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Algebraic Bethe ansatz for the $s\ell(2)$ Gaudin model with boundary

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Abstract

Following Sklyanin's proposal in the periodic case, we derive the generating function of the Gaudin Hamiltonians with boundary terms. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the XXX Heisenberg spin chain and the central element, the so-called Sklyanin determinant. The corresponding Gaudin Hamiltonians with boundary terms are obtained as the residues of the generating function. By defining the appropriate Bethe vectors which yield strikingly simple off shell action of the generating function, we fully implement the algebraic Bethe ansatz, obtaining the spectrum of the generating function and the corresponding Bethe equations.

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1. Introduction

A model of interacting spins in a chain was first considered by Gaudin [1,2]. Gaudin derived these models as a quasi-classical limit of the quantum chains. Sklyanin studied the rational $s\ell(2)$ model in the framework of the quantum inverse scattering method using the $s\ell(2)$ invariant classical r-matrix [3]. A generalisation of these results to all cases when skew-symmetric r-matrix satisfies the classical Yang–Baxter equation [4] was relatively straightforward [5,6]. Therefore, considerable attention has been devoted to Gaudin models corresponding to the classical r-matrices of simple Lie algebras [7–12] and Lie superalgebras [13–17].

Hikami, Kulish and Wadati showed that the quasi-classical expansion of the transfer matrix of the periodic chain, calculated at the special values of the spectral parameter, yields the Gaudin Hamiltonians [18,19]. Hikami showed how the quasi-classical expansion of the transfer matrix, calculated at the special values of the spectral parameter, yields the Gaudin Hamiltonians in the case of non-periodic boundary conditions [20]. Then the ABA was applied to open Gaudin model in the context of the Vertex-IRF correspondence [21–23]. Also, results were obtained for the open Gaudin models based on Lie superalgebras [24]. An approach to study the open Gaudin models based on the classical reflection equation [25–27] and the non-unitary r-matrices was developed recently, see [28,29] and the references therein. For a review of the open Gaudin model see [30]. Progress in applying Bethe ansatz to the Heisenberg spin chain with non-periodic boundary conditions compatible with the integrability of the quantum systems [31–41] had recent impact on the study of the corresponding Gaudin model [41,42]. The so-called $T - Q$ approach to implementation of Bethe ansatz [35,36] was used to obtain the eigenvalues of the associated Gaudin Hamiltonians and the corresponding Bethe ansatz equations [42]. In [41] the off shell action of the generating function of the Gaudin Hamiltonians on the Bethe vectors was obtained through the so-called quasi-classical limit.

Here we derive the generating function of the Gaudin Hamiltonians with boundary terms following Sklyanin's approach in the periodic case [3]. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the inhomogeneous XXX Heisenberg spin chain and the central element, the so-called Sklyanin determinant. The essential step in this derivation is the expansion of the monodromy matrix in powers of the quasi-classical parameter. Moreover, we show how the representation of the relevant Lax matrix in terms of local spin operators yields the partial fraction decomposition of the generating function. Consequently, the Gaudin Hamiltonians with the boundary terms are obtained from the residues of the generating function at poles. We derive the relevant linear bracket for the Gaudin Lax operator and certain classical r-matrix, obtained form the $s\ell(2)$ invariant classical r-matrix and the corresponding K-matrix. The local realisation of the Lax matrix together with the linear bracket provide the necessary structure for the implementation of the algebraic Bethe ansatz. In this framework, the Bethe vectors, defined as the symmetric functions of its arguments, have a remarkable property that the off shell action of the generating function on them is strikingly simple. Actually, it is as simple as it can be since it practically coincide with the corresponding formula in the case when the boundary matrix is diagonal [20]. The off shell action of the generating function of the Gaudin Hamiltonians with the boundary terms yields the spectrum of the system and the corresponding Bethe equations. As usual, when the Bethe equations are imposed on the parameters of the Bethe vectors, the unwanted terms in the action of the generating function are annihilated.

However, more compact form of the Bethe vector $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, requires further studies. As it is evident form the formulas for the Bethe vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$ given in Appendix B, the problem lies in the definition the scalar coefficients

$c_M^{(m)}(\mu_1, \ldots \mu_m; \mu_{m+1}, \ldots, \mu_M)$, with $m = 1, 2, \ldots, M$. Some of them are straightforward to obtain but, in particular, the coefficient $c_M^{(M)}(\mu_1, \mu_2 \ldots, \mu_M)$ still represents a challenge, at least in the present form.

This paper is organised as follows. In Section 2 we review the $SL(2)$-invariant Yang R-matrix and provide fundamental tools for the study of the inhomogeneous XXX Heisenberg spin chain and the corresponding Gaudin model. Moreover, we outline Sklyanin's derivation of the rational $s\ell(2)$ Gaudin model. The general solutions of the reflection equation and the dual reflection equation are given in Section 3. As one of the main results of the paper, the generating function of the Gaudin Hamiltonians with boundary terms is derived in Section 4, using the quasi-classical expansion of the linear combination of the transfer matrix of the inhomogeneous XXX spin chain and the so-called Sklyanin determinant. The relevant algebraic structure, including the classical reflection equation, is given in Section 5. The implementation of the algebraic Bethe ansatz is presented in Section 6, including the definition of the Bethe vectors and the formulae of the off shell action of the generating function of the Gaudin Hamiltonians. Our conclusions are presented in Section 7. Finally, in Appendix A are given some basic definitions for the convenience of the reader.

