# Decontraction formula for $sl(n, \mathbb{R})$ algebras and applications in theory of gravity *

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**Abstract**

Special linear group $SL(n, \mathbb{R})$, as well as its covering group $\overline{SL}(n, \mathbb{R})$ in quantum domain, appear as relevant symmetry groups in many physical models based on spacetime symmetries. Applications of these symmetries and their representations in physics problems require knowledge of the $sl(n, \mathbb{R})$ algebra representations. Spinorial $sl(n, \mathbb{R})$ representations are of particular importance in various problems of quantum field theory, quantum and alternative theories of gravity, and theories of extended objects (strings, branes, etc.). Construction of the unitary and spinorial representations of the $sl(n, \mathbb{R})$ algebras is further involved by the fact that these representations are necessarily infinite dimensional. Moreover, transformation properties of physical entities, as well as their correct physical interpretation, require knowledge of the relevant $sl(n, \mathbb{R})$ algebra and $\overline{SL}(n, \mathbb{R})$ group representations in the basis of the orthogonal subgroup $Spin(n)$.

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The method used to derive expressions of the $\overline{SL}(n,\mathbb{R})$ generators is based on the so called decontraction, also known as the Gell-Mann, formula that is in the focus of this work. This formula, in our case, determines the $sl(n,\mathbb{R})$ algebra elements in terms of the algebra elements obtained by the Inönü-Wigner algebra contraction with respect to the $so(n)$ subalgebra. It is shown that this formula is valid only for some particular classes of irreducible representations that are insufficient for applications in physical models.

Next we demonstrate how the Gell-Mann formula can be generalized. The obtained generalized formula is valid for all $sl(n,\mathbb{R})$ irreducible representations: finite and infinite, unitary and non unitary, tensorial and spinorial. All expressions of the matrix elements of the $\overline{SL}(n,\mathbb{R})$, $n \geq 2$, generators are obtained explicitly by making use of the generalized decontraction formula. They are given in a closed form, in terms of the Hilbert space functions over the $Spin(n)$ subgroup, for an arbitrary irreducible representation characterized by the corresponding set of labels. This result provides, due to $sl(n,\mathbb{R})$ and $su(n)$ algebras relation, expressions of the matrix elements of the $SU(n)$ generators in the $SO(n)$ basis for all irreducible representations. An example that illustrate applications of the generalized decontraction formula in models of alternative theories of gravity based on a local affine symmetry is also presented.

# 1 Introduction

Special linear group over the field of real numbers $SL(n,\mathbb{R})$ is defined as a set of unit determinant $n \times n$ real matrices, equipped with usual matrix multiplication and inversion operations. In modern physics, this group appears in many different context. It appears either independently, or as the essential part of the general linear group $GL(n,\mathbb{R})$ – as it is well known that basically all mathematical problems related to the general linear group reduce to the corresponding problems for the case of the special linear subgroup: representations, topology, Clebsch-Gordan coefficients... Accordingly, special linear group also plays an important role in a number of contemporary attempts to solve existing problems in the theory of gravity.

In the first place, it is the case for the affine theories of gravity, both metric affine [1, 2] and gauge affine [3, 4]. In this context, special linear group plays the role that the Lorentz group has within the standard, Poincaré based theory of gravity. Flat space-time symmetry here is $R_n \wedge GL(n, \mathbb{R})$, i.e. semidirect product of translations in $n$ dimensions with a subgroup of all homogenous linear transformations of these $n$ dimensions. Already at this level understanding of the particle content of the models requires knowledge of the special linear group representations (which are the key step for finding representations of the entire affine symmetry). The gravitational interaction is introduced by localization of the affine symmetry, while the necessary symmetry breaking can be induced in various ways. Matrix elements of $SL(n, \mathbb{R})$ group and of the $SL(n, \mathbb{R})$ double covering group $\overline{SL}(n, \mathbb{R})$ appear in the interaction vertex terms. The detailed knowledge of the special linear group representations is particularly needed for the construction of concrete symmetry breaking scenarios [5].

Knowledge of representations of the special linear group, in particular of the, so called, spinorial representations is needed already in the analysis of the classical Einstein's theory of gravity. Namely, general linear group is the homogenous part of the diffeomorphism group $Diff(n, \mathbb{R})$ and the (world) tensors in the general theory of gravity transform according to the corresponding representations of exactly $GL(n, \mathbb{R})$. Therefore, the most straightforward and natural way to introduce spinorial matter in general relativity would be via fields that transform with respect to spinorial representations of the general linear group (so called *world spinors*) [6, 7]. Spinorial representations are those which, after symmetry reduction to the (pseudo)orthogonal subgroup of the general linear group (one corresponding to physical rotations), decompose into spinorial representations of the orthogonal group. The very existence of the spinorial representaions of the general linear, i.e. of the special linear group has been long unknown (up to eighties, [6]) due to the fact that all such representations are necessarily infinite dimensional. This fact certainly complicates finding and application of this type of representations, but does not diminish its physical relevance.

In last decade or two, general linear group appears in the papers also as an important subgroup in supersymmetry models with, so called, tensorial central charges [8, 9, 10, 11, 12, 13, 14, 15, 16]. In these models super-Poincaré symmetry in $n$ dimensions is extended by adding of $n(n-1)/2$

generalized momenta ("tensorial charges") to the set of $n$ standard generators of the space-time translations. Altogether they comprise a set of $n(n+1)/2$ operators, transforming as a second order symmetric tensor w.r.t. (with respect to) the general linear group $GL(n,\mathbb{R})$ that here extends and replaces the Lorentz group.[1] Such a generalization of the Poincaré superalgebra is additionally important as it corresponds to the symmetry of the M-theory [17, 18, 19, 20, 21, 22], so that it is also called M-algebra. Gravitational interaction ca be introduced in these models also by localization of $GL(n,\mathbb{R})$ symmetry. Thus the importance of detailed knowledge of representations of general (special) linear group in this context is also clear.

Special linear group corresponds to volume conserving transformations (i.e. area conserving in the two dimensional case). This makes the group relevant also in all cases where dynamics of the system is such that some volume (area) is conserved. Such situations occur in the context of strings and branes [23, 24].

To summarize, knowledge of $SL(n,\mathbb{R})$ group/$sl(n,\mathbb{R})$ algebra and its representations is of extreme importance in theory of gravity and theories of extended objects (in these examples it is usually enough to know representations of $sl(n,\mathbb{R})$ algebra, i.e. of the generators of $SL(n,\mathbb{R})$ group). These representations, as well as the context of their physical applications, have many specific properties.

First of all, in physics we are often interested in unitary representations, and since we are here dealing with a noncompact group, it is well known that all such representations are infinite dimensional. Next, $SL(n,\mathbb{R})$ group has its double cover group $\overline{SL}(n,\mathbb{R})$. This is most easily seen via Iwasawa decomposition $SL(n,\mathbb{R}) = NAK$: nilpotent ($N$) and abelian ($A$) subgroups are simply connected, so the covering of the entire group is determined by the covering of the maximal compact subgroup $K$ – in this case $SO(n)$ that, as it is well known, has double cover $Spin(n) \equiv \overline{SO}(n)$. If we are interested in models that include fermionic matter, the representations of the double covering group $\overline{SL}(n,\mathbb{R})$ (that is $\overline{GL}(n,\mathbb{R})$) are of the utmost

[1] We note that this symmetry significantly differs from the affine symmetry even in the bosonic part, since the abelian subgroup generators, i.e. generalized momenta, transform as symmetric tensors of the second order in one case and as the $n$-dimensional vector representations of $GL(n,\mathbb{R})$ in the second case.

interest, since among these representations are those that decompose into spinorial representations of the (pseudo)orthogonal subgroups. Additional mathematical difficulty is the fact that all these spinorial representations are infinite dimensional, irrespectively of their unitarity properties [25, 6, 26].

Physical context also determines the basis of the representations space: to differentiate between fields of different spins, we usually need to know form of the symmetry generators in basis of the (pseudo)orthogonal subgroup $SO(m, n-m)$ ( $m$ depends on the signature of residual symmetry metrics in the model). This requirement reduces the number of available mathematical methods for finding of representations: for example, standard "canonical" approach (induction from maximal parabolic subgroup [27]) would provide representations in a basis of Cartan subalgebra weight vectors, and subsequent change of (infinite dimensional) basis is a difficult task. Additional problem is due to the fact that representations of the special linear group in general have nontrivial multiplicity with respect to the decomposition into (pseudo)orthogonal representations. Therefore, in this basis it is also necessary to take care of the multiplicity label.

Finally, dimension of the physical space varies from one model to another: (Kaluza-Klein theories, strings, branes) so that a generic (for arbitrary $n$) result in a closed form is highly preferred.

The listed technical requirements make the problem of finding the special linear group representations in this context very difficult. One possible solution to this problem, that satisfies all the above criteria, is considered in this paper. The solution is based on the generalization of the, so called, Gell-Mann (decontraction) formula.

Gell-Mann decontraction formula [28, 29, 30, 31, 32, 33] is a transformation aimed to serve as an "inverse" to the Inönü-Wigner contraction [34]. More precisely, while the Inönü-Wigner contraction is a singular transformation, more concretely a limiting procedure, that yields "contracted algebra" operators from the operators of the original algebra, the goal of the Gell-Mann formula is to provide a way to express the operators of the starting "non-contracted" algebra as functions of the contracted algebra elements. The concrete expression of the Gell-Mann formula will be written in the next section.

We are here interested in the case of $SL(n, \mathbb{R})$ group. In this context, important is its contraction contraction w.r.t. its maximal compact subgroup $SO(n)$. This procedure takes $sl(n,\mathbb{R})$ algebra into a semidirect sum of abelian subalgebra of generalized translations and a special orthogonal algebra: $r_{n(n+1)/2-1} \uplus so(n)$. Representations of this contracted group/algebra are much easier to find than the representations of the starting group/algebra (especially since the representations should be given in an $SO(n)$ adapted basis). Therefore, one approach to obtain representations of $sl(n,\mathbb{R})$ would be to convert, using the Gell-Mann formula, representations of contracted $r_{n(n+1)/2-1} \uplus so(n)$ algebra into representations of $sl(n,\mathbb{R})$.

The problem with this approach comes from the fact that the Gell-Mann formula is actually only a prescription that is not valid always, i.e. not for all algebras and all their representations. Moreover, this formula is entirely valid – i.e. as an algebraic identity – only in the case of (pseudo)orthogonal algebras, that is, for contractions $so(m+1,n) \rightarrow r_{m+n} \uplus so(m,n)$ $so(m,n+1) \rightarrow r_{m+n} \uplus so(m,n)$ [35, 36]. In other cases, including the $sl(n,\mathbb{R})$ algebra case, the Gell-Mann formula is valid only for a certain subset of representations (the validity conditions will be the subject of the third section). Thus, by using the Gell-Mann formula we can obtain only some of the $sl(n,\mathbb{R})$ representations, amongst whom, for example, there are neither spinorial nor representations with multiplicity.

On the other hand, in the $sl(n,\mathbb{R})$ case, it is possible to generalize Gell-Mann formula so to broaden its domain of applicability to all representations, including both unitary and nonunitary, both multiplicity free and with multiplicity, both tensorial and spinorial. As a direct mathematical application of the generalized formula, a closed expression for matrix elements of $SL(n,\mathbb{R})$ generators can be given – for an arbitrary (irreducible) representation, for arbitrary $n$, in the basis of the orthogonal subgroup. Due to the close relation of $sl(n,\mathbb{R})$ and $su(n)$ algebra, the same can be done also in the case of special unitary group/algebra.

This paper is organized as follows: the subjects of the next section are Inönü-Wigner contraction and the original form of the Gell-Mann decontraction formula; in the third section we will concentrate on the particular case of $sl(n,\mathbb{R})$ algebras and discuss the domain of validity of the formula for these algebras; the fourth section deals with the generalization of the formula; the fifth section contains discussion of the applicability of the formula in the context of affine gravity models; sixth section contains a summary of the paper; Appendix contains Clebsch-Gordan coefficients of the $SO(5)$ group necessary for finding explicit matrix values in the considered five dimensional case.

# 2 Inönü-Wigner contraction and the Gell-Mann decontraction formula

## 2.1 Contraction

Inönü and Wigner have long ago introduced the notion of algebra contraction, in order to mathematically describe transition from the relativistic Poincare symmetry to non-relativistic Galilei symmetry in the limit of infinite velocity of light [34]. At the basic of the contraction idea is the observation that change of basis $A_i$ of algebra $\mathcal{A}$:

$$A_i \to A'_i = X_i^{\ j} A_j \tag{1}$$

can transform algebra $\mathcal{A}$ into a non-isomorphic algebra $\mathcal{A}'$ if the transformation coefficients $X_i^{\ j}$ are singular. This type of transformation is called Inönü-Wigner contraction if the singular transition coefficients can be obtained as a limit, when some parameter $\epsilon$ approaches zero, of otherwise non-singular transformation coefficients linear in $\epsilon$: $X_i^{\ j} = X_i^{\ j}(\epsilon)$. In such a case, new structural constants of algebra $\mathcal{A}'$ have well defined limit if and only if algebra $\mathcal{A}$ contains a subalgebra $\mathcal{M}$ with respect to which the contraction is done in the following way:

$$\mathcal{M} \to \mathcal{M}' = \mathcal{M}, \qquad \mathcal{T} \to \mathcal{T}' = \epsilon \mathcal{T}$$

where $\mathcal{A} = \mathcal{M} + \mathcal{T}$ and $\mathcal{A}' = \mathcal{M}' + \mathcal{T}'$. We say that the algebra was contracted with respect to subalgebra $\mathcal{M}$ (that remained unaltered), and the elements $\mathcal{T}'$ we call "contracted". Contracted elements form an abelian ideal $\mathcal{T}'$ of algebra $\mathcal{A}'$, since in the limit $\epsilon \to 0$ it holds $[T'_i, T'_j] = \epsilon^2 C^a_{ij} M'_a + \epsilon C^k_{ij} T'_k = 0$, where $M'_a \in \mathcal{M}'$, $T'_i \in \mathcal{T}'$, and $C^i_{jk}$ are structural constants of algebra $\mathcal{A}$.

When the limit of structural constants is well defined, some other properties of the contracted algebra can be also found as a limit of the properties of the starting algebra – eg. group parameters, matrix elements of the operators, representation space basis vectors, Casimir operators. [34].

In this way, simultaneous Inönü-Wigner contraction of spatial momenta and boost generators transforms Poincare algebra into Galilei one (that is, by contraction w.r.t. subgroup generated by spatial rotations and time translation). Another example of Inönü-Wigner contraction is a transformation of three dimensional rotation algebra into Euclidean algebra in two dimension, or contraction of (anti)de Sitter algebra into Poincare algebra (by contraction of four generalized rotations into four-momenta: $P_\mu = \epsilon M_{4\mu}$, where $M_{\mu\nu} \in so(3,2)$ or $M_{\mu\nu} \in so(4,1)$).

In the case of $sl(n,\mathbb{R})$ algebras we are interested in the contraction w.r.t. the maximal compact subalgebra $so(n)$. Algebra $sl(n,\mathbb{R})$ contains $n(n-1)/2$ elements of rotational subalgebra $M_{ab} \in \mathcal{M} = so(n)$, $a,b = 1,2,...,n$ (corresponding to antisymmetric real matrices , $M_{ab} = -M_{ba}$) and $n(n+1)/2-1$ noncompact generators $T_{ab} \in \mathcal{T}$, $a,b = 1,2,...,n$ (corresponding to traceless symmetric real matrices $T_{ab} = T_{ba}$). In the context of space-time symmetries and deformations of rigid bodies, the latter are known as shear generators.

Structural relations of the special linear algebra, in Cartesian basis, are:

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}), && (2)\\ [M_{ab}, T_{cd}] &= i(\delta_{ac}T_{bd} + \delta_{ad}T_{cb} - \delta_{bc}T_{ad} - \delta_{bd}T_{ca}), && (3)\\ [T_{ab}, T_{cd}] &= i(\delta_{ac}M_{db} + \delta_{ad}M_{cb} + \delta_{bc}M_{da} + \delta_{bd}M_{ca}). && (4) \end{aligned}$$

Inönü-Wigner contraction w.r.t. the maximal compact subgroup is given by the following limit:

$$U_{ab} \equiv \lim_{\epsilon\to 0}(\epsilon T_{ab}). \tag{5}$$

Relations of the contracted algebra are:

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}) && (6)\\ [M_{ab}, U_{cd}] &= i(\delta_{ac}U_{bd} + \delta_{ad}U_{cb} - \delta_{bc}U_{ad} - \delta_{bd}U_{ca}) && (7)\\ [U_{ab}, U_{cd}] &= 0. && (8) \end{aligned}$$

(Above we used the notation $U$ instead of $T'$ for the contracted elements, to avoid excessive use of prime symbols.)

The connection of the two algebras, established by this contraction procedure, can be used to obtain certain classes of representations of the contracted algebra from the known representations of the starting algebra. However, more often it would be of a greater practical merit to establish the opposite type of relation, which is the subject of the next subsection.

## 2.2 Decontraction

As we saw, the Inönü-Wigner contraction of the $sl(n,\mathbb{R})$ algebra yields a semidirect sum $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$, where $r_{\frac{n(n+1)}{2}-1}$ is abelian subalgebra (ideal) of "translations" in $\frac{n(n+1)}{2}-1$ dimensions. If we knew representations of the special linear algebra, by contraction procedure we could obtain certain classes of representations of this semidirect sum algebra. However, in this case (and in most of the others, as the matter in fact) this is of not much practical use for finding representations. Namely, it is here much more easy to find, by using direct methods of representation theory, representations of the contracted algebra than of the starting special linear one. Therefore, of a great utility would be a method that would allow us the opposite: to get representations of the $sl(n,\mathbb{R})$ starting from known representations of the contracted semidirect sum. It this sense, instead of the limit (5) that expresses elements of the contracted algebra as functions of the starting one, it would be good to have expressions for the elements of the $sl(n,\mathbb{R})$ as functions of $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ operators.

An attempt to establish this type of connection resulted in the Gell-Mann formula [31, 32, 33, 37]. This formula, in its basic form, first time appeared in a paper of Dothan and Ne'eman, back in 1966 [28], and was known as the "decontraction" formula at the time [29, 30]. The formula was largely advocated by Hermann [33, 32], who, on the other hand, had learnt about it from Gell-Mann. Not knowing the details of its genesis, he referred to it as the "Gell-Mann formula". Under this latter name the formula is nowadays known in some textbooks [32] and even in a mathematical encyclopedia [31].

As it traveled a long road since its birth, this formula now appears in a few variants and forms. First we give a definition close to the one given in the encyclopedia [31].

Let $\mathcal{A}$ be a symmetric Lie algebra $\mathcal{A} = \mathcal{M} + \mathcal{T}$ with subalgebra $\mathcal{M}$, so that it holds:

$$[\mathcal{M},\mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M},\mathcal{T}] \subset \mathcal{T}, \quad [\mathcal{T},\mathcal{T}] \subset \mathcal{M}. \tag{9}$$

Let $\mathcal{A}'$ be its Inönü-Wigner contraction w.r.t. the subalgebra $\mathcal{M}$. Then $\mathcal{A}' = \mathcal{M}' + \mathcal{U}$ and exists an isomorphism of vector spaces $\pi : \mathcal{A} \to \mathcal{A}'$, given by the Inönü-Wigner contraction, such that $\pi(\mathcal{M}) = \mathcal{M}'$, $\pi(\mathcal{T}) = \mathcal{U}$, $[\pi(M), \pi(A)] = \pi([M, A])$ and $[\pi(T_1), \pi(T_2)] = 0$ where $M \in \mathcal{M}$, $A \in \mathcal{A}$ $T_1, T_2 \in \mathcal{T}$. Let $U^2$ denote quadratic element of the enveloping algebra of subalgebra $\mathcal{U}$ that is invariant w.r.t. action of subalgebra $\mathcal{M}'$. If $D'$ is a representation of $\mathcal{A}'$ such that $D'(U^2)$ is a multiple of the unit operator, then the Gell-Mann formula for the representations $D$ of algebra $\mathcal{A}$ is:

$$D(T) = \alpha[D'(C_2), D'(\pi(T))] + \sigma D'(\pi(T)), \qquad D(M) = D'(\pi(M)), \quad (10)$$

where $T \in \mathcal{T}, M \in \mathcal{M}$, $C_2$ is a second order Casimir operator of the enveloping algebra of $\mathcal{M}'$, $\alpha$ is a constant dependant upon $D'(U^2)$ and $\sigma$ is an arbitrary parameter.

In a mathematically less rigorous way, but closer to the original formulation, the formula can be written as the following operator expression:

$$T_\mu = i\frac{\alpha'}{\sqrt{U^2}}[C_2(\mathcal{M}), U_\mu] + i\sigma U_\mu, \quad (11)$$

where $T_\mu \in \mathcal{T}$, $U_\mu \in \mathcal{U}$ and we assume, just as in the above definition, that the algebra $\mathcal{A} = \mathcal{M} + \mathcal{T}$ is Inönü-Wigner contracted into $\mathcal{A}' = \mathcal{M} + \mathcal{U}$, with $T_\mu \to U_\mu$. $C_2(\mathcal{M})$ is quadratic Casimir of the algebra $\mathcal{M}$, $\alpha'$ is a constant, and $\sigma$ is an arbitrary parameter. By writing the square root of $U^2$ ($U^2$ is defined above) as a normalization in the denominator we cancel the dependance of $\alpha$ on $U^2$, that was present in the formulation (10). In this way we can frite the formula, at least formally, as an operator expresson, unlike the relation (10) that is given at the representations level. This form makes apparent the goal of the formula: to express the operators of the starting algebra as functions of contracted algebra elements.

It is of great importance to establish the domain of validity of the formulas (10) and (11). There is a number of papers on this subject [33, 32, 35, 36]. It is known that the formula is valid for almost all representations in the case of contractions $so(m+1, n) \to r_{m+n} \uplus so(m, n)$ and $so(m, n+1) \to r_{m+n} \uplus so(m, n)$ (problems exist only with representations where $U^2$ is represented as 0) [35, 36]. Namely, in these cases the relation (11) can be checked to satisfy proper commutation relations.

