УНИВЕРЗИТЕТ У БЕОГРАДУ
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Naučnom veću Instituta za fiziku

Predlog za Godišnju nagradu za naučni rad Instituta za fiziku

Sa posebnim zadovoljstvom predlažemo dr Branislava Cvetkovića, višeg naučnog saradnika, za godišnju nagradu Instituta za fiziku za naučni rad za njegov doprinos razumevanju *Hamiltonove strukture i čestičnog spektra opšte lokalne Poenkareove teorije*. Navedeni rezultati ostvareni su u periodu od 01.01.2017. do 31.12.2018. godine u okviru projekta OI 171031 ,,Fizičke implikacije modifikovanog prostor-vremena" na Institutu za fiziku u Beogradu.

Tema istraživanja. Osnovna tema istraživačkog rada Branislava Cvetkovića odnosi se na *Poenkareovu gradijentu teoriju* (skraćeno PG), ili *teoriju gravitacije sa torzijom*. Ova teorija nastala je 60-tih godina prošlog veka kao alternativa Ajnštajnovoj Opštoj teoriji relativnosti (OTR). Alternativne teorije su nastale iz potrebe da se prevazidju ozbiljne slabosti OTR, kao što su postojanje klasičnih singularnosti (kod crnih rupa i u kosmologiji) i nemogućnost njene konzistentne kvantizacije. Polazeći od Poenkareove simetrije prostor-vremena bez gravitacionog polja, koja je u skladu sa svim poznatim eksperimentima u fizici osnovnih interakcija, PG teorija uvodi gravitaciju lokalizacijom ove simetrije, što joj daje istaknuto mesto medju alternativnim teorijama gravitacije.

Dosadašnja istraživanja u okviru PG uglavnom su bila ograničena na klasu Lagranžijana koji čuvaju parnost i koji su kvadratični po jačinama polja. Sezgin i Nivenhojzen su još osamdesetih godina analizirali čestični spektar PG^+ u aproksimaciji slabog polja oko prostora Minkovskog M_4, koristeći metod spinskih projektora i zaključili da odsustvo duhova i tahiona postavlja niz restrikcija na parametre PG^+. Najveći mogući broj torzionih modova koji mogu istovremeno da propagiraju je tri.

Opšti dinamički aspekti PG^+, uključujući i identifikaciju propagirajućih stepeni slobode se najbolje mogu razumeti u okviru Dirakovog pristupa za sisteme sa vezama . Blagojević i Nikolić su započeli sistematsku Hamiltonovu analizu PG^+, fokusirajući se na generičke aspekte teorije, identifikujući podskup primarnih veza koje su uvek prisutne, i koje su povezane sa Poenkareovom simetrijom. Ako su odredjeni kritični parametri jednaki nuli Blagojević i Nikolić su pronašli dodatne primarne veze – *ako veze* , konstruisali totalni Hamiltonijan i razmotrili odredjene aspekte procedure konzistentnosti.

Ne postoje fizički argumenti u korist očuvanja parnosti u gravitacionoj interakciji. Modeli koji *narušavaju parnost* sa svim mogućim kvadratičnim članovima u Lagranžijanu razmatrani su još osamdesetih godina, ali je nedavno porastao interes za razumevanje kako osnovne strukture tako i raznih dinamičkih aspekata ovih modela uključujući i kosmološke primene i talasna rešenja. Posebnu pažnju zavredjuje analiza čestičnog spektra koju je izveo Karananas u radu:

- G. K. Karananas, The particle spectrum of parity-violating Poincaré gravitational theory, Classical Quantum Gravity **32**, 055012 (2015); Erratum, Classical Quantum Gravity **32**, 089501 (2015).

koristeći formalizam spinskih projektora koji su Sezgin i Nivenhojzen uspešno primenili u PG$^+$. Iz njegovih rezultata sledi da je skup ,,dobrih" propagirajućih modova značajno uvećan u odnosu na PG$^+$.

Opis rezultata i ličnog doprinosa kandidata. Branislav Cvetković (u saradnji sa Milutinom Blagojevićem) razmatrao je generičke apekte Hamiltonove strukture opšte PG sa narušenjem parnosti i iskoristio ih da ispita čestični spektar tordiona. Koristeći Dirakov Hamiltonov pristup identifikovali su skupove svih mogućih ako-veza, izraza koji postaju prave veze ako su odgovarajući kritični parametri jednaki nuli. I ako-veze i njima pridruženi kritični parametri imaju ključni uticaj na dinamiku PG. Konstruisan je i opšti oblik kanonskog Hamiltonijana, koji je odredjen u slučaju kada su svi kritični parametri različiti od nule. Takodje, razmotreno je i proširenje procedure za slučaj kada su kritični parametri jednaki nuli.

Osim što je sama po sebi značajna kanoska struktura je veoma blisko povezana sa čestičnim spektrom PG, čije istraživanje je započeto računanjem svojstvenih vrenosti masa $m^2_\pm(J)$ torzionih modova spinova $J = 0, 1$, i 2, zasnovanim na aproksimaciji slabog polja u gravitacionim jednačinama kretanja oko M_4. Kao test konzistentnosti verifikovano je da su $m^2_\pm(J)$ proporcionalni inverznim vrednostima kritičnih parametara $1/c_n$, što za posledicu ima da kadgod je neki od parametara c_n jednak nuli odgovarajuće mase $m^2_\pm(J)$ postaju beskonačne, što sprečava propagaciju odgovarajućeg torzionog moda. Poredjenje masenih formula sa Karananasovim rezultatima dovelo je do sledećih zaključaka:

□ za modove spina-0 i spina-1 rezultati se slažu sa Karananasovim.

□ za modove spina-2 postoje značajne razlike koje su najpre uočene u radu:

  – M. Blagojević, B. Cvetković, and Y. N. Obukhov, Generalized plane waves in Poincaré gauge theory of gravity, Phys. Rev. D **96**, 064031 (2017).

Odsustvo duhova i tahiona u čestičnom spektru je osigurano pozitivnošću specifičnih spin-J članova u kanonskom Hamiltonijani, dok se odsustvo tahiona postiže zahtevom

$m^2_{\pm}(J) > 0$. Ovi uslovi postavljaju niz ograničenja na parametre PG koja su detaljno analizirana i uporedjena sa Karananasovim uslovima. Uslovi za odsustvo duhova se manje više slažu sa Karananasovim, dok se uslovima za odsutvo tahiona opažaju izvesne razlike. Najvažnija pronadjena je u sektoru spina-2, gde su dva uslova za odsustvo tahiona medjusobno *kontradiktorna* – nasuprot Karananasovom zaključku.

Analiza koju je izveo Branislav Cvetković razjasnila je strukturu čestičnog spektra opšte PG i poboljšala i ispravila rezultate koje je našao Karanas. Posebno su značajne korekcije u sektoru spina-2, gde je dobijeno da oba moda ne mogu biti *istovremeno propagirajuća* čak ni u slučaju opšte PG sa članovima koji narušavaju parnost. Ovo je zanimljiv i pomalo neočekivan rezultat nasuprot očekivanjima da će se stroge restrikcije na broj propagirajućih tordiona u PG^+ ,,relaksirati" uvodjenjem novih članova (i parametara) u Lagranžijan. S druge strane uvedeni elementi Hamiltonove strukture, uključujući i ekstenziju za slučaj kada su kritični parametri jednaki nuli, predstavljaju dobru polaznu osnovu za dalja istraživanja nelinearne dinamike PG.

Rezultati su objavljeni u radu:

- M. Blagojević and B. Cvetković, General Poincaré gauge theory: Hamiltonian structure and particle spectrum, Phys. Rev. D **98**, 024014 (2018) ,

i u potpunosti su ostvareni na Insitutu za fiziku u Beogradu.

Statistika radova i impakt rezultata na naučnu oblast. U recenziji rada, M. Blagojević and B. Cvetković, General Poincaré gauge theory: Hamiltonian structure and particle spectrum, Phys. Rev. D **98**, 024014 (2018), u koju smo imali uvid i koju prilažemo uz ovaj predlog referi izmedju ostalog kaže:

"This is a very well written comprehensive paper on an important topic. Such a work has been awaited every since the renewed interest that began with some papers in 2011 (Refs [21,22,23]). It was hoped that there would be an investigation into the Hamiltonian structure of the PG, the general Poincaré gauge theory of gravity (including both even and odd parity). This work goes beyond the expectations...This is a good foundation for further investigation of the full nonlinear dynamics of PG."

Rad je izabran za *Editors suggestion* za mesec jul 2018. godine i objavljen je ,,naslovnoj" internet strani časopisa Physical Review D.

U toku kalendarske 2017. i 2018. godine Branislav Cvetkovć osim pomenutog rada kategorije M21 objavio još 5 radova: 1 kategorije M21a i 4 kategorije M21, koji su do sada citirani 22 puta prema bazi InSpire. Ukupan impakt faktor radova je 26.64. Impakt dobijenih rezultata se ogleda u kvalitetu časopisa i kroz njihovu citiranost. Značaj radova dr Cvetkovića odnosi se na razumevanje Hamiltonove strukture i dinamike PG, kao i uopšte alternativnih teorija gravitacije.

Branislav Cvetković je bio mentor doktorske disertacije Dejana Simića odbranjene 2018. godine, kao i dva master rada odbranjena 2017. i 2018. godine. Svojim radom i razvijenom medjunarodnom Branislav Cvetković doprinosi prepoznatiljivosti Grupe za fiziku čestica i gravitaciju Insituta za fiziku.

Zbog svega navedenog smatramo da je dr Branislav Cvetković postigao izuzetne naučne rezultate u poslednje dve godine u oblasti alternativnih teorija gravitacije i zadovoljstvo nam je da ga predložimo za Godišnju nagradu Instituta za fiziku.

Beograd, 08.03.2019.

dr Milovan Vasilić
naučni savetnik
Institut za fiziku u Beogradu

prof. dr Branislav Sazdović
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Curriculum Vitae

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Ime i prezime Branislav Cvetković

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Obrazovanje Matematička gimnazija u Beogradu, 1994–1998, djak generacije, učestvovao na Medjunarodnoj olimpijadi iz matematike (Argetina, 1997) i Medjunarodnoj olimpijadi iz fizike (Island 1998)

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Osnovne studije: Fizički fakultet Univerziteta u Beogradu, 1998–2002, student generacije Diplomski rad: *Nekomutativna* ϕ^4 *teorija*
Mentor: Prof. Maja Burić
Magistarski rad: Fizički fakultet Univerziteta u Beogradu, 2005
Teza: *Kanonska struktura 3D gravitacije sa torzijom*
Mentor: Prof. Milutin Blagojević
Doktorat: Fizički fakultet Univerziteta u Beogradu, 2008
Teza: *Asimptotska struktura 3D gravitacije sa torzijom*
Mentor: Prof. Milutin Blagojević

Akademska karijera 2003–2008, Istraživač pripravnik i istraživač saradnik, Institut za fiziku, Beograd
2008–2013, Naučni saradnik, Institut za fiziku, Beograd
2013– Viši naučni saradnik, Institut za fiziku, Beograd

Studijske posete 2010, Institute for theoretical physics, TU Vienna, Austria
2009, 2012, 2015 Institute of physics, PUCV, Valparaiso, Chile

Jezici Srpski (materni)
Enlgleski(čitanje, pisanje, konverzacija)
Nemački (čitanje, pisanje, konverzacija)
Francuski (čitanje, pisanje, konverzacija)

Istraživanje Poenkareova gradijentna teorije gravitacije
Trodimenziona 3D (super)gravitacija
Hamiltonova dinamika sistema sa vezama
AdS/CFT i Kerr/CFT korespondencija

Rad sa studentima 2014-2018 Mentor Dejana Simića
Teza: *Lavlokova gravitacija sa torzijom: egzaktna rešenja, kanonka i holografska struktura*, odbranjena 2018. na Fizičkom fakultetu Univerziteta u Beogradu

Predavanja 2012– Kurs na doktorskim studijama "Teorija gravitacije II", Fizički fakultet Univerziteta u Beogradu
2004– Fizika, Matematička gimnazija

Ostalo Referi za medjunarodne časopise:
Physical Review Letters
Physical Review D
Classical and Quantum Gravity
Journal of physics A: Mathematical and Theoretical
International journal of physics A
European Journal of Physics
Referi za FONDECYT, Čileansku nacionalnu fondaciju za nauku
Član Organizacionih komiteta medjunarodnih konferencija:
2018 Workshop on Gravity, Holography, Strings and Noncommutative Geometry (Beograd 2018),
Gravity: new ideas for unsovled problems (Divč ibare 2011),
Gravity: new ideas for unsovled problems II (Divčibare 2013),
7th MATHEMATICAL PHYSICS MEETING, 9 - 19 September 2012, Belgrade, Serbia
Gravity:New Ideas for Unsolved Problems, September 12-14, 2011, Divibare, Serbia 6th MATHEMATICAL PHYSICS MEETING, 14-23 September 2010, Belgrade, Serbia
5th MATHEMATICAL PHYSICS MEETING, 6 - 17 July 2008, Belgrade, Serbia
IV Summer School in Modern Mathematical Physics, 3 - 14 September 2006, Belgrade, Serbia

Lične informacija Supruga, Sanja, Inženjer prehrambene tehnologije
Ćerka, Lenka, rodjena 2011

Publikacije

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2. B. Cvetković and D. Simić, Near horizon OTT black hole asymptotic symmetries and soft hair, Chin. Phys. C **43**, 013109 (2019).

3. M. Blagojević and B. Cvetković, General Poincaré gauge theory: Hamiltonian structure and particle spectrum, Phys. Rev. D **98**, 024014 (2018).

4. B. Cvetković and D. Simić, A black hole with torsion in 5D Lovelock gravity, Class. Quantum Grav. **35** (2018) 055005 (13pp).

5. M. Blagojević, B. Cvetković and Y. N. Obukhov, Generalized plane waves in Poincaré gauge theory of gravity, Phys. Rev. D **96**, 064031 (2017).

6. M. Blagojević and B. Cvetković, Generalized *pp* waves in Poincaré gauge theory, Phys. Rev D **95**, 104018 (2017).

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9. M. Blagojević and B. Cvetković, Conformally flat black holes in Poincaré gauge theory, Phys. Rev D **93**, 044018 (2016).

10. M. Blagojević and B. Cvetković, Vaidya-like exact solutions with torsion, JHEP**05**(2015)101.

11. M. Blagojević and B. Cvetković, Siklos waves in Poincaré gauge theory, Phys. Rev. D **92**, 024047 (2015).

12. M. Blagojević and B. Cvetković, Siklos waves with torsion in 3D, JHEP**11**(2014)141.

13. M. Blagojević and B. Cvetković, Gravitational waves with torsion in 3D, Phys. Rev. D **90**, 044006 (2014).

14. M. Blagojević and B. Cvetković, Three-dimensional gravity with propagating torsion: Hamiltonian structure of the scalar sector, Phys.Rev. D **88**, 104032 (2013).

15. M. Blagojević, B. Cvetković, M. Vasilić, Exotic black holes with torsion, Phys.Rev. D **88**, 101501(R) (2013).

16. M. Blagojevic, B. Cvetkovic, O. Miskovic and R. Olea, Holography in 3D AdS gravity with torsion, JHEP **1305** (2013) 103, [arXiv:1301.1237 [gr-qc]]

17. M. Blagojevic and B. Cvetkovic, 3D gravity with propagating torsion: the AdS sector, Phys.Rev. D **85** (2012) 104003, [arXiv:1201.4277 [gr-qc]]

18. H. Afshar, B. Cvetkovic, S. Ertl, D. Grumiller and N. Johansson, Conformal Chern-Simons holography - lock, stock and barrel, Phys. Rev. D **85** (2012) 0464033 [arXiv:1110.5644 [hep-th]].

19. H. Afshar, B. Cvetkovic, S. Ertl, D. Grumiller and N. Johansson, Holograms of Conformal Chern-Simons Gravity, Phys. Rev. D **84** (2011) 041502 [arXiv:1106.6299 [hep-th]].

20. M. Blagojevic and B. Cvetkovic, Extra gauge symmetries in BHT gravity, JHEP **1103** (2011) 139 [arXiv:1103.2388 [gr-qc]].

21. M. Blagojevic and B. Cvetkovic, Hamiltonian analysis of BHT massive gravity, JHEP **1101** (2011) 082 [arXiv:1010.2596 [gr-qc]].

22. M. Blagojevic and B. Cvetkovic, Conserved charges in 3D gravity, Phys. Rev. D **81** (2010) 124024 [arXiv:1003.3782 [gr-qc]].

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24. M. Blagojevic and B. Cvetkovic, Asymptotic structure of topologically massive gravity in spacelike stretched AdS sector, JHEP **0909** (2009) 006 [arXiv:0907.0950 [gr-qc]].

25. M. Blagojevic, B. Cvetkovic and O. Miskovic, Nonlinear electrodynamics in 3D gravity with torsion, Phys. Rev. D **80** (2009) 024043 [arXiv:0906.0235 [gr-qc]].

26. M. Blagojevic and B. Cvetkovic, Canonical structure of topologically massive gravity with a cosmological constant, JHEP **0905** (2009) 073 [arXiv:0812.4742 [gr-qc]].

27. M. Blagojevic and B. Cvetkovic, Self-dual Maxwell field in 3D gravity with torsion, Phys. Rev. D **78** (2008) 044037 [arXiv:0805.3627 [gr-qc]].

28. M. Blagojevic and B. Cvetkovic, Electric field in 3D gravity with torsion, Phys. Rev. D **78** (2008) 044036 [arXiv:0804.1899 [gr-qc]].

29. B. Cvetkovic and M. Blagojevic, Stability of 3D black hole with torsion, Mod. Phys. Lett. A **22** (2007) 3047 [arXiv:0712.1434 [gr-qc]].

30. B. Cvetkovic and M. Blagojevic, Supersymmetric 3D gravity with torsion: Asymptotic symmetries, Class. Quant. Grav. **24** (2007) 3933 [arXiv:gr-qc/0702121].

31. M. Blagojevic and B. Cvetkovic, Covariant description of the black hole entropy in 3D gravity, Class. Quant. Grav. **24** (2007) 129 [arXiv:gr-qc/0607026].

32. M. Blagojevic and B. Cvetkovic, Black hole entropy from the boundary conformal structure in 3D gravity with torsion, JHEP **0610** (2006) 005 [arXiv:gr-qc/0606086].

33. M. Blagojevic and B. Cvetkovic, Black hole entropy in 3D gravity with torsion, Class. Quant. Grav. **23** (2006) 4781 [arXiv:gr-qc/0601006].

34. M. Blagojevic and B. Cvetkovic, Canonical structure of 3D gravity with torsion, *Progress in General Relativity and Quantum Cosmology*, vol. 2, ed. Ch. Benton (Nova Science Publishers, New York, 2006), 103. [arXiv:gr-qc/0412134]

Izabrana predavanja na medjunarodnim konferencijama

1. M. Blagojević and B. Cvetković, *Asymptotic charges in 3d gravity with torsion*, talk "Fourth Meeting on Constrained Dynamics and Quantum Gravity" (Sardinia, Italy, 12–16 September 2005), J. Phys. Conf. Ser. **33** (2006) 248.

2. M. Blagojević and B. Cvetković, *The influence of torsion on the black hole entropy in 3d gravity*, talk at "4th Summer School in Mathematical Physics" (Belgrade, 3–14. September 2006), SFIN A **1** (2007) 51–66.

3. B. Cvetković and M. Blagojević, Supersymmetric 3D gravity with torsion: asymptotic symmetries and black hole stability, talk at "V International Symposium on Quantum Theory and Symmetries" (Valladolid, Špain 22–28 July 2007.), J. Phys. Conf. Series **128** (2008) 012001.

4. M. Blagojević and B. Cvetković, Self-dual Maxwell field in 3D gravity with torsion and dynamical role of central charges, talk at "Recent Developments in Gravity (NEB XIII)" (Solun, Grčka 4–06. jun 2008.), J. Phys. Conf. Ser. **189** (2009) 012010.

5. M. Blagojević and B. Cvetković, Asymptotic symmetries of spacelike stretched gravity, talk at "The Twelfth Marcel Grossmann Meeting" (Paris, France 12-18. July 2009.), Proceedings of the MG12 Meeting on General Relativity, pp. 1823-1825.

6. M. Blagojević and B. Cvetković, Canonical structure of new massive gravity, Salcburg 6-10 Septembar 2010. talk at *60th Annual Meeting of the Austrian Physical Society*, 69.

7. M. Blagojević and B. Cvetković, Vaidya-like exact solutions with torsion, talk at talk at "The Fourteenth Marcel Grossmann Meeting" (Rome, Italy 12-18. July 2015.), The Fourteenth Marcel Grossmann Meeting, pp. 2597-2602 (2017).

Objavljeni radovi u 2017. i 2018. godini i propratni materijal

Subject	**To_author DR12060 Blagojevi\'c PRD Editors' Suggestion**
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Dr. Branislav Cvetkovic
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Re: DR12060
General Poincar\'e gauge theory: Hamiltonian structure and particle spectrum
By: M. Blagojevi\'c and B. Cvetkovi\'c

Dear Dr. Cvetkovic,

We are pleased to inform you that we have selected your recently accepted manuscript to be a PRD Editors' Suggestion. A small fraction of papers which we judge to be particularly important, interesting, and well written is chosen for an Editors' Suggestion. Congratulations on your outstanding paper!

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Please email your image to prd@aps.org with the manuscript code number in the subject and "Suggestion image," or simply reply to this email.

Sincerely,

The PRD Editors

Referee Report: Manuscript Number DR12060
Title: General Poincaré gauge theory: Hamiltonian structure and particle spectrum
Authors: M. Blagojević and B. Cvetković.

This is a very well written comprehensive paper on an important topic. Such a work has been awaited every since the renewed interest that began with some papers in 2011 (Refs [21,22,23]). It was hoped that there would be an investigation into the Hamiltonian structure of the PG, the general Poincaré gauge theory of gravity (including both even and odd parity). This work goes beyond the expectations. The authors have done even more than was hoped for. Using the recent (very suitable) general choice of parameters of Ref [27], they developed the generic Hamiltonian formulation for the PG. All the critical parameter combinations and all the associated "if constraints" were identified. From an analysis of the linearized equations the mass eigenvalues for the linearized modes were identified, along with the associated positivity and reality conditions (no ghosts, no tachyons). These results were compared in detail with those found by Karananas Ref [25], and the differences discussed in detail. It is clear that not all the modes can be dynamic. The analysis clarifies the structure of the particle spectrum of the general PG by improving the results found by Karananas, in particular the status of the spin-2 sector. All the appropriate references have been cited. The work also includes some very useful technical information, e.g., (i) the comparison of parameters with Karananas given in eqs 6.18, (ii) Appendix B has an alternative form of the Lagrangian that allows an easier comparison to the literature, Refs [22, 25], (iii) Appendix D has a nice remarkably short and simple, discussion of how to handle *if*-constraints, and (iv) Appendix F has nice arguments for simpler sufficient conditions for reality of the masses. This is a good foundation for further investigation of the full nonlinear dynamics of PG.

I would be very happy to see this work published in PRD.

Here is a list of problems that were noticed; they all seem to be minor.

1. Eq 2.1a: Change the index m to μ.
2. Eq 2.2: Missing exterior product wedge symbol. Perhaps the authors did this intentionally, then it should be explained.
3. Eq 2.5: Change ν to μ.
4. Eq 2.9ab: Missing wedge symbols.
5. Eq 3.13c, last line: The $(\bar{b}_2 - \bar{b}_5)$ term seems out of place, this is probably a typo. As far as I can see there is no way for this term to arise here for spin 2 (such a coefficient belongs to spin 1, see eqs 3.17) and this coefficient is not propagated to the formulas 3.19, 3.20 or to section 6.3.
6. Eq 4.13bc: Replace $=$ by $\approx$.

7. Eq 5.3: Here and in many subsequent places the notation $\partial\mathcal{A}$ is used for apparently $\partial_i\mathcal{A}^i$. Although it is not difficult to guess the meaning, maybe it would be better to explain it at this point.

8. Eq 6.4, first line: The $(\mathrm{tr}M_0)$ in the radical should be squared.

9. Eq 6.5, first equation: The right hand side should be $G(\Box U_i - \partial_i\partial U)$, only then will the rhs have vanishing divergence as does the lhs. It seems that this is just a typo here, as the corresponding terms do appear in eq 6.7a.

10. Sec 7.3, line 1: misspelling: positivity

11. Eq between C.1.b and C.2: n_l to n^l.

Editors' Suggestion

General Poincaré gauge theory: Hamiltonian structure and particle spectrum

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Basic aspects of the Hamiltonian structure of the parity-violating Poincaré gauge theory are studied. We found all possible primary constraints, identified the corresponding critical parameters, and constructed the generic form of the canonical Hamiltonian. In addition to being important in their own right, these results offer dynamical information that is essential for a proper understanding of the particle spectrum of the theory, calculated in the weak field approximation around the Minkowski background.

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I. INTRODUCTION

Weyl's idea of gauge invariance [1] turned out to be a key principle underlying the dynamical structure of all the fundamental physical interactions. Following this idea and the subsequent works of Yang, Mills, and Utiyama [2], Kibble and Sciama [3] formulated a new theory of gravity, the Poincaré gauge theory (PG, aka PGT), based on gauging the Poincaré group of spacetime symmetries. In PG, spacetime is characterized by a Riemann-Cartan geometry, in which the torsion and curvature are the field strengths associated with the translation and Lorentz subgroups of the Poincaré group; for more details, see [4–10].

Earlier investigations of PG were mostly focused on the class of *parity preserving* Lagrangians quadratic in the field strengths; see, for instance, Hayashi and Shirafuji [5] or Obukhov [11]. We denote this class of models as PG^+. Sezgin and Niuwenhuizen [12] analyzed the particle spectrum of PG^+ in the weak field approximation around the Minkowski background M_4. Using the absence of ghosts and tachyons as physical requirements, they found a number of restrictions on the PG^+ parameters that ensure the propagating torsion modes to be well behaved.

General dynamical aspects of PG^+, including the identification of its physical degrees of freedom (d.o.f.), are most naturally understood in Dirac's Hamiltonian approach for constrained dynamical systems [13]. Blagojević and Nikolić [14,15] started a systematic Hamiltonian analysis of PG^+, focusing on its generic aspects. They identified a subset of the primary constraints that are always present ("sure" constraints, associated to the local Poincaré symmetry). Moreover, if certain critical parameters vanished, they found additional primary constraints ("if-constraints"), constructed the total Hamiltonian, and discussed certain aspects of the consistency procedure. Further advances in this direction were made by Cheng *et al.* [16] and Chen *et al.* [17], who found that the nonlinear nature of constraints may drastically change the number of propagating modes obtained in the linearized analysis. Yo and Nester [18] made a detailed study of this phenomenon in PG^+, concluding that there are apparently only two good propagating torsion modes. For an interesting application of this result to cosmology, see Shie *et al.* [19].

There are no physical arguments that favor the conservation of parity in the gravitational interaction. *Parity violating models* based on the general PG, with all possible quadratic invariants in the Lagrangian, were considered already in the 1980s [20], but the subject remained without wider response. Recently, there has been increased interest in a better understanding of both the basic structure and various dynamical aspects of these models, including cosmological applications and wave solutions [21–27]. In particular, one should mention the analysis of the particle spectrum carried out by Karananas [25], who made a suitable extension of the weak field approximation method used earlier in PG^+ [12] and applied it to the general PG. According to his results, it seems that the set of good modes that can coexist is significantly enlarged in comparison to PG^+.

The objective of the present work is to examine the Hamiltonian structure of the general PG, based on the if-constraint formalism [14,15,18], and use it to clarify the physical content of its particle spectrum, calculated in the weak field approximation around M_4. In this regard, a particularly important role is played by both the critical parameters appearing in the analysis of the primary constraints, and the structure of the canonical Hamiltonian. By comparing the properties of the particle spectrum to those found in Ref. [25], we noted certain differences. On the other side, elements of the Hamiltonian structure developed

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here can be a good starting point for studying the nonlinear dynamics of PG.

The paper is organized as follows. In Sec. II, we give a short account of the Lagrangian formalism for the general PG. In Secs. III and IV, we find the canonical critical parameters, identify the related if-constraints, and construct the generic, "most dynamical" canonical Hamiltonian, determined by the nonvanishing critical parameters. Then, in Secs. V and VI, we derive the linearized gravitational field equations and use them to identify the mass eigenvalues of the torsion modes. The conditions for the absence of ghosts and tachyons, as well as the reality conditions of the mass eigenvalues, are examined in Sec. VII. Essential features of the particle spectrum are either tested by or derived from the Hamiltonian structure of PG. In contrast to the results obtained in [25], we show that the two spin-2 torsion modes cannot propagate simultaneously. In Sec. VIII, we give a short summary of our results, and six appendices contain useful technical details, including an outline of the Hamiltonian formalism describing the case of vanishing critical parameters.

Our conventions are as follows. The Latin indices $(i, j, \dots)$ are the local Lorentz indices, the Greek indices $(\mu, \nu, \dots)$ are the coordinate indices, and both run over 0,1,2,3; the orthonormal frame (tetrad) is $b^i{}_\mu$, the inverse tetrad is $h_i{}^\mu$, the Lorentz connection is $\omega^{ij}{}_\mu$, and $\eta_{ij} = (1, -1, -1, -1)$ and $g_{\mu\nu} = \eta_{ij} b^i{}_\mu b^j{}_\nu$ are the metric components in the local Lorentz and coordinate frame, respectively; a totally antisymmetric tensor ε_{ijkl} is normalized to $\varepsilon_{0123} = +1$, and the dual of an antisymmetric tensor X_{ij} is ${}^\star X_{ij} = (1/2)\varepsilon_{ij}{}^{mn} X_{mn}$.

II. LAGRANGIAN FORMALISM

In this section, we give a short account of the Lagrangian formalism for the general parity–violating PG. Basic dynamical variables are the tetrad field $b^i = b^i{}_\mu dx^\mu$ and the antisymmetric spin connection $\omega^{ij} = \omega^{ij}{}_\mu dx^\mu = -\omega^{ji}$ (1-forms), which represent the gauge potentials associated with translations and Lorentz transformations, respectively. The corresponding field strengths are the torsion and the curvature (2-forms),

$$\begin{aligned} T^i &:= db^i + \omega^i{}_k \wedge b^k = \frac{1}{2} T^i{}_{\mu\nu} dx^\mu \wedge dx^\nu, \\ R^{ij} &:= d\omega^{ij} + \omega^i{}_k \wedge \omega^{kj} = \frac{1}{2} R^{ij}{}_{\mu\nu} dx^\mu \wedge dx^\nu, \end{aligned} \tag{2.1}$$

which satisfy the Bianchi identities

$$\nabla T^i = R^i{}_k \wedge b^k, \qquad \nabla R^{ij} = 0. \tag{2.2}$$

The underlying spacetime continuum is described by Riemann-Cartan geometry [7–9].

A. Field equations

The PG dynamics is determined by a Lagrangian $L = L_M + L_G$, where L_M describes matter and its interaction with gravity, and L_G is the pure gravitational part. In the framework of tensor calculus, the gravitational field equations in vacuum are obtained by varying the action $I_G = \int d^4x L_G(b^i{}_\mu, T_{ijk}, R_{ijkl})$ with respect to $b^i{}_\mu$ and $\omega^{ij}{}_\mu$. After introducing the covariant gravitational momenta

$$H_i{}^{\mu\nu} := \frac{\partial L_G}{\partial T^i{}_{\mu\nu}}, \qquad H_{ij}{}^{\mu\nu} := \frac{\partial L_G}{\partial R^{ij}{}_{\mu\nu}}, \tag{2.3a}$$

and the associated energy-momentum and spin currents

$$E_i{}^\nu := \frac{\partial L_G}{\partial b^i{}_\mu}, \qquad E_{ij}{}^\mu := \frac{\partial L_G}{\partial \omega^{ij}{}_\mu}, \tag{2.3b}$$

the gravitational field equations take a compact form:

$$\text{(1ST)} \quad \mathcal{E}_i{}^\nu := -\frac{\delta L_G}{\delta b^i{}_\mu} = \nabla_\mu H_i{}^{\mu\nu} - E_i{}^\nu = 0, \tag{2.4a}$$

$$\text{(2ND)} \quad \mathcal{E}_{ij}{}^\nu := -\frac{\delta L_G}{\delta \omega^{ij}{}_\mu} = \nabla_\mu H_{ij}{}^{\mu\nu} - E_{ij}{}^\nu = 0. \tag{2.4b}$$

The explicit expressions for the energy-momentum and spin currents are given by

$$\begin{aligned} E_i{}^\nu &= h_i{}^\nu L_G - T^m{}_{ki} H_m{}^{k\nu} - \frac{1}{2} R^{mn}{}_{ki} H_{mn}{}^{k\nu}, \\ E_{ij}{}^\mu &= -2H_{[ij]}{}^\mu. \end{aligned} \tag{2.5}$$

In the presence of matter, the right-hand sides of (2.4a) and (2.4b) contain the corresponding matter currents.

B. Quadratic PG models

We assume the Lagrangian density L_G to contain all possible quadratic invariants, constructed out of the three irreducible components of the torsion and the six irreducible components of the curvature (Appendix A). Relying on the Lagrangian 4-form given in Ref. [27], one finds that the corresponding Lagrangian density has the form $L_G = b\mathcal{L}_G$, where $b := \det(b^i{}_\mu)$ and

$$\begin{aligned} \mathcal{L}_G = &-(a_0 R + 2\Lambda_0) - \bar{a}_0 X \\ &+ \frac{1}{2} T^{ijk} \sum_{n=1}^{3} (a_n {}^{(n)}T_{ijk} - \bar{a}_n {}^{\star(n)}T_{ijk}), \\ &+ \frac{1}{4} R^{ijkl} \sum_{n=1}^{6} (b_n {}^{(n)}R_{ijkl} - \bar{b}_n {}^{\star(n)}R_{ijkl}). \end{aligned} \tag{2.6}$$

Here, the irreducible components of the field strengths are defined in Appendix A, the parity even and parity odd sectors are described by the parameters (a_n, b_n, Λ_0) and $(\bar{a}_n, \bar{b}_n)$, respectively, and the star symbol denotes the duality operation with respect to the frame indices of the field strengths. Another form of $\mathcal{L}_G$, useful for comparison with the literature, is given in Appendix B. Knowing $\mathcal{L}_G$, one finds that the covariant momentum densities (2.3a) can be written in the form $H_{imn} = b\mathcal{H}_{imn}$ and $H_{ijmn} = b\mathcal{H}_{ijmn}$, where

$$\mathcal{H}_{imn} = 2\sum_{n=1}^{3}(a_n{}^{(n)}T_{imn} - \bar{a}_n{}^\star T_{imn}),$$
$$\mathcal{H}_{ijmn} = {}^L\mathcal{H}_{ijmn} + \mathcal{H}'_{ijmn}, \tag{2.7a}$$

and

$${}^L\mathcal{H}_{ijmn} = -2a_0(\eta_{im}\eta_{jn} - \eta_{jm}\eta_{\rm in}) + 2\bar{a}_0\varepsilon_{ijmn},$$
$$\mathcal{H}'_{ijmn} = 2\sum_{n=1}^{6}(b_n{}^{(n)}R_{ijmn} - \bar{b}_n{}^{\star(n)}R_{ijmn}). \tag{2.7b}$$

C. On the choice of Lagrangian parameters

In the Lagrangian (2.6), the two parity sectors are presented in a very symmetric way, but the set of three identities (A3a) implies that not all of the parameters $(\bar{a}_n, \bar{b}_n)$ are independent. To resolve this issue, we choose the conditions

$$\bar{a}_2 = \bar{a}_3, \qquad \bar{b}_2 = \bar{b}_4, \qquad \bar{b}_3 = \bar{b}_6, \tag{2.8}$$

which reduce the number of Lagrangian parameters to $21 - 3 = 18$. Note that the above conditions are not unique.

Further freedom in the choice of parameters follows from the existence of three topological invariants. The Euler and Pontryagin invariants are defined by the 4-forms

$$I_E \coloneqq R^{ij} \wedge R^{mn}\varepsilon_{mnij}, \qquad I_P \coloneqq R^{ij} \wedge R_{ij}, \tag{2.9a}$$

respectively, whereas the third invariant is based on the Nieh-Yan identity,

$$I_{NY} \coloneqq T^i \wedge T_i - R_{ij} \wedge b^i \wedge b^j \equiv d(b^i \wedge T_i). \tag{2.9b}$$

These invariants produce vanishing contributions to the field equations, which implies that not all of the Lagrangian parameters are dynamically independent. Indeed, they can be used to eliminate three more terms from the Lagrangian, leaving us with the final number of $18 - 3 = 15$ independent parameters; see Ref. [23]$_3$ for more details. In this paper, we use only the conditions in Eq. (2.8), allowing thereby for an easier comparison to the literature.

For a clear understanding of the physical content of PG, it is convenient to use dimensionless parameters (coupling constants). The Lagrangian parameters in Eq. (2.6) are not dimensionless, but the transition to their dimensionless counterparts can be easily realized by suitable rescalings; see, for instance, Ref. [27]. However, to make the general exposition simpler and more compact, we find it useful to keep the Lagrangian parameters in the form Eq. (2.6), which corresponds to using the units $c = \hbar = 2\kappa = 1$. The true dimensionless parameters can be reintroduced later whenever needed.

III. PRIMARY CONSTRAINTS

Hamiltonian structure is by itself a particularly important aspect of PG as a gauge theory [13]. Moreover, it also offers dynamical information that is essential for a proper understanding of the particle spectrum of PG.

We begin the subject by analyzing the primary constraints (PC) of PG. The canonical momenta $(\pi_i{}^\mu, \Pi_{ij}{}^\mu)$ associated to the basic Lagrangian variables $(b^i{}_\mu, \omega^{ij}{}_\mu)$ are

$$\pi_i{}^\mu \coloneqq \frac{\partial L_G}{\partial(\partial_0 b^i{}_\mu)} = b\mathcal{H}_i{}^{0\mu}, \qquad \Pi_{ij}{}^\mu \coloneqq \frac{\partial L_G}{\partial(\partial_0 \omega^{ij}{}_\mu)} = b\mathcal{H}_{ij}{}^{0\mu}. \tag{3.1}$$

Since the field strengths do not depend on the velocities $\partial_0 b^i{}_0$ and $\partial_0\omega^{ij}{}_0$, the above relations define ten constraints that are always present in the theory ("sure" PCs), regardless of the values of the coupling constants. They read

$$\pi_i{}^0 \approx 0, \qquad \Pi_{ij}{}^0 \approx 0, \tag{3.2}$$

and their existence is directly related to the local Poincaré symmetry of PG.

Before we proceed, let us note that at each point of a spatial hypersurface $\Sigma_0 : x^0 = \text{const}$, one can define the unit timelike vector $\boldsymbol{n}$, normal to Σ_0. Then, any spacetime vector V_k can be decomposed into a component $V_\perp$ along $\boldsymbol{n}$ and a component $V_{\bar{k}}$ in the tangent space of ("parallel" to) Σ_0; that is, $V_k = n_k V_\perp + V_{\bar{k}}$, where $V_\perp = n^k V_k$ and $n^k V_{\bar{k}} = 0$ (Appendix C).

To find additional constraints that may appear in Eq. (3.1), it is useful to define the parallel gravitational momenta

$$\hat{\pi}_{i\bar{k}} \coloneqq \pi_i{}^\alpha b_{k\alpha} = J\mathcal{H}_{i\perp\bar{k}}, \tag{3.3a}$$

$$\hat{\Pi}_{ij\bar{k}} \coloneqq \Pi_{ij}{}^\alpha b_{k\alpha} = J\mathcal{H}_{ij\perp\bar{k}}, \tag{3.3b}$$

such that $\hat{\pi}_{i\bar{k}} n^k = 0$, $\Pi_{ij\bar{k}} n^k = 0$, and J is defined by $b = NJ$, with $N = n_k b^k{}_0$. Depending on the values of the coupling constants, these relations may produce additional constraints (primary "if-constraints"). In analogy to the above orthogonal-parallel decomposition of a vector V_k, one can introduce a similar decomposition of the field strengths,

$$T_{ikm} = T_{i\bar{k}\bar{m}} + (n_k T_{i\perp\bar{m}} + n_m T_{i\bar{k}\perp}) = \bar{T}_{ikm} + \mathcal{T}_{ikm}, \tag{3.4a}$$

$$R_{ijkm} = R_{ij\bar{k}\bar{m}} + (n_k R_{ij\perp\bar{m}} + n_m R_{ij\bar{k}\perp}) = \bar{R}_{ijkm} + \mathcal{R}_{ijkm}. \tag{3.4b}$$

It is very useful for further analysis to know that the parallel components $\bar{T}_{ikm} := T_{i\bar{k}\bar{m}}$ and $\bar{R}_{ijkm} := R_{ij\bar{k}\bar{m}}$ are independent not only of the "velocities" $T_{i\perp\bar{m}}, R_{ij\perp\bar{m}}$, but also of the unphysical variables $(b^i{}_0, \omega^{ij}{}_0)$; for more details, see Refs. [7,14,15].

A. Torsion sector

The torsion piece of the Lagrangian in Eq. (2.6) depends on the velocities $\partial_0 b^i{}_\alpha$ only through $T_{i\perp\bar{k}}$. The linearity of $\mathcal{H}_{i\perp\bar{k}}$ in $\bar{T}$ and $\mathcal{T}$ allows us to rewrite Eq. (3.3a) in the form

$$\phi_{i\bar{k}} := \frac{\hat{\pi}_{i\bar{k}}}{J} - \mathcal{H}_{i\perp\bar{k}}(\bar{T}) = \mathcal{H}_{i\perp\bar{k}}(\mathcal{T}), \tag{3.5}$$

where all possible velocity terms are moved to the right-hand side. Now, we decompose this equation into irreducible parts with respect to the group of rotations in Σ_0 (Appendix C):

$${}^S\phi := \frac{\hat{\pi}_{\bar{k}}{}^{\bar{k}}}{J} + \bar{a}_2 \varepsilon^{\bar{k}\bar{m}\bar{m}} T_{\bar{k}\bar{m}\bar{n}} = -2a_2 T^{\bar{k}}{}_{\bar{k}\perp}, \tag{3.6a}$$

$$\begin{aligned}\phi_{\perp\bar{k}} &:= \frac{\hat{\pi}_{\perp\bar{k}}}{J} + \frac{2}{3}(a_1 - a_2) T^{\bar{m}}{}_{\bar{m}\bar{k}} + \frac{1}{3}(2\bar{a}_1 + \bar{a}_2)\varepsilon_{\bar{k}}{}^{\bar{m}\bar{n}} T_{\perp\bar{m}\bar{n}} \\ &= \frac{2}{3}(2a_1 + a_2) T_{\perp\perp\bar{k}} + \frac{2}{3}(\bar{a}_1 - \bar{a}_2)\varepsilon_{\bar{k}}{}^{\bar{m}\bar{n}} T_{\bar{m}\bar{n}\perp},\end{aligned} \tag{3.6b}$$

$$\begin{aligned}{}^A\phi_{\bar{\imath}\bar{k}} &:= \frac{{}^A\hat{\pi}_{\bar{\imath}\bar{k}}}{J} - \frac{2}{3}(a_1 - a_3) T_{\perp\bar{\imath}\bar{k}} - \frac{1}{3}(\bar{a}_1 + 2\bar{a}_3)\varepsilon_{\bar{\imath}\bar{k}}{}^{\bar{n}} T^{\bar{m}}{}_{\bar{m}\bar{n}} \\ &= -\frac{2}{3}(a_1 + 2a_3) T_{[\bar{\imath}\bar{k}]\perp} - \frac{2}{3}(\bar{a}_1 - \bar{a}_2)\varepsilon_{\bar{\imath}\bar{k}}{}^{\bar{n}} T_{\perp\perp\bar{n}},\end{aligned} \tag{3.6c}$$

$$\begin{aligned}{}^T\phi_{\bar{\imath}\bar{k}} &:= \frac{{}^T\hat{\pi}_{\bar{\imath}\bar{k}}}{J} + \bar{a}_1 \left[\varepsilon_{(\bar{\imath}}{}^{\bar{m}\bar{n}} T_{\bar{k})\bar{m}\bar{n}} - \frac{1}{3}\eta_{\bar{\imath}\bar{k}}\varepsilon^{\bar{k}\bar{m}\bar{n}} T_{\bar{k}\bar{m}\bar{n}}\right] \\ &= -2a_1 {}^T T_{\bar{\imath}\bar{k}\perp}.\end{aligned} \tag{3.6d}$$

Here, the set $({}^S\phi, \phi_{\perp\bar{k}}, {}^A\phi_{\bar{\imath}\bar{k}}, {}^T\phi_{\bar{\imath}\bar{k}})$, defined by the scalar, vector, antisymmetric, and traceless-symmetric parts of $\phi_{i\bar{k}}$, represents the set of all possible new constraints. The mechanism by which these if-constraints become true constraints is simply explained in the parity even case, characterized by four critical parameters: a_2, $(2a_1 + a_2)$, $(a_1 + 2a_3)$, and a_1. When some of these parameters vanish, the corresponding velocity terms on the right-hand sides of (3.6) also vanish, and consequently, the associated if-constraints become true constraints. However, if none of the critical parameters vanishes, there are no new constraints.

The same mechanism works also in the general PG. Whereas the critical parameters for ${}^S\phi$ and ${}^T\phi_{\bar{\imath}\bar{k}}$ remain the same as in PG$^+$, a_2 and a_1, the structure of the if-constraints $\phi_{\perp\bar{k}}$ and ${}^A\phi_{\bar{\imath}\bar{k}}$ is more complicated, as the right-hand sides of Eqs. (3.6b) and (3.6c) depend on two velocities, $T_{[\bar{m}\bar{n}]\perp}$ and $T_{\perp\perp\bar{k}}$. To find the related critical parameters, we first transform ${}^A\phi_{\bar{\imath}\bar{k}}$ into the axial 3-vector ${}^A\phi_{\bar{k}} := \varepsilon_{\bar{k}}{}^{\bar{m}\bar{n}A}\phi_{\bar{m}\bar{n}}$, so that Eq. (3.6c) goes over into

$${}^A\phi_{\bar{k}} = \frac{4}{3}(\bar{a}_1 - \bar{a}_2) T_{\perp\perp\bar{k}} - \frac{2}{3}(a_1 + 2a_3)\varepsilon_{\bar{k}}{}^{\bar{m}\bar{n}} T_{\bar{m}\bar{n}\perp}. \tag{3.7}$$

Then, the set of equations involving $(\phi_{\perp\bar{k}}, {}^A\phi_{\bar{k}})$ can be written in the matrix form as

$$\begin{pmatrix}\phi_{\perp\bar{k}} \\ {}^A\phi_{\bar{k}}\end{pmatrix} = \frac{2}{3} A \begin{pmatrix} T_{\perp\perp\bar{k}} \\ \varepsilon_{\bar{k}}{}^{\bar{m}\bar{n}} T_{\bar{m}\bar{n}\perp}\end{pmatrix}, \tag{3.8a}$$

where

$$A := \begin{pmatrix} 2a_1 + a_2 & \bar{a}_1 - \bar{a}_2 \\ 2(\bar{a}_1 - \bar{a}_2) & -(a_1 + 2a_3)\end{pmatrix},$$
$$\det A = -[(2a_1 + a_2)(a_1 + 2a_3) + 2(\bar{a}_1 - \bar{a}_2)^2]. \tag{3.8b}$$

If the matrix A has two distinct eigenvalues, one can construct the invertible matrix P that transforms A into a diagonal form, $D_A := P^{-1}AP$. Then, Eq. (3.8a) implies

$$\phi_{\bar{k}} := P^{-1}\begin{pmatrix}\phi_{\perp\bar{k}} \\ {}^A\phi_{\bar{k}}\end{pmatrix} = \frac{2}{3} D_A P^{-1} \begin{pmatrix} T_{\perp\perp\bar{k}} \\ \varepsilon_{\bar{k}}{}^{\bar{m}\bar{n}} T_{\bar{m}\bar{n}\perp}\end{pmatrix}, \tag{3.9}$$

where the column $\phi_{\bar{k}}$ represents two diagonalized if-constraints, and the diagonal elements of D_A are the critical parameters,

$$c_\pm(A) = \frac{1}{2}\left(\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}\right). \tag{3.10}$$

More details on this construction can be found in Appendix D. In general, the number of true constraints in Eq. (3.9) is equal to the number of vanishing critical parameters.

(i) The critical parameters of the torsion sector are a_2, $c_\pm(A)$, and a_1.

