Nonlinear Excitations in Bose-Einstein Condensates: Parametric and Geometric Resonances, Faraday Waves*

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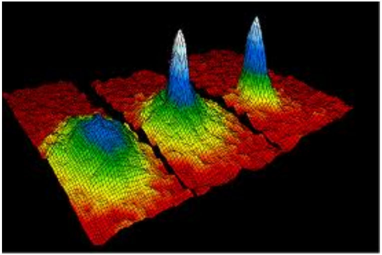
Overview

- Introduction
- Parametric resonances
  - Modulation of the interaction
  - Condensate dynamics
  - Excitation spectra
  - Frequency shifts
- Geometric resonances
  - Linear stability analysis
  - Condensate dynamics
  - Frequency shifts
  - Resonant mode coupling
- Faraday waves in two-component BECs
  - Non-resonant Faraday waves
  - Periods of Faraday waves
  - Resonant waves
- Conclusions and outlook

Bose-Einstein condensation

- Intensive progress in the field of ultracold atoms has been recognized by 2001 Nobel prize for experimental realization of Bose-Einstein condensation in 1995
- Cold alkali atoms:
  Rb, Na, Li, K . . .
  $T \sim 1\,\mathrm{nK}$, $\rho \sim 10^{14}\mathrm{cm}^{-3}$



- Cold bosons, cold fermions
- Optical lattices
- Short-range interactions,
  long-range dipolar interactions
- Tunable quantum systems concerning dimensionality, type and strength of interactions

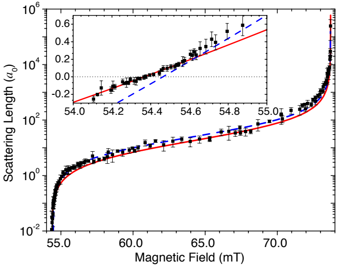
Feshbach resonance

- Scattering length depends on the external magnetic field

- For ^{7}Li: PRL **102**, 090402 (2009)

$$a(B) = a_{\mathrm{BG}} \left(1 + \frac{\Delta}{B - B_\infty} \right)$$

$a_{\mathrm{BG}} = -24.5\, a_0$, $B_\infty = 73.68\,\mathrm{mT}$,
$\Delta = 19.2\,\mathrm{mT}$



- The interaction can be in principle tuned to any small or large, positive or negative value

Mean-field theory

- At $T = 0$ (no thermal excitations), order parameter ψ satisfies mean-field Gross-Pitaevskii equation:

$$\imath\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m}\Delta + V(\vec{r}) + g_2 N|\psi(\vec{r},t))|^2 + g_3 N^2|\psi(\vec{r},t))|^4\right]\psi(\vec{r},t)$$

- $\psi(\vec{r},t)$ is a condensate wave-function
- $V(\vec{r}) = \frac{1}{2}m\omega_\rho^2(\rho^2 + \lambda^2 z^2)$ is a harmonic trap potential
- $\ell = \sqrt{\hbar/m\omega_\rho}$ is a characteristic harmonic oscillator length
- Effective contact interaction between atoms is $g\delta(\vec{r})$
- $g_2 = \frac{4\pi\hbar^2}{m}a$, where a is the s-wave scattering length
- g_3 is a three-body interaction strength
- N is the number of atoms in a BEC

Variational approach

- Starting point for analytical calculations is the Gaussian variational approach: PRL **77**, 5320 (1996)
- For an axially symmetric trap:

$$\psi(\rho, z, t) = \mathcal{N}(t) \exp\left[-\frac{1}{2}\frac{\rho^2}{u_\rho(t)^2} + i\rho^2\phi_\rho(t)\right] \exp\left[-\frac{1}{2}\frac{z^2}{u_z(t)^2} + iz^2\phi_z(t)\right]$$

- By extremizing the corresponding action, we obtain four equations
- Two equations (for the phases) are algebraic and can be solved
- This then leads to two ordinary differential equations of the second order

Variational equations

- In terms of dimensionless condensate widths, the equations are

$$\ddot{u}_\rho(t) + u_\rho(t) - \frac{1}{u_\rho(t)^3} - \frac{p(t)}{u_\rho(t)^3 u_z(t)} - \frac{k(t)}{u_\rho(t)^5 u_z(t)^2} = 0$$

$$\ddot{u}_z(t) + \lambda^2 u_z(t) - \frac{1}{u_z(t)^3} - \frac{p(t)}{u_\rho(t)^2 u_z(t)^2} - \frac{k(t)}{u_\rho(t)^4 u_z(t)^3} = 0$$

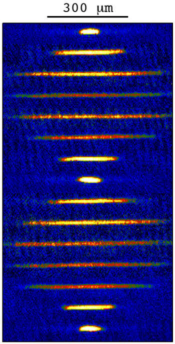
where dimensionless interaction are given by

$$p = \sqrt{2/\pi}\, Na/\ell \quad \text{and} \quad k = 4g_3 N^2/9\sqrt{3}\pi^3\hbar\omega_\rho \ell^6$$

- 3-body interaction can be expressed as $k = \frac{32 g_3 \hbar\omega_\rho}{9\sqrt{3} g_2^2} p^2$
- For $N = 10^5$ ^{87}Rb atoms, in a trap with $\omega_\rho = 2\pi \times 112$ Hz:
  $g_2 = 5\hbar \times 10^{-11}\text{cm}^3\text{s}^{-1} \Rightarrow p = 426$
  $g_3 \approx \hbar \times 10^{-26}\text{cm}^6\text{s}^{-1} \Rightarrow k = 1050.$

Modulation of the interaction

- Nonlinear effects lead to rich resonant phenomena in BEC
- Interaction modulation in a recent experiment by groups of R. Hulet (Rice Univ.) and V. Bagnato (São Paulo Univ.) PRA **81**, 053627 (2010)
- BEC of ^{7}Li is confined in a cylindrical harmonic trap
- Time-dependent modulation of atomic interactions via a Feshbach resonance
- Excitation of the lowest-lying quadrupole mode
- Shift of the quadrupole mode frequency
- Interesting setup for studying nonlinear BEC dynamics