For example, let the operators $M_{ab}, a, b = 1, 2, ..., n$ satisfy $so(n)$ algebra commutation relation and let define contraction w.r.t. $so(n-1)$ sublagebra as a limit of transformation $P_i = \epsilon M_{ni}, i = 1, 2, ..., n-1$. Contracted algebra satisfies:

$$\begin{aligned} [M_{ij}, M_{kl}] &= i(\delta_{ik} M_{jl} + \delta_{il} M_{jk} - \delta_{jk} M_{il} - \delta_{jl} M_{ki}), && (12) \\ [M_{ij}, P_k] &= i(\delta_{ik} P_j - \delta_{jk} P_i) && (13) \\ [P_i, P_j] &= 0. && (14) \end{aligned}$$

We can explicitly check that the Gell-Mann formula will, in this case, be indeed inverse to the contraction: operators $\overline{M}_{ni}$, *defined* as in (11) as:

$$\overline{M}_{ni} = \frac{i}{2\sqrt{\sum_{j=1}^{n-1}(P_j)^2}}[\tfrac{1}{2}\sum_{j,k=1}^{n-1}(M_{jk})^2, P_i] + i\sigma P_i, \qquad i = 1, 2, ..., n-1, \quad (15)$$

together with subalgebra elements $M_{ij}, i, j = 1, 2, ..., n-1$ will again satisfy structural relations of $so(n)$ algebra (we had to fix the value of constant $\alpha' = \frac{1}{2}$). As the matter in fact, since $P_i$ obviously transform according the the vector representation of the $so(n-1)$ subalgebra, it remains to check:

$$\begin{aligned} [\overline{M}_{ni}, \overline{M}_{nj}] &= -\tfrac{i}{16\sqrt{P^2}}[\{M^{kl}, \delta_{ik} P_l - \delta_{il} P_k\}, \{M^{k'l'}, \delta_{ik'} P_{l'} - \delta_{il'} P_{k'}\}] \\ &= \cdots = \tfrac{i}{\sqrt{P^2}} P_k P_l \delta^{kl} M_{ij} \stackrel{P^2 \neq 0}{=} i M_{ij}. \qquad (16) \end{aligned}$$

In the above expressions we implied summation convention and Euclidian metrics $\delta_{ij}$ with respect to the first $n-1$ coordinates and the curly brackets denote anticommutator. Therefore, we see that in the case of this algebra, the Gell-Mann formula is completely valid, that is, as an algebraic identity (apart from the special case of contracted algebra representations satisfying $P^2 = 0$, when the very formula expression is ill defined).

Unfortunately, the (pseudo)orthogonal algebras are also the only class of of algebras where the Gell-Mann formula is valid in such, algebraic sense. For example, we can try to apply the same Gell-Mann prescription in the $sl(n, \mathbb{R})$ case. In that case, the Gell-Mann formula tells us to look for the sheer generators as the following functions of the contracted algebra elements (6)-(8):

$$T_{ab} = \frac{i\alpha}{\sqrt{\sum(U_{cd})^2}}[C_2(so(n)), U_{ab}] + \sigma U_{ab}. \qquad (17)$$

For the sake of later comparison, we mention that the same expression can be also written as:

$$T_{ab} = -\frac{2\alpha}{\sqrt{\sum (U_{cd})^2}} \sum_c U_{c\{a} M_{b\}c} + \sigma' U_{ab}, \tag{18}$$

where $\sigma'$ differs from $\sigma$, and $\{\ \}$ denotes antisymmetrization of the indices in the bracket.

However, if we calculate commutators of so defined shear generators, it will turn out that they do not satisfy $sl(n, \mathbb{R})$ commutation relation (4), more precisely, that additional terms appear on the righthand side. These additional terms are, in general, nonzero, rendering the formula inapplicable. Only in certain representations of the contracted algebra these terms vanish, and for that subclass of representations of the contracted algebra the Gell-Mann formula is valid, resulting in the corresponding subset of representations of the special linear algebra.

The situation is similar in the case of other algebras and their contractions – although the formula is not satisfied algebraically, it can still be valid for a certain subclass of representations. A partial answer to the question of what subclasses these precisely are was given by Hermann in [33] and [32]. However, he did not even attempt to give the complete answer, concentrating, as he said, to "what seems to be the simplest case" and ignoring the cases when the little group (in Wigner's terminology) is nontrivially represented.

On the other hand, this question (for the case of $sl(n, \mathbb{R})$ algebras) is of extreme importance for us, since in the cases when the formula is applicable we have an extremely simple and convenient expression for representation of operators of the special linear algebra. Therefore, the conditions for validity of the Gell-Mann formula in the $sl(n, \mathbb{R})$ case will be discussed in the next section.

# 3 Domain of validity of the Gell-Mann formula for $sl(n, \mathbb{R})$

## 3.1 Mathematical framework

In order to make use of the Gell-Mann formula to obtain the $sl(n, \mathbb{R})$ representations, the first necessary step is to determine representation matrix elements of the contracted algebra operators. The corresponding contracted group is a semidirect product of $SO(n)$ and an Abelian group, and it is well known that the usual group induction method provides the complete set of all inequivalent irreducible representations [38]. Nevertheless, we will not pursue the induction approach here. Instead, we will rather proceed to work in the representation space of square integrable functions $\mathcal{L}^2(Spin(n))$ over the $Spin(n)$ group (in accord with the $SL(n, \mathbb{R})$ topological properties), with the standard invariant Haar measure. As for our final goal, this approach ensures certain advantages: (i) The generalized Gell-Mann formula is expressed in terms of tensor operators w.r.t. the maximal compact subgroup basis (instead w.r.t. the eigenvector basis of the Abelian subgroup), (ii) This representation space contains all inequivalent irreducible representations of the contracted group (some of the irreducible representations are multiply contained, i.e. each such representation appears as many times as is the dimension of the corresponding little group representation and all of them, irrespectively of the corresponding stabilizer, can be treated in an unified manner), and (iii) this space is rich enough to contain all representatives from equivalence classes of the $\overline{SL}(n, \mathbb{R})$ group, i.e. $sl(n, \mathbb{R})$ algebra representations [40]. The last feature provides the necessary requirement of a framework needed for generalization of the Gell-Mann formula, i.e. a unique framework providing for all $sl(n, \mathbb{R})$ (unitary) irreducible representations.

The generators of the contracted group are generically represented in this space as follows.

Space $\mathcal{L}^2(Spin(n))$ is the space of the vectors:

$$|\phi\rangle = \int_{Spin(n)} \phi(g)\, |g\rangle\, dg, \qquad g \in Spin(n), \tag{19}$$

where $\phi(g)$ denotes a square integrable function on the $Spin(n)$ group, $|g\rangle$ are (generalized) basis vectors of group elements, and $dg$ is a (normalized) Haar measure.

Operators of $so(n)$ subalgebra act on these vectors in a natural way:

$$M_{ab}\,|\phi\rangle = -i\frac{d}{dt}\exp(itM_{ab})\Big|_{t=0}\,|\phi\rangle\,,$$

where the action of element $g'$ of the $Spin(n)$ group on an arbitrary vector $|\phi\rangle \in \mathcal{L}^2(Spin(n))$ is determined by right group action on basis vectors $|g\rangle$ of this space:

$$g'\,|\phi\rangle = g'\int \phi(g)\,|g\rangle\,dg = \int \phi(g)\,|g'g\rangle\,dg, \qquad g', g \in Spin(n). \tag{20}$$

The abelian operators $U_{ab}$ (5) of the contracted algebra in this basis act multiplicatively as Wigner's $D$-functions ($SO(n)$ group matrix elements as functions of the group parameters):

$$U_{ab} \to |u| D^{\square\!\square}_{w(ab)}(g^{-1}) \equiv |u| \left\langle \begin{array}{c}\square\!\square\\ w\end{array} \middle| (D^{\square\!\square}(g))^{-1} \middle| \begin{array}{c}\square\!\square\\ ab\end{array} \right\rangle, \tag{21}$$

$|u|$ being a constant norm, $g$ being an $SO(n)$ element, and in order to simplify notation we denote by $\square\!\square$ (in a parallel to the Young tableaux) the symmetric second rank tensor representation of $SO(n)$. The vector $\left| \begin{array}{c}\square\!\square\\ ab\end{array} \right\rangle$ from the $\square\!\square$ representation space is determined by the $ab$ "double" index of $U_{ab}$, whereas the vector $\left| \begin{array}{c}\square\!\square\\ v\end{array} \right\rangle$ can be an arbitrary vector belonging to the $\frac{1}{2}n(n+1)-1$ dimensional $\square\!\square$ representation (the choice of $v$ is determined, in Wigner's terminology, by the little group of the obtained representation). Taking an inverse of $g$ in (21) ensures the correct transformation properties. The form of the representation of the Abelian operators merely reflects the fact that they transform as symmetric second rank tensor w.r.t $so(n)$ (7) and that they mutually commute.

A natural discrete orthonormal basis in the $\mathcal{L}^2(Spin(n))$ representation space is given by properly normalized Wigner $D$-functions:

$$\left\{ \left| \begin{array}{c} \{J\} \\ \{k\}\{m\} \end{array} \right\rangle \equiv \int \sqrt{dim(\{J\})} D^{\{J\}}_{\{k\}\{m\}}(g^{-1}) dg \, |g\rangle \right\}, \tag{22}$$

$$\left\langle \begin{array}{c} \{J'\} \\ \{k'\}\{m'\} \end{array} \middle| \begin{array}{c} \{J\} \\ \{k\}\{m\} \end{array} \right\rangle = \delta_{\{J'\}\{J\}} \delta_{\{k'\}\{k\}} \delta_{\{m'\}\{m\}},$$

where $D^{\{J\}}_{\{k\}\{m\}}$ are matrix elements of $Spin(n)$ irreducible representations:

$$D^{\{J\}}_{\{k\}\{m\}}(g) \equiv \left\langle \begin{array}{c} \{J\} \\ \{k\} \end{array} \middle| D^{\{J\}}(g) \middle| \begin{array}{c} \{J\} \\ \{m\} \end{array} \right\rangle. \tag{23}$$

Here, $\{J\}$ stands for a set of the $Spin(n)$ irreducible representation labels, while $\{k\}$ and $\{m\}$ labels enumerate the $dim(D^{\{J\}})$ representation basis vectors.

An action of the $so(n)$ operators in this basis is well known, and it can be written in terms of the Clebsch-Gordan coefficients of the $Spin(n)$ group as follows:

$$\left\langle \begin{array}{c} \{J'\} \\ \{k'\}\{m'\} \end{array} \middle| M_{ab} \middle| \begin{array}{c} \{J\} \\ \{k\}\{m\} \end{array} \right\rangle = \delta_{\{J'\}\{J\}} \sqrt{C_2(\{J\})} \; C^{\{J\} \; \begin{smallmatrix}\square\\\square\end{smallmatrix} \; \{J'\}}_{\{m\}(ab)\{m'\}} \; , \tag{24}$$

where $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ denotes $Spin(n)$ representations of second order antisymmetric tensors.

The matrix elements of the $U_{ab}$ operators in this basis are readily found to read:

$$\begin{aligned} &\left\langle \begin{array}{c} \{J'\} \\ \{k'\}\{m'\} \end{array} \middle| U^{w}_{ab} \middle| \begin{array}{c} \{J\} \\ \{k\}\{m\} \end{array} \right\rangle \\ &= |u| \left\langle \begin{array}{c} \{J'\} \\ \{k'\}\{m'\} \end{array} \middle| D^{-1 \, \square\square}_{w(ab)} \middle| \begin{array}{c} \{J\} \\ \{k\}\{m\} \end{array} \right\rangle \\ &= |u| \sqrt{dim(\{J'\}) dim(\{J\})} \int D^{\{J'\}*}_{\{k'\}\{m'\}}(g) D^{\square\square}_{w(ab)}(g) D^{\{J\}}_{\{k\}\{m\}}(g) dg \\ &= |u| \sqrt{\tfrac{dim(\{J\})}{dim(\{J'\})}} C^{\{J\} \, \square\square \, \{J'\}}_{\{k\} \;\; w \;\; \{k'\}} \; C^{\{J\} \, \square\square \, \{J'\}}_{\{m\}(ab)\{m'\}} \; . \end{aligned} \tag{25}$$

A closed form of the matrix elements of the whole contracted algebra $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ representations is thus explicitly given in this space by (24) and (25).

Moreover, we introduce the so called, left action generators $K$ as:

$$K_{ab} \equiv g^{(a''b'')(a'b')} D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{(ab)(a''b'')} M_{a'b'}, \tag{26}$$

where $g^{(a''b'')(a'b')}$ is the Cartan metric tensor of $SO(n)$ and $D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ are multiplicative operators analogous to operators $D^{\square\square}$, but that correspond Wigner $D$-functions of representation of antisymmetric second order tensors.

The $K_\mu$ operators behave exactly as the rotation generators $M_\mu$, apart from that that they act on the lower left-hand side indices of the basis (22):

$$\left\langle \begin{matrix}\{J'\}\\\{k'\}\end{matrix}\{m'\} \middle| K_{ab} \middle| \begin{matrix}\{J\}\\\{k\}\end{matrix}\{m\} \right\rangle = \delta_{\{J'\}\{J\}} \sqrt{C_2(\{J\})}\; C^{\{J\}\;\begin{smallmatrix}\square\\\square\end{smallmatrix}\;\{J'\}}_{\{k\}(ab)\{k'\}} \;. \tag{27}$$

Due to the fact that the mutually contragradient $SO(n)$ representations are equivalent, the $K_{ab}$ operators are directly related to the "left" action of the $SO(n)$ subgroup on $\mathcal{L}^2(|g(\theta)\rangle)$: $g'\,|g\rangle = \left|g g'^{-1}\right\rangle$. For this reason we will refer to the group generated by $K_{ab}$ simply as the left orthogonal (sub)group. The $K_{ab}$ and $M_{ab}$ operators mutually commute, however, the corresponding Casimir operators match, i.e.:

$$\frac{1}{2}\sum_{a,b} K_{ab}^2 = \frac{1}{2}\sum_{a,b} M_{ab}^2. \tag{28}$$

Commutators of $K$ and $D^{\square\square}$ operators are:

$$[K_{ab}, D^{\square\square}_{(cd)(ef)}] = i(\delta_{ac} D^{\square\square}_{(bd)(ef)} + \delta_{ad} D^{\square\square}_{(cb)(ef)} - \delta_{bc} D^{\square\square}_{(ad)(ef)} - \delta_{bd} D^{\square\square}_{(ca)(ef)}). \tag{29}$$

We note that, in complete analogy with operators $D^{\square\square}$ and $D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$, it is possible to introduce also operators $D^{\{J\}}_{\{k\}\{m\}}$ that act multiplicatively in the space $\mathcal{L}^2(Spin(n))$ as corresponding Wigner $D$-functions of representation $\{J\}$. Due to their multiplicative action, these operators obey the same identities that are standardly fulfilled by the Wigner $D$-functions.

## 3.2 Condition for the validity of the Gell-Mann formula

The problem with validity of the Gell-Mann formula lies in the fact that commutator of two operators (from the subalgebra $\mathcal{T}$) constructed by using this formula does not always belong to the subalgebra $\mathcal{M}$ with respect to which the contraction has been performed (9), as it should. In the $sl(n,\mathbb{R})$ case that means that the commutator of two shear generators, constructed by using (17) is not equal to a linear combination of operators from $so(n)$ subalgebra. That is, the problem is in the relation (4) which is not satisfied *a priori*, i.e. without imposing additional conditions. On the other hand, relation (3) is automatically satisfied by the construction, due to obvious transformation properties of the Gell-Mann formula constructed operators $T_{ab}$ w.r.t. the $so(n)$ subalgebra.

To investigate circumstances in which relation (4) holds, we will evaluate this commutator using relations and mathematical framework from the previous subsection. For the sake of generality of the results, we do not wish to fix the basis for algebra elements – to stress this, we will use a single letter indices (e.g. $T_\mu$) instead of Cartesian basis double indices ($T_{ab}$)

Using (21) and (28), the Gell-Mann formula (17) now reads:

$$T_\mu = i\alpha[C_2(so(n))_K, D^{\square\square}_{w\mu}] + i\sigma D^{\square\square}_{w\mu}, \tag{30}$$

where $C_2(so(n))_K$ is quadratic Casimir operator of the $so(n)$ subalgebra expressed using $K$ operators (28):

$$C_2(so(n))_K = K_i K_i. \tag{31}$$

Starting from the expression (30) and using known properties of Wigner $D$-functions, we find:

$$\begin{aligned}
[T_\mu, T_\nu] &= -2\alpha^2[K_{\{i}, [K_{j\}}, D^{\square\square}_{w\nu}]][K_j, D^{\square\square}_{w\mu}]K_i - (\mu \leftrightarrow \nu) \\
&= \cdots = -\alpha^2 \textstyle\sum_J \sum_{\lambda,\lambda'} \left(C^{\square\square\ \square\square\ J}_{\mu\ \ \ \nu\ \ \ \lambda} - C^{\square\square\ \square\square\ J}_{\nu\ \ \ \mu\ \ \ \lambda}\right)\cdot \\
&\Big(2(C_2(J) - 2C_2(\square\square))\langle\langle{}^{J}_{\lambda'}|1 \otimes K_i|{}^{\square\square}_{w}\rangle|{}^{\square\square}_{w}\rangle + \\
&\langle\langle{}^{J}_{\lambda'}|[1 \otimes K_i, C_{2(I+II)_K}]|{}^{\square\square}_{w}\rangle|{}^{\square\square}_{w}\rangle\Big) D^J_{\lambda'\lambda} K_i.
\end{aligned} \tag{32}$$

The $C_{2(I+II)_K}$ operator here denotes the second order Casimir operator acting in the tensor product of two $\Box\!\Box$ representations, i.e. $C_{2(I+II)_K} = \sum_i (K_i \otimes 1 + 1 \otimes K_i)^2$.

The summation index $J$ in (32) runs over all irreducible representations of the $Spin(n)$ group that appear in the tensor product $\Box\!\Box \otimes \Box\!\Box$, and $\lambda, \lambda'$ count the vectors of these representations. Since all irreducible representations terms, apart those for which the Clebsch-Gordan coefficient $C^{\Box\!\Box\ \Box\!\Box\ J}_{\mu\ \ \nu\ \ \lambda}$ is antisymmetric w.r.t. $\mu \leftrightarrow \nu$ vanish, we are left with only two values that $J$ takes: one corresponding to the antisymmetric second order tensor $\substack{\Box\\ \Box}$ and the other one corresponding to the representation that we denote as $\begin{smallmatrix}\Box\Box\\ \Box\Box\end{smallmatrix}$. The fact that in the case of $sl(n,\mathbb{R})$ algebras, there is another representation term, in addition to $\substack{\Box\\ \Box}$, in the antisymmetric product of two $\Box\!\Box$ representations (i.e. representations that correspond to abelian $U$ operators), is in the root of the Gell-Mann formula validity problem. Note that in the case of the $so(m+1,n) \to iso(m,n)$, i.e. $so(m,n+1) \to iso(m,n)$) contractions, where the Gell-Mann formula works on the algebraic level, the contracted $U$ operators transform as $\Box$ and the antisymmetric product of two such representations certainly belongs to the $\substack{\Box\\ \Box}$ representation (i.e. to the representation that corresponds to $\mathcal{M} = so(m,n)$ subalgebra operators).

The $so(n)$ Casimir operator values satisfy $C^2(\begin{smallmatrix}\Box\Box\\ \Box\Box\end{smallmatrix}) = 2C^2(\Box\!\Box) = 4n$, implying that one of the two terms vanishes in (32) when $J = \begin{smallmatrix}\Box\Box\\ \Box\Box\end{smallmatrix}$, leaving us with:

$$
\begin{aligned}
\frac{1}{2\alpha^2}[T_\mu, T_\nu] = 4(n+2) & \textstyle\sum_{\lambda,\lambda'} C^{\Box\!\Box\ \Box\!\Box\ \substack{\Box\\ \Box}}_{\mu\ \ \nu\ \ \lambda} \langle\!\langle {}^{\substack{\Box\\ \Box}}_{\lambda'} | 1 \otimes K_i | {}^{\Box\!\Box}_{w} \rangle | {}^{\Box\!\Box}_{w} \rangle D^{\substack{\Box\\ \Box}}_{\lambda'\lambda} K_i - \\
& \textstyle\sum_{\lambda,\lambda'} C^{\Box\!\Box\ \Box\!\Box\ \substack{\Box\\ \Box}}_{\mu\ \ \nu\ \ \lambda} \langle\!\langle {}^{\substack{\Box\\ \Box}}_{\lambda'} | [1 \otimes K_i, C_{2(I+II)_K}] | {}^{\Box\!\Box}_{w} \rangle | {}^{\Box\!\Box}_{w} \rangle D^{\substack{\Box\\ \Box}}_{\lambda'\lambda} K_i - \qquad (33) \\
& \textstyle\sum_{\lambda,\lambda'} C^{\Box\!\Box\ \Box\!\Box\ \begin{smallmatrix}\Box\Box\\ \Box\Box\end{smallmatrix}}_{\mu\ \ \nu\ \ \lambda} \langle\!\langle {}^{\begin{smallmatrix}\Box\Box\\ \Box\Box\end{smallmatrix}}_{\lambda'} | [1 \otimes K_i, C_{2(I+II)_K}] | {}^{\Box\!\Box}_{w} \rangle | {}^{\Box\!\Box}_{w} \rangle D^{\begin{smallmatrix}\Box\Box\\ \Box\Box\end{smallmatrix}}_{\lambda'\lambda} K_i,
\end{aligned}
$$

where we used that $C^2(\substack{\Box\\ \Box}) = 2n - 4$.

As the coefficient $\alpha$ can be adjusted freely, all that is needed for the Gell-Mann formula to be valid is that (33) is proportional to the appropriate linear combination of the $Spin(n)$ generators, as determined by the Wigner-

Eckart theorem, i.e.:

$$[T_\mu, T_\nu] \sim \sum_\lambda C^{\Box\Box\ \Box\Box\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_{\mu\ \ \nu\ \ \lambda} M_\lambda = \sum_{\lambda,i} C^{\Box\Box\ \Box\Box\ \begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_{\mu\ \ \nu\ \ \lambda} D^{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_{i\lambda} K_i. \tag{34}$$

We now analyze these requirements, skipping some straightforward technical details. The third term in (33), containing $D$ functions of the representation $\begin{smallmatrix}\Box\Box\\\Box\Box\end{smallmatrix}$, is to vanish. Since it is not possible to choose vectors $w$ so that this term vanishes identically as an operator, the remaining possibility is to restrain the the space (22) of its domain to some subspace $V = \{|v\rangle\} \subset \mathcal{L}^2(Spin(n))$. More precisely, for this term to vanish, there must exist a subalgebra $\mathbf{L} \subset so(n)_K$, spanned by some $\{K_\alpha\}$, such that $K_\alpha \in \mathbf{L} \Rightarrow K_\alpha |v\rangle = 0$. Requiring additionally that this subspace $V$ ought to close under an action of the shear generators, and that the first two terms of (33) ought to yield (34), we arrive at the following two necessary conditions:

1. The algebra $\mathbf{L}$, must be a symmetric subalgebra of $so(n)$, i.e.

$$[\mathbf{L}, \mathbf{N}] \subset \mathbf{N}, [\mathbf{N}, \mathbf{N}] \subset \mathbf{L}; \mathbf{N} = \mathbf{L}^\perp. \tag{35}$$

2. The vector $\left| \begin{smallmatrix}\Box\Box\\ w\end{smallmatrix} \right\rangle$ ought to be invariant under the $L$ subgroup action (subgroup of $Spin(n)$ corresponding to $\mathbf{L}$), i.e.

$$K_\alpha \in \mathbf{L} \Rightarrow K_\alpha \left| \begin{smallmatrix}\Box\Box\\ w\end{smallmatrix} \right\rangle = 0. \tag{36}$$

The space $V$ is thus $Spin(n)/L$. In Wigner's terminology, this means that $L$ is the little group of the contracted algebra representation, and that necessarily it is to be represented trivially. Besides, the little group is to be a symmetric subgroup of the $Spin(n)$ group. This coincides with one class of the solutions found by Hermann [33]. However, now we demonstrated that there are no other solutions in the $sl(n, \mathbb{R})$ algebra cases, in particular, there are no solutions with little group represented non trivially.

