the Husimi function $\lambda^2 Q(\lambda q; \lambda p)$ the stretched state and investigate the properties of the stretched Fock states. These states can be obtained as a result of applying the scaling transform to the Fock states of the harmonic oscillator. The harmonic-oscillator Fock states are pure states, but the stretched Fock states are mixed states. We find the density matrices of stretched Fock states in an explicit form. Their structure can be described with the help of negative binomial distributions. We present the graphs of distributions of negative binomial coefficients for different stretched Fock states and show the von Neumann entropy of the simplest stretched Fock state.

Keywords: Husimi function, harmonic oscillator, scaling transform, Fock states, stretched states.

1. Introduction

The usual formulation of quantum mechanics is based on the concept of Hilbert space and related structures. In this approach, quantum states are associated with vectors of this space and the observables, with operators in this space. However, there exists the formulation of quantum mechanics similar to the classical statistical mechanics. Quantum mechanics in the statistical theory, and only in it, can predict the probabilities of measurements and, in this context, it is similar to the classical statistical mechanics. The main object of investigation in this theory is the distribution function $\rho(q,p)$. In the classical statistical mechanics, in order to find the mean value of any function $F(q,p)$ defined on the phase space, one has to integrate this function over the phase space weighted with an appropriate probability density function, i.e.,

$$\langle F \rangle = \int F(q,p)\rho(q,p)\,dq\,dp. \tag{1}$$

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In order to describe quantum phenomena in a similar way, one associates a quasidistribution function $D(q,p)$ with each quantum state; the other function $A_D(q,p)$ is called the symbol of the operator $\hat{A}$. We calculate the mean value analogously to (1) and obtain

$$\langle \hat{A} \rangle = \int A_D(q,p) D_\rho(q,p)\, dq\, dp. \tag{2}$$

There exist three well-known quasidistributions: the Wigner function $W(q,p)$ [1], the Husimi–Kano $Q(q,p)$-function [2,3], and the Glauber–Sudarshan $P(q,p)$-function [4,5]. If the operator $\hat{A}$ has the form of the bivariate polynomial of the creation and annihilation operators, its W, Q, and P symbols are obtained using operations of symmetrization and antinormal and normal orderings [6–8].

In this paper, we use the Husimi functions for investigating the properties of stretched Fock states. These states can be obtained as a result of applying the scaling transform $(q;p) \to (\lambda q; \lambda p)$ to usual Fock states of the harmonic oscillator.

2. Husimi Function and the Mean Value Problem

We consider a state described by the density operator $\hat{\rho}$. The Husimi function for this state is determined by the set of coherent states $\langle x|\alpha\rangle$ of the harmonic oscillator [2,3],

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \int \langle \alpha|x\rangle \rho(x,y) \langle y|\alpha\rangle\, dx\, dy, \tag{3}$$

where $\alpha = \alpha_r + i\alpha_i$ is an arbitrary complex number, and $\rho(x,y)$ is the kernel of the density operator in the coordinate representation. For the complex number α, which determines the coherent state, we employ the expression $\alpha = (q+ip)/\sqrt{2}$ and consider the Husimi function Q as the function of p and q,

$$Q(q,p) = \frac{1}{2\pi\hbar} \langle q,p|\hat{\rho}|q,p\rangle, \tag{4}$$

where $|q,p\rangle$ is the coherent state given in terms of the variables q and p as follows:

$$\langle x|q,p\rangle = \left(\frac{1}{\pi}\right)^{1/4} \exp\left[-\frac{1}{2}(x-q)^2 + ipx - \frac{i}{2}qp\right], \tag{5}$$

The Husimi function of the state can be described by the wave function $\psi(x)$,

$$Q(q,p) = \frac{1}{2}\left(\frac{1}{\pi}\right)^{3/2} \int \exp\left[-\frac{1}{2}(y-q)^2 - \frac{1}{2}(x-q)^2 + ip(y-x)\right] \psi^*(y)\psi(x)\, dx\, dy. \tag{6}$$

Using the Husimi function, one can evaluate the mean value of an operator by applying formula (2).

In the standard approach, the Q-symbol of the operator $\hat{A}$ can be found, in view of antinormal ordering of the creation and annihilation operators in the expression for the operator $\hat{A}$. The other method of obtaining the Husimi symbols was proposed in [9].

It was proved in [10, 11] that, if $Q(q,p)$ is the Husimi function of the physical state, the value $\lambda^2 Q(\lambda q; \lambda p)$ is also the Husimi function of some physical state. We call the state with the Husimi function $\lambda^2 Q(\lambda q; \lambda p)$ the stretched state.

3. Stretched Fock States

Now as an example, we apply the results obtained to the harmonic oscillator.

In [10, 11], it was shown that the Fock state of the harmonic oscillator is converted under the scaling transform into a mixed state, which is described by the density matrix,

$$\hat{\rho}_N = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\frac{(N+k)!}{k!}(1-\lambda^2)^k|N+k\rangle\langle N+k|, \qquad \lambda^2 < 1. \tag{7}$$

The stretched Fock states contain pure states $|N+k\rangle, \quad k = 0, 1, 2, \dots, \infty$, and the probability for each pure state $|N+k\rangle$ to enter the stretched Fock states is

$$D_k^N(\lambda) = \frac{\lambda^{2N+2}(N+k)!}{N!k!}(1-\lambda^2)^k. \tag{8}$$

The distribution of pure states is described by the negative binomial distribution [12]:

$$f(k,r,p) = \begin{pmatrix} r+k-1 \\ k \end{pmatrix} p^r q^k, \quad p+q=1, \quad k = 0, 1, 2, \dots \tag{9}$$

In Fig. 1, we show the distribution of negative binomial coefficients for different values of N and λ.

One can find the mean photon number and its dispersion in the stretched Fock state (7); they read

$$\langle n\rangle = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}(N+k)\frac{(N+k)!}{k!}(1-\lambda^2)^k = \frac{N+1}{\lambda^2} - 1, \tag{10}$$

$$\sigma_n = \langle n^2\rangle - (\langle n\rangle)^2 = \frac{(N+1)(1-\lambda^2)}{\lambda^4}. \tag{11}$$

The dispersions σ_{qq}, σ_{pp}, and σ_{qp} for the stretched Fock states (7) can be found in an explicit form, in view of (7); they are

$$\langle n|q^2|n\rangle = n + 1/2, \tag{12}$$

$$\sigma_{qq\lambda} = \frac{\lambda^{2N+2}}{N!}\sum_{k=0}^{\infty}\left(N+k+\frac{1}{2}\right)\frac{(N+k)!}{k!}(1-\lambda^2)^k = \frac{\sigma_{qq}}{\lambda^2} + \frac{1-\lambda^2}{2\lambda^2}. \tag{13}$$

A measure of uncertainty associated with the probability distribution is entropy.

For quantum systems, the von Neumann entropy H is used; in terms of the density matrix $\hat{\rho}$, the von Neumann entropy H reads

$$H = -\sum_k p_k \log_2 p_k, \tag{14}$$

where p_k are the diagonal matrix elements of the density matrix ρ.

It is clear that the von Neumann entropy of the pure Fock state $|N\rangle$ is zero. The entropy of a superposition of the Fock states is defined by the moduli of the coefficients of the states constituting the superposition.

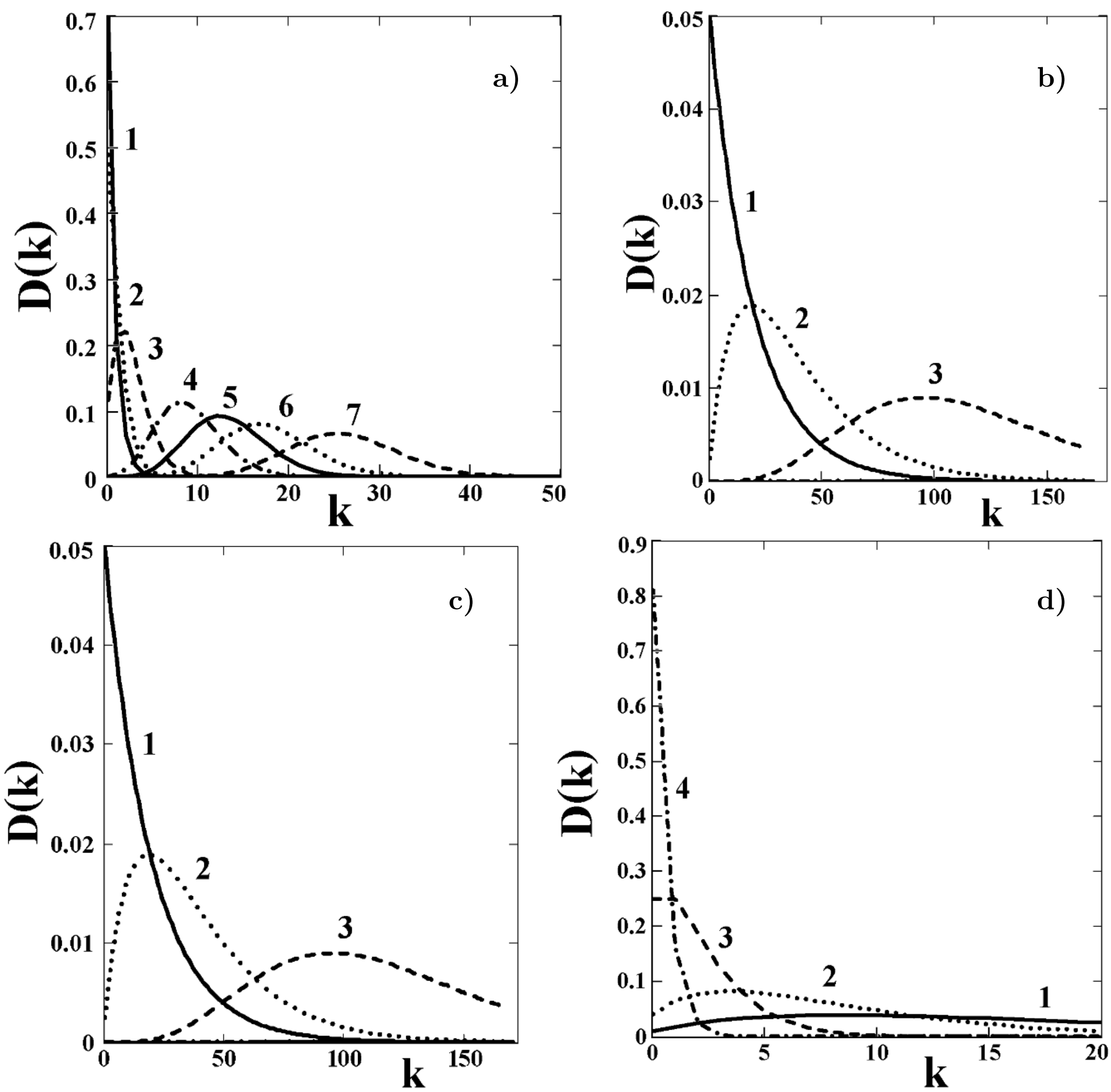

Fig. 1. The negative binomial distribution, where $N = 0$ (curve 1), 1 (curve 2), 5 (curve 3), 20 (curve 4), 30 (curve 5), 40 (curve 6), 60 (curve 7) and $\lambda^2 = 0.1$ (a) and 0.7 (b), $\lambda^2 = 0.05$ and $N = 0$ (curve 1), 1 (curve 2), and 5 (curve 3) (c), and $N = 1$ and $\lambda^2 = 0.1$ (curve 1), 0.2 (curve 2), 0.5 (curve 3), and 0.9 (curve 4) (d).

Now we calculate the von Neumann entropy H of the stretched Fock state and show the result in Fig. 2.

A stretched state is described by the density matrix

$$\hat{\rho}_N = \frac{\lambda^{2N+2}}{N!} \sum_{k=0}^{\infty} \frac{(N+k)!}{k!} (1-\lambda^2)^k |N+k\rangle\langle N+k|, \qquad \lambda^2 < 1. \tag{15}$$

Its diagonal matrix elements read

$$D_k^N(\lambda) = \frac{\lambda^{2N+2}(N+k)!}{N!k!} (1-\lambda^2)^k. \tag{16}$$

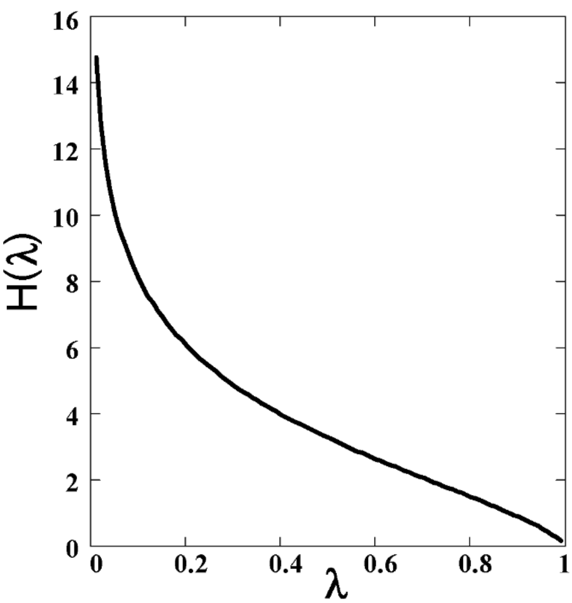

Fig. 2. The von Neumann entropy of the stretched Fock state $\hat{\rho}_0^\lambda$.