2. $s\ell(2)$ Gaudin model

The XXX Heisenberg spin chain is related to the $SL(2)$-invariant Yang R-matrix [43]

$$R(\lambda) = \lambda \mathbb{1} + \eta \mathcal{P} = \begin{pmatrix} \lambda+\eta & 0 & 0 & 0 \\ 0 & \lambda & \eta & 0 \\ 0 & \eta & \lambda & 0 \\ 0 & 0 & 0 & \lambda+\eta \end{pmatrix}, \tag{2.1}$$

where λ is a spectral parameter, η is a quasi-classical parameter, $\mathbb{1}$ is the identity operator and we use $\mathcal{P}$ for the permutation in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The Yang R-matrix satisfies the Yang–Baxter equation [43–46] in the space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$R_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) R_{12}(\lambda - \mu), \tag{2.2}$$

we use the standard notation of the quantum inverse scattering method to denote spaces on which corresponding R-matrices R_{ij}, $ij = 12, 13, 23$ act nontrivially and suppress the dependence on the quasi-classical parameter η [45,46].

The Yang R-matrix also satisfies other relevant properties such as

unitarity	$R_{12}(\lambda) R_{21}(-\lambda) = (\eta^2 - \lambda^2)\mathbb{1}$;
parity invariance	$R_{21}(\lambda) = R_{12}(\lambda)$;
temporal invariance	$R_{12}^t(\lambda) = R_{12}(\lambda)$;
crossing symmetry	$R(\lambda) = \mathcal{J}_1 R^{t_2}(-\lambda - \eta) \mathcal{J}_1$,

where t_2 denotes the transpose in the second space and the entries of the two-by-two matrix $\mathcal{J}$ are $\mathcal{J}_{ab} = (-1)^{a-1} \delta_{a,3-b}$.

Here we study the inhomogeneous XXX spin chain with N sites, characterised by the local space $V_m = \mathbb{C}^{2s+1}$ and inhomogeneous parameter α_m. For simplicity, we start by considering the periodic boundary conditions. The Hilbert space of the system is

$$\mathcal{H} = \bigotimes_{m=1}^{N} V_m = (\mathbb{C}^{2s+1})^{\otimes N}. \tag{2.3}$$

Following [3] we introduce the Lax operator [41]

$$\mathbb{L}_{0m}(\lambda) = \mathbb{1} + \frac{\eta}{\lambda}\left(\vec{\sigma}_0 \cdot \vec{S}_m\right) = \frac{1}{\lambda}\begin{pmatrix} \lambda + \eta S_m^3 & \eta S_m^- \\ \eta S_m^+ & \lambda - \eta S_m^3 \end{pmatrix}. \tag{2.4}$$

Notice that $\mathbb{L}(\lambda)$ is a two-by-two matrix in the auxiliary space $V_0 = \mathbb{C}^2$. It obeys

$$\mathbb{L}_{0m}(\lambda)\mathbb{L}_{0m}(\eta - \lambda) = \left(1 + \frac{\eta^2 c_{2,m}}{\lambda(\eta - \lambda)}\right)\mathbb{1}_0, \tag{2.5}$$

where $c_{2,m}$ is the value of the Casimir operator on the space V_m [41].

When the quantum space is also a spin $\frac{1}{2}$ representation, the Lax operator becomes the R-matrix, $\mathbb{L}_{0m}(\lambda) = \frac{1}{\lambda} R_{0m}(\lambda - \eta/2)$.

Due to the commutation relations (A.1), it is straightforward to check that the Lax operator satisfies the RLL-relations

$$R_{00'}(\lambda - \mu)\mathbb{L}_{0m}(\lambda)\mathbb{L}_{0'm}(\mu) = \mathbb{L}_{0'm}(\mu)\mathbb{L}_{0m}(\lambda)R_{00'}(\lambda - \mu). \tag{2.6}$$

The so-called monodromy matrix

$$T(\lambda) = \mathbb{L}_{0N}(\lambda - \alpha_N)\cdots\mathbb{L}_{01}(\lambda - \alpha_1) \tag{2.7}$$

is used to describe the system. For simplicity we have omitted the dependence on the quasi-classical parameter η and the inhomogeneous parameters $\{\alpha_j, j = 1, \ldots, N\}$. Notice that $T(\lambda)$ is a two-by-two matrix acting in the auxiliary space $V_0 = \mathbb{C}^2$, whose entries are operators acting in $\mathcal{H}$. From RLL-relations (2.6) it follows that the monodromy matrix satisfies the RTT-relations

$$R_{00'}(\lambda - \mu)T_0(\lambda)T_{0'}(\mu) = T_{0'}(\mu)T_0(\lambda)R_{00'}(\lambda - \mu). \tag{2.8}$$

The periodic boundary conditions and the RTT-relations (2.8) imply that the transfer matrix

$$t(\lambda) = \mathrm{tr}_0 T(\lambda), \tag{2.9}$$

commute at different values of the spectral parameter,

$$[t(\mu), t(\nu)] = 0, \tag{2.10}$$

here we have omitted the nonessential arguments.