As for the first requirement, an inspection of the tables of symmetric spaces, yields two possibilities: $L = Spin(m) \times Spin(n-m)$, where $Spin(1) \equiv 1$, and, for $n = 2k$, $L = U(k)$ ($U$ is the unitary group). However, this second possibility certainly does not imply another solution, since it turns out that there is no vector satisfying the second above property.

Thus, *the only remaining possibility* is as follows,

$$L = Spin(m) \times Spin(n-m), \quad m = 1, 2, \ldots, n-1, \quad Spin(1) \equiv 1. \tag{37}$$

It is rather straightforward but somewhat lengthy to show that proportionality of (33) and (34) really holds in this case. The vector $\left| \begin{array}{c} \square\square \\ w \end{array} \right\rangle$ exists, and it is the one corresponding to traceless diagonal $n \times n$ matrix $diag(\frac{1}{m}, \ldots, \frac{1}{m}, -\frac{1}{n-m}, \ldots, -\frac{1}{n-m})$.

The analysis accomplished above can not be applied directly to the $n = 2$ case, thus the $sl(2, \mathbb{R})$ case must be treated separately. The maximal compact subgroup $SO(2)$, that is, its double cover $Spin(2)$, has only one generator $M$, and therefore it has only one-dimensional irreducible representations. In this case, there are two Abelian generators $U_\pm$ of the contracted group:

$$[M, U_\pm] = \pm U_\pm, \qquad [U_+, U_-] = 0. \tag{38}$$

Based on these relations, it is easy to verify that the $T_\pm$ operators obtained by the Gell-Mann construction as:

$$T_\pm = i[M^2, U_\pm] + i\sigma U_\pm \tag{39}$$

automatically satisfy the $sl(2, \mathbb{R})$ commutation relation:

$$[T_+, T_-] = -2M. \tag{40}$$

Therefore, we demonstrate that the Gell-Mann formula applies to the $sl(2, \mathbb{R})$ case as well.

The above results can be summarized into a conclusion that the formula is valid only in Hilbert spaces over $Spin(n)/(Spin(m) \times Spin(n-m))$, $m = 1, 2, \ldots, n-1, Spin(1) \equiv 1$.

## 3.3 Matrix elements

The presented approach allows us also to write down explicitly the matrix elements of the $sl(n,\mathbb{R})$ generators in the cases when the Gell-Mann formula is valid. The possible cases are determined by the numbers $n$ and $m$. The corresponding representation space (not irreducible in general) is the one over the coset space $Spin(n)/Spin(m) \times Spin(n-m)$. The proportionality factor $\alpha$ is determined to be:

$$\alpha = \frac{1}{2}\sqrt{\frac{m(n-m)}{n}}, \tag{41}$$

and, in a matrix notation for $\square\square$ representation:

$$\left|\begin{array}{c}\square\square\\ w\end{array}\right\rangle = \sqrt{\tfrac{m(n-m)}{n}}\,diag(\frac{1}{m},\ldots,\frac{1}{m},-\frac{1}{n-m},\ldots,-\frac{1}{n-m}). \tag{42}$$

The Gell-Mann formula (30), and the matrix representation of the contracted Abelian generators $U$ (25) yield:

$$\begin{gathered}\left\langle\begin{array}{c}J'\\ m'\end{array}\right| T_\mu \left|\begin{array}{c}J\\ m\end{array}\right\rangle = \\ i\sqrt{\tfrac{m(n-m)}{4n}}\sqrt{\tfrac{dim(J)}{dim(J')}}(C_2(J')-C_2(J)+\sigma)C^{J\,\square\square\,J'}_{0\;\;0\;\;0}\;C^{J\,\square\square\,J'}_{m\;\;\mu\;\;m'}\,.\end{gathered} \tag{43}$$

The zeroes in the indices of Clebsch-Gordan coefficients here denote vectors that are invariant w.r.t. $Spin(m) \times Spin(n-m)$ transformations (in that spirit $\left|\begin{array}{c}\square\square\\ w\end{array}\right\rangle = \left|\begin{array}{c}\square\square\\ 0\end{array}\right\rangle$). In the formula (43), the space reduction from $\mathcal{L}^2(Spin(n))$ to $\mathcal{L}^2(Spin(n)/Spin(m) \times Spin(n-m))$ implies a reduction of the basis (22), i.e. $\left|\begin{array}{c}J\\ 0\,m\end{array}\right\rangle \to \left|\begin{array}{c}J\\ m\end{array}\right\rangle$, i.e. only the vectors invariant w.r.t. left $Spin(m) \times Spin(n-m)$ action remain: $\left|\begin{array}{c}J\\ 0\,m\end{array}\right\rangle$. By fixing value of the left index to be zero in the basis (22), we effectively lose multiplicity of the representation w.r.t. the $Spin(n)$ subgroup.

The expression (43), together with the action of the $Spin(n)$ generators (24) provides an explicit form of the $SL(n,\mathbb{R})$ generators representation, that is labelled by a free parameter $\sigma$. Such representations are multiplicity free w.r.t. the maximal compact $Spin(n)$ subgroup, and all of them are *a priori* tensorial. One can obtain from these representations, for certain $\sigma$ parameter values, the $sl(n, \mathbb{R})$ spinorial representations as well by explicitly evaluating the Clebsch-Gordan coefficient and performing an appropriate analytic continuation in terms of the $Spin(n)$ labels [39, 30].

## 3.4 Conclusion on the original Gell-Mann formula for $sl(n, \mathbb{R})$

In this section, we clarified the issue of the Gell-Mann formula validity for the $sl(n, \mathbb{R}) \to r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ algebra contraction. We have shown that the only $sl(n, \mathbb{R})$ representations obtainable in this way are given in Hilbert spaces over the symmetric spaces $Spin(n)/Spin(m) \times Spin(n-m)$, $m = 1, 2, \ldots, n-1$. Moreover, by making use of the Gell-Mann formula in these spaces, we have obtained a closed form expressions of the noncompact operators (generating $SL(n, \mathbb{R})/SO(n)$ cosets) irreducible representations matrix elements. The matrix elements of both compact and noncompact operators of the $sl(n, \mathbb{R})$ algebra are given by (24) and (43), respectively.

In particular, it turns out that, due to Gell-Mann's formula validity conditions, no representations with $so(n)$ subalgebra representations multiplicity can be obtained in this way. Moreover, the matrix expressions of the noncompact operators as given by (43) do not account *a priori* for the $sl(n, \mathbb{R})$ spinorial representations.

Due to mutual connection of the $sl(n, \mathbb{R})$ and $su(n)$ algebras, the results apply to the corresponding $su(n)$ case as well. The $SU(n)/SO(n)$ generators differ from the corresponding $sl(n, \mathbb{R})$ operators by the imaginary unit multiplicative factor, while the spinorial representations issue in the $su(n)$ case is pointless due to the fact that the $SU(n)$ is a simply connected (there exists no double cover) group.

In many physics applications one is interested in the unitary irreducible representations. The unitarity question goes beyond the scope of the present work, and it relates to the Hilbert space properties, i.e. the vector space scalar product. An efficient method to study unitarity is to start with a Hilbert space $L^2(Spin(n), \kappa)$ of square integrable functions with a scalar product given in terms of an arbitrary kernel $\kappa$, and to impose the unitarity constraints both on the scalar products itself and on the noncompact operators matrix elements in that scalar product (cf. [41]). The simplest series of the $sl(n, \mathbb{R})$ unitary irreducible representations, the Principal series, of the representations constructed above are obtained when $\sigma = i\sigma_I$, $\sigma_I \in \mathbb{R}\backslash\{0\}$, i.e. when $\sigma$ takes an arbitrary nonzero pure imaginary value.

# 4 Generalization of the Gell-Mann formula in the $sl(n, \mathbb{R})$ case

## 4.1 Low dimensional cases $sl(3, \mathbb{R})$ and $sl(4, \mathbb{R})$

In the previous section we have shown that the Gell-Mann formula in $sl(n, \mathbb{R})$ case is of a very limited domain of validity. The reason why the formula (17) is not valid in entire space $\mathcal{L}^2(Spin(n))$ can be understood in the following way. While the $sl(n, \mathbb{R})$ operators $M_{ab}$ are invariant w.r.t. the left action of the $Spin(n)$ group in this space, i.e. they commute with operators $K_{ab}$, it is not the case with the shear generators $T_{ab}$, as constructed by using the Gell-Mann formula (17). These transformation properties of shear generators $T_{ab}$ are inherited from the corresponding operators $U_{ab}$ of the contracted algebra. Their nontrivial transformation properties w.r.t. the left action of the $Spin(n)$ group are determined by the choice of the vector $w$ in (21). As a consequence, a commutator of two such operators *a priori* will also have nontrivial transformation properties w.r.t. $SO(n)_K$ group (the one generated by $K_\mu$).

Therefore, unlike $M_\mu$, this commutator is not scalar w.r.t. $SO(n)_K$ action, so that commutation relation (4) is not satisfied. In certain cases it is possible to restrict representations space to a subspace in such a way that only $SO(n)_K$ invarian part of commutator $[T_\mu, T_\nu]$ remains. That is exactly what happens in the cases discussed in the last section, when the Gell-Mann formula is valid.

This analysis gives a motivation to attempt to modify the formula by adding some terms proportional to the generators of the left $SO(n)_K$ group, in such a way to cancel the unwanted terms in the commutator $[T_\mu, T_\nu]$. Indeed, such a generalization of the Gell-Mann formula can be effectively read out from the known form of the matrix elements for $sl(3, \mathbb{R})$ representations with multiplicity. Namely, in the form of $sl(3, \mathbb{R})$ matrix element expression from the paper [41] it is possible to recognize terms that correspond to the Gell-Mann formula, together with certain additional terms. Therefore, in the $n = 3$ case, using the results of [41] we can directly write:

$$T_\mu = \sigma D^2_{0\mu} + \frac{i}{\sqrt{6}}[C_2(so(3)), D^2_{0\mu}] + i(D^2_{2\mu} - D^2_{-2\mu})K_0 + \delta(D^2_{2\mu} + D^2_{-2\mu}). \quad (44)$$

We used the standard spherical basis for $Spin(3)$ representations, with $\square\square$ representations here corresponding to $J = 2$, and the vectors within this representation are labeled by $\mu = 0, \pm 1, \pm 2$. Complex numbers $\sigma$ and $\delta$ are parameterizing the representations of the $sl(3,\mathbb{R})$ group. The first two terms we recognize as the original Gell-Mann formula (30), with vector $w$ chosen to be invariant w.r.t. the action of the $K_3$ (i.e. chosen is vector $|J = 2, \mu = 0\rangle$. The additional terms to the "original" Gell-Mann formula secure that the $T_\mu$ operators satisfy the commutation relation (4) in the entire representation space. Note that there are two $sl(3,\mathbb{R})$ representation labels $\sigma$ and $\delta$, matching the algebra rank, contrary to the case of the original Gell-Mann formula whose single free parameter cannot account for the entire representation labeling. Notice also that additional terms in the formula (44) change value of projection $K_3$, that is, action of these terms on vector form the basis (22) will change the multiplicity label $k$ (in general leading to nontrivial multiplicity of the representations).

The generalized expression (44) contains the original formula as a special case: by restricting the representation space $\mathcal{L}^2(Spin(3))$ to the subspace of $k = 0$ (that is the subspace $\mathcal{L}^2(Spin(3)/Spin(2))$), and choosing $\delta = 0$ one arrives at the multiplicity free representations that were obtained by using the original formula. Moreover, the generalized Gell-Mann formula allows one to obtain some $sl(3,\mathbb{R})$ multiplicity free representations that cannot be reached by making use of the original formula. For example, with the choice $\sigma = \frac{3}{2}$, and $\delta = -\frac{1}{2}$ [42], a subspace spanned by the following vectors (linear combinations of basis vectors with different $k$ values):

$$\begin{gathered}
\Big\{ \left| \begin{smallmatrix} \frac{1}{2} \\ m \end{smallmatrix} \right>' = \left| \begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \;\; m \end{smallmatrix} \right> + \left| \begin{smallmatrix} \frac{1}{2} \\ -\frac{1}{2} \;\; m \end{smallmatrix} \right>, \\
\left| \begin{smallmatrix} \frac{5}{2} \\ m \end{smallmatrix} \right>' = \left| \begin{smallmatrix} \frac{5}{2} \\ \frac{5}{2} \;\; m \end{smallmatrix} \right> + \sqrt{\tfrac{5}{2}} \left| \begin{smallmatrix} \frac{5}{2} \\ \frac{1}{2} \;\; m \end{smallmatrix} \right> + \sqrt{\tfrac{5}{2}} \left| \begin{smallmatrix} \frac{5}{2} \\ -\frac{1}{2} \;\; m \end{smallmatrix} \right> + \left| \begin{smallmatrix} \frac{5}{2} \\ -\frac{5}{2} \;\; m \end{smallmatrix} \right>, \\
\left| \begin{smallmatrix} \frac{9}{2} \\ m \end{smallmatrix} \right>' = \left| \begin{smallmatrix} \frac{9}{2} \\ \frac{9}{2} \;\; m \end{smallmatrix} \right> + \left| \begin{smallmatrix} \frac{9}{2} \\ \frac{5}{2} \;\; m \end{smallmatrix} \right> + \sqrt{\tfrac{7}{2}} \left| \begin{smallmatrix} \frac{9}{2} \\ \frac{1}{2} \;\; m \end{smallmatrix} \right> + \sqrt{\tfrac{7}{2}} \left| \begin{smallmatrix} \frac{9}{2} \\ -\frac{1}{2} \;\; m \end{smallmatrix} \right> + \left| \begin{smallmatrix} \frac{9}{2} \\ -\frac{5}{2} \;\; m \end{smallmatrix} \right> + \left| \begin{smallmatrix} \frac{9}{2} \\ -\frac{9}{2} \;\; m \end{smallmatrix} \right>, \dots \Big\}
\end{gathered} \tag{45}$$

is invariant w.r.t. the action of $sl(3,\mathbb{R})$ operators. At the same time, subspaces with fixed value of  are here one dimensional and the values of  are half-integer. Therefore, this is an example of a spinorial multiplicity free $sl(3,\mathbb{R})$ representation. More precisely, this is the unique unitary $sl(3,\mathbb{R})$ spinorial multiplicity free representation $sl(3,\mathbb{R})$, and this representation cannot be obtained by application of the original Gell-Mann formula (without resorting to certain analytical continuation of the Clebsch-Gordan coefficient expressions [41]).

Matrix elements of the $sl(4,\mathbb{R})$ representations with multiplicity are also known ($n = 4$ is the largest dimension with known matrix elements). The Gell-Mann formula thus can similarly be generalized in the case of the $sl(4,\mathbb{R})$ algebra. Again, by extracting from the known matrix elements of the $sl(4,\mathbb{R})$ representations with multiplicity [30], we find:

$$
\begin{aligned}
T_{\mu_1\mu_2} = i\Big(&\sigma D^{11}_{00\mu_1\mu_2} + \tfrac{1}{2}[C_2(so(4)), D^{11}_{00\mu_1\mu_2}] \\
&+\delta_1(D^{11}_{11\mu_1\mu_2} + D^{11}_{-1-1\mu_1\mu_2}) + (D^{11}_{11\mu_1\mu_2} - D^{11}_{-1-1\mu_1\mu_2})(K^{10}_{00} + K^{01}_{00}) \\
&+\delta_2(D^{11}_{-11\mu_1\mu_2} + D^{11}_{1-1\mu_1\mu_2}) + (D^{11}_{-11\mu_1\mu_2} - D^{11}_{1-1\mu_1\mu_2})(K^{10}_{00} - K^{01}_{00})\Big),
\end{aligned} \tag{46}
$$

where we used $Spin(4) \supset (Spin(2) \times Spin(2))$ basis and $\mu_1, \mu_2 = 0, \pm 1$. As the rank of the $sl(4,\mathbb{R})$ algebra is three, there are precisely three representation labels $\sigma$, $\delta_1$, and $\delta_2$ (if complex, only three real are independent).

As in the $sl(3,\mathbb{R})$ case, the generalized formula reduces, for certain values of the labels $\delta_1 = \delta_2 = 0$, in a representation subspace defined by $K^{10}_{00} = K^{01}_{00} = 0$ (i.e. in the subspace $\mathcal{L}^2(Spin(4)/(Spin(2) \times Spin(2)))$) to the original Gell-Mann formula.

If we express the generalized Gell-Mann formula for $sl(4,\mathbb{R})$ in a basis that corresponds to the subgroup chain $Spin(4) \supset Spin(3) \supset Spin(2)$, we obtain:

$$
\begin{aligned}
T_{j,\mu} = \gamma_1 D^{\;11}_{\substack{0\,j\\0\,\mu}} - \tfrac{i\sqrt{3}}{4}[C_2(so(4)), D^{\;11}_{\substack{0\,j\\0\,\mu}}]+ \\
\gamma_2 D^{\;11}_{\substack{2\,j\\0\,\mu}} + \tfrac{i}{\sqrt{6}}[C_2(so(3))_K, D^{\;11}_{\substack{2\,j\\0\,\mu}}]+ \\
\gamma_3(D^{\;11}_{\substack{2\,j\\2\,\mu}} + D^{\;11}_{\substack{2\;\;j\\-2\mu}}) + (D^{\;11}_{\substack{2\,j\\2\,\mu}} - D^{\;11}_{\substack{2\;\;j\\-2\mu}})(K^{\;10}_{\substack{1\\0}} + K^{\;01}_{\substack{1\\0}}),
\end{aligned} \tag{47}
$$

where $j = 0, 1, 2$, $|\mu| \leq j$, and $C_2(so(3))_K$ denotes quadratic Casimir operator of the left $SO(3)$ subgroup, generated by

$$
\{K^{\;10}_{\substack{1\\-1}} + K^{\;01}_{\substack{1\\-1}}, K^{\;10}_{\substack{1\\0}} + K^{\;01}_{\substack{1\\0}}, K^{\;10}_{\substack{1\\1}} + K^{\;01}_{\substack{1\\1}}\}
$$

(in Cartesian basis: $\{K_{12}, K_{23}, K_{31}\}$). We note that in subspace $\mathcal{L}^2(Spin(4))$ it holds $C_2(so(4)) \equiv C_2(so(4))_M = C_2(so(4))_K$, but this connection does not exist for the Casimir operators of the subalgebras, i.e. $C_2(so(3))_M \neq C_2(so(3))_K$! Relation of the parameters used in (46) and (47) is $i\sigma = -\frac{1}{\sqrt{3}}\gamma_1 + \sqrt{\frac{2}{3}}\gamma_2 - 2i$, $\delta_1 = \gamma_3$, and $\delta_2 = \frac{1}{\sqrt{3}}\gamma_1 + \frac{1}{\sqrt{6}}\gamma_2 - 2i$. From this form of the expression can be easily seen that, if we choose $\gamma_2 = \gamma_3 = 0$, the generalized formula reduces to the original one in the subspace $\mathcal{L}^2(Spin(4)/Spin(3))$, since in that subspace the additional terms vanish (and only the first row of the expression remains).

## 4.2 Generalization of the Gell-Mann formula for $sl(5,\mathbb{R})$

In the previous subsection, thanks to the known matrix elements of the $sl(n,\mathbb{R})$, $n = 3, 4$ representations with multiplicity, we generalized Gell-Mann formula for the $\overline{SL}(3,\mathbb{R})$ and $\overline{SL}(4,\mathbb{R})$ groups, finding a formula that is valid in entire space of square integrable functions over the compact subgroup. In this subsection we will construct a generalization of the Gell-Mann formula for $sl(5,\mathbb{R})$ case and, as a direct application, we will derive matrix elements of $sl(5,\mathbb{R})$ generators for arbitrary irreducible representation. This approach to the problem of finding matrix elements is particularly important since the matrix elements for $sl(3,\mathbb{R})$ and $sl(4,\mathbb{R})$ were in [41] and [30] found in a very computationally involved way, that would be difficult to repeat for the higher dimensional $sl(5,\mathbb{R})$ case.

As a hint toward a way to generalize the formula in $n = 5$ case we note a certain recursive pattern in transition from $n = 3$ to $n = 4$ can be seen by comparing the expressions (44) and (47). Namely, in the formula (47) all additional terms coincide with the generalized formula for $sl(3,\mathbb{R})$ itself (44), where only the quadratic Casimir operator $C_2(so(3))$ is replaced by $C_2(so(3))_K$. It turns out that transition from $n = 4$ to $n = 5$ can be obtained in a similar manner.