Modulation of the interaction

- We study effects of harmonic modulation of the s-wave scattering length on collective modes:

$$B(t) = B_{\rm av} + \delta B \cos \Omega t, \quad a(t) \simeq a_{\rm av} + \delta a \cos \Omega t$$

$$a_{\rm av} = a(B_{\rm av}), \quad \delta a = -\frac{a_{\rm BG}\Delta \delta B}{(B_{\rm av} - B_\infty)^2}$$

$$B_{\rm av} = 56.5\,{\rm mT}\,, \quad \delta B = 1.4\,{\rm mT}\,, \quad a_{\rm av} \sim 3a_0\,, \quad \delta a \sim 2a_0$$

$$\Rightarrow p(t) \simeq p + q \cos \Omega t$$

- For now, we neglect three-body interactions
- Nonlinear form of GP equation induces shifts in the frequencies of low-lying modes (beyond linear response)
- For Ω close to some of BEC eigenmodes, we expect resonances - large amplitude oscillations, for which nonlinear terms becomes crucial
- Vidanović, Balaž, Al-Jibbouri, Pelster, PRA **84**, 013618 (2011)

Linear stability analysis

- Equilibrium widths: $u_{\rho 0} = \frac{1}{u_{\rho 0}^3} + \frac{p}{u_{\rho 0}^3 u_{z0}}$, $\quad \lambda^2 u_{z0} = \frac{1}{u_{z0}^3} + \frac{p}{u_{\rho 0}^2 u_{z0}^2}$
- Linearized equations:

$$u_\rho(t) = u_{\rho 0} + \delta u_\rho(t),\ u_z(t) = u_{z0} + \delta u_z(t)$$

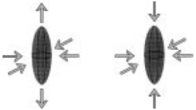
$$\delta\ddot{u}_\rho + \delta u_\rho \left(1 + \frac{3}{u_{\rho 0}^4} + \frac{3p}{u_{\rho 0}^4 u_{z0}}\right) + \delta u_z \frac{p}{u_{\rho 0}^3 u_{z0}^2} = 0$$

$$\delta\ddot{u}_z + \delta u_z \left(\lambda^2 + \frac{3}{u_{z0}^4} + \frac{2p}{u_{\rho 0}^2 u_{z0}^3}\right) + \delta u_\rho \frac{2p}{u_{\rho 0}^3 u_{z0}^2} = 0$$

- Breathing mode ω_{B0} and quadrupole mode ω_{Q0}:

$$\sqrt{2}\left[\left(1 + \lambda^2 - \frac{p}{4u_{\rho 0}^2 u_{z0}^3}\right) \pm \sqrt{\left(1 - \lambda^2 + \frac{p}{4u_{\rho 0}^2 u_{z0}^3}\right)^2 + 8\left(\frac{p}{4u_{\rho 0}^3 u_{z0}^2}\right)^2}\right]^{1/2}$$

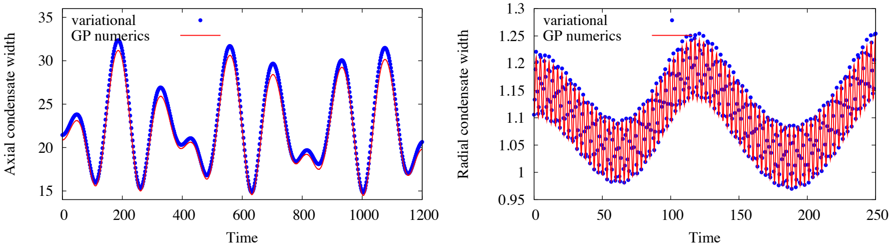
Quadrupole mode ω_{Q0}



Breathing mode ω_{B0}

Condensate dynamics

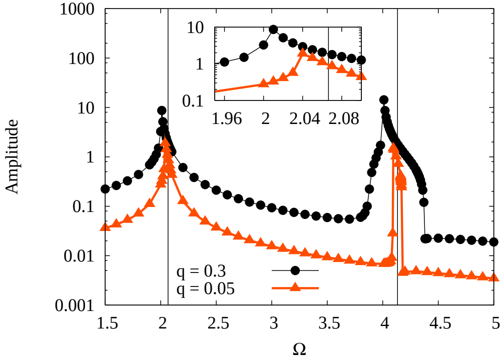
- Numerical results for cylindrically symmetric BEC $p = 15$, $q = 10$, $\lambda = 0.021$ and $\Omega = 0.05$



- Vudragović, Vidanović, Balaž, Muruganandam, Adhikari, Comput. Phys. Commun. **183**, 2021 (2012)

Excitation spectra

- Resonant behaviour for $\Omega \sim \omega_0$ and $\Omega \sim 2\omega_0$ for spherically symmetric BEC



- Collective modes exhibit shifts close to resonances

Poincaré-Lindstedt method

- To calculate collective modes to higher orders, we rescale time as $s = \omega t$ and use expansions:

$$\begin{aligned} u(s) &= u_0 + q\, u_1(s) + q^2\, u_2(s) + q^3\, u_3(s) + \ldots \\ \omega &= \omega_0 + q\, \omega_1 + q^2\, \omega_2 + q^3\, \omega_3 + \ldots \end{aligned}$$

- This leads to a hierarchical system of equations for spherically symmetric BEC:

