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On the Structure of Green's Ansatz

Igor Salom

Abstract It is well known that the symmetric group has an important role (via Young tableaux formalism) both in labelling of the representations of the unitary group and in construction of the corresponding basis vectors (in the tensor product of the defining representations). We show that orthogonal group has a very similar role in the context of positive energy representations of $osp(1|2n,\mathbb{R})$. In the language of parabose algebra, we essentially solve, in the parabosonic case, the long standing problem of reducibility of Green's Ansatz representations.

1 Introduction

The $osp(1|2n,\mathbb{R})$ superalgebra attracts nowadays significant attention, primarily as a natural generalization of the conformal supersymmetry in higher dimensions [1–9]. In the context of space-time supersymmetry, knowing and understanding unitary irreducible representations (UIR's) of this superalgebra is of extreme importance, as these should be in a direct relation with the particle content of the corresponding physical models.

And the most important from the physical viewpoint are certainly, so called, positive energy UIR's, which are the subject of this paper. More precisely, the goal of the paper is to clarify how these representations can be obtained by essentially tensoring the simplest nontrivial positive energy UIR (the one that corresponds to oscillator representation). This parallels the case of the UIR's of the unitary group $U(n)$ constructed within the tensor product of the defining (i.e. "one box") representations. In both cases the tensor product representation is reducible, and while this reduction in the $U(n)$ case is governed by the action of the commuting group of permutations, in the osp case,[1] as we will show, the role of permutations is played by an orthogonal group. We will clarify the details of this reduction.

[1] We will often write shortly $osp(1|2n)$ or osp for the $osp(1|2n,\mathbb{R})$.

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The $osp(1|2n)$ superalgebra is also known by its direct relation to parabose algebra [10, 11]. In the terminology of parastatistics, the tensor product of oscillator UIR's is known as the Green's Ansatz [12]. The problem of the decomposition of parabose Green's Ansatz space to parabose (i.e. $osp(1|2n)$) UIR's is an old one [12], that we here solve by exploiting additional orthogonal symmetry of a "covariant" version of the Green's Ansatz.

2 Covariant Green's Ansatz

Structural relations of $osp(1|2n)$ superalgebra can be compactly written in the form of trilinear relations of odd algebra operators a_α and $a_\alpha^\dagger$:

$$[\{a_\alpha, a_\beta^\dagger\}, a_\gamma] = -2\delta_{\beta\gamma} a_\alpha, \qquad [\{a_\alpha^\dagger, a_\beta\}, a_\gamma^\dagger] = 2\delta_{\beta\gamma} a_\alpha^\dagger, \tag{1}$$

$$[\{a_\alpha, a_\beta\}, a_\gamma], \qquad [\{a_\alpha^\dagger, a_\beta^\dagger\}, a_\gamma^\dagger] = 0, \tag{2}$$

where operators $\{a_\alpha, a_\beta^\dagger\}$, $\{a_\alpha, a_\beta\}$ and $\{a_\alpha^\dagger, a_\beta^\dagger\}$ span the even part of the superalgebra and Greek indices take values $1, 2, \ldots n$ (relations obtained from these by use of Jacobi identity are also implied). This compact notation emphasises the direct connection [11] of $osp(1|2n)$ superalgebra with the parabose algebra of n pairs of creation/annihilation operators [10].

If we (in the spirit of original definition of parabose algebra [10]) additionally require that the dagger symbol $\dagger$ above denotes hermitian conjugation in the algebra representation Hilbert space (of positive definite metrics), then we have effectively constrained ourselves to the, so called, positive energy UIR's of $osp(1|2n)$.[2] Namely, in such a space, "conformal energy" operator $E \equiv \frac{1}{2}\sum_\alpha \{a_\alpha, a_\alpha^\dagger\}$ must be a positive operator. Operators a_α reduce the eigenvalue of E, so the Hilbert space must contain a subspace that these operators annihilate. This subspace is called vacuum subspace: $V_0 = \{|v\rangle, a_\alpha|v\rangle = 0\}$. If the positive energy representation is irreducible, all vectors from V_0 have the common, minimal eigenvalue ϵ_0 of E: $E|v\rangle = \epsilon_0|v\rangle, |v\rangle \in V_0$. Representations with one dimensional subspace V_0 are called "unique vacuum" representations.