The RTT-relations admit a central element

$$\Delta\left[T(\lambda)\right] = \mathrm{tr}_{00'} P^-_{00'} T_0\left(\lambda - \eta/2\right) T_{0'}\left(\lambda + \eta/2\right), \tag{2.11}$$

where

$$P^-_{00'} = \frac{\mathbb{1} - \mathcal{P}_{00'}}{2} = -\frac{1}{2\eta} R_{00'}\left(-\eta\right). \tag{2.12}$$

A straightforward calculation shows that

$$\Big[\Delta\big[T(\mu)\big], T(\nu)\Big] = 0. \tag{2.13}$$

As the first step toward the study of the Gaudin model we consider the expansion of the monodromy matrix (2.7) with respect to the quasi-classical parameter η

$$T(\lambda) = \mathbb{1} + \eta \sum_{m=1}^{N} \frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} + \frac{\eta^2}{2} \sum_{\substack{n,m=1 \\ n \neq m}}^{N} \frac{\mathbb{1}_0 \left(\vec{S}_m \cdot \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)}$$
$$+ \frac{\eta^2}{2} \sum_{m=1}^{N} \left(\sum_{n>m}^{N} \frac{\imath \vec{\sigma}_0 \cdot \left(\vec{S}_n \times \vec{S}_m\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \sum_{n<m}^{N} \frac{\imath \vec{\sigma}_0 \cdot \left(\vec{S}_m \times \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} \right) + \mathcal{O}(\eta^3). \quad (2.14)$$

If the Gaudin Lax matrix is defined by [3]

$$L_0(\lambda) = \sum_{m=1}^{N} \frac{\vec{\sigma}_0 \cdot \vec{S}_m}{\lambda - \alpha_m} \quad (2.15)$$

and the quasi-classical property of the Yang R-matrix [3]

$$\frac{1}{\lambda} R(\lambda) = \mathbb{1} - \eta r(\lambda), \qquad \text{where} \qquad r(\lambda) = -\frac{\mathcal{P}}{\lambda} \quad (2.16)$$

is taken into account, then substitution of the expansion (2.14) into the RTT-relations (2.8) yields the so-called Sklyanin linear bracket [3]

$$[L_1(\lambda), L_2(\mu)] = [r_{12}(\lambda - \mu), L_1(\lambda) + L_2(\mu)]\,. \quad (2.17)$$

Using the expansion (2.14) it is evident that

$$t(\lambda) = 2 + \eta^2 \sum_{m=1}^{N} \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \mathcal{O}(\eta^3). \quad (2.18)$$

The same expansion (2.14) leads to

$$\Delta\,[T(\lambda)] = \mathbb{1} + \eta\,\mathrm{tr} L(\lambda) + \frac{\eta^2}{2} \mathrm{tr}_{00'} P^{-}_{00'} \sum_{m=1}^{N} \left(\frac{\vec{\sigma}_0 \cdot \vec{S}_m}{(\lambda - \alpha_m)^2} - \frac{\vec{\sigma}_{0'} \cdot \vec{S}_m}{(\lambda - \alpha_m)^2} \right)$$
$$+ \frac{\eta^2}{2} \mathrm{tr}_{00'} P^{-}_{00'} \sum_{m=1}^{N} \sum_{n \neq m}^{N} \left(\frac{\mathbb{1}_0 \left(\vec{S}_m \cdot \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \frac{\mathbb{1}_{0'} \left(\vec{S}_m \cdot \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} \right)$$
$$+ \frac{\eta^2}{2} \mathrm{tr}_{00'} P^{-}_{00'} \sum_{m=1}^{N} \left(\sum_{n>m}^{N} \frac{\imath \vec{\sigma}_0 \cdot \left(\vec{S}_n \times \vec{S}_m\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \sum_{n<m}^{N} \frac{\imath \vec{\sigma}_0 \cdot \left(\vec{S}_m \times \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} \right)$$
$$+ \frac{\eta^2}{2} \mathrm{tr}_{00'} P^{-}_{00'} \sum_{m=1}^{N} \left(\sum_{n>m}^{N} \frac{\imath \vec{\sigma}_{0'} \cdot \left(\vec{S}_n \times \vec{S}_m\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} + \sum_{n<m}^{N} \frac{\imath \vec{\sigma}_{0'} \cdot \left(\vec{S}_m \times \vec{S}_n\right)}{(\lambda - \alpha_m)(\lambda - \alpha_n)} \right)$$
$$+ \eta^2 \mathrm{tr}_{00'} P^{-}_{00'} L_0(\lambda) L_{0'}(\lambda) + \mathcal{O}(\eta^3), \quad (2.19)$$

where $L(\lambda)$ is given in (2.15). The final expression for the expansion of $\Delta\,[T(\lambda)]$ is obtained after taking all the traces

$$\Delta\,[T(\lambda)] = \mathbb{1} + \eta^2 \left(\sum_{m=1}^{N} \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{(\lambda - \alpha_m)(\lambda - \alpha_n)} - \frac{1}{2} \mathrm{tr}\, L^2(\lambda) \right) + \mathcal{O}(\eta^3). \quad (2.20)$$