Thus, in order to find the von Neumann entropy of the state (15), we need to calculate the value

$$H_N = -\sum_{k=0}^{\infty} D_k^N(\lambda) \log_2 D_k^N(\lambda). \tag{17}$$

If $N = 0$, we have

$$\begin{aligned} H_0 &= -\lambda^2 \sum_{k=0}^{\infty} (1-\lambda^2)^k \log_2 \left(\lambda^2 (1-\lambda^2)^k\right) \\ &= -\log_2 \lambda^2 - \log_2 (1-\lambda^2) \frac{1-\lambda^2}{\lambda^2}. \end{aligned} \tag{18}$$

If $\lambda \to 1$, the value $H_0 \to 0$, and this means that the von Neumann entropy of the N-particle pure state is zero. If $\lambda \to 0$, then $H_0 \to \infty$, and this means that the uncertainty to observe an arbitrary n-particle state increases.

From the viewpoint of information theory, this means that for $\lambda = 1$ we can state that there is only one pure state in the mixed state (15) at $N = 0$, and this state is the vacuum state $|0\rangle$.

If $N > 0$, the von Neumann entropy H_N has a more complicated form and cannot be represented in a closed form. For example, when $N = 1$, the von Neumann entropy H_1 is

$$\begin{aligned} H_1 &= -\lambda^4 \left[\log_2 \lambda^4 \sum_{k=0}^{\infty} (k+1)(1-\lambda^2)^k + \log_2 (1-\lambda^2) \sum_{k=0}^{\infty} k(k+1)(1-\lambda^2)^k \right. \\ &\quad \left. + \sum_{k=0}^{\infty} \log_2 (k+1)(k+1)(1-\lambda^2)^k \right]. \end{aligned} \tag{19}$$

It is obvious that, for any $N > 0$, the behavior of the von Neumann entropy H_N at $\lambda \to 1$ and $\lambda \to 0$ is the same as the behavior of the von Neumann entropy H_0.

4. Conclusions

We investigated the statistical properties of stretched Fock states. These states are constructed as a result of the scaling transform of the Husimi functions of usual Fock states of the harmonic oscillator, and these states are mixed states.

We found the explicit form of the density matrices of the stretched Fock states. The distribution of the pure states in these mixed states is described by the negative binomial distribution.

The stretched Fock states can be employed while studying the quantum tunneling phenomenon [13]. Also the scaling transform arises in the problem of quantum-state amplification [14]; in this case, the parameter λ is equal to the inverse value of the amplification coefficient $G = 1/\lambda$.

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LINEAR LIGHT AMPLIFIER AND AMPLIFICATION OF N-PHOTON STATES

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Abstract

We consider a linear quantum amplifier consisting of N_A two-level atoms and study the problem of amplification of N-photon states. The N-photon states are associated with N-quantum states of the harmonic oscillator. We show that the process of interaction of the electromagnetic field with atoms can be associated with some transformation of the phase space and functions defined on this phase space. We consider the Husimi functions $Q_N(q,p)$ of N-quantum states of the harmonic oscillator, which are defined on the phase space, investigate transformation of these functions, and find an explicit form of the density matrix of the amplified N-photon state.

Keywords: quantum amplifier, Husimi function, harmonic oscillator, scaling transform, density matrix, phase space, evolution equation.

1. Introduction

We consider the process of interaction of a single-mode electromagnetic field with a two-level atomic medium. We assume that the medium is described by the Jaynes–Cummings model. We are interested in the dynamics of the electromagnetic field only; thus we assume that the state of the medium in the process of interaction does not change. In fact, the complete system is separated into two subsystems. Such separated systems are described by the density matrix [1–3]. The evolution equation satisfying the density matrix is known. It is solved by replacing the density operator with quasiprobability distributions.

The quasiprobability distribution functions such as Wigner functions [4], Glauber–Sudarshan functions [5,6], and Husimi–Kano functions [7,8] are often used in quantum optics. The Glauber–Sudarshan P-distribution was employed in [9–11] in order to obtain information on the properties of the field at the amplifier output. In [12], it was suggested to use the Husimi–Kano Q-distribution instead of the Glauber–Sudarshan P-distribution, the evolution equation for the Q-function was obtained, its solution was found for a particular case, and its properties were studied.

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In this paper, we use the original method, which was developed in [13–17] to analyze the solutions of the equation for the Husimi function. The essence of the method consists in the fact that a certain transformation of the phase space of the system is related to the dynamical process occurring in it. The transformations of the phase space generate the transformations of the functions defined on this phase space. Transformations of functions have different forms, depending on both the nature of the dynamical process and the type of functions defined on the phase space. In this paper, we consider the Husimi Q-functions. In [17], we have shown that in the case of the complete inverse population of two-level atomic systems forming the amplifier, the transformation of the phase space, corresponding to the amplification of the field passing through the amplifier reads

$$(q,p) \to (\lambda q, \lambda p), \qquad |\lambda|^2 \leq 1. \tag{1}$$

It corresponds to scaling of the phase space, and the corresponding transformation of functions is

$$Q(q,p) \to \widetilde{Q}(q,p) = \lambda^2 Q(\lambda q, \lambda p). \tag{2}$$

In [13], we showed that if $Q(q,p)$ is the Husimi function of quantum state, then $\lambda^2 Q(\lambda q, \lambda p)$ is also the Husimi function of another quantum state, provided that $|\lambda|^2 \leq 1$.

Here, we consider a more general situation, namely, we assume that a part of atoms forming the amplifier are in the ground state, and the rest of them are in the excited state. In this case, the dynamics of the process of interaction can also be associated with transformation of the phase space, but this transformation and the corresponding transformation of functions have a more complex form than (1) and (2). We describe these transformations and find an explicit form of the density matrix of an amplified N-photon state.

We treat the states of the harmonic oscillator only, but they can be identified as photons, which is a standard practice in quantum optics. In this way, the analysis of the amplification process and its results are also applicable for photons.

2. Quantum Amplifier Structure

We consider a system of N_A two-level atoms, N_1 of which are excited, and N_0 atoms are in the ground state, with $N_0 < N_1$. These atoms are interacting with the one-mode quantum field, for which we assume that it is an eigenmode of a free field, and that its frequency is resonant with the atomic frequency. We assume also that the populations N_1 and N_0 are kept constant in time due to some pump and loss mechanism.

Let $\hat{\rho}$ be the density operator of the electromagnetic field. The master equation for $\hat{\rho}$ reads [1,12]

$$\frac{\partial \hat{\rho}}{\partial t} = -\gamma N_1(\hat{a}\hat{a}^\dagger \hat{\rho} - 2\hat{a}^\dagger \hat{\rho}\hat{a} + \hat{\rho}\hat{a}\hat{a}^\dagger) - \gamma N_0(\hat{a}^\dagger \hat{a}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger \hat{a}). \tag{3}$$

Here, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators of the electromagnetic field, N_1 and N_0 are the populations of the upper and lower levels of the two-level atom, and γ is the amplification coefficient.

Using the relation between the density matrix and the Husimi function, we can pass from the operator equation (3) to an ordinary differential equation for the Husimi function. Using this equation, the expression for the Husimi function for the state at the quantum amplifier output was obtained in [12]; it is

$$Q(\alpha,t)=\int d^2\beta Q(\beta)\frac{1}{\pi m}\exp\left[-\frac{|\alpha-\beta G|^2}{m}\right]. \tag{4}$$

Here

$$G(t)=\exp[2(N_1-N_0)\gamma t], \qquad m=\frac{N_0}{N_1-N_0}(G^2-1). \tag{5}$$

In [17], we considered the case $m=0$ corresponding to the situation where all atoms are in the excited state. In this case, the Husimi function of the state after the amplification is given by a simple expression,

$$Q_{\rm out}(\alpha,t)=\frac{1}{G^2}Q_{\rm in}(\alpha/G)=\left\langle\frac{\alpha}{G}|\hat\rho_{\rm in}|\frac{\alpha}{G}\right\rangle. \tag{6}$$

We found an explicit form of the density matrix of the amplified state in the case where the input state is a superposition of an arbitrary number of Fock states.

The simplest case takes place if the input state is the pure N-photon state. In this case, at the output of the amplifier one has the state described by the following density matrix:

$$\hat\rho_N^\lambda=\sum_{j=0}^{\infty}\lambda^{2N+2}\frac{(1-\lambda^2)^j(N+j)!}{j!N!}|N+j\rangle\langle N+j|, \tag{7}$$

where $\lambda=G^{-1}$. The density matrix (7) is diagonal. Its first N diagonal elements with labels $0,1,\ldots,N-1$ are equal to zero, and the remaining diagonal elements are

$$F_{N+j}^N=\frac{(1-\lambda^2)^j(N+j)!}{j!N!}\lambda^{2N+2}, \qquad j=0,1,\ldots \tag{8}$$

The values (8) form a negative binomial distribution, and the elements of this distribution are given by [19]

$$f(k,r,p)=\begin{pmatrix} r+k-1\\ k\end{pmatrix}p^rq^k=\frac{(r+k-1)!}{(r-1)!(k)!}p^r(1-p)^k, \qquad p+q=1, \quad k=0,1,2,\ldots \tag{9}$$

They are defined by two parameters r and p, and the value k is the element label in the distribution. In our case, $r=N+1$, $k=j$, and $p=\lambda^2$, i.e., we have

$$F_{N+j}^N=f(j,N+1,\lambda^2). \tag{10}$$

The density matrix (7) of the amplified N-photon state was found in [17]. In this paper, we consider the case where the population levels N_1 and N_0 ($N_1>N_0$) can take arbitrary values. As an input state, we take the N-photon state and find the density matrix of the corresponding amplified state. We use formula (7) in our subsequent calculations. In order to distinguish the density matrices of amplified states obtained by an amplifier with $N_0=0$ from the density matrices of amplified states obtained by an amplifier with arbitrary N_0 and N_1, we denote the former by $\hat\rho$, and the latter by $\widetilde{\hat\rho}$.

3. The Density Matrix of Amplified State

In order to find the Husimi function $\widetilde{Q}(\alpha)$ of the amplified state, obtained due to passing trough a linear quantum amplifier with arbitrary values of populations N_1 and N_0 ($N_0 < N_1$), one should calculate the integral (4). The function $Q(\beta)$ is the Husimi function of the input state. For the input state, we take the quantum state of the harmonic oscillator $|N\rangle$; its Husimi function has the following form:

$$Q_N(\beta) = \langle\beta|N\rangle\langle N|\beta\rangle = e^{-|\beta|^2}\frac{|\beta|^{2N}}{N!}. \tag{11}$$

Substituting the above Husimi function into the integral (4) one obtains

$$\widetilde{Q}_N^{\lambda}(\alpha) = \frac{1}{\pi m N!}\int d^2\beta\, e^{-|\beta|^2}\,|\beta|^{2N}\,\exp\left[-\frac{|\alpha-\beta G|^2}{m}\right], \tag{12}$$

which after integration becomes

$$\widetilde{Q}_N^{\lambda}(\alpha) = \frac{1}{N!(m+G^2)}\frac{m^N}{(m+G^2)^N}L_N\left(-\frac{|\alpha|^2G^2}{m(m+G^2)}\right)\exp\left(-\frac{|\alpha|^2}{m+G^2}\right). \tag{13}$$

Here $L_N(x)$ is the Laguerre polynomial

$$L_N(x) = N!\sum_{k=0}^{N}(-1)^k\binom{N}{k}\frac{x^k}{k!}. \tag{14}$$

In [17], we developed a method that allows one to derive the density matrix from the Husimi function of the form similar to (13). Applying this method, we obtain the density matrix $\widetilde{\hat{\rho}_N}^{\lambda}$ of the amplified $|N\rangle$ state; it reads

$$\begin{aligned}\widetilde{\hat{\rho}_N}^{\lambda} &= \frac{m^N}{(m+G^2)^{N+1}}\sum_{k=0}^{\infty}\sum_{l=0}^{N}\binom{N}{N-l}\binom{k+N-l}{n-l}\\ &\times\Big(\frac{G^2}{m(m+G^2)}\Big)^{N-l}\Big(1-\frac{1}{m+G^2}\Big)^k|k+N-l\rangle\langle k+n-l|,\end{aligned} \tag{15}$$

where $\lambda^{-2} = m+G^2$.