B. Curvature sector

For the curvature sector, we use the linearity of $\mathcal{H}'_{ij\perp k}$ in $\bar{R}$ and $\mathcal{R}$ to rewrite Eq. (3.3b) in the form

$$\Phi_{ij\bar{k}} := \frac{\Pi_{ij\bar{k}}}{J} - \mathcal{H}_{ij\perp\bar{k}}(\bar{R}) = \mathcal{H}'_{ij\perp\bar{k}}(\mathcal{R}). \tag{3.11}$$

The content of the object $\Phi_{ij\bar{k}}$ is described by two three-dimensional (3d) tensors, $\Phi_{ij\bar{k}} = (\Phi_{\perp\bar{j}\bar{k}}, \Phi_{\bar{\imath}\bar{j}\bar{k}})$. The irreducible decomposition of $\Phi_{\perp\bar{j}\bar{k}}$ takes the form defined in (C4):

$$\begin{aligned} {}^S\Phi &\equiv \frac{\Pi_{\perp\bar{k}}{}^{\bar{k}}}{J} + 6a_0 + \frac{1}{2}(b_4 - b_6)\underline{R} + \frac{1}{2}(\bar{b}_2 + \bar{b}_3)\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\perp\bar{k}\bar{m}\bar{n}} \\ &= (b_4 + b_6)R_{\perp\perp} - \frac{1}{2}(\bar{b}_2 - \bar{b}_3)\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{k}\bar{m}\bar{n}\perp}, \end{aligned} \tag{3.12a}$$

$$\begin{aligned} {}^A\Phi_{\perp\bar{j}\bar{k}} &\equiv \frac{{}^A\Pi_{\perp\bar{j}\bar{k}}}{J} + (b_2 - b_5)^A\underline{R}_{\bar{j}\bar{k}} - \frac{1}{2}(\bar{b}_2 + \bar{b}_5)^A(\varepsilon_{\bar{j}}{}^{\bar{m}\bar{n}}R_{\perp\bar{k}\bar{m}\bar{n}}) \\ &= (b_2 + b_5)^A R_{\perp\bar{j}\perp\bar{k}} - \frac{1}{2}(\bar{b}_2 - \bar{b}_5)^A(\varepsilon_{\bar{j}}{}^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}\bar{k}\perp}), \end{aligned} \tag{3.12b}$$

$$\begin{aligned} {}^T\Phi_{\perp\bar{j}\bar{k}} &\equiv \frac{{}^T\Pi_{\perp\bar{j}\bar{k}}}{J} + (b_1 - b_4)^T\underline{R}_{\bar{j}\bar{k}} + \frac{1}{2}(\bar{b}_1 + \bar{b}_2)^T(\varepsilon_{\bar{j}}{}^{\bar{m}\bar{n}}R_{\perp\bar{k}\bar{m}\bar{n}}) \\ &= (b_1 + b_4)^T R_{\perp\bar{j}\perp\bar{k}} - \frac{1}{2}(\bar{b}_1 - \bar{b}_2)^T(\varepsilon_{\bar{j}}{}^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}\perp\bar{k}}). \end{aligned} \tag{3.12c}$$

The irreducible parts of $\Phi_{\bar{\imath}\bar{j}\bar{k}} = -\Phi_{\bar{j}\bar{\imath}\bar{k}}$ are the pseudoscalar, the vector, and the traceless symmetric part; see (C5):

$$\begin{aligned} {}^P\Phi &\equiv \frac{{}^P\Pi}{J} + 12\bar{a}_0 + (b_2 - b_3)\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\perp\bar{k}\bar{m}\bar{n}} - (\bar{b}_1 + 2\bar{b}_2 + \bar{b}_3)\underline{R} \\ &= -(b_2 + b_3)\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{k}\bar{m}\bar{n}\perp} - 2(\bar{b}_2 - \bar{b}_3)R_{\perp\perp}, \end{aligned} \tag{3.13a}$$

$$\begin{aligned} {}^V\Phi_{\bar{\imath}} &\equiv \frac{{}^V\Pi_{\bar{\imath}}}{J} - (b_4 - b_5)R_{\perp\bar{\imath}} + \frac{1}{2}(\bar{b}_2 + \bar{b}_5)\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{\imath}\bar{k}\bar{m}\bar{n}} \\ &= (b_4 + b_5)R_{\bar{\imath}\perp} - (\bar{b}_2 - \bar{b}_5)\varepsilon_{\bar{\imath}}{}^{\bar{k}\bar{n}}R_{\perp\bar{k}\perp\bar{n}}, \end{aligned} \tag{3.13b}$$

$$\begin{aligned} {}^T\Phi_{\bar{\imath}\bar{j}\bar{k}} &\equiv \frac{{}^T\Pi_{\bar{\imath}\bar{j}\bar{k}}}{J} + (b_1 - b_2)^T R_{\perp\bar{j}\bar{k}\bar{\imath}} - {}^T\mathcal{H}'^{-}_{\bar{\imath}\bar{j}\perp\bar{k}}(\bar{R}) \\ &= (b_1 + b_2)^T R_{\bar{\imath}\bar{j}\perp\bar{k}} - (\bar{b}_1 - \bar{b}_2)^T(\varepsilon_{\bar{\imath}\bar{j}}{}^{\bar{n}\Sigma}R_{\perp\bar{n}\perp\bar{k}}). \end{aligned} \tag{3.13c}$$

In Eqs. (3.12) and (3.13), the underlined objects do not contain velocities, $\underline{R} := R^{\bar{m}\bar{n}}{}_{\bar{m}\bar{n}}$ and $\underline{R}_{\bar{\imath}\bar{j}} := R_{\bar{\imath}\bar{n}\bar{j}}{}^{\bar{n}}$, the superscript Σ denotes symmetrization, and $\mathcal{H}'^{-}_{\bar{\imath}\bar{j}\perp\bar{k}}$ in Eq. (3.13c) denotes the parity odd part of the covariant momentum,

$${}^T\mathcal{H}'^{-}_{\bar{\imath}\bar{j}\perp\bar{k}}(\bar{R}) = -{}^T\left\{\frac{1}{4}\varepsilon_{\bar{k}}{}^{\bar{m}\bar{n}}[(\bar{b}_1 + 2\bar{b}_2 + \bar{b}_5)R_{\bar{\imath}\bar{j}\bar{m}\bar{n}} + (\bar{b}_1 - \bar{b}_5)R_{\bar{m}\bar{n}\bar{\imath}\bar{j}}]\right\}.$$

Looking at the type of velocities appearing in the above equations, one can see that the critical parameters can be found by grouping these equations into suitably chosen pairs.

1. Spin-0 pair

Consider first Eqs. (3.12a) and (3.13a), which contain the same set of velocities, $R_{\perp\perp}$ and $\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{k}\bar{m}\bar{n}\perp}$. They can be written in the matrix form as

$$\begin{pmatrix} {}^S\Phi \\ {}^P\Phi \end{pmatrix} = B_0 \begin{pmatrix} R_{\perp\perp} \\ \varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{k}\bar{m}\bar{n}\perp} \end{pmatrix}, \tag{3.14a}$$

where

$$\begin{aligned} B_0 &:= \begin{pmatrix} b_4 + b_6 & -\frac{1}{2}(\bar{b}_2 - \bar{b}_3) \\ -2(\bar{b}_2 - \bar{b}_3) & -(b_2 + b_3) \end{pmatrix}, \\ \det B_0 &= -(b_4 + b_6)(b_2 + b_3) - (\bar{b}_2 - \bar{b}_3)^2. \end{aligned} \tag{3.14b}$$

In analogy to what we found in the previous subsection, the critical parameters are the eigenvalues of B_0, $c_\pm(B_0)$, and the related column of the if-constraints reads

$${}^0\Phi := P_0^{-1}\begin{pmatrix} {}^S\Phi \\ {}^P\Phi \end{pmatrix} = D_0 P_0^{-1}\begin{pmatrix} R_{\perp\perp} \\ \varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\bar{k}\bar{m}\bar{n}\perp} \end{pmatrix}, \tag{3.15}$$

where P_0 is the matrix that diagonalizes B_0, $D_0 = P_0^{-1}B_0P_0$.

2. Spin-1 pair

Similarly, after transforming ${}^A\Phi_{\perp\bar{\imath}\bar{j}}$ into ${}^A\Phi_{\bar{k}} := \varepsilon_{\bar{k}}{}^{\bar{m}\bar{n}}\Phi_{\perp\bar{m}\bar{n}}$, Eq. (3.12b) becomes

$${}^A\Phi_{\bar{\imath}} = (b_2 + b_5)\varepsilon_{\bar{\imath}}{}^{\bar{m}\bar{n}}R_{\perp\bar{m}\perp\bar{n}} + (\bar{b}_2 - \bar{b}_5)R_{\bar{\imath}\perp}, \tag{3.16}$$

and the matrix form of Eqs. (3.16) and (3.13b) reads

$$\begin{pmatrix} {}^A\Phi_{\bar{\imath}} \\ {}^V\Phi_{\bar{\imath}} \end{pmatrix} = B_1 \begin{pmatrix} \varepsilon_{\bar{\imath}}{}^{\bar{m}\bar{n}}R_{\perp\bar{m}\perp\bar{n}} \\ R_{\bar{\imath}\perp} \end{pmatrix}, \tag{3.17a}$$

where

$$\begin{aligned} B_1 &:= \begin{pmatrix} b_2 + b_5 & \bar{b}_2 - \bar{b}_5 \\ -(\bar{b}_2 - \bar{b}_5) & b_4 + b_5 \end{pmatrix}, \\ \det B_1 &= (b_4 + b_5)(b_2 + b_5) + (\bar{b}_2 - \bar{b}_5)^2. \end{aligned} \tag{3.17b}$$

As before, the critical parameters are $c_\pm(B_1)$, and the if-constraints are determined by the matrix P_1 that diagonalizes B_1,

$${}^1\Phi_{\bar{\imath}} := P_1^{-1}\begin{pmatrix} {}^A\Phi_{\bar{\imath}} \\ {}^V\Phi_{\bar{\imath}} \end{pmatrix} = D_1 P_1^{-1}\begin{pmatrix} \varepsilon_{\bar{\imath}}{}^{\bar{m}\bar{n}}R_{\perp\bar{m}\perp\bar{n}} \\ R_{\bar{\imath}\perp} \end{pmatrix}. \tag{3.18}$$

3. Spin-2 pair

To find the critical parameters in the spin-2 sector, it is convenient to replace ${}^T\Phi_{\bar\imath\bar\jmath\bar k}$ with the expression ${}^T\Phi_{\bar\imath\bar k} \coloneqq {}^T(\varepsilon_{\bar\imath}{}^{\bar m\bar n}\Phi_{\bar m\bar n\bar k})$. Indeed, ${}^T\Phi_{\bar\imath\bar k}$ refers to the same set of velocities that appears in Eq. (3.12c),

$$ {}^T\Phi_{\bar\imath\bar k} = (b_1+b_2){}^T(\varepsilon_{\bar\imath}{}^{\bar m\bar n}R_{\bar m\bar n\perp\bar k}) + 2(\bar b_1-\bar b_2){}^T R_{\perp\bar\imath\perp\bar k}, \tag{3.19} $$

which allows Eqs. (3.12c) and (3.19) to be written in the matrix form

$$ \begin{pmatrix} {}^T\Phi_{\perp\bar\jmath\bar k} \\ {}^T\Phi_{\bar\jmath\bar k} \end{pmatrix} = B_2 \begin{pmatrix} {}^T R_{\perp\bar\jmath\perp\bar k} \\ {}^T(\varepsilon_{\bar\jmath}{}^{\bar m\bar n}R_{\bar m\bar n\perp\bar k}) \end{pmatrix}, \tag{3.20a} $$

where

$$ B_2 \coloneqq \begin{pmatrix} b_1+b_4 & -\frac{1}{2}(\bar b_1-\bar b_2) \\ 2(\bar b_1-\bar b_2) & b_1+b_2 \end{pmatrix}, $$
$$ \det B_2 = (b_1+b_2)(b_1+b_4) + (\bar b_1-\bar b_2)^2. \tag{3.20b} $$

Hence, the critical parameters are $c_\pm(B_2)$, and the column of if-constraints has the form

$$ {}^2\Phi_{\bar\jmath\bar k} \coloneqq P_2^{-1}\begin{pmatrix} {}^T\Phi_{\perp\bar\jmath\bar k} \\ {}^T\Phi_{\bar\jmath\bar k} \end{pmatrix} = D_2P_2^{-1}\begin{pmatrix} {}^T R_{\perp\bar\jmath\perp\bar k} \\ {}^T(\varepsilon_{\bar\jmath}{}^{\bar m\bar n}R_{\bar m\bar n\perp\bar k}) \end{pmatrix}. \tag{3.21} $$

(ii) The critical parameters in the curvature sector are $c_\pm(B_0), c_\pm(B_1)$, and $c_\pm(B_2)$.

C. Critical parameters and if-constraints

Since the if-constraints belong to irreducible representations of 3d rotations, they are characterized by a specific spin content. Their structure is best understood by grouping them into pairs with definite spin, as shown in Table I. In this classification, the parity eigenvalues are absent since parity is not conserved.

The generic set of the critical parameters $c_\pm(F)$, $F = A, B_0, B_1, B_2$, is defined provided the parity odd parameters in F do not vanish; see Appendix D. Hence, the limit of the final expressions $c_\pm(F)$ when these parameters tend to zero is not well defined. However, since in that case F is already diagonal, one can identify $c_\pm$ directly from F.

TABLE I. Critical parameters and if-constraints.

Spin	Critical parameters	If-constraints
0	$a_2,\ c_\pm(B_0)$	${}^S\phi,\ ({}^0\Phi)_\pm$
1	$c_\pm(A), c_\pm(B_1)$	$(\phi_{\bar k})_\pm, ({}^1\Phi_{\bar k})_\pm$
2	$a_1,\ c_\pm(B_2)$	${}^T\phi_{\bar\imath\bar k}, ({}^2\Phi_{\bar\imath\bar k})_\pm$

The total number of the primary if-constraints is $10\times 3 = 30$, the same as the number of the parallel canonical momenta (3.3). The if-constraints and the associated critical parameters have a decisive influence on the structure of the canonical Hamiltonian.

IV. HAMILTONIAN

The procedure for constructing the canonical (and total) Hamiltonian in PG$^+$ is well known [7,14,15,18], but its extension to PG, although in principle straightforward, is technically rather complicated.

Starting with the standard definition of the canonical Hamiltonian density,

$$ \mathcal{H}_c = \pi_i{}^\alpha\partial_0 b^i{}_\alpha + \frac{1}{2}\Pi_{ij}{}^\alpha\partial_0\omega^{ij}{}_\alpha - b\mathcal{L}, \tag{4.1} $$

one can rewrite it in the Dirac-ADM form,

$$ \mathcal{H}_c = N\mathcal{H}_\perp + N^\alpha\mathcal{H}_\alpha - \frac{1}{2}\omega^{ij}{}_0\mathcal{H}_{ij} + \partial_\alpha D^\alpha, \tag{4.2} $$

where N and N^α are the lapse and shift functions (see Appendix C), and

$$ \begin{aligned} \mathcal{H}_{ij} &= 2\pi_{[i}{}^\alpha b_{j]\alpha} + \nabla_\alpha\Pi_{ij}{}^\alpha, \\ \mathcal{H}_\alpha &= \pi_i{}^\beta T^i{}_{\alpha\beta} + \frac{1}{2}\pi_{ij}{}^\beta R^{ij}{}_{\alpha\beta} - b^k{}_\alpha\nabla_\beta\pi_k{}^\beta, \\ \mathcal{H}_\perp &= \pi_i{}^{\bar k}T^i{}_{\perp\bar k} + \frac{1}{2}\Pi_{ij}{}^{\bar k}R^{ij}{}_{\perp\bar k} - J\mathcal{L} - n^k\nabla_\beta\pi_k{}^\beta, \\ D^\alpha &= b^i{}_0\pi_i{}^\alpha + \frac{1}{2}\omega^{ij}{}_0\Pi_{ij}{}^\alpha. \end{aligned} \tag{4.3} $$

Since $\mathcal{H}_\perp$ is the only term that depends on the form of the Lagrangian, explicit construction of the whole $\mathcal{H}_c$ reduces just to the construction of its dynamical piece $\mathcal{H}_\perp$. In this process, we focus our attention on the "most dynamical" case when all the critical parameters are nonvanishing (that is, when none of the if-constraints becomes a true constraint). Such an assumption is sufficient for our study of the particle spectrum of PG. Extension of the formalism to include vanishing critical parameters is outlined in Appendix D.

A. Torsion sector

Isolating the torsion contribution to $\mathcal{L}_G$, one finds the corresponding part of $\mathcal{H}_\perp$,

$$ \mathcal{H}_\perp^T = \frac{1}{2}\phi_{i\perp\bar k}T^{i\perp\bar k} - J\bar{\mathcal{L}}_{T^2} - n^k\nabla_\beta\pi_k{}^\beta, \tag{4.4a} $$

where $\bar{\mathcal{L}}_{T^2} = \mathcal{L}_{T^2}(\bar T)$ does not contain velocities. In order to express the velocities in terms of the phase-space variables, we decompose the first term into four irreducible parts:

$$\phi_{i\perp\bar{k}}T^{i\perp\bar{k}} = \phi_{\perp\bar{k}}T^{\perp\perp\bar{k}} + \frac{1}{2}{}^{A}\phi_{\bar{\imath}}\varepsilon^{\bar{\imath}}{}_{\bar{m}\bar{n}}T^{\bar{m}\bar{n}\perp} + {}^{T}\phi_{\bar{\imath}\bar{k}}{}^{T}T^{\bar{\imath}\perp\bar{k}} + \frac{1}{3}{}^{S}\phi T_{\bar{k}\perp}{}^{\bar{k}}. \tag{4.4b}$$

If a_1, $a_2 \neq 0$, the velocities from the last two terms can be directly eliminated using Eqs. (3.6a) and (3.6d),

$$\frac{1}{3}{}^{S}\phi T_{\bar{k}\perp}{}^{\bar{k}} + {}^{T}\phi_{\bar{\imath}\bar{k}}{}^{T}T^{\bar{\imath}\perp\bar{k}} = \frac{1}{6a_2}{}^{S}\phi^{S}\phi + \frac{1}{2a_1}{}^{T}\phi_{\bar{\imath}\bar{k}}{}^{T}\phi^{\bar{\imath}\bar{k}}. \tag{4.5a}$$

Continuing with the first two terms in (4.4b), we note that for $\det A \neq 0$, one can use the relation $A^{-1} \times (3.8a)$ to eliminate the velocities. Introducing the notation $\varphi_{\bar{k}} \coloneqq (\phi_{\perp\bar{k}}, {}^{A}\phi_{\bar{k}})^T$, the result takes a compact matrix form,

$$\begin{pmatrix} \phi_{\perp\bar{k}}, & \frac{1}{2}{}^{A}\phi_{\bar{k}} \end{pmatrix} \begin{pmatrix} T^{\perp\perp\bar{k}} \\ \varepsilon^{\bar{k}}{}_{\bar{m}\bar{n}}T^{\bar{m}\bar{n}\perp} \end{pmatrix} = \frac{3}{2\det A}\varphi_{\bar{k}}^T T\varphi^{\bar{k}},$$
$$T \coloneqq \begin{pmatrix} 2a_1 + a_2 & \bar{a}_1 - \bar{a}_2 \\ \bar{a}_1 - \bar{a}_2 & -(a_1 + 2a_3)/2 \end{pmatrix}, \qquad \det T = \frac{1}{2}\det A. \tag{4.5b}$$

Hence, the resulting form of $\mathcal{H}_\perp^T$ reads

$$\mathcal{H}_\perp^T = \frac{1}{2}J\phi_T^2 - J\bar{\mathcal{L}}_{T^2} - n^k\nabla_\beta \pi_k{}^\beta,$$
$$\phi_T^2 \coloneqq \frac{1}{6a_2}{}^{S}\phi^{S}\phi + \frac{1}{2a_1}{}^{T}\phi_{\bar{\imath}\bar{k}}{}^{T}\phi^{\bar{\imath}\bar{k}} + \frac{3}{2\det A}\varphi_{\bar{k}}^T T\varphi^{\bar{k}}. \tag{4.6}$$

B. Curvature sector

In a similar manner, one finds the curvature contribution to $\mathcal{H}_\perp$:

$$\mathcal{H}_\perp^R = \frac{1}{4}\Phi_{ij\bar{k}}R^{ij\perp\bar{k}} - J\bar{\mathcal{L}}_{R^2} - a_0 R^{\bar{m}\bar{n}}{}_{\bar{m}\bar{n}} + \bar{a}_0\varepsilon^{\bar{m}\bar{n}\bar{k}}R_{\perp\bar{m}\bar{n}\bar{k}}, \tag{4.7a}$$

where $\bar{\mathcal{L}}_{R^2} = \mathcal{L}_{R^2}(\bar{R})$ does not contain velocities, and

$$\Phi_{ij\bar{k}}R^{ij\perp\bar{k}} = \frac{2}{3}{}^{S}\Phi R_{\perp\perp} - {}^{A}\Phi_{\bar{k}}\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\perp\bar{m}\perp\bar{n}} + 2{}^{T}\Phi_{\perp\bar{\jmath}\bar{k}}{}^{T}R^{\perp\bar{\jmath}\perp\bar{k}}, - \frac{1}{6}{}^{P}\Phi\varepsilon_{\bar{\imath}\bar{\jmath}\bar{k}}R^{\bar{\imath}\bar{\jmath}\perp\bar{k}} + {}^{V}\Phi^{\bar{\imath}}R_{\bar{\imath}\bar{k}\perp}{}^{\bar{k}} - \frac{1}{2}{}^{T}\Phi^{\bar{\imath}\bar{k}T}(\varepsilon_{\bar{\imath}}{}^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}\perp\bar{k}}). \tag{4.7b}$$

Summing up the scalar and pseudoscalar term from the expression (4.7b) and using the relation $B_0^{-1} \times (3.14a)$ to eliminate the velocities, one obtains

$$\frac{2}{3}{}^{S}\Phi R_{\perp\perp} + \frac{1}{6}{}^{P}\Phi\varepsilon_{\bar{\imath}\bar{\jmath}\bar{k}}R^{\bar{\imath}\bar{\jmath}\bar{k}\perp} = J\frac{1}{6\det B_0}{}^{(0)}\Phi^T R_0{}^{(0)}\Phi,$$
$$R_0 = \begin{pmatrix} -4(b_2 + b_3) & 2(\bar{b}_2 - \bar{b}_3) \\ 2(\bar{b}_2 - \bar{b}_3) & b_4 + b_6 \end{pmatrix}, \quad \det R_0 = 4\det B_0, \tag{4.8}$$

where ${}^{(0)}\Phi^T \coloneqq ({}^{S}\Phi, {}^{P}\Phi)$.

Similarly, the sum of the axial vector and vector term, combined with $B_1^{-1} \times (3.17a)$, yields

$$-{}^{A}\Phi_{\bar{k}}\varepsilon^{\bar{k}\bar{m}\bar{n}}R_{\perp\bar{m}\perp\bar{n}} + {}^{V}\Phi^{\bar{\imath}}R_{\bar{\imath}\perp} = -J\frac{1}{\det B_1}{}^{(1)}\Phi_{\bar{\imath}}^T R_1{}^{(1)}\Phi^{\bar{\imath}},$$
$$R_1 = \begin{pmatrix} b_4 + b_5 & -(\bar{b}_2 - \bar{b}_5) \\ -(\bar{b}_2 - \bar{b}_5) & -(b_2 + b_5) \end{pmatrix}, \quad \det R_1 = -\det B_1, \tag{4.9}$$

where ${}^{(1)}\Phi_{\bar{\imath}}^T = ({}^{A}\Phi_{\bar{\imath}}, {}^{V}\Phi_{\bar{\imath}})$.

Finally, using $B_2^{-1} \times (3.20a)$, the sum of the two tensor terms is given by

$$2{}^{T}\Phi_{\perp\bar{\jmath}\bar{k}}{}^{T}R^{\perp\bar{\jmath}\perp\bar{k}} - \frac{1}{2}{}^{T}\Phi^{\bar{\imath}\bar{k}T}(\varepsilon_{\bar{\imath}}{}^{\bar{m}\bar{n}}R_{\bar{m}\bar{n}\perp\bar{k}}) = J\frac{1}{4\det B_2}{}^{(2)}\Phi_{\bar{\imath}\bar{k}}^T R_2{}^{(2)}\Phi^{\bar{\imath}\bar{k}},$$
$$R_2 = \begin{pmatrix} 4(b_1 + b_2) & 2(\bar{b}_2 - \bar{b}_1) \\ 2(\bar{b}_2 - \bar{b}_1) & -(b_1 + b_4) \end{pmatrix}, \quad \det R_2 = -4\det B_2, \tag{4.10}$$

where ${}^{(2)}\Phi_{\bar{\imath}\bar{k}}^T \coloneqq ({}^{T}\Phi_{\perp\bar{\imath}\bar{k}}, {}^{T}\Phi_{\bar{\imath}\bar{k}})$.

Summing up the above three contributions, one obtains the expression for $\mathcal{H}_\perp^R$ as

$$\mathcal{H}_\perp^R = \frac{1}{4}J\Phi_R^2 - J\mathcal{L}_{R^2}(\bar{R}) - a_0 R^{\bar{m}\bar{n}}{}_{\bar{m}\bar{n}} + \bar{a}_0\varepsilon^{\bar{m}\bar{n}\bar{k}}R_{\perp\bar{m}\bar{n}\bar{k}},$$
$$\Phi_R^2 \coloneqq \frac{1}{6\det B_0}{}^{(0)}\Phi^T R_0{}^{(0)}\Phi - \frac{1}{\det B_1}{}^{(1)}\Phi_{\bar{\imath}}^T R_1{}^{(1)}\Phi^{\bar{\imath}} + \frac{1}{4\det B_2}{}^{(2)}\Phi_{\bar{\imath}\bar{k}}^T R_2{}^{(2)}\Phi^{\bar{\imath}\bar{k}}. \tag{4.11}$$

The complete expression $\mathcal{H}_\perp = \mathcal{H}_\perp^T + \mathcal{H}_\perp^R$ will be used in Sec. VII to formulate the conditions for the positivity of energy of the isolated spin modes.

C. Consistency conditions

The complete canonical Hamiltonian of PG, with $\mathcal{H}_\perp = \mathcal{H}_\perp^T + \mathcal{H}_\perp^R$, is calculated by assuming that none of the critical parameters is vanishing. In the next step, one can construct the total Hamiltonian that generates the temporal evolution of dynamical variables. Since the only primary constraints are the sure constraints (3.2), the total Hamiltonian is given by

$$\mathcal{H}_T = \mathcal{H}_c + u^i\pi_i{}^0 + \frac{1}{2}u^{ij}\Pi_{ij}{}^0, \tag{4.12}$$

where u^i and u^{ij} are canonical multipliers.

By construction, the components $\mathcal{H}_{ij}$, $\mathcal{H}_\alpha$, and $\mathcal{H}_\perp$ of the canonical Hamiltonian do not depend on the unphysical variables $b^i{}_0$ and $\omega^{ij}{}_0$. Hence, by demanding that the primary constraints be preserved during the time evolution, one finds the set of secondary constraints,

$$\mathcal{H}_\perp \approx 0, \qquad \mathcal{H}_\alpha \approx 0, \qquad \mathcal{H}_{ij} \approx 0. \tag{4.13}$$

General arguments, based on the existence of local Poincaré invariance, show that these constraints are first class [13]; see also [28]. Hence, the Dirac consistency algorithm is completed at the level of the secondary constraints (4.13).

The present PG model has $N_1 = 20$ first-class constraints and $N_2 = 0$ second-class constraints. Since the number of the Lagrangian variables is $N = 40$ (16 tetrad, plus 24 connection components), the number of the Lagrangian d.o.f. is $N^* = (2N - 2N_1 - N_2)/2 = 20$. They are the same as those found in the weak field approximation of PG: 2 massless spin-2 modes and 18 massive torsion modes (two spin-0, six spin-1, and ten spin-2 modes). However, we shall show that not all of these d.o.f. are physically acceptable, in contrast to earlier expectations [25]. To do that, we will first calculate the mass eigenvalues $m^2_\pm(J)$ for the torsion modes with spin $J = 0$, 1, 2.

V. LINEARIZED FIELD EQUATIONS

In this section, we start our analysis of the particle spectrum of PG by deriving the weak field approximation of the gravitational field equations (2.4) around the Minkowski background M_4; for consistency, we assume $\Lambda_0 = 0$. Such an approximation is based on the following weak field expansion of the basic dynamical variables,

$$b^i{}_\mu = \delta^i_\mu + \tilde{b}^i{}_\mu + O_2, \qquad \omega^{ij}{}_\mu = \tilde{\omega}^{ij}{}_\mu + O_2.$$

To simplify the notation, we omit writing the tilde sign and the symbol O_2, with an implicit understanding of their effects. Furthermore, we find it technically convenient to use the following abbreviations:

$$\begin{aligned} A_n &= a_n - a_1, \qquad & B_n &= b_n - b_1, \\ \bar{A}_n &= \bar{a}_n - \bar{a}_1, \qquad & \bar{B}_n &= \bar{b}_n - \bar{b}_1. \end{aligned} \tag{5.1}$$

A. First field equation

In the first field equation (2.4a), the covariant momentum associated to torsion has the form

$$\begin{aligned} \mathcal{H}_{imn} &= 2a_1 T_{imn} + \frac{2}{3}A_2(\eta_{im}\mathcal{V}_n - \eta_{in}\mathcal{V}_m) + 2A_3\varepsilon_{imnl}\mathcal{A}^l, \\ &\quad - \bar{a}_1 T_{irs}\varepsilon^{rs}{}_{mn} - \frac{2}{3}\bar{A}_2\varepsilon_{imns}\mathcal{V}^s \\ &\quad + 2\bar{A}_3(\eta_{im}\mathcal{A}_n - \eta_{in}\mathcal{A}_m), \end{aligned} \tag{5.2}$$

where $\bar{a}_2 = \bar{a}_3$ yields $\bar{A}_2 = \bar{A}_3$. Then, after calculating the linearized form of $E_i{}^\nu$,

$$\begin{aligned} E_i{}^\nu &= 2a_0 G^\nu{}_i - \bar{a}_0(R_{mnki}\varepsilon^{mnk\nu} + h_i{}^\nu X) \\ &= 2a_0 G^\nu{}_i - 2\bar{a}_0 X_i{}^\nu, \end{aligned}$$

the linearized (1ST) takes the form

$$\begin{aligned} \mathcal{E}_{in} &= \partial^m\mathcal{H}_{imn} - 2a_0 G_{ni} + 2\bar{a}_0 X_{in} \\ &= -2a_1\partial^m T_{inm} + \frac{2}{3}A_2(\partial_i\mathcal{V}_n - \eta_{in}\partial\mathcal{V}) \\ &\quad - 2A_3\varepsilon_{inmk}\partial^m\mathcal{A}^k + \frac{2}{3}\bar{A}_2\varepsilon_{inmk}\partial^m\mathcal{V}^k \\ &\quad + 2\bar{A}_2(\partial_i\mathcal{A}_n - \eta_{in}\partial\mathcal{A}) - 2a_0 G_{ni} \\ &\quad + 2(\bar{a}_0 - \bar{a}_1)X_{in} = 0, \end{aligned} \tag{5.3}$$

where we used (E3), and $\partial\mathcal{V} \coloneqq \partial_i\mathcal{V}^i$, $\partial\mathcal{A} \coloneqq \partial_i\mathcal{A}^i$.

B. Second field equation

Using the formulas obtained in the weak field approximation,

$$\begin{aligned} \nabla_\mu{}^L H_{ij}{}^{\mu n} &= 2a_0(T^n{}_{ij} - \delta^n_i\mathcal{V}_j + \delta^n_j\mathcal{V}_i) \\ &\quad - \bar{a}_0\varepsilon_{ij}{}^{rs}(T^n{}_{rs} - \delta^n_r\mathcal{V}_s + \delta^n_s\mathcal{V}_r), \\ 2\mathcal{H}_{[ij]n} &= -\frac{4}{3}(2a_1 + a_2)\eta_{n[i}\mathcal{V}_{j]} + 2(a_1 + 2a_3)\varepsilon_{ijnk}\mathcal{A}^k \\ &\quad - \frac{4}{3}\bar{A}_2\varepsilon_{ijnk}\mathcal{V}^k - 4\bar{A}_3\eta_{n[i}\mathcal{V}_{n]}, \end{aligned} \tag{5.4}$$

the linearized form of (2ND) reads

$$\begin{aligned} \mathcal{E}_{ijn} &= \partial^m\mathcal{H}'_{ijmn} + 2a_0(T^n{}_{ij} - \delta^n_i\mathcal{V}_j + \delta^n_j\mathcal{V}_i) \\ &\quad - \bar{a}_0\varepsilon_{ij}{}^{rs}(T^n{}_{rs} - \delta^n_r\mathcal{V}_s + \delta^n_s\mathcal{V}_r) + 2\mathcal{H}_{[ij]n} = 0. \end{aligned} \tag{5.5a}$$

Using the double duality relations for the curvature, see Appendix C in Ref. [29], the term $\partial^m\mathcal{H}'_{ijmn}$ is found to have the form

$$\begin{aligned}\partial^m\mathcal{H}'_{ijmn} = {}& (b_2+b_1)\partial^m(\eta_{ir}\Psi_{js}-\eta_{jr}\Psi_{is})\varepsilon^{rs}{}_{mn} + \frac{1}{6}B_3\varepsilon_{ijmn}\partial^m X \\ & + (b_4+b_1)[(\partial_i\Phi_{jn}-\eta_{in}\partial^m\Phi_{jm}) - (i\leftrightarrow j)] + \frac{1}{6}B_6(\eta_{jn}\partial_i - \eta_{in}\partial_j)R \\ & + B_5[(\partial_i\hat{R}_{[jn]} - \eta_{in}\partial^m\hat{R}_{[jm]}) - (i\leftrightarrow j)] - \bar{B}_5\partial^m(\eta_{ir}\hat{R}_{[js]} - \eta_{jr}\hat{R}_{[is]})\varepsilon^{rs}{}_{mn} \\ & + (\bar{b}_2-\bar{b}_1)[(\partial_i\Psi_{jn}-\eta_{in}\partial^m\Psi_{jm}) - (i\leftrightarrow j)] + \frac{1}{6}\bar{B}_3(\eta_{jn}\partial_i - \eta_{in}\partial_j)X \\ & - (\bar{b}_4-\bar{b}_1)\partial^m(\eta_{ir}\Phi_{js}-\eta_{jr}\Phi_{is})\varepsilon^{rs}{}_{mn} - \frac{1}{6}\bar{B}_6\varepsilon_{ijmn}\partial^m R. \end{aligned} \tag{5.5b}$$

VI. PARTICLE SPECTRUM

The particle spectrum of PG contains important information of its physical content. Recently, Karananas [25] made a detailed analysis of this problem by extending the spin-projection operator formalism, used earlier in the context of PG^+ [12], and applying it to study the PG field excitations around the Minkowski background. His work resulted in the mass formulas for the spin-0, spin-1, and spin-2 massive torsion modes, together with the related restrictions on the parameter space, stemming from the requirements for the absence of ghosts and tachyons.

In this section, we study the same problem by analyzing the linearized field equations along the lines presented in [5]. The obtained results are tested by verifying their compatibility with the expressions for the critical parameters found in the canonical analysis, whereas the absence of ghosts and tachyons is studied in the next section.

A. Spin-0 modes

The spin-0 sector is determined by the traces of the field equations $\mathcal{E}_{in}$, $\partial^i\mathcal{E}_{ijn}$, and $\partial^k({}^\star\mathcal{E})_{kln}$, where ${}^\star\mathcal{E}_{kln} \coloneqq (1/2)\varepsilon_{kl}{}^{ij}\mathcal{E}_{ijn}$ is the dual of $\mathcal{E}_{ijn}$:

$$\begin{aligned} & -a_2\partial V - 3\bar{a}_2\partial A + a_0 R + \bar{a}_0 X = 0, \\ & (b_4+b_6)\Box R + (\bar{b}_3-\bar{b}_2)\Box X + 4(2a_0+a_2)\partial V \\ & \quad + 12(\bar{a}_2-\bar{a}_0)\partial A = 0, \\ & (b_2+b_3)\Box X - (\bar{b}_3-\bar{b}_2)\Box R - 12(a_0+2a_3)\partial A \\ & \quad + 8(\bar{a}_2-\bar{a}_0)\partial V = 0. \end{aligned} \tag{6.1}$$

With $X = 3\partial A$, the first equation can be used to express R in terms of $\partial\mathcal{V}$ and $\partial\mathcal{A}$, whereupon the remaining two equations are written in the matrix form as

$$(K_0\Box + 4a_0N_0)U = 0, \tag{6.2a}$$

$$\begin{aligned} K_0 &= \begin{pmatrix} a_2(b_4+b_6) & -3a_0(\bar{b}_2-\bar{b}_3)-3(\bar{a}_0-\bar{a}_2)(b_4+b_6) \\ a_2(\bar{b}_2-\bar{b}_3) & 3a_0(b_2+b_3)-3(\bar{a}_0-\bar{a}_2)(\bar{b}_2-\bar{b}_3) \end{pmatrix}, \\ N_0 &= \begin{pmatrix} (2a_0+a_2) & -3(\bar{a}_0-\bar{a}_2) \\ -2(\bar{a}_0-\bar{a}_2) & -3(a_0+2a_3) \end{pmatrix}, \qquad U = \begin{pmatrix} \partial\mathcal{V} \\ \partial\mathcal{A} \end{pmatrix}. \end{aligned} \tag{6.2b}$$

The determinants of K_0 and N_0 are given by

$$\begin{aligned} \det K_0 &= 3a_0a_2[(b_4+b_6)(b_2+b_3) + (\bar{b}_2-\bar{b}_3)^2], \\ \det N_0 &= -3[(2a_0+a_2)(a_0+2a_3) + 2(\bar{a}_0-\bar{a}_2)^2]. \end{aligned} \tag{6.2c}$$

For $\det K_0 \neq 0$, one can multiply (6.2a) by K_0^{-1}, and obtain the Klein-Gordon equation for the massive spin-0 torsion modes,

$$(\Box + M_0)U = 0, \qquad M_0 = 4a_0K_0^{-1}N_0. \tag{6.3}$$

The masses of these modes are given by the eigenvalues of the mass matrix M_0,

$$\begin{aligned} m_\pm^2(0) &= \frac{1}{2}\left(\mathrm{tr}M_0 \pm \sqrt{(\mathrm{tr}M_0)^2 - 4(\det M_0)}\right) \\ &= \frac{2a_0}{\det K_0}\left(\mathrm{tr}f_0 + \sqrt{(\mathrm{tr}f_0)^2 - 4\det f_0}\right), \end{aligned} \tag{6.4a}$$

where $f_0 \coloneqq (\det K_0)K_0^{-1}N_0$, and

$$\begin{aligned} \mathrm{tr}f_0 &= 3a_0(2a_0+a_2)(b_2+b_3) - 12a_0(\bar{a}_0-\bar{a}_2)(\bar{b}_2-\bar{b}_3) \\ &\quad - 3[a_2(a_0+2a_3) + 2(\bar{a}_0-\bar{a}_2)^2](b_4+b_6), \\ \det f_0 &= (\det K_0)(\det N_0). \end{aligned} \tag{6.4b}$$

It is interesting to note that $\det K_0$ is proportional to the product of two critical parameters, a_2 and $\det B_0$, characterizing the spin-0 sector of the set of if-constraints (see Table I). Hence, when the critical parameters vanish, we have $\det K_0 = 0$, the mass eigenvalues (6.4) become infinite, and consequently, the spin-0 modes do not propagate. In the linear regime, this mechanism provides a Lagrangian description of the dynamical role of if-constraints.

As a further test of our mass formula (6.4), we calculated its form in the parity-even sector $(\bar{a}_0, \bar{a}_n, \bar{b}_n) = 0$, and found the well-known result for the spin-$0^\pm$ torsion modes:

$$m_+^2(0) = \frac{4a_0(2a_0+a_2)}{a_2(b_4+b_6)}, \qquad m_-^2(0) = -\frac{4(a_0+2a_3)}{(b_2+b_3)}.$$

B. Spin-1 modes

To understand the linearized dynamics of the spin-1 sector, it is convenient to start with the antisymmetric part of (1ST), $\mathcal{E}_{[ij]}$, and its dual, ${}^\star\mathcal{E}_{ij}$. Taking derivatives of these equations yields

$$\frac{1}{3}(2a_1+a_2)(\Box\mathcal{V}_j-\partial_j\partial\mathcal{V})+\bar{A}_3(\Box\mathcal{A}_j-\partial_j\partial\mathcal{A})+2A_0\partial^i\hat{R}_{[ij]}+2\bar{A}_0\partial^iX_{[ij]}=0,$$
$$(a_1+2a_3)(\Box\mathcal{A}_j-\partial_j\partial\mathcal{A})-\frac{2}{3}\bar{A}_2(\Box\mathcal{V}_j-\partial_j\partial\mathcal{V})-2A_0\partial^iX_{[ij]}+2\bar{A}_0\partial^i\hat{R}_{[ij]}=0.$$

Then, the solutions for $\partial^m\hat{R}_{[mi]}$ and $\partial^mX_{[mi]}$ are found to be given in the matrix form as

$$2\begin{pmatrix}-\partial^m\hat{R}_{[mi]}\\ \partial^mX_{[mi]}\end{pmatrix}=G(\Box U_i-\partial_i(\partial U)),\qquad U_i=\begin{pmatrix}\mathcal{V}_i\\ \mathcal{A}_i\end{pmatrix},\qquad g:=A_0^2+\bar{A}_0^2,$$
$$G:=\frac{1}{g}\begin{pmatrix}\frac{1}{3}[A_0(2a_1+a_2)-2\bar{A}_0\bar{A}_2] & [A_0\bar{A}_2+\bar{A}_0(a_1+2a_3)]\\ -\frac{1}{3}[\bar{A}_0(2a_1+a_2)+2A_0\bar{A}_2] & -[\bar{A}_0\bar{A}_2-A_0(a_1+2a_3)]\end{pmatrix}. \tag{6.5}$$

Next, consider the trace of (2ND), $\eta^{jk}\mathcal{E}_{ijk}$, and of its dual, $\eta^{jk\star}\mathcal{E}_{ijk}$. Using the identities (E4), these trace components take the form

$$-2(b_4+b_5)\partial^m\hat{R}_{[mi]}+2(\bar{b}_2-\bar{b}_5)\partial^mX_{[mi]}+\frac{1}{2}(b_4+b_6)\partial_iR-\frac{1}{2}(\bar{b}_2-\bar{b}_3)\partial_iX$$
$$+2(2a_0+a_2)\mathcal{V}_i-6(\bar{a}_0-\bar{a}_2)\mathcal{A}_i=0, \tag{6.6a}$$

$$-4(\bar{b}_2-\bar{b}_5)\partial^m\hat{R}_{[mi]}-4(b_2+b_5)\partial^mX_{[mi]}+(\bar{b}_2-\bar{b}_3)\partial_iR+(b_2+b_3)\partial_iX$$
$$-8(\bar{a}_0-\bar{a}_2)\mathcal{V}_i-12(a_0+2a_3)\mathcal{A}_i=0. \tag{6.6b}$$

Using the expressions for $\partial^m\hat{R}_{[mi]}$ and $\partial^mX_{[mi]}$ found in (6.5), and the expression for R determined by the trace of (1ST), Eqs. (6.6) multiplied by $-2g$ can be written in the matrix form as

$$(K_1\Box-4gN_1)U_i+(L_1-K_1)\partial_i(\partial U)=0, \tag{6.7a}$$

where

$$K_1=B_1'(gG),\qquad B_1':=-2\begin{pmatrix}b_4+b_5 & \bar{b}_2-\bar{b}_5\\ 2(\bar{b}_2-\bar{b}_5) & -2(b_2+b_5)\end{pmatrix},$$
$$N_1=\begin{pmatrix}(2a_0+a_2) & -3(\bar{a}_0-\bar{a}_2)\\ -4(\bar{a}_0-\bar{a}_2) & -6(a_0+2a_3)\end{pmatrix},\qquad U_i=\begin{pmatrix}\mathcal{V}_i\\ \mathcal{A}_i\end{pmatrix},$$
$$L_1=-\frac{g}{a_0}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}K_0=-4gN_1M_0^{-1}. \tag{6.7b}$$

The determinants of K_1 and N_1 are given by

$$\det K_1=\frac{8}{3}g(\det A)(\det B_1),$$
$$\det N_1=-6[(a_0+2a_3)(2a_0+a_2)+2(\bar{a}_0-\bar{a}_2)^2]. \tag{6.7c}$$

When $\det K_1\neq 0$, one can multiply Eq. (6.7a) by K_1^{-1} and obtain the matrix Klein-Gordon equation for the massive spin-1 torsion modes,

$$
\begin{aligned}
(\Box + M_1)\tilde{U}_i = 0, \qquad M_1 &:= -4gK_1^{-1}N_1, \\
\tilde{U}_i := U_i + M_0^{-1}\partial_i U, \qquad \partial^i\tilde{U}_i &= 0.
\end{aligned} \tag{6.8}
$$

The eigenvalues of the mass matrix M_1 are given by

$$
m_\pm^2(1) = \frac{-2g}{\det K_1}\left(\mathrm{tr} f_1 \pm \sqrt{(\mathrm{tr} f_1)^2 - 4\det f_1}\right), \tag{6.9a}
$$

where $f_1 := (\det K_1)K_1^{-1}N_1$, and

$$
\begin{aligned}
\det f_1 &= (\det N_1)(\det K_1), \\
\mathrm{tr} f_1 &= 4(b_2+b_5)[(2a_0+a_2)[(a_0-a_1)(a_1+2a_3) - (\bar{a}_0-\bar{a}_1)^2] + 2(a_0-a_1)(\bar{a}_0-\bar{a}_2)^2] \\
&\quad + 4(b_4+b_5)[(a_0+2a_3)[(a_0-a_1)(2a_1+a_2) - 2(\bar{a}_0-\bar{a}_1)^2] + 2(a_0-a_1)(\bar{a}_0-\bar{a}_2)^2] \\
&\quad + 8(\bar{b}_2-\bar{b}_5)[-(2a_0+a_2)(a_0+2a_3)(\bar{a}_0-\bar{a}_1) + 2[(a_0-a_1)^2 + (\bar{a}_0-\bar{a}_1)^2](\bar{a}_0-\bar{a}_2) \\
&\quad - 2(\bar{a}_0-\bar{a}_1)(\bar{a}_0-\bar{a}_2)^2].
\end{aligned} \tag{6.9b}
$$

The determinant of K_1 is the product of two critical parameters associated to the spin-1 sector (see Table I). A discussion of what happens when at least one of these parameters vanishes is given in Appendix D.

In the parity even sector, our mass formula (6.9) yields the familiar result for the spin-$1^\pm$ torsion modes:

$$
\begin{aligned}
m_+^2(1) &= \frac{6(a_0-a_1)(a_0+2a_3)}{(a_1+2a_3)(b_2+b_5)}, \\
m_-^2(1) &= \frac{6(a_0-a_1)(2a_0+a_2)}{(2a_1+a_2)(b_4+b_5)}.
\end{aligned}
$$

C. Spin-2 modes

Although, in principle, the analysis of the spin-2 sector is not much more complicated than the one for the spin-1 case, the fact that there are lots of variables makes the general procedure rather complex and difficult to follow. In Ref. [27], the mass eigenvalues of the spin-2 torsion modes were found by studying a class of exact wave solutions, defined by an ansatz that creates only the tensorial irreducible part of the torsion, whereas the vector and axial vector parts vanish. This motivates us to simplify the present discussion by considering a dynamical system with vanishing spin-0 and spin-1 modes, $\mathcal{V}_i = 0$ and $\mathcal{A}_i = 0$. The physical content of such a system is described solely by the spin-2 tensor t_{ijk} (Appendix A). Such a technical simplification does not influence the final result for the spin-2 mass eigenvalues.

The adopted assumptions have two additional consequences: $X = 0$, which follows from $X = 3\partial\mathcal{A}$; and $R = 0$, which follows from the trace of (1ST). To analyse the spin-2 sector, we need the symmetrized version of (1ST),

$$
-a_1\Theta_{ik} - a_0\Phi_{ik} + \bar{A}_0\Psi_{ik} = 0, \tag{6.10}
$$

where $\Theta_{ik} := \partial^m t_{ikm} = \partial^m T_{(ik)m}$, as follows from the definition (A1) of t_{ikm}. Moreover, we also need two equations that follow from (2ND), $\partial^m\mathcal{E}_{m(ik)}$, and $\partial^m({}^\star\mathcal{E})_{m(ik)}$:

$$
\begin{aligned}
&(b_1+b_4)[\Box\Phi_{ik} - 2\partial_{(i}\partial^m\Phi_{k)m}] + \bar{B}_2[\Box\Psi_{ik} - 2\partial_{(i}\partial^m\Psi_{k)m}] \\
&\quad - 2A_0\Theta_{ik} - 2\bar{A}_0\Psi_{ik} = 0,
\end{aligned} \tag{6.11a}
$$

$$
\begin{aligned}
&(b_1+b_2)[\Box\Psi_{ik} - 2\partial_{(i}\partial^m\Psi_{k)m}] - \bar{B}_4[\Box\Phi_{ik} - 2\partial_{(i}\partial^m\Phi_{k)m}] \\
&\quad - 2\bar{A}_0\Theta_{ik} + 2A_0\Psi_{ik} = 0.
\end{aligned} \tag{6.11b}
$$

Since Φ_{ik} has a nontrivial Riemannian part associated to the massless graviton, a proper description of the torsion spin-2 modes is obtained by using Eq. (6.10) to eliminate Φ_{ik} from Eqs. (6.11):

$$
\begin{aligned}
&(b_1+b_4)\Box(-a_1\Theta_{ik} + \bar{A}_0\Psi_{ik}) + a_0\bar{B}_2\Box\Psi_{ik} \\
&\quad - 2a_0(A_0\Theta_{ik} + \bar{A}_0\Psi_{ik}) = 0,
\end{aligned} \tag{6.12a}
$$

$$
\begin{aligned}
&a_0(b_1+b_2)\Box\Psi_{ik} - \bar{B}_2\Box(-a_1\Theta_{ik} + \bar{A}_0\Psi_{ik}) \\
&\quad - 2a_0(\bar{A}_0\Theta_{ik} - A_0\Psi_{ik}) = 0.
\end{aligned} \tag{6.12b}
$$

These equations can be compactly represented in the matrix form as

$$
(K_2\Box + 2a_0N_2)U_{ik} = 0, \tag{6.13}
$$

where

$$
\begin{aligned}
K_2 &:= \begin{pmatrix} a_1(b_1+b_4) & -\bar{A}_0(b_1+b_4) - a_0(\bar{b}_2-\bar{b}_1) \\ -a_1(\bar{b}_2-\bar{b}_1) & \bar{A}_0(\bar{b}_2-\bar{b}_1) - a_0(b_2+b_1) \end{pmatrix}, \\
N_2 &:= \begin{pmatrix} A_0 & \bar{A}_0 \\ \bar{A}_0 & -A_0 \end{pmatrix}, \qquad U_{ik} := \begin{pmatrix} \Theta_{ik} \\ \Psi_{ik} \end{pmatrix}.
\end{aligned}
$$

For $\det K_2 \neq 0$, Eq. (6.13) is equivalent to

$$(\Box + M_2)U_{ik} = 0, \qquad M_2 := 2a_0 K_2^{-1} N_2, \tag{6.14}$$

where M_2 is the mass matrix of the spin-2 torsion mode.