In this paper we will constrain our analysis to UIR's with integer and half-integer values of ϵ_0 (in principle, ϵ_0 has also continuous part of the spectrum—above the, so called, first reduction point of the Verma module). It turns out that all representations from this class can be obtained by representing the odd superalgebra operators a and $a^\dagger$ as the following sum:

$$a_\alpha = \sum_{a=1}^p b_\alpha^a\, e^a, \qquad a_\alpha^\dagger = \sum_{a=1}^p b_\alpha^{a\dagger}\, e^a. \tag{3}$$

[2] Omitting a short proof, we note that in such a Hilbert space all superalgebra relations actually follow from one single relation—the first or the second of (1).

In this expression integer p is known as the order of the parastatistics, e^a are elements of a real Clifford algebra:

$$\{e^a, e^b\} = 2\delta^{ab} \tag{4}$$

and operators b^a_α together with adjoint $b^{a\dagger}_\alpha$ satisfy ordinary bosonic algebra relations. There are total of $n \cdot p$ mutually commuting pairs of bosonic annihilation-creation operators $(b^a_\alpha, b^{a\dagger}_\alpha)$:

$$[b^a_\alpha, b^{b\dagger}_\beta] = \delta_{\beta\alpha}\delta^{ab}; \quad [b^a_\alpha, b^b_\beta] = 0. \tag{5}$$

Indices $a, b, \ldots$ from the beginning of the Latin alphabet will, throughout the paper, take values $1, 2, \ldots p$. Relation (3) is a slight variation, more precisely, realization, of a more common form of the Green's Ansatz [10, 13].

The representation space of operators (3) can be seen as tensor product of p multiples of Hilbert spaces $\mathcal{H}_a$ of ordinary linear harmonic oscillator in n-dimensions multiplied by the representation space of the Clifford algebra:

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p \otimes \mathcal{H}_{CL}. \tag{6}$$

A single factor Hilbert space $\mathcal{H}_a$ is the space of unitary representation of n dimensional bose algebra of operators $(b^a_\alpha, b^{a\dagger}_\alpha), \alpha = 1, 2, \ldots n$: $\mathcal{H}_a \cong \mathcal{U}(b^{a\dagger})|0\rangle_a$, where $|0\rangle_a$ is the usual Fock vacuum of factor space $\mathcal{H}_a$. The representation space $\mathcal{H}_{CL}$ of real Clifford algebra (4) is of dimension $2^{[p/2]}$, i.e. isomorphic with $\mathbb{C}^{2^{[p/2]}}$ (matrix representation). Positive definite scalar product is introduced in usual way in each of the factor spaces, endowing entire space $\mathcal{H}$ also with positive definite scalar product. The space is spanned by the vectors:

$$\mathcal{H} = l.s.\{\mathcal{P}(b^\dagger)|0\rangle \otimes \omega\}, \tag{7}$$

where $\mathcal{P}(b^\dagger)$ are monomials in mutually commutative operators $b^{a\dagger}_\alpha$, $|0\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots \otimes |0\rangle_p$ and $w \in \mathcal{H}_{CL}$.

In the case $p = 1$ (the Clifford part becomes trivial) we obtain the simplest positive energy UIR of $osp(1|2n)$—the n dimensional harmonic oscillator representation. The order p Green's Ansatz representation of $osp(1|2n)$ is, effectively, representation in the p-fold tensor product of oscillator representations [12], with the Clifford factor space taking care of the anticommutativity properties of odd superalgebra operators. It is easily verified that even superalgebra elements act trivially in the Clifford factor space and that their action is simply sum of actions in each of the factor spaces.