When $m=0$, the summation over l reduces to only one term ($l=0$), so that the density matrix becomes

$$\begin{aligned}\widetilde{\hat{\rho}_N}^{\lambda}(m\to 0) &= \frac{1}{G^{2(N+1)}}\sum_{k=0}^{\infty}\binom{k+N}{N}\Big(1-\frac{1}{G^2}\Big)^k|k+N\rangle\langle k+N|\\ &= \frac{1}{G^{2(N+1)}}\sum_{k=0}^{\infty}\frac{(n+k)!}{N!k!}\Big(1-\frac{1}{G^2}\Big)^k|k+N\rangle\langle k+N|.\end{aligned} \tag{16}$$

Now we consider some special cases. Let the input state at the entrance of the amplifier be the vacuum state $|0\rangle$. The Husimi function of $|0\rangle$ has the form

$$Q_0(\alpha) = e^{-|\alpha|^2}. \tag{17}$$

Substituting this expression into (12), we obtain the expression for the Husimi function of the amplified state $|0\rangle_{\rm out}$,

$$\widetilde{Q}_0^\lambda(\alpha) = \frac{1}{m+G^2}\exp\left(-\frac{|\alpha|^2}{m+G^2}\right). \tag{18}$$

We see that expression (18) coincides with the expression for the Husimi function of the state $|0\rangle_{\rm out}$ obtained in [17] assuming $\lambda^{-2} = m + G^2$. Therefore,

$$\widetilde{\hat\rho_0}^{\lambda} = \hat\rho_0^\lambda. \tag{19}$$

Let now the input state be $|1\rangle$. In this case, the state $|1\rangle_{\rm out}$ appearing at the amplifier output has the following density matrix:

$$\widetilde{\hat\rho_1}^{\lambda} = \frac{m}{(m+G^2)^2}\left[\sum_{k=0}^{\infty}\left(\frac{(k+1)G^2}{m(m+G^2)}\right)\left(1-\frac{1}{m+G^2}\right)^k |k+1\rangle\langle k+1|\right] + \frac{m}{(m+G^2)^2}\left[\sum_{k=0}^{\infty}\left(1-\frac{1}{m+G^2}\right)^k |k\rangle\langle k|\right]. \tag{20}$$

Comparing expression (20) with the density matrix of the amplified $|N\rangle$ state (7), we see that

$$\widetilde{\hat\rho_1}^{\lambda} = \frac{m}{m+G^2}\hat\rho_0^\lambda + \frac{G^2}{m+G^2}\hat\rho_1^\lambda. \tag{21}$$

In the general case, one has

$$\begin{aligned}\widetilde{\hat\rho_N}^{\lambda} &= m^N \sum_{l=0}^{N}\frac{N!\lambda^{2N+2}}{l!(N-l)!}\left(\frac{G^2\lambda^2}{m}\right)^{N-l}\sum_{k=0}^{\infty}(1-\lambda^2)^k\frac{(N-l+k)!}{k!(N-l)!}|N-l+k\rangle\langle N-l+k| \\ &= m^N\lambda^{2N}\sum_{l=0}^{N}\frac{N!}{l!(N-l)!}\left(\frac{G^2}{m}\right)^{N-l}\hat\rho_{N-l}^\lambda.\end{aligned} \tag{22}$$

We see that the density matrix (22) of the state, which is obtained from the N-photon state $|N\rangle$, after passing through the amplifier with arbitrary values of the population levels N_1 and N_0, is the sum of the density matrices $\hat\rho_{N-l}^\lambda$ of $(N-l)$-photon states passed through the amplifier, whose population of the ground level is $N_0 = 0$, i.e., trough the completely excited medium. Properties of such states were studied in [17] – they have the form (7). These density matrices are diagonal, with either zeros or values from the negative binomial distribution on the main diagonal. So in the density matrix $\hat\rho_{N-l}^\lambda$, the first $(N-l-1)$ numbers on the diagonal are zeros followed by values F_{N+j}^N, $j = 0, 1, \ldots$ given in (16). The total density matrix (22) is the sum of such matrices multiplied by the coefficients that are elements of the binomial distribution $\left(1 + G^2/m\right)^N$. Thus, the matrix elements of the density matrix (22) have the form of sums of the products of the elements of the binomial distribution and the elements of the negative binomial distribution.

Let us note that in the case of a completely excited medium, when the population of the ground level is $N_0 = 0$, the N-photon amplified state, $|N\rangle_{\rm out}$, contains only states with photon number equal to or greater than N, namely, $|N\rangle$, $|N+1\rangle$, $|N+2\rangle, \ldots$ However, if the population of the ground level $N_0 > 0$, then the output state contains every k-photon state, including the vacuum state ($k = 0$).

Now we find the form of the diagonal elements of matrix (22); for this, we present expression (22) as

$$\widetilde{\hat{\rho}_N}^{\lambda} = \sum_{p=0}^{N} S_p^N \rho_p^{\lambda}, \tag{23}$$

where the coefficients S_p^N are

$$S_p^N = m^N \lambda^{2N} \left(\frac{G^2}{m}\right)^p \frac{N!}{p!(N-p)!} = m^{(N-p)} \lambda^{2N} G^{2p} \binom{N}{p}. \tag{24}$$

The density operator $\hat{\rho}_p^{\lambda}$ given in Eq. (16) can be rewritten as follows:

$$\hat{\rho}_p^{\lambda} = \sum_{j=0}^{\infty} F_{p+j}^{p} |p+j\rangle\langle p+j|, \tag{25}$$

where F_{p+j}^p are elements of the negative binomial distribution.

Now we can write the density operator given by (22) as

$$\widetilde{\hat{\rho}_N}^{\lambda} = \sum_{k=0}^{\infty} H_k^N |k\rangle\langle k|, \tag{26}$$

Here, the coefficients H_k^N are

$$H_k^N = \sum_{j=0}^{k} S_j^N F_k^j, \quad k = 0, 1, \ldots, N, \qquad H_k^N = \sum_{j=0}^{N} S_j^N F_k^j, \quad k = N+1, N+2, \ldots \infty. \tag{27}$$

Let us evaluate the trace of the density operator (22)

$$\begin{aligned}
\mathrm{Tr}(\widetilde{\hat{\rho}}_N^{\lambda}) &= \mathrm{Tr}\left(m^N \lambda^{2N} \sum_{l=0}^{N} \frac{N!}{l!(N-l)!} \left(\frac{G^2}{m}\right)^{N-l} \rho_{N-l}^{\lambda} \right) \\
&= m^N \lambda^{2N} \sum_{l=0}^{N} \frac{N!}{l!(N-l)!} \left(\frac{G^2}{m}\right)^{N-l} \mathrm{Tr}\left(\sum_{j=0}^{\infty} F_{p+j}^{p} |p+j\rangle\langle p+j|, \right) \\
&= m^N \lambda^{2N} \sum_{l=0}^{N} \frac{N!}{l!(N-l)!} \left(\frac{G^2}{m}\right)^{N-l} \sum_{j=0}^{\infty} F_{p+j}^{p} \\
&= \left(\frac{m}{m+G^2}\right)^N \sum_{l=0}^{N} \binom{N}{l} \left(\frac{G^2}{m}\right)^{N-l} = 1.
\end{aligned} \tag{28}$$

Thus, by direct calculation we showed that the trace of the operator (22) is equal to unity. Operator (22) is the diagonal operator and the sum of its diagonal elements $\sum_{k=0}^{\infty} H_k^N = 1$. The trace of the operator $(\widetilde{\hat{\rho}_N}^{\lambda})^2$ reads

$$\mathrm{Tr}\left[(\widetilde{\hat{\rho}_N}^{\lambda})^2\right] = \sum_{k=0}^{\infty} (H_k^N)^2 \leq 1. \tag{29}$$

Therefore, operator (22) is actually the density operator of the quantum state.

4. Conclusions

The aim of this paper was to investigate the relation between the dynamics of physical processes and the transformations of the phase space and the functions defined on the phase space. The problem was addressed by considering a linear quantum amplifier. In this case, we found explicitly interrelations between the elements describing the dynamics and the transformations in the phase space.

We showed that the state of the quantum amplifier has a significant influence on the state of the electromagnetic field which passed trough it. In the case where the quantum amplifier is completely inverted ($N_0 = 0$), the transformation of the phase space and Husimi functions defined on it are determined by formulas (1) and (2). If both levels of the quantum amplifier, ground and excited, have nonzero population, the transformation of Husimi functions has a more complex form. It is determined by formula (4) which, in the particular case where the input state is the ground state of harmonic oscillator, after calculating the integral, takes the form (18). Expression (18) is structurally similar to the Husimi function transformed by (2), but the fundamental difference between transformations (2) and (4) consists in the fact that (2) is a local transformation, and (4) is a nonlocal transformation. Thus, the general structure of transformations of the phase space and the functions specified on the phase space is much more complex than considered in [17].

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Weakly curved background T-duals*

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Abstract

We discuss the generalized T-dualization procedure, its connection to the standard procedure, and the results of its application to the arbitrary set of coordinates of the closed string moving in the weakly curved background. This background consists of a constant metric and linearly coordinate dependent Kalb-Ramond field with infinitesimal strength. We unite all the results into a T-dualization diagram, representing all T-dual theories, the ways to obtain the theories from one another and the T-dual coordinate transformation laws connecting the corresponding coordinates.

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T-duality is a symmetry seen in the string spectrum. It was observed that for toroidal compactifications [1], where one dimension is compactified on a circle of radius R and the corresponding dual dimension is compactified on a circle of radius $1/R$, one obtains the discription of a string with the same physical properties. So, there exist different string theories, describing the string in the geometrically different backgrounds, with the same predictions. Such a symmetry is not present in any point particle theory [2, 3, 4, 5], and because of that the explanation for T-duality was sought for in a fact that strings can wrap around compactified dimensions. Investigation of T-duality lead to a discovery of a Buscher T-dualization procedure [6, 7], which gave a prescription how to find the T-dual theory for same known theory. The procedure is applicable along isometry directions, what allowed the investigation of a backgrounds which do not depend on some coordinates. This procedure enabled the investigation of the properties of the background connected by T-duality. It was discovered that geometric backgrounds transform to the non-geometric backgrounds and these to different non-geometric backgrounds, which differ in a form of the background fluxes, some of which are not locally well defined [8, 9]. T-duality is also investigated for the double string theories, where T-duality is a manifest symmetry [10, 11, 12, 13].

In this talk we will discuss the results of T-dualizations done for the closed string moving in the weakly curved background, using the generalized T-dualization procedure, defined in our paper [14]. This background depends on all the space-time coordinates and as such was not a candidate for T-dualization using the standard T-dualization procedures. In paper [14], we presented generalized T-dualization procedure applicable to all space-time directions regardless of the possible background coordinate dependence. We obtained the T-dual theory which is a result of T-dualizing all the initial coordinates. In paper [15], we broadened the investigation by considering T-dualization of an arbitrary set of coordinates of both initial and its completely T-dual theory. We will recapitulate the results here and discuss further investigations. We obtained the T-dualization diagram describing the relation between all string theories T-dual to the string moving in a weakly curved background, their backgrounds and giving the T-duality laws connecting the corresponding coordinates.

So, let us start by the action describing a closed string moving in a coordinate dependent background

$$S[x] = \kappa \int_{\Sigma} d^2\xi \; \partial_+ x^\mu \Pi_{+\mu\nu}(x) \partial_- x^\nu, \quad \partial_\pm = \partial_\tau \pm \partial_\sigma \tag{1}$$

given in the conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$. The background field composition is defined by

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x). \tag{2}$$

It consists of a symmetric metric tensor $G_{\mu\nu} = G_{\nu\mu}$ and an antisymmetric Kalb-Ramond field $B_{\mu\nu} = -B_{\nu\mu}$. The background must obey the following space-time equations of motion

$$R_{\mu\nu} - \frac{1}{4}B_{\mu\rho\sigma}B_{\nu}{}^{\rho\sigma} = 0, \quad D_\rho B^\rho{}_{\mu\nu} = 0, \tag{3}$$

in order to have a consistent quantum theory. We will consider one of the simplest coordinate dependent solutions, the weakly curved background, composed of a constant metric and linearly coordinate dependent Kalb-Ramond field which has an infinitesimal field strength

$$G_{\mu\nu}(x) = const,\ B_{\mu\nu}(x) = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho, \quad b_{\mu\nu}, B_{\mu\nu\rho} = const. \tag{4}$$

What are the backgrounds T-dual to this background? As, the standard T-dualization procedure is applicable to coordinate directions which do not appear as background field arguments, and the weakly curved background depends on all space-time coordinates, this procedure could not provide the answer to this question. So, a generalization of a T-dualization procedure which does not have this limitation had to be made. The main difference between the procedures obviously must be connected to background fields argument. We presented the new T-dualization procedure in [14].

Both procedures are built as a localization of a global coordinate shift symmetry $\delta x^\mu = \lambda^\mu = const$. One introduces the gauge fields v^μ_α and substitutes the ordinary derivatives with the covariant ones

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha. \tag{5}$$

Imposing the following transformation law for the gauge fields

$$\delta v^\mu_\alpha = -\partial_\alpha \lambda^\mu, \quad (\lambda^\mu = \lambda^\mu(\tau,\sigma)) \tag{6}$$

one obtains that $\delta D_\alpha x^\mu = 0$. If the background does not depend on the coordinates which are T-dualized the gauge invariant action is already obtained. But, what if the background depends on all the coordinates? The additional step must be introduced. It consists of a substitution of background field argument (the coordinate x^μ), by the invariant argument (invariant coordinate) defined as a line integral of the covariant derivatives of the original coordinate

$$\Delta x^\mu_{inv} \equiv \int_P d\xi^\alpha\, D_\alpha x^\mu = x^\mu - x^\mu(\xi_0) + \Delta V^\mu, \tag{7}$$

where

$$\Delta V^\mu \equiv \int_P d\xi^\alpha v^\mu_\alpha. \tag{8}$$

Consequently the arguments of the background fields will be nonlocal. Here, they are defined as the line integrals of the gauge fields, and as such are nonlocal. Later, once the explicit form of T-dual theories is obtained the non locality will appear as dependence on double coordinates.