To obtain the generation function of the Gaudin Hamiltonians notice that (2.18) and (2.20) yield

$$t(\lambda) - \Delta[T(\lambda)] = \mathbb{1} + \frac{\eta^2}{2} \mathrm{tr}\, L^2(\lambda) + \mathcal{O}(\eta^3). \tag{2.21}$$

Therefore

$$\tau(\lambda) = \frac{1}{2} \mathrm{tr}\, L^2(\lambda) \tag{2.22}$$

commute for different values of the spectral parameter,

$$[\tau(\lambda), \tau(\mu)] = 0. \tag{2.23}$$

Moreover, from (2.15) it is straightforward to obtain the expansion

$$\tau(\lambda) = \sum_{m=1}^{N} \frac{2H_m}{\lambda - \alpha_m} + \sum_{m=1}^{N} \frac{\vec{S}_m \cdot \vec{S}_m}{(\lambda - \alpha_m)^2} = \sum_{m=1}^{N} \frac{2H_m}{\lambda - \alpha_m} + \sum_{m=1}^{N} \frac{s_m(s_m+1)}{(\lambda - \alpha_m)^2}, \tag{2.24}$$

and the Gaudin Hamiltonians, in the periodic case, are

$$H_m = \sum_{n \neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n}. \tag{2.25}$$

This shows that $\tau(\lambda)$ is the generating function of Gaudin Hamiltonians when the periodic boundary conditions are imposed [3].

3. Reflection equation

A way to introduce non-periodic boundary conditions which are compatible with the integrability of the bulk model, was developed in [27]. Boundary conditions on the left and right sites of the system are encoded in the left and right reflection matrices K^- and K^+. The compatibility condition between the bulk and the boundary of the system takes the form of the so-called reflection equation. It is written in the following form for the left reflection matrix acting on the space $\mathbb{C}^2$ at the first site $K^-(\lambda) \in \mathrm{End}(\mathbb{C}^2)$

$$R_{12}(\lambda - \mu) K_1^-(\lambda) R_{21}(\lambda + \mu) K_2^-(\mu) = K_2^-(\mu) R_{12}(\lambda + \mu) K_1^-(\lambda) R_{21}(\lambda - \mu). \tag{3.1}$$

Due to the properties of the Yang R-matrix the dual reflection equation can be presented in the following form

$$\begin{aligned} &R_{12}(\mu - \lambda) K_1^+(\lambda) R_{21}(-\lambda - \mu - 2\eta) K_2^+(\mu) \\ &\quad = K_2^+(\mu) R_{12}(-\lambda - \mu - 2\eta) K_1^+(\lambda) R_{21}(\mu - \lambda). \end{aligned} \tag{3.2}$$

One can then verify that the mapping

$$K^+(\lambda) = K^-(-\lambda - \eta) \tag{3.3}$$

is a bijection between solutions of the reflection equation and the dual reflection equation. After substitution of (3.3) into the dual reflection equation (3.2) one gets the reflection equation (3.1) with shifted arguments.

The general, spectral parameter dependent solutions of the reflection equation (3.1) can be written as follows [47]

$$K^{-}(\lambda)=\begin{pmatrix}\xi-\lambda & \psi\lambda\\ \phi\lambda & \xi+\lambda\end{pmatrix}. \tag{3.4}$$

It is straightforward to check the following useful identities

$$K^{-}(-\lambda)K^{-}(\lambda)=\Big(\xi^2-\lambda^2\,(1+\phi\psi)\Big)\,\mathbb{1}=\det\big(K^{-}(\lambda)\big)\,\mathbb{1}, \tag{3.5}$$

$$K^{-}(-\lambda)=\operatorname{tr}K^{-}(\lambda)-K^{-}(\lambda). \tag{3.6}$$

4. $s\ell(2)$ Gaudin model with boundary terms

With the aim of describing the inhomogeneous XXX spin chain with non-periodic boundary condition it is instructive to recall some properties of the Lax operator (2.4). The identity (2.5) can be rewritten in the form [41]

$$\mathbb{L}_{0m}(\lambda-\alpha_m)\mathbb{L}_{0m}(-\lambda+\alpha_m+\eta)=\left(1+\frac{\eta^2\,s_m(s_m+1)}{(\lambda-\alpha_m)(-\lambda+\alpha_m+\eta)}\right)\mathbb{1}_0. \tag{4.1}$$