The space (6) is highly reducible under action of osp superalgebra. It necessarily decomposes into direct sum of positive energy representations (both unique vacuum and non unique vacuum representations) and thus, from the aspect of osp transformation properties, space $\mathcal{H}$ is spanned by:

$$\mathcal{H} = l.s.\{|(\Lambda, l), \eta_\Lambda\rangle\}, \tag{8}$$

where Λ labels $osp(1|2n)$ positive energy UIR, l uniquely labels a concrete vector within the UIR Λ, and $\eta_\Lambda = 1, 2, \dots N_\Lambda$ labels possible multiplicity of UIR Λ in the representation space $\mathcal{H}$. If some UIR Λ does not appear in decomposition of $\mathcal{H}$, then the corresponding N_Λ is zero. Label Λ in (8) runs through all (integer and halfinteger positive energy) UIR's of $osp(1|2n)$ such that $N_\Lambda > 0$ and l runs through all vectors from UIR Λ.

3 Gauge Symmetry of the Ansatz

Green's Ansatz in the form (3) possesses certain intrinsic symmetries. First, we note that hermitian operators

$$G^{ab} \equiv \sum_{\alpha=1}^{n} i(b_\alpha^{a\dagger} b_\alpha^b - b_\alpha^{b\dagger} b_\alpha^a) + \frac{i}{4}[e^a, e^b] \tag{9}$$

commute with entire osp superalgebra, which immediately follows after checking that $[G^{ab}, a_\alpha] = 0$. Operators G^{ab} themselves satisfy commutation relations of $so(p)$ algebra. The second term in (9) acts in the Clifford factor space, generating a faithful representation of $Spin(p)$ (i.e. spinorial representation of double cover of $SO(p)$ group). Action of the first terms from (9) generate $SO(p)$ group action in the space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p$. In the entire space $\mathcal{H}$ operators G generate $Spin(p)$ group and all vectors belong to spinorial unitary representations of this symmetry group. The two terms in (9) thus resemble orbital and spin parts of rotation generators and we will often use that terminology. In particular $\mathcal{H} \equiv \mathcal{H}^{\mathbf{o}} \otimes \mathcal{H}^{\mathbf{s}}$, where $\mathcal{H}^{\mathbf{o}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p$ and $\mathcal{H}^{\mathbf{s}} = \mathcal{H}_{CL}$. Furthermore, due to existence of operators $I^a \equiv -iexp(i\pi \sum_\alpha b_\alpha^{a\dagger} b_\alpha^a)\overline{e}e^a$ where $\overline{e} \equiv i^{[p/2]} e^1 e^2 \cdots e^p$, for even values of p, the symmetry can be extended to $Pin(p)$ group (the double cover of orthogonal group $O(p)$). We will refer to the symmetry group of the Green's ansatz as the gauge group.

Vectors in space $\mathcal{H}$ carry quantum numbers also according to their transformation properties under the gauge group. As the gauge group commutes with $osp(1|2n)$, these numbers certainly remove at least a part of degeneracy of osp representations in $\mathcal{H}$, in the sense that relation (8) can be rewritten as:

$$\mathcal{H} = l.s.\{|(\Lambda, l), (M, m), \eta_{(\Lambda,M)}\rangle\}, \tag{10}$$

where (Λ, l) uniquely label vector l within $osp(1|2n)$ positive energy UIR Λ, (M, m) uniquely label vector m within finite dimensional UIR M of the gauge group, and $\eta_{(\Lambda,M)} = 1, 2, \dots N_{(\Lambda,M)}$ labels possible remaining multiplicity of tensor product of these two representations $\mathcal{D}_\Lambda^{osp} \otimes \mathcal{D}_M^{gauge}$ in the space $\mathcal{H}$. Again, if some combination (Λ, M) does not appear in decomposition of $\mathcal{H}$, then the corresponding $N_{(\Lambda,M)}$ is zero.