Let us recall first some basic $so(5)$ algebra representation notions. The $so(5)$ algebra is of rang two, and its irreducible representations are labeled by a pair of labels $(\overline{J}_1, \overline{J}_2)$, resembling the $so(4)$ labeling. The complete labeling of the representation space vectors can be achieved by making use of the subalgebra chain: $so(5) \supset so(4) = so(3) \oplus so(3) \supset so(2) \oplus so(2)$. The basis of the $so(5)$ algebra representation space can be taken as in [43, 44]:

$$\left\{ \left| \begin{array}{cc} \overline{J}_1 & \overline{J}_2 \\ J_1 & J_2 \\ m_1 & m_2 \end{array} \right\rangle, \quad \overline{J}_i = 0, \frac{1}{2}, \ldots; \quad \overline{J}_1 \geq \overline{J}_2; \quad |m_i| \leq J_i, \quad i = 1,2 \right\}. \tag{48}$$

The admissible values of $J_1$ and $J_2$, within an irreducible representation $(\overline{J}_1, \overline{J}_2)$ are given in [45]. Now, the basis of the $so(5)$ algebra, i.e. the $Spin(5)$ group, representation space vectors (22) is given as follows:

$$\left\{ \left| \begin{array}{cccc} \overline{J}_1 & \overline{J}_2 & & \\ K_1 & K_2 & J_1 & J_2 \\ k_1 & k_2 & m_1 & m_2 \end{array} \right\rangle \right\}. \tag{49}$$

The ten $so(5)$ algebra operators, generating the adjoint representation of $Spin(5)$, transform, in notation (48), under the representation $(\overline{1}, \overline{0})$. Their $so(4)$ subalgebra representation content is: $(\overline{1}, \overline{0}) \rightarrow (1,0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (0,1)$. The shear operators transform under the 14-dimensional $so(5)$ irreducible representation $(\overline{1}, \overline{1})$ of $so(5)$ which contains $(1,1)$, $(\frac{1}{2}, \frac{1}{2})$ and $(0,0)$ representation upon reduction to $so(4)$:

$$\left\{ T^{j_1 j_2}_{\mu_1 \mu_2} \right\} = \left\{ T^{1\,1}_{\mu_1 \mu_2}, T^{\frac{1}{2}\,\frac{1}{2}}_{\mu_1 \mu_2}, T^{00}_{00} \right\}.$$

Now, by analogy to the transition from $n = 3$ to $n = 4$ (44, 47) we will make the following educated guess for the form of the $sl(5, \mathbb{R})$ generalized Gell-Menn formula:

$$\begin{aligned} T^{j_1 j_2}_{\mu_1 \mu_2} \; = \; & \sigma_1 D^{\substack{\overline{1}\overline{1} \\ 00\, j_1 j_2}}_{00\, \mu_1 \mu_2} + i\alpha_5 \left[ C_2(so(5)), D^{\substack{\overline{1}\overline{1} \\ 00\, j_1 j_2}}_{00\, \mu_1 \mu_2} \right] \\ & + i \Bigg( \sigma_2 D^{\substack{\overline{1}\overline{1} \\ 11\, j_1 j_2}}_{00\, \mu_1 \mu_2} + \tfrac{1}{2} \left[ C_2(so(4)_K), D^{\substack{\overline{1}\overline{1} \\ 11\, j_1 j_2}}_{00\, \mu_1 \mu_2} \right] \\ & - D^{\substack{\overline{1}\;\overline{1} \\ 1\;1\, j_1 j_2}}_{1\,-1\, \mu_1 \mu_2} \left( \delta_1 + K^{\substack{\overline{1}\overline{0} \\ 10}}_{00} - K^{\substack{\overline{1}\overline{0} \\ 01}}_{00} \right) - D^{\substack{\overline{1}\;\overline{1} \\ 1\;1\, j_1 j_2}}_{-1\,1\, \mu_1 \mu_2} \left( \delta_1 - K^{\substack{\overline{1}\overline{0} \\ 10}}_{00} + K^{\substack{\overline{1}\overline{0} \\ 01}}_{00} \right) \\ & + D^{\substack{\overline{1}\overline{1} \\ 11\, j_1 j_2}}_{11\, \mu_1 \mu_2} \left( \delta_2 + K^{\substack{\overline{1}\overline{0} \\ 10}}_{00} + K^{\substack{\overline{1}\overline{0} \\ 01}}_{00} \right) + D^{\substack{\overline{1}\;\overline{1} \\ 1\;1\, j_1 j_2}}_{-1\,-1\, \mu_1 \mu_2} \left( \delta_2 - K^{\substack{\overline{1}\overline{0} \\ 10}}_{00} - K^{\substack{\overline{1}\overline{0} \\ 01}}_{00} \right) \Bigg), \end{aligned} \tag{50}$$

where $j_i = 0, \frac{1}{2}, 1,\ |\mu_i| \leq j_i,\ i = 1, 2$, the representation labels $\sigma_1, \sigma_2, \delta_1$ and $\delta_2$ are arbitrary (complex) parameters (four real are independent), and $C_2(so(4)_K)$ denotes the quadratic Casimir operator of the left action $so(4)_K$ algebra. Coefficient $\alpha_5$ is determined from the requirement that that commutation relations of the $[T, T] \subset M$ type should be satisfied (4). And indeed, with the choice $\alpha_5 = \frac{1}{\sqrt{5}}$ this relation can be checked to hold.

Unlike the $sl(n, \mathbb{R})$, $n = 3, 4$ cases where we started from the known expressions for matrix elements of the shear oeprators, here, in the case of $sl(5, \mathbb{R})$ algebra, using the obtained generalization of the Gell-Mann formula (50) we can now derive matrix elements for arbitrary $sl(5, \mathbb{R})$ representation (given by the parameters $\sigma_1, \sigma_2, \delta_1$ and $\delta_2$).

Matrix elements of the $sl(5, \mathbb{R})$ shear generators are:

$$
\begin{aligned}
&\left\langle \begin{smallmatrix} \overline{J}'_1\, \overline{J}'_2 & \\ K'_1 K'_2 J'_1\ J'_2 \\ k'_1\ k'_2\ m'_1 m'_2 \end{smallmatrix} \right| T_{\mu_1\mu_2}^{j_1 j_2} \left| \begin{smallmatrix} \overline{J}_1\, \overline{J}_2 & \\ K_1 K_2 J_1\ J_2 \\ k_1\ k_2\ m_1 m_2 \end{smallmatrix} \right\rangle = \sqrt{\tfrac{dim(\overline{J}_1, \overline{J}_2)}{dim(\overline{J}'_1, \overline{J}'_2)}}\, C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\ \overline{1} & \overline{J}'_1 \overline{J}'_2 \\ J_1\ J_2 & j_1 j_2 & J'_1\ J'_2 \\ m_1 m_2 & \mu_1\mu_2 & m'_1 m'_2 \end{smallmatrix}} \\
&\times \Bigg( \Big(\sigma_1 + i\sqrt{\tfrac{4}{5}}(\overline{J}'_1(\overline{J}'_1+2) + \overline{J}'_2(\overline{J}'_2+1) - \overline{J}_1(\overline{J}_1+2) - \overline{J}_2(\overline{J}_2+1))\Big) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 00 & K'_1 K'_2 \\ k_1\ k_2 & 00 & k'_1\ k'_2 \end{smallmatrix}} \\
&+ i\Big(\sigma_2 + K'_1(K'_1+1) + K'_2(K'_2+1) - K_1(K_1+1) - K_2(K_2+1)\Big) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 11 & K'_1 K'_2 \\ k_1\ k_2 & 00 & k'_1\ k'_2 \end{smallmatrix}} \\
&- i(\delta_1 + k_1 - k_2) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 11 & K'_1 K'_2 \\ k_1\ k_2 & 1 -1 & k'_1\ k'_2 \end{smallmatrix}} - i(\delta_1 - k_1 + k_2) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\ \overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 1\ 1 & K'_1 K'_2 \\ k_1\ k_2 & -1 1 & k'_1\ k'_2 \end{smallmatrix}} \\
&+ i(\delta_2 + k_1 + k_2) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 11 & K'_1 K'_2 \\ k_1\ k_2 & 11 & k'_1\ k'_2 \end{smallmatrix}} + i(\delta_2 - k_1 - k_2) C^{\begin{smallmatrix} \overline{J}_1 \overline{J}_2 & \overline{1}\ \overline{1} & \overline{J}'_1 \overline{J}'_2 \\ K_1 K_2 & 1\ 1 & K'_1 K'_2 \\ k_1\ k_2 & -1 -1 & k'_1\ k'_2 \end{smallmatrix}} \Bigg).
\end{aligned}
\tag{51}
$$

$dim(\overline{J}_1, \overline{J}_2) = (2\overline{J}_1 - 2\overline{J}_2 + 1)(2\overline{J}_1 + 2\overline{J}_2 + 3)(2\overline{J}_1 + 2)(2\overline{J}_2 + 1)/6$ is the dimension of the $so(5)$ irreducible representation $(\overline{J}_1, \overline{J}_2)$ [45].

To summarize: matrix elements of (noncompact) shear generators (51), together with the known matrix elements of the (compact) $so(5)$ operators (24), give an action of the $sl(5, \mathbb{R})$ algebra on the basis vectors (49) space over the maximal compact subgroup $Spin(5)$ of the group $\overline{SL}(5, \mathbb{R})$. This result is general due to Corollary from the Harish-Chandra paper [40], that is directly applicable to the case of $sl(5, \mathbb{R})$ algebra.

## 4.3 Generalization of the Gell-Mann formula for arbitrary $n$

The generalized Gell-Mann formulas for $sl(3,\mathbb{R})$, $sl(4,\mathbb{R})$ and $sl(5,\mathbb{R})$ [25] are given by rather cumbersome expressions. However, when these formulas are expressed in the Cartesian basis (like formulas (2)-(4)) in terms of the $K_{ab}$ operators and anti-commutators rather than commutators the resulting expressions become extremely simple. Moreover, this form allows for an immediate generalization to the case of an arbitrary $n$. We prove below that the generalized Gell-Mann formula for any $sl(n,\mathbb{R})$ algebra w.r.t its $so(n)$ subalgebra takes the following form::

$$T_{ab}^{\sigma_2\ldots\sigma_n} = -\sum_{c>d}^{n}\{K_{cd}, D^{\square\square}_{(cd)(ab)}\} + i\sum_{c=2}^{n}\sigma_c D^{\square\square}_{(cc)(ab)}, \tag{52}$$

where $\sigma_c$ is a set of $n-1$ arbitrary parameters that essentially (up to some discrete parameters) label $sl(n,\mathbb{R})$ irreducible representations. Note that the sum in the first term goes only over pairs $(c,d)$ where $c>d$ i.e. it is not symmetric in $c,\ d$.

Let us begin the proof that the expressions (52) satisfy the $sl(n,\mathbb{R})$ commutation relation (4) by introducing operators:

$$T_{ab}^{[c]} = -\sum_{d=1}^{c-1}\{K_{cd}, D^{\square\square}_{(cd)(ab)}\} + i\sigma_c D^{\square\square}_{(cc)(ab)}, \quad c=2,\ldots,n \tag{53}$$

and expressing the generalized expression (52) as:

$$T_{ab} = \sum_{c=2}^{n} T_{ab}^{[c]}. \tag{54}$$

Next we calculate the commutator $[T^{[c]}_{ab}, T^{[d]}_{a'b'}]$ for $c < d$:

$$
\begin{aligned}
[T^{[c]}_{ab}, T^{[d]}_{a'b'}] &= \sum_{e=1}^{c-1} [-\{K_{ce}, D^{\square\square}_{(ce)(ab)}\}, -\sum_{f=1}^{d-1}\{K_{df}, D^{\square\square}_{(df)(a'b')}\}] \\
&+ \sum_{e=1}^{c-1} i\sigma_d [-\{K_{ce}, D^{\square\square}_{(ce)(ab)}\}, D^{\square\square}_{(dd)(a'b')}] \\
&+ \sum_{f=1}^{d-1} i\sigma_c [D^{\square\square}_{(cc)(ab)}, -\{K_{df}, D^{\square\square}_{(df)(a'b')}\}] \qquad (55) \\
&= \sum_{e=1}^{c-1} [\{K_{ce}, D^{\square\square}_{(ce)(ab)}\}, \{K_{dc}, D^{\square\square}_{(dc)(a'b')}\} + \{K_{de}, D^{\square\square}_{(de)(a'b')}\}] \\
&+ 0 - i\sigma_c [D^{\square\square}_{(cc)(ab)}, \{K_{dc}, D^{\square\square}_{(dc)(a'b')}\}] \\
&= \cdots = i \sum_{e=1}^{c-1} \{K_{ce}, \{D^{\square\square}_{(ed)(ab)}, D^{\square\square}_{(cd)(a'b')}\} + \{D^{\square\square}_{(cd)(ab)}, D^{\square\square}_{(ed)(a'b')}\}\} \\
&+ 2\sigma_c \{D^{\square\square}_{(dc)(ab)}, D^{\square\square}_{(dc)(a'b')}\}.
\end{aligned}
$$

The result is symmetric under the change of pair of indices $(ab) \leftrightarrow (a'b')$, and similarly can be concluded when $c > d$. We conclude that for $c \neq d$ it holds $[T^{[c]}_{ab}, T^{[d]}_{a'b'}] = [T^{[c]}_{a'b'}, T^{[d]}_{ab}]$, and thus we find::

$$
[T_{ab}, T_{a'b'}] = \sum_c [T^{[c]}_{ab}, T^{[c]}_{a'b'}] = i \sum_{c,d,d'} \{K_{dd'}, \{D^{\square\square}_{(cd)(ab)}, D^{\square\square}_{(cd')(a'b')}\}\}. \qquad (56)
$$

By making use of the identity:

$$
\begin{aligned}
&\textstyle\sum_c (D^{\square\square}_{(cd)(ab)} D^{\square\square}_{(cd')(a'b')} - D^{\square\square}_{(cd')(ab)} D^{\square\square}_{(cd)(a'b')}) \qquad (57) \\
&= \tfrac{1}{2}(\delta_{aa'} D^{\substack{\square\\\square}}_{(dd')(bb')} + \delta_{bb'} D^{\substack{\square\\\square}}_{(dd')(aa')} + \delta_{ab'} D^{\substack{\square\\\square}}_{(dd')(ba')} + \delta_{ba'} D^{\substack{\square\\\square}}_{(dd')(ab')}),
\end{aligned}
$$

and the fact that the $M$ generators are given in terms of the $K$ operators via the $D^{\substack{\square\\\square}}$ operators (cf. (26)), one verifies the desired expression (4).

Note that the first equality in (56) implies that the overall sign of operators $T^{[c]}_{ab}$ is inessential. Moreover, any left rotation (generated by the

$K$ operators) of the generalized formula (52) will preserve the $[T, T]$ commutator (4) and thus lead to another valid expression for the generalized Gell-Mann formula. The generalized formulas related in this way form an equivalence class of formulas that yield the same set of $sl(n, \mathbb{R})$ irreducible representations. Besides this class there are a few alternative useful expressions of the generalized Gell-Mann formula. We point out explicitly two cases below.

Let us consider operators:

$$U_{ab}^{(cd)} \equiv D^{\square\square}_{(cd)(ab)}, \tag{58}$$

stressing that $D^{\square\square}_{(cd)(ab)}$ is just a particular representation of the $U_{ab}$ operators (21), characterized by the choice of the vector $v$ to be $v = (cd)$ and $|u|= 1$. Then, by making use of the commutation relations to shift the $K$ operators to the right in (52) we find :

$$T_{ab} = -2 \sum_{c>d}^{n} U_{ab}^{(cd)} K_{cd} + i \sum_{c=2}^{n} \sigma'_c U_{ab}^{(cc)}. \tag{59}$$

The parameters in the two forms of the formula are connected by relation: $\sigma'_c = \sigma_c - 2(c-1)$.

The last expression for the generalized formula can now be directly compared to the original formula in the form (18). It is as simple as the original Gell-Mann formula, with a crucial advantage of being valid in the whole representation space over $\mathcal{L}^2(Spin(n))$. General validity of the new formula is reflected in the fact that there are now $n-1$ free parameters, i.e. representation labels, matching the $sl(n, \mathbb{R})$ algebra rank, compared to just one parameter of the original Gell-Mann formula.

Another notable form of the generalized formula relies on the fact that the operators $T^{[c]}$ (53) can be written as:

$$T_{ab}^{[c]} = \frac{i}{2}[C_2(so(c)_K), U_{ab}^{(cc)}] + i\sigma_c U_{ab}^{(cc)}, \quad c = 2, \ldots, n \tag{60}$$

where $C_2(so(c)_K)$ is the second order Casimir of the $so(c)$ left action subalgebra, i.e. $C_2(so(c)_K) = \frac{1}{2}\sum_{a,b=1}^{c}(K_{ab})^2$. The generalized Gell-Mann formula can now be written as:

$$T_{ab}^{\sigma_2 \ldots \sigma_n} = i \sum_{c=2}^{n} \frac{1}{2}[C_2(so(c)_K), U_{ab}^{(cc)}] + \sigma_c U_{ab}^{(cc)}, \tag{61}$$

which is to be compared with the original formula in the form (17). Again, the generalized formula matches, by simplicity of the expression, the original one. Besides, the very term when $c = n$ is, essentially, the original Gell-Mann formula (since $C_2(so(n)_K) = C_2(so(n)_M)$), whereas the rest of the terms can be seen as necessary corrections securing the formula validity in the entire representation space. The additional terms vanish for some representations yielding the original formula.

The generalized Gell-Mann formula expression for the noncompact "shear" generators $T_{ab}$ holds for all cases of $sl(n, \mathbb{R})$ irreducible representations, irrespective of their $so(n)$ subalgebra multiplicity (multiplicity free of the original Gell-Mann formula, and nontrivial multiplicity) and whether they are tensorial or spinorial. The price paid is that the Generalized Gell-Mann formula is no longer solely a Lie algebra operator expression, but an expression in terms of representation dependant operators $K_{ab}$ and $U^{(cd)_{ab}}$.

## 4.4 Direct application – martix elements of $SL(n, \mathbb{R})$ generators for arbitrary irreducible representation

The generalized Gell-Mann formula, as given by (61), can be directly applied to yield all matrix elements of the $\overline{SL}(n, \mathbb{R})$ generators for all irreducible representations, characterized by a complete set of labels $\sigma_i$, $i = 2, 3, \ldots, n$ (the invariant Casimir operators are analytic functions of solely these labels), in the basis of the maximal compact subgroup $Spin(n)$. Taking the matrix elements of (61) we get:

$$
\begin{aligned}
&\left\langle \begin{matrix}\{J'\} & \\ \{k'\} & \{m'\}\end{matrix} \right| T_{ab}^{\sigma_2\ldots\sigma_n} \left| \begin{matrix}\{J\} & \\ \{k\} & \{m\}\end{matrix} \right\rangle \qquad (62)\\
=\ &\left\langle \begin{matrix}\{J'\} & \\ \{k'\} & \{m'\}\end{matrix} \right| i\sum_{c=2}^{n}\frac{1}{2}[C_2(so(c)_K), U_{ab}^{(cc)}] + \sigma_c U_{ab}^{(cc)} \left| \begin{matrix}\{J\} & \\ \{k\} & \{m\}\end{matrix} \right\rangle \\
=\ &\frac{i}{2}\sum_{c=2}^{n}(C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \sigma_c) \left\langle \begin{matrix}\{J'\} & \\ \{k'\} & \{m'\}\end{matrix} \right| U_{ab}^{(cc)} \left| \begin{matrix}\{J\} & \\ \{k\} & \{m\}\end{matrix} \right\rangle \\
=\ &\tfrac{i}{2}\sqrt{\tfrac{dim(\{J\})}{dim(\{J'\})}}\sum_{c=2}^{n}(C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \sigma_c) C^{\{J\}\,\square\square\,\{J'\}}_{\{k\}\,(cc)\,\{k'\}}\ C^{\{J\}\,\square\square\,\{J'\}}_{\{m\}\,(ab)\,\{m'\}}\ ,
\end{aligned}
$$

where, in the last equality, the expression (25) for the matrix elements of the $U$ operators is used. The second Clebsch-Gordan coefficient, that is merely reflecting the Wigner-Eckart theorem, can be evaluated in any suitable basis, not necessarily the Cartesian one, due to the fact that the expression is covariant with respect to the free index $(ab)$. Note, that this is not the case for the first Clebsch-Gordan coefficient – it is necessary in order to evaluate it to express the specific vector $\left| \begin{matrix}\square\square \\ (cc)\end{matrix} \right\rangle$ in some basis that spans the entire vector space over $Spin(n)$.

The final expression is simplified by choosing the indexes of the generalized Gell-Mann formula matrix elements to be given by labels of the $Spin(n) \supset Spin(n-1) \supset \cdots \supset Spin(2)$ group chain representation labels. In this notation, the basis vectors of the $Spin(n)$ irreducible representations are written as:

$$
\left| \begin{matrix}\{J\} \\ \{m\}\end{matrix} \right\rangle = \left| \begin{matrix} J_{Spin(n),1}\ J_{Spin(n),2}\ J_{Spin(n),3}\cdots \\ J_{Spin(n-1),1}\ J_{Spin(n-1),2}\cdots \\ \cdots \\ J_{Spin(2)} \end{matrix} \right\rangle . \qquad (63)
$$

Likewise, the set of indices $\{k\}$ of (22) is thus given by the labels of the irreducible representations $\{J_{Spin(n-1),1}, J_{Spin(n-1),2}, \cdots; J_{Spin(n-2),1}, J_{Spin(n-2),2}, \cdots; \ldots; J_{Spin(2)}\}$ of the $Spin(n) \supset Spin(n-1) \supset \cdots \supset Spin(2)$ group chain.

To express the vector $\left| \begin{matrix}\square\square \\ (cc)\end{matrix} \right\rangle$ in such a basis we notice first that it corresponds to a diagonal traceless $n$ by $n$ matrix of the form $diag(-\frac{1}{n}, \ldots, -\frac{1}{n}$

, $\frac{n-1}{n}$, $-\frac{1}{n}, \dots, -\frac{1}{n}$), with $\frac{n-1}{n}$ positioned at the $c$-th row and column. On the other hand, the diagonal traceless matrix $\sqrt{\frac{1}{c(c-1)}}\, diag(-1, \dots, -1, c-1, 0, \dots, 0)$, with first $c-1$ occurrences of $-1$, corresponds to a vector that belongs to a second order symmetric tensor ($\square\square$ representation) with respect to $Spin(c), Spin(c+1), \dots, Spin(n)$ subgroups, and it is invariant under $Spin(c-1)$:

$$\left| \begin{array}{l} \{\square\square\}_{Spin(n)} \\ \cdots \\ \{\square\square\}_{Spin(c)} \\ \{0\}_{Spin(c-1)} \\ \cdots \\ 0 \end{array} \right\rangle . \tag{64}$$

This vector has $n-c+1$ double-boxes followed by $c-2$ zeros underneath – in shorthand notation: $\left| \begin{array}{l} \{\square\square\}^{n-c+1} \\ \{0\}^{c-2} \end{array} \right\rangle$. Somewhat peculiar is the matrix $\sqrt{\frac{1}{2}}\, diag(-1, 1, 0, 0, \dots)$ that corresponds to:

$$\left| \begin{array}{l} \{\square\square\}^{n-1} \\ \{0\}^{0} \end{array} \right\rangle \equiv \frac{1}{\sqrt{2}} \left| \begin{array}{l} \{\square\square\}^{Spin(n)} \\ \cdots \\ \{\square\square\}^{Spin(4)} \\ 2 \\ 2 \end{array} \right\rangle + \frac{1}{\sqrt{2}} \left| \begin{array}{l} \{\square\square\}^{Spin(n)} \\ \cdots \\ \{\square\square\}^{Spin(4)} \\ 2 \\ -2 \end{array} \right\rangle , \tag{65}$$

where the standard labelling for $SO(n)$, $n \leq 3$ is implied, in particular the $\square\square$ representation corresponds to $J_{Spin(3)} = 2$).

By combining these facts we find:

$$\left| \begin{array}{c} \square\square \\ (cc) \end{array} \right\rangle + \frac{1}{c} \sum_{d=c+1}^{n} \left| \begin{array}{c} \square\square \\ (dd) \end{array} \right\rangle = \sqrt{\frac{c-1}{c}} \left| \begin{array}{l} \{\square\square\}^{n-c+1} \\ \{0\}^{c-2} \end{array} \right\rangle . \tag{66}$$

However, when evaluating the $U^{(cc)}$ operators of (61) in this basis, only the first term on the left-hand side is relevant due to the fact that:

$$d > c \quad \Rightarrow \quad [C_2(so(c)_K), U^{(dd)}_{ab}] = 0. \tag{67}$$

Having this in mind, we make use of (66) to recast, in the first equality of (62), the $U^{(cc)}$ operators accordingly. Taking into account arbitrariness of the $\sigma_c$ coefficients and following the same steps as in (62), we finally obtain a rather simple expression for the shear generator matrix elements for an arbitrary $sl(n,\mathbb{R})$ representation (labelled now by parameters $\tilde{\sigma}_c$):

$$\left\langle \begin{array}{l} \{J'\} \\ \{k'\}\{m'\} \end{array} \middle| T_{\{w\}} \middle| \begin{array}{l} \{J\} \\ \{k\}\{m\} \end{array} \right\rangle = \tfrac{i}{2}\sqrt{\tfrac{dim(\{J\})}{dim(\{J'\})}}\, C^{\{J\}\square\square\{J'\}}_{\{m\}\{w\}\{m'\}}$$
$$\times \sum_{c=2}^{n} \sqrt{\tfrac{c-1}{c}} \Big( C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \tilde{\sigma}_c \Big) C^{\{J\}(\square\square)^{n-c+1}\{J'\}}_{\{k\}\quad (0)^{c-2} \quad \{k'\}} \; . \tag{68}$$

The relation of the labelling of (43) and the one of (52), i.e. (61), is achieved provided $\sigma_c = \tilde{\sigma}_c + \sum_{d=2}^{c-1} \tilde{\sigma}_d/d$. The Clebsch-Gordan coefficient with indices $\{m\}, \{w\}, \{m'\}$ in (43) can be evaluated in an arbitrary basis (which is stressed by denoting the appropriate index by $w$ instead by $ab$). The other Clebsch-Gordan coefficient can be evaluated in any basis labelled according to the $Spin(n) \supset Spin(n-1) \supset \cdots \supset Spin(2)$ subgroup chain (e.g. Gel'fand-Tsetlin basis) and can be, nowadays, rather easily evaluated, at least numerically.