It follows from the equation above and the RLL-relations (2.6) that the RTT-relations (2.8) can be recast as follows

$$\widetilde{T}_{0'}(\mu)R_{00'}(\lambda+\mu)T_0(\lambda)=T_0(\lambda)R_{00'}(\lambda+\mu)\widetilde{T}_{0'}(\mu), \tag{4.2}$$

$$\widetilde{T}_0(\lambda)\widetilde{T}_{0'}(\mu)R_{00'}(\mu-\lambda)=R_{00'}(\mu-\lambda)\widetilde{T}_{0'}(\mu)\widetilde{T}_0(\lambda), \tag{4.3}$$

where

$$\widetilde{T}(\lambda)=\mathbb{L}_{01}(\lambda+\alpha_1+\eta)\cdots\mathbb{L}_{0N}(\lambda+\alpha_N+\eta). \tag{4.4}$$

The Sklyanin monodromy matrix $\mathcal{T}(\lambda)$ of the inhomogeneous XXX spin chain with non-periodic boundary consists of the two matrices $T(\lambda)$ (2.7) and $\widetilde{T}_0(\lambda)$ (4.4) and a reflection matrix $K^{-}(\lambda)$ (3.4),

$$\mathcal{T}_0(\lambda)=T_0(\lambda)K_0^{-}(\lambda)\widetilde{T}_0(\lambda). \tag{4.5}$$

Using the RTT-relations (2.8), (4.2), (4.3) and the reflection equation (3.1) it is straightforward to show that the exchange relations of the monodromy matrix $\mathcal{T}(\lambda)$ in $V_0\otimes V_{0'}$ are [41]

$$R_{00'}(\lambda-\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda+\mu)\mathcal{T}_{0'}(\mu)=\mathcal{T}_{0'}(\mu)R_{00'}(\lambda+\mu)\mathcal{T}_0(\lambda)R_{0'0}(\lambda-\mu). \tag{4.6}$$

The open chain transfer matrix is given by the trace of $\mathcal{T}(\lambda)$ over the auxiliary space V_0 with an extra reflection matrix $K^{+}(\lambda)$ [27],

$$t(\lambda)=\operatorname{tr}_0\big(K^{+}(\lambda)\mathcal{T}(\lambda)\big). \tag{4.7}$$

The reflection matrix $K^{+}(\lambda)$ (3.3) is the corresponding solution of the dual reflection equation (3.2). The commutativity of the transfer matrix for different values of the spectral parameter

$$[t(\lambda),t(\mu)]=0, \tag{4.8}$$

is guaranteed by the dual reflection equation (3.2) and the exchange relations (4.6) of the monodromy matrix $\mathcal{T}(\lambda)$.

The exchange relations (4.6) admit a central element

$$\Delta\,[\mathcal{T}(\lambda)] = \mathrm{tr}_{00'}\, P^{-}_{00'}\, \mathcal{T}_0(\lambda-\eta/2) R_{00'}(2\lambda)\mathcal{T}_{0'}(\lambda+\eta/2). \tag{4.9}$$

For the study of the open Gaudin model we impose

$$\lim_{\eta\to 0}\Big(K^{+}(\lambda)K^{-}(\lambda)\Big) = \Big(\xi^2 - \lambda^2\,(1+\phi\psi)\Big)\,\mathbb{1}. \tag{4.10}$$

In particular, this implies that the parameters of the reflection matrices on the left and on the right end of the chain are the same. In general this not the case in the study of the open spin chain. However, this condition is essential for the Gaudin model. Then we will write

$$K^{-}(\lambda) \equiv K(\lambda), \tag{4.11}$$

so that

$$K^{+}(\lambda) = K(-\lambda-\eta) = K(-\lambda) - \eta\, M \quad \text{with} \quad M = \begin{pmatrix} -1 & \psi \\ \phi & 1 \end{pmatrix}. \tag{4.12}$$

Remark that the matrix M obeys $M^2 = (1+\psi\phi)\mathbb{1}$.

The expansion of $T(\lambda)$ is given in (2.14). It is easy to get the expansion for $\widetilde{T}(\lambda)$ as introduced in (4.4) and then, the one for $\mathcal{T}(\lambda)$. Using the relation (4.12), we deduce the expansion of $t(\lambda)$ (6.24) in powers of η:

$$\begin{aligned} t(\lambda) &= 2\Big(\xi^2-\lambda^2\,(1+\phi\psi)\Big) - 2\eta\lambda\,(1+\phi\psi) \\ &\quad - \eta^2\,\mathrm{tr}_0\,(M_0\,(L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda))) \\ &\quad + \eta^2\Big(\xi^2-\lambda^2\,(1+\phi\psi)\Big) \sum_{\substack{m,n=1\\ n\neq m}}^{N} \left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right) \\ &\quad - \eta^2\mathrm{tr}_0\, L_0(\lambda)K_0(\lambda)L_0(-\lambda)K_0(-\lambda) + \mathcal{O}(\eta^3). \end{aligned} \tag{4.13}$$