Important property of the gauge symmetry is that it actually removes all degeneracy in decomposition of $\mathcal{H}$ to $osp(1|2n)$ UIR's, i.e. that the multiplicity of $osp(1|2n)$ UIR's is fully taken into account by labeling transformation properties of the vector w.r.t. the gauge symmetry group. Furthermore, there is one-to-one correspondence between UIR's of $osp(1|2n)$ and of the gauge group that appear in the decomposition, meaning that transformation properties under the gauge group action automatically fix the $osp(1|2n)$ representation. We formulate this more precisely in the following theorem.

Theorem 1. *The following statements hold for the basis (10) of the Hilbert space $\mathcal{H}$:*

1. *All multiplicities $N_{(\Lambda,M)}$ are either 1 or 0.*
2. *Let the $\mathcal{N}$ be the set of all pairs (Λ, M) for which $N_{(\Lambda,M)} = 1$, i.e. $\mathcal{N} = \{(\Lambda, M)|N_{(\Lambda,M)} = 1\}$ and let the $\mathcal{L}$ and $\mathcal{M}$ be sets of all Λ and M, respectively, that appear in any of the pairs from $\mathcal{N}$. Then pairs from $\mathcal{N}$ naturally define bijection from $\mathcal{L}$ to $\mathcal{M}$, $\mathcal{N}: \mathcal{L} \to M$.*

The theorem is proved by explicit construction of the bijection $\mathcal{N}$. First we must go through some preliminary definitions and lemmas.

Corollary 1. *If $osp(1|2n)$ representation Λ appears in the decomposition of the space $\mathcal{H}$, then its multiplicity in the decomposition is given by the dimension of the gauge group representation $\mathcal{N}(\Lambda)$.*

4 Root Systems

At this point we must introduce root systems, both for $osp(1|2n)$ superalgebra and for the $so(p)$ algebra of the gauge group.

We choose basis of a Cartan subalgebra $\mathfrak{h}_{osp}$ of (complexified) $osp(1|2n)$ as:

$$\mathfrak{h}_{osp} = l.s.\left\{ \frac{1}{2}\{a_\alpha^\dagger, a_\alpha\}, \alpha = 1, 2, \ldots n \right\}. \tag{11}$$

Positive roots, expressed using elementary functionals, are:

$$\begin{aligned} \Delta_{osp}^+ = \{&+\delta_\alpha, 1 \le \alpha \le n; +\delta_\alpha + \delta_\beta, 1 \le \alpha < \beta \le n; \\ &+\delta_\alpha - \delta_\beta, 1 \le \alpha < \beta \le n; +2\delta_\alpha, 1 \le \alpha \le n\} \end{aligned} \tag{12}$$

and the corresponding positive root vectors, spanning subalgebra $\mathfrak{g}_{osp}^+$, are (in the same order):

$$\begin{gathered} \Big\{a_\alpha^\dagger, 1 \le \alpha \le n; \{a_\alpha^\dagger, a_\beta^\dagger\}, 1 \le \alpha < \beta \le n; \\ \{a_\alpha^\dagger, a_\beta\}, 1 \le \alpha < \beta \le n; \{a_\alpha^\dagger, a_\alpha^\dagger\}, 1 \le \alpha \le n\Big\}. \end{gathered} \tag{13}$$

Simple root vectors are:

$$\Big\{\{a_1^\dagger, a_2\}, \{a_2^\dagger, a_3\}, \ldots, \{a_{n-1}^\dagger, a_n\}, a_n^\dagger\Big\}. \tag{14}$$

With this choice of positive roots, positive energy UIR's of $osp(1|2n)$ become lowest weight representations. Thus, we will label positive energy UIR's of $osp(1|2n)$ either by their lowest weight

$$\underline{\lambda} = (\underline{\lambda}_1, \underline{\lambda}_2, \ldots, \underline{\lambda}_n), \tag{15}$$

or by its signature

$$\Lambda = [d; \Lambda_1, \Lambda_2, \ldots, \Lambda_{n-1}] \tag{16}$$

related to the lowest weight $\underline{\lambda}$ by $d = \underline{\lambda}_1$, $\Lambda_\alpha = \underline{\lambda}_{\alpha+1} - \underline{\lambda}_\alpha$. Λ_α are nonnegative integers [14] and spectrum of d is positive and dependant of Λ_α values.