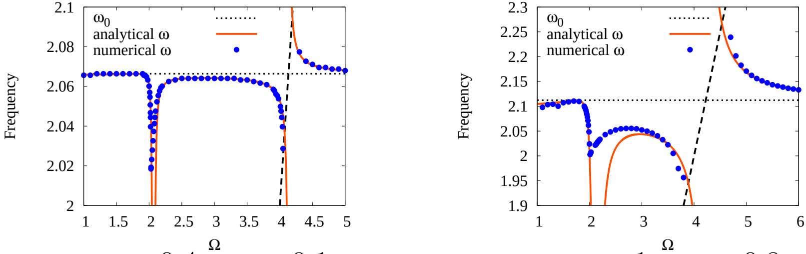
$$\begin{aligned} \omega_0^2\, \ddot{u}_1(s) + \omega_0^2\, u_1(s) &= \frac{1}{u_0^4} \sin \frac{\Omega s}{\omega} \\ \omega_0^2\, \ddot{u}_2(s) + \omega_0^2\, u_2(s) &= -2\omega_0\, \omega_1\, \ddot{u}_1(s) - \frac{4}{u_0^5}\, u_1(s)\, \sin \frac{\Omega s}{\omega} + \alpha\, u_1(s)^2 \\ \omega_0^2\, \ddot{u}_3(s) + \omega_0^2\, u_3(s) &= -2\omega_0\, \omega_2\, \ddot{u}_1(s) - 2\beta\, u_1(s)^3 + 2\alpha\, u_1(s) u_2(s) - \omega_1^2\, \ddot{u}_1(s) \\ &\quad + \frac{10}{u_0^6}\, u_1(s)^2\, \sin \frac{\Omega s}{\omega} - \frac{4}{u_0^5}\, u_2(s)\, \sin \frac{\Omega s}{\omega} - 2\omega_0\, \omega_1\, \ddot{u}_2(s) \end{aligned}$$

- $\omega_1, \omega_2, \ldots$ are obtained so as to cancel secular terms

Frequency shift for spherically symmetric BEC

- Frequency of the breathing mode vs. driving frequency Ω for spherically symmetric BEC
- Result in second order of perturbation theory

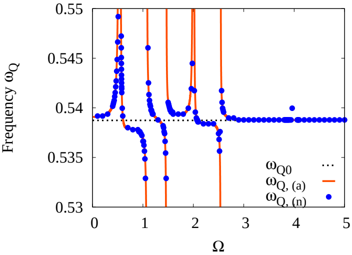
$$\omega = \omega_0 + q^2 \frac{\text{Polynomial}(\Omega)}{(\Omega^2 - \omega_0^2)^2 (\Omega^2 - 4\omega_0^2)}$$



$p = 0.4, \quad q = 0.1$ $\qquad$ $p = 1, \quad q = 0.8$

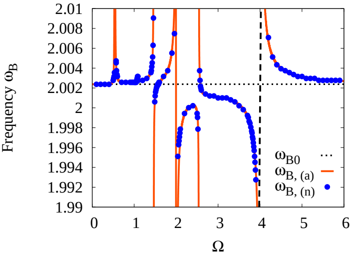
Frequency shift for cylindrically symmetric BEC

- Frequency of quadrupole mode ω_Q versus driving frequency Ω



- Poles: ω_{Q0}, $\omega_{B0} - \omega_{Q0}$, $2\omega_{Q0}$, $\omega_{Q0} + \omega_{B0}$, ω_{B0}

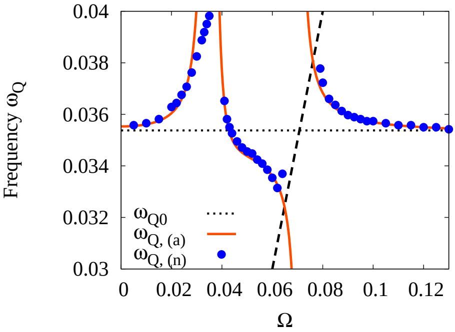
- Frequency of breathing mode ω_B versus driving frequency Ω



- Poles: ω_{Q0}, $\omega_{B0} - \omega_{Q0}$, ω_{B0}, $\omega_{Q0} + \omega_{B0}$, $2\omega_{B0}$

Frequency shift for the experimental setup

- In the experiment:
  - $\omega_B >> \omega_Q$, $\Omega \in (0, 3\omega_Q)$, large modulation amplitude
  - Strong excitation of quadrupole mode
  - Excitation of breathing mode in the radial direction
  - Frequency shifts of quadrupole mode of about 10% are present



Geometric resonances

- Modulation of the interaction is one way to induce strong nonlinear regime and generate resonances
- However, this is possible to achieve even without the dynamical manipulation of the system
- Geometry of the system (trap aspect ratio) essentially influences its behavior
- It is known that systems exhibits resonances for particular values of the trap aspect ratios:
  Stringari, PRL **77**, 2360 (1996)
  Dalfovo, Minniti, Pitaevskii, PRA **56**, 4855 (1997)
- We study variationally and numerically geometric resonances in systems with 2-body and 3-body interactions
- Al-Jibbouri, Vidanović, Balaž, Pelster, arXiv:1208.0991 (2012)

Variational equations

- Gaussian variational approach:

$$\ddot{u}_\rho(t) + u_\rho(t) - \frac{1}{u_\rho(t)^3} - \frac{p(t)}{u_\rho(t)^3 u_z(t)} - \frac{k(t)}{u_\rho(t)^5 u_z(t)^2} = 0$$

$$\ddot{u}_z(t) + \lambda^2 u_z(t) - \frac{1}{u_z(t)^3} - \frac{p(t)}{u_\rho(t)^2 u_z(t)^2} - \frac{k(t)}{u_\rho(t)^4 u_z(t)^3} = 0$$

Linear stability analysis

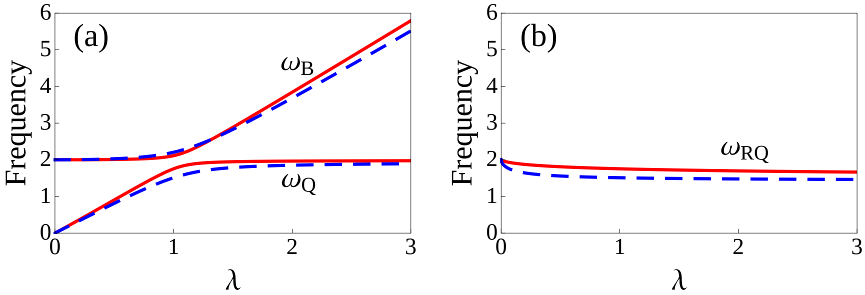
- For $N = 10^5$ ^{87}Rb atoms, in a trap with $\omega_\rho = 2\pi \times 112$ Hz, we earlier calculated $p = 426$ and $k = 1050$.
- Although k is larger than p, the corresponding 3-body terms are suppressed by $u_\rho^2 u_z$ compared to 2-body terms
- Equilibrium widths:

$$u_{\rho 0} = \frac{1}{u_{\rho 0}^3} + \frac{p}{u_{\rho 0}^3 u_{z0}} + \frac{k}{u_{\rho 0}^5 u_{z0}^2}$$

$$\lambda^2 u_{z0} = \frac{1}{u_{z0}^3} + \frac{p}{u_{\rho 0}^2 u_{z0}^2} + \frac{k}{u_{\rho 0}^4 u_{z0}^3}$$