Our next step is to obtain the expansion of $\Delta\,[\mathcal{T}(\lambda)]$ (4.9) in powers of η. We follow the analogous steps as for the periodic case, and after some tedious but straightforward calculations we get

$$\begin{aligned} \Delta\,[\mathcal{T}(\lambda)] &= \lambda\left(\mathrm{tr}_0^2 K_0(\lambda) - \mathrm{tr}_0 K_0^2(\lambda)\right) + 2\eta\lambda\mathrm{tr}_0 K_0(\lambda)\ \mathrm{tr}_0\,(L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)) \\ &\quad - 2\eta\lambda\Big(\mathrm{tr}_0\left\{L_0(\lambda)K_0^2(\lambda)\right\} - \mathrm{tr}_0\left\{L_0(-\lambda)K_0^2(\lambda)\right\}\Big) - \frac{\eta}{2}\mathrm{tr}_0\,K_0^2(\lambda) \\ &\quad + \eta^2\lambda \sum_{\substack{m,n=1\\ n\neq m}}^{N} \left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right) \mathrm{tr}_0\,K_0(-\lambda)K_0(\lambda) \\ &\quad - 2\eta^2\lambda\,\mathrm{tr}_0\,L_0(\lambda)K_0(\lambda)L_0(-\lambda)K_0(-\lambda) \\ &\quad - \eta^2\,\mathrm{tr}_0\left\{\Big(L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)\Big)K_0(\lambda)\right\} \\ &\quad + \eta^2\lambda\,\Big(\mathrm{tr}_0\,\{L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)\}\Big)^2 \\ &\quad - \eta^2\lambda\,\mathrm{tr}_0\left\{\Big(L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)\Big)\Big(L_0(\lambda)K_0(\lambda) - K_0(\lambda)L_0(-\lambda)\Big)\right\} \\ &\quad + \frac{\eta^2\lambda}{4}\mathrm{tr}_0\,M_0^2 + \mathcal{O}(\eta^3). \end{aligned} \tag{4.14}$$

Using the relations (3.5) and (3.6) the first term of the expansion above simplifies and the second and third term together turn out to be propositional to the trace of $L(\lambda)$ (2.15) and therefore vanish,

$$\begin{aligned}\Delta\left[\mathcal{T}(\lambda)\right] = {} & 2\lambda\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)-\eta\left(\xi^2+\lambda^2\left(1+\phi\psi\right)\right)+2\eta^2\lambda\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)\\ & \times\sum_{\substack{m,n=1\\ n\neq m}}^{N}\left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)}+\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right)\\ & -2\eta^2\lambda\,\mathrm{tr}_0\,L_0(\lambda)K_0(\lambda)L_0(-\lambda)K_0(-\lambda)\\ & -\eta^2\,\mathrm{tr}_0((L_0(\lambda)K_0(\lambda)-K_0(\lambda)L_0(-\lambda))K_0(\lambda))\\ & +\eta^2\lambda\,\mathrm{tr}_0\Big\{\Big(\mathrm{tr}_{0'}\{L_{0'}(\lambda)K_{0'}(\lambda)-K_{0'}(\lambda)L_{0'}(-\lambda)\}-L_0(\lambda)K_0(\lambda)\\ & +K_0(\lambda)L_0(-\lambda)\Big)\Big(L_0(\lambda)K_0(\lambda)-K_0(\lambda)L_0(-\lambda)\Big)\Big\}+\frac{\eta^2\lambda}{2}(1+\phi\psi)\\ & +\mathcal{O}\big(\eta^3\big).\end{aligned} \tag{4.15}$$

In order to simplify some formulae we introduce the following notation

$$\mathcal{L}_0(\lambda)=L_0(\lambda)-K_0(\lambda)L_0(-\lambda)K_0^{-1}(\lambda). \tag{4.16}$$

Using the formulas (4.13) and (4.15) we calculate the expansion in powers of η of the difference

$$\begin{aligned}2\lambda t(\lambda)-\Delta\left[\mathcal{T}(\lambda)\right] = {} & 2\lambda\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)+\eta\left(\xi^2-3\lambda^2\left(1+\phi\psi\right)\right)\\ & -2\eta^2\lambda\,\mathrm{tr}_0\left(M_0\mathcal{L}_0(\lambda)K_0(\lambda)\right)+\eta^2\mathrm{tr}_0\left(\mathcal{L}_0(\lambda)K_0^2(\lambda)\right)\\ & -\eta^2\lambda\,\mathrm{tr}_0\left(\left(\mathrm{tr}_{0'}\left(\mathcal{L}_{0'}(\lambda)K_{0'}(\lambda)\right)\mathbb{1}_0-\mathcal{L}_0(\lambda)K_0(\lambda)\right)\mathcal{L}_0(\lambda)K_0(\lambda)\right)\\ & -\frac{\eta^2\lambda}{2}\left(1+\phi\psi\right)+\mathcal{O}(\eta^3).\end{aligned} \tag{4.17}$$