In order to obtain the physically equivalent theories, one must make the introduced gauge fields nonphysical which is done by requiring that there field strength

$$F^\mu_{\alpha\beta} \equiv \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha \tag{9}$$

must be zero. This is achieved by adding the Lagrange multiplier y_μ term to the Lagrangian. Finally, the gauge invariant action, physically equivalent to the initial action is

$$S_{inv} = \kappa \int d^2\xi \Big[D_+ x^\mu \Pi_{+\mu\nu}(\Delta x_{inv}) D_- x^\nu + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu) \Big]. \tag{10}$$

Fixing the gauge $x^\mu(\xi) = x^\mu(\xi_0)$, one obtains

$$S_{fix}[y, v_\pm] = \kappa \int d^2\xi \Big[v^\mu_+ \Pi_{+\mu\nu}(\Delta V) v^\nu_- + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu) \Big]. \tag{11}$$

The gauge fixed action is the main crossway of the procedure, for an appropriate equation of motion it can transform both to initial action and to the T-dual action. For the equation of motion obtained varying the action over the Lagrange multipliers $\partial_+ v^\mu_- - \partial_- v^\mu_+ = 0$, with solution $v^\mu_\pm = \partial_\pm x^\mu$, the gauge fixed action reduces to the initial action. For the equation of motion obtained varying the action over the gauge fields $\Pi_{\mp\mu\nu}[\Delta V] v^\nu_\pm + \frac{1}{2}\partial_\pm y_\mu = \mp \beta^\mp_\mu [V]$, where $\beta^\alpha_\mu[V] \equiv \partial_\mu B_{\nu\rho} \epsilon^{\alpha\beta} V^\nu \partial_\beta V^\rho$, one obtains the T-dual theory. Comparing the solutions for the gauge fields in these two directions, one obtains the T-dual coordinate transformation laws. So, for application along all directions we obtain the following connection

$$S[x] = \kappa \int_\Sigma d^2\xi \, \partial_+ x^\mu \Pi_{+\mu\nu}(x) \partial_- x^\nu \Leftrightarrow {}^\star S[y] = \frac{\kappa^2}{2} \int d^2\xi \, \partial_+ y_\mu \Theta^{\mu\nu}_-(\Delta V) \partial_- y_\nu, \tag{12}$$

with $\Delta V^\mu = V^\mu(\xi) - V^\mu(\xi_0)$, $\quad V^\mu = (g^{-1})^{\mu\nu}[(2bG^{-1})_\nu{}^\rho y_\rho + \tilde{y}_\nu]$. The dual background field composition is defined by

$$\Theta^{\mu\nu}_\pm = -\frac{2}{\kappa}(G^{-1}_E \Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G^{-1}_E)^{\mu\nu}, \tag{13}$$

and consequently the T-dual background are

$$\begin{aligned} G_{\mu\nu} \quad &\Leftrightarrow \quad {}^\star G^{\mu\nu}(y, \tilde{y}) = (G^{-1}_E)^{\mu\nu}(\Delta V), \\ B_{\mu\nu}(x) \quad &\Leftrightarrow \quad {}^\star B^{\mu\nu}(y, \tilde{y}) = \frac{\kappa}{2}\theta^{\mu\nu}(\Delta V), \end{aligned} \tag{14}$$

where $G_{E\mu\nu}$ and $\theta^{\mu\nu}$ are the effective metric and the noncommutativity parameter for open bosonic string, which are

$$G_{E\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \, \theta^{\mu\nu} = -\frac{2}{\kappa}(G^{-1}_E B G^{-1})^{\mu\nu}. \tag{15}$$

But what if one does not consider T-dualization over all coordinates, but only some set of coordinates. To investigate this problem, let us mark a T-dualization along direction x^μ by T^μ and a T-dualization along dual direction y_μ by T_μ. Also mark the T-dualizations along some d initial directions, all other $D-d$ initial directions, and all initial directions by

$$\mathcal{T}^a = \circ_{n=1}^{d} T^{\mu_n}, \qquad \mathcal{T}^i = \circ_{n=d+1}^{D} T^{\mu_n}, \qquad \mathcal{T} = \circ_{n=1}^{D} T^{\mu_n} \tag{16}$$

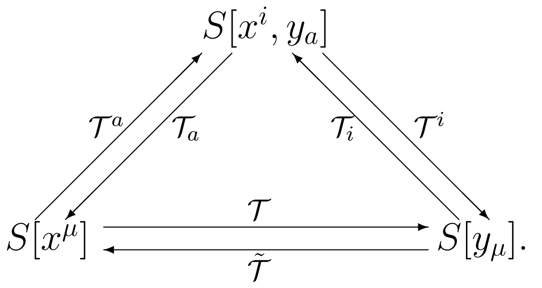
and T-dualizations along corresponding dual directions by

$$\mathcal{T}_a = \circ_{n=1}^{d} T_{\mu_n}, \qquad \mathcal{T}_i = \circ_{n=d+1}^{D} T_{\mu_n}, \qquad \tilde{\mathcal{T}} = \circ_{n=1}^{D} T_{\mu_n} \tag{17}$$

$\mu_n \in (0, 1, \ldots, D-1)$. We showed in [15] that these T-dualizations form an Abelian group

$$\mathcal{T}^i \circ \mathcal{T}^a = \mathcal{T}, \quad \mathcal{T}_i \circ \mathcal{T}_a = \tilde{\mathcal{T}}, \quad \mathcal{T}_a \circ \mathcal{T}^a = 1. \tag{18}$$

We showed that all the theories T-dual to the theory of the closed bosonic string are the part of the T-dualization diagram, given by



This diagram clearly describes the connection between arbitrary theory and the initial and completely T-dual theory. The explicit form of the theory obtained T-dualizing some set (marked by a) of the initial coordinates is the following

$$\begin{aligned} S[x^i, y_a] \;=\; \kappa \int d^2\xi \Big[& \partial_+ x^i \overline{\Pi}_{+ij}(x^i, \Delta V^a(x^i, y_a)) \partial_- x^j \\ & -\kappa\, \partial_+ x^i \Pi_{+ia}(x^i, \Delta V^a(x^i, y_a)) \tilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a)) \partial_- y_b \\ & +\kappa\, \partial_+ y_a \tilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a)) \Pi_{+bi}(x^i, \Delta V^a(x^i, y_a)) \partial_- x^i \\ & +\frac{\kappa}{2}\, \partial_+ y_a \tilde{\Theta}_-^{ab}(x^i, \Delta V^a(x^i, y_a)) \partial_- y_b \Big]. \end{aligned} \tag{19}$$

The new background field compositions $\overline{\Pi}_{\pm ij}$ and $\tilde{\Theta}_\pm^{ab}$ are defined as the inveres of the ordinary background field compositions $\Theta_\mp^{jk}$ and $\Pi_{\mp bc}$ reduced to the appropriate d and $D-d$ dimensional subspaces

$$\overline{\Pi}_{\pm ij} \Theta_\mp^{jk} = \Theta_\mp^{kj} \overline{\Pi}_{\pm ji} = \frac{1}{2\kappa} \delta_i^k, \tag{20}$$

$$\tilde{\Theta}^{ab}_{\pm}\Pi_{\mp bc} = \Pi_{\mp cb}\tilde{\Theta}^{ba}_{\pm} = \frac{1}{2\kappa}\delta^a_c. \tag{21}$$

It can be shown that

$$\overline{\Pi}_{+ij} \equiv \Pi_{+ij} - 2\kappa\Pi_{+ia}\tilde{\Theta}^{ab}_{-}\Pi_{+bj}. \tag{22}$$

The argument of the background fields is

$$\begin{aligned} \Delta V^{(0)a}(x^i, y_a) &= -\kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} + \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta x^{(0)i} \\ &- \kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} - \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta\tilde{x}^{(0)i} \\ &- \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} + \tilde{\Theta}^{ab}_{0-}\Big]\Delta y^{(0)}_b - \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} - \tilde{\Theta}^{ab}_{0-}\Big]\Delta\tilde{y}^{(0)}_b, \end{aligned} \tag{23}$$

where $\Delta x^\mu(\xi) = x^\mu(\xi) - x^\mu(\xi_0)$ and $\Delta y_\mu(\xi) = y_\mu(\xi) - y_\mu(\xi_0)$ while $\Delta\tilde{x}^\mu(\xi)$ and $\Delta\tilde{y}_\mu(\xi)$ are their duals, defined by

$$\Delta\tilde{x}^\mu(\xi) = \int_P d\xi^\alpha\, \varepsilon^\beta{}_\alpha\, \partial_\beta x^\mu, \quad \Delta\tilde{y}(\xi) = \int_P d\xi^\alpha\, \varepsilon^\beta{}_\alpha\, \partial_\beta y_\mu. \tag{24}$$

Calculating the symmetric and antisymmetric part of the background fields we obtain the T-dual metric and Kalb-Ramond field:

$$\begin{aligned} &\bullet G_{ij} = \overline{G}_{ij} = G_{ij} - G_{ia}(\tilde{G}^{-1}_E)^{ab}G_{bj} \\ &\quad -2\kappa\Big(B_{ia}\tilde{\theta}^{ab}G_{bj} + G_{ia}\tilde{\theta}^{ab}B_{bj}\Big) - 4B_{ia}(\tilde{G}^{-1}_E)^{ab}B_{bj} \\ &\bullet B_{ij} = \overline{B}_{ij} = B_{ij} - \frac{\kappa}{2}G_{ia}\tilde{\theta}^{ab}G_{bj} - B_{ia}(\tilde{G}^{-1}_E)^{ab}G_{bj} \\ &\quad -G_{ia}(\tilde{G}^{-1}_E)^{ab}B_{bj} - 2\kappa B_{ia}\tilde{\theta}^{ab}B_{bj} \\ &\bullet G^{ab} = (\tilde{G}^{-1}_E)^{ab} \\ &\bullet B^{ab} = \frac{\kappa}{2}\tilde{\theta}^{ab} \\ &\bullet G^a{}_i = \kappa\tilde{\theta}^{ab}G_{bi} + 2(\tilde{G}^{-1}_E)^{ab}B_{bi} \\ &\bullet B^a{}_i = \kappa\tilde{\theta}^{ab}B_{bi} + \frac{1}{2}(\tilde{G}^{-1}_E)^{ab}G_{bi}. \end{aligned} \tag{25}$$

As the constituents of the dual background field there appear the effective metric in the d-dimensional subspace a, defined by

$$\tilde{G}_{Eab} \equiv G_{ab} - 4B_{ac}(\tilde{G}^{-1})^{cd}B_{db}, \tag{26}$$

the non-commutativity parameter in the same subspace

$$\tilde{\theta}^{ab} \equiv -\frac{2}{\kappa}(\tilde{G}^{-1}_E)^{ac}B_{cd}(\tilde{G}^{-1})^{db}, \tag{27}$$

which combined give the new theta function $\tilde{\Theta}^{ab}_{\pm} = \tilde{\theta}^{ab} \mp \frac{1}{\kappa}(\tilde{G}^{-1}_{E})^{ab}$.

Every arrow in the T-duality diagram is accompanied with the appropriate T-dual coordinate transformation law. These are obtained comparing the solutions for the gauge fields in a T-dualization procedures performed between two actions in both directions. The laws for transitions

$$\mathcal{T}^a : S[x^\mu] \rightarrow S[x^i, y_a], \quad \mathcal{T}_a : S[x^i, y_a] \rightarrow S[x^\mu],$$

which are inverse to each other, are given by

$$\begin{aligned} \partial_\mp x^a &\cong -2\kappa \tilde{\Theta}^{ab}_{\mp}(x^i, \Delta V^a(x^i, y_a)) \cdot \\ &\quad \cdot \Big[\Pi_{\pm bi}(x^i, \Delta V^a(x^i, y_a))\partial_\mp x^i + \frac{1}{2}\partial_\mp y_b \mp \beta^\pm_b(x^i, V^a(x^i, y_a))\Big] \\ x^{(0)a} &\cong V^{(0)a}(x^i, y_a) \end{aligned} \tag{28}$$

and its inverse

$$\begin{aligned} \partial_\mp y_a &\cong -2\Pi_{\pm a\mu}(x)\partial_\mp x^\mu \pm 2\beta^\pm_a(x), \\ y^{(0)}_a &\cong U^{(0)}_a(x). \end{aligned} \tag{29}$$

These relations enable an investigation of the closed string non-commutativity and other geometric properties of the T-dual backgrounds. One can determine the geometric structure for an arbitrary sigma model in a T-duality diagram, find the connection between the Poisson structures of T-dual theories and the relations between non-commutativity parameters. The coordinates of the closed string are commutative when the string moves in a constant background. In a three dimensional space with the Kalb-Ramond field depending on one of the coordinates, successive T-dualizations along isometry directions lead to a theory with Q flux and the non-commutative coordinates [16, 17, 18]. Using the generalized T-dualization procedure, we found the non-commutativity characteristics of a closed string moving in the weakly curved background [15] comparing the initial and completely T-daul theory. One can expect the further investigations will reveal novelties regarding the form of the fluxes of all T-dual background forming a diagram. For now it is known that all fluxes are of type R.

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Closed string noncommutativity in the weakly curved background*

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Abstract

We consider the closed bosonic string moving in the weakly curved background. Using T-duality transformation laws we calculate the Poisson brackets of the coordinates in the T-dual space assuming that initial theory is geometric one, which means that standard Poisson algebra is obeyed. The result is that the commutative initial theory is equivalent to the non-commutative T-dual theory. All noncommutativity parameters are infinitesimal and proportional to the $B_{\mu\nu\rho}$, field strength of Kalb-Ramond field $B_{\mu\nu}$. In addition we find the algebra of the T-dual winding numbers and momenta in terms of the winding numbers and momenta of the initial theory.