The matrices K_2 and N_2 are of the same form as those found in Ref. [27], Eq. (4.50), up to inessential differences in conventions. Hence, the mass eigenvalues are also the same. Expressed in terms of the matrix $f_2 = (\det K_2)K_2^{-1}N_2$, they are given by

$$m_\pm^2(2) = \frac{a_0}{\det K_2}\left(\mathrm{tr} f_2 \pm \sqrt{(\mathrm{tr} f_2)^2 - 4\det f_2}\right), \tag{6.15a}$$

where

$$\begin{aligned}\det f_2 &= (\det K_2)(\det N_2),\\ \mathrm{tr} f_2 &= -a_0(a_0 - a_1)(b_1 + b_2) - 2a_0(\bar a_0 - \bar a_1)(\bar b_2 - \bar b_1)\\ &\quad + [-a_1(a_0 - a_1) + (\bar a_0 - \bar a_1)^2](b_1 + b_4).\end{aligned} \tag{6.15b}$$

As expected, the determinant of K_2 is proportional to the product of the critical parameters given in the third line of Table I,

$$\det K_2 = -a_0 a_1 \det B_2, \quad \det N_2 = -(A_0^2 + \bar A_0^2). \tag{6.16}$$

In the parity-even sector, the above formulas produce the well-known result,

$$m_+^2(2) = \frac{2a_0(a_0 - a_1)}{a_1(b_1 + b_4)}, \qquad m_-^2(2) = \frac{2(a_0 - a_1)}{b_1 + b_2}.$$

The above procedure can be extended to the case with nonvanishing spin-0 and spin-1 terms. After a straightforward but rather clumsy calculation, we found that the new terms do not influence the mass eigenvalues, they only modify the spin-2 state U_{ik}. A compact form of the result reads

$$\begin{aligned}U_{ik} &\to \tilde U_{ik}\\ &:= Z\left[\bar U_{ik} - G\partial_{(i}U_{k)} + \frac{1}{3}H(M_0^{-1}\partial_i\partial_k U + \eta_{ik}U)\right],\end{aligned} \tag{6.17a}$$

where

$$Z := \frac{1}{a_1}\begin{pmatrix} -a_0 & \bar A_0 \\ 0 & a_1 \end{pmatrix}, \qquad \bar U_{ik} := \begin{pmatrix} \Phi_{ik} \\ \Psi_{ik} \end{pmatrix},$$
$$H := \frac{1}{a_0}\begin{pmatrix} a_2 & 3(\bar a_2 - \bar a_0) \\ 0 & -2a_0 \end{pmatrix}. \tag{6.17b}$$

The role of Z is to replace Φ_{ik} in $\bar U_{ik}$ by its form obtained from the symmetrized (1ST). The spin-2 nature of $\tilde U_{ik}$ is ensured by the properties $\partial^i \tilde U_{ik} = 0$, $\eta^{ik}\tilde U_{ik} = 0$. In fact, these properties are sufficient to completely determine $\tilde U_{ik}$.

D. Comparison with Karananas' mass formulas

Our mass formulas are found to be consistent with the expressions for the canonical critical parameters, displayed in Table I. A more detailed test can be conducted by comparing them to the recent calculations of Karananas [25]. The first step in this direction is to compare the Lagrangian (5) in Ref. [25] with our expression (B1). Although the procedure is straightforward, a number of misprints found in Ref. [25] complicate the process. Nevertheless, we established the following correspondence between the related parameters:

$$\begin{aligned}
&a_0 = \lambda, \qquad \bar a_0 = \Lambda = 0,\\
&a_1 = \lambda + t_1, \qquad a_2 = 2(-\lambda + t_3),\\
&a_3 = (-\lambda + t_2)/2,\\
&\bar a_1 = -2t_5, \qquad \bar a_2 = \bar a_3 = -t_4,\\
&b_1 = 4(r_1 - r_3), \qquad b_2 = 4r_3,\\
&b_3 = 4(r_2 - r_3), \qquad b_4 = 4(r_1 - r_3 + r_4),\\
&b_5 = 4(r_3 + r_5), \qquad b_6 = 4(r_1 - r_3 + 3r_4),\\
&\bar b_1 = r_7 - 3r_8, \qquad \bar b_2 = \bar b_4 = r_7 + r_8,\\
&\bar b_3 = \bar b_6 = -4r_6 + r_7 + r_8, \qquad \bar b_5 = -3r_7 + r_8.
\end{aligned} \tag{6.18}$$

The remaining part of the comparison is rather simple. By substituting the above expressions into Eqs. (6.4) and (6.9), one finds that the resulting mass eigenvalues for the spin-0 and spin-1 torsion modes exactly reproduce the respective result that Karananas gives in his Appendix A. Moreover, we also found that, with the exception of minor differences, our mass formula (6.15) for the spin-2 modes is in agreement with his result (A.3.5); see also subsection IV.E in Ref. [27]. Although the difference is small, it might be responsible for more serious discrepancies in the physical properties of the spin-2 modes, found in the next section.

VII. PHYSICAL RESTRICTIONS ON THE SPACE OF PARAMETERS

In this section, we study the physical requirements of the absence of ghosts ($E > 0$), the absence of tachyons ($m^2 > 0$), and the reality (m^2 real) in the spectrum of torsion modes. Our approach is based on the Hamiltonian analysis developed in Secs. III and IV, subject to the assumption that all the critical parameters are nonvanishing, or equivalently, that all the torsion modes are propagating. In what follows, we shall examine whether such an assumption is compatible with the adopted physical requirements.

Our general strategy is the following. The conditions of the *positivity of energy* can be read from the dynamical component $\mathcal{H}_\perp^R$ of the canonical Hamiltonian; see Eq. (4.11). By introducing the matrices

$$F_J := \frac{1}{\det B_J} R_J, \qquad J = 0, 1, 2,$$

these conditions can be expressed by demanding that the eigenvalues of F_j be positive. Using the general formula for the eigenvalues of a 2×2 matrix [see (3.9)], one can express these conditions in a more practical form as

$$E_J > 0\colon \ \det F_J > 0, \mathrm{tr} F_j > 0. \tag{7.1}$$

The absence of tachyons is effectively described by the conditions of *positivity* of the eigenvalues $m_\pm^2(J)$ of the mass matrices M_J:

$$m_\pm^2(J) > 0\colon \ \det M_J > 0, \mathrm{tr} M_J > 0. \tag{7.2}$$

Moreover, the presence of square roots in the mass eigenvalues requires to check their *reality*:

$$m_\pm^2(J)\,\mathrm{real}\colon \ (\mathrm{tr} M_J)^2 - 4\det M_J > 0. \tag{7.3}$$

By applying these general physical criteria to the specific spin-J sectors, one obtains a set of restrictions on the original Lagrangian parameters. An important goal of our analysis is to clarify the issue of their mutual (in)consistency. We shall always use $a_0 > 0$, the condition that ensures the correct limit to GR.

A. Spin-0 sector

1. Positivity of energy

The energy of the spin-0 modes is positive if the eigenvalues of the matrix $F_0 = R_0/\det B_0$ are positive. Since $\det R_0 = 4\det B_0$, the first condition $\det F_0 > 0$ implies that $\det B_0 > 0$, or, equivalently,

$$(b_2 + b_3)(b_4 + b_6) + (\bar{b}_2 - \bar{b}_3)^2 < 0, \tag{7.4a}$$

$$\Rightarrow (b_2 + b_3)(b_4 + b_6) < 0. \tag{7.4b}$$

Then, the second condition takes the form tr $R_0 > 0$. In combination with Eq. (7.4b), it yields the relations

$$b_2 + b_3 < 0, \qquad b_4 + b_6 > 0, \tag{7.5}$$

which coincide with those appearing in PG$^+$. The independent conditions are the condition Eq. (7.4a) and, for instance, the first one in Eq. (7.5),

$$(b_2 + b_3)(b_4 + b_6) + (\bar{b}_2 - \bar{b}_3)^2 < 0, \qquad b_2 + b_3 < 0. \tag{7.6}$$

These two conditions coincide with the first two relations found in Eq. (48) of Ref. [25] (the third relation is redundant).

2. Positivity of $m_\pm^2(0)$

The mass matrix M_0 of the spin-0 torsion modes has the form (6.3),

$$M_0/4a_0 = K_0^{-1} N_0 = \frac{1}{\det K_0} f_0, \quad \det K_0 = -3a_0 a_2 \det B_0. \tag{7.7}$$

The positivity of its eigenvalues is expressed by the conditions $\det M_0 > 0$ and $\mathrm{tr} M_0 > 0$:

$$\frac{\det N_0}{\det K_0} > 0, \qquad \frac{1}{\det K_0}\mathrm{tr} f_0 > 0. \tag{7.8}$$

Since $\det B_0 > 0$, they take the form

$$a_2 \det N_0 < 0, \tag{7.9a}$$

$$a_2 \mathrm{tr} f_0 < 0. \tag{7.9b}$$

As shown in Appendix F, these general conditions can be transformed into an unexpectedly simple form, in which the parameters $(b_n \bar{b}_n)$ are completely absent:

$$\begin{aligned} &a_2[(2a_0 + a_2)(a_0 + 2a_3) + 2(\bar{a}_0 - \bar{a}_2)^2] > 0, \\ &a_2(2a_0 + a_2) > 0. \end{aligned} \tag{7.10}$$

Returning to the parameters introduced in Eq. (6.18), this result takes the form

$$(t_3 - \lambda)(t_2 t_3 + t_4^2) > 0, \qquad (t_3 - \lambda) t_3 > 0.$$

The first formula is equivalent to Karananas's result [25], but the second one is different.

B. Spin-1 sector

1. Positivity of energy

Starting with $F_1 := R_1/\det B_1$ and using $\det R_1 = -\det B_1$, the first condition for the positivity of energy, $\det F_1 > 0$, reads

$$\det B_1 \equiv (b_2 + b_5)(b_4 + b_5) + (\bar{b}_2 - \bar{b}_5)^2 < 0, \tag{7.11a}$$

$$\Rightarrow (b_2 + b_5)(b_4 + b_5) < 0. \tag{7.11b}$$

The second condition, written as $\mathrm{tr}R_1 < 0$ and combined with Eq. (7.11b), yields

$$b_2 + b_5 > 0, \qquad b_4 + b_5 < 0, \tag{7.12}$$

which is the PG^+ result. As the two independent conditions, we choose

$$(b_2 + b_5)(b_4 + b_5) + (\bar{b}_2 - \bar{b}_5)^2 < 0, \qquad b_4 + b_5 < 0. \tag{7.13}$$

Again, there is a complete agreement with the first two relations in Eq. (49) of [25], whereas the third relation is redundant.

2. Positivity of $m^2_\pm(1)$

To make the technical exposition more compact, we introduce the following notation:

$$\mu_2 \coloneqq 2a_0 + a_2, \qquad \mu_3 \coloneqq a_0/2 + a_3,$$
$$k_2 \coloneqq 2a_1 + a_2, \qquad k_3 \coloneqq a_1/2 + a_3.$$
$$\det A = -2[k_2 k_3 + (\bar{a}_1 - \bar{a}_2)^2],$$
$$\det N_1 = -12[\mu_2\mu_3 + (\bar{a}_0 - \bar{a}_2)^2].$$

The mass matrix of the spin-1 torsion modes was found in subsection VI B,

$$M_1 = -4gK_1^{-1}N_1 = -\frac{4g}{\det K_1} f_1,$$
$$\det K_1 = \frac{8}{3} g(\det A)(\det B_1), \tag{7.14}$$

with $g \equiv A_0^2 + \bar{A}_0^2$. The positivity of the mass eigenvalues is expressed by the requirements

$$\frac{\det N_1}{\det K_1} > 0, \qquad \frac{1}{\det K_1}\mathrm{tr} f_1 < 0. \tag{7.15}$$

Since $\det B_1 < 0$, these conditions are equivalent to

$$(\det A)(\det N_1) < 0, \tag{7.16a}$$

$$(\det A)\mathrm{tr} f_1 > 0. \tag{7.16b}$$

The expression for $\mathrm{tr} f_1$ is given in subsection VI B; see also Appendix F.

A simple inspection of Eq. (7.16a) shows that it can be realized by $\det A < 0$, $\det N_1 > 0$, or vice versa, whereas, as shown in Appendix F, Eq. (7.16b) can be replaced by a much simpler expression. The resulting conditions, equivalent to Eq. (7.16), are defined in (F7):

(i) $k_2k_3 + (\bar{a}_1 - \bar{a}_2)^2 < 0, \qquad \mu_2\mu_3 + (\bar{a}_0 - \bar{a}_2)^2 > 0,$
$\mu_3 k_2 A_0 - 2\mu_3 \bar{A}_0^2 + A_0(\bar{a}_0 - \bar{a}_2)^2 < 0,$

(ii) $k_2k_3 + (\bar{a}_1 - \bar{a}_2)^2 > 0, \qquad \mu_2\mu_3 + (\bar{a}_0 - \bar{a}_2)^2 < 0,$
$\mu_3 k_2 A_0 - 2\mu_3 \bar{A}_0^2 + A_0(\bar{a}_0 - \bar{a}_2)^2 > 0. \qquad (7.17)$

As before, they do not depend on the parameters $(b_n, \bar{b}_n)$. Going over to the parameters defined in Eq. (6.18), the relations (i) read

$$(t_1 + t_2)(t_1 + t_3) + (t_4 - 2t_5)^2 < 0, \qquad t_2t_3 + t_4^2 > 0,$$
$$t_2(t_1^2 + 4t_5^2) + t_1(t_2t_3 + t_4^2) > 0.$$

The first two inequalities in the set (i) coincide with those found in Ref. [25]; the third one is a bit different, but the whole complementary set (ii) is missing.

C. Spin-2 sector

1. Positivity of energy

The first condition for the positivity of the eigenvalues of $F_2 = R_2/\det B_2$, $\det F_2 > 0$, combined with $\det R_2 = -4\det B_2$, takes the form

$$\det B_2 \equiv (b_1 + b_2)(b_1 + b_4) + (\bar{b}_2 - \bar{b}_1)^2 < 0, \tag{7.18a}$$

$$\Rightarrow (b_1 + b_2)(b_1 + b_4) < 0. \tag{7.18b}$$

The second condition combined with Eq. (7.18b) yields relations that are also valid in PG^+,

$$b_1 + b_2 < 0, \qquad b_1 + b_4 > 0. \tag{7.19}$$

The two independent conditions are

$$(b_1 + b_2)(b_1 + b_4) + (\bar{b}_2 - \bar{b}_1)^2 < 0, \qquad b_1 + b_2 < 0. \tag{7.20}$$

Comparing these conditions to the first two relations in Eq. (50) of Ref. [25], one finds a complete agreement (the third relation is redundant).

2. Positivity of $m^2_\pm(2)$

The mass matrix for the spin-2 modes is found in subsection VI C:

$$M_2 = 2a_0 K_2^{-1} N_2 = \frac{2a_0}{\det K_2} f_2,$$
$$\det K_2 = -a_0 a_1 \det B_2,$$
$$\det N_2 = -[(a_0 - a_1)^2 + (\bar{a}_0 - \bar{a}_1)^2].$$

The positivity of the mass eigenvalues is expressed by the requirements

$$\frac{\det N_2}{\det K_2} > 0, \qquad \frac{2a_0}{\det K_2}\mathrm{tr}f_2 > 0. \tag{7.21}$$

The condition $\det N_2 < 0$ implies $\det K_2 < 0$, whereupon, relying on $\det B_2 < 0$, one obtains

$$a_1 < 0, \tag{7.22a}$$

$$\mathrm{tr}f_2 < 0, \tag{7.22b}$$

where $\mathrm{tr}f_2$ is calculated in subsection VI C.

3. Is the spin-2 sector free of ghosts and tachyons?

Let us recall that in PG$^+$, the conditions $a_1 < 0$ and $b_1 + b_2 < 0$ imply $\mathrm{tr}f_2 > 0$, so that one of the two spin-$2^\pm$ modes is always a tachyon, as is well known. In what follows, we will prove, somewhat unexpectedly, that the same conclusion also holds in the general PG.

To show this, we rewrite $\mathrm{tr}f_2$ in a compact notation as

$$\mathrm{tr}f_2 = \alpha_2(b_1+b_2) + \beta_2(\bar b_2 - \bar b_1) + \gamma_2(b_1+b_4),$$
$$\alpha_2 < 0, \tag{7.23a}$$

where the coefficients α_2, β_2 and γ_2 can be read from Eq. (6.15b),

$$\alpha_2 = -a_0(a_0-a_1), \qquad \beta_2 = -2a_0(\bar a_0 - \bar a_1)^2,$$
$$\gamma_2 = -a_1(a_0-a_1) + (\bar a_0 - \bar a_2)^2,$$

and $\alpha_2 <$ follows from Eq. (7.22a). Since $b_1 + b_4 > 0$, one finds

$$\frac{\mathrm{tr}f_2}{b_1+b_4} = \alpha_2\frac{b_1+b_2}{b_1+b_4} + \beta_2\frac{\bar b_2 - \bar b_1}{b_1+b_4} + \gamma_2. \tag{7.23b}$$

Having in mind the first relation in Eq. (7.20), written as

$$\frac{b_1+b_2}{b_1+b_4} + x^2 < 0, \qquad x \coloneqq \frac{\bar b_2 - \bar b_1}{b_1+b_4},$$

we find it useful to rewrite Eq. (7.23b) in an equivalent form,

$$\frac{\mathrm{tr}f_2}{b_1+b_4} = \alpha_2\left(\frac{b_1+b_2}{b_1+b_4} + x^2\right) + F_2(x),$$
$$F_2(x) \coloneqq -\alpha_2 x^2 + \beta_2 x + \gamma_2. \tag{7.23c}$$

A critical argument in our analysis comes from the observation that the discriminant of the quadratic function $F_2(x)$, $\Delta_2 = \beta_2^2 + 4\alpha_2\gamma_2$, is automatically negative,

$$\Delta_2 = 4a_0a_1[(\bar a_0 - \bar a_1)^2 + (a_0-a_1)^2]$$
$$= -4a_0a_1\det N_2 < 0. \tag{7.24}$$

Next, since $\alpha_2 < 0$ (the parabola F_2 opens upward) and $\Delta_2/\alpha_2 > 0$ (minimum of F_2 is positive), it follows that $F_2(x) > 0$ for any x. Hence, using Eq. (7.20), one obtains the result

$$\mathrm{tr}f_2 > (b_1+b_4)F_2(x) > 0, \tag{7.25}$$

which contradicts to (7.22b). Thus,

S2. The two no-tachyon conditions in Eqs. (7.22a) and (7.22b) are mutually exclusive; hence, the two spin-2 torsion modes cannot be simultaneously physical.

Such a conclusion is not in agreement with the result found by Karananas [25].

4. No-ghost conditions: spin-2 versus spin-1 sector

The no-ghost conditions for spin-1 and spin-2 sectors are in contradiction to each other. Indeed, Eq. (7.12) implies that $b_2 > b_4$, whereas Eq. (7.19) implies that $b_4 > b_2$. Hence, only one of these two sectors can be physical. The result is in agreement with the Corrigendum in [25].

D. Reality conditions

The structure of the general reality conditions in Eq. (7.3) looks rather cumbersome. However, after replacing $|\mathrm{tr}f_0|$, $|\mathrm{tr}f_1|$, and $|\mathrm{tr}f_2|$ with their minimal values, calculated from the inequalities (F3), (F6), and (7.25), respectively, the reality conditions in Eq. (7.3) transform into

$$\text{spin}\,0\colon\ (b_4+b_6)^2 a_2\det N_0 + 12a_0(2a_0+a_2)^2\det B_0 < 0,$$
$$\text{spin}\,1\colon\ g(b_2+b_5)^2(\det A)(\det N_1) - 24\alpha_1^2\det B_1 < 0,$$
$$\text{spin}\,2\colon\ (b_1+b_4)^2 a_1\det N_2 + 4a_0(a_0-a_1)^2\det B_2 > 0; \tag{7.26}$$

see Sec. VI and Appendix F. These formulas are much simpler than Eq. (7.3), but they represent only sufficient conditions for the reality of the corresponding mass eigenvalues.

VIII. SUMMARY AND CONCLUSIONS

In this paper, we investigated generic aspects of the Hamiltonian structure of the general parity-violating PG, and used them to study the torsion particle spectrum [30].

Making use of Dirac's Hamiltonian approach, we identified the set of all *if-constraints*, the expressions that become true constraints if the corresponding *critical parameters* c_n vanish. Both the if-constraints and the associated critical parameters have a crucial influence on the PG dynamics. Then, we constructed the generic form of the *canonical Hamiltonian* $\mathcal{H}_c$, determined by taking all the critical parameters to be nonvanishing. An extension of the procedure to allow for a proper treatment of the vanishing critical parameters is outlined in Appendix D.

Apart from being important by itself, the Hamiltonian structure introduced here turns out to be intrinsically related to the particle spectrum of PG. To examine that subject, we first calculated the *mass eigenvalues* $m^2_\pm(J)$ of the torsion modes with spin $J = 0$, 1, and 2, relying on the weak field approximation of the gravitational field equations around M_4. As a test of the results, we verified that $m^2_\pm(J)$ are proportional to the inverse critical parameters $1/c_n$. As a consequence, whenever some of c_n vanish, the corresponding values of $m^2_\pm(J)$ become infinite, thereby preventing the associated torsion modes from propagating. This is consistent with the canonical effects of the vanishing critical parameters in PG$^+$ (in the weak field approximation). A comparison of our mass formulas to those found by Karananas [25] leads to the following conclusions:

(k1) For the spin-0 and spin-1 torsion modes, our results confirm those of Karananas.

(k2) For the spin-2 modes, there are certain differences, noted already in Ref. [27].

The *absence of ghosts* (positivity of energy) in the particle spectrum is ensured by demanding the positivity of the specific spin-J terms in the canonical Hamiltonian, whereas the conditions for the *absence of tachyons* are defined by the requirement $m^2_\pm(J) > 0$. A detailed analysis shows that these requirements can be formulated as follows:

$$\begin{aligned}
&\text{Spin 0:}\ (b_2+b_3)(b_4+b_6)+(\bar b_2-\bar b_3)^2<0, \qquad b_2+b_3<0,\\
&\qquad a_2[(2a_0+a_2)(a_0/2+a_3)+(\bar a_0-\bar a_2)^2]<0, \qquad a_2(2a_0+a_2)>0.\\
&\text{Spin 1:}\ (b_2+b_5)(b_4+b_5)+(\bar b_2-\bar b_5)^2<0, \qquad b_4+b_5<0,\\
&\quad \text{(i)}\ (2a_1+a_2)(a_1/2+a_3)+(\bar a_1-\bar a_2)^2<0, \qquad (2a_0+a_2)(a_0/2+a_3)+(\bar a_0-\bar a_2)^2>0,\\
&\qquad (a_0-a_1)[(a_0/2+a_3)(2a_1+a_2)+(\bar a_0-\bar a_2)^2]-2(a_0/2+a_3)(\bar a_0-\bar a_1)^2<0;\\
&\quad \text{(ii) an alternative set of conditions, obtained by } (i)\to(-1)\times(i).\\
&\text{Spin 2:}\ (b_1+b_2)(b_1+b_4)+(\bar b_2-\bar b_1)^2<0, \qquad b_1+b_2<0,\\
&\qquad \text{the conditions for the absence of tachyons are } \textit{mutually exclusive}.
\end{aligned} \tag{8.1}$$

The results for the absence of ghosts (first line in each spin sector) are identical to those of Karananas, whereas the formulas describing the absence of tachyons show a number of less or more serious differences. In particular, the whole set of conditions (ii) in the spin-1 sector is missing in Karananas's analysis, but the most important difference is found in the spin-2 sector, where the two conditions for the absence of tachyons are in contradiction to each other, in contrast to Karananas's conclusion.

The presence of square roots in the expressions for the mass eigenvalues $m^2_\pm(J)$ requires us to verify their reality. A sufficient form of the *reality conditions*, compactly presented at the end of Sec. VII, is much simpler than their general form.

In conclusion, our analysis clarifies the structure of the particle spectrum of the general PG by improving the results found by Karananas, in particular the status of the spin-2 sector. On the other hand, elements of the Hamiltonian structure introduced here, including its extension to the case of vanishing critical parameters outlined in Appendix D, are a good starting point for further investigation of the full nonlinear dynamics of PG.

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APPENDIX A: IRREDUCIBLE DECOMPOSITION OF THE FIELD STRENGTHS

The torsion tensor has three irreducible pieces:

$$\begin{aligned}
{}^{(2)}T_{imn} &= \frac{1}{3}(\eta_{im}\mathcal{V}_n-\eta_{\rm in}\mathcal{V}_m),\\
{}^{(3)}T_{imn} &= \varepsilon_{imnk}\mathcal{A}^k,\\
{}^{(1)}T_{imn} &= T_{imn}-{}^{(2)}T_{imn}-{}^{(3)}T_{imn}=\frac{4}{3}t_{i[mn]},
\end{aligned} \tag{A1a}$$

where

$$\begin{aligned}
&\mathcal{V}_n := T^k{}_{kn}, \qquad \mathcal{A}_k := \frac{1}{6}\varepsilon_{krst}T^{rst},\\
&t_{imn} := T_{(im)n}+\frac{1}{3}\eta_{n(i}\mathcal{V}_{m)}-\frac{1}{3}\eta_{im}\mathcal{V}_n.
\end{aligned} \tag{A1b}$$

The Riemann-Cartan curvature tensor can be decomposed into six irreducible pieces:

$$
\begin{aligned}
{}^{(2)}R_{ijmn} &= \frac{1}{2}(\eta_{ik}\Psi_{jl} - \eta_{jk}\Psi_{il})\varepsilon^{kl}{}_{mn},\\
{}^{(3)}R_{ijmn} &= \frac{1}{12}X\varepsilon_{ijmn},\\
{}^{(4)}R_{ijmn} &= \frac{1}{2}(\eta_{im}\Phi_{jn} - \eta_{jm}\Phi_{in}) - (m \leftrightarrow n),\\
{}^{(5)}R^{ij} &= \frac{1}{2}(\eta_{im}\hat{R}_{[jn]} - \eta_{jm}\hat{R}_{[in]}) - (m \leftrightarrow n),\\
{}^{(6)}R_{ijmn} &= \frac{1}{12}R(\eta_{im}\eta_{jn} - \eta_{jm}\eta_{im}),\\
{}^{(1)}R_{ijmn} &= R_{ijmn} - \sum_{a=2}^{6}{}^{(a)}R_{ijmn},
\end{aligned}
\tag{A2a}
$$

where

$$
\begin{aligned}
&\hat{R}_{im} := Ric_{im} = R_{inm}{}^{n}, \qquad R := Ric^{m}{}_{m},\\
&X_{ij} := \frac{1}{2}R_{ikmn}\varepsilon^{kmn}{}_{j}, \qquad X := X^{n}{}_{n},\\
&\Phi_{ij} := Ric_{(ij)} - \frac{1}{4}\eta_{ij}R, \qquad \Psi_{ij} := X_{(ij)} - \frac{1}{4}\eta_{ij}X.
\end{aligned}
\tag{A2b}
$$

The above definitions are the tensor counterparts of the corresponding formulas given in terms of the differential forms; see [27,29]. They imply the following relations characterizing the parity-odd sector:

$$
\begin{aligned}
T^{ijk\star(2)}T_{ijk} &= T^{ijk\star(3)}T_{ijk},\\
R^{ijkl\star(2)}R_{ijkl} &= R^{ijkl\star(4)}R_{ijkl},\\
R^{ijkl\star(3)}R_{ijkl} &= R^{ijkl\star(6)}R_{ijkl},
\end{aligned}
\tag{A3a}
$$

and also

$$
\begin{aligned}
T^{ijk\star(1)}T_{ijk} &= {}^{(1)}T^{ijk\star(1)}T_{ijk},\\
R^{ijkl\star(1)}R_{ijkl} &= {}^{(1)}R^{ijkl\star(1)}R_{ijkl},\\
R^{ijkl\star(5)}R_{ijkl} &= {}^{(5)}R^{ijkl\star(5)}R_{ijkl}.
\end{aligned}
\tag{A3b}
$$

APPENDIX B: ALTERNATIVE FORM OF THE LAGRANGIAN

In this appendix, we rewrite our Lagrangian (2.6) in an equivalent form that allows an easier comparison to the literature [22,25]:

$$
\begin{aligned}
\mathcal{L}_G &= -(a_0R + 2\Lambda_0 + \bar{a}_0X) + \mathcal{L}_{T^2} + \mathcal{L}_{R^2},\\
\mathcal{L}_{T^2} &= h_1T^{ijk}T_{ijk} + h_2T^{imn}T_{nmi} + h_3V_mV^n\\
&\quad + \varepsilon^{mnkl}(\bar{h}_4T^{i}{}_{mn}T_{ikl} + \bar{h}_5T_{mn}{}^{i}T_{kli}),
\end{aligned}
\tag{B1a}
$$

$$
\begin{aligned}
\mathcal{L}_{R^2} &= \frac{1}{2}(f_1R^{ijmn}R_{ijmn} + f_2R^{ijmn}R_{imjn} + f_3R^{ijmn}R_{mnij}\\
&\quad + f_4Ric^{im}Ric_{im} + f_5Ric^{im}Ric_{mi} + f_6R^2)\\
&\quad + \frac{1}{2}\varepsilon^{mnkl}(\bar{f}_7R_{mnkl}R + \bar{f}_8R_{ijmn}R^{ij}{}_{kl}\\
&\quad + \bar{f}_9R_{mnij}R_{kl}{}^{ij} + \bar{f}_{10}R_{mnij}R^{ij}{}_{kl}).
\end{aligned}
\tag{B1b}
$$

The parameters $(h_n, \bar{h}_n)$ and $(f_n, \bar{f}_n)$ can be expressed in terms of the "irreducible" parameters appearing in Eq. (2.6), as follows:

$$
\begin{aligned}
&h_1 = \frac{1}{6}(2a_1 + a_3), \qquad h_2 = \frac{1}{3}(a_1 - a_3),\\
&h_3 = -\frac{1}{3}(a_1 - a_2),\\
&\bar{h}_4 := -\frac{1}{24}(4\bar{a}_1 + \bar{a}_2 + \bar{a}_3), \qquad \bar{h}_5 := -\frac{1}{6}(2\bar{a}_1 - \bar{a}_2 - \bar{a}_3),
\end{aligned}
\tag{B2a}
$$

and

$$
\begin{aligned}
&f_1 := \frac{1}{12}(2b_1 + 3b_2 + b_3), \qquad f_2 := \frac{1}{3}(b_1 - b_3),\\
&f_3 := \frac{1}{12}(2b_1 - 3b_2 + b_3), \qquad f_4 := -\frac{1}{2}(b_1 + b_2 - b_4 - b_5),\\
&f_5 := -\frac{1}{2}(b_1 - b_2 - b_4 + b_5), \qquad f_6 := \frac{1}{12}(2b_1 - 3b_4 + b_6),\\
&\bar{f}_7 := \frac{1}{24}(2\bar{b}_1 - \bar{b}_3 - \bar{b}_6), \qquad \bar{f}_8 := -\frac{1}{16}(\bar{b}_1 + \bar{b}_2 + \bar{b}_4 + \bar{b}_5),\\
&\bar{f}_9 := -\frac{1}{16}(\bar{b}_1 - \bar{b}_2 - \bar{b}_4 + \bar{b}_5), \qquad \bar{f}_{10} := -\frac{1}{8}(\bar{b}_1 - \bar{b}_5).
\end{aligned}
\tag{B2b}
$$

Relying on the existence of three topological invariants (2.9), Karananas [25] imposed three conditions on the Lagrangian parameters in (B1): $\bar{a}_0$, f_6, $\bar{f}_8 = 0$.

APPENDIX C: (3+1) DECOMPOSITION OF SPACETIME

The dynamical content of canonical constraints is greatly clarified by using a decomposition of tensor fields with respect to the subgroup of 3d rotations in the spatial hypersurface Σ_0: $x^0 = \text{const}$.

Let $\boldsymbol{e}_\alpha$ be a basis of three coordinate tangent vectors in Σ_0, $\boldsymbol{e}_\alpha = \partial_\alpha (\alpha = 1, 2, 3)$, and $\boldsymbol{n}$ the unit normal to Σ_0, $n_k = h_k{}^0/\sqrt{g^{00}}$. The four vectors $(\boldsymbol{n}, \boldsymbol{e}_\alpha)$ define the so-called ADM basis of tangent vectors in spacetime. The decomposition of the vector $\boldsymbol{e}_0$ in the ADM basis is given by

$$\boldsymbol{e}_0 = N\boldsymbol{n} + N^\alpha \boldsymbol{e}_\alpha, \tag{C1a}$$

where N and N^α, known as the lapse and shift functions, respectively, are linear in $b^k{}_0$:

$$N = \boldsymbol{e}_0\boldsymbol{n} = n_k b^k{}_0 = 1/\sqrt{g^{00}},$$
$$N^\alpha = \boldsymbol{e}_0\boldsymbol{e}_\beta{}^3g^{\beta\alpha} = h_{\bar k}{}^\alpha b^k{}_0 = -g^{0\alpha}/g^{00}. \tag{C1b}$$

Introducing the projectors on $\boldsymbol{n}$ and $\boldsymbol{e}_\alpha$, given respectively by

$$(P_\perp)^l_k = n_k n^l, \qquad (P_{||})^l_k = \delta^l_k - n_k n^l,$$

one can express a spacetime vector V_k in terms of its orthogonal (to Σ_0) and "parallel" (living in the tangent space of Σ_0) components:

$$V_k = n_k V_\perp + V_{\bar k}, \tag{C2}$$

where $V_\perp \coloneqq n^k V_k$ and $V_{\bar k} \coloneqq V_k - n_k V_\perp$. Here, we use a convention that a bar over an index k denotes its parallel projection, so that $n^k V_{\bar k}$ vanishes. The objects $V_\perp$ and $V_{\bar k}$ are respectively a scalar and a vector with respect to 3d rotations in Σ_0.

Consider now a second-rank tensor, X_{ik}. Its orthogonal-parallel decomposition reads

$$X_{ik} = n_i X_{\perp\bar k} + n_i n_k X_{\perp\perp} + n_k X_{\bar\imath\perp} + X_{\bar\imath\bar k}. \tag{C3}$$

Here, $X_{\perp\bar k}$ is a vector and $X_{\perp\perp}$ a scalar with respect to 3d rotations, whereas the irreducible parts of $X_{\bar\imath\bar k}$ are its trace, antisymmetric, and traceless symmetric parts:

$${}^SX \coloneqq X^{\bar k}{}_{\bar k}, \qquad {}^AX_{\bar\imath\bar k} \coloneqq X_{[\bar\imath\bar k]}, \quad {}^TX_{\bar\imath\bar k} \coloneqq X_{(\bar\imath\bar k)} - \frac{1}{3}\eta_{\bar\imath\bar k}X^{\bar m}{}_{\bar m},$$
$$X_{\bar\imath\bar k} = {}^AX_{\bar\imath\bar k} + {}^TX_{\bar\imath\bar k} + \frac{1}{3}\eta_{\bar\imath\bar k}{}^SX. \tag{C4a}$$

As a consequence,

$$X^{\bar\imath\bar k}Y_{\bar\imath\bar k} = {}^AX_{\bar\imath\bar k}{}^AY_{\bar\imath\bar k} + {}^TX_{\bar\imath\bar k}{}^TY_{\bar\imath\bar k} + \frac{1}{3}{}^SX^SY. \tag{C4b}$$

Now, it is straightforward to extend these considerations to any tensor. As a particularly interesting example, we consider the spacetime tensor $X_{ijk} = -X_{jik}$, which is decomposed into the set of spatial tensors $(X_{\perp\bar\jmath\perp}, X_{\perp\bar\jmath\bar k}, X_{\bar\imath\bar\jmath\perp}, X_{\bar\imath\bar\jmath\bar k})$. The irreducible parts of $X_{\bar\imath\bar\jmath\bar k} = -X_{\bar\jmath\bar\imath\bar k}$ are the pseudoscalar, the vector, and the traceless symmetric tensor, respectively:

$${}^PX \coloneqq \varepsilon^{\bar\imath\bar\jmath\bar k}X_{\bar\imath\bar\jmath\bar k}, \qquad {}^VX_{\bar\imath} \coloneqq X_{\bar\imath\bar k}{}^{\bar k},$$
$${}^TX_{\bar\imath\bar\jmath\bar k} \coloneqq X_{\bar\imath(\bar\jmath\bar k)} + \frac{1}{2}\eta_{\bar\imath(\bar\jmath}{}^VX_{\bar k)} - \frac{1}{2}\eta_{\bar\jmath\bar k}{}^VX_{\bar\imath}. \tag{C5a}$$

The tensor part satisfies the cyclic identity ${}^TX_{\bar\imath\bar\jmath\bar k} + {}^TX_{\bar k\bar\imath\bar\jmath} + {}^TX_{\bar\jmath\bar k\bar\imath} = 0$. The epsilon tensor $\varepsilon_{\bar\imath\bar\jmath\bar k}$ is defined by $\varepsilon_{\bar\imath\bar\jmath\bar k} \coloneqq \varepsilon_{\perp\bar\imath\bar\jmath\bar k}$ and satisfies the identities

$$\varepsilon_{\bar\imath\bar\jmath\bar k}\varepsilon^{\bar m\bar n\bar k} = -(\delta^{\bar m}_{\bar\imath}\delta^{\bar n}_{\bar\jmath} - \delta^{\bar m}_{\bar\jmath}\delta^{\bar n}_{\bar\imath}),$$
$$\varepsilon_{\bar\imath\bar n\bar k}\varepsilon^{\bar m\bar n\bar k} = -3\delta^{\bar m}_{\bar\imath}, \qquad \varepsilon_{\bar m\bar n\bar k}\varepsilon^{\bar m\bar n\bar k} = -6.$$

The related decomposition formulas read:

$$X_{\bar\imath\bar\jmath\bar k} = \frac{4}{3}{}^TX_{[\bar\imath\bar\jmath]\bar k} - \eta_{\bar k[\bar\imath}{}^VX_{\bar\jmath]} - \frac{1}{6}\varepsilon_{\bar\imath\bar\jmath\bar k}{}^PX,$$
$$X^{\bar\imath\bar\jmath\bar k}Y_{\bar\imath\bar\jmath\bar k} = \frac{4}{3}{}^TX^{\bar\imath\bar\jmath\bar k T}Y_{\bar\imath\bar\jmath\bar k} + {}^VX^{\bar\imath V}Y_{\bar\imath} - \frac{1}{6}{}^PX^PY. \tag{C5b}$$

APPENDIX D: GENERAL CONSTRUCTION OF $\mathcal{H}_\perp$

In this appendix, we discuss the general structure of $\mathcal{H}_\perp$, including the case when some of the critical parameters vanish. In a simplified but self-evident notation, the relations that define critical parameters have the following typical form (see Sec. III):

$$\varphi = FV, \tag{D1}$$

where

$$\varphi \coloneqq \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \qquad F \coloneqq \begin{pmatrix} a & \bar b\\ \bar c & d\end{pmatrix}, \qquad V \coloneqq \begin{pmatrix}V_1\\ V_2\end{pmatrix}.$$

Here, φ represents the if-constraints, V are the corresponding velocities, and F is the matrix with eigenvalues c_1, c_2. Since F is chosen to represent A, B_0, B_1 or B_2, the parameter $\bar c$ is proportional to $\bar b$, $\bar c = \kappa\bar b$. If $\bar b = 0$, the matrix F is already diagonal, and the construction of $\mathcal{H}_\perp$ is quite simple. When $\bar b \neq 0$, which is typical for the parity-violating PG, the matrix F needs first to be diagonalized. The diagonal form D of F is constructed as

$$D = P^{-1}FP, \qquad P \coloneqq \begin{pmatrix} -\bar b & -\bar b\\ a - c_1 & a - c_2\end{pmatrix},$$
$$D = \begin{pmatrix} c_1 & 0\\ 0 & c_2\end{pmatrix}, \tag{D2}$$

where P is invertible provided $\det P = \bar b(c_2 - c_1) \neq 0$, and

$$P^{-1} = \frac{1}{\det P}\begin{pmatrix} a - c_2 & \bar b\\ -a + c_1 & -\bar b\end{pmatrix}.$$

Left multiplication of (D1) by P^{-1} yields

$$\varphi' = DV', \tag{D3a}$$

where $\varphi' \coloneqq P^{-1}\varphi$ and $V' \coloneqq P^{-1}V$, or, equivalently,

$$\varphi'_1 = c_1 V'_1, \qquad \varphi'_2 = c_2 V'_2. \tag{D3b}$$

To construct the related F-part of $\mathcal{H}_\perp$, note that its typical form reads

$$\mathcal{H}^F_\perp = \varphi^T Q V \equiv \varphi'^T (P^T Q P) V', \qquad Q \coloneqq \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}; \tag{D4}$$

see Sec. IV. Further discussion depends on the specific values of c_1 and c_2.

(1) When c_1, $c_2 \neq 0$, Eq. (D1) implies $V = F^{-1}\varphi$, and $\mathcal{H}^F_\perp = \varphi^T Q F^{-1}\varphi$ coincides with the result found in Sec. IV. (2) The case $c_1 = c_2 = 0$ is rather trivial: both if-constraints φ'_n become true constraints that appear in the total Hamiltonian, but $\mathcal{H}^F_\perp = 0$. (3) Finally, when only one critical parameter vanishes (which requires $\det F = 0$), say $c_2 = 0$, then $\varphi'_2 = 0$ (a new constraint), V'_2 remains undetermined, and $\varphi'_1 = c_1 V'_1$. Hence, Eq. (D4) implies that

$$\mathcal{H}^F_\perp = (\bar b^2 q_1 + d^2 q_2)\frac{1}{c_1}(\varphi'_1)^2 + \varphi'_1(\bar b^2 q_1 - a d q_2)V'_2. \tag{D5}$$

The result can be also expressed in terms of the original if-constraints φ_n by noting that $\varphi'_2 = 0$ implies $\varphi'_1 = -\varphi_1/\bar b$. The factor $1/c_1$ in the first term shows a typical dependence on the critical parameters, known from PG$^+$, whereas the second term, linear in the undetermined velocity V'_2, can be absorbed into the total Hamiltonian; see [14,15,23]. The presence of an extra constraint φ'_2 requires us to complete the whole consistency procedure.

In the context of the weak field approximation, the form of $\mathcal{H}^F_\perp$ in Eq. (D5) determines the no-ghost conditions for the case (3):

$$\det F = ad - \bar b\,\bar c = 0, \qquad \sigma c_1 > 0, \tag{D6}$$

where σ is the sign of $(\bar b^2 q_1 + d^2 q_2)$ and $c_1 = a + d$.

Now, we have a comment on kind of “non-analiticity” of the above results. Since the assumption $\bar b \neq 0$ ensures the regularity of the matrix P, the diagonal matrix D in Eq. (D2) has no valid limit for $\bar b \to 0$. Hence, the expressions for c_n when $\bar b = 0$ cannot be obtained by taking the limit $\bar b \to 0$ of the generic result. However, since the matrix F for $\bar b = 0$ is already diagonal, the critical parameters c_n can be obtained directly from F. The same conclusion also holds for the form of $\mathcal{H}^F_\perp$.

APPENDIX E: LINEARIZED BIANCHI IDENTITIES

In Secs. V and VI, many technical simplifications were obtained with the help of the linearized Bianchi identities,

$$\varepsilon^{\mu\nu\lambda\rho} R^{ij}{}_{\nu\lambda\rho} = 0, \qquad \varepsilon^{\mu\nu\lambda\rho}\partial_\nu T_{i\lambda\rho} = \varepsilon^{\mu\nu\lambda\rho} R_{i\nu\lambda\rho}, \tag{E1}$$

and their consequences. In particular, the first identity implies that

$$\partial^k X_{ik} = 0, \qquad \partial^i G_{ik} = 0, \tag{E2}$$

where $G_{ik} \coloneqq Ric_{ik} - (1/2)\eta_{ik} R$, and the second identity yields

$$\begin{aligned} X_i{}^j &= -\frac{1}{2}\varepsilon^{jkmn}\partial_k T_{imn}, \qquad X = 3\partial\mathcal{A}, \\ \varepsilon^{ijmn} R_{ijmk} &= 2X_k{}^n - \delta^n_k X, \\ 2Ric_{[mn]} &= -\partial^k T_{kmn} + 2\partial_{[m}\mathcal{V}_{n]}. \end{aligned} \tag{E3}$$

As a consequence,

$$\partial^m \Phi_{im} = \partial^m \hat R_{[im]} + \frac{1}{4}\partial_i R, \qquad \partial^m \Psi_{im} = \partial^m X_{[mi]} - \frac{1}{4}\partial_i X. \tag{E4}$$

APPENDIX F: SIMPLIFIED CONDITIONS FOR THE ABSENCE OF TACHYONS

In this appendix, we derive a simplified form of the conditions (7.9) and (7.16), describing the absence of tachyons in the spin-0 and spin-1 sectors, respectively; the spin-2 sector is discussed in subsection VII C.

1. Spin-0 sector

The expression for $\mathrm{tr} f_0$, found in subsection VI A, can be represented in a suitable form as

$$\frac{1}{3}\mathrm{tr} f_0 = \alpha_0(b_2 + b_3) + \beta_0(\bar b_2 - \bar b_3) + \gamma_0(b_4 + b_6), \tag{F1a}$$

where

$$\begin{aligned} \alpha_0 &= a_0(2a_0 + a_2), \qquad \beta_0 = 4a_0(\bar a_2 - \bar a_0), \\ \gamma_0 &= -[a_2(a_0 + 2a_3) + 2(\bar a_0 - \bar a_2)^2]. \end{aligned}$$

After dividing this equation by $(b_4 + b_6) > 0$, one obtains

$$\frac{\mathrm{tr} f_0}{3(b_4 + b_6)} = \alpha_0 \frac{b_2 + b_3}{b_4 + b_6} + \beta_0 \frac{\bar b_2 - \bar b_3}{b_4 + b_6} + \gamma_0.$$

By noting that the first relation in Eq. (7.6) can be written as

$$\frac{b_2 + b_3}{b_4 + b_6} + x^2 < 0, \qquad x \coloneqq \frac{\bar b_2 - \bar b_3}{b_4 + b_6},$$

we find it useful to rewrite Eq. (F1a) in the form

$$\begin{aligned} \frac{\mathrm{tr} f_0}{3(b_4 + b_6)} &= \alpha_0\left(\frac{b_2 + b_3}{b_4 + b_6} + x^2\right) + F_0(x), \\ F_0(x) &\coloneqq -\alpha_0 x^2 + \beta_0 x + \gamma_0. \end{aligned} \tag{F1b}$$

Further analysis is based on an important property of the quadratic function $F_0(x)$, based on Eq. (7.9a); its discriminant, $\Delta_0 = \beta_0^2 + 4\alpha_0\gamma_0$, is always negative,

$$\Delta_0 = -4a_0a_2[(2a_0+a_2)(a_0+2a_3)+2(\bar a_0-\bar a_2)^2] \equiv (4/3)a_0a_2\det N_0 < 0. \tag{F2}$$

Similar considerations applied to $a_2\mathrm{tr}f_0$ modify Eq. (F1b) by an overall multiplicative factor a_2. To simplify the discussion, we introduce a suitable notation: $\alpha_0' := a_2\alpha_0$ and $F_0'(x) := a_2F_0(x)$. Note that the discriminant $\Delta_0{}'$ of the new function $F_0'(x)$ remains negative. Now, we are ready to prove the following statement:

S0. Given $\Delta_0 < 0$, the condition $\alpha_0' \equiv a_2\alpha_0 > 0$ is equivalent to $a_2\mathrm{tr}f_0 < 0$.

To prove this equivalence, we start by assuming $\alpha_0' > 0$, which implies

$$a_2\mathrm{tr}f_0 < 3(b_4+b_6)F_0'(x). \tag{F3}$$

Moreover, the parabola $F_0'(x)$ opens downward, and $\Delta_0'/\alpha_0' < 0$ (negative at vertex) ensures that $F_0'(x) < 0$ for any x. Hence, $a_2\mathrm{tr}f_0 < 0$, what was to be shown.

The reverse statement $a_2\mathrm{tr}f_0 < 0 \Rightarrow \alpha_0' > 0$ can be easily proven by reductio ad absurdum, that is, by showing that $\alpha_0' < 0$ implies $a_2\mathrm{tr}f_0 > 0$, which is a contradiction.

The statement **S0** allows us to replace Eq. (7.9b) with the much simpler condition $a_2 > 0$.