As a basis of Cartan subalgebra $\mathfrak{h}_{so}$ of $so(p)$ we take:

$$\mathfrak{h}_{so} = l.s.\Big\{G^{(k)} \equiv G^{2k-1,2k}, k = 1, 2, \ldots q\Big\}, \tag{17}$$

where $q = [p/2]$ is the dimension of Cartan subalgebra (indices $k, l, \ldots$ from the middle of alphabet will take values $1, 2, \ldots, q$). Positive roots in case of even p are:

$$\Delta_{so}^{+} = \{+\delta_k + \delta_l, 1 \leq k < l \leq q; +\delta_k - \delta_l, 1 \leq k < l \leq q\}, \tag{18}$$

while in the odd case we additionally have $\{+\delta_k, 1 \leq k \leq q\}$.

In accordance with the choice of Cartan subalgebra $\mathfrak{h}_{so}$ it is more convenient to use the following linear combinations:

$$B_{\alpha\pm}^{(k)\dagger} \equiv \frac{1}{\sqrt{2}}(b_\alpha^{2k-1\dagger} \pm i b_\alpha^{2k\dagger}), \qquad B_{\alpha\pm}^{(k)} = \frac{1}{\sqrt{2}}(b_\alpha^{2k-1} \mp i b_\alpha^{2k}), \tag{19}$$

instead of $b^\dagger$ and b, as $[G^{(k)}, B_{\alpha\pm}^{(l)\dagger}] = \pm\delta^{kl} B_{\alpha\pm}^{(l)\dagger}$ and $[G^{(k)}, B_{\alpha\pm}^{(l)}] = \mp\delta^{kl} B_{\alpha\pm}^{(l)}$. Similarly, we introduce $e_\pm^{(k)} \equiv \frac{1}{\sqrt{2}}(e^{2k-1} \pm i e^{2k})$ that satisfy:

$$[G^{(k)}, e_\pm^{(l)}] = \pm\delta^{kl} e_\pm^{(l)}. \tag{20}$$

Odd superalgebra operators take form:

$$a_\alpha^\dagger = \Big(\sum_{k=1}^{q} B_{\alpha+}^{(k)\dagger} e_-^{(k)} + B_{\alpha-}^{(k)\dagger} e_+^{(k)}\Big) + \epsilon\, b_\alpha^{p\dagger} e^p, \tag{21}$$

$$a_\alpha = \Big(\sum_{k=1}^{q} B_{\alpha+}^{(k)} e_+^{(k)} + B_{\alpha-}^{(k)} e_-^{(k)} \Big) + \epsilon\, b_\alpha^p e^p, \tag{22}$$

where $\epsilon = p \mod 2$.

The space $\mathcal{H}$ decomposes to spinorial UIR's of $so(p)$ with the highest weight $\overline{\mu} = (\overline{\mu}^1, \overline{\mu}^2, \ldots, \overline{\mu}^q)$ satisfying $\overline{\mu}^1 \geq \overline{\mu}^2 \geq \cdots \geq \overline{\mu}^{q-1} \geq |\overline{\mu}^q| \geq \frac{1}{2}$ with all $\overline{\mu}^q$ taking half-integer values ($\overline{\mu}^q$ can take negative values when p is even). However, since the gauge symmetry group in the case of even p is enlarged to $Pin(p)$ group, any highest weight of UIR of the gauge group satisfies: $\overline{\mu}^1 \geq \overline{\mu}^2 \geq \cdots \geq \overline{\mu}^q \geq 0$. As the gauge group representation in $\mathcal{H}$ is spinorial, all $\overline{\mu}^k$ take half-integer values greater or equal to $\frac{1}{2}$. To label UIR's of the gauge group we will also use signature

$$M = [M^1, M^2, \ldots, M^q] \tag{23}$$

with $M^k = \overline{\mu}^k - \overline{\mu}^{k+1}, k < q$ and $M^q = \overline{\mu}^q - \frac{1}{2}$. All M^k are nonnegative integers.