## 4.5 A comment on the generalized formula

As already stated, the matrix elements of the $sl(n,\mathbb{R})/so(n)$ operators, as given by the Generalized Gell-Mann formula, apply to all tensorial, spinorial, unitary, nonunitary (both finite a infinite-dimensional) $sl(n,\mathbb{R})$ irreducible representations. In many physics applications one is interested in the unitary irreducible representations. The unitarity question goes beyond the scope of the present paper, and it relates to the Hilbert space properties, i.e. the vector space scalar product. An efficient method to study unitarity is to start with a Hilbert space $L^2(Spin(n),\kappa)$ of square integrable functions with a scalar product in terms of an arbitrary kernel $\kappa$, and to impose the unitarity constraints both on the scalar products itself and on the $sl(n,\mathbb{R})/so(n)$ operators matrix elements in that scalar product (cf. [41]).

We note that the results of the previous subsection can be directly conveyed to the case of special unitary group $SU(n)$. Namely, operators of $su(n)$ algebra can be, similarly as in the case of $sl(n,\mathbb{R})$ algebra, split w.r.t. its $so(n)$ subalgebra into $M_{ab}$ operators and $T^{su(n)}_{ab}$, $a,b = 1,2,...,n$ operators. Relation of $T^{su(n)}_{ab}$ and $T_{ab}$ operators is a direct one: $T^{su(n)}_{ab} = iT_{ab}$, and the commutator $[T^{su(n)}_{ab}, T^{su(n)}_{cd}]$ differs from the commutator (4) only by an overall minus sign. Therefore, all formulas obtained for $SL(n,\mathbb{R})$ shear generators $T_{ab}$ (44, 46, 47, 52, 59, 61, 68), are after multiplying by imaginary unit also applicable to $SU(n)$ generators $T_{ab}^{su(n)}$, with the following remark: since the $SU(n)$ group is its own covering group, space $L^2(Spin(n))$ has to be reduced to space $L^2(SO(n))$.

To sum up, the expressions (24) and (43) fully determine the action of the $sl(n,\mathbb{R})$ operators for an arbitrary irreducible representation given by the set of $n-1$ invariant Casimir operators labels $\tilde{\sigma}_c$. This action is given in the basis (22) of the representation spaces of the maximal compact subgroup $Spin(n)$ of the $\overline{SL}(n,\mathbb{R})$ group. This result is general due to a Corollary of Harish-Chandra [40] that explicitly applies to the case of the $sl(n,\mathbb{R})$ algebras.

# 5 Application in affine theories of gravity

## 5.1 Affine gravity models

In the introduction we have listed a few models of space-time symmetries and gravity where special linear group plays a substantial role. In this section we will briefly illustrate where the Gell-Mann formula can be applied in the context of affine theories of gravity.

Mostly due to the difficulties encountered in attempts to quantize Einstein's theory of gravity, a new ways to generalize and expand the basic concepts of Riemannian geometry and general relativity were sought. Affine theory of gravity is one of the possible directions to take. It is interesting to mention that even Einstein himself has considered affine generalizations of general relativity [51].

In affine gravity models the flat space-time symmetry of the theory (prior to any symmetry breaking) is given by the General Affine Group $GA(n,\mathbb{R}) = T^n \wedge GL(n,\mathbb{R})$ (or, sometimes, by the Special Affine Group $SA(n,\mathbb{R}) = T^n \wedge SL(n,\mathbb{R})$). In the quantum case, the General Affine Group is replaced by its double cover counterpart $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$, which contains double cover of $\overline{GL}(n,\mathbb{R})$ as a subgroup. This subgroup here plays the role that Lorentz group has in the Poincaré symmetry case. Thus it is clear that knowledge of $\overline{GL}(n,\mathbb{R})$ representations is a must-know for any serious analysis of Affine Gravity models. On the other hand, the essential part of the $\overline{GL}(n,\mathbb{R}) = R_+ \otimes \overline{SL}(n,\mathbb{R})$ group is its $\overline{SL}(n,\mathbb{R})$ subgroup, and that is where $\overline{SL}(n,\mathbb{R})$ generators matrix elements, obtained by using the generalized Gell-Mann formula, come into play ($R_+$ is subgroup of dilatations). We will apply expression for these matrix elements in order to obtain coefficients for some of the gauge field–matter interaction vertices.

Gravitational interaction is into these models usually introduced by gauging the global affine symmetry $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$. Since in the tensor product of two defining representation of $GL(n,\mathbb{R})$ group (also of $SL(n,\mathbb{R})$ group) does not appear any $GL(n,\mathbb{R})$ (or $SL(n,\mathbb{R})$ respectively) invariant tensor, there is also no equivalent of Minkovski metrics $\eta_{\mu\nu}$, and connection will not preserve length of vectors. Actually, as the transition from Riemannian to Riemann-Cartan space can be seen as a result of introduction of torsion, similarly, the transition from the Riemann-Cartan geometry to the affine geometry is related to abandoning of the requirement of metricity.

## 5.2 Gauge Affine action

A standard way to introduce interactions into affine gravity models is by localization of the global affine symmetry $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$. Thus, quite generally, affine Lagrangian consists of a gravitational part (i.e. kinetic terms for gauge potentials) and Lagrangian of the matter fields: $L = L_g + L_m$. Gravitational part $L_g$ is a function of gravitational gauge potentials and their derivatives, and also of the dilaton field $\varphi$ (that ensures action invariance under local dilatations). In the case of the standard Metric affine gravity [1, 2], gravitational potentials are tetrads $e^a{}_\mu$, metrics $g_{ab}$ and affine connection $\Gamma^a{}_{b\mu}$, so that we can write: $L_g = L_g(e, \partial e, g, \partial g, \Gamma, \partial\Gamma, \varphi)$. More precisely, due to action invariance under local affine transformations, gravitational part of Lagrangian must be a function of the form $L_g = L_g(e, g, T, R, N, \varphi)$, where $T^a{}_{\mu\nu} = \partial_\mu e^a{}_\nu + \Gamma^a{}_{b\mu} e^b{}_\nu - (\mu \leftrightarrow \nu)$, $R^a{}_{b\mu\nu} = \partial_\mu \Gamma^a{}_{b\nu} + \Gamma^c{}_{b\mu}\Gamma^a{}_{c\nu} - (\mu \leftrightarrow \nu)$, $N_{\mu ab} = D_\mu g_{ab}$ are, respectively, torsion, curvature and nonmetricity. Assuming, as usual, that equations of motion are linear in second derivatives of gauge fields, we are confined to no higher than quadratic powers of the torsion, curvature and nonmetricity. Covariant derivative is of the form $D_\mu = \partial_\mu - i\Gamma_a{}^b{}_\mu Q_b{}^a$, where $Q_b{}^a$ denote generators of $\overline{GL}(n,\mathbb{R})$ group. The matter Lagrangian (assuming minimal coupling for all fields except the dilaton one) is a function of some number of affine fields $\phi^I$ and their covariant derivatives, together with metrics and tetrads (affine connection enters only through covariant derivative): $L_m = L_m(\phi^I, D\phi^I, e, g)$.

With all these general remarks, we will consider a class of affine Lagrangians, in arbitrary number of dimensions $n$, of the form:

$$L(e_{\mu}{}^{a}, \partial_{\nu} e_{\mu}{}^{a}, \Gamma_{b\mu}{}^{a}, \partial_{\nu}\Gamma_{b\mu}{}^{a}, g_{ab}, \Psi_{A}, \partial_{\nu}\Psi_{A}, \Phi_{A}, \partial_{\nu}\Phi_{A}, \varphi, \partial_{\nu}\varphi) =$$
$$e\Big[\varphi^2 R - \varphi^2 T^2 - \varphi^2 N^2 +$$
$$\bar{\Psi} i g^{ab}\gamma_a e_b{}^{\mu} D_{\mu}\Psi + \tfrac{1}{2} g^{ab} e_a{}^{\mu} e_b{}^{\nu} (D_{\mu}\Phi)^{+}(D_{\nu}\Phi) + \tfrac{1}{2} g^{ab} e_a{}^{\mu} e_b{}^{\nu} D_{\mu}\varphi D_{\nu}\varphi +$$
$$- L_g(n) + L_m(n)\Big]. \tag{69}$$

The terms in the first row represent general gravitational part of the Lagrangian, that is invariant w.r.t. affine transformations (dilatational invariance is obtained with the aid of field $\varphi$, of mass dimension $n/2-1$). Here $T^2$ and $N^2$ stand for linear combination of terms quadratic in torsion and nonmetricity, respectively, formed by irreducible components of these fields (a discussion of available possibilities can be found in Appendix B of [2]). For the scope of this paper, we need not fix these terms any further. This is a general form of gravitational kinetic terms, invariant for an arbitrary space-time dimension $n \geq 3$.

The Lagrangian matter terms, invariant w.r.t. the local $\overline{GA}(n, \mathbb{R})$, $n \geq 3$, transformations, are written in the second row. The field $\Psi$ denotes a spinorial $\overline{GL}(n, \mathbb{R})$ field – components of that field transform under some appropriate spinorial $\overline{GL}(n, \mathbb{R})$ irreducible representations. All spinorial $\overline{GL}(n, \mathbb{R})$ representations are necessarily infinite dimensional [6], and thus the field $\Psi$ will have infinite number of components. The concrete spinorial irreducible representation of field $\Psi$ is given by a set of $n-1$ $\overline{SL}(n, \mathbb{R})$ labels $\{\sigma_c^{\Psi}\}$ together with the dilatation charge $d_{\Psi}$. The field $\Phi$ is a representative of a tensorial $\overline{GL}(n, \mathbb{R})$ field, transforming under a tensorial $\overline{GL}(n, \mathbb{R})$ representation (i.e. one transforming w.r.t. single-valued representation of the $SO(n)$ subgroup) labelled by parameters $\{\sigma_c^{\Phi}\}$ and $d_{\Phi}$. Since, as it is argued in the following section, the noncompact $\overline{SL}(n-1, \mathbb{R})$ affine subgroup is to be represented unitarily, the tensorial field $\Phi$ is also to transform under an infinite-dimensional representation and to have an infinite number of components. The remaining dilaton field $\varphi$ is scalar with respect to $\overline{SL}(n, \mathbb{R})$ subgroup, and thus has only one component.

Finally, the third row contains possible additional gravitational and matter terms, denoted respectively by $L_g(n)$ and $L_m(n)$, that, due to restrictions imposed by the dilatational invariance requirement, can appear only for some concrete values of $n$. (E.g., in [5] dealing with the four dimensional case, authors take $L_g(4) = \alpha_1 R_{[abcd]} R^{[abcd]} + \alpha_2 R_{[ab[c]d]} R^{[ab[c]d]} + \alpha_3 R_{[a(b][c)d]} R^{[a(b][c)d]} + \alpha_4 R_{(a[b)cd]} R^{(a[b)cd]} + \alpha_5 R_{(ab[c)d]} R^{(ab[c)d]}$, and $L_m(4) = \mu \bar{\Psi}\Phi\Psi - \lambda_\Phi (\Phi^+\Phi)^2 - \lambda(\Phi^+\Phi)\varphi^2 - \lambda_\varphi \varphi^4$.)

Interaction of affine connection with matter fields is determined by terms containing covariant derivatives. We write these terms in a component notation, where the component labelling is done with respect to the physically important Lorenz $Spin(1, n-1)$ subgroup of $\overline{GL}(n, \mathbb{R})$. Such a labelling allows, in principle, to identify affine field components with Lorentz fields of models based on the Poincaré symmetry. Namely, the affine models of gravity necessarily imply existence of some symmetry breaking mechanism that reduces the global symmetry to the Poincaré one, reflecting the subgroup structure $T^n \wedge \overline{SO}(1, n-1) \subset T^n \wedge \overline{GL}(n, \mathbb{R})$. Therefore, we consider the field $\Psi$ (and similarly for $\Phi$ field) as a sum of its Lorentz components:

$$\sum_{\substack{\{J\}\\ \{k\}\{m\}}} \Psi^{\{J\}}_{\{k\}\{m\}} \Big|^{\{J\}}_{\{k\}\{m\}}\Big\rangle.$$

Ket vectors in this decomposition are basis vectors of the $\{\sigma_c^\Psi\}$ representation of $\overline{SL}(n, \mathbb{R})$ group [26]. Sets of labels $\{J\}$ and $\{m\}$ determine transformation properties of a basis vector under the Lorentz $Spin(1, n-1)$ subgroup: $\{J\}$ label irreducible representation of $Spin(1, n-1)$, while numbers $\{m\}$ label particular vector within that representation. The set of parameters $\{k\}$ enumerate $Spin(1, n-1)$ multiplicity of representation $\{J\}$ within the $\{\sigma_c^\Psi\}$ representation of $\overline{SL}(n, \mathbb{R})$. These parameters $\{k\}$ are mathematically related to the left action of $Spin(n)$ subgroup in representation space $\mathcal{L}^2(Spin(n))$ of square integrable functions over the $Spin(n)$ group (for more details c.f. [26]).

The interaction term connecting fields $g^{cd}$, $e_d^{\ \mu}$, $\Gamma^{ab}_\mu$, $\bar{\Psi}^{\{J\}}_{\{k\}\{m\}}$, $\Psi^{\{J'\}}_{\{k'\}\{m'\}}$ is now:

$$g^{cd} e_d^{\ \mu} \Gamma^{ab}_\mu \bar{\Psi}^{\{J\}}_{\{k\}\{m\}} \Psi^{\{J'\}}_{\{k'\}\{m'\}} \sum_{\substack{\{J''\}\\ \{k''\}\{m''\}}} \Big\langle^{\{J\}}_{\{k\}\{m\}} \Big| \gamma_c \Big|^{\{J''\}}_{\{k''\}\{m''\}}\Big\rangle \Big\langle^{\{J''\}}_{\{k''\}\{m''\}} \Big| Q_{ab} \Big|^{\{J'\}}_{\{k'\}\{m'\}}\Big\rangle, \tag{70}$$

while the interaction of tensorial field with connection is given by:

$$-\tfrac{i}{2}g^{cd}e_c{}^{\mu}e_d{}^{\nu}\Gamma^{ab}_{\nu}\partial_{\mu}\Phi^{\dagger\{J\}}_{\{k\}\{m\}}\Phi^{\{J'\}}_{\{k'\}\{m'\}}\langle^{\{J\}}_{\{k\}\{m\}}|Q_{ab}|^{\{J'\}}_{\{k'\}\{m'\}}\rangle + \tag{71}$$

$$\tfrac{i}{2}g^{cd}e_c{}^{\mu}e_d{}^{\nu}\Gamma^{ab}_{\nu}\Phi^{\dagger\{J\}}_{\{k\}\{m\}}\partial_{\mu}\Phi^{\{J'\}}_{\{k'\}\{m'\}}\langle^{\{J'\}}_{\{k'\}\{m'\}}|Q_{ab}|^{\{J\}}_{\{k\}\{m\}}\rangle^{*} + \tag{72}$$

$$\tfrac{1}{2}g^{cd}e_c{}^{\mu}e_d{}^{\nu}\Gamma^{ab}_{\mu}\Gamma^{a'b'}_{\nu}\Phi^{\dagger\{J\}}_{\{k\}\{m\}}\partial_{\mu}\Phi^{\{J'\}}_{\{k'\}\{m'\}}\cdot$$

$$\sum_{\substack{\{J''\}\\ \{k''\}\{m''\}}}\langle^{\{J\}}_{\{k\}\{m\}}|Q_{ab}|^{\{J''\}}_{\{k''\}\{m''\}}\rangle\langle^{\{J''\}}_{\{k''\}\{m''\}}|Q_{a'b'}|^{\{J'\}}_{\{k'\}\{m'\}}\rangle. \tag{73}$$

The scalar dilaton field interact only with the trace of affine connection:

$$\tfrac{1}{2}g^{ab}e_a{}^{\mu}e_b{}^{\nu}(\partial_{\mu}-i\Gamma_a{}^{a}{}_{\mu}d_{\varphi})\varphi(\partial_{\nu}-i\Gamma_a{}^{a}{}_{\nu}d_{\varphi})\varphi, \tag{74}$$

where $d_{\varphi}$ denotes dilatation charge of $\varphi$ field.

In the above interaction terms we note an appearance of matrix elements of $\overline{GL}(n,\mathbb{R})$ generators, written in a basis of the Lorenz subgroup $Spin(1,n-1)$. The dilatation generator (that is, the trace $Q^a_a$) acts merely as multiplication by dilatation charge, so it is really the $\overline{SL}(n,\mathbb{R})$ matrix elements that should be calculated. (An infinite dimensional generalization of Dirac's gamma matrices also appear in the term (70); more on these matrices can be found in papers of Šijački [53].) However, before we illustrate how to evaluate these matrix elements, and thus how to calculate vertex coefficients, we must make some additional general remarks on $\overline{GL}(n,\mathbb{R})$ representations that correspond to physical fields.

## 5.3 Deunitarizing automorphism

We will briefly discuss the matter of unitarity of the representations corresponding to fields in affine models. In standard, Poincaré symmetric models, gauge and matter fields have finite number of components and this fits well the experimental data. However, since the Lorenz group is a non compact one, this is made possible by the fact that the fields transform under the non-unitary representations of the Lorenz group. Note that it is only the compact $\overline{SO}(n-1)$ part of the Lorentz group that is represented unitary. If the unitary, so called Gelfand-Naimark, representations of the Lorenz group were used [52], the boosts would mix infinitely many field components, in contrary to observations.

For the same physical reasons, the Lorenz subgroup of $\overline{GL}(n,\mathbb{R})$ should act in an analogous way on $\overline{GL}(n,\mathbb{R})$ fields: boosts should be represented non unitarily and the Lorenz subgroup should reduce in finite dimensional subspaces of field components. On the other hand, much in the same way as spatial rotation part of the Lorenz group acts unitarily on Poincaré fields, it is physically favorable that the spatial "little group" $\overline{GL}(n-1,\mathbb{R})$, a subgroup of $\overline{GL}(n,\mathbb{R})$, acts unitarily on field components.

This can be elegantly accomplished by using a so called deunitarizing automorphism. Namely, there exists an inner automorphism [6], which leaves the $R_+ \otimes \overline{SL}(n-1,\mathbb{R})$ subgroup intact, and which maps the $Q_{(0k)}$, $Q_{[0k]}$ generators into $iQ_{[0k]}$, $iQ_{(0k)}$ respectively ($k = 1, 2, \ldots n-1$). Here $Q_{[ab]} = \frac{1}{2}(Q_{ab} - Q_{ba})$ denote the antisymmetric operators that generate the Lorentz subgroup $Spin(1, n-1)$, whereas $Q_{(ab)} = \frac{1}{2}(Q_{ab} + Q_{ba}) - \frac{1}{n} g_{ab} Q_c{}^c$ are the symmetric traceless operators that generate the proper $n$-volume-preserving deformations (shears).

The deunitarizing automorphism thus allows us to start with the unitary representations of the $\overline{SL}(n,\mathbb{R})$ subgroup, and upon its application, to identify the finite (unitary) representations of the abstract $\overline{SO}(n)$ compact subgroup with nonunitary representations of the physical Lorentz group, while the infinite (unitary) representations of the abstract $\overline{SO}(1,n-1)$ group now represent (non-unitarily) the compact $\overline{SO}(n)/\overline{SO}(n-1)$ generators.

## 5.4 Gauge affine symmetry vertex coefficients evaluation

Now we return to evaluation of vertex coefficients for interaction between various Lorentz components of the $\overline{GL}(n,\mathbb{R})$ fields. The nontrivial part of the problem is to find matrix elements of $\overline{SL}(n,\mathbb{R})$ shear generators in expressions (70)-(73). We will do that by using formula (68).

However, formula (68) is given in the basis of the compact $Spin(n)$ subgroup, and not in the basis of the physically important Lorentz group $Spin(1, n-1)$. On the other hand, it turns out that taking into account deunitarizing automorphism exactly amounts to keeping reduced matrix element from (68) and replacing the remaining Clebsch-Gordan coefficient of the $Spin(n)$ group by the corresponding coefficient of the Lorenz group

$Spin(1, n-1)$.

Now, as a concrete example, we will consider tensorial affine field $\Phi$ in $n = 5$ dimensions. For example, let the field $\Phi$ correspond to an unitary multiplicity free $\overline{SL}(5, \mathbb{R})$ representation, defined by labels $\sigma_2 = -4, \delta_1 = \delta_2 = 0$, with $\sigma_1$ arbitrary real. The representation space is spanned by vectors (49) satisfying $\overline{J}_1 = \overline{J}_2 = \overline{J} \in N_0 + \frac{1}{2}; K_1 = K_2 = 0; J_1 = J_2 = J \leq \overline{J}$. This is a simplest class of multiplicity free representations that is unitary assuming usual scalar product. If we denote $\Phi^a, a = 1 \ldots 5$ the five $\Phi$ components with $\overline{J}_1 = \overline{J}_2 = \frac{1}{2}$ (in this sense $\Phi^a$ corresponds to a Lorenz 5-vector) then the interaction vertex (71) connecting fields $\Phi^{a\dagger}$, $\partial_\mu \Phi^d$ and affine shear connection $\Gamma^{bc}_\nu$ is:

$$\frac{i}{2} g^{ef} {e_e}^\mu {e_f}^\nu \Phi^{a\dagger} \Gamma^{bc}_\nu \partial_\mu \Phi^d \frac{\sqrt{5}}{14}\, \sigma_1 (\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{n}\eta_{ad}\eta_{bc}). \tag{75}$$

To obtain this result we used an easily derivable formula for Clebsch-Gordan coefficient connecting Lorentz vector and symmetric second order Lorenz tensor representations:

$$C^{L\,\square\,\square\square\,\square}_{\;\;a\;(bc)\;d} = \sqrt{\tfrac{n}{2(n+2)(n-1)}} (\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{n}\eta_{ad}\eta_{bc}), \tag{76}$$

where we labelled $Spin(1, n-1)$ irreducible representations by Young diagrams, as in [26]. More importantly, we also used value of the reduced matrix element:

$$\left\langle \begin{smallmatrix} \frac{1}{2}\,\frac{1}{2} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \middle\| Q \middle\| \begin{smallmatrix} \frac{1}{2}\,\frac{1}{2} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \right\rangle = \sqrt{\frac{2}{7}} \sigma_1, \tag{77}$$

that we obtained by using formula (51) (based on this formula, a Mathematica program was generated that directly calculates $sl(5, \mathbb{R})$ matrix elements, taking into account relevant $Spin(5)$ Clebsch-Gordan coefficients given in the Appendix).