1. Introduction

In order to obtain noncommutativity in the open string case it is enough to consider the open string in the presence of the *constant* gravitational $G_{\mu\nu}$ and Kalb-Ramond field $B_{\mu\nu}$ and use the boundary conditions [1, 2]. Treating boundary conditions as canonical constraints and solving them, one gets the initial coordinates expressed in terms of the Ω even effective coordinates and momenta, where Ω is world-sheet parity transformation $\Omega : \sigma \to -\sigma$. Because effective variables have nonzero Poisson bracket (PB), the PB between initial coordinates is also nonzero. The noncommutativity parameter is proportional to the Kalb-Ramond field $B_{\mu\nu}$.

There is one interesting thing which we noted in the open string case. The effective metric and the noncommutativity parameter are (up to some constants) the backgroud fields of the T-dual theory. As we know T-dual theory is physically equivalent to the initial one in the sense they have the same degrees of freedom - one at the scale R and the T-dual one at the scale $1/R$. The mathematical realization of the T-duality goes through Buscher procedure [3]. As a result of the procedure we get the relation between initial and T-dual variables which we call *transformation laws.*

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The closed strings do not have endpoints, so in the constant background there are no boundary conditions. To obtain noncommutativity in the closed string case we have to use T-duality as a helping tool. But, in the constant background case, T-duality relates σ-derivatives of the coordinates of one theory with the momenta of its T-dual one. Assuming that momenta of the initial theory commute (geometric theory) it follows that the T-dual coordinates commute as well. Consequently, in the constant background case there is no closed string non-commutativity.

It is obvious that T-duality is just one part of the solution in order to get the closed string noncommutativity. The second part is coordinate dependent background obeying the space-time field equations [4, 5]. Considering the closed string in the constant gravitational field $G_{\mu\nu}$ and Kalb-Ramond field depending on one coordinate, the closed string non-commutativity was first observed in the paper [6], and investigated further in [7, 8, 9]. In these articles 3-torus is considered, where $B_{\mu\nu}$ depends on one coordinate and T-dualization is performed along two other coordinates (isometry directions) using standard Buscher procedure [3].

One can ask if it is possible to do that in the background where $B_{\mu\nu}$ depends on all space-time coordinates. The answer is affirmative but in order to achieve that we have to use the generalized T-duality procedure presented in details in [10] and to apply it to the weakly curved background. The weakly curved background used in the present article is defined by constant gravitational $G_{\mu\nu} = const$ and the linear Kalb-Ramond field $B_{\mu\nu} = b_{\mu\nu} + \frac{1}{3}B_{\mu\nu\rho}x^\rho$, where the field strength $B_{\mu\nu\rho}$ is supposed to be infinitesimal. Such background obeys space-time field equations [4, 5] in the linear approximation in $B_{\mu\nu\rho}$.

We perform the generalized T-dualization procedure [10] along all the coordinates and obtain the T-duality transformation law, $\partial_\pm y_\mu = \partial_\pm y_\mu(\partial_\pm x)$, where $\partial_\pm$ are world-sheet partial derivatives. Using canonical formalism, the T-dual coordinates are expressed in terms of the original variables, $y'_\mu \cong \frac{1}{\kappa}\pi_\mu - \beta^0_\mu[x]$, where π_μ are canonically conjugated momenta to the coordinates x^μ. The infinitesimal expression β^0_μ is the correction in comparison to the flat background case. Assuming that the coordinates and momenta of the original theory satisfy standard Poisson algebra (initial theory is geometric one), we get the coordinate noncommutativity relations in the T-dual picture. In addition, we obtain the complete algebra of the T-dual winding numbers and momenta.

2. Generalized T-duality and noncommutativity

We consider the closed bosonic string moving in the D-dimensional space-time described by the action

$$S[x] = \kappa \int_\Sigma d^2\xi \; \partial_+ x^\mu \left(B_{\mu\nu}[x] + \frac{1}{2}G_{\mu\nu}[x] \right) \partial_- x^\nu \,, \tag{1}$$

where the light-cone coordinates are defined as $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$ and the corresponding derivatives $\partial_\pm = \partial_\tau \pm \partial_\sigma$. In order to keep conformal invariance on the quantum level, the background fields have to obey the following one-loop consistency conditions [4, 5]

$$R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} = 0, \quad D_\rho B^\rho{}_{\mu\nu} = 0\,. \tag{2}$$

Here $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are Ricci tensor and the covariant derivative with respect to the space-time metric.

The solution of the equations in the first order in $B_{\mu\nu\rho}$, so called the weakly curved background, [7, 10, 11, 12], is defined by

$$\begin{aligned} G_{\mu\nu}[x] &= const, \\ B_{\mu\nu}[x] &= b_{\mu\nu} + h_{\mu\nu}[x] = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho, \qquad b_{\mu\nu}, B_{\mu\nu\rho} = const. \end{aligned} \tag{3}$$

Here, the field strength $B_{\mu\nu\rho}$ is infinitesimal.

Applying the generalized T-dualization procedure [10] on the closed string propagating in the weakly curved background, we obtain the T-dual action

$${}^\star S[y] = \frac{\kappa^2}{2} \int d^2\xi \; \partial_+ y_\mu \Theta_-^{\mu\nu}[\Delta V[y]] \partial_- y_\nu, \tag{4}$$

where

$$\begin{aligned} \Theta_\pm^{\mu\nu} &\equiv -\frac{2}{\kappa} (G_E^{-1} \Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa} (G_E^{-1})^{\mu\nu}\,, \\ G_{E\mu\nu} &\equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}\,, \quad \Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2} G_{\mu\nu}\,. \end{aligned} \tag{5}$$

The argument ΔV is defined nonlocally as

$$\Delta V^\mu[y] = -\kappa \theta_0^{\mu\nu} \Delta y_\nu + (g^{-1})^{\mu\nu} \Delta \tilde{y}_\nu\,, \tag{6}$$

where

$$\Delta y_\mu = \int_P (d\tau \dot{y}_\mu + d\sigma y'_\mu) = y_\mu(\xi) - y_\mu(\xi_0), \quad \Delta \tilde{y}_\mu = \int_P (d\tau y'_\mu + d\sigma \dot{y}_\mu), \tag{7}$$

and

$$g_{\mu\nu} = G_{\mu\nu} - 4(bG^{-1}b)_{\mu\nu}, \quad \theta_0^{\mu\nu} = -\frac{2}{\kappa} (g^{-1} b G^{-1})^{\mu\nu}\,. \tag{8}$$

It is obvious from the definitions (7) that these two coordinates are related by the following expressions, $\dot{y}_\mu = \tilde{y}'_\mu$, $y'_\mu = \dot{\tilde{y}}_\mu$.

The transformation laws connecting initial and T-dual coordinates play the key role in our considerations. To be more precise, we obtain from T-dualization procedure the relations between world-sheet derivatives of the initial and T-dual coordinates

$$\partial_\pm x^\mu \cong -\kappa \Theta_\pm^{\mu\nu}[\Delta V] \Big[\partial_\pm y_\nu \pm 2\beta_\nu^\mp[V] \Big], \tag{9}$$

where

$$\beta^{\pm}_\mu[x] = \frac{1}{2}(\beta^0_\mu \pm \beta^1_\mu) = \mp\frac{1}{2}h_{\mu\nu}[x]\partial_\mp x^\nu\,,$$
$$\beta^0_\mu[x] = h_{\mu\nu}[x]x'^\nu,\ \beta^1_\mu[x] = -h_{\mu\nu}[x]\dot{x}^\nu\,. \tag{10}$$

Because we use the canonical formalism, we must have these transformation laws in the canonical form

$$x'^\mu \cong \frac{1}{\kappa}\,{}^\star\pi^\mu - \kappa\theta_0^{\mu\nu}\beta^0_\nu[V], \tag{11}$$
$$\pi_\mu \cong \kappa y'_\mu + \kappa\beta^0_\mu[V]\,, \tag{12}$$

where π_μ and ${}^\star\pi^\mu$ are canonically conjugated momenta to the coordinates x^μ and y_μ, respectively. It is shown in Ref. [10] that the T-dual of the T-dual action is the initial one. If we want to have T-dual coordinates in terms of the initial ones, we just have to invert the relation (9)

$$\partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}[\Delta x]\partial_\pm x^\nu \mp 2\beta^\mp_\mu[x]. \tag{13}$$

The canonical form of the T-dual transformations is

$$y'_\mu \cong \frac{1}{\kappa}\pi_\mu - \beta^0_\mu[x], \tag{14}$$
$${}^\star\pi^\mu \cong \kappa x'^\mu + \kappa^2\theta_0^{\mu\nu}\beta^0_\nu[x]. \tag{15}$$

Our intention is to calculate the PB's of the T-dual variables y_μ and $\tilde{y}_\mu$ using PB algebra of the initial variables. Consequently, we assume that initial theory is geometric which means that coordinates x^μ and momenta π_ν satisfy standard PB algebra

$$\{x^\mu(\sigma), \pi_\nu(\bar\sigma)\} = \delta^\mu_\nu\delta(\sigma-\bar\sigma), \quad \{x^\mu(\sigma), x^\nu(\bar\sigma)\} = 0, \quad \{\pi_\mu(\sigma), \pi_\nu(\bar\sigma)\} = 0. \tag{16}$$

In this article we will calculate, besides already mentioned PB algebra of the T-dual coordinates, also the algebra of the T-dual winding numbers and momenta. For both purposes, the first step is introducing the quantity

$$\Delta Y_\mu(\sigma,\sigma_0) = \int_{\sigma_0}^{\sigma} d\eta\, Y'_\mu(\eta) = Y_\mu(\sigma) - Y_\mu(\sigma_0), \tag{17}$$

where $Y_\mu = (y_\mu, \tilde{y}_\mu)$. The second step is to calculate their PB's. It is obvious that key relation which we have to calculate is PB between σ derivatives of Y's. When we calculate it in three possible cases it turns out that it can be written in the form

$$\{X'_\mu(\sigma), Y'_\nu(\bar\sigma)\} \cong K'_{\mu\nu}(\sigma)\delta(\sigma-\bar\sigma) + L_{\mu\nu}(\sigma)\delta'(\sigma-\bar\sigma)\,. \tag{18}$$

Integrating this relation by parts over σ and $\bar{\sigma}$, after straightforward calculation, we extract PB we are searching for

$$\{X_\mu(\tau,\sigma), Y_\nu(\tau,\bar{\sigma})\} \cong -\left[K_{\mu\nu}(\sigma) - K_{\mu\nu}(\bar{\sigma}) + L_{\mu\nu}(\bar{\sigma})\right]\theta(\sigma-\bar{\sigma})\,, \tag{19}$$

where $\theta(\sigma)$ is the step function defined as

$$\theta(\sigma) = \begin{cases} 0 & \text{if } \sigma = 0 \\ 1/2 & \text{if } 0 < \sigma < 2\pi, \\ 1 & \text{if } \sigma = 2\pi \end{cases} \quad \sigma \in [0, 2\pi]. \tag{20}$$

This is a general form of the relation. Using transforation laws we calculate PB's in three cases: $\{y'_\mu(\sigma), y'_\nu(\bar{\sigma})\}$, $\{y'_\mu(\sigma), \tilde{y}'_\nu(\bar{\sigma})\}$ and $\{\tilde{y}'_\mu(\sigma), \tilde{y}'_\nu(\bar{\sigma})\}$, and express them in the form of (18). Reading the corresponding values of K and L and using (19), we get the noncommutativity relations for T-dual closed string coordinates

$$\{y_\mu(\sigma), y_\nu(\bar{\sigma})\} \cong -\frac{1}{\kappa}B_{\mu\nu\rho}[x^\rho(\sigma) - x^\rho(\bar{\sigma})]\theta(\sigma-\bar{\sigma}), \tag{21}$$

$$\{y_\mu(\sigma), \tilde{y}_\nu(\bar{\sigma})\} \cong -\Big\{\frac{1}{\kappa}B_{\mu\nu\rho}[\tilde{x}^\rho(\sigma) - \tilde{x}^\rho(\bar{\sigma})] - \frac{3}{2\kappa}\Gamma^E_{\rho,\mu\nu}[x^\rho(\sigma) - x^\rho(\bar{\sigma})]$$
$$+\frac{1}{\kappa}g_{\mu\nu} - \frac{3}{2\kappa}\Gamma^E_{\rho,\mu\nu}\, x^\rho(\bar{\sigma})\Big\}\theta(\sigma-\bar{\sigma}), \tag{22}$$

$$\{\tilde{y}_\mu(\sigma), \tilde{y}_\nu(\bar{\sigma})\} \cong -\Big\{-\frac{1}{\kappa}\Big[B_{\mu\nu\rho} - 6g_{\mu\alpha}Q^{\alpha\beta}{}_\rho g_{\beta\nu}\Big][x^\rho(\sigma) - x^\rho(\bar{\sigma})] \tag{23}$$
$$+\Big[-\frac{3}{2\kappa}\Big(\Gamma^E_{\mu,\nu\rho} - \Gamma^E_{\nu,\mu\rho}\Big) + \frac{4}{\kappa}B_{\mu\nu\sigma}(G^{-1}b)^\sigma{}_\rho\Big][\tilde{x}^\rho(\sigma) - \tilde{x}^\rho(\bar{\sigma})]\Big\}\theta(\sigma-\bar{\sigma})\,,$$

where

$$\tilde{x}'^\mu = \frac{1}{\kappa}(G^{-1})^{\mu\nu}\pi_\nu + 2(G^{-1}B)^\mu{}_\nu x'^\nu\,. \tag{24}$$