2. Spin-1 sector

For the spin-1 sector, we first rewrite $\mathrm{tr}f_1$ in the form

$$\frac{1}{4}\mathrm{tr}f_1 = \alpha_1(b_4+b_5)+\beta_1(\bar b_2-\bar b_5)+\gamma_1(b_2+b_5), \tag{F4a}$$

where

$$\alpha_1 := 2\mu_3k_2A_0 - 4\mu_3\bar A_0^2 + 2A_0(\bar a_0-\bar a_2)^2,$$
$$\beta_1 := -4\mu_2\mu_3\bar A_0 + 4(A_0^2+\bar A_0^2)(\bar a_0-\bar a_2) - 4\bar A_0(\bar a_0-\bar a_2)^2,$$
$$\gamma_1 := 2A_0\mu_2k_3 - \mu_2\bar A_0^2 + 2A_0(\bar a_0-\bar a_2)^2.$$

After dividing by $(b_2+b_5) > 0$, one can rewrite Eq. (F4a) in a suitable form

$$\frac{1}{4(b_2+b_5)}\mathrm{tr}f_1 = \alpha_1\left(\frac{b_4+b_5}{b_2+b_5}+x^2\right)+F_1(x),$$
$$F_1(x) := -\alpha_1x^2+\beta_1x+\gamma_1, \qquad x := \frac{\bar b_2-\bar b_5}{b_2+b_5}. \tag{F4b}$$

As a consequence of Eq. (7.16a), the discriminant Δ_1 of the quadratic function $F_1(x)$ is automatically negative,

$$\Delta_1 := 16(A_0^2+\bar A_0^2)[\mu_2\mu_3+(\bar a_0-\bar a_2)^2][k_2k_3+(\bar a_1-\bar a_2)^2] \equiv \frac{2}{3}(A_0^2+\bar A_0^2)(\det N_1)(\det A) < 0. \tag{F5}$$

To relate our considerations to the properties of $(\det A)\mathrm{tr}f_1$, we multiply Eq. (F4b) by $\det A$, and introduce a suitable notation $\alpha_1' := (\det A)\alpha_1$ and $F_1'(x) := (\det A)\times F_1(x)$. The new discriminant Δ_1' is also negative. Now, one can prove the following statement:

S1. For $\Delta_1 < 0$, the condition $\alpha_1' \equiv (\det A)\alpha_1 < 0$ is equivalent to $(\det A)\mathrm{tr}f_1 > 0$.

The proof goes as follows. Starting with $\alpha_1' < 0$, one obtains

$$(\det A)\mathrm{tr}f_1 > 4(b_2+b_5)F_1'(x). \tag{F6}$$

Then, by noting that the parabola opens upward ($\alpha_1' < 0$) and $\Delta_1'/\alpha_1' > 0$ (positive at vertex), one concludes that $F_1'(x) > 0$. Hence, $(\det A)\mathrm{tr}f_1 > 0$.

As before, the reverse statement $(\det A)\mathrm{tr}f_1 > 0 \Rightarrow \alpha_1' < 0$ can be proven by showing that $\alpha_1' > 0$ leads to $(\det A)\mathrm{tr}f_1 < 0$, which is a contradiction.

The condition $\Delta_1 < 0$, combined with $(\det A)\alpha_1 < 0$, can be realized in two ways:

$$\text{(i)}\ \det A > 0, \det N_1 < 0, \alpha_1 < 0.$$
$$\text{(ii)}\ \det A < 0, \det N_1 > 0, \alpha_1 > 0. \tag{F7}$$

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Generalized plane waves in Poincaré gauge theory of gravity

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A family of exact vacuum solutions, representing generalized plane waves propagating on the (anti-)de Sitter background, is constructed in the framework of Poincaré gauge theory. The wave dynamics is defined by the general Lagrangian that includes all parity even and parity odd invariants up to the second order in the gauge field strength. The structure of the solution shows that the wave metric significantly depends on the spacetime torsion.

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I. INTRODUCTION

The gauge principle, which was originally formulated by Weyl in the context of electrodynamics [1], now belongs to the key concepts which underlie the modern understanding of dynamical structure of fundamental physical interactions. Development of Weyl's idea, most notably in the works of Yang, Mills and Utiyama [2,3], resulted in the construction of the general gauge-theoretic framework for arbitrary non-Abelian groups of *internal* symmetries. Sciama and Kibble extended this formalism to the *spacetime* symmetries, and proposed a theory of gravity [4,5] based on the Poincaré group—a semidirect product of the group of spacetime translations times the Lorentz group. The importance of the Poincaré group in particle physics strongly supports the Poincaré gauge theory (PGT) as the most appropriate framework for description of the gravitational phenomena.

The "translational" gauge field potentials (corresponding to the subgroup of the spacetime translations) can be consistently identified with the spacetime coframe field, whereas the "rotational" gauge field potentials (corresponding to the local Lorentz subgroup) can be interpreted as the spacetime connection. This introduces the Riemann–Cartan geometry on the spacetime manifold, since one naturally recovers the torsion and the curvature as the Poincaré gauge field strengths [6–16] ("translational" and "rotational" one, respectively). The gravitational dynamics in PGT is determined by a Lagrangian that is assumed to be the function of the field strengths, the curvature and the torsion, and the dynamical setup is completed by including a suitable matter Lagrangian.

In the past, investigations of PGT were mostly focused on the class of models with quadratic parity symmetric Lagrangians of the Yang-Mills type, expecting that the results obtained for such a class should be sufficient to reveal essential dynamical features of the more complex general theory, for an overview see [17]. Recently, however, there has been a growing interest for the extended class of models with a general Lagrangian that includes both parity even and parity odd quadratic terms, see for instance [18–23]. An important difference between these two classes of PGT models is manifest in their particle spectra. Generically, the particle spectrum of the parity conserving PGT model consists of the massless graviton and eighteen massive torsion modes. The conditions for the absence of ghosts and tachyons impose serious restrictions on the propagation of these modes [24–29]. In contrast, a recent analysis of the general PGT [30] shows that the propagation of torsion modes is much less restricted. This is a new and physically interesting dynamical effect of the parity odd sector.

Based on the experience stemming from general relativity (GR), it is well known that exact solutions play an important role in understanding gravitational dynamics. An important class of these solutions consists of the gravitational waves [31–35], one of the best known families of exact solutions in GR. For many years, investigation of gravitational waves has been an interesting subject also in the framework of PGT [36–45], as well as in the metric-affine gravity theory which is obtained in the gauge-theoretic approach when the Poincaré group is extended to the general affine symmetry group [46–54]. Noticing that dynamical effects of the parity odd sector of PGT are not sufficiently well known, recently one of us [55] has studied exact plane wave solutions with torsion in vacuum, propagating on the flat background, for the case of the vanishing cosmological constant Λ. In another recent work [56] complementary results have been obtained, when the

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generalized pp waves with torsion were derived as exact vacuum solutions of the parity even PGT, but for the case of a nontrivial $\Lambda \neq 0$. In the present paper, we merge and extend these investigations by constructing the generalized plane waves with torsion as vacuum solutions of the general quadratic PGT with nonvanishing cosmological constant. The resulting structure offers a deeper insight into the dynamical role of the parity odd sector of PGT.

The paper is organized as follows. In the next Sec. II we present a condensed introduction to the Poincaré gauge gravity theory, giving the basic definitions and describing the main structures; more details can be found in [6–9]. In Sec. III we start with representing an (anti)-de Sitter spacetime as a gravitational wave and use the properties of the plane-fronted electromagnetic and gravitational waves discussed in [57] to formulate an ansatz for the gravitational wave in the Poincaré gauge gravity. The properties of the resulting curvature and torsion 2-forms are studied. In Sec. IV the set of differential equations for the wave variables is derived. It is worthwhile to note that the functions which describe the wave's profile satisfy a system of linear equations, even though the original field equations of the Poincaré gauge theory are highly nonlinear. Solutions of this system are constructed, and their properties are discussed. We demonstrate the consistency of the results obtained with the particle spectrum of the general Poincaré gauge gravity model. Finally, the conclusions are outlined in Sec. V.

Our basic notation and conventions are consistent with [7]. In particular, Greek indices $\alpha,\beta,\ldots = 0,\ldots,3$, denote the anholonomic components (for example, of a coframe ϑ^α), while the Latin indices $i,j,\ldots = 0,\ldots,3$, label the holonomic components (dx^i, e.g.). The anholonomic vector frame basis e_α is dual to the coframe basis in the sense that $e_\alpha \rfloor \vartheta^\beta = \delta^\beta_\alpha$, where $\rfloor$ denotes the interior product. The volume 4-form is denoted η, and the η-basis in the space of exterior forms is constructed with the help of the interior products as $\eta_{\alpha_1\ldots\alpha_p} := e_{\alpha_p}\rfloor\ldots e_{\alpha_1}\rfloor\eta$, $p = 1,\ldots,4$. They are related to the ϑ-basis via the Hodge dual operator *, for example, $\eta_{\alpha\beta} = {}^*(\vartheta_\alpha \wedge \vartheta_\beta)$. The Minkowski metric $g_{\alpha\beta} = \mathrm{diag}(+1,-1,-1,-1)$. All the objects related to the parity-odd sector (coupling constants, irreducible pieces of the curvature, gravitational wave potentials, etc) are marked by an overline, to distinguish them from the corresponding parity-even objects.

II. BASICS OF POINCARÉ GAUGE GRAVITY

The gravitational field is described by the coframe $\vartheta^\alpha = e_i^\alpha dx^a$ and connection $\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta dx^i$ 1-forms. The translational and rotational field strengths read

$$T^\alpha = D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta, \tag{2.1}$$

$$R_\alpha{}^\beta = d\Gamma_\alpha{}^\beta + \Gamma_\gamma{}^\beta \wedge \Gamma_\alpha{}^\gamma. \tag{2.2}$$

As usual, the covariant differential is denoted D.

The gravitational Lagrangian 4-form is (in general) an arbitrary function of the geometrical variables:

$$V = V(\vartheta^\alpha, T^\alpha, R_\alpha{}^\beta). \tag{2.3}$$

Its variation with respect to the gravitational (translational and Lorentz) potentials yields the field equations

$$\mathcal{E}_\alpha := \frac{\delta V}{\delta\vartheta^\alpha} = -DH_\alpha + E_\alpha = 0, \tag{2.4}$$

$$\mathcal{C}^\alpha{}_\beta := \frac{\delta V}{\delta\Gamma_\alpha{}^\beta} = -DH^\alpha{}_\beta + E^\alpha{}_\beta = 0. \tag{2.5}$$

Here, the Poincaré *gauge field momenta* 2-forms are introduced by

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha}, \qquad H^\alpha{}_\beta := -\frac{\partial V}{\partial R_\alpha{}^\beta}, \tag{2.6}$$

and the 3–forms of the *canonical* energy–momentum and spin for the gravitational gauge fields are constructed as

$$E_\alpha := \frac{\partial V}{\partial\vartheta^\alpha} = e_\alpha\rfloor V + (e_\alpha\rfloor T^\beta)\wedge H_\beta + (e_\alpha\rfloor R_\beta{}^\gamma)\wedge H^\beta{}_\gamma, \tag{2.7}$$

$$E^\alpha{}_\beta := \frac{\partial V}{\partial\Gamma_\alpha{}^\beta} = -\vartheta^{[\alpha}\wedge H_{\beta]}. \tag{2.8}$$

The field equations (2.4) and (2.5) are written here for the vacuum case. In the presence of matter, the right-hand sides of (2.4) and (2.5) contain the canonical energy-momentum and the canonical spin currents of the physical sources, respectively.

A. Quadratic Poincaré gravity models

The torsion 2-form can be decomposed into the 3 irreducible parts, whereas the curvature 2-form has 6 irreducible pieces. Their definition is presented in the Appendix.

The general quadratic model is described by the Lagrangian 4-form that contains all possible quadratic invariants of the torsion and the curvature:

$$V = \frac{1}{2\kappa c}\Big\{(a_0\eta_{\alpha\beta} + \bar a_0\vartheta_\alpha\wedge\vartheta_\beta)\wedge R^{\alpha\beta} - 2\lambda_0\eta - T^\alpha\wedge\sum_{I=1}^{3}[a_I\,{}^*({}^{(I)}T_\alpha) + \bar a_I\,{}^{(I)}T_\alpha]\Big\} - \frac{1}{2\rho}R^{\alpha\beta}\wedge\sum_{I=1}^{6}[b_I\,{}^*({}^{(I)}R_{\alpha\beta}) + \bar b_I\,{}^{(I)}R_{\alpha\beta}]. \tag{2.9}$$

The Lagrangian has a clear structure: the first line is *linear* in the curvature, the second line collects *torsion quadratic* terms, whereas the third line contains the *curvature quadratic* invariants. Furthermore, each line is composed of the parity even pieces (first terms on each line), and the parity odd parts (last terms on each line). The dimensionless constant $\bar{a}_0 = \frac{1}{\xi}$ is inverse to the so-called Barbero-Immirzi parameter ξ, and the linear part of the Lagrangian —the first line in (2.9)—describes what is known in the literature as the Einstein-Cartan-Holst model. A special case $a_0 = 0$ and $\bar{a}_0 = 0$ describes the purely quadratic model without the Hilbert-Einstein linear term in the Lagrangian. In the Einstein-Cartan model, one puts $a_0 = 1$ and $\bar{a}_0 = 0$.

Besides that, the general PGT model contains a set of the coupling constants which determine the structure of quadratic part of the Lagrangian: ρ, a_1, a_2, a_3 and $\bar{a}_1, \bar{a}_2, \bar{a}_3$, $b_1, \ldots, b_6$ and $\bar{b}_1, \ldots, \bar{b}_6$. The overbar denotes the constants responsible for the parity odd interaction. We have the dimension $[\frac{1}{\rho}] = [\hbar]$, whereas a_I, $\bar{a}_I$, b_I and $\bar{b}_I$ are dimensionless. Moreover, not all of these constants are independent: we take $\bar{a}_2 = \bar{a}_3$, $\bar{b}_2 = \bar{b}_4$ and $\bar{b}_3 = \bar{b}_6$ because some of terms in (2.9) are the same in view of (A14)–(A16).

For the Lagrangian (2.9) from (2.6)–(2.8) we derive the gauge gravitational field momenta

$$H_\alpha = \frac{1}{\kappa c} h_\alpha, \tag{2.10}$$

$$H^\alpha{}_\beta = -\frac{1}{2\kappa c}(a_0 \eta^\alpha{}_\beta + \bar{a}_0 \vartheta^\alpha \wedge \vartheta_\beta) + \frac{1}{\rho} h^\alpha{}_\beta, \tag{2.11}$$

and the canonical energy-momentum and spin currents of the gravitational field

$$E_\alpha = \frac{1}{2\kappa c}(a_0 \eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} + 2\bar{a}_0 R_\alpha{}^\beta \wedge \vartheta_\beta - 2\lambda_0 \eta_\alpha + q_\alpha^{(T)}) + \frac{1}{\rho} q_\alpha^{(R)}, \tag{2.12}$$

$$E^\alpha{}_\beta = \frac{1}{2}(H^\alpha \wedge \vartheta_\beta - H_\beta \wedge \vartheta^\alpha). \tag{2.13}$$

For convenience, we introduced here the 2-forms which are linear functions of the torsion and the curvature, respectively, by

$$h_\alpha = \sum_{I=1}^{3} [a_I {}^*({}^{(I)}T_\alpha) + \bar{a}_I {}^{(I)}T_\alpha], \tag{2.14}$$

$$h^\alpha{}_\beta = \sum_{I=1}^{6} [b_I {}^*({}^{(I)}R^\alpha{}_\beta) + \bar{b}_I {}^{(I)}R^\alpha{}_\beta], \tag{2.15}$$

and the 3-forms which are quadratic in the torsion and in the curvature, respectively:

$$q_\alpha^{(T)} = \frac{1}{2}[(e_\alpha \rfloor T^\beta) \wedge h_\beta - T^\beta \wedge e_\alpha \rfloor h_\beta], \tag{2.16}$$

$$q_\alpha^{(R)} = \frac{1}{2}[(e_\alpha \rfloor R_\beta{}^\gamma) \wedge h^\beta{}_\gamma - R_\beta{}^\gamma \wedge e_\alpha \rfloor h^\beta{}_\gamma]. \tag{2.17}$$

By construction, (2.14) has the dimension of a length, $[h_\alpha] = [\ell]$, whereas the 2-form (2.15) is obviously dimensionless, $[h^\alpha{}_\beta] = 1$. Similarly, we find for (2.16) the dimension of length $[q_\alpha^{(T)}] = [\ell]$, and the dimension of the inverse length, $[q_\alpha^{(R)}] = [1/\ell]$ for (2.17).

The resulting *vacuum* Poincaré gravity field equations (2.4) and (2.5) then read:

$$\frac{a_0}{2}\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} + \bar{a}_0 R_\alpha{}^\beta \wedge \vartheta_\beta - \lambda_0 \eta_\alpha + q_\alpha^{(T)} + \ell_\rho^2 q_\alpha^{(R)} - Dh_\alpha = 0, \tag{2.18}$$

$$a_0 \eta^\alpha{}_{\beta\gamma} \wedge T^\gamma + \bar{a}_0 (T^\alpha \wedge \vartheta_\beta - T_\beta \wedge \vartheta^\alpha) + h^\alpha \wedge \vartheta_\beta - h_\beta \wedge \vartheta^\alpha - 2\ell_\rho^2 Dh^\alpha{}_\beta = 0. \tag{2.19}$$

The contribution of the curvature square terms in the Lagrangian (2.9) to the gravitational field dynamics in the Eqs. (2.18) and (2.19) is characterized by the parameter

$$\ell_\rho^2 = \frac{\kappa c}{\rho}. \tag{2.20}$$

Since $[\frac{1}{\rho}] = [\hbar]$, this new coupling parameter has the dimension of the area, $[\ell_\rho^2] = [\ell^2]$.

III. GRAVITATIONAL WAVES IN POINCARÉ GAUGE GRAVITY

Gravitational waves are of fundamental importance in physics, and recently the purely theoretical research in this area was finally supported by the first experimental evidence [58–60]. A general overview of the history of this fascinating subject can be found in [61–63].

A. (Anti)-de Sitter spacetime as a wave

Let us now discuss the four-dimensional manifold which can be viewed as an “(anti)-de Sitter spacetime in the wave disguise”. As before [55], we use the same local coordinates which are divided into two groups: $x^i = (x^a, x^A)$, where $x^a = (x^0 = \sigma, x^1 = \rho)$ and $x^A = (x^2, x^3)$. Hereafter the indices from the beginning of the Latin alphabet label the coordinates σ and ρ parametrizing the wave rays, $a, b, c\ldots = 0,\ 1$, whereas the capital Latin indices, $A, B, C\ldots = 2, 3$, refer to coordinates x^A on the wave front.

The coframe 1-form is chosen as a direct generalization of the ansatz used in [55,57]:

$$\hat{\vartheta}^{\hat{0}} = \frac{q}{2p}[(\hat{U} + 1)d\sigma + d\rho], \tag{3.1}$$

$$\hat{\vartheta}^{\hat{1}} = \frac{q}{2p}[(\hat{U}-1)d\sigma + d\rho], \tag{3.2}$$

$$\hat{\vartheta}^{\hat{A}} = \frac{1}{p}dx^A, \qquad A = 2,3. \tag{3.3}$$

Here the three functions are given by the following expressions:

$$\hat{U} = -\frac{\lambda}{4}\rho^2, \tag{3.4}$$

$$p = 1 + \frac{\lambda}{4}\delta_{AB}x^Ax^B, \tag{3.5}$$

$$q = 1 - \frac{\lambda}{4}\delta_{AB}x^Ax^B. \tag{3.6}$$

The constant parameter λ is an arbitrary real number (which can be positive, negative or zero). As a result, the line element reads

$$ds^2 = \frac{1}{p^2}\{q^2(d\sigma d\rho + \hat{U}d\sigma^2) - \delta_{AB}dx^Adx^B\}. \tag{3.7}$$

The key object for the description of the wave configurations is the wave 1-form. On the basis of the earlier results [55], we introduce a wave 1-form k as

$$k \coloneqq d\sigma = \frac{p}{q}(\hat{\vartheta}^{\hat{0}} - \hat{\vartheta}^{\hat{1}}). \tag{3.8}$$

By construction, we have $k \wedge {}^*k = 0$. As before, the wave covector is $k_\alpha = e_\alpha \rfloor k$. Its (anholonomic) components are thus $k_\alpha = \frac{p}{q}(1,-1,0,0)$ and $k^\alpha = \frac{p}{q}(1,1,0,0)$. Hence, this is a null vector field, $k_\alpha k^\alpha = 0$.

The corresponding Riemannian connection $\hat{\Gamma}_\beta{}^\alpha$ is determined from

$$d\hat{\vartheta}^\alpha + \hat{\Gamma}_\beta{}^\alpha \wedge \hat{\vartheta}^\beta = 0, \tag{3.9}$$

and it reads explicitly (recall that $a,b,\ldots = 0$, 1 and $A,B,\ldots = 2$, 3)

$$\hat{\Gamma}_{\hat{0}}{}^{\hat{1}} = \hat{\Gamma}_{\hat{1}}{}^{\hat{0}} = -\frac{\lambda\rho}{2}k, \tag{3.10}$$

$$\hat{\Gamma}_B{}^a = \frac{p}{q}\hat{\vartheta}^a e_B \rfloor d\left(\frac{q}{p}\right), \tag{3.11}$$

$$\hat{\Gamma}_B{}^A = \frac{1}{p}(\hat{\vartheta}_B e^A \rfloor dp - \hat{\vartheta}^A e_B \rfloor dp). \tag{3.12}$$

Substituting (3.4)–(3.6), we straightforwardly find the curvature:

$$\hat{R}_\beta{}^\alpha = \lambda \hat{\vartheta}_\beta \wedge \hat{\vartheta}^\alpha. \tag{3.13}$$

Thus, the coframe and connection $(\hat{\vartheta}^\alpha, \hat{\Gamma}_\beta{}^\alpha)$, described by (3.1)–(3.3) and (3.10)–(3.12), represent the geometry of a torsionless (3.9) spacetime of constant curvature (3.13). Depending on the sign of λ, we have either a de Sitter or an anti-de Sitter space.

We mark the corresponding geometrical quantities by the hat over the symbols. This geometry will be used as a starting point for the construction of the plane wave solutions in the Poincaré gauge gravity with nontrivial cosmological constant.

It is worthwhile to note that the wave vector field k is a null geodesic in this geometry:

$$k \wedge {}^*k = 0, \qquad k \wedge {}^*\hat{D}k^\alpha = 0. \tag{3.14}$$

B. Generalized plane wave ansatz

We will construct new gravitational wave solutions in Poincaré gauge gravity theory by making use of the ansatz for the coframe and for the local Lorentz connection

$$\vartheta^\alpha = \hat{\vartheta}^\alpha + \frac{U}{2}\frac{q}{p}k^\alpha k, \tag{3.15}$$

$$\Gamma_\alpha{}^\beta = \hat{\Gamma}_\alpha{}^\beta + \frac{q}{p}(k_\alpha W^\beta - k^\beta W_\alpha)k. \tag{3.16}$$

Here the function $U = U(\sigma, x^A)$ determines the wave profile. The ansatz for the local Lorentz connection is postulated as a direct analogue of the construction used earlier in [55], and the vector variable $W^\alpha = W^\alpha(\sigma, x^A)$ satisfies the same orthogonality property, $k_\alpha W^\alpha = 0$, which is guaranteed by the choice

$$W^\alpha = \begin{cases} W^a = 0, & a = 0,1, \\ W^A = W^A(\sigma, x^B), & A = 2,3. \end{cases} \tag{3.17}$$

Consequently, the generalized ansatz for the Poincaré gauge potentials—coframe (3.15) and connection (3.16) —is described by the three variables $U = U(\sigma, x^B)$ and $W^A = W^A(\sigma, x^B)$. These should be determined from the gravitational field equations.

The ansatz (3.15) and (3.16) can be viewed as a non-Riemannian extension of the Kerr-Schild-Kundt construction developed recently [64–67] in general relativity and in modified gravity models. The original Kerr-Schild construction [34] in GR is underlain by the existence of preferred null directions. In our approach, the metric defined by the coframe (3.15) can be written in a typical Kerr-Schild form

$$g_{ij} = \hat{g}_{ij} + \frac{q}{p}Uk_ik_j, \tag{3.18}$$

where $\hat{g}_{ij}$ is the spacetime metric of the (anti)-de Sitter line element (3.7), and $k_i = \partial_i \rfloor k = \partial_i \rfloor d\sigma = (1,0,0,0)$ is the null vector with respect to both $\hat{g}_{ij}$ and g_{ij}. On the other hand, the orthogonality property of the vector W^α that defines the radiation piece of the connection (3.17), $k_\alpha W^\alpha = 0$, ensures typical radiation structure of the Riemann-Cartan field strengths, the torsion and the curvature.

The line element for this ansatz has the same form (3.7), with a replacement

$$\hat{U} \to \hat{U} + \frac{p}{q} U. \tag{3.19}$$

It is important to stress that the wave 1-form k is still defined by (3.8), which however can be recast into

$$k = d\sigma = \frac{p}{q}(\vartheta^{\hat{0}} - \vartheta^{\hat{1}}). \tag{3.20}$$

Consequently, the anholonomic components of the wave covector $k_\alpha = e_\alpha \rfloor k$ still have the values $k_\alpha = \frac{p}{q}(1, -1, 0, 0)$ and $k^\alpha = \frac{p}{q}(1, 1, 0, 0)$. As before, this is a null vector field, $k_\alpha k^\alpha = 0$.

One may wonder why does the factor $\frac{q}{p}$ appear in the ansatz (3.15) and (3.16). After all, it is always possible to absorb it by redefining U and W^A. However, it is convenient to keep this factor explicitly by noticing that the combination $\frac{q}{p} k^\alpha = (1, 1, 0, 0)$ has the constant values. It becomes clear then that the following differential relations are valid:

$$dk = 0, \qquad d\left(\frac{q}{p} k_\alpha\right) = 0. \tag{3.21}$$

Moreover, although Dk_α no longer vanishes, we find

$$k \wedge D\left(\frac{q}{p} k_\alpha\right) = k \wedge \hat{D}\left(\frac{q}{p} k_\alpha\right) = 0. \tag{3.22}$$

Taking this into account, we straightforwardly compute the torsion 2-form

$$T^\alpha = -k \wedge \frac{q}{p} k^\alpha \Theta, \tag{3.23}$$

where we introduced the 1-form

$$\Theta = \frac{1}{2} \underline{d} U + W_\alpha \vartheta^\alpha, \tag{3.24}$$

with the differential $\underline{d} := \vartheta^A e_A \rfloor d = dx^A \partial_A$ that acts in the transversal 2-space spanned by $x^A = (x^2, x^3)$.

The structure of the torsion is qualitatively the same as in the case of the vanishing parameter λ, see [55]. The structure of curvature is more nontrivial, though. A direct computation yields a 2-form

$$R_\alpha{}^\beta = \lambda \vartheta_\alpha \wedge \vartheta^\beta - k \wedge \frac{q}{p}(k_\alpha \Omega^\beta - k^\beta \Omega_\alpha), \tag{3.25}$$

where we introduced the vector-valued 1-form with the components

$$\Omega^\alpha = \begin{cases} \Omega^a = 0, & a = 0, 1, \\ \Omega^A = \hat{D} W^A + \frac{\lambda}{2} U \vartheta^A, & A = 2, 3. \end{cases} \tag{3.26}$$

The transversal covariant derivative is defined by

$$\hat{D} W^A = \underline{d} W^A + \hat{\Gamma}_B{}^A W^B. \tag{3.27}$$

Note that the Riemannian de Sitter connection (3.12) appears here (more exactly, the corresponding components of the Riemann-Cartan connection (3.16) coincide with the Riemannian components: $\Gamma_B{}^A = \hat{\Gamma}_B{}^A$).

Let us describe the geometry of the transversal 2-space spanned by $x^A = (x^2, x^3)$ explicitly. The volume 2-form reads $\underline{\eta} = \frac{1}{2} \eta_{AB} \vartheta^A \wedge \vartheta^B = \frac{1}{p^2} dx^2 \wedge dx^3$, where $\eta_{AB} = -\eta_{BA}$ is the 2-dimensional Levi-Civita tensor (with $\eta_{23} = 1$). Obviously this is a non-flat space. The corresponding Riemannian connection (3.12) yields a nontrivial curvature $\hat{R}_B{}^A = \lambda \vartheta_B \wedge \vartheta^A$ of a 2-dimensional de Sitter space. The volume 4-form of the spacetime manifold reads $\eta = \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}} = \frac{q^2}{2p^2} k \wedge d\rho \wedge \underline{\eta}$. For the wave 1-form we find the remarkable relation

$$^*k = -k \wedge \underline{\eta}. \tag{3.28}$$

We will denote the geometrical objects on the transversal 2-space by underlining them; for example, a 1-form $\underline{\phi} = \phi_A \vartheta^A$. The Hodge duality on this space is defined as usual via $\underline{*}\vartheta_A = \underline{\eta}_A = e_A \rfloor \underline{\eta} = \eta_{AB} \vartheta^B$. With the help of (3.28), we can verify

$$^*(k \wedge \underline{\phi}) = k \wedge \underline{*}\underline{\phi}. \tag{3.29}$$

The new 1-forms (3.24) and (3.26) have the obvious transversality properties:

$$k \wedge {}^*\Theta = 0, \qquad k \wedge {}^*\Omega^\alpha = 0, \qquad k_\alpha \Omega^\alpha = 0. \tag{3.30}$$

In accordance with (3.17) and (3.26), we have explicitly:

$$\Theta = \vartheta^A \left(\frac{1}{2} \hat{D}_A U - \delta_{AB} W^B\right), \tag{3.31}$$

$$\Omega^A = \vartheta^B \left(\hat{D}_B W^A + \frac{\lambda}{2} U \delta_B^A\right). \tag{3.32}$$

Here we denoted $\hat{D}_A = e_A \rfloor \hat{D}$. Applying the transversal differential to (3.24), and making use of (3.26), we find

$$\underline{d}\Theta = \Omega_\alpha \wedge \vartheta^\alpha. \tag{3.33}$$

In essence, this is equivalent to the Bianchi identity $DT^\alpha = R_\beta{}^\alpha \wedge \vartheta^\beta$ which is immediately checked by applying the covariant differential D to (3.23) and using (3.25). Note that it is crucial to use (3.21).

A further refinement of the generalized wave ansatz will be considered in Sec. IV C.

C. Irreducible decomposition of gravitational field strengths

Irreducible parts of the torsion and the curvature are as follows. The second (trace) and third (axial trace) irreducible part of the torsion are trivial, ${}^{(2)}T^\alpha = 0$ and ${}^{(3)}T^\alpha = 0$, and the first (pure tensor) piece is nontrivial:

$${}^{(1)}T^\alpha = T^\alpha = -k \wedge \frac{q}{p} k^\alpha \Theta. \tag{3.34}$$

At the same time, the curvature pieces ${}^{(3)}R^{\alpha\beta} = {}^{(5)}R^{\alpha\beta} = 0$, whereas

$${}^{(6)}R^{\alpha\beta} = \lambda \vartheta^\alpha \wedge \vartheta^\beta, \tag{3.35}$$

and for $I = 1, 2, 4$:

$${}^{(I)}R^{\alpha\beta} = 2k \wedge {}^{(I)}\Omega^{[\alpha} k^{\beta]} \frac{q}{p}. \tag{3.36}$$

Here ${}^{(1)}\Omega^\alpha + {}^{(2)}\Omega^\alpha + {}^{(4)}\Omega^\alpha = \Omega^\alpha$, and explicitly we have

$${}^{(1)}\Omega^\alpha = \frac{1}{2}(\Omega^\alpha - \vartheta^\alpha e_\beta \rfloor \Omega^\beta + \vartheta^\beta e^\alpha \rfloor \Omega_\beta), \tag{3.37}$$

$${}^{(2)}\Omega^\alpha = \frac{1}{2}(\Omega^\alpha - \vartheta^\beta e^\alpha \rfloor \Omega_\beta), \tag{3.38}$$

$${}^{(4)}\Omega^\alpha = \frac{1}{2}\vartheta^\alpha e_\beta \rfloor \Omega^\beta. \tag{3.39}$$

The transversal components of these objects are constructed in terms of the irreducible pieces of the 2×2 matrix $\hat{D}_B W^A$: symmetric traceless part, skew-symmetric part and the trace, respectively. Using (3.32), we derive ${}^{(I)}\Omega^A = {}^{(I)}\Omega^A{}_B \vartheta^B$, with

$${}^{(1)}\Omega^A{}_B = \frac{1}{2}(\hat{D}_B W^A + \hat{D}^A W_B - \delta^A_B \hat{D}_C W^C), \tag{3.40}$$

$${}^{(2)}\Omega^A{}_B = \frac{1}{2}(\hat{D}_B W^A - \hat{D}^A W_B), \tag{3.41}$$

$${}^{(4)}\Omega^A{}_B = \frac{1}{2}\delta^A_B(\hat{D}_C W^C + \lambda U). \tag{3.42}$$

One can demonstrate the following properties of these 1-forms:

$$\vartheta_\alpha \wedge {}^{(1)}\Omega^\alpha = 0, \qquad \vartheta_\alpha \wedge {}^{(2)}\Omega^\alpha = \vartheta_\alpha \wedge \Omega^\alpha, \tag{3.43}$$

$$\vartheta_\alpha \wedge {}^{(4)}\Omega^\alpha = 0, \qquad e_\alpha \rfloor {}^{(1)}\Omega^\alpha = -e_\alpha \rfloor \Omega^\alpha, \tag{3.44}$$

$$e_\alpha \rfloor {}^{(2)}\Omega^\alpha = 0, \qquad e_\alpha \rfloor {}^{(4)}\Omega^\alpha = 2e_\alpha \rfloor \Omega^\alpha, \tag{3.45}$$

$$k_\alpha {}^{(1)}\Omega^\alpha = -\frac{1}{2}k e_\alpha \rfloor \Omega^\alpha, \qquad k_\alpha {}^{(2)}\Omega^\alpha = 0, \tag{3.46}$$

$$k_\alpha {}^{(4)}\Omega^\alpha = \frac{1}{2}k e_\alpha \rfloor \Omega^\alpha, \qquad k \wedge {}^{*(2)}\Omega^\alpha = 0, \tag{3.47}$$

$$k \wedge {}^{*(1)}\Omega^\alpha = -k \wedge {}^{*(4)}\Omega^\alpha = -\frac{1}{2}k^\alpha \vartheta_\beta \wedge {}^*\Omega^\beta. \tag{3.48}$$

IV. FIELD EQUATIONS

Let us now turn to the quadratic Poincaré gauge model with the general Lagrangian (2.9), and allow for a nontrivial cosmological constant λ_0.

Substituting the torsion (3.34) and the curvature (3.35), (3.36), into (2.14) and (2.15), we find

$$h^\alpha = -k^\alpha Z \frac{q}{p}, \tag{4.1}$$

$$h^{\alpha\beta} = \lambda b_6 \eta^{\alpha\beta} + \lambda \bar{b}_6 \vartheta^\alpha \wedge \vartheta^\beta - 2k^{[\alpha} Z^{\beta]} \frac{q}{p}, \tag{4.2}$$

where we introduced the 2-forms

$$Z = a_1 {}^*(k \wedge \Theta) + \bar{a}_1 k \wedge \Theta, \tag{4.3}$$

$$Z^\alpha = \sum_{I=1,2,4} [b_I {}^*(k \wedge {}^{(I)}\Omega^\alpha) + \bar{b}_I k \wedge {}^{(I)}\Omega^\alpha]. \tag{4.4}$$

Making use of (3.30) and (3.43)–(3.48) we can show that

$$k \wedge h^\alpha = 0, \qquad k \wedge {}^*h^\alpha = 0, \qquad k_\alpha h^\alpha = 0. \tag{4.5}$$

As a result, substituting (4.2) into (2.16) and (2.17), we find $q_\alpha^{(T)} = 0$ and

$$q_\alpha^{(R)} = 2\lambda \frac{q}{p} k_\alpha \{-(b_4 + b_6){}^*k e_\beta \rfloor \Omega^\beta + (\bar{b}_2 - \bar{b}_6) k \wedge \vartheta_\beta \wedge \Omega^\beta\}. \tag{4.6}$$

With an account of the properties (4.5), one can check that

$$Dh_\alpha = -\hat{D}\left(k_\alpha Z \frac{q}{p}\right), \tag{4.7}$$

$$Dh_{\alpha\beta} = -\hat{D}\left(2k_{[\alpha} Z_{\beta]} \frac{q}{p}\right) + \lambda b_6 \eta_{\alpha\beta\mu} \wedge T^\mu + \lambda \bar{b}_6 (T_\alpha \wedge \vartheta_\beta - T_\beta \wedge \vartheta_\alpha). \tag{4.8}$$

The transversal nature of Θ and Ω^A leads to a further simplification. In particular, using (3.29), we recast (4.3) and (4.4) into

$$Z = k \wedge \Xi, \qquad Z^A = k \wedge \Xi^A, \tag{4.9}$$

where we have introduced the 1-forms

$$\Xi = a_1 \underline{^*}\Theta + \bar{a}_1 \Theta, \tag{4.10}$$

$$\Xi^A = \sum_{I=1,2,4} [b_I \underline{^*}{}^{(I)}\Omega^A + \bar{b}_I {}^{(I)}\Omega^A]. \tag{4.11}$$

A. Wave equations

After all these preparations, we are in a position to write down the gravitational field equations for the quadratic Poincaré gauge model (2.9). Substituting the gravitational wave ansatz (3.15)–(3.16) into (2.18), we derive the first equation

$$(3a_0\lambda - \lambda_0)\eta_\alpha + \frac{q}{p} k_\alpha {}^*k(e_\beta \rfloor \Omega^\beta)[a_0 - 2\lambda\ell_\rho^2(b_4 + b_6)] + \frac{q}{p} k_\alpha k \wedge \{\vartheta_\beta \wedge \Omega^\beta[\bar{a}_0 + 2\lambda\ell_\rho^2(\bar{b}_2 - \bar{b}_6)] - \underline{d}\Xi\} = 0. \tag{4.12}$$

Contracting this with k^α, we find the value of the constant parameter in the wave ansatz:

$$\lambda = \frac{\lambda_0}{3a_0}, \tag{4.13}$$

and with an account of (3.28) and (4.10) we recast (4.12) into

$$[a_0 - 2\lambda\ell_\rho^2(b_4 + b_6)]\vartheta_A \wedge \underline{^*}\Omega^A + a_1\underline{d}\underline{^*}\Theta - [\bar{a}_0 + \bar{a}_1 + 2\lambda\ell_\rho^2(\bar{b}_2 - \bar{b}_6)]\vartheta_A \wedge \Omega^A = 0. \tag{4.14}$$

The first two terms describe the parity-even model, whereas the last term accounts for the parity-odd sector.

Similarly, by gravitational wave ansatz (3.15)–(3.16) in (2.19), we obtain the second equation

$$k_a \frac{q}{p} k \wedge \{(a_0 + a_1 - 2\lambda\ell_\rho^2 b_6)\vartheta_B \wedge \underline{^*}\Theta + (\bar{a}_0 + \bar{a}_1 - 2\lambda\ell_\rho^2\bar{b}_6)\vartheta_B \wedge \Theta - 2\ell_\rho^2\hat{D}\Xi_B\} = 0. \tag{4.15}$$

Note here that the $[ab]$ and $[AB]$ components in (2.19) are satisfied identically, and only the mixed $[aB]$ components give rise to the result (4.15).

Equation (4.14) and the expression inside the curly bracket in (4.15) are both 2-forms on the 2-dimensional transversal space spanned by $x^A = (x^2, x^3)$, and thus (4.14) and (4.15) describe a system of three partial differential equations for the three variables $U = U(\sigma, x^B)$ and $W^A = W^A(\sigma, x^B)$. Substituting (3.31) and (3.32), we recast (4.14) and (4.15) into the final tensorial form

$$A_0(\hat{D}_A W^A + \lambda U) + a_1\left(\hat{D}_A W^A - \frac{1}{2}\hat{\Delta}U\right) - \bar{A}_0\eta^{AB}\hat{D}_A\underline{W}_B = 0, \tag{4.16}$$

$$-A_1\left(\underline{W}_A - \frac{1}{2}\hat{D}_A U\right) + \bar{A}_1\eta_{AB}\left(W^B - \frac{1}{2}\hat{\underline{D}}^B U\right) + \ell_\rho^2(\bar{b}_1 - \bar{b}_2)[\hat{D}_A(\eta^{BC}\hat{D}_B\underline{W}_C) + \eta_{AB}\hat{\underline{D}}^B(\hat{D}_C W^C + \lambda U)] + \ell_\rho^2(b_1 + b_4)\left[-\hat{\Delta}\left(\underline{W}_A - \frac{1}{2}\hat{D}_A U\right) + \lambda\left(\underline{W}_A - \frac{1}{2}\hat{D}_A U\right) - \hat{D}_A(\hat{D}_B W^B + \lambda U) + \hat{D}_A\left(\hat{D}_B W^B - \frac{1}{2}\hat{\Delta}U\right)\right] = 0. \tag{4.17}$$

The 2-dimensional transversal space has the (anti)-de Sitter geometry and the corresponding covariant Laplacian reads

$$\hat{\Delta} = \delta^{AB}\hat{D}_A\hat{D}_B = p^2\underline{\Delta}, \tag{4.18}$$

where $\underline{\Delta} = \delta^{AB}\partial_A\partial_B$ is the usual Laplace operator.

Note that $\bar{b}_4 = \bar{b}_2$. Here we denoted $\underline{W}_A = \delta_{AB}W^B$ and $\hat{\underline{D}}^A = \delta^{AB}\hat{D}_B$, and introduced the convenient abbreviations for the combinations of the coupling constants,

$$A_0 = a_0 - 2\lambda\ell_\rho^2(b_4 + b_6), \tag{4.19}$$

$$\bar{A}_0 = \bar{a}_0 + \bar{a}_1 + 2\lambda\ell_\rho^2(\bar{b}_2 - \bar{b}_6), \tag{4.20}$$

$$A_1 = a_0 + a_1 + 2\lambda\ell_\rho^2(b_1 - b_6), \tag{4.21}$$

$$\bar{A}_1 = \bar{a}_0 + \bar{a}_1 + 2\lambda\ell_\rho^2(\bar{b}_1 - \bar{b}_6). \tag{4.22}$$

The transversal covariant derivatives do not commute,

$$(\hat{D}_A\hat{D}_B - \hat{D}_B\hat{D}_A)W^C = \hat{R}_{ABD}{}^C W^D = 2\lambda\delta^C_{[A}\underline{W}_{B]}, \tag{4.23}$$

and we used this fact when deriving (4.16) and (4.17). Direct consequences of (4.23) are:

$$\eta^{BC}\hat{D}_B\hat{D}_C\underline{W}_A = \lambda\eta_{AB}W^B, \tag{4.24}$$

$$(\hat{\Delta}\hat{D}_A - \hat{D}_A\hat{\Delta})U = \lambda\hat{D}_A U. \qquad (4.25)$$

It is worthwhile to notice that the derivatives of W^A appear in (4.16)–(4.17) only in combinations

$$\Omega := e_\alpha \rfloor \Omega^\alpha = \hat{D}_A W^A + \lambda U, \qquad (4.26)$$

$$\Phi := {}^*\underline{d}\,{}^*\Theta = \hat{D}_A W^A - \frac{1}{2}\hat{\Delta}U, \qquad (4.27)$$

$$\bar{\Phi} := {}^*\underline{d}\Theta = -\eta^{AB}\hat{D}_A \underline{W}_B, \qquad (4.28)$$

which have a clear geometrical meaning in terms of the curvature and the torsion.

The system (4.16)–(4.17) always admits a nontrivial solution for arbitrary quadratic Poincaré gauge model with any choices of the coupling constants. There are some interesting special cases.

B. Torsionless gravitational waves

The torsion (3.23) vanishes when $\Theta = 0$ which is realized, see (3.24) and (3.31), for

$$W^A = \frac{1}{2}\delta^{AB}\hat{D}_B U. \qquad (4.29)$$

Substituting this into (4.16), we find

$$A_0\{\hat{\Delta}U + 2\lambda U\} = 0, \qquad (4.30)$$

whereas (4.17) reduces to

$$\begin{aligned} &\ell_\rho^2(\bar{b}_1 - \bar{b}_2)\eta_{AB}\underline{\hat{D}}^B\{\hat{\Delta}U + 2\lambda U\} \\ &-\ell_\rho^2(b_1 + b_4)\hat{D}_A\{\hat{\Delta}U + 2\lambda U\} = 0. \end{aligned} \qquad (4.31)$$

Accordingly, we conclude that the well-known torsionless wave solution of GR with the function U satisfying

$$p^2\underline{\Delta}U + 2\lambda U = 0 \qquad (4.32)$$

is an exact solution of the generic quadratic Poincaré gauge gravity model. This is consistent with our earlier results on the torsion-free solutions in Poincaré gauge theory [16].

Moreover, the torsionless wave (4.29)–(4.30) represents a general solution for the purely torsion quadratic class of Poincaré models, since this is the only configuration admitted by the system (4.16)–(4.17) for $b_I = \bar{b}_I = 0$.

C. Torsion gravitational waves

The torsion-free ansatz (3.9) can be generalized to

$$W^A = \frac{1}{2}\delta^{AB}\hat{D}_B(U + V) + \frac{1}{2}\eta^{AB}\hat{D}_B\bar{V}, \qquad (4.33)$$

with $V \neq 0$. The two scalar functions $V = V(\sigma, x^A)$ and $\bar{V} = \bar{V}(\sigma, x^A)$ define the non-Riemannian piece of the connection, stemming from torsion:

$$\begin{aligned} \Theta &= -\frac{1}{2}(\underline{d}V + {}^*\underline{d}\,\bar{V}) \\ &= -\frac{1}{2}\vartheta^A(\hat{D}_A V - \eta_{AB}\underline{\hat{D}}^B\bar{V}). \end{aligned} \qquad (4.34)$$

For the above choice, the metric and torsion contributions to the connection are described in a rather symmetric way, in terms of the three potentials $(U, V, \bar{V})$. In particular, we find for (4.26)–(4.28):

$$\Omega = \frac{1}{2}(\hat{\Delta}V + \hat{\Delta}U + 2\lambda U), \qquad (4.35)$$

$$\Phi = \frac{1}{2}\hat{\Delta}V, \qquad \bar{\Phi} = \frac{1}{2}\hat{\Delta}\,\bar{V}. \qquad (4.36)$$

Substituting (4.33) into (4.16) and (4.17), we derive

$$A_0\Omega + a_1\Phi + \bar{A}_0\bar{\Phi} = 0, \qquad (4.37)$$

$$\begin{aligned} &\hat{D}_A\left\{-\frac{1}{2}A_1 V - \frac{1}{2}\bar{A}_1\bar{V} - \ell_\rho^2(b_1 + b_4)\Omega - \ell_\rho^2(\bar{b}_1 - \bar{b}_2)\bar{\Phi}\right\} \\ &+\eta_{AB}\underline{\hat{D}}^B\left\{-\frac{1}{2}A_1 V + \frac{1}{2}\bar{A}_1\bar{V} - \ell_\rho^2(b_1 + b_4)\bar{\Phi} + \ell_\rho^2(\bar{b}_1 - \bar{b}_2)\Omega\right\} = 0. \end{aligned} \qquad (4.38)$$

One needs to pay attention to the noncommutativity of the covariant derivatives and use (4.23)–(4.25).

As a result, we obtain the system of the three linear second order differential equations for the three functions U, V, $\bar{V}$:

$$A_0(\hat{\Delta}V + \hat{\Delta}U + 2\lambda U) + a_1\hat{\Delta}V + \bar{A}_0\hat{\Delta}\,\bar{V} = 0, \qquad (4.39)$$

$$-\ell_\rho^2(b_1 + b_4)(\hat{\Delta}V + \hat{\Delta}U + 2\lambda U) - A_1 V - \ell_\rho^2(\bar{b}_1 - \bar{b}_2)\hat{\Delta}\,\bar{V} - \bar{A}_1\bar{V} = 0, \qquad (4.40)$$

$$\ell_\rho^2(\bar{b}_1 - \bar{b}_2)(\hat{\Delta}V + \hat{\Delta}U + 2\lambda U) + \bar{A}_1 V - \ell_\rho^2(b_1 + b_2)\hat{\Delta}\,\bar{V} - A_1\bar{V} = 0. \qquad (4.41)$$

D. Solution for potentials

Before starting the analysis of solutions, one can notice that the system (4.40) and (4.41) is actually not equivalent to the original equation (4.38). Indeed, by taking the covariant divergence (applying $\hat{D}^A$) and by taking the covariant curl (applying $\eta^{AB}\hat{D}_B$) of (4.38), we derive the pair of equations where on the right-hand sides of (4.40) and (4.41) one finds not zeros but arbitrary functions, say, $\alpha(\sigma, x^A)$ and $\beta(\sigma, x^A)$, which are *harmonic*, in the sense that they both satisfy equations $\hat{\Delta}\alpha = \hat{\Delta}\beta = 0$. However, one then immediately notices that with the help of redefinitions

$$V \to V + v, \qquad \hat{\Delta}v = 0, \tag{4.42}$$

$$\bar{V} \to \bar{V} + \bar{v}, \qquad \hat{\Delta}\,\bar{v} = 0, \tag{4.43}$$

we can always remove these nontrivial right-hand sides and come to the system (4.40) and (4.41).

In other words, a solution of the system (4.39)–(4.41) admits the *gauge* transformation (4.42)–(4.43), under which the potentials V and $\bar{V}$ can be shifted by arbitrary harmonic functions. Such gauge transformed potentials are of course still solutions of the Poincaré gauge field equations (4.37) and (4.38). What is important, however, the curvature and the torsion remain invariant under the redefinition (4.42)–(4.43) of potentials: (4.35) and (4.36) obviously are not affected by the arbitrary harmonic functions.