The "spin" factor space $\mathcal{H}^{\mathrm{s}}$ is irreducible w.r.t. action of the gauge group. Gauge group representation in the space $\mathcal{H}^{\mathrm{s}}$ has the highest weight $\overline{\mu}_{\mathbf{s}} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Weight spaces of this representation are one dimensional, meaning that basis vectors can be fully specified by weights $\mu_{\mathbf{s}}$:

$$\mathcal{H}^{\mathrm{s}} = l.s.\{\omega_{\mu_{\mathrm{s}}} \equiv \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \ldots, \mu_{\mathbf{s}}^q) | \mu_{\mathbf{s}}^k = \pm\frac{1}{2}\}. \tag{24}$$

An action of operators $e_+^{(k)}$, $e_-^{(k)}$ and e^p in this basis is given by:

$$e_\pm^{(k)} \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \ldots, \mu_{\mathbf{s}}^q) = \sqrt{2} \left(\prod_{l=1}^{k-1} 2\mu_{\mathbf{s}}^l \right) \omega(\mu_{\mathbf{s}}^1, \ldots, \mu_{\mathbf{s}}^{k-1}, \mu_{\mathbf{s}}^k \pm 1, \mu_{\mathbf{s}}^{k+1}, \ldots, \mu_{\mathbf{s}}^q) \tag{25}$$

and, when p is odd, also:

$$e^p \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \ldots, \mu_{\mathbf{s}}^q) = \left(\prod_{l=1}^{q} 2\mu_{\mathbf{s}}^l \right) \omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \ldots, \mu_{\mathbf{s}}^q). \tag{26}$$

In these definitions it is implied that $\omega(\mu_{\mathbf{s}}^1, \mu_{\mathbf{s}}^2, \ldots, \mu_{\mathbf{s}}^q) \equiv 0$ if any $|\mu_{\mathbf{s}}^k| > \frac{1}{2}$.

Gauge group representation in "orbital" factor space $\mathcal{H}^{\mathbf{o}}$ decomposes to highest weight $\overline{\mu}_{\mathbf{o}}$ UIR's such that all $\overline{\mu}_{\mathbf{o}}^k$ are nonnegative integers. Besides, it is not difficult to verify that, if $n < q$, then

$$\overline{\mu}_{\mathbf{o}}^{n+1} = \overline{\mu}_{\mathbf{o}}^{n+2} = \cdots = \overline{\mu}_{\mathbf{o}}^q = 0 \tag{27}$$

(since maximally n operators (19) can be antisymmetrized).

5 Decomposition of the Green's Ansatz Space

Now we can formulate the following lemma that is the remaining step necessary for the proof of Theorem 1.

Lemma 1. *The vector* $|(\underline{\lambda}, \underline{\lambda}), (\overline{\mu}, \overline{\mu}), \eta_{(\underline{\lambda},\overline{\mu})}\rangle \in \mathcal{H}$ *that is the lowest weight vector of* $osp(1|2n)$ *positive energy UIR* $\underline{\lambda}$ *and the highest weight vector of the gauge group UIR* $\overline{\mu}$ *exists if and only if signatures* Λ *and* M *(16, 23) satisfy:*

$$M_k = \Lambda_{n-k}, \tag{28}$$

where $\Lambda_0 \equiv d - p/2$ *and it is implied that* $M_k = 0, k > q$ *and* $\Lambda_\alpha = 0, \alpha < 0$*. In that case this vector has the following explicit form (up to multiplicative constant) in the basis (7):*

$$|(\underline{\lambda}, \underline{\lambda}), (\overline{\mu}, \overline{\mu}), \eta_{(\underline{\lambda},\overline{\mu})}\rangle = \left(B^{(1)\dagger}_{n+}\right)^{\Lambda_{n-1}} \left(B^{(1)\dagger}_{n+} B^{(2)\dagger}_{n-1+} - B^{(2)\dagger}_{n+} B^{(1)\dagger}_{n-1+}\right)^{\Lambda_{n-2}} \cdots$$

$$\cdot\left(\sum_{k_1,k_2,\ldots k_n=1}^{\min(n,q)} \varepsilon_{k_1 k_2 \ldots k_n} B^{(k_1)\dagger}_{n+} B^{(k_2)\dagger}_{n-1+} \cdots B^{(k_n)\dagger}_{1+}\right)^{\Lambda_0} |0\rangle \otimes \omega\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right). \tag{29}$$

We will omit a rather lengthy proof of the lemma.