Here the infinitesimal fluxes are defined as

$$\Gamma^E_{\mu,\nu\rho} = \frac{1}{2}\Big(\partial_\nu G^E_{\mu\rho} + \partial_\rho G^E_{\mu\nu} - \partial_\mu G^E_{\nu\rho}\Big) = -\frac{4}{3}\Big(B_{\mu\sigma\nu}(G^{-1}b)^\sigma{}_\rho + B_{\mu\sigma\rho}(G^{-1}b)^\sigma{}_\nu\Big), \tag{25}$$

$$Q^{\mu\nu}{}_\rho = -\frac{1}{3}\Big[(g^{-1})^{\mu\sigma}(g^{-1})^{\nu\tau} - \kappa^2\theta_0^{\mu\sigma}\theta_0^{\nu\tau}\Big]B_{\sigma\tau\rho}. \tag{26}$$

For $\sigma = \bar{\sigma}$ we obtain that all PB's vanish, and consequently, coordinates commute. Also we can consider $\sigma = \bar{\sigma} + 2\pi$, which is the same point on the world-sheet as our first choice $\sigma = \bar{\sigma}$. Taking $\sigma = \bar{\sigma} + 2\pi$, three noncommutativity relations take the form

$$\{y_\mu(\sigma + 2\pi), y_\nu(\sigma)\} \cong -\frac{2\pi}{\kappa}B_{\mu\nu\rho}N^\rho\,, \tag{27}$$

$$\begin{aligned}
&\{y_\mu(\sigma+2\pi),\tilde{y}_\nu(\sigma)\}+\{y_\mu(\sigma),\tilde{y}_\nu(\sigma+2\pi)\}\cong-\frac{4\pi}{\kappa^2}B_{\mu\nu\rho}p^\rho \\
+\quad&\frac{\pi}{\kappa}\left(3\Gamma^E_{\rho,\mu\nu}-8B_{\mu\nu\lambda}b^\lambda{}_\rho\right)N^\rho,
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
&\{\tilde{y}_\mu(\sigma+2\pi),\tilde{y}_\nu(\sigma)\}\cong \\
\cong\quad&\frac{2\pi}{\kappa}\left[-B_{\mu\nu\rho}-6g_{\mu\alpha}Q^{\alpha\beta}{}_\rho g_{\beta\nu}+2B_{\mu\nu}{}^\lambda g_{\lambda\rho}+3\left(\Gamma^E_{\mu,\nu\lambda}-\Gamma^E_{\nu,\mu\lambda}\right)b^\lambda{}_\rho\right]N^\rho \\
+\quad&\frac{\pi}{\kappa^2}\left[3\left(\Gamma^E_{\mu,\nu\rho}-\Gamma^E_{\nu,\mu\rho}\right)p^\rho-8B_{\mu\nu\lambda}b^\lambda{}_\rho\right]p^\rho\,,
\end{aligned} \tag{29}$$

where $N^\mu=\frac{1}{2\pi}\left[x^\mu(\sigma+2\pi)-x^\mu(\sigma)\right]$ is winding number of the initial coordinates and

$$p_\mu=\frac{1}{2\pi}\int_\sigma^{\sigma+2\pi}d\eta\pi_\mu(\eta)\,, \tag{30}$$

is mean value of the momentum π_μ. Note that all three PB's are proportional to the Kalb-Ramond field strength which means they are infinitesimal.

In addition we can obtain the algebra of the T-dual winding number and momenta defined as

$$\Delta y_\mu(2\pi,0)=2\pi^\star N_\mu\,,\quad \Delta\tilde{y}_\mu(2\pi,0)=2\pi^\star P_\mu\,, \tag{31}$$

while we introduced earlier

$$\Delta x^\mu(2\pi,0)=2\pi N^\mu\,,\quad \Delta\tilde{x}^\mu(2\pi,0)=2\pi P^\mu\,. \tag{32}$$

Using (17), (18), transformation laws and above definitions we have

$$\{{}^\star N_\mu,{}^\star N_\nu\}=\frac{1}{\pi\kappa}B_{\mu\nu\rho}N^\rho\,, \tag{33}$$

$$\{{}^\star N_\mu,{}^\star P_\nu\}=\frac{1}{\pi\kappa}B_{\mu\nu\rho}P^\rho-\frac{3}{4\pi\kappa}\Gamma^E_{\rho,\mu\nu}N^\rho\,, \tag{34}$$

$$\begin{aligned}
\{{}^\star P_\mu,{}^\star P_\nu\}\quad=&\quad-\frac{1}{\pi\kappa}\left(B_{\mu\nu\rho}-6g_{\mu\alpha}Q^{\alpha\beta}{}_\rho g_{\beta\nu}\right)N^\rho \\
&+\quad\frac{1}{\pi}\left[-\frac{3}{2\kappa}(\Gamma^E_{\mu,\nu\rho}-\Gamma^E_{\nu,\mu\rho})+\frac{4}{\kappa}B_{\mu\nu\sigma}(G^{-1}b)^\sigma{}_\rho\right]P^\rho\,.
\end{aligned}$$

3. Concluding remarks

In the present article we considered the theory describing the closed bosonic string moving in the weakly curved background and derived the non-commutativity relations using canonical approach.

We applied generalized T-duality procedure and obtained the transformation laws connecting the initial and T-dual variables. They, expressed in the canonical form, have the central role in calculation of the PB's of the T-dual coordinates y_μ and $\tilde{y}_\mu$. Infinitesimal Kalb-Ramond field strength, as a part of the function β_μ, gives the main contribution to the noncommutativity parameters. The result is that we showed the physical equivalence of the commutative initial theory and noncommutative T-dual one in linear approximation in the field strength $B_{\mu\nu\rho}$.

The general structure of the non-commutativity relations is

$$\{Y_\mu(\sigma), Y_\nu(\bar{\sigma})\} = \{F_{\mu\nu\rho}\left[x^\rho(\sigma) - x^\rho(\bar{\sigma})\right] + \tilde{F}_{\mu\nu\rho}\left[\tilde{x}^\rho(\sigma) - \tilde{x}^\rho(\bar{\sigma})\right]\}\theta(\sigma - \bar{\sigma})\,, \tag{35}$$

where $Y_\mu = (y_\mu, \tilde{y}_\nu)$ and $F_{\mu\nu\rho}$ and $\tilde{F}_{\mu\nu\rho}$ are the constant and infinitesimally small fluxes. At the same points, for $\sigma = \bar{\sigma}$ all PB's are zero. In the important particular case for $\sigma = \bar{\sigma} + 2\pi$ we get

$$\{Y_\mu(\sigma + 2\pi), Y_\nu(\sigma)\} = 2\pi\left[(F_{\mu\nu\rho} + 2\tilde{F}_{\mu\nu\alpha}b^\alpha_\rho)N^\rho + \frac{1}{\kappa}\tilde{F}_{\mu\nu}{}^\rho p_\rho\right]\,, \tag{36}$$

where N^μ and p_μ are winding numbers and momenta of the original theory. In addition we calculated the PB algebra of the T-dual winding numbers and momenta in terms of the initial ones.

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T-dualization of a weakly curved background

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T-dualization of a weakly curved background

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Abstract. We consider a string moving in a weakly curved background, composed of a constant metric and a linearly coordinate dependent Kalb-Ramond field with an infinitesimal strength. We discuss the T-dualization procedure which we developed for a closed bosonic string moving in a weakly curved background. The procedure is a generalization of a Buscher T-dualization procedure and enables the T-dualization of the nonisometry directions. The same procedure is used to investigate the T-duals of an open bosonic string as well. The generalized T-dualizations give insight to the connection between the geometrical properties of the T-dual spaces.

1. Introduction

In string theory there exists a symmetry, T-duality, which allows the physical equivalence of the string living on the different geometrical structures of the compactified dimensions. The string living in a space with one dimension compactified on a radius R, has the same physical features as a string leaving in a space with one dimension compactified on a radius $\frac{\alpha'}{R}$, where α' is a Regge slope parameter. T-duality was first described in the context of toroidal compactification in [1] (thoroughly explained in [2]), it can be generalized to the arbitrary toroidal compactification [3], and extended to the non-flat conformal backgrounds [4]. The origin of T-duality is seen in a possibility that, unlike a point particle, the string can wrap around compactified dimensions.

The first T-dualization procedure, the prescription for obtaining a theory which is T-dual of a given theory, was defined by Buscher [5]. The procedure was done for a string sigma model, describing a string moving in a background composed of a metric $G_{\mu\nu}$, an antisymmetric field $B_{\mu\nu}$ and a dilaton field Φ. It is required that the metric admits at least one continuous abelian isometry which leaves the action invariant. The procedure is founded in gauging the isometry by introducing the gauge fields v^μ_α. In order to preserve the physical content of the original theory, one requires that the new fields v^μ_α are nonphysical, which is achieved by the requirement that the gauge fields have a vanishing field strength $F^\mu_{\alpha\beta} = \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha$. This requirement is included in the theory by adding the Lagrange multiplier term $y_\mu F^\mu_{01}$ into the Lagrangian. Fixing the gauge one obtains the gauge fixed Lagrangian which carries the information on both initial and a T-dual theory. The integration over the Lagrange multipliers y_μ, simply recovers the original theory. The integration over the gauge fields v^μ_α, produces the T-dual theory.

The standard T-dualization procedure is applicable along directions which do not appear as the background field arguments. The generalized T-dualization procedure which is applicable along an arbitrary coordinate was done in Refs. [6, 7, 8]. The procedure is founded in the standard procedure and keeps the main rules of the standard procedure. In order to gauge the global isometry, one introduces the gauge fields v^μ_α, as usual. The replacement of the derivatives $\partial_\alpha x^\mu$ with the covariant ones $D_\alpha x^\mu$, does not as before make the whole action invariant. The

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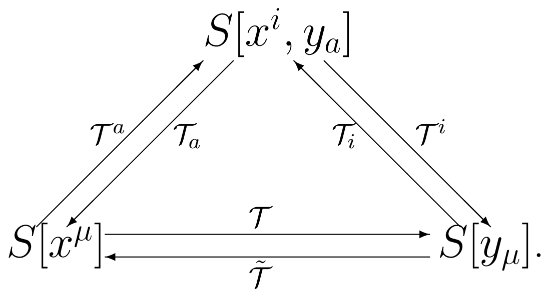
obstacle is the background field $B_{\mu\nu}$ depending on x^μ, which is not locally gauge invariant. So, as a new rule we substitute the argument of the background fields by an invariant argument Δx^μ_{inv}, defined as the line integral of the covariant derivatives of the original argument. As before, in order to obtain the theory physically equivalent to the original one, we add the Lagrange multiplier term. Using the local gauge freedom we fix the gauge taking $x^\mu(\xi) = x^\mu(\xi_0)$. The obtained gauge fixed action reduces to to the original action for the equations of motion for the Lagrange multiplier. The T-dual theory is obtained for the equations of motion for the gauge fields v^μ_α.

The generalized T-dualization procedure was investigated for a string moving in a weakly curved background composed of a constant metric, a linearly coordinate dependent Kalb-Ramond field with an infinitesimal filed strength and a constant dilaton field. It was first applied to all space-time coordinates in Ref. [6], and a T-dual was obtained. In Ref. [8], the procedure was applied to an arbitrary set of the initial coordinates. Choosing d arbitrary directions, we denote $\mathcal{T}^a = \circ_{n=1}^d T^{\mu_n}$, $\mathcal{T}^i = \circ_{n=d+1}^D T^{\mu_n}$, and $\mathcal{T} = \circ_{n=1}^D T^{\mu_n}$, where T^μ stands for a T-dualization along direction x^μ and $\mathcal{T}_a = \circ_{n=1}^d T_{\mu_n}$, $\mathcal{T}_i = \circ_{n=d+1}^D T_{\mu_n}$, $\tilde{\mathcal{T}} = \circ_{n=1}^D T_{\mu_n}$, where T_μ stands for the T-dualization along a dual direction y_μ. Performing the generalized procedure we proved the following composition laws:

$$\mathcal{T}^i \circ \mathcal{T}^a = \mathcal{T}, \quad \mathcal{T}_i \circ \mathcal{T}_a = \tilde{\mathcal{T}}, \quad \mathcal{T}_a \circ \mathcal{T}^a = 1, \tag{1}$$

where 1 denotes the identical transformation (T-dualization not performed). We found the explicit forms of the resulting theories and the corresponding T-dual coordinate transformation laws. These results complete the T-dualization diagram connecting all the theories T-dual to the initial theory.



The initial theory, describing the bosonic string moving in the weakly curved background is defined on the geometrical space. All its T-dual theories are non-geometric and non-local because they depend on variable V^μ, which is a line integral of the derivatives of the dual coordinates. To all of these theories there corresponds a flux which is of the same type as the R flux unlike the non-geometric theories with Q flux, which have a local geometric description.

2. Bosonic string action

Let us consider the action [9, 10] describing the propagation of the bosonic string in a background composed of a space-time metric $G_{\mu\nu}$, a Kalb-Ramond field $B_{\mu\nu}$ and a dilaton field Φ

$$S[x] = \kappa \int_\Sigma d^2\xi \sqrt{-g}\Big[\Big(\frac{1}{2}g^{\alpha\beta}G_{\mu\nu}(x) + \frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}}B_{\mu\nu}(x)\Big)\partial_\alpha x^\mu \partial_\beta x^\nu + \frac{1}{4\pi\kappa}\Phi(x)R^{(2)}\Big]. \tag{2}$$

The integration goes over two-dimensional world-sheet Σ parametrized by ξ^α ($\xi^0 = \tau,\ \xi^1 = \sigma$), $g_{\alpha\beta}$ is intrinsic world-sheet metric, $R^{(2)}$ corresponding 2-dimensional scalar curvature, $x^\mu(\xi)$, $\mu = 0, 1, ..., D-1$ are the coordinates of the D-dimensional space-time, $\kappa = \frac{1}{2\pi\alpha'}$ and $\varepsilon^{01} = -1$.