Now, as a first step, we substitute $(\hat{\Delta}V + \hat{\Delta}U + 2\lambda U)$ from (4.39) into (4.40) and (4.41). The resulting system reads

$$\ell_\rho^2\hat{\Delta}\{a_1(b_1+b_4)V + [-A_0(\bar{b}_1-\bar{b}_2) + \bar{A}_0(b_1+b_4)]\bar{V}\} - A_0A_1V - A_0\bar{A}_1\bar{V} = 0, \tag{4.44}$$

$$\ell_\rho^2\hat{\Delta}\{a_1(\bar{b}_1-\bar{b}_2)V + [A_0(b_1+b_2) + \bar{A}_0(\bar{b}_1-\bar{b}_2)]\bar{V}\} - A_0\bar{A}_1V + A_0A_1\bar{V} = 0. \tag{4.45}$$

After solving this system, we can use the potentials V and $\bar{V}$ to substitute them into (4.39) which then becomes an inhomogeneous differential equation for the metric potential U:

$$A_0(\hat{\Delta}U + 2\lambda U) = -(a_1 + A_0)\hat{\Delta}V - \bar{A}_0\hat{\Delta}\,\bar{V}. \tag{4.46}$$

For the parity-even models with $\bar{a}_I = 0$, $\bar{b}_I = 0$, hence $\bar{A}_0 = 0$ and $\bar{A}_1 = 0$, the system (4.44)–(4.45) reduces to the two uncoupled equations

$$a_1(b_1+b_4)\ell_\rho^2\hat{\Delta}V - A_0A_1V = 0, \tag{4.47}$$

$$(b_1+b_2)\ell_\rho^2\hat{\Delta}\,\bar{V} + A_1\bar{V} = 0, \tag{4.48}$$

recovering the result of [56].

To analyze the system (4.44)–(4.45), let us rewrite it in matrix form

$$\hat{\Delta}\mathcal{V} - M\mathcal{V} = 0, \qquad M \coloneqq \frac{A_0}{\ell_\rho^2}F, \tag{4.49}$$

where we combined the potentials into a single object, a "2-vector" $\mathcal{V} = \binom{V}{\bar{V}}$, and the 2×2 matrix $F = K^{-1}N$ is constructed from

$$K = \left(\begin{array}{c|c} a_1(b_1+b_4) & \bar{A}_0(b_1+b_4) - A_0(\bar{b}_1-\bar{b}_2) \\ \hline a_1(\bar{b}_1-\bar{b}_2) & A_0(b_1+b_2) + \bar{A}_0(\bar{b}_1-\bar{b}_2) \end{array}\right),$$

$$N = \left(\begin{array}{c|c} A_1 & \bar{A}_1 \\ \hline \bar{A}_1 & -A_1 \end{array}\right). \tag{4.50}$$

One immediately notices the simple structure of the matrix N which is manifest in the properties

$$N^2 = (A_1^2+\bar{A}_1^2)\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right), \qquad \det N = -(A_1^2+\bar{A}_1^2). \tag{4.51}$$

One can solve the matrix differential equation (4.49) by diagonalizing this system. To achieve this, one needs to find the eigenvalues of the matrix M and to construct the corresponding eigenvectors. Let m^2 be an eigenvalue of the matrix M. It is determined from the corresponding characteristic equation $\det(M - m^2) = 0$ which has the meaning of the dispersion relation for the mass:

$$\ell_\rho^4 m^4 \det K + \ell_\rho^2 m^2 A_0 \mathrm{tr}(NK) - A_0^2(A_1^2 + \bar{A}_1^2) = 0. \tag{4.52}$$

The coefficients of the quadratic equation (4.52) are constructed from the coupling constants of the gauge gravity model. From (4.50) we have explicitly:

$$\det K = a_1A_0[(b_1+b_4)(b_1+b_2) + (\bar{b}_1-\bar{b}_2)^2], \tag{4.53}$$

$$\mathrm{tr}(NK) = (a_1A_1 + \bar{A}_0\bar{A}_1)(b_1+b_4) - A_0A_1(b_1+b_2) + (a_1\bar{A}_1 - A_0\bar{A}_1 - \bar{A}_0A_1)(\bar{b}_1-\bar{b}_2). \tag{4.54}$$

For the parity-even models with $\bar{a}_I = 0$, $\bar{b}_I = 0$, hence $\bar{A}_0 = 0$ and $\bar{A}_1 = 0$, the dispersion equation (4.52) reduces to

$$[\ell_\rho^2 m^2 a_1(b_1+b_4) - A_0A_1] \times [\ell_\rho^2 m^2 A_0(b_1+b_2) + A_0A_1] = 0, \tag{4.55}$$

and hence we recover the result (4.47)–(4.48).

General case with parity-odd terms in the Lagrangian is more complicated. No obvious simplification of (4.52) is visible.

Having found the eigenvalues m_1^2 and m_2^2 of the mass matrix M as the two roots of the quadratic equation (4.49), one can construct the matrix P that transforms M to its diagonal form. For $M_{12} \neq 0$, the latter reads

$$P = \begin{pmatrix} -M_{12} & -M_{12} \\ M_{11} - m_1^2 & M_{11} - m_2^2 \end{pmatrix}. \tag{4.56}$$

Multiplying Eq. (4.49) by P^{-1}, one then obtains

$$\hat{\Delta}\mathcal{V}' - M'\mathcal{V}' = 0, \tag{4.57}$$

where

$$M' \coloneqq P^{-1}MP = \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}, \tag{4.58}$$

and $\mathcal{V}'$ is the eigenvector of M, corresponding to the eigenvalues m_1^2 and m_2^2:

$$\begin{aligned} \mathcal{V}' &= \begin{pmatrix} \mathcal{V}'_1 \\ \mathcal{V}'_2 \end{pmatrix} = P^{-1}\mathcal{V} \\ &= \frac{1}{\det P}\begin{pmatrix} (M_{11} - m_2^2)V + M_{12}\bar{V} \\ -(M_{11} - m_1^2)V - M_{12}\bar{V} \end{pmatrix}. \end{aligned} \tag{4.59}$$

Recalling $\hat{\Delta} = p^2\underline{\Delta}$, we thus recast the system of the field equations (4.44) and (4.45) into a diagonal form

$$p^2\underline{\Delta}\mathcal{V}'_n - m_n^2\mathcal{V}'_n = 0, \tag{4.60}$$

with $n = 1, 2$. The solutions for $\mathcal{V}'_n$ are given in terms of the hypergeometric functions ${}_2F_1(a, b, c, z)$, see [56]. Similar construction exists in the case $M_{21} \neq 0$.

Now, we can return to (4.46) to find the solution for U. Each solution for $\mathcal{V}'_n$ defines the corresponding solution

$$\mathcal{V} = P\mathcal{V}' \tag{4.61}$$

of (4.49). Inserting these solutions for V and $\bar{V}$ on the right-hand side of (4.46), this equation becomes an inhomogeneous differential equation for U. Its general solution is given as a general solution of the homogeneous equation plus a particular solution of the inhomogeneous equation, $U = U_{\mathrm{h}} + U_{\mathrm{p}}$. Note that U_{h} coincides with the general vacuum solution of GR, see (4.32). The solution for U obtained by choosing $U_{\mathrm{h}} = 0$ has a very interesting interpretation. Indeed, in that case U reduces just to the particular solution U_{p}, the form of which is completely determined by the torsion potentials $(V, \bar{V})$. A similar mechanism was found also in the parity even sector [56]. Clearly, there are many other solutions for U_{h}, and consequently, for U. In each of them, the influence of torsion on the metric is quite clear.

E. Masses of the torsion modes

In order to get a deeper understanding of the role of the torsion in our gravitational wave solution, it is important to examine the mass spectrum of the associated torsion modes. Having found the matrix $F = K^{-1}N$ with the help of (4.50), the solutions of the characteristic equation (4.52) can be conveniently represented in terms of the matrix $f = (\det K)F$ as

$$m_\pm^2 = \frac{A_0}{2\ell^2 \det K}(\mathrm{tr}f \pm \sqrt{(\mathrm{tr}f)^2 - 4\det f}). \tag{4.62}$$

This is an exact formula for the mass eigenvalues $m_\pm^2$ associated to the gravitational wave. It is worthwhile to notice that $\mathrm{tr}f = -\mathrm{tr}(NK)$, and $\det f = (\det N)(\det K)$.

The particle spectrum of PGT has been calculated only with respect to the Minkowski background [24–29], and never for the (anti)-de Sitter spacetime. Accordingly, we can compare the result (4.62) with those existing in the literature only for the values of $m_\pm^2$ in the limit of the vanishing cosmological constant. In the limit of $\lambda \to 0$, we have

$$\begin{aligned} \mathrm{tr}f &= -[a_1(a_0 + a_1) + (\bar{a}_0 + \bar{a}_1)^2](b_1 + b_4) \\ &\quad + a_0(a_0 + a_1)(b_1 + b_2) + 2a_0(\bar{a}_0 + \bar{a}_1)(\bar{b}_1 - \bar{b}_2), \\ \det f &= -a_0a_1[(a_0 + a_1)^2 + (\bar{a}_0 + \bar{a}_1)^2] \\ &\quad \times [(b_1 + b_2)(b_1 + b_4) + (\bar{b}_1 - \bar{b}_2)^2], \\ \det K &= a_0a_1[(b_1 + b_2)(b_1 + b_4) + (\bar{b}_1 - \bar{b}_2)^2]. \end{aligned} \tag{4.63}$$

As a first test, we apply the formula (4.62) to the parity even sector of PGT. One can straightforwardly see that the corresponding values of $m_\pm^2$ coincide with the masses of the spin-$2^\pm$ torsion modes, known from the literature [24]; compare also with [56]. This is consistent with (4.55).

A more complete verification can be done by comparing (4.62) with the recent work of Karananas [30], which presently offers the only existing calculation of the complete mass spectrum for the most general PGT with both parity even and parity odd sectors included. A comparison of the Lagrangian (5) of Ref. [30] with our expression (2.9) is straightforward, although one should be careful since the paper [30] contains numerous misprints. As a result, we establish the following relations between our and Karananas' coupling constants (we use the notation t_0 instead of Karananas' λ to distinguish it from our cosmological term):

$$a_0 = 2\kappa c t_0, \qquad \bar{a}_0 = 0, \tag{4.64}$$

$$a_1 = 2\kappa c(-t_1 - t_0), \tag{4.65}$$

$$a_2 = 4\kappa c(-t_3 + t_0), \tag{4.66}$$

$$a_3 = \kappa c(-t_2 + t_0), \tag{4.67}$$

$$\bar{a}_1 = 4\kappa c t_5, \tag{4.68}$$

$$\bar{a}_2 = \bar{a}_3 = 2\kappa c t_4. \tag{4.69}$$

$$b_1 = 4\rho(-r_1 + r_3), \tag{4.70}$$

$$b_2 = 4\rho(-r_3), \tag{4.71}$$

$$b_3 = 4\rho(-r_2 + r_3), \tag{4.72}$$

$$b_4 = 4\rho(-r_1 + r_3 - r_4), \tag{4.73}$$

$$b_5 = 4\rho(-r_3 - r_5), \tag{4.74}$$

$$b_6 = 4\rho(-r_1 + r_3 - 3r_4), \tag{4.75}$$

$$\bar{b}_1 = \rho(-r_7 + 3r_8), \tag{4.76}$$

$$\bar{b}_2 = \bar{b}_4 = \rho(-r_7 - r_8), \tag{4.77}$$

$$\bar{b}_3 = \bar{b}_6 = \rho(4r_6 - r_7 - r_8), \tag{4.78}$$

$$\bar{b}_5 = \rho(3r_7 - r_8). \tag{4.79}$$

Substituting the expressions for a_I, b_I and $\bar{a}_I$, $\bar{b}_I$ into (4.63), one finds that the resulting values of $m^2_\pm$ in (4.62) exactly reproduce the result (A.3.5) of Karananas' paper [30] (after correcting a number of his misprints), which displays the spin-$2^\pm$ torsion modes.

Thus, we conclude that the massive spin-$2^\pm$ torsion modes turn out to be an essential ingredient of our gravitational wave, in the sense that these massive torsion modes determine the structure of the wave profile encoded in the functions V, $\bar{V}$ and U. This is a remarkable result if one recalls that the particle spectrum of PGT is derived from the linearized equations of motion, whereas our gravitational waves are exact solutions of the full nonlinear field equations.

V. DISCUSSION AND CONCLUSION

In this paper, we have found a family of the exact vacuum solutions of the most general PGT model with all possible parity even and parity odd linear and quadratic invariants in the Lagrangian (2.9), and with a nontrivial cosmological constant $\lambda_0 \neq 0$. This family represents generalized plane waves with torsion, propagating on the (anti)-de Sitter background. The present paper extends the results obtained recently in [55,56].

The underlying construction can be understood as a generalization of the Kerr-Schild-Kundt ansatz from the Riemannian to the Riemann-Cartan geometry of PGT. An essentially new element in this extended formalism is the ansatz for the local Lorentz connection $\Gamma_\alpha{}^\beta$, the radiation piece of which is constructed in terms of the null covector field k. The generalized plane wave ansatz (3.15)–(3.16) ensures that the torsion 2-form T^α and the radiation piece of the curvature 2-form $S^{\alpha\beta} \coloneqq R^{\alpha\beta} - \lambda\vartheta^\alpha \wedge \vartheta^\beta$ satisfy the radiation conditions

$$k \wedge {}^*T^\alpha = 0, \qquad k \wedge {}^*S^{\alpha\beta} = 0, \tag{5.1}$$

$$k \wedge T^\alpha = 0, \qquad k \wedge S^{\alpha\beta} = 0, \tag{5.2}$$

$$T^\alpha \wedge {}^*T^\beta = 0, \qquad S^{\alpha\beta} \wedge {}^*S^{\rho\sigma} = 0. \tag{5.3}$$

These relations represent an extension of the well-known Lichnerowicz criterion for identifying gravitational waves [68] (see also [32]), based on analogy with the electromagnetic waves, to the framework of the PGT.

In the limit of vanishing torsion, the generalized plane waves with torsion reduce to the family of the Riemannian pp waves on the (anti)-de Sitter background. The pp waves are classified as solutions of Petrov type N, since the corresponding Weyl tensor satisfies the special algebraic condition $k^\alpha C_{\alpha\beta\mu\nu} = 0$, see [34,35]. This criterion can be naturally extended to a Riemann-Cartan geometry of PGT as

$$k^{\alpha\,(1)}R_{\alpha\beta\mu\nu} = 0, \tag{5.4}$$

where ${}^{(1)}R_{\alpha\beta\mu\nu}$ is the first irreducible part of the curvature tensor, see [55,56]. The validity of (5.4) for the generalized plane waves with torsion confirms that they are also of type N.

The spacetime torsion is an essential ingredient of the generalized gravitational wave solution; its dynamical characteristics are described by the two potentials V and $\bar{V}$, satisfying the matrix equation (4.49). The mass matrix M is of particular importance for the physical interpretation of the torsion. We demonstrate that, in the limit of $\lambda \to 0$, the eigenvalues of M coincide with the values of the mass square the spin-$2^\pm$ torsion modes, identified in the work of Karananas [30]. Generically, wave front profile of a generalized plane wave with torsion is thus determined by two spin-2 massive torsion modes and the massless graviton, produced by the third, coframe potential U (which enters the spacetime metric).

It is interesting to note that there exist particular solutions for which the metric potential is completely determined by the torsion. For such solutions, the motion of a spinless test particle is effectively determined by the spacetime torsion.

The results obtained in this work were checked with the help of the computer algebra systems *Reduce* and *Mathematica*.

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APPENDIX: IRREDUCIBLE DECOMPOSITION OF THE TORSION AND CURVATURE

The torsion 2-form can be decomposed into the three irreducible pieces, $T^\alpha = {}^{(1)}T^\alpha + {}^{(2)}T^\alpha + {}^{(3)}T^\alpha$, where

$$ {}^{(2)}T^\alpha = \frac{1}{3}\vartheta^\alpha \wedge (e_\nu \rfloor T^\nu), \tag{A1} $$

$$ {}^{(3)}T^\alpha = \frac{1}{3}e^\alpha \rfloor (T^\nu \wedge \vartheta_\nu), \tag{A2} $$

$$ {}^{(1)}T^\alpha = T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha. \tag{A3} $$

The Riemann-Cartan curvature 2-form is decomposed $R^{\alpha\beta} = \sum_{I=1}^{6} {}^{(I)}R^{\alpha\beta}$ into the 6 irreducible parts

$$ {}^{(2)}R^{\alpha\beta} = -{}^*(\vartheta^{[\alpha} \wedge \bar\Psi^{\beta]}), \tag{A4} $$

$$ {}^{(3)}R^{\alpha\beta} = -\frac{1}{12}{}^*(\bar X \vartheta^\alpha \wedge \vartheta^\beta), \tag{A5} $$

$$ {}^{(4)}R^{\alpha\beta} = -\vartheta^{[\alpha} \wedge \Psi^{\beta]}, \tag{A6} $$

$$ {}^{(5)}R^{\alpha\beta} = -\frac{1}{2}\vartheta^{[\alpha} \wedge e^{\beta]} \rfloor (\vartheta^\gamma \wedge X_\gamma), \tag{A7} $$

$$ {}^{(6)}R^{\alpha\beta} = -\frac{1}{12} X \vartheta^\alpha \wedge \vartheta^\beta, \tag{A8} $$

$$ {}^{(1)}R^{\alpha\beta} = R^{\alpha\beta} - \sum_{I=2}^{6} {}^{(I)}R^{\alpha\beta}, \tag{A9} $$

where

$$ X^\alpha := e_\beta \rfloor R^{\alpha\beta}, \qquad X := e_\alpha \rfloor X^\alpha, \tag{A10} $$

$$ \bar X^\alpha := {}^*(R^{\beta\alpha} \wedge \vartheta_\beta), \qquad \bar X := e_\alpha \rfloor \bar X^\alpha, \tag{A11} $$

and

$$ \Psi_\alpha := X_\alpha - \frac{1}{4}\vartheta_\alpha X - \frac{1}{2}e_\alpha \rfloor (\vartheta^\beta \wedge X_\beta), \tag{A12} $$

$$ \bar\Psi_\alpha := \bar X_\alpha - \frac{1}{4}\vartheta_\alpha \bar X - \frac{1}{2}e_\alpha \rfloor (\vartheta^\beta \wedge \bar X_\beta). \tag{A13} $$

Directly from the definitions (A1)–(A3) and (A4)–(A9), one can prove the relations

$$ T^\alpha \wedge {}^{(2)}T_\alpha = T^\alpha \wedge {}^{(3)}T_\alpha = {}^{(2)}T^\alpha \wedge {}^{(3)}T_\alpha, \tag{A14} $$

$$ R^{\alpha\beta} \wedge {}^{(2)}R_{\alpha\beta} = R^{\alpha\beta} \wedge {}^{(4)}R_{\alpha\beta} = {}^{(2)}R^{\alpha\beta} \wedge {}^{(4)}R_{\alpha\beta}, \tag{A15} $$

$$ R^{\alpha\beta} \wedge {}^{(3)}R_{\alpha\beta} = R^{\alpha\beta} \wedge {}^{(6)}R_{\alpha\beta} = {}^{(3)}R^{\alpha\beta} \wedge {}^{(6)}R_{\alpha\beta}, \tag{A16} $$

whereas $T^\alpha \wedge {}^{(1)}T_\alpha = {}^{(1)}T^\alpha \wedge {}^{(1)}T_\alpha$ and $R^{\alpha\beta} \wedge {}^{(1)}R_{\alpha\beta} = {}^{(1)}R^{\alpha\beta} \wedge {}^{(1)}R_{\alpha\beta}$ and $R^{\alpha\beta} \wedge {}^{(5)}R_{\alpha\beta} = {}^{(5)}R^{\alpha\beta} \wedge {}^{(5)}R_{\alpha\beta}$.

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Holography in Lovelock Chern-Simons AdS gravity

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We analyze holographic field theory dual to Lovelock Chern-Simons anti–de Sitter (AdS) gravity in higher dimensions using first order formalism. We first find asymptotic symmetries in the AdS sector showing that they consist of local translations, local Lorentz rotations, dilatations and non-Abelian gauge transformations. Then, we compute 1-point functions of energy-momentum and spin currents in a dual conformal field theory and write Ward identities. We find that the holographic theory possesses Weyl anomaly and also breaks non-Abelian gauge symmetry at the quantum level.

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I. INTRODUCTION

The AdS/CFT correspondence [1] relates the fields in $(d+1)$-dimensional asymptotically anti–de Sitter (AAdS) space and correlators in a d-dimensional conformal field theory (CFT). These two theories are dual in the asymptotic sector of gravity, such that the weak coupling regime of one is related to the strong coupling regime of another. For a weak gravitational coupling, the bulk theory is well described by its semiclassical approximation, leading to the form of the duality most often used.

Since its discovery, the correspondence tools have been applied to many strongly coupled systems, giving rise to new insights into their dynamics, for example in hydrodynamics [2] and condensed matter systems such as superconductors [3].

On the other hand, much effort has been invested in analyzing the duality in semiclassical approximation of a bulk theory, with twofold purpose. First, it enables us to test the conjecture itself. Second, it helps us to gain the knowledge about strongly coupled systems which are nonperturbative and not very well understood. However, most of this investigation deals with Riemannian geometry of bulk spacetime, see for example [3–8], while a more general structure based on Riemann-Cartan geometry, where both torsion and curvature determine gravitational dynamics, is mostly underinvestigated. One of the first studies of Riemann-Cartan holography used first order formalism to obtain a holographic dual of Chern-Simons AdS gravity in five dimensions [9]. After that, in three dimensions, holographic dual to the Mielke-Baekler model was analyzed in [10], and to the most general parity-preserving three-dimensional gravity with propagating torsion in [11]. The physical interpretation of torsional degrees of freedom as carriers of a nontrivial gravitational magnetic field in 4D Einstein-Cartan gravity was discussed in [12].

Studying holographic duals of gravity with torsion has many benefits. Since its setup is more general, it also contains the results of torsion-free gravity. One of the very important features is that treating vielbein and spin connection as independent dynamical variables simplifies calculations to great extent. In Ref. [11], it was shown that for three-dimensional bulk gravity conservation laws of the boundary theory take the same form in Riemann-Cartan and Riemannian theory when the boundary torsion is set to zero. Thus, it is possible to treat vielbein and spin connection as independent dynamical variables and reproduce Riemannian results in the limit of zero torsion. In this work, we extend the results of [11] to all odd dimensions in case of holographic theory dual to Lovelock-Chern-Simons AdS gravity, by reproducing the conservation laws with respect to diffeomorphisms, Weyl and local Lorentz symmetry using first order formalism after taking a Riemannian limit.

Working in the framework of gravity with torsion also leads to richer boundary non-Abelian symmetries, as it is explicitly demonstrated for the particular model studied in this paper.

We analyze a holographic structure of Lovelock Chern-Simons AdS Gravity [13,14] in asymptotically AdS spaces. The key feature of this model is that it possesses a unique AdS vacuum, which is multiply degenerate in odd $D \geq 5$ dimensions. Unlike general Lovelock-Lanczos [15] gravity, it contains only two free parameters—gravitational constant κ and the AdS radius ℓ. This theory also features a symmetry enhancement from local Lorentz to AdS gauge symmetry. Degenerate vacuum makes the linear perturbation analysis not applicable around the AdS background. The holographic study in AAdS spacetimes, however, is nonperturbative, because the gravitational fields in a dual theory are not dynamical but they play the role of external sources for the CFT matter. Indeed, the holographic theory will remain fully nonlinear in gravitational fields, which

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will be explicitly shown in Sec. IV. On the other hand, these theories couple successfully to external sources [16], which are stable in the framework of Lovelock Chern-Simons (LCS) supergravities [17].

The paper is organized as follows. In Sec. II we introduce the holographic ansatz for the fundamental dynamical variables and we arrive to their radial expansion in the asymptotic sector. Expressed in terms of the metric, it reduces to Fefferman-Graham expansion [18]. We also analyze corresponding residual gauge symmetries which leave this ansatz invariant. In Sec. III we focus to the holographic quantum theory and derive the Noether-Ward identities. In Sec. IV we focus on Chern-Simons–AdS gravity in arbitrary odd dimensions and compute 1-point functions in the corresponding dual theory, which are energy-momentum and spin currents. We show that translational and Lorentz symmetries are present also at the quantum level, but the Weyl anomaly and non-Abelian anomaly arise, breaking the conformal and non-Abelian symmetries quantically, the former being proportional to the Euler density up to a divergence. Our results generalize the ones of [9] from five to arbitrary dimensions. Our calculations are simplified to great extent by using the results of [19]. Section V contains concluding remarks, while appendices deal with some technical details.

Our conventions are given by the following rules. On a $D = d + 1$-dimensional spacetime manifold M, the latin indices $(i, j, k, \ldots)$ refer to the local Lorentz frame, the greek indices $(\mu, \nu, \rho, \ldots)$ refer to the coordinate frame. The symmetric and antisymmetric parts of a tensor X_{ij} are $X_{(ij)} = \frac{1}{2}(X_{ij} + X_{ji})$ and $X_{[ij]} = \frac{1}{2}(X_{ij} - X_{ji})$, respectively. The $d + 1$ decomposition of spacetime is described in terms of the suitable coordinates $x^\mu = (\rho, x^\alpha)$, where ρ is a radial coordinate and x^α are local coordinates on the boundary ∂M. In the local Lorentz frame, this decomposition is expressed by $i = (1, a)$.

II. HOLOGRAPHIC ANSATZ

We are interested in a gravitational theory which possesses a local AdS symmetry. The presence of local spacetime translations and spacetime rotations introduces naturally the vielbein and the spin connection as the fundamental fields. Our goal is to gauge fix this symmetry by imposing a set of conditions on the fundamental fields in a such a way that it singles out a particular coordinate frame which is suitable for a description of a holographically dual theory. This frame should be consistent with the known Fefferman-Graham coordinate choice used on the Riemannian manifold. All the properties that follow from this gauge-fixing are purely kinematical and can be applied to any gravity invariant under local AdS group. To include the dynamics we focus, in particular, on Lovelock-Chern-Simons gravity.

A. AdS gauge transformations

In a theory with local AdS symmetry, the fundamental fields are components of a gauge field (1-form) for the AdS group $SO(D-1, 2)$ (see Appendix A) and is defined by

$$A = \frac{1}{\ell}\hat{e}^A P_A + \frac{1}{2}\hat{\omega}^{AB} J_{AB}, \tag{2.1}$$

where ℓ is the AdS radius. For the sake of simplicity, we set $\ell = 1$. Gauge transformations, parametrized by $\lambda \coloneqq \eta^A P_A + \frac{1}{2}\lambda^{AB} J_{AB}$, act on the gauge field as

$$\delta_0 A = D\lambda = d\lambda + [A, \lambda], \tag{2.2}$$

wherefrom we get the transformation law of the fundamental fields,

$$\delta_0 \hat{e}^A = \hat{\nabla}\eta^A - \lambda^{AB}\hat{e}_B,$$
$$\delta_0 \hat{\omega}^{AB} = \hat{\nabla}\lambda^{AB} + 2e^{[A}\eta^{B]}. \tag{2.3}$$

Here, the $\hat{\omega}$-covariant derivative is $\hat{\nabla}\eta^A \coloneqq d\eta^A + \hat{\omega}^{AB}\eta_B$. The AdS field strength $F = dA + A \wedge A$ has components

$$F = \hat{T}^A P_A + \frac{1}{2}F^{AB} J_{AB}, \tag{2.4}$$

which are the torsion 2-form $\hat{T}^A$ and AdS curvature F^{AB},

$$\hat{T}^A = \frac{1}{2}\hat{T}^A{}_{\mu\nu} dx^\mu \wedge dx^\nu = d\hat{e}^A + \hat{\omega}^{AB} \wedge \hat{e}_B,$$
$$F^{AB} = \frac{1}{2}F^{AB}{}_{\mu\nu} dx^\mu \wedge dx^\nu = d\hat{\omega}^{AB} + \hat{\omega}^{AC} \wedge \hat{\omega}_C{}^B + \hat{e}^A \wedge \hat{e}^B. \tag{2.5}$$

The wedge product sign is going to be omitted for simplicity from now on in the text. The global AdS space is described by a Riemannian manifold ($\hat{T}^A = 0$), whose AdS curvature vanishes ($F^{AB} = 0$), and where the Riemannian curvature $\hat{R}^{AB} = d\hat{\omega}^{AB} + \hat{\omega}^{AC} \wedge \hat{\omega}_C{}^B$ becomes explicitly constant, $\hat{R}^{AB} = -\hat{e}^A \wedge \hat{e}^B$.

B. Radial expansion and residual gauge transformations

We use the radial foliation with the local coordinates $x^\mu = (\rho, x^\alpha)$ and the Lorentz indices decomposed correspondingly as $A = (1, a)$. The asymptotic boundary of the manifold is located at the constant radius $\rho = \rho_0$. For convenience we set $\rho_0 = 0$.

1. Gauge fixing

There are two types of local symmetries, small and large, depending on how they behave asymptotically. Small local symmetries are characterized by the parameters which go to zero at infinity and all other local symmetries are large.

Small gauge symmetries act trivially on boundary fields and must be considered as redundancies in the theory, i.e., they must be gauge fixed. A good gauge choice should fix all small gauge transformations and should lead to a well-posed boundary value problem, meaning that the form of a residual symmetry in the bulk is completely determined by the boundary values of the symmetry parameters. Note that the large gauge transformations do not have to be fixed by a gauge choice. For more details, see Ref. [20].

Local transformations at our disposal are spacetime diffeomorphisms and local AdS transformations. Let us first focus on local AdS symmetry. A good gauge fixing for our purposes is the one where the spacetime is AAdS and where residual gauge transformations contain conformal transformations on the boundary.

The last condition is introduced because we want to have a CFT as a holographic theory. Too strong gauge fixing can overkill all residual transformations and give rise to a trivial holographic theory. Since the bulk theory is gauge invariant only up to boundary terms, different gauge fixings can lead to nonequivalent boundary theories.

Another important observation is that, in the metric formulation of Riemann gravity, according to the theorem of Fefferman-Graham (FG) [18], in any AAdS space there is a coordinate choice so that the metric can be cast in the FG form, that is, with $\hat{g}_{\rho\rho} = 1/(2\rho)^2$, $\hat{g}_{\rho\alpha} = 0$ and $\rho\hat{g}_{\alpha\beta}(\rho, x)$ regular on the boundary $\rho = 0$. Thus, a gauge-fixing choice of the vielbein and spin connection must be such that the corresponding metric acquires the FG form.

The number of gauge parameters of AdS group is $\frac{D(D+1)}{2}$, implying that we need the same number of gauge conditions. We impose the following D conditions on the vielbeins $\hat{e}^A{}_\rho$ and $\frac{D(D-1)}{2}$ conditions on connection $\hat{\omega}^{AB}{}_\rho$:

$$\hat{e}^A{}_\rho = -\frac{1}{2\rho}\delta_1{}^A, \qquad \hat{\omega}^{AB}{}_\rho = 0. \tag{2.6}$$

In the choice of the gauge fixing one has to keep in mind the invertibility of vielbein. Therefore, all $\hat{e}^A{}_\rho$ components cannot be set to zero. Furthermore, although in principle a choice of the radial coordinate is arbitrary, we want to have the Fefferman-Graham coordinate frame, where the metric component $g_{\rho\rho}$ behaves as $1/4\rho^2$, generalized to first order formalism, which implies the above behavior of the radial component of the vielbein.

To find residual transformations, we look at the restrictions on gauge parameters imposed by the gauge conditions (2.6) and we find that they have to satisfy

$$\begin{aligned} &\partial_\rho\eta^1 = 0, \qquad \partial_\rho\eta^a - \frac{1}{2\rho}\lambda^{1a} = 0, \\ &\partial_\rho\lambda^{ab} = 0, \qquad \partial_\rho\lambda^{1a} - \frac{1}{2\rho}\eta^a = 0. \end{aligned} \tag{2.7}$$

The equations in η^1 and λ^{ab} are straightforward to solve. To find η^a and λ^{1a}, we combine the corresponding differential equations and obtain for the parameter η^a

$$\rho^2\partial_\rho^2\eta^a + \rho\partial_\rho\eta^a - \frac{1}{4}\eta^a = 0. \tag{2.8}$$

This is the Euler-Fuchs equation which solution takes the form $\eta^a(\rho) \sim \rho^k$. Hence, from (2.8) we get $k^2 = \frac{1}{4}$ and consequently the general solution is given by

$$\begin{aligned} &\eta^1(\rho,x) = u(x), \qquad \eta^a(\rho,x) = \frac{1}{\sqrt{\rho}}\alpha^a(x) + \sqrt{\rho}\beta^a(x), \\ &\lambda^{ab}(\rho,x) = \lambda^{ab}(x), \qquad \lambda^{1a}(\rho,x) = -\frac{1}{\sqrt{\rho}}\alpha^a(x) + \sqrt{\rho}\beta^a(x). \end{aligned} \tag{2.9}$$

We see that our gauge choice is good, as desired, because symmetry parameters in the whole bulk are determined by a few arbitrary functions u, α^a, β^a and λ^{ab} defined on the boundary. We still have to identify an asymptotic symmetry group defined by these parameters.

The residual gauge parameters which describe asymptotic symmetry group naturally induce a change of the basis in the Lie algebra $J_a^\pm = P_a \pm J_{1a}$, so that the Lie-algebra valued gauge parameter has the form

$$\lambda = u(x)P_1 + \frac{1}{\sqrt{\rho}}\alpha^a(x)J_a^- + \sqrt{\rho}\beta^a(x)J_a^+ + \frac{1}{2}\lambda^{ab}(x)J_{ab}. \tag{2.10}$$

The AdS algebra in terms of the new generators reads

$$\begin{aligned} &[J_a^+, J_b^-] = 2J_{ab} + 2\eta_{ab}P_1, \qquad [J_a^\pm, J_b^\pm] = 0, \\ &[J_{ab}, J_c^\pm] = -\eta_{ac}J_b^\pm + \eta_{bc}J_a^\pm, \qquad [P_1, J_{ab}] = 0, \\ &[P_1, J_a^\pm] = \pm J_a^\pm. \end{aligned} \tag{2.11}$$

2. Radial decomposition of gauge field and field strength

Up to now the results are valid for any theory possessing AdS gauge symmetry. From now on we concentrate on Chern-Simons AdS gravity. For holography, one needs to know how the fields evolve along the radial direction and to study their near-boundary behavior. Since the radial components are already fixed by the gauge condition (2.6), now we have to determine the behavior of the spatial components.

To this end, we can use invariance of gravity under general coordinate transformations. In Ref. [21], it was shown that only $D-1$ spatial diffeomorphisms are linearly independent on gauge generators, in a physical system where time evolution was analyzed. In our case, we look at the radial quantization of a Hamiltonian, because we are interested in radial evolution of the fields from the bulk to the boundary. Thus, our independent diffeomorpisms act only in the transversal section of spacetime, that is, as $x^\alpha \to x^\alpha + \xi^\alpha(\rho, x)$. Furthermore, we know that the radial diffeomorphisms are broken by the boundary set at constant radii, so this choice of quantization is natural in our case.

Thus, we have $D-1$ transversal diffeomorphisms to gauge fix. In Ref. [21] it was shown that, in any generic Chern-Simons gauge theory (AdS in our case), there is an on-shell identity $F_{\rho\alpha} = F_{\alpha\beta}N^\beta$, with $D-1$ arbitrary functions N^β related to the transversal diffeomorphisms $\xi^\alpha(\rho, x)$. Therefore, to gauge fix them, we can just set the $D-1$ functions to zero, $N^\beta = 0$. As a consequence, we also get $F_{\rho\alpha} = 0$ or, equivalently, $\hat{T}^A{}_{\rho\alpha} = F^{AB}{}_{\rho\alpha} = 0$. These conditions are particular for Chern-Simons theory and they arise from its dynamics. Interestingly, they can be exactly solved using the gauge fixing (2.6), also written as $A_\rho = -\frac{1}{2\rho}P_1$. Rewriting the AdS Lie-algebra valued condition $F_{\rho\alpha} = 0$ as $(dA + A^2)_{\rho\alpha} = 0$, we get

$$\partial_\rho A_\alpha - \frac{1}{2\rho}\hat{e}^a{}_\alpha J_{a1} + \frac{1}{2\rho}\hat{\omega}^{1a}{}_\alpha P_A = 0.$$

This first order differential equation in $A_\alpha(\rho, x)$ can be exactly solved, given the initial condition

$$A_\alpha(0, x) \equiv e^a{}_\alpha(x)J_a^+ + k^a{}_\alpha(x)J_a^- + \frac{1}{2}\omega^{ab}{}_\alpha(x)J_{ab}. \quad (2.12)$$

The solution is

$$A_\alpha(\rho, x) = \frac{1}{\sqrt{\rho}}e^a{}_\alpha(x)J_a^+ + \sqrt{\rho}k^a{}_\alpha(x)J_a^- + \frac{1}{2}\omega^{ab}{}_\alpha(x)J_{ab}. \quad (2.13)$$

In components, this solution leads to the radial expansion of the gravitational fields expressed in terms of the boundary fields $e^a{}_\alpha$, $k^a{}_\alpha$ and $\omega^{ab}{}_\alpha$,

$$\begin{aligned}\hat{e}^a{}_\alpha &= \frac{1}{\sqrt{\rho}}(e^a{}_\alpha + \rho k^a{}_\alpha),\\ \hat{\omega}^{1a}{}_\alpha &= -\frac{1}{\sqrt{\rho}}(e^a{}_\alpha - \rho k^a{}_\alpha),\\ \hat{\omega}^{ab}{}_\alpha &= \omega^{ab}{}_\alpha.\end{aligned} \quad (2.14)$$

Thus, this is a generalization of the FG expansion of the bulk metric. Indeed, the metric $\hat{g}_{\mu\nu} = \hat{e}^A{}_\mu \hat{e}^B{}_\nu \eta_{AB}$ takes the FG form since the line element can be written as

$$ds^2 = \frac{1}{4\rho^2}d\rho^2 + \frac{1}{\rho}(g_{\alpha\beta} + 2\rho k_{(\alpha\beta)} + \rho^2 k^a{}_\alpha k_{a\beta})dx^\alpha dx^\beta, \quad (2.15)$$

where $g_{\alpha\beta} \coloneqq \eta^{ab}e^a{}_\alpha e^b{}_\beta$ and $k_{\alpha\beta} \coloneqq e_{a\alpha}k^a{}_\beta$. We conclude that the FG expansion is *finite*. Finite FG expansion is typical for Chern-Simons gravity [9] and also for general relativity when the Weyl tensor vanishes [8].

The induced metric $\gamma_{\alpha\beta}$ is defined by $\gamma_{\alpha\beta} = \rho\hat{g}_{\alpha\beta}$. The coefficients in the radial expansion of $\gamma_{\alpha\beta}$ are

$$\begin{aligned}\gamma^{(0)}_{\alpha\beta} &= g_{\alpha\beta}, & \gamma^{(1)}_{\alpha\beta} &= 2k_{(\alpha\beta)},\\ \gamma^{(2)}_{\alpha\beta} &= k^a{}_\alpha k_{a\beta}, & \gamma^{(n)}_{\alpha\beta} &= 0, \quad n \geq 3.\end{aligned} \quad (2.16)$$

From the radial expansion of the field strength we get on the boundary

$$\begin{aligned}F^{a1} &= \frac{1}{\sqrt{\rho}}(T^a - \rho\nabla k^a), & \hat{T}^1 &= -2e^a k_a,\\ F^{ab} &= R^{ab} + 4e^{[a}k^{b]}, & \hat{T}^a &= \frac{1}{\sqrt{\rho}}(T^a + \rho\nabla k^a),\end{aligned} \quad (2.17)$$

where $T^a = \nabla e^a$ and $R^{ab} = d\omega^{ab} + \omega^a{}_c\omega^{cb}$.

Physical interpretation of the boundary fields can be found from their transformation law under the residual (boundary) gauge transformations.

3. Residual gauge transformations

The complete transformation law of the basic dynamical variables in the bulk that include the spacetime diffeomorphisms is given by

$$\begin{aligned}\delta_0\hat{e}^A{}_\mu &= \hat{\nabla}_\mu\eta^A - \lambda^{AB}\hat{e}_{B\mu} - \partial_\mu\xi^\nu\hat{e}^A{}_\nu - \xi^\nu\partial_\nu\hat{e}^A{}_\mu,\\ \delta_0\hat{\omega}^{AB}{}_\mu &= \hat{\nabla}_\mu\lambda^{AB} + 2\hat{e}^{[A}{}_\mu\eta^{B]} - \partial_\mu\xi^\nu\hat{\omega}^{AB}{}_\nu - \xi^\nu\partial_\nu\hat{\omega}^{AB}{}_\mu,\end{aligned} \quad (2.18)$$

where the last two terms of each line are the Lie derivatives with respect to ξ^μ. If we make the following redefinition of parameters,

$$\begin{aligned}\eta^A &\to \eta^A + \xi^\mu\hat{e}^A{}_\mu,\\ \lambda^{AB} &\to \lambda^{AB} + \xi^\mu\hat{\omega}^{AB}{}_\mu,\end{aligned} \quad (2.19)$$

transformations (2.18) take the following form:

$$\begin{aligned}\delta_0\hat{e}^A{}_\mu &= \hat{\nabla}_\mu\eta^A - \lambda^{AB}\hat{e}_{B\mu} + \xi^\nu\hat{T}^A{}_{\mu\nu},\\ \delta_0\hat{\omega}^{AB}{}_\mu &= \hat{\nabla}_\mu\lambda^{AB} + 2\hat{e}^{[A}{}_\mu\eta^{B]} + \xi^\nu F^{AB}{}_{\mu\nu}.\end{aligned} \quad (2.20)$$

Due to the condition $F_{\rho\alpha} = 0$, the transformation laws (2.20) of $\hat{e}^A{}_\rho$ and $\hat{\omega}^{AB}{}_\rho$ with redefined parameters (2.19) take the *same form* as in the case when diffeomorphisms are absent in the transformation law (2.18). Therefore, introduction of diffeomorphisms *does not effectively change* the result (2.9).

From the transformation law for $\omega^{ab}{}_\alpha$, it follows that ξ^α *does not depend on* ρ. The complete transformation law of the gauge fields under residual transformations reads

$$\begin{aligned}
\delta_0 e^a{}_\alpha &= \nabla_\alpha \alpha^a - \lambda^{ab} e_{b\alpha} + u e^a{}_\alpha - \xi^\beta{}_{,\alpha} e^a{}_\beta - \xi^\beta \partial_\beta e^a{}_\alpha, \\
\delta_0 k^a{}_\alpha &= \nabla_\alpha \beta^a - \lambda^{ab} k_{b\alpha} - u k^a{}_\alpha - \xi^\beta{}_{,\alpha} k^a{}_\beta - \xi^\beta \partial_\beta k^a{}_\alpha, \\
\delta_0 \omega^{ab}{}_\alpha &= \nabla_\alpha \lambda^{ab} + 4 e^{[a}{}_\alpha \beta^{b]} + 4 k^{[a}{}_\alpha \alpha^{b]} \\
&\quad - \xi^\beta{}_{,\alpha} \omega^{ab}{}_\beta - \xi^\beta \partial_\beta \omega^{ab}{}_\alpha,
\end{aligned} \tag{2.21}$$

with

$$\eta^1 + \frac{\xi^\rho}{2\rho} = u(x), \qquad \xi^\alpha = \xi^\alpha(x). \tag{2.22}$$

Let us note that the residual diffeomorphisms do not change the condition $F_{\rho\alpha} = 0$, as expected. Their form shows that our gauge choice is good.

In holography it is important for the boundary to be orthogonal to the radial direction. That is why we shall impose an additional condition $\hat{e}^1{}_\alpha = 0$, which puts the bulk vielbein in the block-diagonal form with the only one boundary component $e^a{}_\alpha(x)$. The extra condition reduces the asymptotic symmetries because the parameter β^a is not independent any longer,

$$\beta^a = e^{a\alpha} \left(\frac{1}{2} \partial_\alpha u + k^b{}_\alpha \alpha_b \right). \tag{2.23}$$

The generators of the asymptotic group cannot be determined straightforwardly because a change of the basis of the Lie algebra necessary to identify this subgroup is nonlinear, that is, it depends on the point of spacetime. We shall deduce the algebra directly from the action on the fields.

Independent transformations acting on the fields are transversal diffeomorphisms or local translations $\delta_T(\xi)$, local Lorentz rotations $\delta_L(\lambda)$, local Weyl or conformal transformations $\delta_C(u)$ and non-Abelian gauge transformations $\delta_G(\alpha)$. Each transformation can be seen as generated by some generator T_a through the commutator, for example $[\delta_G(\alpha'), \delta_G(\alpha'')] = \alpha'^a \alpha''^b [T_a, T_b]$, and similarly for all other transformations. In that way, the asymptotic algebra closes as

$$\begin{aligned}
&[\delta_T(\xi'), \delta_T(\xi'')] = \delta_T([\xi', \xi'']), &&[\delta_C(u), \delta_G(\alpha)] = \delta_C(\alpha \cdot \partial u) - \delta_L(\tilde{\lambda}) - \delta_G(u\alpha), \\
&[\delta_T(\xi), \delta_L(\lambda)] = \delta_L(\xi \cdot \partial \lambda), &&[\delta_G(\alpha'), \delta_G(\alpha'')] = -\delta_C(\tilde{u}) - \delta_L(\Lambda), \\
&[\delta_T(\xi), \delta_C(u)] = \delta_C(\xi \cdot \partial u), &&[\delta_L(\lambda, \delta_G(\alpha)] = \delta_G(\lambda \cdot \alpha), \\
&[\delta_T(\xi), \delta_G(\alpha)] = \delta_G(\xi \cdot \partial \alpha), &&[\delta_L(\lambda), \delta_C(u)] = 0, \\
&[\delta_L(\lambda'), \delta_L(\lambda'')] = \delta_L([\lambda', \lambda'']), &&[\delta_C(u'), \delta_C(u'')] = 0,
\end{aligned} \tag{2.24}$$

where $[\xi', \xi'']^\alpha = \xi' \cdot \partial \xi''^\alpha - \xi' \cdot \partial \xi''^\alpha$ is the Lie bracket and $[\lambda', \lambda'']^{ab} = \lambda'^{ac} \lambda''^b_c - \lambda''^{ac} \lambda'^{\ b}_c$ is the group commutator. We also introduced the contraction $\xi \cdot \partial = \xi^\beta \partial_\beta$ and the matrix multiplication $(\lambda \cdot \alpha)^a = \lambda^{ab} \alpha_b$, and defined the auxiliary Lorentz parameters $\tilde{\lambda}^{ab} = 2\alpha^{[a} \partial^{b]} u$ and $\Lambda^{ab} = 4k^{c[a}(\alpha'_c \alpha''^{b]} - \alpha''_c \alpha'^{b]})$, as well as the Weyl parameter $\tilde{u} = 4k^{[ab]} \alpha'_a \alpha''_b$.

The above brackets are computed by acting on $e^a{}_\alpha$, but their form is field independent. The boundary diffeomorphisms, Lorentz rotations and Weyl dilatations close in the standard way and they form the Weyl subgroup. Furthermore, the non-Abelian extension is realized nonlinearly, because the parameters Λ and $\tilde{u}$ explicitly depend on the field k^{ab}. To understand better the origin of such non-Abelian transformations, let us note that

$$\delta_G(\alpha) e^a{}_\alpha = (\partial_\alpha \alpha^\beta) e^a{}_\beta + \alpha^\beta \partial_\beta e^a{}_\alpha + \alpha^\beta \omega^{ab} e_{b\alpha} + \alpha^\beta T^a{}_{\alpha\beta}, \tag{2.25}$$

where $\alpha^\beta = \alpha^a e_a{}^\beta$. Therefore, the gauge transformations can be cast in the form

$$\delta_G(\alpha) e^a{}_\alpha = -\delta_T(\alpha^\beta) - \delta_L(\omega^{ab}{}_\beta \alpha^\beta) + \alpha^\beta T^a{}_{\alpha\beta}. \tag{2.26}$$

Shifting the parameters as $\xi^\beta \to \xi^\beta + \alpha^\beta$ and $\lambda^{ab} \to \lambda^{ab} + \omega^{ab}{}_\beta \alpha^\beta$ helps us identify the independent non-Abelian gauge transformations $\delta_G(\alpha) e^a{}_\alpha = \alpha^\beta T^a_{\alpha\beta}$. From (2.25) and the above relation we easily conclude that non-Abelian gauge transformations act on the boundary vielbein independently if and only if torsion is nonvanishing. In the case of vanishing torsion non-Abelian gauge transformations stop to be independent and they can be represented as composition of local translations and local Lorentz rotations with the suitable redefinition of parameters. Similar conclusion holds when one acts on the boundary spin connection because it is an independent field only if the torsion is nonvanishing.

Let us now, for completeness, inspect the action of the transformations (2.21) on the metric $g_{\alpha\beta} = e^a{}_\alpha e_{a\beta}$. We obtain

$$\begin{aligned}
\delta_0 g_{\alpha\beta} &= -\xi^\gamma{}_{,\alpha} g_{\gamma\beta} - \xi^\gamma{}_{,\beta} g_{\alpha\gamma} - \xi^\gamma \partial_\gamma g_{\alpha\beta} + 2u g_{\alpha\beta} \\
&\quad + e_{a\beta} \nabla_\alpha \alpha^a + e_{a\alpha} \nabla_\beta \alpha^a.
\end{aligned}$$

Similarly, as in the case of vielbein, the action of the non-Abelian gauge transformations on the metric reads

$$\delta_G(\alpha) g_{\alpha\beta} = -\delta_T(\alpha) g_{\alpha\beta} + 2\alpha^\gamma T_{(\alpha\beta)\gamma}. \tag{2.27}$$

Again, we conclude that in the case when torsion vanishes the action of non-Abelian gauge transformations on the metric reduces to local translations with the already mentioned redefinition of parameters [4]. The above transformation law of the metric is not usual in field theories, but is not surprising because we started with local AdS symmetry which mixes vielbein and spin connection.