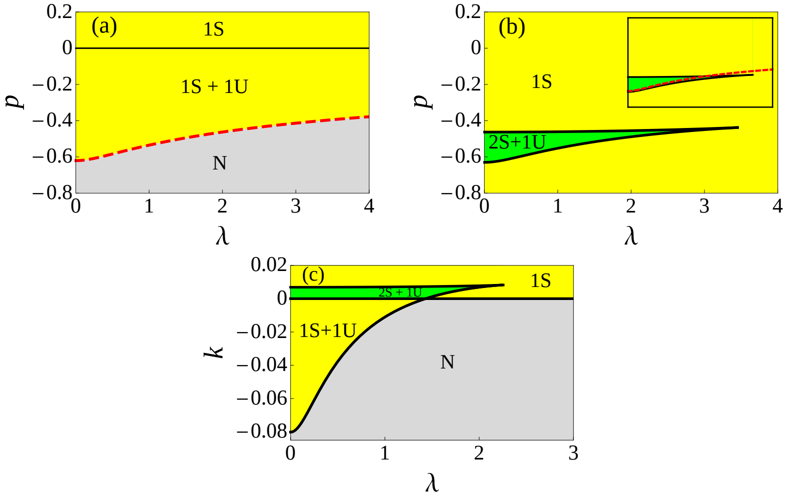
- For $\lambda = 3/2$, we obtain numerically $u_{\rho 0} \approx 3.69$ and $u_{z0} \approx 2.47$, i.e. $u_{\rho 0}^2 u_{z0} \approx 33.6$
- 3-body terms thus have effective coupling of $k/33.6 \approx 31.2$, which makes them small corrections of the order of 7%
- However, close to resonances, they may be more significant

Frequencies of collective modes

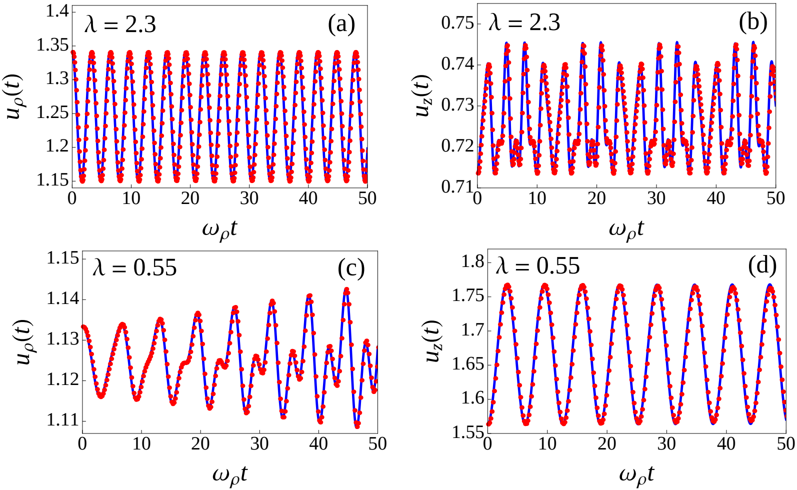


Frequencies of the breathing, quadrupole, and radial quadrupole modes versus trap aspect ratio λ for $p = 1$, $k = 0.001$ (solid lines) and $p = 10$, $k = 0.1$ (dashed lines)

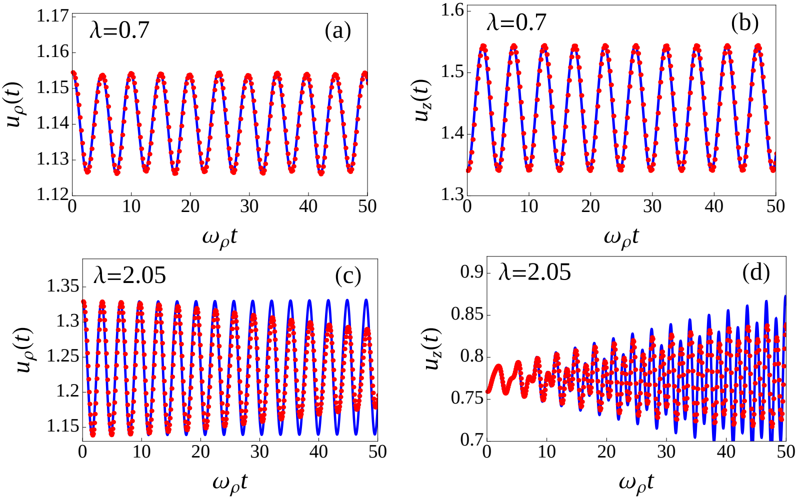
Stability diagram



Condensate dynamics (1)



Condensate dynamics (2)



Poincaré-Lindstedt method

- To calculate collective modes to higher orders, we rescale time as $s = \omega t$ and use expansions:

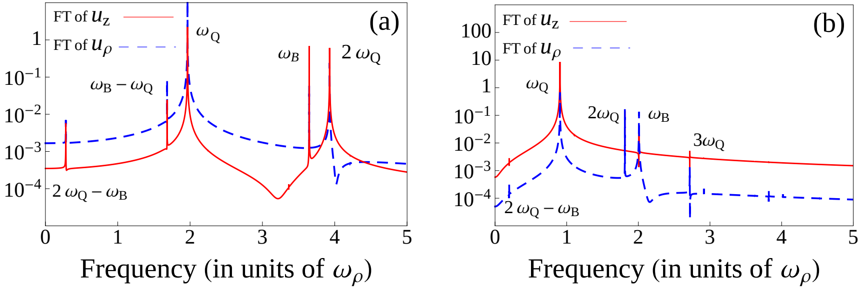
$$\begin{aligned} u_\rho &= u_{\rho 0} + \varepsilon u_{\rho 1} + \varepsilon^2 u_{\rho 2} + \varepsilon^3 u_{\rho 3} + \dots \\ u_z &= u_{z0} + \varepsilon u_{z1} + \varepsilon^2 u_{z2} + \varepsilon^3 u_{z3} + \dots \\ \omega &= \omega_0 + \varepsilon\,\omega_1 + \varepsilon^2\,\omega_2 + \varepsilon^3\,\omega_3 + \dots \end{aligned}$$