It is no more difficult to obtain coefficients of the vertices of the form (73). Lagrangian term (73) connecting Lorenz 5-vector $\Phi$ components $\Phi_5$, $\Phi_5^\dagger$ and affine connection component $\Gamma_{(55)\mu}$ is:

$$\frac{1}{15} \left(\sigma_1^2 - 25\right) g^{cd} {e_c}^\mu {e_d}^\nu \Gamma^{55}_\mu \Gamma^{55}_\nu \Phi_5^\dagger \partial_\mu \Phi_5. \tag{78}$$

In a similar fashion, we can find vertex coefficients for more complex representations with nontrivial multiplicity.

# 6 Conclusion

Gell-Mann decontraction formula is, at the algebraic level, applicable only in the case of the (pseudo)orthogonal algebras. In the case of other algebras this formula is not applicable for all representations. As for the case of $sl(n, \mathbb{R})$ algebras, contracted w.r.t. the $so(n)$ subalgebra, we saw that it can be applied only to certain classes of tensorial representations without multiplicity. More specifically, we have shown that the formula is valid only in Hilbert spaces over $Spin(n)/(Spin(m) \times Spin(n-m))$, $m = 1, 2, \ldots, n-1$. When the formula is applicable, it directly yields matrix element expressions of the $sl(n, \mathbb{R})$ operators: (24) and (43).

Starting from the known expression for generator matrix elements of $sl(3, \mathbb{R})$ and $sl(4, \mathbb{R})$ representations with multiplicity, it was possible to easily obtain expressions for the generalized Gell-Mann formulas in the corrsponding cases, and then to follow a similar pattern and obtain generalized formula in the $sl(5, \mathbb{R})$ case. By expressing the obtained formulas in Cartesian basis, the Gell-Mann formula was generalized for arbitrary dimension $n$. The generalized formula is given by the expression (52). As the most direct and important application of the formula, we obtained closed form expressions for matrix elements of $sl(n, \mathbb{R})$ operators in arbitrary irreducible representation (finite, infinite, tensorial, spinorial, multiplicity free or not). The form of the generalized formula is quite elegant and comparable by simplicity to the form of the original formula.

We have also considered an application of the Gell-Mann formula in the context of affine models of gravity.

# 7 Appendix: Clebsch-Gordan coefficients for the 14 dimensional uinitary irreducible representation of $Spin(5)$

Analytical expressions for the $Spin(5)$ Clebsch-Gordan coefficients involving the 14-dimensional representations are a must know for obtaining and confirming all of the results pertaining to the 5 dimensional case. These coefficients were published long ago [58]. However, in attempt to use these coefficients, it turned out that some 30% of the expressions in the paper are incorrect. Therefore, a detailed analysis of the polynomial expressions had to be carried out, that led to their correction. Additionally, an algorithm for numerical evaluation of $Spin(5)$ Clebsch-Gordan coefficients was developed in order to compare their values in a vast number of points.

The obtained results are given in this appendix. More details can be found in [44].

Any $Spin(5)$ Clebsch-Gordan coefficient can be written as a multiple of two $Spin(3)$ Clebshc-Gordan coefficients and one reduced $Spin(5)$ Clebsch-Gordan coefficient:

$$\left(\begin{array}{cc|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}'_1 & \bar{j}'_2 & \bar{j}''_1 & \bar{j}''_2 \\ j_1 & j_2 & j'_1 & j'_2 & j''_1 & j''_2 \\ m_1 & m_2 & m'_1 & m'_2 & m''_1 & m''_2 \end{array}\right) = \left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}'_1 & \bar{j}'_2 & \bar{j}''_1 & \bar{j}''_2 \\ j_1 & j_2 & j'_1 & j'_2 & j''_1 & j''_2 \end{array}\right) \left(\begin{array}{c|cc} j_1 & j'_1 & j''_1 \\ m_1 & m'_1 & m''_1 \end{array}\right) \left(\begin{array}{c|cc} j_2 & j'_2 & j''_2 \\ m_2 & m'_2 & m''_2 \end{array}\right). \tag{79}$$

Since the $Spin(3)$ coefficients are well known, we will list only the reduced $Spin(5)$ Clebsch-Gordan coefficients.

The direct product of a representation $(\bar{j}_1, \bar{j}_2)$ with 14-dimensional representation $(\bar{1}, \bar{1})$, decompose into the following representations:

$$\begin{aligned}(\bar{j}_1, \bar{j}_2) \otimes (\bar{1}, \bar{1}) = {} & (\bar{j}_1 + 1, \bar{j}_2 + 1) \oplus (\bar{j}_1, \bar{j}_2 + 1) \oplus (\bar{j}_1 - 1, \bar{j}_2 + 1) \\ & \oplus (\bar{j}_1 + 1, \bar{j}_2) \oplus (\bar{j}_1 - 1, \bar{j}_2) \oplus (\bar{j}_1 + 1, \bar{j}_2 - 1) \oplus (\bar{j}_1, \bar{j}_2 - 1) \\ & \oplus (\bar{j}_1 - 1, \bar{j}_2 - 1) \oplus (\bar{j}_1 + \frac{1}{2}, \bar{j}_2 + \frac{1}{2}) \oplus (\bar{j}_1 - \frac{1}{2}, \bar{j}_2 + \frac{1}{2}) \\ & \oplus (\bar{j}_1 + \frac{1}{2}, \bar{j}_2 - \frac{1}{2}) \oplus (\bar{j}_1 - \frac{1}{2}, \bar{j}_2 + \frac{1}{2}) \oplus 2(\bar{j}_1, \bar{j}_2).\end{aligned} \tag{80}$$

The reduced coefficients follow:

Clebsch-Gordan coefficients $(\bar{j}_1, \bar{j}_2) \otimes (\bar{1}, \bar{1}) \to (\bar{j}_1 + 1, \bar{j}_2 + 1)$ are:

$$N_a(\bar{j}_1, \bar{j}_2) = ((2\bar{j}_1 + 2)\,(2\bar{j}_1 + 3)\,(\bar{j}_1 + \bar{j}_2 + 2)\,(\bar{j}_1 + \bar{j}_2 + 3)\,(2\bar{j}_2 + 1)\,(2\bar{j}_2 + 2)\,(2\bar{j}_1 + 2\bar{j}_2 + 3)\,(2\bar{j}_1 + 2\bar{j}_2 + 5))^{-\frac{1}{2}}\,, \tag{81}$$

$$\left(\begin{matrix} \bar{j}_1 + 1 & \bar{j}_2 + 1 \\ j_1 + 1 & j_2 + 1 \end{matrix}\right\Vert \left.\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big(N_a(\bar{j}_1, \bar{j}_2)\ ((j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 2)\ (j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 3)\ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2 + 1)\ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2 + 2)\ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 2)\ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 3)\ (-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 2)\,(-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 3)\,(j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 3)\ (j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 4)\ (j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 5)\ (j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 6))^{1/2}\Big)\Big/\Big(4\,((j_1 + 1)\,(2j_1 + 3)\,(j_2 + 1)\,(2j_2 + 3))^{1/2}\Big)\,, \tag{82}$$

$$\left(\begin{matrix} \bar{j}_1 + 1 & \bar{j}_2 + 1 \\ j_1 - 1 & j_2 - 1 \end{matrix}\right\Vert \left.\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big(N_a(\bar{j}_1, \bar{j}_2)\ ((j_1 + j_2 + \bar{j}_1 - \bar{j}_2)\ (j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 1)\ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2 - 1)\ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2)\ (-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 1)\,(-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 2)\,(-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 3)\ (-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 4)\ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 2)\ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 3)\ (-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 2)\,(-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 3))^{1/2}\Big)\Big/\Big(4\,(j_1\,(2j_1 - 1)\,j_2\,(2\,j_2 - 1))^{1/2}\Big)\,, \tag{83}$$

$$\left(\begin{matrix} \bar{j}_1 + 1 & \bar{j}_2 + 1 \\ j_1 - 1 & j_2 + 1 \end{matrix}\right\Vert \left.\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big(N_a(\bar{j}_1, \bar{j}_2)\ ((j_1 - j_2 + \bar{j}_1 - \bar{j}_2 - 1)\ (j_1 - j_2 + \bar{j}_1 - \bar{j}_2)\ (-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 1)\,(-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 2)\,(-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 1)\ (-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 2)\,(-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 2)\,(-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 3)\ (-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 4)\,(-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 5)\,(j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 3)\ (j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 4))^{1/2}\Big)\Big/\Big(4\,\big(j_1\,(2\,j_1 - 1)\,\big(2j_2^2 + 5j_2 + 3\big)\big)^{1/2}\Big)\,, \tag{84}$$

$$\begin{pmatrix}\bar{j}_1+1 & \bar{j}_2+1 \\ j_1+1 & j_2-1\end{pmatrix}\!\!\begin{Vmatrix}\bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1\end{pmatrix} = \Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2 + 2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2-1)\,(-j_1+j_2+\bar{j}_1 - \bar{j}_2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 5)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 4)\big)^{1/2}\Big)/\Big(4\left(\left(2j_1^2+5\ j_1+3\right)j_2\,(2j_2-1)\right)^{1/2}\Big), \tag{85}$$

$$\begin{pmatrix}\bar{j}_1+1 & \bar{j}_2+1 \\ j_1+1 & j_2\end{pmatrix}\!\!\begin{Vmatrix}\bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1\end{pmatrix} = \Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1 - \bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 + 1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 4)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 5)\big)^{1/2}\Big)/\Big(4\left((j_1+1)\,(2j_1+3)\,j_2\,(j_2+1)\right)^{1/2}\Big), \tag{86}$$

$$\begin{pmatrix}\bar{j}_1+1 & \bar{j}_2+1 \\ j_1-1 & j_2\end{pmatrix}\!\!\begin{Vmatrix}\bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1\end{pmatrix} = -\Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2 +\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 +3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1 +j_2+\bar{j}_1+\bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2 +\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\Big)/\Big(4\left(j_1\,(2\ j_1-1)\,j_2\,(j_2+1)\right)^{1/2}\Big), \tag{87}$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2+1 \\ j_1 & j_2+1\end{matrix}\right\Vert\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1\end{matrix}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)(-j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+5)\big)^{1/2}\Big)/\Big(4\,(j_1\,(j_1+1)\,(j_2+1)\,(2j_2+3))^{1/2}\Big), \tag{88}$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2+1 \\ j_1 & j_2-1\end{matrix}\right\Vert\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1\end{matrix}\right) = -\Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)(-j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+4)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\Big)/\Big(4\,(j_1\,(j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{89}$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2+1 \\ j_1 & j_2\end{matrix}\right\Vert\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1\end{matrix}\right) = -\Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\big)^{1/2}\ \big(j_1^2+j_1+j_2^2-\bar{j}_1^2-\bar{j}_2^2+j_2-\bar{j}_1+2\,\bar{j}_1\bar{j}_2+\bar{j}_2\big)\Big)/\Big(4\,(j_1\,(j_1+1)\ j_2\,(j_2+1))^{1/2}\Big), \tag{90}$$

$$\left(\begin{matrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\right\| \left.\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1 + \bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1 + \bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1 + \bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(j_1+j_2+\bar{j}_1 + \bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 5)\big)^{1/2}\Big)\Big/\Big(2\,((j_1+1)\,(j_2+1))^{1/2}\Big)\,, \tag{91}$$

$$\left(\begin{matrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\| \left.\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = -\Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1 + \bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2 + 3)\big)^{1/2}\Big)\Big/\Big(2\ (j_1 j_2)^{1/2}\Big)\,, \tag{92}$$

$$\left(\begin{matrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{matrix}\right\| \left.\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1 - \bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+4)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 4)\big)^{1/2}\Big)\Big/\Big(2\,((j_1+1)\,j_2)^{1/2}\Big)\,, \tag{93}$$

$$\left(\begin{matrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{matrix}\right\| \left.\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_a(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 + 1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 + 2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 + 4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 4)\big)^{1/2}\Big)\Big/\Big(2\,(j_1\,(j_2+1))^{1/2}\Big)\,, \tag{94}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2+1 \\ j_1 & j_2 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 0 & 0 \end{matrix} \right) = \frac{1}{2}\sqrt{5}N_a(\bar{j}_1,\bar{j}_2)\ \big((-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2 +\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1 +\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 +3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\big)^{1/2}, \tag{95}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1+1,\bar{j}_2)$ are:

$$N_b(\bar{j}_1,\bar{j}_2) = \big((2\bar{j}_1+2)\,(2\bar{j}_1+3)\,(2\bar{j}_1-2\bar{j}_2+1)\,(\bar{j}_1-\bar{j}_2+1)\,\bar{j}_2\,(\bar{j}_1+\bar{j}_2 +2)\,(2\bar{j}_2+2)\,(2\bar{j}_1+2\bar{j}_2+3)\big)^{-\frac{1}{2}} \tag{96}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1+1 & j_2+1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix} \right) = -\Big(N_b(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1 -\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+4)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1 +j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +5)\big)^{1/2}\Big)\Big/\Big(4\,\big((j_1+1)\,(2j_1+3)\,(j_2+1)\,(2j_2+3)\big)^{1/2}\Big), \tag{97}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1-1 & j_2-1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix} \right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-2)\ (j_1 +j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1 -j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1 +\bar{j}_2+2)\big)^{1/2}\Big)\Big/\Big(4\,\big(j_1\,(2\ j_1-1)\,j_2\,(2j_2-1)\big)^{1/2}\Big), \tag{98}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 & \| & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1-1 & j_2+1 & \| & j_1 & j_2 & 1 & 1 \end{pmatrix} = \Big( N_b(\bar{j}_1,\bar{j}_2) \ \big( (j_1-j_2+\bar{j}_1-\bar{j}_2) \ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+3) \ (j_1 +j_2+\bar{j}_1-\bar{j}_2+2) \ (j_1+j_2-\bar{j}_1+\bar{j}_2) \ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1) \ (j_1-j_2+\bar{j}_1+\bar{j}_2+1) \ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1 +j_2+\bar{j}_1+\bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+4) \ (j_1+j_2+\bar{j}_1 +\bar{j}_2+3)\big)^{1/2}\Big) \Big/ \Big(4\left(j_1\,(2\,j_1-1)\,\left(2j_2^2+5j_2+3\right)\right)^{1/2}\Big), \tag{99}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 & \| & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+1 & j_2-1 & \| & j_1 & j_2 & 1 & 1 \end{pmatrix} = \Big( N_b(\bar{j}_1,\bar{j}_2) \ \big( (j_1-j_2+\bar{j}_1-\bar{j}_2+1) \ (j_1-j_2+\bar{j}_1-\bar{j}_2 +2) \ (j_1-j_2+\bar{j}_1-\bar{j}_2+3) \ (-j_1+j_2+\bar{j}_1-\bar{j}_2) \ (j_1+j_2 +\bar{j}_1-\bar{j}_2+2) \ (j_1+j_2-\bar{j}_1+\bar{j}_2) \ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1) \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2) \ (j_1-j_2+\bar{j}_1+\bar{j}_2+3) \ (j_1 -j_2+\bar{j}_1+\bar{j}_2+4) \ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1) \ (j_1+j_2+\bar{j}_1 +\bar{j}_2+3)\big)^{1/2}\Big) \Big/ \Big(4\left(\left(2j_1^2+5\,j_1+3\right)j_2\,(2j_2-1)\right)^{1/2}\Big), \tag{100}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 & \| & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+1 & j_2 & \| & j_1 & j_2 & 1 & 1 \end{pmatrix} = \Big( N_b(\bar{j}_1,\bar{j}_2) \ \big( (j_1-j_2+\bar{j}_1-\bar{j}_2+1) \ (j_1-j_2+\bar{j}_1-\bar{j}_2+2) \ (j_1 + j_2+\bar{j}_1-\bar{j}_2+2) \ (j_1+j_2+\bar{j}_1-\bar{j}_2+3) \ (j_1-j_2+\bar{j}_1 +\bar{j}_2+2) \ (j_1-j_2+\bar{j}_1+\bar{j}_2+3) \ (j_1+j_2+\bar{j}_1+\bar{j}_2+3) \ (j_1 +j_2+\bar{j}_1+\bar{j}_2+4)\big)^{1/2} \left(-j_1^2+(2\,\bar{j}_1+1)\ j_1+j_2^2-\bar{j}_1^2+\bar{j}_2^2 +j_2-\bar{j}_1+\bar{j}_2\right) \Big) \Big/ \Big(4\ \left(\left(2j_1^2+5j_1+3\right)j_2\,(j_2+1)\right)^{1/2}\Big), \tag{101}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 & \| & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1-1 & j_2 & \| & j_1 & j_2 & 1 & 1 \end{pmatrix} = -\Big( N_b(\bar{j}_1,\bar{j}_2) \ \big( (-j_1+j_2+\bar{j}_1-\bar{j}_2+1) \ (-j_1+j_2+\bar{j}_1-\bar{j}_2+2) \ (j_1+j_2-\bar{j}_1 +\bar{j}_2-1) \ (j_1+j_2-\bar{j}_1+\bar{j}_2) \ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 +2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+2) \ (-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2} (2j_2\ \bar{j}_2 +(j_1-j_2+\bar{j}_1-\bar{j}_2+1) \ (j_1+j_2+\bar{j}_1+\bar{j}_2+2)) \Big) \Big/ \Big(4\,(j_1\ (2j_1-1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{102}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1 & j_2+1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right. = \Big(N_b(\bar{j}_1,\bar{j}_2)\ \left((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\right)^{1/2}\ \left(j_1^2+j_1-j_2^2-\bar{j}_1^2+\bar{j}_2^2-\bar{j}_1+j_2\,(2\,\bar{j}_1+1)+\bar{j}_2\right)\Big)\Big/\Big(4\left(j_1\,(j_1+1)\ \left(2j_2^2+5j_2+3\right)\right)^{1/2}\Big), \tag{103}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right. = \Big(N_b(\bar{j}_1,\bar{j}_2)\ \left((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\right)^{1/2}\ \left(j_1^2+j_1-j_2^2-\bar{j}_1^2+\bar{j}_2^2-3\bar{j}_1-j_2\,(2\,\bar{j}_1+3)+\bar{j}_2-2\right)\Big)\Big/\Big(4\,(j_1\,(j_1+1)\ j_2\,(2j_2-1))^{1/2}\Big), \tag{104}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right. = -\Big(N_b(\bar{j}_1,\bar{j}_2)\ \left((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\right)^{1/2}\ \left(j_1^2+j_1+j_2^2-\bar{j}_1^2+\bar{j}_2^2+j_2-3\,\bar{j}_1+\bar{j}_2-2\right)\Big)\Big/\Big(4\,(j_1\,(j_1+1)\,j_2\ (j_2+1))^{1/2}\Big), \tag{105}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right. = \Big(N_b(\bar{j}_1,\bar{j}_2)\ (j_1+j_2-\bar{j}_1)\ \left((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\right)^{1/2}\Big)\Big/\Big(2\,((j_1+1)\,(j_2+1))^{1/2}\Big), \tag{106}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = -\Big(N_b(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+\bar{j}_1+2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)/\Big(2\,(j_1j_2)^{1/2}\Big), \tag{107}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ (-j_1+j_2+\bar{j}_1+1)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)/\Big(2\,((j_1+1)\,j_2)^{1/2}\Big), \tag{108}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_b(\bar{j}_1,\bar{j}_2)\ (j_1-j_2+\bar{j}_1+1)\ ((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)/\Big(2\,(j_1\,(j_2+1))^{1/2}\Big), \tag{109}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2 \\ j_1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 0 & 0 \end{matrix}\right) = \frac{1}{2}\sqrt{5}N_b(\bar{j}_1,\bar{j}_2)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}. \tag{110}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1,\bar{j}_2+1)$ are:

$$N_c(\bar{j}_1,\bar{j}_2) = ((2\bar{j}_1+1)\,(2\bar{j}_1+3)\,(2\bar{j}_1-2\bar{j}_2+1)\,(\bar{j}_1-\bar{j}_2)\,(\bar{j}_1+\bar{j}_2+2)\,(2\bar{j}_2 + 1)\,(2\bar{j}_2+2)\,(2\bar{j}_1+2\bar{j}_2+3))^{-\frac{1}{2}} \tag{111}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1+1 & j_2+1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix} \right. = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2)\ \ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2 + 2)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2+2)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2 + 3)\ \ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ \ (j_1 + j_2+\bar{j}_1+\bar{j}_2+3)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 5))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\ \ ((j_1+1)\,(2j_1+3)\,(j_2+1)\ \ (2j_2+3))^{1/2}\Big), \tag{112}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1-1 & j_2-1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix} \right. = -\Big(N_c(\bar{j}_1,\bar{j}_2)\quad ((j_1-j_2+\bar{j}_1-\bar{j}_2)\quad (-j_1+j_2+\bar{j}_1 - \bar{j}_2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2-1)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2)\ \ (j_1 + j_2+\bar{j}_1-\bar{j}_2+1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ \ (-j_1-j_2 + \bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2+2)\,(-j_1-j_2 + \bar{j}_1+\bar{j}_2+3)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2 + \bar{j}_1+\bar{j}_2+2)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 2))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ \ (2j_1-1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{113}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1-1 & j_2+1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix} \right. = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2-2)\ \ (j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ \ (j_1-j_2+\bar{j}_1 - \bar{j}_2)\ \ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2 + 1)\ \ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2)\,(-j_1+j_2+\bar{j}_1+\ \bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 3))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ \ (2j_1-1)\,(j_2+1)\,(2j_2+3))^{1/2}\Big), \tag{114}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix}$$