III. NOETHER-WARD IDENTITIES

The AdS/CFT correspondence between the D-dimensional AdS space and d-dimensional CFT identifies the quantum effective action in CFT with the classical gravitational action in AdS space for given boundary conditions. Thus, let us assume that the *renormalized* effective action in a holographic theory, $I_{\rm ren}[e,\omega]$, has an extremum for Dirichlet boundary conditions on the independent fields, which are the vielbein, $e^a{}_\alpha$, and the spin connection, $\omega^{ab}{}_\alpha$, so that its variation takes the form

$$\delta I_{\rm ren}[e,\omega] = -\int d^d x \left(\tau^\alpha{}_a \delta_0 e^a{}_\alpha + \frac{1}{2}\sigma^\alpha{}_{ab}\delta_0\omega^{ab}{}_\alpha\right). \tag{3.1a}$$

The tensor densities,

$$\tau^\alpha{}_a = -\frac{\delta I_{\rm ren}}{\delta e^a{}_\alpha}, \qquad \sigma^\alpha{}_{ab} = -\frac{\delta I_{\rm ren}}{\delta \omega^{ab}{}_\alpha}, \tag{3.1b}$$

are the energy-momentum and spin currents of our dynamical system.

The holographic theory is invariant under d-dimensional diffeomorphisms with the parameter ξ^α and the local Lorentz transformations with the parameter λ^{ab}. The conservation law of the corresponding Noether current reads

$$\begin{aligned} e^a{}_\beta \nabla_\alpha \tau^\alpha{}_a + \tau^\alpha{}_a T^a{}_{\alpha\beta} + \frac{1}{2}\sigma^\alpha{}_{ab} R^{ab}{}_{\alpha\beta} \\ + \frac{1}{2}\omega^{ab}{}_\beta (\nabla_\alpha \sigma^\alpha{}_{ab} - 2\tau_{[ab]}) = 0, \end{aligned} \tag{3.2a}$$

$$\nabla_\alpha \sigma^\alpha{}_{ab} - 2\tau_{[ab]} = 0, \tag{3.2b}$$

which is also known as the generalized conservation laws of $\tau_\alpha{}^a$ and $\sigma^\alpha{}_{ab}$. Note that if the second Noether identity (3.2b) is fulfilled, the last term in (3.2a) can be omitted. We shall keep this term, however, because it modifies the conservation law in cases when there are quantum anomalies.

The invariance of $I_{\rm ren}$ under Weyl transformations leads to the additional conservation law,

$$\tau - \nabla_\beta \sigma^a{}_a{}^\beta = 0, \tag{3.2c}$$

where $\tau := \tau^a{}_a$ is the trace of the energy-momentum tensor.

Finally, invariance under the non-Abelian gauge transformations leads to

$$\nabla_\alpha \tau^\alpha{}_a - 2\sigma^b{}_{bc} k_a{}^c - 2\sigma_{bca} k^{cb} = 0. \tag{3.2d}$$

In Ref. [9], it was proposed that these residual gauge transformations contain the information about the chiral anomaly of the fermions in holographic CFT, encoded in the completely antisymmetric part of the spin current.

Gravitational dynamics in the bulk is described by nonvanishing torsion, but it may happen that some solutions on the boundary are Riemannian. For such solutions, the boundary connection $\omega^{ab}{}_\alpha$ takes its Riemannian value $\tilde\omega^{ab}{}_\alpha = \tilde\omega^{ab}{}_\alpha(e)$ and can be expressed in terms of the vielbein $e^a{}_\alpha$ in the following way:

$$\begin{aligned} \tilde\omega_{ab\alpha} &= \frac{1}{2}(c_{abc} - c_{cab} + c_{bca}) e^c{}_\alpha, \\ c_{a\alpha\beta} &:= \partial_\alpha e_{a\beta} - \partial_\beta e_{a\alpha}. \end{aligned} \tag{3.3}$$

Although boundary connection is no more independent dynamical variable, the Noether-Ward identities keep the form (3.2), but now $\omega_{ab\alpha}$ takes on the Riemannian value $\tilde\omega_{ab\alpha}$.

From the Riemannian renormalized action $\tilde I_{\rm ren} = I_{\rm ren}[e^a{}_\alpha, \tilde\omega_\alpha]$, we get that the related spin current $\Sigma^\alpha := -\delta\tilde I_{\rm ren}/\delta\omega_\alpha$ vanishes, while the energy-momentum current $\Theta^\alpha{}_a := -\delta \tilde I_{\rm ren}/\delta e^a{}_\alpha$ acquires an additional contribution

$$\Theta^\alpha{}_a = \tilde\tau^\alpha{}_a - \frac{1}{2}\tilde\nabla_\beta(\tilde\sigma^{\beta\alpha}{}_a - \tilde\sigma_a{}^{\beta\alpha} + \tilde\sigma^\alpha{}_a{}^\beta), \tag{3.4}$$

where $\tilde X$ denotes the Riemannian limit of a tensor X. The Noether identities for the action $\tilde I_{\rm ren}$ are found to be

$$e^a{}_\beta \tilde\nabla_\alpha \Theta^\alpha{}_a - \tilde\omega^{ab}{}_\beta \Theta_{[ab]} = 0, \tag{3.5a}$$

$$\Theta_{ab} = \Theta_{ba}, \tag{3.5b}$$

$$\Theta = 0. \tag{3.5c}$$

Let us remind that, as we concluded at the end of the previous section, the non-Abelian gauge transformations are not independent for Riemannian solutions, thus in this case there are only three independent Noether identities (3.5).

When the Lorentz invariance is fulfilled, (3.5a) reduces to the usual form $D_\alpha(e^{-1}\Theta^\alpha{}_\beta) = 0$, where D_α is the

Riemannian covariant derivative. The relations (3.5b) and (3.5c) are the standard Riemannian conditions for the Lorentz and Weyl invariance, respectively.

After using the condition of vanishing torsion, $T_{abc} = 0$, the identity $[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta] f_a = \tilde{R}_{ab\alpha\beta} f^b$ and the Bianchi identity, $\tilde{R}_{abcd} + \tilde{R}_{acdb} + \tilde{R}_{adbc} = 0$, enable us to write the expressions (3.5) as

$$e^a{}_\beta \tilde{\nabla}_\alpha \tilde{\tau}^\alpha{}_a + \frac{1}{2}\tilde{\sigma}^\alpha{}_{ab}\tilde{R}^{ab}{}_{\alpha\beta} + \frac{1}{2}\tilde{\omega}^{ab}{}_\beta(\nabla_\alpha \tilde{\sigma}^\alpha{}_{ab} - 2\tilde{\tau}_{[ab]}) = 0, \tag{3.6a}$$

$$\tilde{\nabla}_\alpha \tilde{\sigma}^\alpha{}_{ab} - 2\tilde{\tau}_{[ab]} = 0, \tag{3.6b}$$

$$\tilde{\tau} - \tilde{\nabla}_\beta \tilde{\sigma}^a{}_a{}^\beta = 0. \tag{3.6c}$$

Hence, the Riemannian identities (3.5a), (3.5b) and (3.5c) coincide with those obtained from (3.2a), (3.2b) and (3.2c) in the limit $T_{abc} \to 0$, as expected. Therefore, taking torsionless limit and calculating Noether-Ward identities gives an equivalent result as first calculating the Ward identities and taking torsion zero [22]. This is important when we do not know whether the torsion vanishes. Therefore, one may safely work in first order formalism assuming the boundary conditions and gauge fixing presented previously.

IV. LOVELOCK-CHERN-SIMONS GRAVITY

A. Action and equations of motion

The Lovelock-Lanczos gravity [15] in first order formulation is described by the action

$$I_{\mathrm{L}} = \sum_{p=0}^{[D/2]} \alpha_p L_p, \tag{4.1a}$$

where α_p are arbitrary coupling constants and L_p is dimensionally continued Euler density in D dimensions,

$$L_p = \varepsilon_{i_1 i_2 \ldots i_D} R^{i_1 i_2} \ldots R^{i_{2p-1} i_{2p}} e^{i_{2p+1}} \ldots e^{i_D}. \tag{4.1b}$$

Here p is the power of the curvature tensor in the polynomial L_p. We omit writing the wedge product for the sake of simplicity.

Lovelock-Lanczos gravity possesses numerous black hole solutions with Riemannean geometry [23–25], although some choices of the coupling constants $\{\alpha_p\}$ exhibit a causality problem in the dual CFT [26], or have instable geometries [27,28]. Generic Lovelock gravity without torsion possesses the same number of degrees of freedom as general relativity [29]. With torsion, or when the parameters take the critical values, the dynamical content of Lovelock-Lanczos gravity might change. Solutions in these cases are known as well, for example the ones with Riemann-Cartan geometry in five-dimensional gravity [30,31] and supergravity [32].

In odd-dimensional case $D = 2n+1$, the special choice of coefficients $\alpha_p = \frac{\kappa}{2n+1-2p}\binom{n}{p}$ defines theory with the *unique* (degenerate) AdS vacuum, known as LCS AdS gravity. Alternatively, LCS action can be constructed as a Chern density by taking the topological invariant, Chern form $dL_{\mathrm{CS}} = \varepsilon_{i_1 j_1 \ldots i_n j_n} F^{i_1 j_1} \ldots F^{i_n j_n}$, and writing L_{CS} by using holonomy operator [14,33]. Then, an equivalent form of LCS action is given by

$$I_{\mathrm{LCS}} = \kappa \int_{\mathcal{M}} \int_0^1 dt \varepsilon_{A_1 B_1 A_2 B_2 \ldots A_n B_n C} \times \prod_{k=1}^{n} (\hat{R}^{A_k B_k} + t^2 \hat{e}^{A_k} \hat{e}^{B_k}) \hat{e}^C. \tag{4.2}$$

Dropping the indices for simplicity, the above expression reads

$$I_{\mathrm{LCS}} = \kappa \int_{\mathcal{M}} \int_0^1 dt \varepsilon (\hat{R} + t^2 \hat{e}^2)^n \hat{e} = \kappa \int_{\mathcal{M}} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2k+1} \varepsilon \hat{R}^{n-k} \hat{e}^{2k+1}, \tag{4.3}$$

where we used the binomial expansion to perform an integration over t.

Equations of motion are obtained from the variation of the action (4.3) with respect to fundamental variables $\hat{e}^A$ and $\hat{\omega}^{AB}$. Variation with respect to $\hat{e}$ yields

$$C_A \coloneqq \varepsilon_{AA_1 B_1 \ldots A_n B_n} \prod_{k=1}^{n} F^{A_k B_k} = 0, \tag{4.4}$$

which can be split into 1 and a components,

$$C \coloneqq \varepsilon_{1 a_1 b_1 \ldots a_n b_n} \prod_{k=1}^{n} F^{a_k b_k} = 0, \tag{4.5a}$$

$$C_a \coloneqq \varepsilon_{a 1 b a_2 b_2 \ldots a_n b_n} F^{1b} \prod_{k=2}^{n} F^{a_k b_k} = 0. \tag{4.5b}$$

Variation with respect to ω yields

$$C_{AB} \coloneqq \varepsilon_{ABA_1 B_1 \ldots A_{n-1} B_{n-1} C} \prod_{k=1}^{n-1} F^{A_k B_k} \hat{T}^C = 0, \tag{4.6}$$

and can be split into $[1a]$ and $[ab]$ components,

$$\bar{C}_a \coloneqq \varepsilon_{1 a a_1 b_1 \ldots a_{n-1} b_{n-1} c} \prod_{k=1}^{n-1} F^{a_k b_k} \hat{T}^c = 0, \tag{4.7a}$$

$$C_{ab} \coloneqq \varepsilon_{1 a b a_1 b_1 \ldots a_{n-1} b_{n-1}} \times \prod_{k=1}^{n-2} F^{a_k b_k} (F^{a_{n-1} b_{n-1}} \hat{T}^1 + (n-1) F^{1 a_{n-1}} \hat{T}^{b_{n-1}}). \tag{4.7b}$$

Let us note that $\hat{T}^a = 0$ is a particular solution of the equations (4.7) belonging to the Riemannian subclass of all solutions of the theory. Also, the global AdS space ($F^{ab} = 0$) is a particular solution of all equations of motion.

B. 1-point functions

In this section we calculate the renormalized gravitational LCS action in the classical approximation. Then we use the AdS/CFT correspondence to promote it to the quantum effective action in a holographic CFT, and compute the holographic 1-point functions.

The variation of the LCS action reads

$$\delta I_{\mathrm{LCS}} = n\kappa \int_{\partial\mathcal{M}} \int_0^1 dt \varepsilon_{ABCA_1B_1\ldots A_{n-1}B_{n-1}} \delta\hat{\omega}^{AB}\hat{e}^C \times \prod_{k=1}^{n-1} (\hat{R}^{A_kB_k} + t^2 \hat{e}^{A_k}\hat{e}^{B_k}). \tag{4.8}$$

To perform a near-boundary expansion of the fields, let us first rewrite the following quantity in terms of the AdS tensor:

$$\hat{R}^{A_kB_K} + t^2\hat{e}^{A_k}\hat{e}^{B_k} = F^{A_kB_k} + (t^2-1)\hat{e}^{A_k}\hat{e}^{B_k}.$$

The first term in the above expression is *independent* of ρ since on the boundary $\hat{e}^1 = 0$, and therefore the particular components expand as

$$\hat{R}^{a_kb_k} + \hat{e}^{a_k}\hat{e}^{a_k} = F^{a_kb_k},$$
$$\hat{R}^{a_k1} + \hat{e}^{a_k}\hat{e}^1 = \frac{1}{\sqrt{\rho}}(T^{a_k} - \rho\nabla k^{a_k}). \tag{4.9}$$

Plugging these expansions in the variation of the action, we find

$$\delta I_{\mathrm{LCS}} = n\kappa \int_{\partial\mathcal{M}} \varepsilon\delta\hat{\omega} \sum_{k=0}^{n-1} \begin{matrix} n-1 \\ k \end{matrix} \frac{(-1)^k(2k)!!}{(2k+1)!!} F^{n-k-1}\hat{e}^{2k+1}, \tag{4.10}$$

where we used the beta function to solve the integral $\int_0^1 dt(t^2-1)^k = \frac{(-1)^k(2k)!!}{(2k+1)!!}$.

Variation (4.10) is divergent on the boundary, that is, in the limit $\rho \to 0$ and extraction of physical quantities requires its renormalization, or removal of divergences. For related work on Riemannian Lovelock gravity, see Ref. [5].

The procedure for obtaining finite results consists in introducing a regulating surface at $\rho = \epsilon$ and adding the counterterms which cancel all divergent contributions as ϵ tends to zero [8,34]. Equivalently said, the divergent terms in a variation of an action have to be represented as total variations of local terms integrated over boundary. In general, the computation of the total variation can be substantially simplified after noting that the conditions for the application of the theorem [19] are fulfilled in our case. For an alternative proof of the theorem [19], see Appendix C. The theorem [19] states that the terms which are asymptotically divergent or zero (when $\rho \to 0$) can always be represented as total variations of local boundary functionals. Therefore, we can discard all ρ^α ($\alpha \neq 0$) terms in the expression (4.10) and keep only the ρ^0-terms. For the form of the ρ^α-terms ($\alpha \neq 0$), see Appendix B. Note that the counterterms can contain arbitrary local finite part which is nonphysical and depends on a renormalization scheme. The divergent counterterms are local and there is finite number of them. They also depend on only one coupling constant κ. Counterterms in Riemannian gemetry were calculated in Ref. [35].

Keeping only the finite terms, we obtain the variation of the regularized action $I_{\mathrm{ren}} = I_{\mathrm{LCS}} + I_{\mathrm{ct}}$ in the form

$$\delta I_{\mathrm{ren}} = -2n\kappa\varepsilon\Bigg[\delta\omega T \sum_{l=0}^{n-2} \binom{n-2}{l} \frac{(-1)^l 2^{2l+1}(n-1)}{l+1} (R+4ek)^{n-2-l} e^l k^{l+1} - \delta e \sum_{l=0}^{n-1} \binom{n-1}{l} \frac{(-1)^l 2^{2l+1}}{l+1} (R+4ek)^{n-1-l} e^l k^{l+1}\Bigg], \tag{4.11}$$

where $T = \nabla e$ is the boundary torsion tensor. Comparing to (3.1), the spin and energy-momentum currents are given by, respectively,

$$\sigma_{ab} = -n\kappa\varepsilon_{1ab} T \sum_{l=1}^{n-1} \binom{n-1}{l} 4^l R^{n-1-l} e^{l-1} k^l, \tag{4.12}$$

$$\tau_a = \kappa\varepsilon_{1a} \sum_{l=1}^{n} \binom{n}{l} 4^l R^{n-l} e^{l-1} k^l, \tag{4.13}$$

and they correspond to the vacuum expectation values of the quantum CFT operators, the spin current $\mathcal{S}_{ab}$ and the energy-momentum of the conformal matter $\mathcal{T}_a$,

$$\sigma_{ab} = \langle \mathcal{S}_{ab} \rangle_{\mathrm{CFT}}, \qquad \tau_a = \langle \mathcal{T}_a \rangle_{\mathrm{CFT}}. \tag{4.14}$$

Using these representations of the 1-point functions of the CFT operators, we can study their quantum conservation laws, that is, the Noether-Ward identities.

C. Anomalies

The equations (3.2) describe classical conservation laws in a holographic theory invariant under diffeomorphisms, conformal transformations and non-Abelian gauge transformations. Since now we know the form of the corresponding quantum currents, we can also check the quantum conservation laws. If the law is not satisfied, then the quantum theory possesses a quantum anomaly.

In this section we explore the Ward identities and check for the existence of quantum anomalies: Lorentz anomaly A_{ab}, diffeomorphism anomaly $\bar{A}_a$, conformal anomaly A and gauge anomaly A_a. It is well known that there are two types of non-Abelian anomalies, covariant and consistent. All the anomalies we compute here are covariant, i.e., they transform covariantly under gauge symmetries.

1. Lorentz Ward identity

The conservation law for Lorentz symmetry is given by Eq. (3.2b), so we have to calculate the quantity

$$A_{ab} = \nabla\sigma_{ab} - 2e_{[a}\tau_{b]}. \tag{4.15}$$

Using the expressions (4.12) and (4.13) for the quantum currents, we find

$$A_{ab} = -4n\kappa\varepsilon_{ab}\Bigg[2(n-1)T\nabla k\sum_{l=0}^{n-2}\sum_{m=0}^{n-2-l}\binom{n-2}{l}\binom{n-2-l}{m}4^m(l+m+1)R^{n-2-l-m}e^{l+m}k^{l+m} + e_c k^c\sum_{l=0}^{n-1}\sum_{m=0}^{n-1-l}\binom{n-1}{l}\binom{n-1-l}{m}\frac{(-1)^l 2^{2l+2m+1}(l+m+1)}{l+1}R^{n-1-l-m}e^{l+m}k^{l+m}\Bigg].$$

It turns out that A_{ab} can be completely expressed in terms of the field equations, that means that it vanishes,

$$A_{ab} = -4n\kappa C_{ab} = 0. \tag{4.16}$$

Therefore, there is no Lorentz anomaly in the holographic theory because the Lorentz symmetry is conserved also quantically. This is an expected result, since the Lorentz symmetry is usually broken in the actions that are not parity invariant.

2. Ward identity for diffeomorphisms.

The conservation law for local translations has the form (3.2a),

$$\bar{A}_a = \nabla\tau_a - \left(I_a T^b\tau_b + \frac{1}{2}I_a R^{bc}\sigma_{bc}\right), \tag{4.17}$$

where I_a is the contraction operator with the spacetime index projected to the tangent manifold using the inverse vielbein $e_a{}^\alpha$. Plugging in the quantum currents (4.12) and (4.13), one can show that the conservation law is satisfied,

$$\bar{A}_a = 4n\kappa(k^b{}_a C_b - \bar{C}_a) = 0. \tag{4.18}$$

Therefore, there is no gravitational anomaly, as expected.

3. Conformal Ward identity

The conservation law for local Weyl transformations can be read off from Eq. (3.2c) as

$$A = e^a\tau_a + \nabla(e^a I^b\sigma_{ab}), \tag{4.19}$$

where $e^a\tau_a$ is the trace of energy-momentum tensor, so A is also called the trace anomaly. Using the field equations and discarding the total divergence, one can show that the trace anomaly has the form

$$e^a\tau_a = \kappa\varepsilon_{a_1b_1a_2b_2\ldots a_nb_n}R^{a_1b_1}R^{a_2b_2}\ldots R^{a_nb_n} = \kappa\mathcal{E}_n(R). \tag{4.20}$$

Thus, the holographic anomaly is nonvanishing and, up to a divergence, proportional to the Euler density $\mathcal{E}_n(R) = \varepsilon R^n$, as expected in a CFT dual to a higher-dimensional AdS gravity [36]. Since the Weyl anomaly is topological invariant, it is of the type A, according to the general classification of conformal anomalies given in Ref. [37].

4. Ward identity for gauge symmetry

The conservation law for non-Abelian gauge transformations is given by Eq. (3.2d) as

$$A_a = \nabla\tau_a - 2(e^b\sigma_{bc}k_a{}^c + k^b\sigma_{ba}). \tag{4.21}$$

Using (4.12) and (4.13), as well as the equations of motion, we can express it as

$$A_a = -2n\kappa\varepsilon_{1b}I_aT^b\sum_{l=0}^{n-1}\binom{n-1}{l}\frac{(-1)^l2^{2l+1}}{l+1}(R+4ek)^{n-1-l}e^lk^{l+1}$$
$$-4n\kappa\varepsilon_{1bc}T\left(\frac{1}{2}I_aR^{bc}-2e^bk_a{}^c\right)\sum_{l=0}^{n-2}\binom{n-2}{l}\frac{(-1)^l2^{2l+1}(n-1)}{l+1}(R+4ek)^{n-2-l}e^lk^{l+1}$$
$$+8n\kappa\varepsilon_{1a}T\sum_{l=0}^{n-2}\binom{n-2}{l}\frac{(-1)^l2^{2l+1}(n-1)}{l+1}(R+4ek)^{n-2-l}e^lk^{l+2}\neq 0. \quad (4.22)$$

The above holographic anomaly is in general nonvanishing, but it cancels out when *the torsion is equal to zero,* as expected. Indeed, when $T^a = 0$, the non-Abelian gauge symmetry is not independent, but it can be expressed in terms of the diffeomorphisms, which are conserved at the quantum level. Another derivation of this result is possible by noting that in this particular case the spin tensor vanishes and both Eqs. (3.2a) and (3.2d) reduce to

$$\tilde{\nabla}_\alpha\tilde{\tau}^\alpha{}_\alpha = 0. \quad (4.23)$$

Again non-Abelian gauge anomaly vanishes since $A_a = 0$.

V. CONCLUDING REMARKS

We analyzed a holographic dual of Lovelock Chern-Simons AdS gravity in an arbitrary odd dimension and calculated corresponding holographic currents and anomalies in the quantum CFT. First part of the work is devoted to the kinematics of gravitational theory with AdS gauge symmetry. After motivating a gauge fixing suitable for a holographc analysis, we calculated residual (asymptotic) symmetries. Then we focused to Chern-Simons AdS gravity. We concluded that the largest asymptotic symmetry consists of local translations and rotations (local Poincaré group), local Weyl rescalings and, in the presence of torsion on the boundary, of non-Abelian gauge symmetry. If the torsion on the boundary is zero, then a non-Abelian symmetry is not independent any longer and reduces to local Poincaré transformations.

We found holographic representations of the energy-momentum and spin tensors in a dual theory, which we identified with the corresponding 1-point functions in CFT, in the presence of sources. We also computed their conservation laws and obtained that some of quantum symmetries are broken, leading to quantum anomalies. Explicitly, we obtained that local translations and rotations are symmetries of the quantum theory, while Weyl rescalings and non-Abelian gauge symmetry are anomalous. Similarly as in five dimensions [9], the trace anomaly is proportional to the Euler density and is therefore of the type A.

Because of nonlinearity of the model and working in higher-dimensional Riemann-Cartan space, the regularization of the action was quite involved. However, with the help of a general renormalization theorem shown in Appendix C, it was possible to circumvent an explicit construction of divergent counterterms and extract directly its finite part. An alternative proof of the theorem is given in Ref. [19].

One of the open questions left for future work is an application on non-Abelian gauge transformations to the calculation of chiral anomaly. Namely, in Ref. [9] it was suggested that the chiral anomaly is related to the completely antisymmetric component of the torsion tensor. Another question would be to find a different gauge fixing of either transversal diffeomorphisms or local AdS symmetry, in order to obtain an *infinite* radial expansion of the fields, and possibly the type B anomaly. This would describe an inequivalent holographic theory. Finally, we are also interested in introducing a gauge fixing which breaks relativistic covariance in an arbitrary Poincaré gauge theory, and is suitable for the formulation of Lifshitz holography. These last topics is the work in progress.

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APPENDIX A: ADS ALGEBRA

The algebra of generators $J_{\bar{A}\bar{B}} = -J_{\bar{B}\bar{A}}$ $(\bar{A},\bar{B} = 0, 1,\ldots,D)$ of AdS group $SO(D-1,2)$ if given by

$$[J_{\bar{A}\bar{B}}, J_{\bar{C}\bar{E}}] = \eta_{\bar{B}\bar{C}}J_{\bar{A}\bar{E}} + \eta_{\bar{B}\bar{C}}J_{\bar{A}\bar{E}} - \eta_{\bar{A}\bar{C}}J_{\bar{B}\bar{E}} - \eta_{\bar{B}\bar{E}}J_{\bar{A}\bar{C}}, \quad (A1)$$

where $\eta_{\bar{A}\bar{B}} = (-1, \underbrace{1,\ldots,1}_{D-1}, -1)$. Introducing the splitting of indices $\bar{A} = (A, D)$ and with

$$P_A = J_{AD},$$
$$J_{AB} = -J_{BA}, \qquad A, B = 0, 1, \ldots, D-1, \quad (A2)$$

the algebra (A1) (after taking into account that $\eta_{DD} = -1$) takes the familiar form

$$\begin{aligned}
[P_A, P_B] &= J_{AB}, \\
[P_A, J_{BC}] &= \eta_{AB} P_C - \eta_{AC} P_B, \\
[J_{AB}, J_{CE}] &= \eta_{BC} J_{AE} + \eta_{AE} J_{BC} - \eta_{AC} J_{BE} - \eta_{BE} J_{AC}.
\end{aligned} \tag{A3}$$

APPENDIX B: VARIATION OF LCS ACTION

In this appendix we present the nonvanishing parts of the variation of LCS action given by Eq. (4.10),

$$\delta I_{\mathrm{LCS}} = \sum_{j=0}^{n} \frac{1}{\rho^j} \delta I_j. \tag{B1}$$

We find the following terms, with $1 \leq j \leq (n-2)$:

$$\begin{aligned}
\delta I_n &= \varepsilon_{a1a_1b_1\ldots a_{n-1}b_{n-1}c} \delta e^a e^c K_{-(n-1)}, \\
\delta I_{n-1} &= \varepsilon_{aba_1b_1\ldots d1c} \delta\omega^{ab} e^c \nabla e^d J_{-(n-2)} \\
&\quad + \varepsilon_{a1a_1b_1\ldots a_{n-1}b_{n-1}c} [\delta e^a e^c K_{-(n-2)} + (\delta e^a k^c - \delta k^a e^c) K_{-(n-1)}], \\
\delta I_j &= \varepsilon_{1abcda_1b_1\ldots} \delta\omega^{ab} [e^c \nabla e^d J_{-(j-1)} - (e^c \nabla k^d - k^c \nabla e^d) J_{-j} - k^c \nabla k^d J_{-(j+1)}] \\
&\quad - \varepsilon_{1aca_1b_1\ldots a_{n-1}b_{n-1}} [\delta e^a e^c K_{-(j-1)} + (\delta e^a k^c - \delta k^a e^c) K_{-j} - \delta k^a k^c K_{-(j+1)}], \\
\delta I_0 &= \varepsilon_{1abcda_1b_1\ldots} \delta\omega^{ab} [e^c \nabla e^d J_1 - (e^c \nabla k^d - k^c \nabla e^d) J_0 - k^c \nabla k^d J_{-1}] \\
&\quad - \varepsilon_{1aca_1b_1\ldots a_{n-1}b_{n-1}} [\delta e^a e^c K_1 + (\delta e^a k^c - \delta k^a e^c) K_0 - \delta k^a k^c K_{-1}],
\end{aligned} \tag{B2}$$

and

$$\begin{aligned}
K_\alpha &= \sum_{l=0}^{n-1} \binom{n-1}{l} (R + 4ek)^{n-l-1} A_{l\alpha} e^{l-\alpha} k^{l+\alpha}, \\
J_\alpha &= (n-1) \sum_{l=0}^{n-2} \binom{n-2}{l} (R + 4ek)^{n-l-2} A_{l\alpha} e^{l-\alpha} k^{l+\alpha},
\end{aligned} \tag{B3}$$

where

$$A_{l\alpha} = \frac{(-1)^l 4^l l!^2}{(2l+1)(l-\alpha)!(l+\alpha)!}. \tag{B4}$$

APPENDIX C: ALTERNATIVE PROOF OF THE RENORMALIZATION THEOREM

In this appendix we show an alternative derivation of the results of Ref. [19].

Theorem 1 A surface counterterm can be added to an action of any classical field theory in the bulk to cancel all the terms which depend on the radial coordinate in an on-shell variation, if any of the following conditions are satisfied:

(i) The bulk has the topology $\mathbb{R} \times \partial M$;

(ii) The boundary has a finite number of disjoint pieces and near each one the bulk looks like $\mathbb{R} \times \partial M$.

Here, ∂M is any manifold without boundary with the coordinates x^α and the radial coordinate is labeled by ρ. If the fields have asymptotic expansion near the boundary of the form $\phi^i = \sum_n f_n^i(\rho) \phi_n^i(x^\alpha)$, where $f_n^i(\rho)$ are functions that depend only on ρ and $\phi_n^i(x^\alpha)$ are (ρ-independent) boundary fields, then the counterterm is a local functional of the boundary fields.

Let the action in $(D+1)$-dimensional bulk M be defined in language of differential forms as

$$S = \int_M L. \tag{C1}$$

A variation of the action (C1) takes the form

$$\delta S = \int_M \delta L = \int_M e.o.m. + \int_M d_{D+1} L_D^B \tag{C2}$$

where $e.o.m$ are the terms proportional to the equation of motion. Formula (C2) is also valid without integral and it will be used in that form later. By using the Stoke's theorem, we can write the last term in (C2) as

$$\int_M d_{D+1} L_D^B = \int_{\partial M} L_D, \tag{C3}$$

where the boundary of M is placed at fixed distance $\rho = \varepsilon$ near (but not equal) zero and $L_D \coloneqq L_D^B|_{\rho=\varepsilon}$. Let ∂M be a boundary at each ρ. The most general D-form L_D near the boundary is

$$L_D^B = L_D + d\rho \wedge V, \tag{C4}$$

where V is an arbitrary $(D-1)$-form. The exterior derivative in the bulk can be decomposed near the boundary as

$$d_{D+1} = \partial_\rho d\rho + d, \tag{C5}$$

where d is the exterior derivative at the boundary and d_ρ is the derivative along the direction ρ. From Eqs. (C2), (C4) and (C5), we get on-shell

$$\delta L = d\rho \wedge \partial_\rho L_D - d\rho \wedge dV. \tag{C6}$$

Equivalently, this can be rewritten as

$$\partial_\rho L_D = \delta U + dV \tag{C7}$$

where $\delta L = d\rho \wedge \delta U$. Hence, from (C7) it follows that

$$L_D = \delta A + dB + R(x^\alpha) \tag{C8}$$

where $A = \int d\rho U$, $B = \int d\rho V$ and $R(x^\alpha)$ does not depend on ρ. This conclusion is valid under the assumption that the right side of Eq. (C7) is integrable and that the derivative and integral mutually commute. Therefore, L_D is a sum of a total variation, exact form and a function which does not depend on ρ.

Consequently, we get

$$\int_{\partial M} L_D = \delta \int_{\partial M} A + \int_{\partial M} R, \tag{C9}$$

where we used the fact that an integral of the exact form dB vanishes due to the Stoke's theorem and because the boundary of a boundary is an empty set. After substituting (C9) into (C2) we obtain on-shell

$$\delta(S - S_{\rm ct}) = \int_{\partial M} R, \tag{C10}$$

where $S_{\rm ct} = \int_{\partial M} A$. Since R is ρ independent, the expression (C10) is well defined at the boundary $\rho = 0$. Thus, all ρ-dependent terms can be eliminated by adding a suitable counterterm. An important observation is that this counterterm is *unique*. Given an asymptotic solution of the field equations, a near-boundary behavior is fixed. Furthermore, the counterterm is obtained from the Lagrangian, thus it depends on the same parameters. In other words, we do not include new parameters in the theory. If the starting Lagrangian has a finite number of parameters, so it does the renormalized Lagrangian.

As the counterterm is obtained as a primitive function of local functions, it is not necessarily local. The near-boundary expansion method is, however, able to determine only local counterterms.

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Generalized *pp* waves in Poincaré gauge theory

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Starting from the generalized pp waves that are exact vacuum solutions of general relativity with a cosmological constant, we construct a new family of exact vacuum solutions of Poincaré gauge theory, the generalized pp waves with torsion. The ansatz for torsion is chosen in accordance with the wave nature of the solutions. For a subfamily of these solutions, the metric is dynamically determined by the torsion.

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I. INTRODUCTION

The principle of gauge symmetry was born in the work of Weyl [1], where he obtained the electromagnetic field by assuming local $U(1)$ invariance of the Dirac Lagrangian. Three decades later, the Poincaré gauge theory (PGT) was formulated by Kibble and Sciama [2]; it is nowadays a well-established gauge approach to gravity, representing a natural extension of general relativity (GR) to the gauge theory of the Poincaré group [3,4]. Basic dynamical variables in PGT are the tetrad field b^i and the Lorentz connection $\omega^{ij} = -\omega^{ji}$ (1-forms), and the associated field strengths are the torsion $T^i = db^i + \omega^i{}_k \wedge b^k$ and the curvature $R^{ij} = d\omega^{ij} + \omega^i{}_k \wedge \omega^{kj}$ (2-forms). By construction, PGT is characterized by a Riemann-Cartan geometry of spacetime, and its physical content is directly related to the existence of mass and spin as basic characteristics of matter at the microscopic level. An up-to-date status of PGT can be found in a recent reader with reprints and comments [5].

General PGT Lagrangian L_G is at most quadratic in the field strengths. The number of independent (parity invariant) terms in L_G is nine, which makes the corresponding dynamical structure rather complicated. As is well known from the studies of GR, exact solutions have an essential role in revealing and understanding basic features of the gravitational dynamics [6–9]. This is also true for PGT, where exact solutions allow us, among other things, to study the interplay between dynamical and geometric aspects of torsion [5].

In the context of GR, one of the best known families of exact solutions is the family of *pp waves*: it describes plane-fronted waves with parallel rays propagating on the Minkowski background M_4; see, for instance, Ehlers and Kundt [6]. There is an important generalization of this family, consisting of the exact vacuum solutions of GR with a cosmological constant (GR_Λ) such that for $\Lambda \to 0$, they reduce to the pp waves in M_4. We will refer to this family as the generalized pp waves, or just pp_Λ waves for short. In contrast to the pp waves in M_4, the wave surfaces of the pp_Λ waves have constant curvature proportional to Λ. The family of the pp_Λ waves belongs to a more general family, known as the Kundt class of type N, labeled $\mathrm{KN}(\Lambda)$. Details on the $\mathrm{KN}(\Lambda)$ spacetimes can be found in the monograph by Griffiths and Podolský [9]; see also Refs. [10–12]. In this paper, we start from the Riemannian pp_Λ waves in GR_Λ and construct a new family of the pp_Λ waves with torsion, representing a new class of exact vacuum solutions of PGT. The torsion is introduced relying on the approach used in our previous paper [13]. The present work is motivated by earlier studies of the exact wave solutions in PGT [14], and is regarded as a complement to them.

The paper is organized as follows. In Sec. II, we give a short account of the Riemannian pp_Λ waves, including the relevant geometric and dynamical aspects, as a basis for their extension to pp_Λ waves with torsion. In Sec. III, we first introduce an ansatz for the new, Riemann-Cartan (RC) connection, the structure of which complies with the wave nature of a RC spacetime. The ansatz is parametrized by a specific 1-form K living on the wave surface, and the related torsion has only one, tensorial irreducible component. Then, we use the PGT field equations to show that the dynamical content of K is described by two torsion modes with the spin-parity values $J^P = 2^+$ and 2^-. In Sec. IV, we find solutions for both the metric function H and the torsion function K, in the spin-2^+ sector and for $\lambda > 0, < 0$ and $= 0$. It is shown that K has a decisive influence on the solution for H, and consequently, on the resulting metric. In Sec. V, we shortly discuss solutions in the spin-2^- sector, which are found to be much less interesting. Section VI concludes the exposition with a few remarks on some issues not covered in the main text, and the Appendices are devoted to certain technical details.

Our conventions are as follows. The latin indices $(i, j, \ldots)$ refer to the local Lorentz (co)frame and run over (0, 1, 2, 3), b^i is the tetrad (1-form), and h_i is the dual basis (frame), such that $h_i b^k = \delta^i_k$. The volume 4-form is $\hat\epsilon = b^0 \wedge b^1 \wedge b^2 \wedge b^3$, the Hodge dual of a form α is ${}^\star\alpha$, with ${}^\star 1 = \hat\epsilon$, and the totally antisymmetric tensor is

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defined by $^\star(b_i \wedge b_j \wedge b_k \wedge b_m) = \varepsilon_{ijkm}$ and normalized to $\varepsilon_{0123} = +1$. The exterior product of forms is implicit, except in Appendix B.

II. RIEMANNIAN pp_Λ WAVES

In this section, we give an overview of Riemannian pp_Λ waves using the tetrad formalism [15], necessary for the transition to PGT.

A. Geometry

The family of pp_Λ waves is a specific subclass of the Kundt spacetimes KN(Λ), labeled by KN(λ)[$\alpha=1,\beta=0$]; for the full classification of the KN(Λ) spacetimes, see Refs. [9,10]. In suitable local coordinates $x^\mu = (u, v, y, z)$ (see Appendix A), the metric of the pp_Λ waves can be written as

$$ds^2 = 2\left(\frac{q}{p}\right)^2 du(Sdu + dv) - \frac{1}{p^2}(dy^2 + dz^2), \qquad (2.1a)$$

where

$$p = 1 + \frac{\lambda}{4}(y^2 + z^2), \qquad q = 1 - \frac{\lambda}{4}(y^2 + z^2),$$
$$S = -\frac{\lambda}{2}v^2 + \frac{p}{2q}H(u, y, z), \qquad (2.1b)$$

with λ being a suitably normalized cosmological constant, and the unknown metric function H is to be determined by the field equations. The coordinate v is an affine parameter along the null geodesics $x^\mu = x^\mu(v)$, and u is retarded time such that $u = \text{const}$ are the spacelike surfaces parametrized by $x^\alpha = (y, z)$. Since the null vector $\xi = \xi(u)\partial_v$ is orthogonal to these surfaces, they are regarded as wave surfaces, and ξ is the null direction (ray) of the wave propagation. The vector ξ is not covariantly constant, and consequently, the wave rays are not parallel and the wave surfaces are not flat. For $\lambda \to 0$, the metric (2.1) reduces to the metric of pp waves on the M_4 background, which explains the term generalized pp waves, or pp_Λ waves.

Next, we choose the tetrad field (coframe) in the form

$$b^0 := du, \qquad b^1 := \left(\frac{q}{p}\right)^2 (Sdu + dv),$$
$$b^2 := \frac{1}{p}dy, \qquad b^3 := \frac{1}{p}dz, \qquad (2.2a)$$

so that $ds^2 = \eta_{ij} b^i \otimes b^j$, where η_{ij} is the half-null Minkowski metric:

$$\eta_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The corresponding dual frame h_i is given by

$$h_0 = \partial_u - S\partial_v, \qquad h_1 = \left(\frac{p}{q}\right)^2 \partial_v,$$
$$h_2 = p\partial_y, \qquad h_3 = p\partial_z. \qquad (2.2b)$$

For the coordinates $x^\alpha = (y, z)$ on the wave surface, we have

$$x^c = b^c{}_\alpha x^\alpha = \frac{1}{p}(y, z), \qquad \partial_c = h_c{}^\alpha \partial_\alpha = p(\partial_y, \partial_z),$$

where $c = 2, 3$.

Starting from the general formula for the Riemannian connection 1-form,

$$\omega^{ij} := -\frac{1}{2}\left[h^i \rfloor db^j - h^j \rfloor db^i - (h^i \rfloor h^j \rfloor db^k) b_k\right],$$

one can find its explicit form; for $i < j$, it reads

$$\omega^{01} = \lambda v b^0 - \frac{1}{q}(\lambda y b^2 + \lambda z b^3), \quad \omega^{02} = \frac{\lambda y}{q} b^0, \quad \omega^{03} = \frac{\lambda z}{q} b^0,$$
$$\omega^{12} = \frac{\lambda y}{q} b^1 - \frac{q^2}{p}\partial_y S b^0, \qquad \omega^{13} = \frac{\lambda z}{q} b^1 - \frac{q^2}{p}\partial_z S b^0,$$
$$\omega^{23} = \frac{1}{2}(\lambda z b^2 - \lambda y b^3). \qquad (2.3a)$$

Introducing the notation $i = (A, a)$, where $A = 0$, 1 and $a = (2, 3)$, one can rewrite ω^{ij} in a more compact form:

$$\omega^{01} = \lambda v b^1 - \frac{2}{qp}(b^c \partial_c p),$$
$$\omega^{Ac} = -\frac{2}{qp} b^A \partial^c p + k^A \frac{q^2}{p^2} b^0 \partial^c S,$$
$$\omega^{23} = -\frac{1}{p}(b^2 \partial^3 p - b^3 \partial^2 p), \qquad (2.3b)$$

where $k^i = (0, 1, 0, 0)$ is a null propagation vector, $k^2 = 0$.

The above connection defines the Riemannian curvature $R^{ij} = d\omega^{ij} + \omega^i{}_m \omega^{mj}$; for $i < j$, it is given by

$$R^{ij} = \begin{cases} -\lambda b^1 b^c + k^1 b^0 Q^c, & \text{for } (i, j) = (1, c) \\ -\lambda b^i b^j, & \text{otherwise,} \end{cases} \qquad (2.4a)$$

where Q^c is a 1-form introduced by Obukhov [15],

$$Q_c = -\underline{\nabla}\left[\left(\frac{q}{p}\right)^2 h_c \rfloor \underline{d}S\right] + \left(\frac{q}{p}\right)^3 h_c \rfloor \left[\underline{d}\left(\frac{p}{q}\right) \wedge \underline{d}S\right],$$

and $\underline{d} = dx^\alpha \partial_\alpha$ is the exterior derivative on the wave surface. In more details

$$Q^2 = \frac{q}{2p}[2qp\partial_{yy}S + (q-4)\lambda y\partial_y S - q\lambda z\partial_z S]b^2 + \frac{q}{2}[2q\partial_{yz}S - \lambda z\partial_y S - \lambda y\partial_z S]b^3,$$

$$Q^3 = \frac{q}{2p}[2qp\partial_{zz}S + (q-4)\lambda z\partial_z S - q\lambda y\partial_y S]b^3 + \frac{q}{2}[2q\partial_{yz}S - \lambda z\partial_y S - \lambda y\partial_z S]b^2.$$

As a consequence, R^{ij} can be represented more compactly as

$$R^{ij} = -\lambda b^i b^j + 2b^0 k^{[i} Q^{j]}. \tag{2.4b}$$

The Ricci 1-form $Ric^i \coloneqq h_m \rfloor Ric^{mi}$ is given by

$$Ric^i = -3\lambda b^i + b^0 k^i Q,$$
$$Q = h_c \rfloor Q^c = \frac{qp}{2}\left[\partial_{yy}H + \partial_{zz}H + \frac{2\lambda}{p^2}H\right], \tag{2.5}$$

and the scalar curvature $R \coloneqq h_i \rfloor Ric^i$ reads

$$R = -12\lambda. \tag{2.6}$$

B. Dynamics

1. pp_Λ waves in GR_Λ

Starting with the action $I_0 = -\int d^4x(a_0 R + 2\Lambda_0)$, one can derive the GR_Λ field equations in vacuum,

$$2a_0 G^n{}_i - 2\Lambda_0 \delta^n_i = 0, \tag{2.7a}$$

where $G^n{}_i$ is the Einstein tensor. The trace and the traceless piece of these equations read

$$\Lambda_0 = 3a_0\lambda, \qquad Ric^i - \frac{1}{4}Rb^i \equiv b^0 k^i Q = 0. \tag{2.7b}$$

As a consequence, the metric function H must obey

$$\partial_{yy}H + \partial_{zz}H + \frac{2\lambda}{p^2}H = 0. \tag{2.8}$$

There is a simple solution of these equations,

$$H_c = \frac{1}{p}(A(u)q + B_\alpha x^\alpha)f(u), \tag{2.9}$$

for which Q^a vanishes. This solution is trivial (or pure gauge), since the associated curvature takes the background form, $R^{ij} = -\lambda b^i b^j$; moreover, it is conformally flat, since its Weyl curvature vanishes. The nontrivial vacuum solutions are characterized by $Q = 0$, but $Q^c \neq 0$; their general form can be found in [10].

2. pp_Λ waves in PGT

To better understand the relation between GR_Λ and PGT, it is interesting to examine whether pp_Λ waves satisfying the GR_Λ field equations in vacuum are also a vacuum solution of PGT. It turns out that a more general version of the problem has been already solved by Obukhov [4]. Studying the PGT field equations for torsion-free configurations, he proved the following important theorem:
T1. In the absence of matter, any solution of GR_Λ is a torsion-free solution of PGT.
It is interesting to note that the inverse statement, that any torsion-free vacuum solution of PGT is also a vacuum solution of GR_Λ, is also true, except for three specific choices of the PGT coupling constants.

III. pp_Λ WAVES WITH TORSION

In this section, we extend the pp_Λ waves that are vacuum solutions of GR_Λ to a new family of the exact vacuum solutions of PGT, characterized by the existence of torsion.

A. Ansatz

The main step in constructing the pp_Λ waves with torsion is to find an ansatz for torsion that is compatible with the wave nature of the solutions. This is achieved by introducing torsion at the level of connection.

Looking at the Riemannian connection (2.3), one can notice that its radiation piece appears only in the ω^{1c} components:

$$(\omega^{1c})^R = \frac{q^2}{p^2}(h^{c\alpha}\partial_\alpha S)b^0.$$

This motivates us to construct a new connection by applying the rule

$$\partial_\alpha S \to \partial_\alpha S + K_\alpha, \qquad K_\alpha = K_\alpha(u, y, z), \tag{3.1a}$$

where K_α is the component of the 1-form $K = K_\alpha dx^\alpha$ on the wave surface. Thus, the new form of $(\omega^{ij})^R$ reads

$$(\omega^{ic})^R \coloneqq k^i \frac{q^2}{p^2} h^{c\alpha}(\partial_\alpha S + K_\alpha)b^0, \tag{3.1b}$$

whereas all the other nonradiation pieces retain their Riemannian form (2.3).

The geometric content of the new connection is found by calculating the torsion:

$$T^i = \nabla b^i + \omega^i{}_m b^m = k^i \frac{q^2}{p} b^0 (b^2 K_y + b^3 K_z)$$
$$= k^i \frac{q^2}{p^2} b^0 b^c K_c. \tag{3.2}$$

The only nonvanishing irreducible piece of T^i is ${}^{(1)}T^i$.

The new connection modifies also the curvature, so that its radiation piece becomes

$$(R^{1c})^R = k^1 b^0 \Omega^c, \qquad \Omega^c \coloneqq Q^c + \Theta^c, \tag{3.3a}$$

where the term Θ^c that represents the contribution of torsion is given by

$$\Theta^2 = \frac{q}{2p}[(2qp\partial_y K_y - pK_y\lambda y - qK_z\lambda z)b^2 + (-2qp\partial_z K_y + pK_y\lambda z - qK_z\lambda y)b^3],$$
$$\Theta^3 = \frac{q}{2p}[(2qp\partial_z K_z - pK_z\lambda z - qK_y\lambda y)b^3 + (-2qp\partial_y K_z + pK_z\lambda y - qK_y\lambda z)b^2].$$

The covariant form of the curvature reads

$$R^{ij} = -\lambda b^i b^j + 2b^0 k^{[i}\Omega^{j]}, \tag{3.3b}$$

and the Ricci curvature takes the form

$$Ric^i = -3\lambda b^i + b^0 k^i \Omega, \qquad \Omega \coloneqq h_c \rfloor \Omega^c = Q + \Theta. \tag{3.3c}$$

The torsion has no influence on the scalar curvature:

$$R = -12\lambda. \tag{3.3d}$$

Thus, our ansatz defines a RC geometry of spacetime.

B. PGT field equations

Having adopted the ansatz for torsion defined in Eq. (3.1), we now wish to find explicit form of the PGT field equations and use them to determine dynamical content of our ansatz.