- Initial conditions

$$\begin{aligned} \mathbf{u}(0) &= \mathbf{u}_0 + \varepsilon \mathbf{u}_Q \\ \dot{\mathbf{u}}(0) &= \mathbf{0} \end{aligned}$$

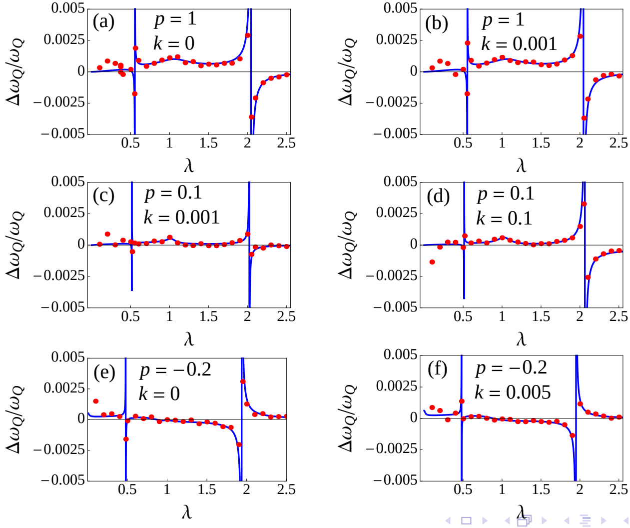
- This leads to a hierarchical system of equations, where ω_1, $\omega_2, \dots$ are obtained so as to cancel secular terms
- Frequency shifts are calculated using the third-order perturbation theory

Far from resonances

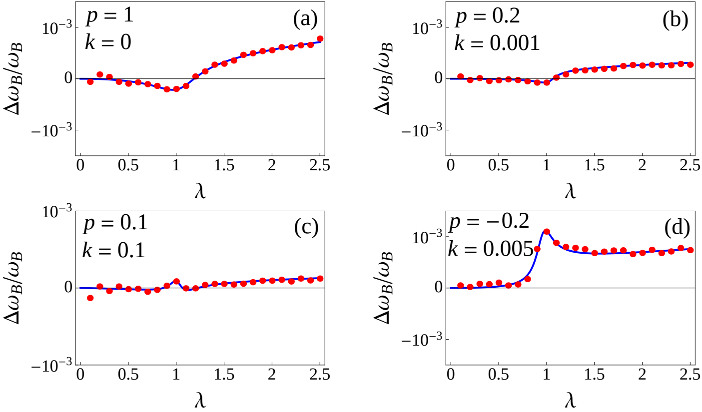


Fourier spectra of the nonlinear BEC dynamics for a repulsive two-body interaction $p = 1$, a repulsive three-body interaction $k = 0.001$, and $\varepsilon = 0.1$ for (a) $\lambda = 1.9$ and (b) $\lambda = 0.5$

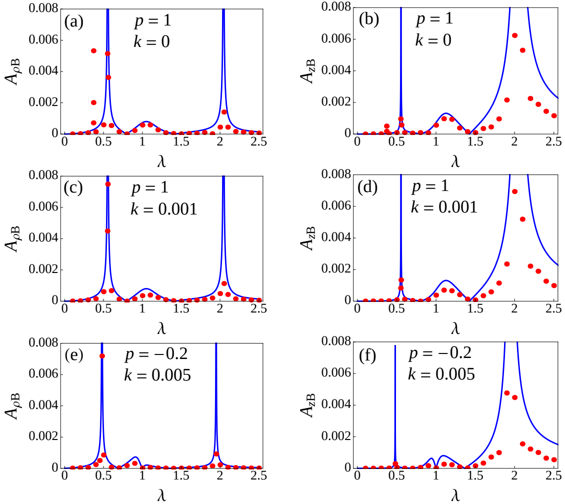
Quadrupole mode



Breathing mode



Resonant mode coupling



What are Faraday waves?

- M. Faraday, Philos. Trans. R. Soc. London **121**, 299 (1831):

  When the upper surface of a plate vibrating so as to produce sound is covered with a layer of water, the water usually presents a beautifully crispated appearance, the crispations being produced more readily and beautifully when there is a certain quantity than when there is less. For small crispations, the water should flow upon the surface freely. Large crispations require more water than small ones. Too much water sometimes interferes with the beauty of the appearance, but the crispation is not incompatible with much fluid, for the depth may amount to eight, ten, or twelve inches, and is probably unlimited.

  [crispation: curled condition; curliness; an undulation. (rare)]

- Faraday patterns became a standard topic in nonlinear physics due to experiments with liquids in the 1980s

Experimental observation

P. Engels, C. Atherton, M. A. Hoefer, PRL **98**, 095301 (2007)

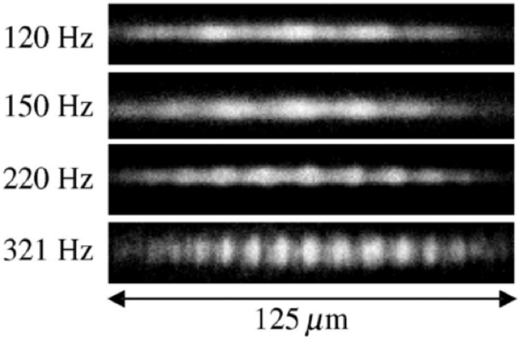


FIG. 1. In-trap absorption images of Faraday waves in a BEC. Frequency labels for each image represent the driving frequency at which the transverse trap confinement is modulated.