In order to have a world-sheet conformal invariance on the quantum level, the background fields have to obey the space-time equations of motion which in the lowest order in slope parameter α', have the following form

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_{\nu}{}^{\rho\sigma} + 2D_\mu \partial_\nu \Phi = 0, \\ &D_\rho B^\rho{}_{\mu\nu} - 2\partial_\rho \Phi B^\rho{}_{\mu\nu} = 0, \\ &4(\partial\Phi)^2 - 4D_\mu \partial^\mu \Phi + \frac{1}{12} B_{\mu\nu\rho} B^{\mu\nu\rho} - R + 4\pi\kappa \frac{D-26}{3} = 0\,, \end{aligned} \tag{3}$$

where $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are Ricci tensor and covariant derivative with respect to space-time metric. We consider the weakly curved background, defined by

$$G_{\mu\nu} = const, \quad B_{\mu\nu}(x) = b_{\mu\nu} + \frac{1}{3} B_{\mu\nu\rho} x^\rho \equiv b_{\mu\nu} + h_{\mu\nu}(x), \quad \Phi = const. \tag{4}$$

The Kalb-Ramond field strength $B_{\mu\nu\rho}$ is taken to be infinitesimal. All the calculations are done in the first order in $B_{\mu\nu\rho}$. In this approximation the weakly curved background is the solution of the space-time equations of motion (3).

Introducing the light-cone coordinates

$$\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$$

and their derivatives $\partial_\pm = \partial_\tau \pm \partial_\sigma$, taking a conformal gauge $g_{\alpha\beta} = e^{2F}\eta_{\alpha\beta}$, the action (2) can be written as

$$S[x] = \kappa \int_\Sigma d^2\xi \; \partial_+ x^\mu \Pi_{+\mu\nu}(x) \partial_- x^\nu, \tag{5}$$

where

$$\Pi_{\pm\mu\nu}(x) = B_{\mu\nu}(x) \pm \frac{1}{2} G_{\mu\nu}(x). \tag{6}$$

3. The Generalized Buscher T-dualization procedure

The standard T-dualization procedure, enables one to find a T-dual of a given theory, applying the procedure to the coordinate directions which do not appear as the background field arguments. The generalized T-dualization procedure does not have this limitation. Both procedures are grounded in a localization of a global coordinate shift symmetry $\delta x^\mu = \lambda^\mu = const$. The first rule of the procedures is the introduction of the gauge fields v^μ_α and the substitution of the ordinary derivatives with the covariant derivatives, defined by

$$\partial_\alpha x^\mu \to D_\alpha x^\mu = \partial_\alpha x^\mu + v^\mu_\alpha. \tag{7}$$

If one imposes the following transformation law for the gauge fields

$$\delta v^\mu_\alpha = -\partial_\alpha \lambda^\mu, \quad (\lambda^\mu = \lambda^\mu(\tau, \sigma)) \tag{8}$$

one obtains $\delta D_\alpha x^\mu = 0$. In the case when the background does not depend on the coordinates, along which the T-dualization is performed, the first step is sufficient to obtain the gauge invariant action. However if the background depends on all the coordinates, an additional rule must be introduced. The new rule reads: *Substitute the background field argument (the* *coordinate x^μ), by the invariant argument (invariant coordinate)*, defined as a line integral of the covariant derivatives of the original coordinate

$$\Delta x^\mu_{inv} \equiv \int_P d\xi^\alpha\, D_\alpha x^\mu = x^\mu - x^\mu(\xi_0) + \Delta V^\mu, \quad \Delta V^\mu \equiv \int_P d\xi^\alpha v^\mu_\alpha. \tag{9}$$

The invariant coordinate is by definition nonlocal. The consequence of this will be a nonlocal T-dual theory, defined on the doubled geometrical space composed of the dual coordinate y_μ and its double $\tilde{y}_\mu$.

The common rule of the procedures is the addition of the Lagrange multiplier term which makes the introduced gauge fields nonphysical, by requiring that there field strength

$$F^\mu_{\alpha\beta} \equiv \partial_\alpha v^\mu_\beta - \partial_\beta v^\mu_\alpha \tag{10}$$

must be zero. This enables the physical equivalence of the theories. Following these rules we built the gauge invariant action.

The main object and the main crossway of the procedure are the gauge fixed action and their equations of motion, because for the equation of motion obtained varying the action over the Lagrange multipliers, one returns to the initial action. On the other hand for the equation of motion obtained varying the gauge fixed action over the gauge fields one obtains the T-dual theory. Comparing the solutions for the gauge fields in these two directions, one obtains the T-dual coordinate transformation laws. These laws are used in investigation of the relations between the non-commutativity characteristics of the spaces connected by T-duality.

The generalized procedure, can be generalized once more in order to allow the T-dualization of the backgrounds which do not have a global symmetry. The generalization was made in Ref. [7] for a bosonic string moving in a weakly curved background of the second order, which consists of the coordinate dependent metric and Kalb-Ramond field. One postulates the auxiliary action which inherits two important features of the gauge fixed action. It reduces to the initial theory for the equations of motion for the Lagrange multipliers and to the T-dual action for the equations of motion for the auxiliary fields.

3.1. Complete T-dualization

If one applies the T-dualization procedure to all coordinates, one obtains a following gauge invariant action

$$S_{inv} = \kappa \int d^2\xi \Big[D_+ x^\mu \Pi_{+\mu\nu}(\Delta x_{inv}) D_- x^\nu + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu)\Big], \tag{11}$$

which is physically equivalent to the initial action. Fixing the gauge by $x^\mu(\xi) = x^\mu(\xi_0)$, one obtains the gauge fixed action

$$S_{fix}[y, v_\pm] = \kappa \int d^2\xi \Big[v^\mu_+ \Pi_{+\mu\nu}(\Delta V) v^\nu_- + \frac{1}{2}(v^\mu_+ \partial_- y_\mu - v^\mu_- \partial_+ y_\mu)\Big]. \tag{12}$$

In order to find a T-dual action one has to integrate out the gauge fields from (12).

The equations of motion with respect to the gauge fields $v^\mu_\pm$ are

$$\Pi_{\mp\mu\nu}(\Delta V) v^\nu_\pm + \frac{1}{2}\partial_\pm y_\mu = \mp \beta^\mp_\mu(V), \tag{13}$$

with the right hand side coming from the variation of the background fields argument, with $\beta^\pm_\mu(x) = \mp\frac{1}{2} h_{\mu\nu}[x]\partial_\mp x^\nu$. The equation of motion can be rewritten as

$$v^\mu_\pm(y) \quad = \quad -\kappa\, \Theta^{\mu\nu}_\pm[\Delta V(y)] \Big[\partial_\pm y_\nu \pm 2\beta^\mp_\nu[V(y)]\Big], \tag{14}$$

where

$$\Theta_{\pm}^{\mu\nu}[\Delta V] = -\frac{2}{\kappa}(G_E^{-1}\Pi_{\pm}G^{-1})^{\mu\nu} = \theta^{\mu\nu}[\Delta V] \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu}[\Delta V], \tag{15}$$

and $G^E_{\mu\nu} \equiv [G - 4BG^{-1}B]_{\mu\nu}$, $\theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu}$ are the open string background fields: the effective metric and the non-commutativity parameter respectively. They are defined in analogy with the flat space-time open string background fields introduced in [11]. Tensors $\Pi_{\mp\mu\nu}$ and $\Theta_{\pm}^{\mu\nu}$ are connected by $\Theta_{\pm}^{\mu\nu}\Pi_{\mp\nu\rho} = \frac{1}{2\kappa}\delta^{\mu}_{\rho}$. Substituting (14) into the action (12), we obtain T-dual action

$$^{\star}S[y] \equiv S_{fix}[y] = \frac{\kappa^2}{2}\int d^2\xi\ \partial_+ y_\mu \Theta_-^{\mu\nu}[\Delta V^{(0)}(y)]\partial_- y_\nu, \tag{16}$$

where we neglected the term $\beta^-_\mu \beta^+_\nu$ as the infinitesimal of the second order, and the argument is given by

$$\Delta V^{(0)\mu}(y) = -\kappa\theta_0^{\mu\nu}\Delta y^{(0)}_\nu + (g^{-1})^{\mu\nu}\Delta\tilde{y}^{(0)}_\nu. \tag{17}$$

Comparing the initial action (5) with the T-dual action (16), we see that they are equal under following transformations $\partial_\pm x^\mu \to \partial_\pm y_\mu$ and $\Pi_{+\mu\nu}[x] \to \frac{\kappa}{2}\Theta_-^{\mu\nu}[\Delta V^{(0)}]$, which implies

$$\begin{aligned} G_{\mu\nu} &\to {}^{\star}G^{\mu\nu} = (G_E^{-1})^{\mu\nu}[\Delta V^{(0)}], \\ B_{\mu\nu}[x] &\to {}^{\star}B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}[\Delta V^{(0)}], \end{aligned} \tag{18}$$

where $(G_E^{-1})^{\mu\nu}$ and $\theta^{\mu\nu}$ are introduced in (15).

The initial background consisted of a constant metric and a linearly coordinate dependent Kalb-Ramond field with an infinitesimal field strength. The T-dual background consists of coordinate dependent metric and Kalb-Ramond field, with the argument ΔV^μ, which is the linear combination of y_μ and its double $\tilde{y}_\mu$. Note that the variable V^μ and consequently T-dual action is not defined on the geometrical space (defined by the coordinate y_μ) but on the so called doubled target space [12] composed of both y_μ and $\tilde{y}_\mu$.

3.2. Partial T-dualization

If one choses only a subset of the initial coordinates, say d coordinates x^a, and performs T-dualization procedure along these coordinates, one obtains the following gauge invariant action

$$\begin{aligned} S_{inv}[x^\mu, x^a_{inv}, y_a] &= \kappa\int d^2\xi\Big[\partial_+ x^i \Pi_{+ij}(x^i, \Delta x^a_{inv})\partial_- x^j \\ &+ \partial_+ x^i \Pi_{+ia}(x^i, \Delta x^a_{inv}) D_- x^a + D_+ x^a \Pi_{+ai}(x^i, \Delta x^a_{inv})\partial_- x^i \\ &+ D_+ x^a \Pi_{+ab}(x^i, \Delta x^a_{inv}) D_- x^b + \frac{1}{2}(v^a_+ \partial_- y_a - v^a_- \partial_+ y_a)\Big]. \end{aligned} \tag{19}$$

This action is obtained localizing the global shift symmetry only for the coordinates x^a, by introducing the gauge fields v^a_α. The ordinary derivatives $\partial_\alpha x^a$ were substituted by the covariant derivatives $D_\alpha x^a = \partial_\alpha x^a + v^a_\alpha$. The covariant derivatives are invariant under the standard gauge transformations $\delta v^a_\alpha = -\partial_\alpha \lambda^a$. The coordinates x^a in the argument of the background fields were substituted by their invariant extension, defined by $\Delta x^a_{inv} \equiv \int_P d\xi^\alpha D_\alpha x^a = x^a - x^a(\xi_0) + \Delta V^a$, where $\Delta V^a \equiv \int_P d\xi^\alpha v^a_\alpha$. The physical equivalence is preserved by adding the Lagrange multiplier term (the last term in the action). Fixing the gauge by $x^a(\xi) = x^a(\xi_0)$ one obtains the gauge fixed action

$$\begin{aligned} S_{fix}[x^i, v^a_\pm, y_a] &= \kappa \int d^2\xi \Big[\partial_+ x^i \Pi_{+ij}(x^i, \Delta V^a)\partial_- x^j \\ &+ \partial_+ x^i \Pi_{+ia}(x^i, \Delta V^a) v^a_- + v^a_+ \Pi_{+ai}(x^i, \Delta V^a)\partial_- x^i \\ &+ v^a_+ \Pi_{+ab}(x^i, \Delta V^a) v^b_- + \frac{1}{2}(v^a_+ \partial_- y_a - v^a_- \partial_+ y_a)\Big]. \end{aligned} \tag{20}$$

This action reduces to the initial one for the equations of motion obtained varying over the Lagrange multipliers. The T-dual action is obtained for the equations of motion for the gauge fields. It reads

$$\begin{aligned} S[x^i, y_a] &= \kappa \int d^2\xi \Big[\partial_+ x^i \overline{\Pi}_{+ij}(x^i, \Delta V^a(x^i, y_a))\partial_- x^j \\ &-\kappa\, \partial_+ x^i \Pi_{+ia}(x^i, \Delta V^a(x^i, y_a))\tilde{\Theta}^{ab}_-(x^i, \Delta V^a(x^i, y_a))\partial_- y_b \\ &+\kappa\, \partial_+ y_a \tilde{\Theta}^{ab}_-(x^i, \Delta V^a(x^i, y_a))\Pi_{+bi}(x^i, \Delta V^a(x^i, y_a))\partial_- x^i \\ &+\frac{\kappa}{2}\, \partial_+ y_a \tilde{\Theta}^{ab}_-(x^i, \Delta V^a(x^i, y_a))\partial_- y_b\Big]. \end{aligned} \tag{21}$$