Actually the third and the fourth term in the expression above vanish

$$\begin{aligned}&\mathrm{tr}_0\left(\mathcal{L}_0(\lambda)K_0^2(\lambda)\right)-2\lambda\,\mathrm{tr}_0\left(M_0\mathcal{L}_0(\lambda)K_0(\lambda)\right)=\mathrm{tr}_0\left(\left(\mathcal{L}_0(\lambda)K_0(\lambda)\right)\left(K_0(\lambda)-2\lambda M_0\right)\right)\\ &\quad=\mathrm{tr}_0\left(\mathcal{L}_0(\lambda)K_0(\lambda)K_0(-\lambda)\right)=\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)\mathrm{tr}_0\mathcal{L}_0=0,\end{aligned} \tag{4.18}$$

due to the fact that the $\mathrm{tr}_0\mathcal{L}_0$ is equal to zero. Therefore the expansion (4.17) reads

$$\begin{aligned}2\lambda t(\lambda)-\Delta\left[\mathcal{T}(\lambda)\right] = {} & 2\lambda\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)+\eta\left(\xi^2-3\lambda^2\left(1+\phi\psi\right)\right)\\ & -\eta^2\lambda\,\mathrm{tr}_0\left(\left(\mathrm{tr}_{0'}\left(\mathcal{L}_{0'}(\lambda)K_{0'}(\lambda)\right)\mathbb{1}_0-\mathcal{L}_0(\lambda)K_0(\lambda)\right)\mathcal{L}_0(\lambda)K_0(\lambda)\right)\\ & -\frac{\eta^2\lambda}{2}\left(1+\phi\psi\right)+\mathcal{O}(\eta^3).\end{aligned} \tag{4.19}$$

It is important to notice that using the following identity

$$\mathrm{tr}_{0'}\left(\mathcal{L}_{0'}(\lambda)K_{0'}(\lambda)\right)\mathbb{1}_0-\mathcal{L}_0(\lambda)K_0(\lambda)=-K_0(-\lambda)\mathcal{L}_0(\lambda), \tag{4.20}$$

the third term in (4.19) can be simplified

$$\mathrm{tr}_0\,K_0(-\lambda)\mathcal{L}_0(\lambda)\mathcal{L}_0(\lambda)K_0(\lambda)=\left(\xi^2-\lambda^2\left(1+\phi\psi\right)\right)\mathrm{tr}_0\,\mathcal{L}_0^2(\lambda). \tag{4.21}$$

Finally, the expansion (4.19) reads

$$2\lambda t(\lambda) - \Delta[\mathcal{T}(\lambda)] = 2\lambda\left(\xi^2 - \lambda^2(1+\phi\psi)\right) + \eta\left(\xi^2 - 3\lambda^2(1+\phi\psi)\right) + \eta^2\lambda\left(\xi^2 - \lambda^2(1+\phi\psi)\right)\operatorname{tr}_0 \mathcal{L}_0^2(\lambda) - \frac{\eta^2\lambda}{2}(1+\phi\psi) + \mathcal{O}(\eta^3). \tag{4.22}$$

This shows that

$$\tau(\lambda) = \operatorname{tr}_0 \mathcal{L}_0^2(\lambda) \tag{4.23}$$

commute for different values of the spectral parameter,

$$[\tau(\lambda), \tau(\mu)] = 0. \tag{4.24}$$

and therefore can be considered to be the generating function of Gaudin Hamiltonians with boundary terms. The multiplicative factor in (6.31), which is equal to the determinant of $K(\lambda)$, will be useful in the partial fraction decomposition of the generating function.

With the aim of obtaining the Gaudin Hamiltonians with the boundary terms from the generating function (6.31), it is instructive to study the representation of $\mathcal{L}_0(\lambda)$ (4.16) in terms of the local spin operators

$$\mathcal{L}_0(\lambda) = \sum_{m=1}^{N}\left(\frac{\vec{\sigma}_0\cdot\vec{S}_m}{\lambda-\alpha_m} + \frac{\left(K_0(\lambda)\vec{\sigma}_0 K_0^{-1}(\lambda)\right)\cdot\vec{S}_m}{\lambda+\alpha_m}\right), \tag{4.25}$$

noticing that

$$\mathcal{L}_0(\lambda) = \sum_{m=1}^{N}\left(\frac{\vec{\sigma}_0\cdot\vec{S}_m}{\lambda-\alpha_m} + \frac{\vec{\sigma}_0\cdot\left(K_m^{-1}(\lambda)\vec{S}_m K_m(\lambda)\right)}{\lambda+\alpha_m}\right). \tag{4.26}$$

Now it is straightforward to obtain the expression for the generating function (6.31) in terms of the local operators

$$\tau(\lambda) = 2\sum_{m,n=1}^{N}\left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{\vec{S}_m\cdot\left(K_n^{-1}(\lambda)\vec{S}_n K_n(\lambda)\right) + \left(K_n^{-1}(\lambda)\vec{S}_n K_n(\lambda)\right)\cdot\vec{S}_m}{(\lambda-\alpha_m)(\lambda+\alpha_n)} + \frac{\left(K_m^{-1}(\lambda)\vec{S}_m K_m(\lambda)\right)\cdot\left(K_n^{-1}(\lambda)\vec{S}_n K_n(\lambda)\right)}{(\lambda+\alpha_m)(\lambda+\alpha_n)}\right). \tag{4.27}$$