$$= \Big(N_c(\bar{j}_1, \bar{j}_2)\ \ ((j_1 - j_2 + \bar{j}_1 - \bar{j}_2 + 1)\ \ (-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 - 2)\,(-j_1 + j_2 + \bar{j}_1 - \bar{j}_2 - 1)\,(-j_1 + j_2 + \bar{j}_1 - \bar{j}_2)\ \ (j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 1)\ \ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2 + 1)\ \ (-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 1)\ \ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 2)\ \ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 3)\ \ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 4)\ \ (-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 1)\ \ (j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 3))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\ \ ((j_1+1)\,(2j_1+3)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{115}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1+1 & j_2 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = \Big(N_c(\bar{j}_1, \bar{j}_2)\ \ ((j_1 - j_2 - \bar{j}_1 + \bar{j}_2)\ \ (j_1 - j_2 - \bar{j}_1 + \bar{j}_2 + 1)\ \ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2 + 1)\ \ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2 + 2)\ \ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 2)\ \ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 3)\ \ (j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 3)\ \ (j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 4))^{1/2}\,(-j_1^2 + 2\bar{j}_2\ j_1 + j_2^2 + \bar{j}_1^2 - \bar{j}_2^2 + j_2 + 2\bar{j}_1 + 1)\ \Big)\Big/\Big(2\ \ (2)^{1/2}\,(\big(2j_1^2 + 5j_1 + 3\big)\,j_2\,(j_2+1))^{1/2}\Big), \tag{116}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1-1 & j_2 \end{pmatrix}\!\!\begin{Vmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = -\Big(N_c(\bar{j}_1, \bar{j}_2)\ \ ((j_1 - j_2 + \bar{j}_1 - \bar{j}_2 - 1)\ \ (j_1 - j_2 + \bar{j}_1 - \bar{j}_2)\ \ (j_1 + j_2 + \bar{j}_1 - \bar{j}_2)\ \ (j_1 + j_2 + \bar{j}_1 - \bar{j}_2 + 1)\ \ (-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 1)\,(-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 2)\,(-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 2)\ \ (-j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 3))^{1/2}\,(j_1^2 + 2\ \ (\bar{j}_2+1)\,j_1 - j_2^2 - \bar{j}_1^2 + \bar{j}_2^2 - j_2 - 2\ \bar{j}_1 + 2\bar{j}_2)\ \Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\,(2j_1-1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{117}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1 & j_2+1 \end{pmatrix}\!\Bigg\|\!\begin{pmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2)\ (j_1 + j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1 + \bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\ \bar{j}_2+3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1 + j_2+\bar{j}_1+\bar{j}_2+4)\big)^{1/2}\ \big(j_1^2+j_1-j_2^2+\bar{j}_1^2-\bar{j}_2^2+2\bar{j}_1+2j_2\,\bar{j}_2 + 1\big)\Big)\Big/\Big(2\,(2)^{1/2}\,\big(j_1\,(j_1+1)\,\big(2\ j_2^2+5j_2+3\big)\big)^{1/2}\Big)\,, \tag{118}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1 & j_2-1 \end{pmatrix}\!\Bigg\|\!\begin{pmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \big((-j_1+j_2+\bar{j}_1-\bar{j}_2-1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1 + j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 + 1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2+2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1 - j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\ \big(j_1^2+j_1-j_2^2+\bar{j}_1^2-\bar{j}_2^2+2\bar{j}_1-2\ \bar{j}_2 - 2j_2\,(\bar{j}_2+1)\big)\Big)\Big/\Big(2\,(2)^{1/2}\,\big(j_1\ \ (j_1+1)\,j_2\,(2j_2-1)\big)^{1/2}\Big)\,, \tag{119}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1 & j_2 \end{pmatrix}\!\Bigg\|\!\begin{pmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{pmatrix} = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1 + j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1 + \bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 +2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\ \big(j_1^2+j_1+j_2^2+\bar{j}_1^2-\bar{j}_2^2+j_2 + 2\bar{j}_1-2\ \bar{j}_2\big)\Big)\Big/\Big(2\,(2)^{1/2}\,\big(j_1\,(j_1+1)\,j_2\ \ (j_2+1)\big)^{1/2}\Big)\,, \tag{120}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\Bigg\|\!\begin{pmatrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\Big(N_c(\bar{j}_1,\bar{j}_2)\,(2\ j_1+2j_2-2\bar{j}_2+1)\ \big((j_1-j_2+\bar{j}_1 -\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1 +j_2-\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2 +\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1 +\bar{j}_2+4)\big)^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\ \ \big((j_1+1)\,(j_2+1)\big)^{1/2}\Big)\,, \tag{121}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \left((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1 -\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1 -j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1-j_2 +\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\right)^{1/2}(2j_1 +2j_2+2\,\bar{j}_2+3)\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1j_2)^{1/2}\Big), \tag{122}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \left((-j_1+j_2+\bar{j}_1-\bar{j}_2-1)\ (-j_1+j_2+\bar{j}_1 -\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1 -j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\right)^{1/2}(-2j_1 +2j_2+2\,\bar{j}_2+1)\Big)\Big/\Big(2\,(2)^{1/2}\,((j_1+1)\,j_2)^{1/2}\Big), \tag{123}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_c(\bar{j}_1,\bar{j}_2)\ \left((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ (j_1-j_2+\bar{j}_1 -\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 +1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 +2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +3)\right)^{1/2}(2j_1-2j_2+2\,\bar{j}_2 +1)\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\,(j_2+1))^{1/2}\Big), \tag{124}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2+1 \\ j_1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 0 & 0 \end{matrix}\right) = -\sqrt{\frac{5}{2}}N_c(\bar{j}_1,\bar{j}_2)\ \left((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1 -\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1 -j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2 +\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\right)^{1/2}. \tag{125}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1+1,\bar{j}_2-1)$ are:

$$N_d(\bar{j}_1,\bar{j}_2)=(2\,(2\bar{j}_1+2)\,(2\bar{j}_1+3)\,(2\bar{j}_1-2\bar{j}_2+1)\,(2\bar{j}_1-2\bar{j}_2+3)\,(\bar{j}_1-\bar{j}_2+1)\,(\bar{j}_1-\bar{j}_2+2)\,\bar{j}_2\,(2\bar{j}_2+1))^{-\frac{1}{2}} \tag{126}$$

$$\begin{pmatrix}\bar{j}_1+1\;\bar{j}_2-1\\ j_1+1\;j_2+1\end{Vmatrix}\begin{matrix}\bar{j}_1\;\bar{j}_2\;\;1\;1\\ j_1\;j_2\;\;1\;1\end{pmatrix}$$
$$=\Big(N_d(\bar{j}_1,\bar{j}_2)\;((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\;(j_1-j_2+\bar{j}_1-\bar{j}_2+2)\;(-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\;(j_1+j_2+\bar{j}_1-\bar{j}_2+3)\;(j_1+j_2+\bar{j}_1-\bar{j}_2+4)\;(j_1+j_2+\bar{j}_1-\bar{j}_2+5)\;(-j_1-j_2+\bar{j}_1+\bar{j}_2-1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\;(j_1+j_2+\bar{j}_1+\bar{j}_2+4))^{1/2}\Big)/\Big(4\,((j_1+1)\,(2j_1+3)\,(j_2+1)\,(2j_2+3))^{1/2}\Big), \tag{127}$$

$$\begin{pmatrix}\bar{j}_1+1\;\bar{j}_2-1\\ j_1-1\;j_2-1\end{Vmatrix}\begin{matrix}\bar{j}_1\;\bar{j}_2\;\;1\;1\\ j_1\;j_2\;\;1\;1\end{pmatrix}=\Big(N_d(\bar{j}_1,\bar{j}_2)\;((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\;(j_1-j_2+\bar{j}_1-\bar{j}_2+2)\;(-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(j_1+j_2-\bar{j}_1+\bar{j}_2-3)\;(j_1+j_2-\bar{j}_1+\bar{j}_2-2)\;(j_1+j_2-\bar{j}_1+\bar{j}_2-1)\;(j_1+j_2-\bar{j}_1+\bar{j}_2)\;(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+1)\;(j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)/\Big(4\,(j_1\,(2\,j_1-1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{128}$$

$$\begin{pmatrix}\bar{j}_1+1\;\bar{j}_2-1\\ j_1-1\;j_2+1\end{Vmatrix}\begin{matrix}\bar{j}_1\;\bar{j}_2\;\;1\;1\\ j_1\;j_2\;\;1\;1\end{pmatrix}=\Big(N_d(\bar{j}_1,\bar{j}_2)\;((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\;(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+4)\;(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\;(j_1+j_2+\bar{j}_1-\bar{j}_2+3)\;(j_1+j_2-\bar{j}_1+\bar{j}_2-1)\;(j_1+j_2-\bar{j}_1+\bar{j}_2)\;(j_1-j_2+\bar{j}_1+\bar{j}_2)\;(j_1-j_2+\bar{j}_1+\bar{j}_2+1)\;(-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)/\Big(4\,\big(j_1\,(2j_1-1)\,\big(2j_2^2+5\,j_2+3\big)\big)^{1/2}\Big), \tag{129}$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2-1\\ j_1+1 & j_2-1\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1 & j_2 & 1 & 1\end{matrix}\right) = \Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+4)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1))^{1/2}\Big)\Big/\Big(4\left(\left(2j_1^2+5\,j_1+3\right)j_2\,(2j_2-1)\right)^{1/2}\Big), \tag{130}$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2-1\\ j_1+1 & j_2\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1 & j_2 & 1 & 1\end{matrix}\right) = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+4)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(4\left(\left(2j_1^2+5\,j_1+3\right)j_2\,(j_2+1)\right)^{1/2}\Big), \tag{131}$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2-1\\ j_1-1 & j_2\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1 & j_2 & 1 & 1\end{matrix}\right) = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(4\,(j_1\,(2\,j_1-1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{132}$$

$$\left(\begin{matrix}\bar{j}_1+1 & \bar{j}_2-1\\ j_1 & j_2+1\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1 & 1\\ j_1 & j_2 & 1 & 1\end{matrix}\right) = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+4)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(4\left(j_1\,(j_1+1)\left(2j_2^2+5j_2+3\right)\right)^{1/2}\Big), \tag{133}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1+1 & \bar{j}_2-1 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2-1 & j_1 & j_2 & 1 & 1 \end{array}\right) = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(4\,(j_1\,(j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big)\,, \tag{134}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1+1 & \bar{j}_2-1 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & j_1 & j_2 & 1 & 1 \end{array}\right) = \Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2))^{1/2}\,(2j_1\ j_2+(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))\Big)\Big/\Big(4\,(j_1\ (j_1+1)\,j_2\,(j_2+1))^{1/2}\Big)\,, \tag{135}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1+1 & \bar{j}_2-1 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+4)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)\Big/\Big(2\,((j_1+1)\,(j_2+1))^{1/2}\Big)\,, \tag{136}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1+1 & \bar{j}_2-1 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = -\Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(2\,(j_1j_2)^{1/2}\Big)\,, \tag{137}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1))^{1/2}\Big)\Big/\Big(2\,((j_1+1)\,j_2)^{1/2}\Big), \tag{138}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_d(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\Big)\Big/\Big(2\,(j_1\,(j_2+1))^{1/2}\Big), \tag{139}$$

$$\begin{pmatrix} \bar{j}_1+1 & \bar{j}_2-1 \\ j_1 & j_2 \end{pmatrix}\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 0 & 0 \end{matrix}\right) = \frac{1}{2}\sqrt{5}N_d(\bar{j}_1,\bar{j}_2)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1-j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2))^{1/2}. \tag{140}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1+\frac{1}{2},\bar{j}_2+\frac{1}{2})$ are:

$$N_e(\bar{j}_1,\bar{j}_2) = ((2\bar{j}_1+2)\,(\bar{j}_1-\bar{j}_2)\,(\bar{j}_1-\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+2)\,(\bar{j}_1+\bar{j}_2+3)\,(2\bar{j}_2+1)\,(2\bar{j}_1+2\bar{j}_2+3))^{-\frac{1}{2}} \tag{141}$$

$$\begin{aligned}&\left(\begin{matrix}\bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2}\\ j_1+1 & j_2+1\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1\,1\\ j_1 & j_2 & 1\,1\end{matrix}\right)\\&=-\Big(N_e(\bar{j}_1,\bar{j}_2)\ (j_1-j_2)\,((j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2\\&\quad-\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2\\&\quad+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\ (j_1\\&\quad+j_2+\bar{j}_1+\bar{j}_2+5))^{1/2}\Big)/\Big(2\,(2)^{1/2}\ ((j_1+1)\,(2j_1+3)\,(j_2+1)\ (2j_2+3))^{1/2}\Big),\end{aligned} \tag{142}$$

$$\begin{aligned}&\left(\begin{matrix}\bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2}\\ j_1-1 & j_2-1\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1\,1\\ j_1 & j_2 & 1\,1\end{matrix}\right)=\Big(N_e(\bar{j}_1,\bar{j}_2)\,(j_1-j_2)\,((j_1+j_2+\bar{j}_1-\bar{j}_2)\,(j_1+j_2+\bar{j}_1\\&\quad-\bar{j}_2+1)\,(j_1+j_2-\bar{j}_1+\bar{j}_2-1)\,(j_1+j_2-\bar{j}_1\\&\quad+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2\\&\quad+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2\\&\quad+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2\\&\quad+2))^{1/2}\Big)/\Big(2\,(2)^{1/2}\,(j_1\ (2j_1-1)\,j_2\,(2j_2-1))^{1/2}\Big),\end{aligned} \tag{143}$$

$$\begin{aligned}&\left(\begin{matrix}\bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2}\\ j_1-1 & j_2+1\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1\,1\\ j_1 & j_2 & 1\,1\end{matrix}\right)\\&=\Big(N_e(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\,(j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2\\&\quad+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1\\&\quad+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2\\&\quad+4)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)/\Big(2\,(2)^{1/2}\left(j_1\ (2j_1-1)\left(2j_2^2+5j_2+3\right)\right)^{1/2}\Big),\end{aligned} \tag{144}$$

$$\begin{aligned}&\left(\begin{matrix}\bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2}\\ j_1+1 & j_2-1\end{matrix}\right\|\left.\begin{matrix}\bar{j}_1 & \bar{j}_2 & 1\,1\\ j_1 & j_2 & 1\,1\end{matrix}\right)\\&=-\Big(N_e(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (-j_1\\&\quad+j_2+\bar{j}_1-\bar{j}_2-1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1\\&\quad+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+4)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2\\&\quad+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\Big)/\Big(2\,(2)^{1/2}\left(\left(2\,j_1^2+5j_1+3\right)j_2\,(2j_2-1)\right)^{1/2}\Big),\end{aligned} \tag{145}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1+1 & j_2 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix} \right)$$

$$= -\Big(N_e(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 +2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1 +j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\big)^{1/2}\,(j_2\,(j_2+1) +(j_1+1)\,(-j_1+\bar{j}_1+\bar{j}_2+1))\Big)/\Big(2\,(2)^{1/2}\,((j_1+1)\,(2j_1+3)\,j_2\ (j_2+1))^{1/2}\Big), \tag{146}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1-1 & j_2 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix} \right)$$

$$= -\Big(N_e(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2 +2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\ \big(j_1^2 +(\bar{j}_1+\bar{j}_2+2)\,j_1-j_2\,(j_2+1)\big)\Big)/\Big(2\,(2)^{1/2}\,(j_1\,(2j_1-1)\,j_2\ (j_2+1))^{1/2}\Big), \tag{147}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1 & j_2+1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix} \right)$$

$$= \Big(N_e(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 +2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\ \bar{j}_2 +3)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\big)^{1/2}\,(j_1\,(j_1+1) +(j_2+1)\,(-j_2+\bar{j}_1+\bar{j}_2+1))\Big)/\Big(2\,(2)^{1/2}\,(j_1\,(j_1+1)\,(j_2+1)\ (2j_2+3))^{1/2}\Big), \tag{148}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1 & j_2-1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix} \right)$$

$$= -\Big(N_e(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2+2)\,(j_1 -j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\big)^{1/2}\ \big(j_1^2+j_1 -j_2\ (j_2+\bar{j}_1+\bar{j}_2+2)\big)\Big)/\Big(2\,(2)^{1/2}\,(j_1\ (j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{149}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1 & j_2 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big(N_e(\bar{j}_1,\bar{j}_2)\,(j_1-j_2\ )\,(j_1+j_2+1)\ \ (-j_1^2-j_1-j_2^2 - j_2+(\bar{j}_1-\bar{j}_2)\ (\bar{j}_1-\bar{j}_2+1))\ \ ((-j_1-j_2+\bar{j}_1 + \bar{j}_2+1)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2+\bar{j}_1 + \bar{j}_2+2)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 3))^{1/2}\Big)/\Big(2\,(2)^{1/2}\,(j_1\ \ (j_1+1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{150}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_e(\bar{j}_1,\bar{j}_2)\,(j_1-j_2\ )\,(2j_1+2j_2-\bar{j}_1-\bar{j}_2 + 1)\ \ ((j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2 + 1)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 4))^{1/2}\Big)/\Big(2\,(2)^{1/2}\ \ ((j_1+1)\,(j_2+1))^{1/2}\Big), \tag{151}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = -\Big(N_e(\bar{j}_1,\bar{j}_2)\ \ (j_1 - j_2)\,((j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ \ (-j_1 - j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2+2)\,(j_1-j_2 + \bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2))^{1/2}\,(2j_1 + 2\ j_2+\bar{j}_1+\bar{j}_2+3)\Big)/\Big(2\,(2)^{1/2}\,(j_1 j_2)^{1/2}\Big), \tag{152}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = -\Big(N_e(\bar{j}_1,\bar{j}_2)\ \ (j_1+j_2 +1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2)\ \ (-j_1 - j_2+\bar{j}_1+\bar{j}_2+1)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (j_1-j_2 + \bar{j}_1+\bar{j}_2+3)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\,(-2j_1 + 2\ j_2+\bar{j}_1+\bar{j}_2+1)\Big)/\Big(2\,(2)^{1/2}\,((j_1+1)\ \ j_2)^{1/2}\Big), \tag{153}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right) = \Big(N_e(\bar{j}_1,\bar{j}_2)\ \ (j_1+j_2+1)\,(2j_1-2j_2+\bar{j}_1+\bar{j}_2 + 1)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2)\ \ (-j_1+j_2+\bar{j}_1-\bar{j}_2 + 1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2)\ \ (-j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 3))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ \ (j_2+1))^{1/2}\Big)\,, \tag{154}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2+\frac{1}{2} \\ j_1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 0\,0 \end{matrix}\right) = \sqrt{\frac{5}{2}}N_e(\bar{j}_1,\bar{j}_2)\ \ (j_1-j_2)\,(j_1+j_2+1)\ \ ((-j_1-j_2 + \bar{j}_1+\bar{j}_2+1)\ \ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ \ (-j_1+j_2 + \bar{j}_1+\bar{j}_2+2)\ \ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\,. \tag{155}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1+\frac{1}{2},\bar{j}_2-\frac{1}{2})$ are:

$$N_f(\bar{j}_1,\bar{j}_2) = ((2\bar{j}_1+2)\,(2\bar{j}_1-2\bar{j}_2+1)\,(\bar{j}_1-\bar{j}_2)\,(\bar{j}_1-\bar{j}_2+1)\,(\bar{j}_1-\bar{j}_2 + 2)\,(\bar{j}_1+\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+2)\,(2\bar{j}_2+1))^{-\frac{1}{2}} \tag{156}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1+1 & j_2+1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right) = \Big(N_f(\bar{j}_1,\bar{j}_2)\,(j_1-j_2\ )\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ \ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ \ (j_1+j_2 + \bar{j}_1-\bar{j}_2+2)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ \ (j_1+j_2+\bar{j}_1-\bar{j}_2+4)\ \ (j_1+j_2-\bar{j}_1+\bar{j}_2 + 1)\ \ (-j_1-j_2+\bar{j}_1+\bar{j}_2-1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+3)\ \ (j_1 + j_2+\bar{j}_1+\bar{j}_2+4))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\ \ ((j_1+1)\,(2j_1+3)\,(j_2+1)\ \ (2j_2+3))^{1/2}\Big)\,, \tag{157}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1-1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right) = \Big(N_f(\bar{j}_1,\bar{j}_2)\,(j_1-j_2\ )\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1 +j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2 -\bar{j}_1+\bar{j}_2-2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\ \bar{j}_2 +2)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +2))^{1/2}\Big)/\Big(2\,(2)^{1/2}\,(j_1\ (2j_1-1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{158}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1-1 & j_2+1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right) = \Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2 +\bar{j}_1-\ \bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1 +j_2+\bar{j}_1+\ \bar{j}_2+3))^{1/2}\Big)/\Big(2\,(2)^{1/2}\,\big(j_1\,(2j_1-1)\ \big(2j_2^2+5j_2+3\big)\big)^{1/2}\Big), \tag{159}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right) = -\Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\,(j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1 -j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2)\ (-j_1 +j_2+\bar{j}_1+\bar{j}_2+1))^{1/2}\Big)/\Big(2\,(2)^{1/2}\ \big(\big(2j_1^2+5j_1+3\big)\,j_2\,(2j_2-1)\big)^{1/2}\Big), \tag{160}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1+1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right) = \Big(N_f(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 +2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1 +j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\,(j_2\,(j_2+1) -(j_1+1)\,(j_1-\bar{j}_1+\bar{j}_2))\Big)/\Big(2\,(2)^{1/2}\,((j_1+1)\,(2j_1+3)\,j_2\ (j_2+1))^{1/2}\Big), \tag{161}$$

$$
\begin{aligned}
&\left(\begin{array}{cc||ccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1-1 & j_2 & j_1 & j_2 & 1\,1 \end{array}\right) = -\Big(N_f(\bar{j}_1,\bar{j}_2)\,\big(j_1^2+\ (\bar{j}_1-\bar{j}_2+1)\,j_1 \\
&\quad - j_2\,(j_2+1)\big)\ \ ((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2 \\
&\quad +\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1 \\
&\quad +\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 \\
&\quad +1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 \\
&\quad +2))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ \ (2j_1-1)\,j_2\,(j_2+1))^{1/2}\Big),
\end{aligned}
\tag{162}
$$

$$
\begin{aligned}
&\left(\begin{array}{cc||ccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2+1 & j_1 & j_2 & 1\,1 \end{array}\right) \\
&= -\Big(N_f(\bar{j}_1,\bar{j}_2)\ \ ((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1 \\
&\quad -\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+3)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 \\
&\quad +1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3))^{1/2}\ \ \big(j_1^2+j_1 \\
&\quad -(j_2+1)\,(j_2-\bar{j}_1+\bar{j}_2)\big)\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\,(j_1+1)\,(j_2+1)\,(2j_2+3))^{1/2}\Big),
\end{aligned}
\tag{163}
$$

$$
\begin{aligned}
&\left(\begin{array}{cc||ccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2-1 & j_1 & j_2 & 1\,1 \end{array}\right) = -\Big(N_f(\bar{j}_1,\bar{j}_2)\ \ \big(j_1^2+j_1-j_2\,(j_2+\bar{j}_1-\bar{j}_2+1)\big)\ \ ((j_1 \\
&\quad -j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2 \\
&\quad -\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1 \\
&\quad +\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1 \\
&\quad +\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 \\
&\quad +2))^{1/2}\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ \ (j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big),
\end{aligned}
\tag{164}
$$

$$
\begin{aligned}
&\left(\begin{array}{cc||ccc} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} & \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & j_1 & j_2 & 1\,1 \end{array}\right) = \Big(N_f(\bar{j}_1,\bar{j}_2)\ \ \big(j_1^2+j_1 \\
&\quad -j_2\,(j_2+1)\big)\ \ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2 \\
&\quad +\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1 \\
&\quad +\bar{j}_2))^{1/2}\ \ \big(-j_1^2-j_1-j_2^2+(\bar{j}_1+\bar{j}_2)^2-j_2+3\ (\bar{j}_1+\bar{j}_2) \\
&\quad +2\big)\Big)\Big/\Big(2\,(2)^{1/2}\,(j_1\ \ (j_1+1)\,j_2\,(j_2+1))^{1/2}\Big),
\end{aligned}
\tag{165}
$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix} \right) = -\Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1-j_2)\,(2j_1+2j_2-\bar{j}_1+\bar{j}_2 +2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 +3)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +3))^{1/2}\Big)/\Big(2\,(2)^{1/2}\ ((j_1+1)\,(j_2+1))^{1/2}\Big), \tag{166}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix} \right) = -\Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1-j_2)\,(2j_1+2j_2+\bar{j}_1-\bar{j}_2 +2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 +2))^{1/2}\Big)/\Big(2\,(2)^{1/2}\,(j_1\ j_2)^{1/2}\Big), \tag{167}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix} \right) = \Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,(2j_1-2j_2-\bar{j}_1 +\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1-\bar{j}_2 +2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 +1))^{1/2}\Big)/\Big(2\,(2)^{1/2}\ ((j_1+1)\,j_2)^{1/2}\Big), \tag{168}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix} \right) = \Big(N_f(\bar{j}_1,\bar{j}_2)\ (j_1+j_2+1)\,(2j_1-2j_2+\bar{j}_1 -\bar{j}_2)\ ((-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1 -\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1 +\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 +2))^{1/2}\Big)/\Big(2\,(2)^{1/2}\,(j_1\ (j_2+1))^{1/2}\Big), \tag{169}$$

$$\begin{pmatrix} \bar{j}_1+\frac{1}{2} & \bar{j}_2-\frac{1}{2} \\ j_1 & j_2 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 0\,0 \end{matrix} \right) = \left(\frac{5}{2}\right)^{1/2} N_f(\bar{j}_1,\bar{j}_2)\ (j_1-j_2)\,(j_1+j_2 \tag{170}$$
$$+1)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2))^{1/2}.$$