Two-component BEC systems

- Experimentally realized with a broad variety of types of atoms and parameters of a system
  - heterogeneous systems: different types of atoms
  - homogeneous systems: same type of atoms, different internal (usually spin) states
- Rich dynamics and interplay of the parameters
- Several possible ground states
- A variety of possible dynamical evolutions
- We focus on the study of Faraday waves and patterns in cigar-shaped two-component ^{87}Rb BECs, with strong radial confinement, which is harmonically modulated
- We also study resonant waves, which appear for specific values of the frequency of radial modulation
- Balaž, Nicolin, PRA **85**, 023613 (2012)

Mean-field description of a 2C BEC

- The system is described by a coupled system of GP equations:

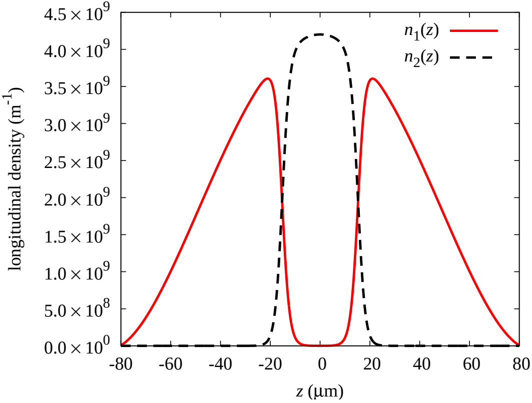
$$i\hbar \frac{\partial \Psi_1(\vec{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m_1} \triangle + V(\vec{r},t) + g_{11}|\Psi_1(\vec{r},t)|^2 + g_{12}|\Psi_2(\vec{r},t)|^2 \right] \Psi_1(\vec{r},t)$$

$$i\hbar \frac{\partial \Psi_2(\vec{r},t)}{\partial t} = \left[-\frac{\hbar^2}{2m_2} \triangle + V(\vec{r},t) + g_{21}|\Psi_1(\vec{r},t)|^2 + g_{22}|\Psi_2(\vec{r},t)|^2 \right] \Psi_2(\vec{r},t)$$

  where the couplings are given by:
  $g_{11} = \frac{4\pi\hbar^2 a_1}{m_1}$, $g_{22} = \frac{4\pi\hbar^2 a_2}{m_2}$, $g_{12} = g_{21} = \frac{2\pi\hbar^2 a_{\rm int}}{m_{\rm eff}}$
- Typical experimental values we consider for two hyperfine states of ^{87}Rb:

$$N_1 = 2.5 \cdot 10^5 \,, \quad N_2 = 1.25 \cdot 10^5$$
$$a_1 = 100.4\, a_0 \,, \quad a_2 = 98.98\, a_0 \,, \quad a_{\rm int} = 100.4\, a_0$$
$$\omega_\rho(t) = \omega_{\rho,0}(1 + \epsilon \sin \omega_{\rm m} t) \,, \quad \omega_{\rho,0} = 160 \cdot 2\pi \,{\rm Hz}$$
$$\omega_{\rm m} = 250 \cdot 2\pi \,{\rm Hz} \,, \quad \epsilon = 0.1 \,, \quad \omega_z = 7 \cdot 2\pi \,{\rm Hz}$$

Ground state - imaginary-time propagation



Density profile of the converged eigenstate obtained by propagation in the imaginary time. Discretization parameters: $N_\rho = N_z = 2000$, $\varepsilon = 10^{-4}/\omega_z$.

Ground state - experimental realization

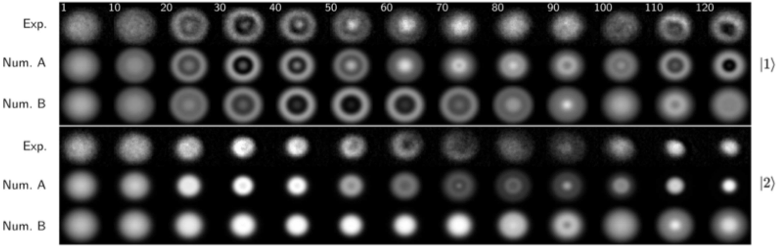
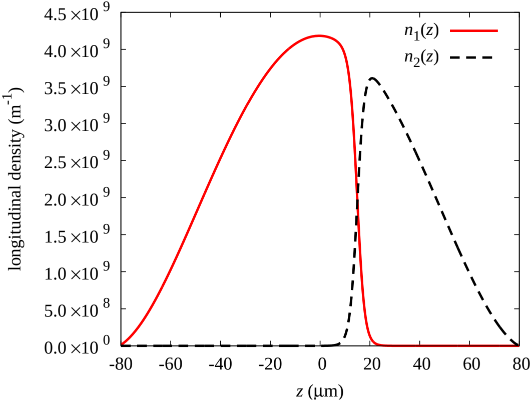


FIG. 1. Top view of a time sequence of experimental and numerical density profiles for $N = 3.50(5) \times 10^5$ ^{87}Rb atoms with equal populations in the $|1\rangle$ and $|2\rangle$ states. The first row shows the measured density profiles for the $|1\rangle$ atoms, while the second and third rows give numerical results including losses and different trap frequencies (Num. A) and without those additional model features (Num. B). A similar arrangement is given for the $|2\rangle$ atoms in the fourth, fifth, and sixth rows. The field of view in all pictures is approximately 100 μm on a side. The evolution time (in ms) for each column is indicated in the top row.

K. M. Mertes, J. W. Merrill, R. Carretero-González, D. J. Frantzeskakis, P. G. Kevrekidis, D. S. Hall, PRL **99**, 190402 (2007)

Segregated state - imaginary-time propagation



First excited eigenstate obtained by imaginary-time propagation.
Discretization parameters: $N_\rho = N_z = 2000$, $\varepsilon = 10^{-4}/\omega_z$.