Note that the Lemma 1 also determines whether an osp representation Λ appears or not in the decomposition of Green's Ansatz of order p: UIR Λ appears in the decomposition if and only if the condition (28) can be satisfied by allowed integer values of M_k. However, if q is not sufficiently high, the first $n - q$ of the Λ components $\Lambda_0, \Lambda_1, \ldots \Lambda_{n-q-1}$ are bound to be zero.

Corollary 2. *All (half)integer positive energy UIR's of* $osp(1|2n)$ *can be constructed in space* $\mathcal{H}$ *with* $p \leq 2n + 1$*.*

Proof. Due to relation (28), values $\Lambda_0, \Lambda_1, \ldots \Lambda_{n-1}$ can be arbitrary integers when $q \geq n$: choice $p = 2n$ contains integer values of d UIR's while $p = 2n + 1$ contains half-integer values. That spaces $\mathcal{H}$ for some $p < 2n$ also contain all UIR's with $d < n$, can be verified by checking the list of all positive energy UIR's of $osp(1|2n)$ given elsewhere [15]. □

In other words, the above corollary states that no additional (half)integer energy UIR's of $osp(1|2n)$ appear when considering $p > 2n + 1$, i.e. it is sufficient to consider only $p \leq 2n + 1$.

The proof of the **Theorem 1** now follows from the Lemma 1.

Proof. Lemma 1 gives the explicit form of the vector that is the lowest weight vector of $osp(1|2n)$ positive energy UIR $\underline{\lambda}$ and the highest weight vector of the gauge group UIR $\overline{\mu}$, when such vector exists. It follows that there can be at most one such vector. Therefore, the multiplicity $N_{(\underline{\lambda},\overline{\mu})}$ can be either 1 or 0. The relation between $\underline{\lambda}$ and $\overline{\mu}$ is given by (28) and it defines bijection $\mathcal{N}$. □

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References

AQ1 1. Fronsdal, C.: Massless particles, orthosymplectic symmetry and another type of Kaluza-Klein theory. In: Essays on Supersymmetry. Reidel, (19[illegible]

AQ2 2. Bandos, I., Lukierski, J., Preitschopf, C., Sorokin, D.P.: Phys. Rev. **D61**, 065009 (2000)
3. Bandos, I., Azcárraga, J.A., Izquierdo, J.M., Lukierski, J.: Phys. Rev. Lett. **86**, 4451–4454 (2001)
4. Lukierski, J., Toppan, F.: Phys. Lett. **B539**, 266 (2002)
5. Vasiliev, M.A.: Phys. Rev. **D66**, 066006 (2002)
6. Plyushchay, M., Sorokin, D., Tsulaia, M.: JHEP **04**, 013 (2003)
7. Vasiliev, M.A.: Nucl. Phys. **B793**, 469 (2008)
8. Salom, I.: Fortschritte der Physik **56**, 505 (2008)
9. Fedoruk, S., Lukierski, J.: JHEP **02**, 128 (2013)
10. Green, H.S.: Phys. Rev. **90**, 270 (1952)
11. Ganchev, A.Ch., Palev, T.D.: J. Math. Phys. **21**, 797–799 (1980)
12. Palev, T.D.: J. Phys. A Math. Gen. **27**, 7373–7387 (1994)
13. Greenberg, O.W., Messiah, A.M.L.: Phys. Rev. **138**, 1155 (1965)
14. Dobrev, V.K., Zhang, R.B.: Phys. Atomic Nuclei **68**, 1660 (2005)

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