Representation $(\bar{j}_1,\bar{j}_2)$ appears twice in the decomposition of the product $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})$. To distinguish between the two, we will use the convention described in [58]. Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1,\bar{j}_2)_1$ are:

$$\begin{aligned} N_g(\bar{j}_1,\bar{j}_2) = 2\,(5)^{1/2}\left(4\bar{j}_2^2\,(\bar{j}_2+1)^2 + 11\left(8\bar{j}_1^2+16\bar{j}_1+5\right)\bar{j}_2\,(\bar{j}_2+1)\right. \\ \left. +\,\bar{j}_1\,(\bar{j}_1+2)\,(2\bar{j}_1-1)\,(2\bar{j}_1+5)\right)^{-\frac{1}{2}} \end{aligned} \tag{171}$$

$$\begin{aligned} &\left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2+1 \end{matrix}\right\|\left.\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right)_1 \\ &= -\Big(N_g(\bar{j}_1,\bar{j}_2)\ ((j_1+j_2-\bar{j}_1-\bar{j}_2)\ (j_1+j_2-\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1 \\ &\quad +j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 \\ &\quad +3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4))^{1/2}\Big)\Big/\Big(8\,((j_1+1)\,(2j_1+3)\,(j_2+1)\,(2j_2+3))^{1/2}\Big), \end{aligned} \tag{172}$$

$$\begin{aligned} \left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j_1-1 & j_2-1 \end{matrix}\right\|\left.\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right)_1 &= -\Big(N_g(\bar{j}_1,\bar{j}_2)\ ((j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2 \\ &+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1 \\ &+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\,(-j_1-j_2+\bar{j}_1+\bar{j}_2 \\ &+2)\,(j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 \\ &+2))^{1/2}\Big)\Big/\Big(8\,(j_1\,(2\,j_1-1)\,j_2\,(2j_2-1))^{1/2}\Big), \end{aligned} \tag{173}$$

$$\begin{aligned} \left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j_1-1 & j_2+1 \end{matrix}\right\|\left.\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1\,1 \\ j_1 & j_2 & 1\,1 \end{matrix}\right)_1 &= -\Big(N_g(\bar{j}_1,\bar{j}_2)\ ((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ (j_1-j_2+\bar{j}_1 \\ &-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\,(-j_1+j_2+\bar{j}_1-\bar{j}_2 \\ &+2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 \\ &+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\,(-j_1+j_2+\bar{j}_1+\bar{j}_2 \\ &+3))^{1/2}\Big)\Big/\Big(8\left(j_1\,(2j_1-1)\left(2j_2^2+5\,j_2+3\right)\right)^{1/2}\Big), \end{aligned} \tag{174}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1 - \bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2-1)\,(-j_1+j_2+\bar{j}_1 - \bar{j}_2)\,(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 1)\big)^{1/2}\Big)/\Big(8\,\big(\big(2j_1^2+5\ j_1+3\big)\,j_2\,(2j_2-1)\big)^{1/2}\Big), \tag{175}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1 - \bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 + 1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1 + j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 3)\big)^{1/2}\Big)/\Big(8\,((j_1+1)\,(2j_1+3)\,j_2\,(j_2+1))^{1/2}\Big), \tag{176}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1-1 & j_2 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right)_1 = \Big(N_g(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1 +j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1 +j_2+\bar{j}_1+\bar{j}_2+2)\big)^{1/2}\Big)/\Big(8\,(j_1\,(2\ j_1-1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{177}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1 & j_2+1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\ \big((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 + 1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2 + 1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1 + j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 + 3)\big)^{1/2}\Big)/\Big(8\,(j_1\,(j_1+1)\,(j_2+1)\,(2j_2+3))^{1/2}\Big), \tag{178}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2-1 & j_1 & j_2 & 1 & 1 \end{array}\right)_1 = \Big(N_g(\bar{j}_1,\bar{j}_2)\ \left((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1 + j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1 +j_2+\bar{j}_1+\bar{j}_2+2)\right)^{1/2}\Big)\Big/\Big(8\,(j_1\,(j_1+1)\,j_2\,(2j_2-1))^{1/2}\Big), \tag{179}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & j_1 & j_2 & 1 & 1 \end{array}\right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\ \ \big(j_1^4+2j_1^3 - \left(10j_2^2+10j_2+2\bar{j}_1^2+2\ \bar{j}_2^2+4\bar{j}_1+2\bar{j}_2+1\right) j_1^2 - 2\left(5j_2^2+5\ j_2+\bar{j}_1^2+\bar{j}_2^2+2\bar{j}_1+\bar{j}_2+1\right) j_1+j_2^4+\ \bar{j}_1^4+\bar{j}_2^4 + 2j_2^3+4\bar{j}_1^3+2\bar{j}_2^3+5\ \bar{j}_1^2-2\bar{j}_1^2\bar{j}_2^2-4\bar{j}_1\ \bar{j}_2^2-\bar{j}_2^2+2\bar{j}_1-2\bar{j}_1^2\bar{j}_2 - 4\ \bar{j}_1\bar{j}_2-2\bar{j}_2-2j_2\left(\bar{j}_1^2+2\ \bar{j}_1+\bar{j}_2^2+\bar{j}_2+1\right)-j_2^2\left(2\bar{j}_1^2 + 4\ \bar{j}_1+2\bar{j}_2^2+2\bar{j}_2+1\right)\big)\Big)\Big/\Big(8\,(j_1\ \ (j_1+1)\,j_2\,(j_2+1))^{1/2}\Big), \tag{180}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+\frac{1}{2} & j_2+\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right)_1 = \Big(N_g(\bar{j}_1,\bar{j}_2)\,(2j_1+2\ j_2+3)\left((j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1 + j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1+j_2 +\bar{j}_1+\bar{j}_2+3)\right)^{1/2}\Big)\Big/\Big(8\,((j_1+1)\,(j_2+1))^{1/2}\Big), \tag{181}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right)_1 = \Big(N_g(\bar{j}_1,\bar{j}_2)\,(2j_1+2\ \ j_2+1)\left((j_1+j_2+\bar{j}_1-\bar{j}_2 +1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2 +1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2)\right)^{1/2}\Big)\Big/\Big(8\,(j_1j_2)^{1/2}\Big), \tag{182}$$

$$\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1+\frac{1}{2} & j_2-\frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right)_1 = \Big(N_g(\bar{j}_1,\bar{j}_2)\,(2j_1-2\ \ j_2+1)\left((j_1-j_2+\bar{j}_1-\bar{j}_2 +1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1 +j_2+\bar{j}_1+\bar{j}_2+1)\right)^{1/2}\Big)\Big/\Big(8\,((j_1+1)\,j_2)^{1/2}\Big), \tag{183}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{matrix}\right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\,(2\quad j_1-2j_2-1)\,((j_1-j_2+\bar{j}_1 - \bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 + 1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2))^{1/2}\Big)/\Big(8\,(j_1\,(j_2+1\ ))^{1/2}\Big)\,, \tag{184}$$

$$\left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j_1 & j_2 \end{matrix}\right\|\left. \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 0 & 0 \end{matrix}\right)_1 = -\Big(N_g(\bar{j}_1,\bar{j}_2)\,\big(5\quad j_1^2+5j_1+5j_2^2+5j_2 - 3\,\big(\bar{j}_1^2+2\ \bar{j}_1+\bar{j}_2^2+\bar{j}_2\big)\big)\Big)/\Big(2\,(5)^{1/2}\Big)\,. \tag{185}$$

Clebsch-Gordan coefficients $(\bar{j}_1,\bar{j}_2)\otimes(\bar{1},\bar{1})\to(\bar{j}_1,\bar{j}_2)_2$ are:

$$\left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j'_1 & j'_2 \end{matrix}\right\|\left. \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & J_1 & J_2 \end{matrix}\right)_2 = \big(H^2-X^2\big)^{1/2}\left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j'_1 & j'_2 \end{matrix}\right\|\left. \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & J_1 & J_2 \end{matrix}\right) - X\left(\begin{matrix} \bar{j}_1 & \bar{j}_2 \\ j'_1 & j'_2 \end{matrix}\right\|\left. \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & J_1 & J_2 \end{matrix}\right)_1\,, \tag{186}$$

where:

$$X = -\frac{1}{10}N_g(\bar{j}_1,\bar{j}_2)\,(\bar{j}_1-\bar{j}_2)\,(\bar{j}_1-\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2 + 2)\,(4\bar{j}_1\,(\bar{j}_1+2)+4\bar{j}_2\,(\bar{j}_2+1)-5)\,, \tag{187}$$

$$H^2 = \frac{1}{5}\,(\bar{j}_1-\bar{j}_2)\,(\bar{j}_1-\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+1)\,(\bar{j}_1+\bar{j}_2+2)\,\big(4\bar{j}_2^4+8\bar{j}_2^3-(8\bar{j}_1\,(\bar{j}_1+2)+9)\,\bar{j}_2^2 - (8\bar{j}_1\,(\bar{j}_1+2)+13)\,\bar{j}_2+(\bar{j}_1+1)^{\,2}\,(4\bar{j}_1\,(\bar{j}_1+2)-5)\big)\,, \tag{188}$$

and a list of additional coefficients is the following:

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2+1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) \\ = \Big( (j_1-j_2)^2 \left((j_1+j_2-\bar{j}_1-\bar{j}_2)(j_1+j_2-\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1 \right. \\ +j_2+\bar{j}_1-\bar{j}_2+3)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 \\ \left. +3)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+4)\right)^{1/2} \Big) / \Big( 4\left((j_1+1)(2j_1+3)(j_2+1)(2j_2+3)\right)^{1/2} \Big), \tag{189}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1-1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big( (j_1-j_2)^2 \left((j_1+j_2+\bar{j}_1-\bar{j}_2)(j_1+j_2+\bar{j}_1-\bar{j}_2 \right. \\ +1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2-1)\ (j_1+j_2-\bar{j}_1 \\ +\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)(-j_1-j_2+\bar{j}_1+\bar{j}_2 \\ +2)(j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2 \\ \left. +2)\right)^{1/2} \Big) / \Big( 4\left(j_1(2\,j_1-1)\,j_2(2j_2-1)\right)^{1/2} \Big), \tag{190}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1-1 & j_2+1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big( (j_1+j_2+1)^2\ \left((j_1-j_2+\bar{j}_1-\bar{j}_2-1)\ (j_1-j_2+\bar{j}_1 \right. \\ -\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)(-j_1+j_2+\bar{j}_1-\bar{j}_2 \\ +2)(j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 \\ +1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)(-j_1+j_2+\bar{j}_1+\bar{j}_2 \\ \left. +3)\right)^{1/2} \Big) / \Big( 4\left(j_1(2j_1-1)\left(2j_2^2+5\,j_2+3\right)\right)^{1/2} \Big), \tag{191}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2-1 \end{pmatrix}\!\!\left\|\begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix}\right) = \Big( (j_1+j_2+1)^2\ \left((j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1 \right. \\ -\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2-1)(-j_1+j_2+\bar{j}_1 \\ -\bar{j}_2)(j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2 \\ +3)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 \\ \left. +1)\right)^{1/2} \Big) / \Big( 4\left(\left(2j_1^2+5\,j_1+3\right)j_2(2j_2-1)\right)^{1/2} \Big), \tag{192}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1+1 & j_2 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix} \right) = \Bigg( \sqrt{j_2\,(j_2+1)} \left( \frac{(j_1+1)^{\,2}}{j_2\,(j_2+1)} - 1 \right) \big( (j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3) \big)^{1/2} \Bigg) \Big/ \Big( 4\,((j_1+1)\,(2j_1+3))^{1/2} \Big), \tag{193}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1-1 & j_2 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix} \right) = \Big( \big( -j_1^2+j_2^2+j_2 \big)\ \big( (j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2) \big)^{1/2} \Big) \Big/ \Big( 4\,(j_1\,(2\ j_1-1))^{1/2}\,(j_2\,(j_2+1))^{1/2} \Big), \tag{194}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1 & j_2+1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix} \right) = \Bigg( \sqrt{j_1\,(j_1+1)} \left( \frac{(j_2+1)^{\,2}}{j_1\,(j_1+1)} - 1 \right) \big( (j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+2)\ (j_1+j_2-\bar{j}_1+\bar{j}_2+1)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+2)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+3) \big)^{1/2} \Bigg) \Big/ \Big( 4\,((j_2+1)\,(2j_2+3))^{1/2} \Big), \tag{195}$$

$$\begin{pmatrix} \bar{j}_1 & \bar{j}_2 \\ j_1 & j_2-1 \end{pmatrix}\!\!\left\| \begin{matrix} \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & 1 & 1 \end{matrix} \right) = \Big( \big( j_1^2+j_1-j_2^2 \big)\ \big( (j_1-j_2+\bar{j}_1-\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2)\ (j_1+j_2+\bar{j}_1-\bar{j}_2+1)\ (j_1+j_2-\bar{j}_1+\bar{j}_2)\ (-j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+2)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2+1)\ (j_1+j_2+\bar{j}_1+\bar{j}_2+2) \big)^{1/2} \Big) \Big/ \Big( 4\,(j_1\,(j_1+1))^{1/2}\,(j_2\,(2j_2-1))^{1/2} \Big), \tag{196}$$

$$\begin{aligned}
\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & j_1 & j_2 & 1 & 1 \end{array}\right) = & -\Big(j_1^6 + 3j_1^5 - \left(j_2^2 + j_2 + 2\bar{j}_1^2 + 2\ \bar{j}_2^2 + 4\bar{j}_1 + 2\bar{j}_2 - 1\right) j_1^4 \\
& - \left(2j_2^2 + 2\ \ j_2 + 4\bar{j}_1^2 + 4\bar{j}_2^2 + 8\bar{j}_1 + 4\bar{j}_2 + 3\right)\ \ j_1^3 \\
& + \left(-j_2^4 - 2j_2^3 + \left(4\bar{j}_1^2 + 8\bar{j}_1 + 4\ \ \bar{j}_2^2 + 4\bar{j}_2 + 2\right) j_2^2\right. \\
& + \left(4\bar{j}_1^2 + 8\ \bar{j}_1 + 4\bar{j}_2^2 + 4\bar{j}_2 + 3\right)\ j_2 + \bar{j}_1^4 + \bar{j}_2^4 + 4\bar{j}_1^3 + 2\bar{j}_2^3 - 3\ \bar{j}_2^2 \\
& \quad \left. - 4\bar{j}_2 + \bar{j}_1^2\left(-2\bar{j}_2^2 - 2\ \bar{j}_2 + 3\right) - 2\bar{j}_1\left(2\bar{j}_2^2 + 2\bar{j}_2 + 1\right) - 2\right) j_1^2 \\
& + \left(-j_2^4 - 2j_2^3 + \left(4\bar{j}_1^2 + 8\ \ \bar{j}_1 + 4\bar{j}_2^2 + 4\bar{j}_2 + 3\right) j_2^2\right. \\
& +4\ \ \left(\bar{j}_1^2 + 2\bar{j}_1 + \bar{j}_2^2 + \bar{j}_2 + 1\right)\ j_2 + \bar{j}_1^4 + 4\bar{j}_1^3 + \bar{j}_1\left(-4\bar{j}_2^2 - 4\ \bar{j}_2 + 2\right) \\
& \qquad\qquad \left. + \bar{j}_1^2\left(-2\bar{j}_2^2 - 2\bar{j}_2 + 5\right) + \bar{j}_2\left(\bar{j}_2^3 + 2\bar{j}_2^2 - \bar{j}_2 - 2\right)\right) j_1 \\
& + j_2\left(j_2 + 1\right)\left(j_2^4 + 2j_2^3 - \left(2\ \bar{j}_1^2 + 4\bar{j}_1 + 2\bar{j}_2^2 + 2\bar{j}_2 + 1\right) j_2^2\right. \\
& \qquad\qquad - 2\ \ \left(\bar{j}_1^2 + 2\bar{j}_1 + \bar{j}_2^2 + \bar{j}_2 + 1\right)\ j_2 + \bar{j}_1^4 + 4\bar{j}_1^3 \\
& \qquad\qquad + \bar{j}_1\left(-4\bar{j}_2^2 - 4\ \bar{j}_2 + 2\right) + \bar{j}_1^2\left(-2\bar{j}_2^2 - 2\bar{j}_2 + 5\right) \\
& \left. + \bar{j}_2\left(\bar{j}_2^3 + 2\bar{j}_2^2 - \bar{j}_2 - 2\right)\right)\Big) \Big/ \Big(4\left(j_1\left(j_1 + 1\right) j_2\left(j_2 + 1\right)\right)^{1/2}\Big),
\end{aligned} \tag{197}$$

$$\begin{aligned}
\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 + \frac{1}{2} & j_2 + \frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = & -\Big(\ (j_1 - j_2)^{\,2}\left(2j_1 + 2j_2 + 3\right)\ \ ((j_1 + j_2 + \bar{j}_1 - \bar{j}_2 \\
& + 2)\ \ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2 + 1)\ \ (-j_1 - j_2 + \bar{j}_1 + \bar{j}_2)\ \ (j_1 \\
& + j_2 + \bar{j}_1 + \bar{j}_2 + 3))^{1/2}\Big) \Big/ \Big(4\left(\left(j_1 + 1\right)\left(j_2 + 1\right)\right)^{1/2}\Big),
\end{aligned} \tag{198}$$

$$\begin{aligned}
\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 - \frac{1}{2} & j_2 - \frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = & -\Big(\ (j_1 - j_2)^{\,2}\left(2j_1 + 2j_2 + 1\right)\ \ ((j_1 + j_2 + \bar{j}_1 - \bar{j}_2 \\
& + 1)\ \ (j_1 + j_2 - \bar{j}_1 + \bar{j}_2)\ \ (-j_1 - j_2 + \bar{j}_1 + \bar{j}_2 \\
& + 1)\ \ (j_1 + j_2 + \bar{j}_1 + \bar{j}_2 + 2))^{1/2}\Big) \Big/ \Big(4\left(j_1 j_2\right)^{1/2}\Big),
\end{aligned} \tag{199}$$

$$\begin{aligned}
\left(\begin{array}{cc\|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1 & \bar{j}_2 & 1 & 1 \\ j_1 + \frac{1}{2} & j_2 - \frac{1}{2} & j_1 & j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = & \Big(\ (j_1 + j_2 + 1)^{\,2}\left(-2j_1 + 2j_2 - 1\right)\ \ ((j_1 - j_2 + \bar{j}_1 - \bar{j}_2 \\
& + 1)\ \ (-j_1 + j_2 + \bar{j}_1 - \bar{j}_2)\ \ (j_1 - j_2 + \bar{j}_1 + \bar{j}_2 + 2)\ \ (-j_1 \\
& + j_2 + \bar{j}_1 + \bar{j}_2 + 1))^{1/2}\Big) \Big/ \Big(4\left(\left(j_1 + 1\right) j_2\right)^{1/2}\Big),
\end{aligned} \tag{200}$$

$$\left(\begin{array}{cc|ccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1\ \bar{j}_2 & 1 & 1 \\ j_1-\frac{1}{2} & j_2+\frac{1}{2} & j_1\ j_2 & \frac{1}{2} & \frac{1}{2} \end{array}\right) = \Big( (2j_1-2j_2-1)\,(j_1+j_2 + 1)^{\,2} \ \left((j_1-j_2+\bar{j}_1-\bar{j}_2)\ (-j_1+j_2+\bar{j}_1-\bar{j}_2 + 1)\ (j_1-j_2+\bar{j}_1+\bar{j}_2+1)\ (-j_1+j_2+\bar{j}_1+\bar{j}_2 + 2)\right)^{1/2}\Big)\Big/\Big(4\,(j_1\,(j_2+1\,))^{1/2}\Big)\,, \tag{201}$$

$$\left(\begin{array}{cc|ccc} \bar{j}_1 & \bar{j}_2 & \bar{j}_1\ \bar{j}_2 & 1 & 1 \\ j_1 & j_2 & j_1\ j_2 & 0 & 0 \end{array}\right) = -\Big(\bar{j}_1^4+4\bar{j}_1^3+\left(-2\bar{j}_2^2-2\,\bar{j}_2+5\right)\bar{j}_1^2+\left(-4\bar{j}_2^2-4\bar{j}_2+2\right)\bar{j}_1+\bar{j}_2^4 + 2\bar{j}_2^3-5\left(j_1^2+j_1-j_2\ \ (j_2+1)\right)^{\,2}-\bar{j}_2^2-2\bar{j}_2\Big)\Big/\Big(2\,(5)^{1/2}\Big)\,. \tag{202}$$

The rest of the coefficients can be obtained by using the symmetries of the coefficients [43]. There are no multiplicity, so the symmetries are simply given by:

$$\left(\begin{array}{cc|cccc} \bar{j}_1 & \bar{j}_2 & \bar{j}'_1 & \bar{j}'_2 & \bar{1} & \bar{1} \\ j_1 & j_2 & j'_1 & j'_2 & j''_1 & j''_2 \end{array}\right) = (-1)^{\bar{j}_1-\bar{j}'_1+\bar{j}'_2-\bar{j}_2+j_1-j'_1+j_2-j'_2+j''_1+j''_2} \times \sqrt{\frac{dim(\bar{j}_1,\bar{j}_2)(2j'_1+1)(2j'_2+1)}{dim(\bar{j}'_1,\bar{j}'_2)(2j_1+1)(2j_2+1)}} \left(\begin{array}{cc|cccc} \bar{j}'_1 & \bar{j}'_2 & \bar{j}_1 & \bar{j}_2 & \bar{1} & \bar{1} \\ j'_1 & j'_2 & j_1 & j_2 & j''_1 & j''_2 \end{array}\right),$$

where $dim(\bar{j}_1,\bar{j}_2)=(2\bar{j}_1-2\bar{j}_2+1)(2\bar{j}_1+2\bar{j}_2+3)(2\bar{j}_1+2)(2\bar{j}_2+1)/6$.

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