As shown in Appendices B and C, the field equations depend on the structure of the irreducible components of the field strengths. For torsion, we already know that the only nonvanishing irreducible component is ${}^{(1)}T_i = T_i$, defined in Eq. (3.2). As for the curvature, we note that our ansatz yields $X = 0$ and $b^m Ric_m = 0$, where X is defined in (B2b). Then, the irreducible decomposition of the curvature implies (see Appendix B)

$${}^{(3)}R_{ij} = 0, \qquad {}^{(5)}R_{ij} = 0, \tag{3.4}$$

whereas the remaining pieces ${}^{(n)}R^{ij}$ are defined by their nonvanishing components as

$${}^{(2)}R^{1c} = \frac{1}{2}{}^\star(\Psi^1 b^c), \qquad {}^{(4)}R^{1c} = \frac{1}{2}(\Phi^1 b^c),$$
$${}^{(6)}R^{ij} = -\lambda b^i b^j, \qquad {}^{(1)}R^{1c} = b^0\left(\Omega^{(ce)} - \frac{1}{2}\eta^{ce}\Omega\right)b_e, \tag{3.5a}$$

where the 1-forms Φ^i and Ψ^i are given by

$$\Phi^i = k^i b^0 (Q + \Theta), \quad \Theta = qp\left[\partial_y\left(\frac{q}{p}K_y\right) + \partial_z\left(\frac{q}{p}K_z\right)\right],$$
$$\Psi^i = X^i = -k^i b^0 \Sigma, \quad \Sigma = qp\left[\partial_z\left(\frac{q}{p}K_y\right) - \partial_y\left(\frac{q}{p}K_z\right)\right]. \tag{3.5b}$$

Having found ${}^{(1)}T_i$ and ${}^{(n)}R_{ij}$, we apply the procedure described in Appendix C to obtain the following form of the two PGT field Eqs. (C3):

$$(1\text{ST}) \quad \Lambda_0 = 3a_0\lambda, \qquad a_1\Theta - A_0(Q + \Theta) = 0, \tag{3.6a}$$

$$(2\text{ND}) \quad -(b_2 + b_1)(\nabla\Psi^1)b^2 - (b_4 + b_1)(\nabla\Phi^1)b^3 - 2(a_0 - A_1)T^1 b^3 = 0,$$
$$-(b_2 + b_1)(\nabla\Psi^1)b^3 + (b_4 + b_1)(\nabla\Phi^1)b^2 + 2(a_0 - A_1)T^1 b^2 = 0, \tag{3.6b}$$

where $A_0 = a_0 + (b_4 + b_6)\lambda$ and $A_1 = a_1 - (b_6 - b_1)\lambda$ [16].

Leaving (1ST) as is, (2ND) can be given a more clear structure as follows:

(i) use (1ST) to express $\Phi^1 = b^0(Q + \Theta)$ in the form $\Phi^1 = (a_1/A_0)b^0\Theta$;

(ii) multiply (2ND) by A_0/q.

As a result, the previous two components of (2ND) transform into

$$A_0(b_2 + b_1)\partial_z(p\Sigma/q) + a_1(b_4 + b_1)\partial_y(p\Theta/q) + 2A_0(A_1 - a_0)(q/p)K_y = 0, \tag{3.7a}$$

$$-A_0(b_2 + b_1)\partial_y(p\Sigma/q) + a_1(b_4 + b_1)\partial_z(p\Theta/q) + 2A_0(A_1 - a_0)(q/p)K_z = 0. \tag{3.7b}$$

Then, calculating $\partial_y(3.7\text{a}) + \partial_z(3.7\text{b})$ and $\partial_z(3.7\text{a}) - \partial_y(3.7\text{b})$ yields the final form of (2ND):

$$(\partial_{yy} + \partial_{zz})(p\Theta/q) - m^2_{2^+}\frac{1}{p^2}(p\Theta/q) = 0,$$
$$m^2_{2^+} \coloneqq \frac{2A_0(a_0 - A_1)}{a_1(b_1 + b_4)}, \tag{3.8a}$$

$$(\partial_{yy} + \partial_{zz})(p\Sigma/q) - m^2_{2^-}\frac{1}{p^2}(p\Sigma/q) = 0,$$
$$m^2_{2^-} \coloneqq \frac{2(a_0 - A_1)}{b_1 + b_2}. \tag{3.8b}$$

The parameters $m^2_{2^\pm}$ have a simple physical interpretation. In the limit $\lambda \to 0$, they represent masses of the spin-$2^\pm$ torsion modes with respect to the M_4 background [17],

$$\bar{m}^2_{2^+} = \frac{2a_0(a_0 - a_1)}{a_1(b_1 + b_4)}, \qquad \bar{m}^2_{2^-} = \frac{2(a_0 - a_1)}{b_1 + b_2},$$

whereas for finite λ, $m^2_{2^\pm}$ are associated to the torsion modes with respect to the (anti)de Sitter [(A)dS] background.

In M_4, the physical torsion modes are required to satisfy the conditions of no ghosts (positive energy) and no tachyons (positive m^2) [17,18]. However, for spin-2^+ and spin-2^- modes, the requirements for the absence of ghosts, given by the conditions $b_1 + b_2 < 0$ and $b_1 + b_4 > 0$, do not allow for both m^2 to be positive. Hence, only one of the two modes can exist as a propagating mode (with finite mass), whereas the other one must be "frozen" (infinite mass). Although these conditions refer to the M_4 background, we assume their validity also for the (A)dS background, in order to have a smooth limit when $\lambda \to 0$.

One should note that the two spin-2 sectors have quite different dynamical structures.

(i) In the spin-2^- sector, the infinite mass of the spin-2^+ mode implies $\Theta = 0$, whereupon (1ST) yields $Q = 0$, which is exactly the GR_Λ field equation for metric. Thus, the existence of torsion has no influence on the metric.

(ii) In the spin-2^+ sector, the infinite mass of the spin-2^- mode implies $\Sigma = 0$, whereas (1ST) yields that Q is proportional to Θ, with $\Theta \neq 0$. Thus, the torsion function Θ has a decisive dynamical influence on the form of the metric.

In the next section, we focus our attention on the spin-2^+ sector, where the metric appears to be a genuine dynamical effect of PGT.

IV. SOLUTIONS IN THE SPIN-2^+ SECTOR

In this section, we first find solutions of Eq. (3.8a) for the spin-2^+ mode $V = (p/q)\Theta$, and then use that V to find the metric function H and the torsion functions K_α, the quantities that completely define the geometry of the pp_Λ waves with torsion.

A. Solutions for $V = (p/q)\Theta$

The field equation for the spin-2^+ sector can be written in a slightly simpler form as

$$(\partial_{yy} + \partial_{zz})V - \frac{m^2}{p^2}V = 0, \tag{4.1}$$

where $V = (p/q)\Theta$ and $m^2 = m^2_{2^+}$. We have seen in Appendix A that local coordinates (y, z) are well defined in the region where p and q do not vanish, which is an open disk of finite radius, $y^2 + z^2 < 4|\lambda|^{-1}$. Since (4.1) is a differential equation with circular symmetry, it is convenient to introduce polar coordinates, $y = \rho\cos\varphi$, $z = \rho\sin\varphi$, in which Eq. (4.1) takes the form

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right)V - \frac{m^2}{p^2}V = 0. \tag{4.2a}$$

Looking for a solution of V in the form of a Fourier expansion,

$$V = \sum_{n=0}^{\infty} V_n(\rho)(c_n e^{in\varphi} + \bar{c}_n e^{-in\varphi}),$$

we obtain

$$V_n'' + \frac{1}{\rho}V_n' - \left(\frac{n^2}{\rho^2} + \frac{m^2}{p^2}\right)V_n = 0, \tag{4.2b}$$

where prime denotes $d/d\rho$.

1. $\lambda/4 \equiv \ell^{-2} > 0$

Let us first consider solutions of the dS type, using a convenient notation:

$$x = \frac{\rho}{\ell}, \qquad \mu = m\ell, \qquad \xi = \frac{1}{2}\left(1 + \sqrt{1 - \mu^2}\right).$$

The general solutions of (4.2b) for $n = 0$ and $n > 0$ are given by

$$V_0 = c_1(1 + x^2)^{1-\xi}{}_2F_1(1 - \xi, 1 - \xi; 2(1 - \xi); -|1 + x^2|) + c_2(1 + x^2)^{\xi}{}_2F_1(\xi, \xi; 2\xi; -|1 + x^2|), \tag{4.3a}$$

$$V_n = c_1(x^2)^{n/2}(1 + x^2)^{\xi}{}_2F_1(\xi, \xi + n; 1 + n, -x^2) + c_2(x^2)^{-n/2}(1 + x^2)^{\xi}{}_2F_1(\xi, \xi - n; 1 - n, -x^2), \tag{4.3b}$$

where $c_n = c_n(u)(n = 1, 2)$ and ${}_2F_1(a, b, c, z)$ is the hypergeometric function [19].

2. $\lambda/4 \equiv -\ell^{-2} < 2$

In the AdS sector, using

$$\bar\xi = \frac{1}{2}\left(1 + \sqrt{1+\mu^2}\right),$$

the solutions for $n = 0$ and $n > 0$ take the following forms:

$$V_0 = c_1(1-x^2)^{1-\bar\xi}{}_2F_1(1-\bar\xi, 1-\bar\xi; 2(1-\bar\xi); |1-x^2|) + c_2(1-x^2)^{\bar\xi}{}_2F_1(\bar\xi,\bar\xi;2\bar\xi;|1-x^2|), \tag{4.4a}$$

$$V_n = c_1(x^2)^{n/2}(x^2-1)^{\bar\xi}{}_2F_1(\bar\xi,\bar\xi+n;1+n,x^2) + c_2(x^2)^{-n/2}(x^2-1)^{\bar\xi}{}_2F_1(\bar\xi,\bar\xi-n;1-n,x^2). \tag{4.4b}$$

These solutions are essentially an analytic continuation by $\ell \to i\ell$ of those in Eq. (4.3).

3. $\lambda = 0$

The general solution of Eq. (4.2b) has the form

$$V_n = c_1 J_n(-im\rho) + c_2 Y_n(-im\rho), \qquad n = 0,1,2,\ldots \tag{4.5}$$

where J_n and Y_n are Bessel functions of the first and second kind, respectively.

B. Solutions for the metric function H

For a given Θ, the first PGT field equation $A_0 Q = (a_1 - A_0)\Theta$, with Q defined in (2.5), represents a differential equation for the metric function H:

$$(\partial_{yy} + \partial_{zz})H + \frac{2\lambda}{p^2}H = \frac{2(a_1-A_0)}{A_0}\frac{1}{p^2}V. \tag{4.6}$$

This is a second order, linear nonhomogeneous differential equation, and its general solution can be written as

$$H = H^h + H^p,$$

where H^h is the general solution of the homogeneous equation, and H^p a particular solution of (4.6). By comparing Eq. (4.6) to Eq. (4.1), one finds a simple particular solution for H:

$$H^p = \sigma V, \qquad \sigma = \frac{2(a_1 - A_0)}{(2\lambda + m^2)A_0}. \tag{4.7a}$$

On the other hand, H^h coincides with the general vacuum solution of GR_Λ; see (2.8). Since our idea is to focus on the genuine torsion effect on the metric, we choose $H^h = 0$ and adopt H^p as the most interesting PGT solution for the metric function H. Thus, we have

$$H_n = \sigma V_n. \tag{4.7b}$$

C. Solutions for the torsion functions K_α

In the spin-2^+ sector, the torsion functions K_α can be determined from Eq. (3.7), combined with the condition $\Sigma = 0$:

$$\partial_y V + m^2\frac{q}{p}K_y = 0, \qquad \partial_z V + m^2\frac{q}{p}K_z = 0. \tag{4.8}$$

Going over to polar coordinates,

$$K_y = K_\rho\cos\varphi - \frac{K_\varphi}{\rho}\sin\varphi, \qquad K_z = K_\rho\sin\varphi + \frac{K_\varphi}{\rho}\cos\varphi,$$

the previous equations are transformed into

$$K_\rho = -\frac{1}{m^2}\frac{p}{q}\partial_\rho V, \qquad K_\varphi = -\frac{1}{m^2}\frac{p}{q}\partial_\varphi V, \tag{4.9a}$$

or equivalently, in terms of the Fourier modes,

$$K_{\rho n} = -\frac{1}{m^2}\frac{p}{q}\partial_\rho V_n, \qquad K_{\varphi n} = -\frac{1}{m^2}\frac{p}{q}nV_n, \tag{4.9b}$$

where $K_\varphi = \sum_{n=1}^\infty (d_n e^{in\varphi} + \bar d_n e^{-in\varphi})$ with $d_n = -ic_n$, and similarly for K_ρ.

D. Graphical illustrations

Here, we illustrate graphical forms of two specific solutions by giving plots of their metric functions H and the typical torsion component $T^1{}_{02}$,

$$H = \sigma V,$$
$$T^1{}_{02} = \frac{q^2}{p^2}K_2 = \frac{q^2}{p}K_y = -\frac{1}{m^2}q(\partial_\rho V\cos\varphi - \rho^{-1}K_\varphi\sin\varphi). \tag{4.10}$$

For $\lambda \neq 0$, it is convenient to use the units in which $\ell = 1$.

In the dS sector (Fig. 1), the zero modes of both H and $T^1{}_{02}(\varphi = 0)$ are regular functions with a clear-cut wavelike behavior in the region $0 < x < 1$. The plots correspond to the pp_Λ geometry for fixed u, and as u increases, the

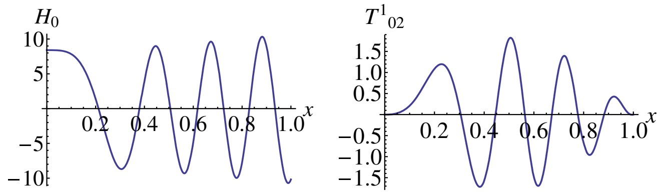

FIG. 1. The plots of a solution in the sector $\lambda > 0$, in units $\ell = 1$, for $n = 0, \mu = 100, c_1 = 1, c_2 = 0, \sigma = 1$. Left: H_0. Right: $T^1{}_{02}(\varphi = 0)$.

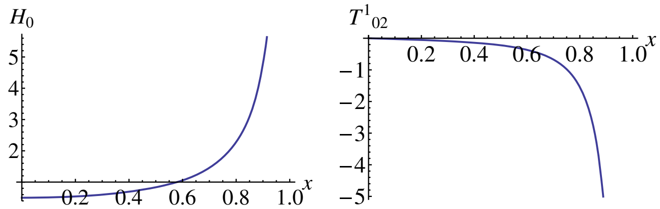

FIG. 2. The plots of a solution in the sector $\lambda < 0$, in units $\ell = 1$, for $n = 0, \mu = \sqrt{8}, c_1 = 0.1, c_2 = 0$. Left: H_0. Right: $T^1{}_{02}(\varphi = 0)$.

pictures change. In the AdS sector (Fig. 2), the solution is singular at $x = 1$, or equivalently at $p = 0$, and it does not have a typical wavelike shape. For a discussion of the singularity at $p = 0$, see Ref. [11]. We also examined a zero mode solution ($n = 0$) in the M_4 sector ($\lambda = 0$); its shape is similar to what we have in Fig. 2, but it remains finite at $x = 1$.

V. SOLUTIONS IN THE SPIN-2^- SECTOR

As we noted at the end of Sec. III, the spin-2^- sector is characterized by $\Theta = 0$ and, as a consequence of (1ST), by $Q = 0$. Equation (3.8b) for Σ reads

$$(\partial_{yy} + \partial_{zz})U - \frac{m^2}{p^2}U = 0, \tag{5.1}$$

where $U = (p/q)\Sigma$ and $m^2 = m^2_{2^-}$. Clearly, the solutions for U coincide with the solutions for $V = (p/q)\Theta$ in Sec. IV A. Furthermore, the metric function H, defined by $Q = 0$, has the GR_Λ form, and the solutions for the torsion functions K_α follow from the two equations

$$\partial_y U + m^2 \frac{q}{p} K_y = 0, \qquad \partial_z U + m^2 \frac{q}{p} K_z = 0, \tag{5.2}$$

the counterparts of those in (4.8).

The fact that the metric of the spin-2^- sector is independent of torsion makes this sector, in general, much less interesting. There is, however, one solution in this sector that should be mentioned: it is the solution with $H = 0$ for which the metric takes the (A)dS/M_4 form, and the complete dynamics is carried solely by the torsion. We skip discussing details of this case, as they can be easily reconstructed from the results given in the previous section, following the procedure outlined above.

VI. CONCLUDING REMARKS

In this paper, we found a new family of the exact vacuum solutions of PGT, the family of the pp_Λ waves with torsion. Here, we wish to clarify a few issues that have not been properly covered in the main text.

The essential step in our construction is the ansatz for the RC connection (3.1), which modifies only the radiation piece of the corresponding Riemannian connection (2.3). A characteristic feature of the resulting solution is the presence of the null vector $k^i = (0, 1, 0, 0)$ in the spacetime geometry. The vector field $k^i\partial_i = (p/q)^2\partial_v$ is orthogonal to the spatial surfaces $u =$ const, and is interpreted as the propagation vector of the pp_Λ wave with torsion. Is such an interpretation justifiable?

Although gravitational waves belong to one of the best known families of exact solutions in GR_Λ, a unique covariant criterion for their precise identification is still missing. One of the early criteria of this type was formulated by Lichnerowicz, based on an analogy with methods used to determine electromagnetic radiation; see Zakharov [7]. This criterion can be formulated as a requirement that the radiation piece of the curvature, $S^{ij} = R^{ij} + \lambda b^i b^j$, satisfies the radiation conditions:

$$k^i S_{ij} = 0, \qquad \varepsilon^{ijkn} k_j S_{kn} = 0. \tag{6.1a}$$

However, when applied to a RC geometry, the Lichnerowicz criterion can be naturally extended to include the torsion 2-form:

$$k^i T_i = 0, \qquad \varepsilon^{ijmn} k_m T_n = 0. \tag{6.1b}$$

A direct calculation based on the expressions (3.2) and (3.3b) shows that both sets of the radiation conditions are satisfied. This result gives a strong support to interpreting the pp_Λ waves with torsion as proper wave solutions of PGT.

Looking at the explicit solutions for the pp_Λ waves with torsion, one should note that, in general, the hypergeometric function ${}_2F_1(a, b, c, x)$ is singular at $x = 1$ ($\rho = \ell$) [19]; moreover, local coordinates we are using are singular at both $x = 1$ and $x = 0$ (Appendix A). To test the nature of these singularities, we calculated the following torsion and curvature invariants:

$$\begin{aligned} &T^i \wedge {}^\star T_i = 0, \\ &R = -12\lambda, \quad R^{ij\star}R_{i,j} = 12\lambda^2\hat{\varepsilon}, \\ &R^{ij}{}_{kl}R^{kl}{}_{mn}R^{mn}{}_{ij} = -48\lambda^3, \end{aligned} \tag{6.2}$$

the fourth order invariant is $96\lambda^4$, and so on. All these invariants are well behaved at $x = 1$, 0, which might be a signal that the singularities in question are just the coordinate singularities. However, according to Wald [20], the geometric singularities are not always visible in the field strength invariants. This issue deserves further clarification.

If the curvature R^{ij} is replaced by its radiation piece S^{ij}, all the invariants in (6.2) are found to vanish. According to Bell's second criterion [7], we have here another result that supports the wave interpretation of our pp_Λ solutions.

In GR_Λ, the pp_Λ waves are algebraically special solutions of Petrov type N; this property can be formulated as an algebraic condition on the Weyl curvature: $W_{ijmn}k^m = 0$ [9,21]. However, one cannot use the same criterion for classifying the solutions of PGT, since W_{ijmn} is not an irreducible part of the RC curvature. The problem can be overcome by replacing W_{ijmn} with ${}^{(1)}R_{ijmn}$, which is a genuine PGT generalization of W_{ijmn} [4]. Using the expression for ${}^{(1)}R_{ijmn}$ from Eq. (3.5), one can directly prove the relation

$$ {}^{(1)}R_{ijmn}k^m = 0, \tag{6.3} $$

which is a natural PGT generalization of the Riemannian condition. The condition (6.3) can be considered as a well-founded criterion for a family of PGT solutions to be of type N.

Finally, we wish to stress that a subfamily of the solutions in the spin-2^+ sector reveals an unexpected dynamical aspect of torsion. Namely, although torsion is introduced by a minor modification of the Riemannian connection [see (3.1)], the metric function H in (4.7) is determined solely by the torsion, and consequently, the related metric is a genuine dynamical effect of PGT. More detailed information could be obtained by analyzing the motion of test particles/fields in the RC spacetimes associated to the pp_Λ waves with torsion.

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APPENDIX A: ON HYPERBOLIC GEOMETRIES

(A)dS space can be simply represented as a 4D hyperboloid H_4 embedded in a 5D Minkowski space M_5 with metric $\eta_{MN} = (+,-,-,-,\sigma)$,

$$ H_4\colon\ X_0^2 - X_1^2 - X_2^2 - X_3^2 - \sigma X_5^2 = -\sigma\ell^2, \tag{A1a} $$

where $\sigma = +1$ for a dS space and $\sigma = -1$ for an AdS space [9,23]. The metric on H_4 reads

$$ ds^2 = dX_0^2 - dX_1^2 - dX_2^2 - dX_3^2 - \sigma dX_5^2, \tag{A1b} $$

and its scalar curvature is $R = -12\sigma/\ell^2$. The group of isometries of the dS/AdS spaces is $SO(1,4)/SO(2,3)$, and the corresponding topologies are $R\times S^3$ for the dS space, and $S^1\times R^3$ for the AdS space (or R^4 for its universal covering).

Going now back to the generalized pp wave metric (2.1), we note that in the limit $H = 0$, it describes the background (A)dS geometry:

$$ \begin{aligned} ds^2 &= 2\left(\frac{q}{p}\right)^2 du(-2\Lambda v^2 du + dv) - \frac{1}{p^2}(dy^2 + dz^2),\\ p &= 1 + \Lambda(y^2+z^2), \qquad q = 1 - \Lambda(y^2+z^2). \end{aligned} \tag{A2} $$

As we shall see below, Λ is related to ℓ by $4\sigma\Lambda = 1/\ell^2$; moreover, $\Lambda > 0$ for dS and $\Lambda < 0$ for AdS. The two forms of the metric associated to the hyperboloid H_4 are related to each other by a coordinate transformation [11],

$$ \begin{aligned} X_0 &= \frac{q}{2p}(u + v + \Lambda u^2 v), \qquad u = 2\sigma\ell\frac{X_5 - \sqrt{-\sigma(X_0^2 - X_1^2 - \sigma X_5^2)}}{X_0 - X_1},\\ X_1 &= \frac{q}{2p}(u - v + \Lambda u^2 v), \qquad v = \frac{X_0 - X_1}{4\ell\sqrt{-\sigma(X_0^2 - X_1^2 - \sigma X_5^2)}},\\ X_2 &= \frac{y}{p}, \qquad X_3 = \frac{z}{p}, \qquad y = \frac{2\ell X_2}{\ell + \sqrt{\ell^2 - \sigma(X_2^2 + X_3^2)}},\\ X_5 &= \frac{1}{2\sqrt{\sigma\Lambda}}\frac{q}{p}(1 + 2\Lambda uv), \qquad z = \frac{2\ell X_3}{\ell + \sqrt{\ell^2 - \sigma(X_2^2 + X_3^2)}}. \end{aligned} \tag{A3} $$

Indeed, the coordinates X_M in M_4 describe the hyperboloid H_4,

$$ (X_0^2 - X_1^2 - \sigma X_5^2) - X_2^2 - X_3^2 = -\frac{1}{4\Lambda}\frac{q^2}{p^2} - \frac{1}{p^2}(y^2 + z^2) = -\frac{1}{4\Lambda} = -\sigma\ell^2, $$

and the corresponding metric (A1b), followed by the rescaling $v \to 2v$, coincides with (A2).

Since local coordinates $x^\mu = (u, v, x, y)$ are introduced by the parametrization (A3), they are well defined for

$$X_0^2 - X_1^2 - \sigma X_5^2 = -\frac{1}{4\Lambda}\frac{q^2}{p^2} > 0.$$

The limiting value $q = 0$ is not allowed, as it represents the singularity of the local coordinate system (u, v, y, z); this singularity is visible only for $\Lambda > 0$. The same conclusion follows from the fact that the determinant of the metric (A2) vanishes for $q = 0$. Furthermore, an inspection of Eq. (A3) reveals the existence of another singularity, located at $p = 0$; it is visible only for $\Lambda < 0$. Thus, local coordinates (u, v, y, z) are restricted to the region where q and/or p do not vanish: $y^2 + z^2 \leq |\Lambda|^{-1}$. More on the geometric interpretation of these singularities can be found in Ref. [11].

APPENDIX B: IRREDUCIBLE DECOMPOSITION OF THE FIELD STRENGTHS

We present here formulas for the irreducible decomposition of the PGT field strengths in a 4D Riemann-Cartan spacetime [4,24].

The torsion 2-form has three irreducible pieces:

$$\begin{aligned} {}^{(2)}T^i &= \frac{1}{3} b^i \wedge (h_m \rfloor T^m), \\ {}^{(3)}T^i &= \frac{1}{3} h^i \rfloor (T^m \wedge b_m), \\ {}^{(1)}T^i &= T^i - {}^{(2)}T^i - {}^{(3)}T^i. \end{aligned} \tag{B1}$$

The RC curvature 2-form can be decomposed into six irreducible pieces:

$$\begin{aligned} &{}^{(2)}R^{ij} = {}^*(b^{[i} \wedge \Psi^{j]}), && {}^{(4)}R^{ij} = b^{[i} \wedge \Phi^{j]}, \\ &{}^{(3)}R^{ij} = \frac{1}{12} X^*(b^i \wedge b^j), && {}^{(6)}R^{ij} = \frac{1}{12} F b^i \wedge b^j, \\ &{}^{(5)}R^{ij} = \frac{1}{2} b^{[i} \wedge h^{j]} \rfloor (b^m \wedge F_m), && {}^{(1)}R^{ij} = R^{ij} - \sum_{a=2}^{6} {}^{(a)}R^{ij}, \end{aligned} \tag{B2a}$$

where

$$\begin{aligned} &F^i := h_m \rfloor R^{mi} = Ric^i, && F := h_i \rfloor F^i = R, \\ &X^i := {}^*(R^{ik} \wedge b_k), && X := h_i \rfloor X^i, \end{aligned} \tag{B2b}$$

and

$$\begin{aligned} \Phi_i &:= F_i - \frac{1}{4} b_i F - \frac{1}{2} h_i \rfloor (b^m \wedge F_m), \\ \Psi_i &:= X_i - \frac{1}{4} b_i X - \frac{1}{2} h_i \rfloor (b^m \wedge X_m). \end{aligned} \tag{B2c}$$

The above formulas differ from those in Refs. [4,24] in two minor details: the definitions of F^i and X^i are taken with an additional minus sign, but at the same time, the overall signs of all the irreducible curvature parts are also changed.

APPENDIX C: CALCULATING THE PGT FIELD EQUATIONS

The gravitational dynamics of PGT is determined by a Lagrangian $L_G = L_G(b^i, T^i, R^{ij})$ (4-form), which is assumed to be at most quadratic in the field strengths (quadratic PGT) and parity invariant [24]. The form of L_G can be conveniently represented as

$$L_G = -{}^\star(a_0 R + 2\Lambda) + \frac{1}{2} T^i H_i + \frac{1}{4} R^{ij} H'_{ij}, \tag{C1}$$

where $H_i := \partial L_G / \partial T^i$ (the covariant momentum) and H'_{ij} define the quadratic terms in L_G:

$$H_i = 2\sum_{n=1}^{3} {}^\star(a_n {}^{(n)}T_i), \qquad H'_{ij} := 2\sum_{n=1}^{6} {}^\star(b_n {}^{(n)}R_{ij}). \tag{C2a}$$

Varying L_G with respect to b^i and ω^{ij} yields the PGT field equations in vacuum. After introducing the complete covariant momentum $H_{ij} := \partial L_G / \partial R^{ij}$ by

$$H_{ij} = -2a_0 {}^\star(b^i b^j) + H'_{ij}, \tag{C2b}$$

these equations can be written in a compact form as [4,24]

$$\begin{aligned} (1ST) \quad \nabla H_i + E_i &= 0, \\ (2ND) \quad \nabla H_{ij} + E_{ij} &= 0, \end{aligned} \tag{C3}$$

where E_i and E_{ij} are the gravitational energy-momentum and spin currents:

$$\begin{aligned} E_i &:= h_i \rfloor L_G - (h_i \rfloor T^m) H_m - \frac{1}{2} (h_i \rfloor R^{mn}) H_{mn}, \\ E_{ij} &:= -(b_i H_j - b_j H_i). \end{aligned} \tag{C4}$$

The above procedure is used in Sec. III B to find the explicit form of the PGT field equations for the pp_Λ waves with torsion, with the result displayed in Eqs. (3.6), (3.7), and (3.8). To simplify calculation of the term $\nabla^{\star(1)} R_{ij}$ in ∇H_{ij}, we used the identity

$$\frac{1}{2} \nabla^\star R_{ij} = \nabla^{\star(2)} R_{ij} + \nabla^{\star(4)} R_{ij}, \tag{C5}$$

that follows from the Bianchi identity $\nabla R^{ij} = 0$ and the double duality properties of the irreducible parts of the curvature.

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A black hole with torsion in 5D Lovelock gravity

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Abstract

We analyze static spherically symmetric solutions of five dimensional (5D) Lovelock gravity in the first order formulation. In the Riemannian sector, when torsion vanishes, the Boulware–Deser black hole represents a unique static spherically symmetric black hole solution for the *generic* choice of the Lagrangian parameters. We show that a *special* choice of the Lagrangian parameters, different from the Lovelock Chern–Simons gravity, leads to the existence of a static black hole solution with *torsion*, the metric of which is asymptotically anti-de Sitter (AdS). We calculate the conserved charges and thermodynamical quantities of this black hole solution.

Keywords: Lovelock gravity, torsion, black holes

1. Introduction

Lovelock gravity [1] represents an intriguing generalization of general relativity, since it is a unique, ghost-free higher derivative extension of Einstein's theory that possesses second order equations of motion. As a higher curvature theory, Lovelock gravity has a considerable number of black hole solutions—see [2–10] and references therein. Many of these possess exotic properties, such as zero mass, peculiar topology of the event horizon etc.

This leads us to an old problem of black hole uniqueness—namely, solutions of general relativity are highly constrained, but the situation changes drastically in the case of higher dimensions. There are new black hole solutions with non-spherical event horizon topology, namely black string, black ring and black brane [11]. Often, these exotic black objects suffer from various instabilities—for example, black strings and branes have Gregory–Laflamme instability [12], and will decay into black holes with spherical horizons. Thus, gravity in higher dimensions represents an interesting area of research, full of surprising discoveries, whose importance stems from its numerous applications.

Lovelock gravity can be also studied within the framework of Poincaré gauge theory (PGT), formulated by Sciamma [13] and Kibble [14] more than half a century ago. PGT is the first modern, gauge-field-theoretic approach to gravity obtained by gauging the Poincaré group of space-time symmetries, the semidirect product of translations and Lorentz transformations. It represents a natural extension of the gauge principle, originally formulated by Weyl within electrodynamics and further developed in the works of Yang, Mills and Utiyama, to the space-time symmetries. The gauge procedure adopted leads directly to a new, Riemann–Cartan geometry of space-time, since torsion and curvature are recovered as the Poincaré gauge field strengths. The Lagrangian in PGT contains a gravitational part, which is a function of the field strengths, the curvature and the torsion, and a suitable matter field Lagrangian.

In the context of Lovelock gravity, this more general setting contains torsionless theory as a limit, and represents a starting point for canonical analysis, coupling with matter fields, supersymmetric extensions of the theory and holographic applications. Interestingly, unlike in the case of Einstein–Cartan theory (first order formulation of general relativity) where all solutions of the equations of motion in vacuum are torsion free, the structure of the vacuum solutions of the Lovelock gravity is more complicated, because there exist solutions with non-vanishing torsion. However, it turns out that exact solutions with torsion are extremely difficult to find, since consistency conditions usually lead to an over-constrained system of equations. Solutions with non-trivial totally antisymmetric torsion have been studied in [8], [15–19]. In this paper, we continue our analysis of the exact solutions of 5D Lovelock gravity solutions with torsion, started in [8], and find a new static, spherically symmetric black hole solution with torsion with zero mass and entropy. The torsion of the solution possesses both tensorial and antisymmetric part. It, unlike the Riemannian Boulware–Deser black hole [20], exists for a specific choice of action parameters. This fine tuning of action parameters was first noticed by Canfora *et al* in their paper [15], and represents a different sector from the highly degenerate Lovelock Chern–Simons gravity.

The paper is organized in the following way. In the second section, we review basics of Poincaré gauge theory and Lovelock gravity in the first order formulation. In section 3 we find the black hole solution of 5D Lovelock gravity with torsion, and analyze its properties. In particular, we find that the quadratic torsional invariant is singular at $r \to 0$. In section 4, we explore the thermodynamics of the previously obtained solution. The appendices contain additional technical details.

We use the following conventions: the Lorentz signature is mostly negative; local Lorentz indices are denoted by the middle letters of the Latin alphabet, while space-time indices are denoted by the letters of the Greek alphabet. Throughout the paper, we mostly use differential forms instead of coordinate notation, and the wedge product is omitted for simplicity.

2. Lovelock gravity

Since the work of Sciamma and Kibble, it has been known that gravity in the first order formulation has the structure of Poincaré gauge theory (PGT)—see [21, 22] for a comprehensive account. For the reader's convenience, we briefly review basics of the PGT.

2.1. PGT in brief

The basic dynamical variables in PGT, playing the role of gauge potentials, are the vielbein e^i 1-form and the spin connection $\omega^{ij} = -\omega^{ji}$ 1-form. In local coordinates x^μ, we can expand the vielbein and the connection 1-forms as $e^i = e^i{}_\mu \mathrm{d}x^\mu$, $\omega^i = \omega^i{}_\mu \mathrm{d}x^\mu$. Gauge symmetries of the theory are local translations (diffeomorphisms) and local Lorentz rotations, parametrized by ξ^μ and ε^{ij} respectively.

From the gauge potentials, we can construct field strengths, namely torsion T^i and curvature R^{ij} (2-forms), which are given as

$$T^i = \nabla e^i \equiv \mathrm{d}e^i + \varepsilon^i{}_{jk}\omega^j \wedge e^k = \frac{1}{2}T^i{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu ,$$
$$R^{ij} = \mathrm{d}\omega^{ij} + \omega^{ik} \wedge \omega_k{}^j = \frac{1}{2}R^{ij}{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu ,$$

where $\nabla = \mathrm{d}x^\mu \nabla_\mu$ is the exterior covariant derivative.

A metric tensor can be constructed from the vielbein and flat metrics: η_{ij}

$$g = \eta_{ij} e^i \otimes e^j = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu ,$$
$$g_{\mu\nu} = \eta_{ij} e^i{}_\mu e^j{}_\nu , \qquad \eta_{ij} = (+,-,-) .$$

The antisymmetry of ω^{ij} in PGT is equivalent to the so-called *metricity condition*, $\nabla g = 0$. A geometry whose connection is restricted by the metricity condition (metric-compatible connection) is called a *Riemann–Cartan geometry*.

The connection ω^{ij} determines the parallel transport in the local Lorentz basis. Because parallel transport is a geometric operation, it is independent of the basis. This property is encoded into PGT via the so-called *vielbein postulate*, which implies

$$\omega_{ijk} = \Delta_{ijk} + K_{ijk} ,$$

where Δ is Levi-Civita connection, and $K_{ijk} = -\frac{1}{2}(T_{ijk} - T_{kij} + T_{jki})$ is the contortion.

2.2. Action and equations of motion

The Lovelock gravity Lagrangian in the first order formulation can be constructed as the linear combination of the dimensionally continued Euler densities L_p, which in D dimensions are defined as

$$L_p = \varepsilon_{i_1 i_2 \ldots i_D} R^{i_1 i_2} \ldots R^{i_{2p-1} i_{2p}} e^{i_{2p+1}} \ldots e^{i_D}.$$

In 5D, there are three Euler densities and the general form of the action of Lovelock gravity [1] is

$$I = \varepsilon_{ijkln} \int \left(\frac{\alpha_0}{5} e^i e^j e^k e^l e^n + \frac{\alpha_1}{3} R^{ij} e^k e^l e^n + \alpha_2 R^{ij} R^{kl} e^n \right) . \tag{2.1}$$

Variation of the action with respect to vielbein e^i and spin connection ω^{ij} yields the gravitational field equations

$$\varepsilon_{ijkln} \left(\alpha_0 e^j e^k e^l e^n + \alpha_1 R^{jk} e^l e^n + \alpha_2 R^{jk} R^{ln} \right) = 0, \tag{2.2}$$

and

$$\varepsilon_{ijkln} \left(\alpha_1 e^k e^l + 2\alpha_2 R^{kl} \right) T^n = 0. \tag{2.3}$$

3. Spherically symmetric solution

3.1. Ansatz

We are looking for a static solution with $SO(4)$ symmetry, which orbits are three-spheres. The most general metric which fulfills these requirements in Schwarzschild-like coordinates $x^\mu = (t, r, \psi, \theta, \varphi)$ is given by

$$\mathrm{d}s^2 = N^2\mathrm{d}t^2 - B^{-2}\mathrm{d}r^2 - r^2(\mathrm{d}\psi^2 + \sin^2\psi\mathrm{d}\theta^2 + \sin^2\psi\sin^2\theta\mathrm{d}\varphi^2), \tag{3.1}$$

where functions N and B depend solely on r, and $r \in [0,\infty)$, $\psi \in [0,\pi)$, $\theta \in [0,\pi)$ and $\varphi \in [0, 2\pi)$. The metric (3.1) possesses seven Killing vectors (see appendix A).

The vielbeins e^i are chosen in a simple diagonal form

$$\begin{aligned} &e^0 = N\mathrm{d}t, \qquad e^1 = B^{-1}\mathrm{d}r, \qquad e^2 = r\mathrm{d}\psi, \qquad e^3 = r\sin\psi\mathrm{d}\theta, \\ &e^4 = r\sin\psi\sin\theta\mathrm{d}\varphi. \end{aligned} \tag{3.2}$$

The most general form of the spin connection compatible with Killing vectors (see appendix A) is given by

$$\begin{aligned} &\omega^{01} = A_0\mathrm{d}t + A_1\mathrm{d}r, \qquad \omega^{02} = A_2\mathrm{d}\psi, \\ &\omega^{03} = A_2\sin\psi\mathrm{d}\theta, \qquad \omega^{04} = A_2\sin\psi\sin\theta\mathrm{d}\varphi, \\ &\omega^{12} = A_3\mathrm{d}\psi, \qquad \omega^{13} = A_3\sin\psi\mathrm{d}\theta, \\ &\omega^{14} = A_3\sin\psi\sin\theta\mathrm{d}\varphi, \qquad \omega^{23} = \cos\psi\mathrm{d}\theta + A_4\sin\psi\sin\theta\mathrm{d}\varphi, \\ &\omega^{24} = -A_4\sin\psi\mathrm{d}\theta + \cos\psi\sin\theta\mathrm{d}\varphi, \qquad \omega^{34} = A_4\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi, \end{aligned} \tag{3.3}$$

where A_i are arbitrary functions of radial coordinate.

3.2. Solution

The sector with vanishing torsion equations of motion for spherically symmetric ansatz has a well-known solution, the Boulware–Deser black hole [20], which exists for the generic choice of action parameters. Another solution, which we construct in this paper, possesses non-vanishing torsion and is given by the following anzatz:

$$\begin{aligned} &A_0 \neq 0, \qquad A_1 = A_2 = A_3 = 0, \qquad A_4 \neq 0 \\ &N = B. \end{aligned} \tag{3.4}$$

By using the adopted anzatz we get that the equations (2.2) reduce to

$$i = 0,1: \qquad 2\alpha_0 r^2 - \alpha_1 + \alpha_1 A_4^2 = 0, \tag{3.5a}$$

$$i = 2,3,4: \qquad \left(2\alpha_2 - 2\alpha_2 A_4^2 - \alpha_1 r^2\right) A_0' + 6\alpha_0 r^2 + \alpha_1(A_4^2 - 1) = 0. \tag{3.5b}$$

The non-vanishing field equations (2.3) take the form

$$ij = 01: \qquad \alpha_1 r^2 + 2\alpha_2 A_4^2 - 2\alpha_2 + 4\alpha_2 r A_4 A_4' = 0, \tag{3.6a}$$

$$ij = 12,13: \qquad \left(\alpha_1 r^2 + 2\alpha_2 A_4^2 - 2\alpha_2\right)(NN' + A_0) + 2\alpha_1 r N^2 = 0, \tag{3.6b}$$

$$ij = 23,24,34: \qquad -2\alpha_2 A_0' + \alpha_1 = 0. \tag{3.6c}$$

From (3.5a) and (3.6c) we get

$$A_4 = \sqrt{1 - \frac{2\alpha_0}{\alpha_1}r^2}, \qquad A_0 = \frac{\alpha_1}{2\alpha_2}r, \tag{3.7}$$

where the integration constant in A_0 is taken to be zero for simplicity. Equation (3.5b) in conjunction with (3.6c) yields to the following constraint between coupling constants:

$$\alpha_1^2 - 12\alpha_0\alpha_2 = 0. \tag{3.8}$$

We consequently get that (3.6*a*) is identically satisfied, while the (3.6*b*) takes the form

$$NN' + \frac{3N^2}{r} - \frac{\alpha_1}{2\alpha_2} r = 0,$$

and can be easily solved for N:

$$N = \sqrt{-\frac{\alpha_1}{8\alpha_2}\left(r^2 - \frac{r_+^8}{r^6}\right)}. \tag{3.9}$$

From (3.8), we conclude that the solution exists in the sector different from the Lovelock Chern–Simons gravity. This is exactly the same fine tuning of parameters found by Canfora *et al* in their paper [15], where the solutions that have the structure of a direct product of a 2D Lorentzian with a 3D Euclidean constant curvature manifold are constructed.

The explicit form of torsion and curvature is given in appendix C. Let us note that both tensorial and antisymmetric part of torsion are non-vanishing unlike in the case of the solution found by Canfora *et al* [16], for which only totally antisymmetric part of torsion is non-vanishing.

Let us now introduce the (anti)-de Sitter ((A)dS) radius ℓ

$$\frac{\alpha_1}{8\alpha_2} = -\frac{\sigma}{\ell^2}, \qquad \sigma = \pm 1. \tag{3.10}$$

By substituting previous relation into (3.7) and (3.9), we get

$$A_4 = \sqrt{1 + \frac{4\sigma r^2}{3\ell^2}},\ N = \sqrt{\sigma\left(\frac{r^2}{\ell^2} - \frac{r_+^8}{\ell^2 r^6}\right)}. \tag{3.11}$$

Note that for the solution to describe a black hole, the following condition must hold:

$$\frac{\alpha_1}{\alpha_2} < 0 \Leftrightarrow \sigma = +1 \tag{3.12}$$

with an *event horizon* located at $r = r_+$.

From the constraint (3.8), it follows that the sign of the ratio $\frac{\alpha_0}{\alpha_1}$ is the same as the sign of $\frac{\alpha_1}{\alpha_2}$

$$\text{sgn}\left(\frac{\alpha_0}{\alpha_1}\right) = \text{sgn}\left(\frac{\alpha_1}{\alpha_2}\right). \tag{3.13}$$

If the ratio is positive, the expression for A_4 implies that we have the maximum value of the radial coordinate, the so called *cosmological horizon*

$$r_0 = \frac{\ell\sqrt{3}}{2}. \tag{3.14}$$

Meanwhile, if the ratio is negative, we have no restriction on the value of the radial coordinate, except that it is positive, and in maximally extended space-time goes to infinity. In this case, the black hole space-time metric is asymptotically AdS.

3.2.1. Invariants. From expressions for curvature and torsion, given in appendix C, we see that quadratic torsional invariant reads

$$T^i \wedge {}^*T_i = -\frac{12\sigma}{\ell^2}\left(1 - \frac{r_+^8}{r^8}\right)\hat{\epsilon}, \tag{3.15}$$

which is obviously divergent in $r = 0$ for r_+ different from zero. Hence, there is a singularity of torsion at $r \to 0$. Scalar Cartan curvature is constant,

$$R = \frac{16\sigma}{\ell^2}, \tag{3.16}$$

while Riemannian scalar curvature is

$$\tilde{R} = \frac{4\sigma}{\ell^2}\left(5 - \frac{3\sigma\ell^2}{2r^2} - \frac{3r_+^8}{r^8}\right), \tag{3.17}$$

and is divergent for $r \to 0$. The quadratic Cartan and Riemannian curvature invariants both vanish:

$$R_{ij} \wedge {}^*R^{ij} = 0, \quad \tilde{R}_{ij} \wedge {}^*\tilde{R}^{ij} = 0. \tag{3.18}$$

We can conclude that the black hole obtained in this article is not of the regular type, and that it possesses singularity at $r = 0$. It is worth noting that solution [16] also possesses singularity of torsion and Riemannian curvature at $r = 0$.

Solving equations of motion (2.2) and (2.3) with seven arbitrary functions is an extremely tedious task, which is facilitated by Mathematica and xAct packages.

3.3. Conserved charges

Conserved charges can be calculated in a number of ways, we decided to make use of Nester's formula [23], the application of which is quite simple in this particular case. In this section, we shall restrict the analysis to the asymptotically AdS case, which corresponds to the black hole. The covariant momenta stemming from the Lovelock action (2.1) are given by

$$\tau_i := \frac{\partial L}{\partial T^i} = 0, \tag{3.19}$$

$$\rho_{ij} = \frac{\partial L}{\partial R^{ij}} = 2\varepsilon_{ijkln}\left(\frac{\alpha_1}{3}e^k e^l + 2\alpha_2 R^{kl}\right)e^n. \tag{3.20}$$

Let us denote the difference between any variable X and its reference value $\bar{X}$ by $\Delta X = X - \bar{X}$. Reference space-time, in respect to which we measure conserved charges, is given for the zero radius of the event horizon $r_+ = 0$. Conserved charges Q_ξ associated to the Killing vector ξ are given by quasi-local surface integrals

$$Q_\xi = \int_{\partial\Sigma} B,$$

where the boundary $\partial\Sigma$ is located at infinity. With a suitable asymptotic behavior of the fields, the proper boundary term reads [23]

$$B = (\xi \rfloor e^i)\Delta\tau_i + \Delta e^i(\xi \rfloor \bar{\tau}_i) + \frac{1}{2}(\xi \rfloor \omega^i{}_j)\Delta\rho_i{}^j + \frac{1}{2}\Delta\omega^i{}_j(\xi \rfloor \bar{\rho}_i{}^j), \tag{3.21}$$

where $\rfloor$ denotes contraction.

For solution (3.9), by making use of the the results of appendix C, we get the covariant momenta

$$
\begin{aligned}
&\rho_{01} = \frac{4\left(\alpha_1^2 - 12\alpha_0\alpha_2\right)}{\alpha_1} e^2 e^3 e^4 \equiv 0, \qquad \rho_{02} = -\frac{8\alpha_1}{3} e^1 e^3 e^4, \qquad \rho_{03} = \frac{8\alpha_1}{3} e^1 e^2 e^4, \\
&\rho_{04} = -\frac{8\alpha_1}{3} e^1 e^2 e^3, \qquad \rho_{12} = \frac{8\alpha_1}{3} e^0 e^3 e^4 - \frac{4\alpha_1 N}{3A_4} e^0 e^1 e^2, \\
&\rho_{13} = -\frac{8\alpha_1}{3} e^0 e^2 e^4 - \frac{4\alpha_1 N}{3A_4} e^0 e^1 e^3, \qquad \rho_{14} = \frac{8\alpha_1}{3} e^0 e^2 e^3 - \frac{4\alpha_1 N}{3A_4} e^0 e^1 e^4, \\
&\rho_{23} = 0, \qquad \rho_{24} = 0, \qquad \rho_{34} = 0.
\end{aligned} \tag{3.22}
$$

From (3.9), we conclude that the connection takes the same form on the background and for $r_+ \neq 0$, $\omega^{ij} = \bar{\omega}^{ij}$. Therefore, formula (3.21) takes the following simpler form:

$$
B = \frac{1}{2}(\xi \rfloor \omega^i{}_j)\Delta\rho_i{}^j.
$$

For the seven Killing vectors $\xi_{(n)}$ (see appendix A) the conserved charges are given by

$$
\begin{aligned}
Q_{(0)} &= \int_{\partial\Sigma} \omega^{01}{}_t \Delta\rho_{01} = 0, \\
Q_{(1)} &= \int_{\partial\Sigma} -\cot\psi \sin\theta \left(\omega^{23}{}_\theta \Delta\rho_{23} + \omega^{24}{}_\theta \Delta\rho_{24}\right) = 0, \\
Q_{(2)} &= \int_{\partial\Sigma} \cot\psi \cos\theta \cos\varphi \left(\omega^{23}{}_\theta \Delta\rho_{23} + \omega^{24}{}_\theta \Delta\rho_{24}\right) \\
&- \frac{\cot\psi}{\sin\theta} \sin\varphi \left(\omega^{14}{}_\varphi \Delta\rho_{14} + \omega^{23}{}_\varphi \Delta\rho_{23} + \omega^{24}{}_\varphi \Delta\rho_{24} + \omega^{34}{}_\varphi \Delta\rho_{34}\right) = 0, \\
Q_{(3)} &= \int_{\partial\Sigma} \cot\psi \cos\theta \sin\varphi \left(\omega^{23}{}_\theta \Delta\rho_{23} + \omega^{24}{}_\theta \Delta\rho_{24}\right) \\
&+ \frac{\cot\psi}{\sin\theta} \cos\varphi \left(\omega^{14}{}_\varphi \Delta\rho_{14} + \omega^{23}{}_\varphi \Delta\rho_{23} + \omega^{24}{}_\varphi \Delta\rho_{24} + \omega^{34}{}_\varphi \Delta\rho_{34}\right) = 0, \\
Q_{(4)} &= \int_{\partial\Sigma} \cos\varphi \left(\omega^{23}{}_\theta \Delta\rho_{23} + \omega^{24}{}_\theta \Delta\rho_{24}\right) \\
&- \cot\theta \sin\varphi \left(\omega^{14}{}_\varphi \Delta\rho_{14} + \omega^{23}{}_\varphi \Delta\rho_{23} + \omega^{24}{}_\varphi \Delta\rho_{24} + \omega^{34}{}_\varphi \Delta\rho_{34}\right) = 0, \\
Q_{(5)} &= \int_{\partial\Sigma} \sin\varphi \left(\omega^{23}{}_\theta \Delta\rho_{23} + \omega^{24}{}_\theta \Delta\rho_{24}\right) \\
&+ \cot\theta \cos\varphi \left(\omega^{14}{}_\varphi \Delta\rho_{14} + \omega^{23}{}_\varphi \Delta\rho_{23} + \omega^{24}{}_\varphi \Delta\rho_{24} + \omega^{34}{}_\varphi \Delta\rho_{34}\right) = 0, \\
Q_{(6)} &= \int_{\partial\Sigma} \left(\omega^{14}{}_\varphi \Delta\rho_{14} + \omega^{23}{}_\varphi \Delta\rho_{23} + \omega^{24}{}_\varphi \Delta\rho_{24} + \omega^{34}{}_\varphi \Delta\rho_{34}\right) = 0.
\end{aligned} \tag{3.23}
$$

Therefore, we conclude that conserved charges for the black hole with torsion (3.9) vanish. In particular, conserved charge $Q_{(0)}$, which corresponds to the energy E of the solution, vanishes due to the *specific choice of the parameters* $\alpha_1^2 = 12\alpha_0\alpha_2$.