First excited state - experimental realization

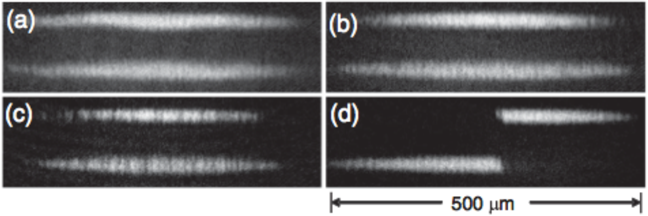
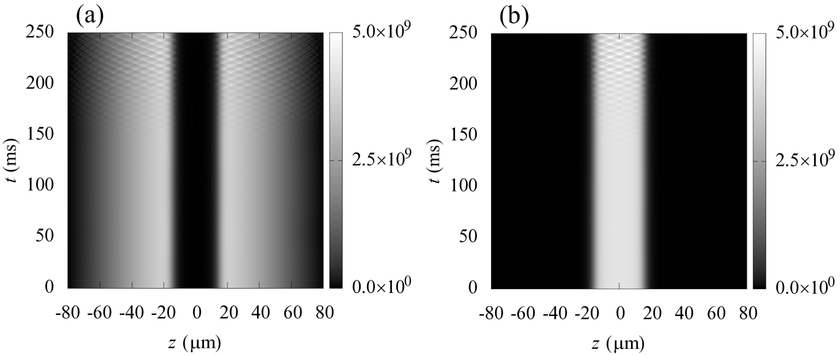


FIG. 1. Time evolution of an initial perfectly overlapped mixture without (a)–(c) and with (d) an applied axial magnetic gradient. Images taken after (a) 100 ms, (b) 1 sec, and (c)–(d) 9 sec of in-trap evolution.

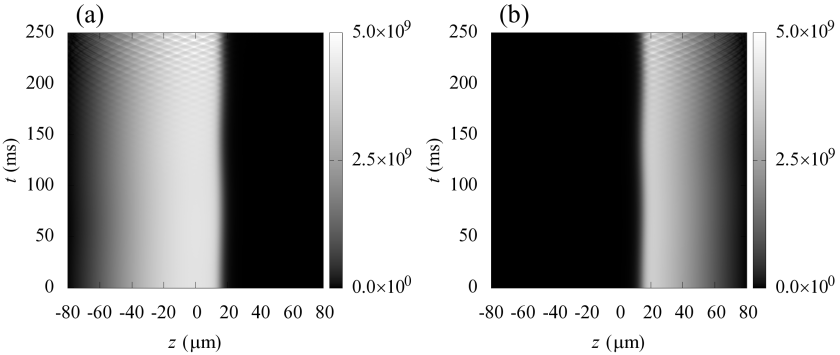
C. Hamner, J. J. Chang, P. Engels, M. A. Hoefer, PRL **106**, 065302 (2011)

Symbiotic pair ground state - Faraday waves



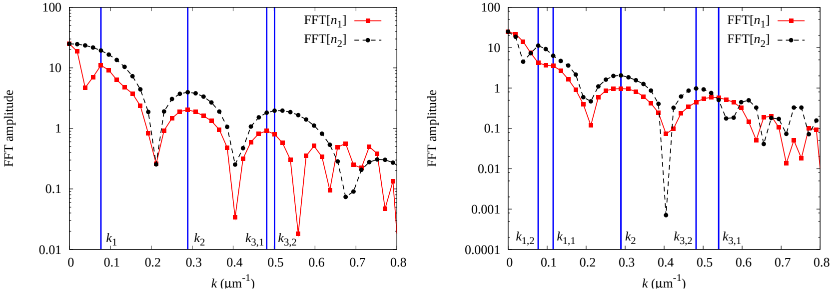
Emergence of Faraday waves as a result of real-time propagation. The radial frequency of the trap is modulated at the non-resonant frequency $\omega_m = 250 \cdot 2\pi$ Hz, $\epsilon = 0.1$.

Segregated state - Faraday waves



Emergence of Faraday waves as a result of real-time propagation. The radial frequency of the trap is modulated at the non-resonant frequency $\omega_m = 250 \cdot 2\pi$ Hz, $\epsilon = 0.1$.

Periods of Faraday waves



FFT of density profiles for the two condensates at $t = 200$ ms. For the symbiotic pair, the periods of waves are found to be 13.0 μm and 12.5 μm, while for the segregated state the periods are 11.6 μm and 13.0 μm.

Variational approach - symbiotic pair (1)

- Variational ansätze for the symbiotic pair wave functions:

$$\psi_1(\rho,z,t) = \mathcal{N}_1 \exp\left(-\frac{\rho^2}{2w_\rho^2(t)} + i\rho^2\alpha^2(t)\right)\left[1-\exp\left(-\frac{z^2}{2w_z^2}\right)\right]$$

$$\psi_2(\rho,z,t) = \mathcal{N}_2 \exp\left(-\frac{\rho^2}{2w_\rho^2(t)} - \frac{z^2}{2w_z^2} + i\rho^2\alpha^2(t)\right)\left[1+(u(t)+iv(t))\cos kz\right]$$

- ψ_1 is considered to be unperturbed, acting as an additional confinement for ψ_2
- Variational analysis leads to a Mathieu-type equation:

$$\ddot{u}(\tau) + u(\tau)\left[a(k,\omega) + \epsilon b\,(k,\omega)\sin 2\tau\right] = 0$$

$$a(k,\omega) = \frac{k^4}{\omega^2} + \frac{k^2}{\omega^2}\Lambda_{\rm sym}\,, \quad b(k,\omega) = \frac{k^2}{\omega^2}\Lambda_{\rm sym}\,, \quad \omega t = 2\tau$$

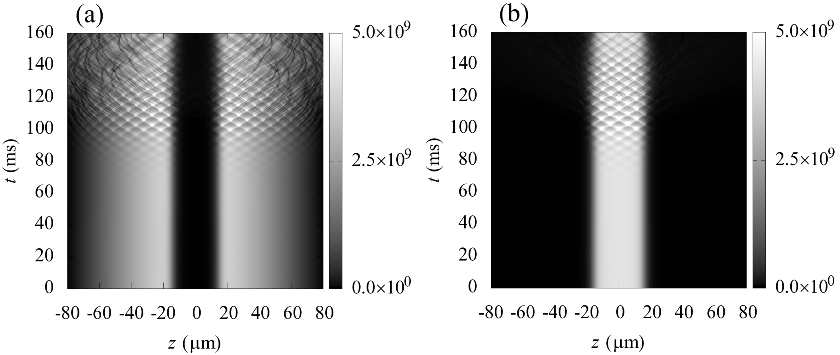
Variational approach - symbiotic pair (2)