The T-dual background fields compositions are the inverses of the already known background compositions, divided into two coordinate subspaces, the subspace formed by the coordinates we T-dualize and the subspace formed by the rest of the coordinates. The background field compositions $\overline{\Pi}_{\pm ij}$ and $\tilde{\Theta}^{ab}_\pm$ are defined as the inverses of the background field compositions $\Theta^{jk}_\mp$ and $\Pi_{\mp bc}$, which are the parts of $\Theta^{\mu\nu}_\mp$ and $\Pi_{\mp\mu\nu}$ in an appropriate subspace

$$\begin{aligned} \overline{\Pi}_{\pm ij}\Theta^{jk}_\mp &= \Theta^{kj}_\mp \overline{\Pi}_{\pm ji} = \frac{1}{2\kappa}\delta^k_i, \\ \tilde{\Theta}^{ab}_\pm \Pi_{\mp bc} &= \Pi_{\mp cb}\tilde{\Theta}^{ba}_\pm = \frac{1}{2\kappa}\delta^a_c. \end{aligned} \tag{22}$$

It can be shown that

$$\overline{\Pi}_{+ij} \equiv \Pi_{+ij} - 2\kappa \Pi_{+ia}\tilde{\Theta}^{ab}_- \Pi_{+bj}. \tag{23}$$

The argument of the background fields is

$$\begin{aligned} \Delta V^{(0)a}(x^i, y_a) &= -\kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} + \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta x^{(0)i} \\ &- \kappa\Big[\tilde{\Theta}^{ab}_{0+}\Pi_{0-bi} - \tilde{\Theta}^{ab}_{0-}\Pi_{0+bi}\Big]\Delta \tilde{x}^{(0)i} \\ &- \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} + \tilde{\Theta}^{ab}_{0-}\Big]\Delta y^{(0)}_b - \frac{\kappa}{2}\Big[\tilde{\Theta}^{ab}_{0+} - \tilde{\Theta}^{ab}_{0-}\Big]\Delta \tilde{y}^{(0)}_b. \end{aligned} \tag{24}$$

Calculating the symmetric and antisymmetric part of the background fields we obtain a T-dual metric and a T-dual Kalb-Ramond field

$$\begin{aligned} &\bullet G_{ij} = \overline{G}_{ij} = G_{ij} - G_{ia}(\tilde{G}^{-1}_E)^{ab}G_{bj} \\ &\quad -2\kappa\Big(B_{ia}\tilde{\theta}^{ab}G_{bj} + G_{ia}\tilde{\theta}^{ab}B_{bj}\Big) - 4B_{ia}(\tilde{G}^{-1}_E)^{ab}B_{bj}, \\ &\bullet B_{ij} = \overline{B}_{ij} = B_{ij} - \frac{\kappa}{2}G_{ia}\tilde{\theta}^{ab}G_{bj} - B_{ia}(\tilde{G}^{-1}_E)^{ab}G_{bj} \\ &\quad -G_{ia}(\tilde{G}^{-1}_E)^{ab}B_{bj} - 2\kappa B_{ia}\tilde{\theta}^{ab}B_{bj}, \\ &\bullet G^{ab} = (\tilde{G}^{-1}_E)^{ab}, \\ &\bullet B^{ab} = \frac{\kappa}{2}\tilde{\theta}^{ab}, \\ &\bullet G^a{}_i = \kappa\tilde{\theta}^{ab}G_{bi} + 2(\tilde{G}^{-1}_E)^{ab}B_{bi}, \\ &\bullet B^a{}_i = \kappa\tilde{\theta}^{ab}B_{bi} + \frac{1}{2}(\tilde{G}^{-1}_E)^{ab}G_{bi}. \end{aligned} \tag{25}$$

As the constituents of the T-dual background field there appear the effective metric in the subspace a, defined by $\tilde{G}_{Eab} \equiv G_{ab} - 4B_{ac}(\tilde{G}^{-1})^{cd}B_{db}$, the non-commutativity parameter in the same subspace $\tilde{\theta}^{ab} \equiv -\frac{2}{\kappa}(\tilde{G}_E^{-1})^{ac}B_{cd}(\tilde{G}^{-1})^{db}$, which combined give the new theta function $\tilde{\Theta}_{\pm}^{ab} = \tilde{\theta}^{ab} \mp \frac{1}{\kappa}(\tilde{G}_E^{-1})^{ab}$.

4. Open string T-dualization

In paper [13] we investigated a T-duality of an open string moving in a weakly curved background. The open string moving in a weakly curved background was a subject of investigation in our papers [14, 15, 16]. Solving the boundary conditions at the open string end-points, one obtains the effective closed string described by the effective closed string theory S^{eff}, defined on the doubled space $(q^\mu, \tilde{q}^\mu)$. As the effective theory is closed string theory, one can try to apply the generalized T-dualization procedure to this theory. The effective theory is defined on the doubled theory, just as the T-duals of the closed string theory moving in the weakly curved background. So, the application in this case resembles the application of the T-dualization procedure to the T-dual theories.

The effective theory of the open string moving in the weakly curved background, obtained for the solution of the boundary conditions equals

$$S^{eff} = \kappa \int d\tau \int_{-\pi}^{\pi} d\sigma\, \partial_+ q^\mu\, \Pi^{eff}_{+\mu\nu}(q, 2b\tilde{q})\, \partial_- q^\nu, \tag{26}$$

where

$$\Pi^{eff}_{\pm\mu\nu}(q, 2b\tilde{q}) \equiv B^{eff}_{\mu\nu}(2b\tilde{q}) \pm \frac{1}{2} G^{eff}_{\mu\nu}(q). \tag{27}$$

The effective variable is $q^\mu(\sigma)$, an even part of the initial coordinate. The effective metric and the Kalb-Ramond field are explicitly given by

$$\begin{aligned} G^{eff}_{\mu\nu}(q) &= G^E_{\mu\nu}(q) := (G - 4B^2(q))_{\mu\nu}, \\ B^{eff}_{\mu\nu}(2b\tilde{q}) &= -\frac{\kappa}{2}\Big(g_E \Delta\theta(2b\tilde{q}) g_E\Big)_{\mu\nu}, \end{aligned} \tag{28}$$

where $\Delta\theta^{\mu\nu}$ is the infinitesimal part of the non-commutativity parameter $\theta^{\mu\nu} = -\frac{2}{\kappa}\Big[G_E^{-1}BG^{-1}\Big]^{\mu\nu} = \theta_0^{\mu\nu} - \frac{2}{\kappa}\Big[g_E^{-1}(h + 4bhb)g_E^{-1}\Big]^{\mu\nu}$. In paper [13] we applied the generalized Buscher T-dualization procedure, to the effective theory along all effective directions q^μ. Following the procedure we find the gauge fixed action

$$\mathcal{S}_{fix} = \kappa \int d^2\xi \Big[v_+^\mu \Pi^{eff}_{+\mu\nu}(\Delta V, 2b\Delta\tilde{V}) v_-^\nu + \frac{1}{2}(v_+^\mu \partial_- \varrho_\mu - v_-^\mu \partial_+ \varrho_\mu)\Big], \tag{29}$$

obtained from the effective action (26), by substituting the light-cone derivatives $\partial_\pm q^\mu$ with the covariant derivatives $D_\pm q^\mu = \partial_\pm q^\mu + v_\pm^\mu$, where $v_\pm^\mu$ are the gauge fields, which transform as $\delta v_\pm^\mu = -\partial_\pm \lambda^\mu$. The argument of the background fields is substituted with an invariant argument, which is obtained substituting the effective coordinate q^μ and its double $\tilde{q}^\mu$ with an invariant effective coordinate and its double, defined by the following line integrals of the gauge fields $\Delta V^\mu = \int_P (d\xi^+ v_+^\mu + d\xi^- v_-^\mu)$, and $\Delta\tilde{V}^\mu = \int_P (d\xi^+ v_+^\mu - d\xi^- v_-^\mu)$. The physical equivalence was achieved by adding the Lagrange multiplier term $\frac{1}{2}(v_+^\mu \partial_- \varrho_\mu - v_-^\mu \partial_+ \varrho_\mu)$ and the gauge is fixed with $q^\mu(\xi) = q^\mu(\xi_0)$.

The T-dual theory was obtained for the equation of motion for the gauge fields. The T-dual action reads

$$^\star\mathcal{S} = \kappa \int d^2\xi \partial_+ \varrho_\mu \frac{\kappa}{2} (\Theta_-^{eff})^{\mu\nu} (\Delta V(\varrho), 2b\Delta\tilde{V}(\varrho)) \partial_- \varrho_\nu, \tag{30}$$

where

$$(\Theta^{eff}_\pm)^{\mu\nu}(x,y) \equiv \Theta^{\mu\nu}_\pm(G_{eff}(x), B_{eff}(y)) = \theta^{\mu\nu}_{eff}(y) \mp \frac{1}{\kappa}(G^{-1}_E)^{\mu\nu}(x), \tag{31}$$

$\theta^{\mu\nu}_{eff} := \theta^{\mu\nu}(G_{eff}(x), B_{eff}(y)) = -\frac{2}{\kappa}\Big(G^{-1}_E(G_{eff}(x), B_{eff}(y))B_{eff}(y)G^{-1}_{eff}(x)\Big)^{\mu\nu}$ and the argument is

$$\begin{aligned} V^\mu_0(\varrho) &= (g^{-1}_E)^{\mu\nu}(G^{eff}, B^{eff})\tilde{\varrho}_\nu = (g^{-1}_E)^{\mu\nu}\tilde{\varrho}_\nu, \\ \tilde{V}^\mu_0(\varrho) &= (g^{-1}_E)^{\mu\nu}(G^{eff}, B^{eff})\varrho_\nu = (g^{-1}_E)^{\mu\nu}\varrho_\nu. \end{aligned} \tag{32}$$

The T-dual metric ${}^\star G^{\mu\nu}$ which depends on the first variable ΔV^μ and the T-dual Kalb-Ramond field ${}^\star B^{\mu\nu}$, which depends on the second variable $2b^\mu{}_\nu \Delta\tilde{V}^\nu$ are

$$\begin{aligned} {}^\star G^{\mu\nu} &= (G^{-1}_E)^{\mu\nu}(\Delta V), \\ {}^\star B^{\mu\nu} &= \frac{\kappa}{2}(\theta^{eff})^{\mu\nu}(2b\Delta\tilde{V}) = \frac{\kappa}{2}\Delta\theta^{\mu\nu}(2b\Delta\tilde{V}). \end{aligned} \tag{33}$$

We see, that the effective metric has transformed to its inverse and that the Kalb-Ramond field has transformed to the infinitesimal part of the non-commutativity parameter.

Finally, we searched for the open string theory $\tilde{S}$ such that its effective theory is ${}^\star S^{eff}$ exactly. We found

$$\widetilde{S}[y] = \kappa \int_\Sigma d^2\xi\ \partial_+ y_\mu \widetilde{\Pi}^{\mu\nu}_+(y)\partial_- y_\nu, \tag{34}$$

with

$$\begin{aligned} \widetilde{G} &= -(C^T)^{-1}GC^{-1}, \\ \widetilde{B}(y) &= \pm(C^T)^{-1}(b - h(C^{-1}y))C^{-1}, \end{aligned} \tag{35}$$

where C makes a connection between the variables of the effective theory of $\tilde{S}$ and the T-dual theory (30)

$$\begin{aligned} q_\mu(y) &= C_{\mu\nu}(g^{-1}_E)^{\nu\rho}\tilde{\varrho}_\rho\,, \\ \bar{q}_\mu(y) &= \mp C_{\mu\nu}2(G^{-1}bg^{-1}_E)^{\nu\rho}\varrho_\rho. \end{aligned} \tag{36}$$

In the closed string moving in the weakly curved background case, the T-duality transforms the geometrical background into a doubled non-geometrical background. It transforms a constant metric to a coordinate dependent effective metric inverse, while the linearly coordinate dependent Kalb-Ramond field is transformed into a coordinate dependent non-commutativity parameter. In the open string case, the T-dual theory remains geometric. T-duality transforms the constant metric of the weakly curved background to a constant T-dual metric, while the coordinate dependent Kalb-Ramond field transforms again to the coordinate dependent field.

In paper [17] a generalization of the standard analysis of the open bosonic string moving in a flat background is addressed. The T-dualization was performed in two ways, first in terms of non-constant vector fields in which case the Buscher T-dualization procedure can not be applied and second in terms of the field strengths of the gauge fields. The role of the gauge fields, which live on the string boundary, is to restore the symmetries of the closed string: the local gauge symmetry of the Kalb-Ramond field and the general coordinate transformations, at the string end-points. The investigation lead to a discovery of the geometrical features of the non-geometry.

Conclusion

The generalized T-dualization procedure, enabled T-dualization over the non isometry directions. It gives the new insights into a connection between the spaces connected by T-duality. It enabled further investigations of the closed string non-commutativity [18]. Comparing the solutions for the gauge fields which transform the gauge fixed actions into the initial or the T-dual actions, one obtains the T-dual coordinate transformation laws. Using these laws one can find how does for example a standard Poisson bracket transform. It is obtained that the original theory which is commutative is equivalent to the non-commutative T-dual theory, whose Poisson brackets are proportional to the background fluxes times winding and momentum numbers. The obtained results add novelty to the form and the origin of different non-commutative structures.

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