4. Thermodynamics

By demanding that Euclidean continuation of the black hole has no conical singularity, we obtain the standard formula for the black hole temperature

$$T = \frac{(N^2)'|_{r=r_+}}{4\pi}. \tag{4.1}$$

In the particular case of the solution (3.9) we get

$$T = \frac{2r_+}{\pi\ell^2}. \tag{4.2}$$

The temperature is positive because solution (3.9) describes black hole iff condition (3.12) is satisfied. Let us note that this type of relation between temperature and the radius of the event horizon is unusual for black holes with spherical horizons. The relation (4.2) is standard in the case of planar black holes (black branes) or black holes in three space-time dimensions.

4.1. Euclidean action

Using the equation of motion (2.2), on-shell Euclidean action takes the form

$$I_{\mathrm{E}} = \varepsilon_{ijklm} \int \left(\frac{2\alpha_1}{3} R^{ij} e^k e^l e^m + \frac{4\alpha_0}{5} e^i e^j e^k e^l e^m \right). \tag{4.3}$$

After substituting the solution (3.9), we get

$$I_{\mathrm{E}} = \int_0^\beta \mathrm{d}t \int_{r_+}^\infty \mathrm{d}r \int \mathrm{d}\psi \mathrm{d}\theta \mathrm{d}\varphi \frac{4(\alpha_1^2 - 12\alpha_0\alpha_2)}{\alpha_2} r^3 \sin^2\psi \sin\theta, \tag{4.4}$$

where the integration over time is performed in the interval $[0, \beta := 1/T]$. By using the constraint on the parameters (3.8), we conclude that

$$I_{\mathrm{E}} = 0. \tag{4.5}$$

From the well-known formula for the entropy

$$S = (\beta\partial_\beta - 1)I_{\mathrm{E}}, \tag{4.6}$$

we obtain

$$S = 0. \tag{4.7}$$

This value of entropy is surprising, but it is not uncommon for Lovelock black holes—see for instance [24], where black holes with zero mass and entropy are obtained. From Euclidean action we can, also, calculate the energy

$$E = \partial_\beta I_{\mathrm{E}}, \tag{4.8}$$

and obtain

$$E = 0, \tag{4.9}$$

in accordance with the results of the previous section.

5. Concluding remarks

We have analyzed static spherically symmetric solutions of Lovelock gravity in five dimensions. For the generic values of the Lagrangian parameters, the theory possesses a well-known solution, the Boulware–Deser black hole, while in the sector $\alpha_1^2 = 12\alpha_0\alpha_2$ we have discovered a new black hole solution with torsion.

We analyzed basic properties of the obtained solution, which torsion possesses non-vanishing tensorial and totally antisymmetric part. The solution has a singularity of torsion and Riemannian curvature for $r \to 0$, while the conserved charges, as well as the entropy, vanish.

It is worth stressing that the black hole metric is asymptotically AdS, which is a crucial condition for holographic investigation. The solution that describes the space-time which is asymptotically dS, with the cosmological horizon located at $r_0 = \frac{\alpha_1}{2\alpha_0}$, is not a black hole.

An interesting property of the solution in the asymptotically AdS case is that, in the semi-classical approximation, its entropy is zero. This means that its number of micro-states is 'small' i.e. it is of order one instead of the expected $\mathcal{O}(\frac{1}{G_N})$. It would be interesting to see what kind of consequences this result has on dual interpretation via gauge/gravity duality.

Acknowledgments

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Appendix A. Killing vectors for metric (3.1)

In addition to the $\frac{\partial}{\partial t}$ Killing vector static and spherically symmetric metric (3.1) possesses six Killing vectors, due to the $SO(4)$ spherical symmetry. The complete set of Killing vectors $\xi^\mu_{(i)}$ of the metric (3.1) is given by:

$$\begin{aligned}
\xi_{(0)} &= \partial_t,\\
\xi_{(1)} &= \cos\theta\partial_\psi - \cot\psi\sin\theta\partial_\theta,\\
\xi_{(2)} &= \sin\theta\cos\varphi\partial_\psi + \cot\psi\cos\theta\cos\varphi\partial_\theta - \frac{\cot\psi}{\sin\theta}\sin\varphi\partial_\varphi,\\
\xi_{(3)} &= \sin\theta\sin\varphi\partial_\psi + \cot\psi\cos\theta\sin\varphi\partial_\theta + \frac{\cot\psi}{\sin\theta}\cos\varphi\partial_\varphi,\\
\xi_{(4)} &= \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi,\\
\xi_{(5)} &= \sin\varphi\partial_\theta + \cot\theta\cos\varphi\partial_\varphi,\\
\xi_{(6)} &= \partial_\varphi.
\end{aligned} \tag{A.1}$$

The independent Killing vectors are $\xi_{(0)}$, $\xi_{(1)}$, $\xi_{(4)}$ and $\xi_{(6)}$, while the others are obtained as their commutators. The invariance conditions of the vielbein under Killing vectors and local Lorentz transformations with parameters $\epsilon^i{}_j$ are

$$\delta_0 e^i_\mu = L_\xi e^i_\mu + \epsilon^i{}_j e^j_\mu = 0, \tag{A.2}$$

where the Lie derivative with respect to ξ is denoted as L_ξ, giving that the only non-zero parameters of the local Lorentz symmetry are

$$\epsilon^{23} = -\frac{\sin\theta}{\sin\psi},\ \epsilon^{34} = -\frac{\sin\varphi}{\sin\theta}. \tag{A.3}$$

Using this and the transformation law for spin connection,

$$\delta_0\omega^{ij}_\mu = L_\xi\omega^{ij}_\mu + \epsilon^i{}_k\omega^{kj}_\mu + \epsilon^j{}_k\omega^{ik}_\mu = 0, \tag{A.4}$$

we can derive the most general form of the spherically symmetric spin connection which is given in the main text, formula (3.3).

Appendix B. Irreducible decomposition of the field strengths

We present here formulas for the irreducible decomposition of the PGT field strengths in a 5D Riemann–Cartan space-time [25].

The torsion 2-form has three irreducible pieces:

$$
\begin{aligned}
{}^{(2)}T^i &= \frac{1}{4} b^i \wedge (h_m \rfloor T^m)\,, \\
{}^{(3)}T^i &= \frac{1}{3} h^i \rfloor (T^m \wedge b_m)\,, \\
{}^{(1)}T^i &= T^i - {}^{(2)}T^i - {}^{(3)}T^i\,.
\end{aligned} \tag{B.1}
$$

The RC curvature 2-form can be decomposed into six irreducible pieces:

$$
\begin{aligned}
&{}^{(2)}R^{ij} = -{}^*(b^{[i} \wedge \Psi^{j]})\,, && {}^{(4)}R^{ij} = \tfrac{2}{3} b^{[i} \wedge \Phi^{j]}\,, \\
&{}^{(3)}R^{ij} = -\tfrac{1}{12} X\,{}^*(b^i \wedge b^j)\,, && {}^{(6)}R^{ij} = \tfrac{1}{20} F\, b^i \wedge b^j\,, \\
&{}^{(5)}R^{ij} = \tfrac{1}{3} b^{[i} \wedge h^{j]} \rfloor (b^m \wedge F_m), && {}^{(1)}R^{ij} = R^{ij} - \textstyle\sum_{a=2}^{6} {}^{(a)}R^{ij}\,.
\end{aligned} \tag{B.2a}
$$

where

$$
\begin{aligned}
&F^i := h_m \rfloor R^{mi} = (Ric)^i\,, \qquad F := h_i \rfloor F^i = R\,, \\
&X^i := {}^*(R^{ik} \wedge b_k)\,, \qquad X := h_i \rfloor X^i,
\end{aligned} \tag{B.2b}
$$

and

$$
\begin{aligned}
\Phi_i &:= F_i - \frac{1}{4} b_i F - \frac{1}{2} h_i \rfloor (b^m \wedge F_m), \\
\Psi_i &:= X_i - \frac{1}{4} b_i X - \frac{1}{2} h_i \rfloor (b^m \wedge X_m)\,.
\end{aligned} \tag{B.2c}
$$

The above formulas differ from those in [25] in two minor details: the definitions of F^i and X^i are taken with an additional minus sign, but at the same time, the overall signs of all the irreducible curvature parts are also changed, leaving their final content unchanged.

Appendix C. Torsion and curvature for the solution (3.9)

In this appendix, we give values of torsion and curvature for the black hole solution.

C.1. Riemannian connection and curvature

The non-vanishing components of the Riemannian connection are given by

$$
\begin{aligned}
&\tilde\omega^{01} = -\frac{\sigma}{\ell^2}\left(\frac{r}{N} + \frac{3r_+^8}{Nr^7}\right) e^0, \qquad \tilde\omega^{12} = \frac{N}{r} e^2, \qquad \tilde\omega^{13} = \frac{N}{r} e^3, \\
&\tilde\omega^{23} = \frac{\cot\psi}{r} e^3, \qquad \tilde\omega^{14} = \frac{N}{r} e^4, \qquad \tilde\omega^{24} = \frac{\cot\psi}{r} e^4, \qquad \tilde\omega^{34} = \frac{\cot\theta}{r\sin\psi} e^4.
\end{aligned} \tag{C.1}
$$

Riemannian curvature reads

$$\tilde{R}^{01} = \frac{\sigma}{\ell^2}\left(1 - \frac{21r_+^8}{r^8}\right)e^0e^1, \qquad \tilde{R}^{02} = \frac{\sigma}{\ell^2}\left(1 + \frac{3r_+^8}{r^8}\right)e^0e^2,$$
$$\tilde{R}^{03} = \frac{\sigma}{\ell^2}\left(1 + \frac{3r_+^8}{r^8}\right)e^0e^3, \qquad \tilde{R}^{04} = \frac{\sigma}{\ell^2}\left(1 + \frac{3r_+^8}{r^8}\right)e^0e^4,$$
$$\tilde{R}^{12} = \frac{\sigma}{\ell^2}\left(1 + \frac{3r_+^8}{r^8}\right)e^1e^2, \qquad \tilde{R}^{13} = \frac{\sigma}{\ell^2}\left(1 + \frac{3r_+^8}{r^8}\right)e^1e^3,$$
$$\tilde{R}^{14} = \frac{\sigma}{\ell^2}\left(1 + \frac{3r_+^8}{r^8}\right)e^1e^4, \qquad \tilde{R}^{04} = \frac{\sigma}{\ell^2}\left(1 + \frac{3r_+^8}{r^8}\right)e^0e^4,$$
$$\tilde{R}^{23} = \frac{\sigma}{\ell^2}\left(1 - \frac{\sigma\ell^2}{r^2} - \frac{r_+^8}{r^8}\right)e^2e^3, \qquad \tilde{R}^{24} = \frac{\sigma}{\ell^2}\left(1 - \frac{\sigma\ell^2}{r^2} - \frac{r_+^8}{r^8}\right)e^2e^4,$$
$$\tilde{R}^{34} = \frac{\sigma}{\ell^2}\left(1 - \frac{\sigma\ell^2}{r^2} - \frac{r_+^8}{r^8}\right)e^3e^4. \tag{C.2}$$

Riemannian scalar curvature is

$$\tilde{R} = -\frac{4\sigma}{\ell^2}\left(-5 + \frac{3\sigma\ell^2}{2r^2} + \frac{3r_+^8}{r^8}\right). \tag{C.3a}$$

The quadratic Riemannian curvature invariant vanishes

$$\tilde{R}_{ij} \wedge {}^*\tilde{R}^{ij} = 0. \tag{C.3b}$$

C.1.1. Torsion and its irreducible decomposition. The non-vanishing components of torsion are given by

$$T^0 = \frac{3N}{r}e^0e^1, \qquad T^2 = \frac{N}{r}e^1e^2 + \frac{2A_4}{r}e^3e^4,$$
$$T^3 = \frac{N}{r}e^1e^3 - \frac{2A_4}{r}e^2e^4, \qquad T^4 = \frac{N}{r}e^1e^4 + \frac{2A_4}{r}e^2e^3. \tag{C.4}$$

The non-vanishing irreducible components of torsion are

$${}^{(1)}T^0 = \frac{3N}{r}e^0e^1,, \qquad {}^{(1)}T^2 = \frac{N}{r}e^1e^2,$$
$${}^{(1)}T^3 = \frac{N}{r}e^1e^3, \qquad {}^{(1)}T^4 = \frac{N}{r}e^1e^4,$$
$${}^{(3)}T^2 = \frac{2A_4}{r}e^3e^4, \qquad {}^{(3)}T^3 = -\frac{2A_4}{r}e^2e^4, \qquad {}^{(3)}T^4 = \frac{2A_4}{r}e^2e^3. \tag{C.5}$$

The 2nd irreducible component of torsion vanishes as in the case of any solution of Lovelock gravity, excluding Lovelock Chern–Simons [8]. Quadratic torsional invariant reads

$$T^i \wedge {}^*T_i = -\frac{12\sigma}{\ell^2}\left(1 - \frac{r_+^8}{r^8}\right)\hat{\epsilon}. \tag{C.6}$$

Non-zero components of the (Cartan) curvature are

$$R^{01} = \frac{4\sigma}{\ell^2}e^0e^1, \qquad R^{23} = \frac{4\sigma}{3\ell^2}\frac{N}{A_4}e^1e^4 + \frac{4\sigma}{3\ell^2}e^2e^3,$$
$$R^{24} = -\frac{4\sigma}{3\ell^2}\frac{N}{A_4}e^1e^3 + \frac{4\sigma}{3\ell^2}e^2e^4, \qquad R^{34} = \frac{4\sigma}{3\ell^2}\frac{N}{A_4}e^1e^2 + \frac{4\sigma}{3\ell^2}e^3e^4. \tag{C.7}$$

Scalar Cartan curvature is constant:

$$R = \frac{16\sigma}{\ell^2}. \tag{C.8}$$

Quadratic Cartan curvature invariant vanishes:

$$R_{ij} \wedge {}^*R^{ij} = 0. \tag{C.9}$$

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Near horizon OTT black hole asymptotic symmetries and soft hair *

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Abstract: We study the near horizon geometry of both static and stationary extremal Oliva Tempo Troncoso (OTT) black holes. For each of these cases, a set of consistent asymptotic conditions is introduced. The canonical generator for the static configuration is shown to be regular. For the rotating OTT black hole, the asymptotic symmetry is described by the time reparametrization, the chiral Virasoro and centrally extended $u(1)$ Kac-Moody algebras.

Keywords: near horizon geometry, OTT black hole, 3D gravity

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1 Introduction

The long-standing problem of the origin of black hole entropy is one of the most important open questions in contemporary physics. There are many proposals for interpreting the black hole entropy and the corresponding micro-states, such as: entanglement entropy [1], fuzz-ball [2] or soft hair on the horizon [3, 4]. The issue has also been the starting point of many ingenious discoveries, the most impressive of which is the holographic nature of gravity [5].

Holographic duality [6] states that the gravitational theory in an asymptotically anti de Sitter (AdS) space-time is dual to a non-gravitational theory defined on the conformal boundary of space-time. Although it is still a conjecture, there is a large number of results supporting this statement. Let us mention, for the purpose of this paper, that holographic duality offers many insights into the black hole physics, including the black hole information paradox and the origin of the black hole micro-states. In fact, holography provides a a way to derive the black hole entropy from the near horizon micro-states via the Cardy formula [7], whose applicability crucially relies on the existence of $2D$ conformal symmetry as a subgroup of the asymptotic symmetry group. In spite of this, the present understanding of the holographic duality is not sufficient for the most general purpose, and we need further generalizations. A notable progress represents the derivation of the Cardy-like formula in the Warped Conformal Field Theory (WCFT), see ref. [8].

A particularly interesting generalization is given in [9], where the authors propose that the extremal Kerr black hole is dual to the chiral $2D$ CFT. There are indications that this chiral CFT should arise as Discrete Light Cone Quantized (DLCQ) [10]. More precisely, the extremal black hole, non necessarily Kerr-like, possesses an intriguing feature that its near horizon geometry is an exact solution of the theory. This allows to study the physics on the horizon by investigating the properties of the near horizon geometry. For a review of the subject see [11].

In this article, we analyse the near horizon limit of a black hole with soft hair known as the Oliva-Tempo-Troncoso (OTT) black hole [12], which is the solution of the BHT gravity [13], as well as of the Poincaré gauge theory of gravity [14] for the special choice of action parameters. The leading idea of this analysis is a study of the influence of the hair parameter on the micro-states of the extremal black hole. The obtained near horizon geometries exist, without any reference to the extremal black hole, as independent solutions and are important on their own.

We first analyse the static OTT black hole, which becomes extremal for the specific value of the hair parameter, and obtain the corresponding near horizon geometry. We then study the asymptotic structure of the near horizon geometry and obtain the asymptotic symmetry group.

We continue with a study of the rotating OTT black hole which can be made extremal in two different ways: either by tuning the hair parameter or the angular momentum. The solution obtained by tuning the hair parameter, surprisingly, leads to the same near horizon geometry as in the non-rotating case. The extremal OTT

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black hole with maximal angular momentum leads to a geometry with a richer structure. We conclude that the asymptotic symmetry is a direct sum of the time reparametrization, the Virasoro algebra and the centrally extended $u(1)$ Kac-Moody algebra. The entropy of the extremal rotating OTT black hole can be expressed in terms of the central extension of the Kac-Moody algebra and the on-shell value of the zero mode Virasoro generator

$$S=2\pi\sqrt{\frac{1}{2}L_0^{\rm on-shell}\kappa}. \qquad (1)$$

Our conventions are the same as in Ref. [14]: the Latin indices $(i,j,k,...)$ refer to the local Lorentz frame, the Greek indices $(\mu,\nu,\rho,...)$ refer to the coordinate frame, e^i is the orthonormal triad (coframe 1-form), ω^{ij} is the Lorentz connection (1-form), the respective field strengths are the torsion $T^i=de^i+\omega^i{}_m\wedge e^m$ and the curvature $R^{ij}=d\omega^{ij}+\omega^i{}_k\wedge\omega^{kj}$ (2-forms), the frame h_i dual to e^j is defined by $h_i\rfloor e^j=\delta_i{}^j$, the signature of the metric is $(+,-,-)$, totally antisymmetric symbol ε^{ijk} is normalized to $\varepsilon^{012}=1$, the Lie dual of an antisymmetric form X^{ij} is $X_i:=-\varepsilon_{ijk}X^{jk}/2$, the Hodge dual of a form α is ${}^\star\alpha$, and the exterior product of forms is implicit.

2 Conformally flat Riemannian solutions in PGT

In the sector with a unique AdS ground state, the BHT gravity possesses an interesting black hole solution, the OTT black hole [12]. One of the key features of this solution is its conformal flatness, such that it is also a Riemannian solution of PGT in vacuum [14], for the special choice of the Lagrangian parameters.

The general parity preserving Lagrangian 3-form of PGT, which is mostly quadratic in field strengths is given by:

$$\begin{aligned} L_G &= -a_0\varepsilon_{ijk}e^iR^{jk}-\frac{1}{3}\Lambda_0\varepsilon_{ijk}e^ie^je^k+L_{T^2}+L_{R^2}, \\ L_{T^2} &= T^{i\star}\left(a_1{}^{(1)}T_i+a_2{}^{(2)}T_i+a_3{}^{(3)}T_i\right), \\ L_{R^2} &= \frac{1}{2}R^{ij\star}\left(b_4{}^{(1)}R_{ij}+b_5{}^{(5)}R_{ij}+b_6{}^{(6)}R_{ij}\right). \end{aligned} \qquad (2)$$

where ${}^{(a)}T_i$ and ${}^{(a)}R_{ij}$ are irreducible components of the torsion and the RC curvature, see [15], $a_0=1/16\pi G$, Λ_0 is a cosmological constant, and (a_1,a_2,a_3) and (b_1,b_2,b_3) are the coupling constants in the torsion and the curvature sector, respectively. In [14], it was shown that any conformally flat solution of the BHT gravity (in particular the OTT black hole) is also a Riemannian solution of PGT, provided that

$$b_4+2b_6=0. \qquad (3)$$

The conformal properties of 3D spacetime, where the Weyl curvature vanishes identically, are characterized by the Cotton 2-form C^i [16], defined by $C^i:=\nabla L^i=dL^i+\omega^i{}_mL^m$ where $L^m:=Ric^m-\frac{1}{4}Re^m$ is the Schouten 1-form. The conformal flatness of space-time is expressed by the condition $C^i=0$.

By using the BHT condition that ensures the existence of the unique maximally symmetric background [14], the identification (3) can be expressed in the following way:

$$\Lambda_0=-a_0/2\ell^2, \qquad b_4=2a_0\ell^2. \qquad (4)$$

3 Canonical generator and conserved charges

The usual construction of the canonical generator of the Poincaré gauge transformations, including diffeomorphisms and Lorentz rotations [17], makes use of the canonical structure of the theory. The construction can be substantially simplified by using the first order formulation of the theory, in which the Lagrangian (3-form) reads:

$$L_G=T^i\tau_i+\frac{1}{2}R^{ij}\rho_{ij}-V(e,\tau,\rho),$$

see [15]. In this formulation, τ^m and ρ_{ij} are independent dynamical variables, the covariant field momenta conjugate to e^i and ω^{ij}. The presence of the potential V ensures the validity of the on-shell relations $\tau_i=H_i$, $\rho_{ij}=H_{ij}$. These relations can be used to transform L_G into its standard quadratic form (2).

The construction of the canonical generator G in the first order formulation can be found in [15]. The action of G on the basic dynamical variables is defined via the Poisson bracket operation, so that G has to be a differentiable phase space functional. The examination of the differentiability of G starts from its variation

$$\begin{aligned} \delta G &= -\int_\Sigma d^2x(\delta G_1+\delta G_2), \\ \delta G_1 &= \varepsilon^{t\alpha\beta}\xi^\mu\left(e^i{}_\mu\partial_\alpha\delta\tau_{i\beta}+\omega^i{}_\mu\partial_\alpha\delta\rho_{i\beta}+\tau^i{}_\mu\partial_\alpha\delta e_{i\beta}\right. \\ &\quad \left.+\rho^i{}_\mu\partial_\alpha\delta\omega_{i\beta}\right)+\mathcal{R}, \qquad (5a) \\ \delta G_2 &= \varepsilon^{t\alpha\beta}\theta^i\partial_\alpha\delta\rho_{i\beta}+\mathcal{R}. \qquad (5b) \end{aligned}$$

Here, Σ is the spatial section of spacetime, the variation is performed in the set of adopted asymptotic states, $\mathcal{R}$ stands for regular (differentiable) terms, and we use ρ^i and ω^i, the Lie duals of ρ_{mn} and ω_{mn}, to simplify the formulas. Diffeomorphisms are parametrized by ξ^μ, and the parameters of local Lorentz rotations are θ^i.

The explicit form of the generator of Lorentz rotations, see [15], implies that there is only one possible non-regular term on the rhs of the variation of the Lorentz rotations generator G_2, which is of the form (5b).

In general $\delta G\neq\mathcal{R}$, so that G is not differentiable. This problem can be, in principle, easily solved by going over to the improved generator $\tilde{G}:=G+\Gamma$, where the boundary term Γ is constructed so that $\delta\tilde{G}=\mathcal{R}$. By making a partial integration in δG, one finds that Γ is defined by the following variational equation

$$\begin{aligned}\delta\Gamma &= \delta\Gamma_1+\delta\Gamma_2,\\ \delta\Gamma_1 &= \int_{\partial\Sigma}\xi^\mu\left(e^i{}_\mu\delta\tau_i+\omega^i{}_\mu\delta\rho_i+\tau^i{}_\mu\delta e_i+\rho^i{}_\mu\delta\omega_i\right), \quad (6a)\\ \delta\Gamma_2 &= \int_{\partial\Sigma}\theta^i\delta\rho_i. \quad (6b)\end{aligned}$$

In many cases the asymptotic conditions ensure the regularity of the Lorentz rotations generator and $\Gamma_2=0$. However, it is worth noting that in the particular problem we are solving the contribution of the surface term of the Lorentz rotations generator is non-trivial, as we shall see in section 5.2.

4 Static OTT black hole orbifold

Extremal static OTT black hole. The metric of the static OTT black hole is given by:

$$\mathrm{d}s^2=N^2\mathrm{d}t^2-N^{-2}\mathrm{d}r^2-r^2\mathrm{d}\varphi^2, \tag{7}$$

where $N^2=-\mu+br+\frac{r^2}{\ell^2}$. Black hole horizons are located at:

$$r_\pm=\frac{1}{2}\left(-b\ell^2\pm\ell\sqrt{b^2\ell^2+4\mu}\right).$$

The black hole is extremal if the horizons coincide, $r_+=r_-$. This condition is satisfied when $b^2\ell^2+4\mu=0$. Let us note that the existence of the extremal black hole horizon implies $b<0$.

Orbifold. Let us now consider the following coordinate transformation:

$$t\to\frac{t}{\varepsilon}, \qquad r\to r_++\varepsilon\rho. \tag{8}$$

The metric now becomes:

$$\mathrm{d}s^2=\frac{\rho^2}{\ell^2}\mathrm{d}t^2-\frac{\ell^2}{\rho^2}\mathrm{d}\rho^2-(r_++\varepsilon\rho)^2\mathrm{d}\varphi^2.$$

In the limit $\varepsilon\to0$, the metric (with the prescription $\rho\to r$) reads:

$$\mathrm{d}s^2=\frac{r^2}{\ell^2}\mathrm{d}t^2-\frac{\ell^2}{r^2}\mathrm{d}r^2-r_+^2\mathrm{d}\varphi^2. \tag{9}$$

It represents a perfectly regular solution, an orbifold.

We choose the triad fields in the simple diagonal form:

$$e^0=\frac{r}{\ell}\mathrm{d}t, \qquad e^1=\frac{\ell}{r}\mathrm{d}r, \qquad e^2=r_+\mathrm{d}\varphi. \tag{10a}$$

The Levi-Civita connection that corresponds to the triad field reads

$$\omega^0=0, \qquad \omega^1=0, \qquad \omega^2=-\frac{e^0}{\ell}. \tag{10b}$$

The curvature 2-form has only one non-vanishing component:

$$R^0=0, \qquad R^1=0, \qquad R^2=\frac{1}{\ell^2}e^0e^1, \tag{11a}$$

the scalar curvature is constant, $R=\frac{2}{\ell^2}$, and the Ricci and Shoutten 1-forms are given by:

$$\begin{aligned}&Ric^0=\frac{e^0}{\ell^2}, \qquad Ric^1=\frac{e^1}{\ell^2}, \qquad Ric^2=0,\\ &L^0=\frac{e^0}{2\ell^2}, \qquad L^1=\frac{e^1}{2\ell^2}, \qquad L^2=-\frac{e^2}{2\ell^2}. \quad (11b)\end{aligned}$$

The solution is conformally flat (as the OTT black hole), i.e. the Cotton 2-form $C^i=\nabla L^i$ vanishes and solves the equations of motion of both BHT gravity and PGT in the sector $b_4+2b_6=0$.

4.1 Asymptotic conditions

Let us consider the following asymptotic conditions for the metric in the region $r\to\infty$:

$$g_{\mu\nu}\sim\begin{pmatrix}\mathcal{O}_{-2} & \mathcal{O}_2 & \mathcal{O}_1\\ \mathcal{O}_2 & -\frac{\ell^2}{r^2}+\mathcal{O}_3 & \mathcal{O}_1\\ \mathcal{O}_1 & \mathcal{O}_1 & \mathcal{O}_0\end{pmatrix}, \tag{12}$$

where $\mathcal{O}_n$ denotes a term with an asymptotic behaviour r^{-n} or faster. In accordance with (12), the the triad fields behave as:

$$e^i{}_\mu\sim\begin{pmatrix}\mathcal{O}_{-1} & \mathcal{O}_3 & \mathcal{O}_2\\ \mathcal{O}_1 & \frac{\ell}{r}+\mathcal{O}_2 & \mathcal{O}_0\\ \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0\end{pmatrix} \tag{13}$$

The condition $T^i=0$, together with (13), gives the following asymptotic behaviour of the spin connection

$$\omega^i{}_\mu\sim\begin{pmatrix}\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_1\\ \mathcal{O}_1 & \mathcal{O}_3 & \mathcal{O}_1\\ \mathcal{O}_{-1} & \mathcal{O}_3 & \mathcal{O}_2\end{pmatrix} \tag{14}$$

The diffeomorphisms that leave the metric (12) invariant are given by:

$$\begin{aligned}\xi^t &= T(t)+\mathcal{O}_3,\\ \xi^r &= rU(\varphi)+\mathcal{O}_0,\\ \xi^\varphi &= S(\varphi)+\mathcal{O}_1. \quad (15)\end{aligned}$$

Lorentz transformations that leave the asymptotic conditions invariant are

$$\theta^0=\mathcal{O}_2, \qquad \theta^1=\mathcal{O}_2, \qquad \theta^2=\mathcal{O}_2. \tag{16}$$

In terms of the Fourier modes $\ell_n:=\delta_0(S=\mathrm{e}^{in\varphi})$ and $j_n:=\delta_0(U=\mathrm{e}^{im\varphi})$, the algebra of the residual gauge transformations takes the form of a semi-direct sum of the

Virasoro and the Kac-Moody algebras:

$$\begin{aligned}[\ell_m,\ell_n] &= -i(m-n)\ell_{m+n},\\ [\ell_m,j_n] &= inj_{m+n},\\ [j_n,j_m] &= 0. \end{aligned} \tag{17}$$

4.2 Algebra of charges

The gauge generator is not a priori well-defined because, for given asymptotic conditions, its functional derivatives may be ill-defined, as already mentioned in section 3. This problem can be solved by constructing an improved generator, which includes suitable surface terms [18]. Since our solution is Riemannian, $\tau_i=0$, relation (6b) reduces to:

$$\delta G=\int_{\partial\Sigma}\xi^\mu(\omega^i{}_\mu\delta\rho_i+\rho^i{}_\mu\delta\omega_i) \tag{18}$$

For the particular asymptotic conditions adopted in this paper, we conclude that the gauge generator is differentiable, so that there is no need for adding any surface term,

$$\Gamma=0. \tag{19}$$

As a consequence, both the central charge of the Virasoro algebra and the level of the $u(1)$ Kac-Moody algebra both vanish.

5 Near-horizon geometry of rotating OTT

Rotating OTT black hole. The rotating OTT black hole is defined by the metric

$$\mathrm{d}s^2=N^2\mathrm{d}t^2-F^{-2}\mathrm{d}r^2-r^2(\mathrm{d}\varphi+N_\varphi\mathrm{d}t)^2, \tag{20a}$$

where

$$\begin{aligned} &F=\frac{H}{r}\sqrt{\frac{H^2}{\ell^2}+\frac{b}{2}H(1+\eta)+\frac{b^2\ell^2}{16}(1-\eta)^2-\mu\eta},\\ &N=AF, \qquad A=1+\frac{b\ell^2}{4H}(1-\eta),\\ &N_\varphi=\frac{\ell}{2r^2}\sqrt{1-\eta^2}(\mu-bH),\\ &H=\sqrt{r^2-\frac{\mu\ell^2}{2}(1-\eta)-\frac{b^2\ell^4}{16}(1-\eta)^2}. \end{aligned} \tag{20b}$$

The roots of $N=0$ are

$$r_\pm=\ell\sqrt{\frac{1+\eta}{2}}\left(-\frac{b\ell}{2}\sqrt{\eta}\pm\sqrt{\mu+\frac{b^2\ell^2}{4}}\right).$$

The metric (20) depends on three free parameters, μ, b and η. For $\eta=1$, it represents the static OTT black hole, and for $b=0$, it reduces to the rotating BTZ black hole with parameters (m,j), such that $4Gm:=\mu$ and $4Gj:=\mu\ell\sqrt{1-\eta^2}$.

The conserved charges of the rotating black hole take the following form:

$$E = \frac{1}{4G}\left(\mu+\frac{1}{4}b^2\ell^2\right), \tag{21a}$$

$$J = \ell\sqrt{1-\eta^2}E. \tag{21b}$$

The rotating OTT black hole is a three-parameter solution, so that the extremal limit can be achieved in two different ways. The first is the same as in the non-rotating case, by requiring $4\mu+b^2\ell^2=0$. As a simple consequence, the resulting geometry is the same as if the black hole were non-rotating. This is not a surprising result if we note that both energy and angular momentum vanish in this case.

The second way to obtain an extremal black hole is to take $\eta=0$, which means that angular momentum takes the maximal possible value. This corresponds to the usual procedure for the Kerr black hole.

The horizon is located at

$$r_0=\frac{\ell\sqrt{b^2\ell^2+4\mu}}{2\sqrt{2}}. \tag{22}$$

The coordinate change is given as

$$\begin{aligned} r&\to r_0+\epsilon r\\ t&\to\frac{t}{\epsilon^2}\\ \varphi&\to\varphi-\frac{t}{\ell\epsilon^2}. \end{aligned} \tag{23}$$

An interesting departure from the usual redefinition of the coordinates in literature is that, in order to obtain a non-singular metric, we have to scale the time coordinate with the same parameter as used in the rescaling of the radial coordinate but to the power of minus two, instead of the standard minus one.

After changing the coordinates and taking the limit $\epsilon\to0$, we obtain the near-horizon metric

$$\mathrm{d}s^2=\frac{32(b^2\ell^2+4\mu)}{b^4\ell^4}\frac{r^4}{\ell^4}\mathrm{d}t^2-\frac{\ell^2}{r^2}\mathrm{d}r^2-r_0^2\left(\mathrm{d}\varphi-\frac{16r^2}{b^2\ell^5}\mathrm{d}t\right)^2, \tag{24}$$

or

$$\mathrm{d}s^2=2r_0^2\frac{16r^2}{b^2\ell^5}\mathrm{d}t\mathrm{d}\varphi-\frac{\ell^2}{r^2}\mathrm{d}r^2-r_0^2\mathrm{d}\varphi^2. \tag{25}$$

It is convenient to further rescale the time coordinate and obtain a more convenient form of the metric

$$\mathrm{d}s^2=\frac{2r^2r_0}{\ell^2}\mathrm{d}t\mathrm{d}\varphi-\frac{\ell^2}{r^2}\mathrm{d}r^2-r_0^2\mathrm{d}\varphi^2. \tag{26}$$

We again choose the triad fields in the diagonal form

$$e^0=\frac{r^2}{\ell^2}\mathrm{d}t,\quad e^1=\frac{\ell}{r}\mathrm{d}r,\quad e^2=\frac{r^2}{\ell^2}\mathrm{d}t-r_0\mathrm{d}\varphi. \tag{27}$$

The Levi-Civita connection is given by:

$$\omega^{01}=-\frac{2e^0}{\ell}+\frac{e^2}{\ell},\quad \omega^{02}=\frac{e^1}{\ell},\quad \omega^{12}=\frac{e^0}{\ell}. \tag{28}$$

The solution is maximally symmetric and therefore we have:

$$R^{ij}=\frac{1}{\ell^2}e^ie^j, \quad Ric^i=\frac{2e^i}{\ell^2}, \quad L^i=\frac{e^i}{2\ell^2}, \quad C^i=0. \tag{29}$$

The rotating OTT black hole for $b=0$ reduces to the rotating BTZ black hole. What can be said about the corresponding near-horizon geometries? If we introduce $\rho=r^2$, we obtain a near-horizon BTZ black hole geometry with two times smaller ℓ, and a different r_0 [11]. The only trace of the hair parameter is hidden in r_0, and it will lead to different values of the central charges. Thus, we are able to recover the results for the near-horizon BTZ black hole geometry from those of the OTT black hole, but not by simply taking $b=0$.

5.1 Asymptotic conditions

We consider the following asymptotic form of the metric

$$g_{\mu\nu}\sim\begin{pmatrix} \mathcal{O}_{-1} & \mathcal{O}_3 & \mathcal{O}_{-2} \\ \mathcal{O}_3 & -\frac{\ell^2}{r^2}+\mathcal{O}_4 & \mathcal{O}_1 \\ \mathcal{O}_{-2} & \mathcal{O}_1 & \mathcal{O}_0 \end{pmatrix}. \tag{30}$$

The asymptotic form of the triad fields is chosen in accordance with the asymptotic behaviour of the metric (30)

$$e^i{}_\mu\sim\begin{pmatrix} \frac{r^2}{\ell^2}+\mathcal{O}_1 & \mathcal{O}_5 & \mathcal{O}_0 \\ \mathcal{O}_1 & \frac{\ell}{r}+\mathcal{O}_3 & \mathcal{O}_0 \\ \frac{r^2}{\ell^2}+\mathcal{O}_1 & \mathcal{O}_5 & \mathcal{O}_0 \end{pmatrix} \tag{31}$$

The asymptotic form of the spin connection reads

$$\omega^i{}_\mu\sim\begin{pmatrix} -\frac{r^2}{\ell^3}+\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0 \\ \mathcal{O}_0 & -\frac{1}{r}+\mathcal{O}_2 & \mathcal{O}_0 \\ -\frac{r^2}{\ell^3}+\mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0 \end{pmatrix} \tag{32}$$

The condition of vanishing torsion $T^i=0$, together with (31) and (32), leads to the following constraints

$$\omega^2_r+\omega^0_r=\mathcal{O}_5, \tag{33a}$$

$$\omega^1_\varphi-\frac{e^1_\varphi}{\ell}=\mathcal{O}_2, \tag{33b}$$

$$\frac{e^0{}_\varphi}{\ell}-\frac{e^2{}_\varphi}{\ell}+\omega^2_\varphi-\omega^0_\varphi=\mathcal{O}_2, \tag{33c}$$

$$\omega^2_\varphi-\frac{e^2_\varphi}{\ell}=\mathcal{O}_1, \tag{33d}$$

$$\omega^0_\varphi-\frac{e^0_\varphi}{\ell}=\mathcal{O}_1. \tag{33e}$$

The diffeomorphisms that leave the metric (30) invariant are given by

$$\begin{aligned} \xi^t &= T(t)+\mathcal{O}_3, \\ \xi^r &= rU(\varphi)+\mathcal{O}_1, \\ \xi^\varphi &= S(\varphi)+\mathcal{O}_4. \end{aligned} \tag{34}$$

Lorentz transformations that leave the asymptotic form of the triads and the spin connection invariant are

$$\begin{aligned} \theta^0 &= \partial_r\xi^t\frac{e^2{}_t}{e^1{}_r}+\mathcal{O}_2, \\ \theta^1 &= -\frac{2\xi^r}{r}+\partial_t\xi^t+\mathcal{O}_4, \\ \theta^2 &= \frac{e^0{}_t}{e^1{}_r}\partial_r\xi^t+\mathcal{O}_2. \end{aligned} \tag{35}$$

5.2 Algebra of charges

The improved generator is given by

$$\tilde{G}=G+\Gamma. \tag{36}$$

A direct calculation yields the surface term

$$\begin{aligned} \Gamma = & -4a_0\int_0^{2\pi}\mathrm{d}\varphi\Big[T(t)\frac{r^2}{\ell^2}\left(\omega^0{}_\varphi-\frac{e^0{}_\varphi}{\ell}-\omega^2{}_\varphi+\frac{e^2{}_\varphi}{\ell}\right) \\ & +S(\varphi)\omega^i{}_\varphi e_{i\varphi}+(2U(\varphi)+\partial_tT(t))e^1{}_\varphi\Big] \end{aligned} \tag{37}$$

The charge is finite due to the conditions that follow from the constraint $T^i=0$. By using the composition law for the local Poincaré transformations

$$\begin{aligned} \xi''^\mu &= \xi^\alpha\partial_\alpha\xi'^\mu-\xi'^\alpha\partial_\alpha\xi^\mu, \\ \theta''^i &= \epsilon^i{}_{jk}\theta^j\theta'^k+\xi^\alpha\partial_\alpha\theta'^i-\xi'^\alpha\partial_\alpha\theta^i \end{aligned} \tag{38}$$

we derive the Poisson bracket algebra for the improved canonical generators (which are also well-defined [19]). The Virasoro algebra is not centrally extended

$$\{L_m,L_n\} = -i(m-n)L_{m+n}, \tag{39}$$

$$\{L_m,J_n\} = inJ_{m+n}, \tag{40}$$

whereas the Kac-Moody algebra does have a central charge κ

$$\{J_m,J_n\}=-i16\pi a_0 m\delta_{m+n,0}, \tag{41}$$

whose value is

$$\kappa=16\pi a_0=\frac{\ell}{G}. \tag{42}$$

For related studies, see [20, 21].

The entropy of the extremal OTT black hole $S=\frac{\pi r_0^2}{G}$ can be reproduced in terms of purely algebraic quantities via a peculiar formula

$$S=2\pi\sqrt{\frac{1}{2}L_0^{\text{on-shell}}\kappa}, \tag{43}$$

where $L_0^{\rm on-shell}$ is the value of the Virasoro generator L_0 on the shell

$$L_0^{\rm on-shell}=\frac{r_0^2}{2\ell G}. \tag{44}$$

The entropy formula has a striking resemblance to the entropy formula of [8]. In our case $J_0^{\rm on-shell}=0$, so that our formula is a consequence of the general expression for entropy in WCFT if

$$L_0^{\rm vac}-\frac{(J_0^{\rm vac})^2}{2\kappa}=-\frac{\kappa}{8}. \tag{45}$$

One might intuitively expect that the formula for the black hole entropy in WCFT should correctly reproduce the entropy of an extremal OTT black hole. The expectation relies on the resemblance of the algebra (39), (40) with the Euclidean WCFT algebra, and it is anticipated that the same derivation as in [8] holds in our case.

6 Sugawara-Sommerfeld construction

It is well-known that the Virasoro algebra can be constructed as a bilinear combination of the elements of the Kac-Moody algebra. We apply this procedure, known as the Sugawara-Sommerfeld construction [22], to the algebra obtained in the previous section.

First we introduce the auxiliary operators

$$K_n=\frac{1}{2\kappa}\sum_i J_i J_{n-i}, \tag{46}$$

which obey the following commutation relations

$$i\{K_m,J_n\} = -nJ_{m+n}, \tag{47}$$

$$i\{K_m,K_n\} = (m-n)K_{m+n}, \tag{48}$$

$$i\{K_m,L_n\} = (m-n)K_{m+n}. \tag{49}$$

Then, we define generators of the first Virasoro algebra as

$$L_n^{\rm R}=L_n-K_n, \tag{50}$$

which satisfy the commutation relations

$$i\{J_m,J_n\} = \kappa m\delta_{m+n,0}, \tag{51}$$

$$i\{J_m,L_n^{\rm R}\} = 0, \tag{52}$$

$$i\{L_m^{\rm R},L_n^{\rm R}\} = (m-n)L_{m+n}^{\rm R}. \tag{53}$$

The generators of the second Virasoro algebra are defined as

$$L_n^{\rm L}=-K_{-n}-in\alpha J_{-n}+\frac{c^{\rm L}}{24}\delta_{n,0}. \tag{54}$$

The generators $L_n^{\rm L}$ and $L_n^{\rm R}$ define the two commuting Virasoro algebras

$$i\{L_m^{\rm L},L_n^{\rm L}\} = (m-n)L_{m+n}^{\rm L}+\frac{c^{\rm L}}{12}m(m^2-1)\delta_{m+n,0}, \tag{55}$$

$$i\{L_m^{\rm L},L_n^{\rm R}\} = 0, \tag{56}$$

$$i\{L_m^{\rm R},L_n^{\rm R}\} = (m-n)L_{m+n}^{\rm R}, \tag{57}$$

with central charges

$$c^{\rm L}=12\kappa\alpha^2, \qquad c^{\rm R}=0. \tag{58}$$

In theories with conformal symmetry, it is well-known that entropy can be reproduced by the Cardy formula. Sugawara-Sommerfeld construction includes an arbitrary parameter α, whose value is fixed by requiring that the Virasoro algebra satisfies certain canonical relations. We shall fix it by requiring that the Cardy formula

$$S=2\pi\sqrt{\frac{L_0^{\rm L}c^{\rm L}}{6}}+2\pi\sqrt{\frac{L_0^{\rm R}c^{\rm R}}{6}}, \tag{59}$$

reproduces entropy correctly. For the orbifold, the values of the Virasoro zero modes are

$$L_0^{\rm L}=\frac{c^{\rm L}}{24}, \qquad L_0^{\rm R}=\frac{r_0^2}{2\ell G}, \tag{60}$$

which implies that the Cardy formula, in combination with (58), gives the entropy

$$S=\frac{\pi c^{\rm L}}{6}=2\pi\kappa\alpha^2. \tag{61}$$

Consequently, we get

$$\alpha^2=\frac{r_0}{2\ell}. \tag{62}$$

7 Thermodynamics at extremality

There is an equivalent Cardy formula in which, instead of using the background values of the Virasoro zero modes, one uses the temperature. Thus, the required additional piece of information is the temperature of the dual CFT, which may be derived from the black hole thermodynamics.

We start from the first law of black hole thermodynamics

$$\delta E=T_{\rm H}\delta S+\Omega\delta J+\Phi_i\delta q^i, \tag{63}$$

where J is the angular momentum, Ω is the angular velocity, q^i are additional conserved charges, and Φ_i are potentials conjugate to q^i. In the case of the extremal black hole (for more details on extremal black holes and the first law of thermodynamics, see [23]), for which the Hawking temperature is zero, $T_{\rm H}=0$, the first law implies that energy is a function of the conserved charges

$$E_{\rm Ext}=E_{\rm Ext}(J_{\rm Ext},q^i_{\rm Ext}). \tag{64}$$

The corresponding generalized temperatures are defined by

$$T_{\rm L}=\frac{\partial S_{\rm Ext}}{\partial J_{\rm Ext}}, \qquad T_i=\frac{\partial S_{\rm Ext}}{\partial q^i_{\rm Ext}}, \tag{65}$$

where $T_{\rm L}$ is the left moving temperature.

The entropy, energy and angular momentum of the extremal OTT black hole are given by:

$$S_{\rm Ext}=\frac{\pi r_0}{G}, \qquad E_{\rm Ext}=\frac{r_0^2}{2\ell^2 G}, \qquad J_{\rm Ext}=\frac{r_0^2}{2\ell G}. \tag{66}$$

From the variation of the entropy of extremal OTT

$$\delta S_{\rm Ext}=\frac{\delta J_{\rm Ext}}{T_{\rm L}},$$

the left moving temperature is determined as

$$T_{\rm L}=\frac{r_0}{\pi\ell}. \tag{67}$$

In the extremal case, the right moving temperature is zero

$$T_{\rm R}=0. \tag{68}$$

By requiring that the alternative form of the Cardy formula,

$$S_{\rm C}=\frac{\pi^2}{3}T_{\rm L}c^{\rm L}+\frac{\pi^2}{3}T_{\rm R}c^{\rm R},$$

reproduces the entropy of the extremal OTT black hole, we conclude that $c^{\rm R}$ is undetermined, and that the left central charge is two times bigger than the Brown-Henneaux central charge

$$c^{\rm L}=\frac{3\ell}{G}. \tag{69}$$

This can be used to fix the constant α appearing in the Sugawara-Sommerfeld construction. From equation (58) and the previous formula, we derive

$$\alpha=\frac{1}{2}. \tag{70}$$

8 Concluding remarks

We investigated the near horizon symmetry of both static and stationary OTT black holes in the quadratic PGT. In the static case, the corresponding asymptotic symmetry is trivial, whereas in the stationary case, the set of consistent asymptotic conditions leads to a symmetry described by time reparametrization and the semidirect sum of the centrally extended $u(1)$ Kac-Moody and the chiral Virasoro algebras. The improved asymptotic conditions that follow from the vanishing of torsion (33c) can be further strengthened, thus making time reparametrization a pure gauge.

The near horizon limit corresponds to deep infrared sector of the theory, which implies that only the soft part of the charge survives. This means that the corresponding charges represent the soft hair on the black hole horizon. Formula

$$S=2\pi\sqrt{\frac{1}{2}L_0^{\rm on-shell}\kappa},$$

shows that there is an intimate relationship between the black hole entropy and the soft hairs on the horizon, but more precise statements require further studies.

Using the Sugawara-Sommerfeld construction, we build the Virasoro algebra as a bilinear combination of the $u(1)$ Kac-Moody and chiral Virasoro algebras. The presence of conformal symmetry enables to use the Cardy formula for entropy, which correctly reproduces the black hole entropy.

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