- For small positive ϵ and positive $b(k,\omega)$, the Mathieu equation has solutions of the form $\exp(\pm i\mu\tau)\sin\sqrt{a}\tau$ and $\exp(\pm i\mu\tau)\cos\sqrt{a}\tau$, where $\mathrm{Im}[\mu]$ consists of a series of lobes positioned around the solution of the equation $a(k,\omega)=n^2$
- The lobe centered around $a(k,\omega)=1$ is the largest, and yields the most unstable solutions, determined by:

$$k_{\mathrm{F,sym}} = \sqrt{-\frac{\Lambda_{\mathrm{sym}}}{2} + \sqrt{\frac{\Lambda_{\mathrm{sym}}^2}{4} + \omega^2}}$$

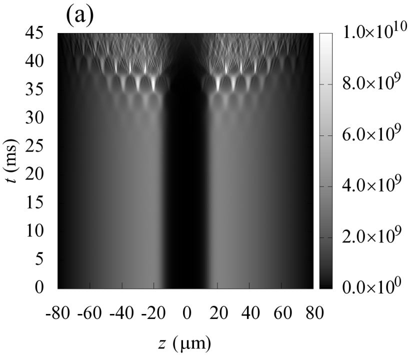
- This dispersion relation yields a period of $12.0\,\mu$m for the Faraday waves, which is in excellent agreement with the numerical results

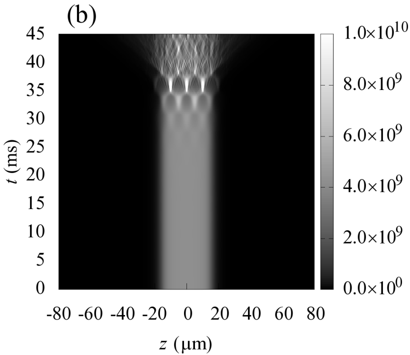
Self-resonance - symbiotic pair



Emergence of Faraday waves as a result of real-time propagation. The radial frequency of the trap is modulated at the resonant frequency $\omega_m = 160 \cdot 2\pi$ Hz, $\epsilon = 0.1$.

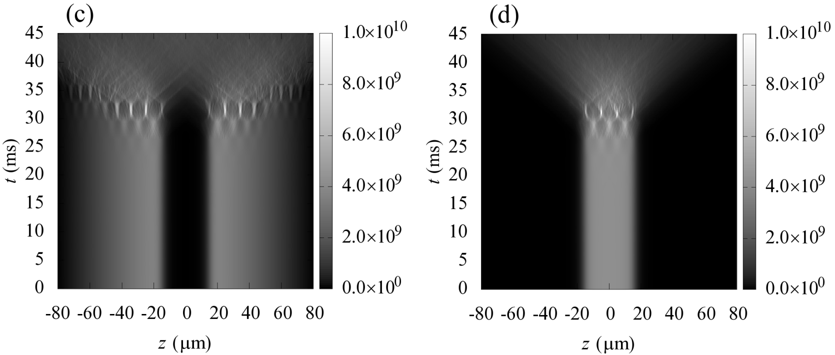
Second resonance - symbiotic pair (1)





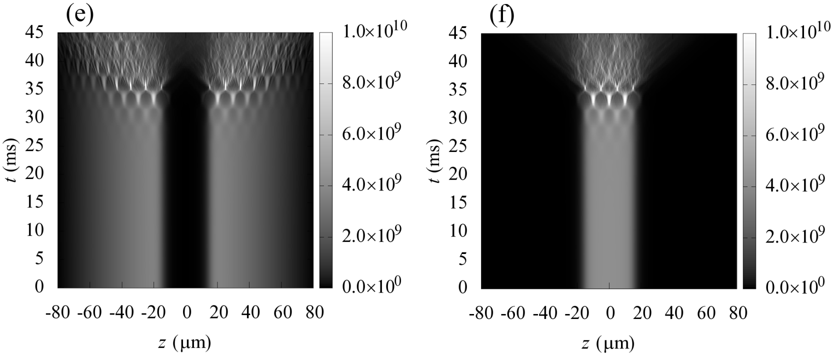
Emergence of resonant waves in a two-component BEC system radially modulated at $\omega_m = 300 \cdot 2\pi$ Hz.

Second resonance - symbiotic pair (2)



Emergence of resonant waves in a two-component BEC system radially modulated at $\omega_m = 320 \cdot 2\pi$ Hz.

Second resonance - symbiotic pair (3)



Emergence of resonant waves in a two-component BEC system radially modulated at $\omega_m = 340 \cdot 2\pi$ Hz.

Conclusions

- We have studied the dynamics and collective excitations of a BEC
  - when interaction is harmonically modulated, and
  - for different trap geometries (aspect ratios)

  and investigated prominent nonlinear effects that arise due to 2-body and 3-body interactions, and their delicate interplay
- We have numerically observed and analytically described
  - significant shifts in the frequencies of collective modes
  - generation of higher harmonics and linear combinations
  - resonant and non-resonant mode coupling
- We have studied the emergence of surface waves in 2C BECs
  - Non-resonant modulation of the radial confinement: Faraday waves with the similar period in both components
  - Self-resonant modulation: resonant waves
  - Second-harmonic resonance: much stronger and faster-emerging resonant waves, turning the non-miscible system miscible

Outlook

- Numerical and variational study of parametric stabilization with and without 3-body interaction
- Interplay of dynamically and geometrically induced resonances
  - Shifts in frequencies of collective modes
  - Resonant mode coupling
  - Suppression of collective modes
- Bistability in BECs with 2-body and 3-body interactions
- Study of miscible two-component systems
- Pancake-shaped two-component systems
- Faraday waves for two-component BEC loaded into an optical lattice