It is important to notice that (4.27) simplifies further

$$\tau(\lambda) = 2\sum_{m,n=1}^{N}\left(\frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda-\alpha_m)(\lambda-\alpha_n)} + \frac{\vec{S}_m\cdot\vec{S}_n}{(\lambda+\alpha_m)(\lambda+\alpha_n)} + \frac{\vec{S}_m\cdot\left(K_n^{-1}(\lambda)\vec{S}_n K_n(\lambda)\right) + \left(K_n^{-1}(\lambda)\vec{S}_n K_n(\lambda)\right)\cdot\vec{S}_m}{(\lambda-\alpha_m)(\lambda+\alpha_n)}\right). \tag{4.28}$$

The Gaudin Hamiltonians with the boundary terms are obtained from the residues of the generating function (4.28) at poles $\lambda = \pm\alpha_m$:

$$\mathrm{Res}_{\lambda=\alpha_m}\tau(\lambda) = 4\, H_m \quad \text{and} \quad \mathrm{Res}_{\lambda=-\alpha_m}\tau(\lambda) = (-4)\, \widetilde{H}_m \tag{4.29}$$

where

$$H_m = \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \sum_{n=1}^{N} \frac{\vec{S}_m \cdot \Big(K_n^{-1}(\alpha_m)\vec{S}_n K_n(\alpha_m)\Big) + \Big(K_n^{-1}(\alpha_m)\vec{S}_n K_n(\alpha_m)\Big)\cdot \vec{S}_m}{2(\alpha_m + \alpha_n)}, \tag{4.30}$$

and

$$\widetilde{H}_m = \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \sum_{n=1}^{N} \frac{\vec{S}_m \cdot \Big(K_n^{-1}(-\alpha_m)\vec{S}_n K_n(-\alpha_m)\Big) + \Big(K_n^{-1}(-\alpha_m)\vec{S}_n K_n(-\alpha_m)\Big)\cdot \vec{S}_m}{2(\alpha_m + \alpha_n)}. \tag{4.31}$$

The above Hamiltonians can be expressed in somewhat a more symmetric form

$$H_m = \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \sum_{n=1}^{N} \frac{\Big(K_m(\alpha_m)\vec{S}_m K_m^{-1}(\alpha_m)\Big)\cdot \vec{S}_n + \vec{S}_n \cdot \Big(K_m(\alpha_m)\vec{S}_m K_m^{-1}(\alpha_m)\Big)}{2(\alpha_m + \alpha_n)}, \tag{4.32}$$

and

$$\widetilde{H}_m = \sum_{n\neq m}^{N} \frac{\vec{S}_m \cdot \vec{S}_n}{\alpha_m - \alpha_n} + \sum_{n=1}^{N} \frac{\Big(K_m(-\alpha_m)\vec{S}_m K_m^{-1}(-\alpha_m)\Big)\cdot \vec{S}_n + \vec{S}_n \cdot \Big(K_m(-\alpha_m)\vec{S}_m K_m^{-1}(-\alpha_m)\Big)}{2(\alpha_m + \alpha_n)}. \tag{4.33}$$

The next step is to study the quasi-classical limit of the exchange relations (4.6) with the aim of deriving relevant algebraic structure for the Lax operator (4.16).

5. Linear bracket relations

The implementation of the algebraic Bethe ansatz requires the commutation relations between the entries of the Lax operator (4.16). Our aim is to derive these relations as the quasi-classical limit of (4.6). As the first step in this direction we observe that using (2.16) the reflection equation (3.1) can be expressed as

$$\begin{aligned}&\big(\mathbb{1} - \eta r_{12}(\lambda-\mu)\big)K_1(\lambda)\big(\mathbb{1} - \eta r_{21}(\lambda+\mu)\big)K_2(\mu)\\ &\quad = K_2(\mu)\big(\mathbb{1} - \eta r_{12}(\lambda+\mu)\big)K_1(\lambda)\big(\mathbb{1} - \eta r_{21}(\lambda-\mu)\big).\end{aligned} \tag{5.1}$$

The conditions obtained from the zero and first order in η are identically satisfied for the matrix $K(\lambda)$. In particular, it obeys the classical reflection equation [25,26]:

$$r_{12}(\lambda-\mu)K_1(\lambda)K_2(\mu)+K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu)$$
$$=K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda)+K_2(\mu)K_1(\lambda)r_{21}(\lambda-\mu). \tag{5.2}$$

The terms of the second order in η in (5.1) are

$$r_{12}(\lambda-\mu)K_1(\lambda)r_{21}(\lambda+\mu)K_2(\mu)=K_2(\mu)r_{12}(\lambda+\mu)K_1(\lambda)r_{21}(\lambda-\mu). \tag{5.3}$$

This equation is also satisfied by the K-matrix (3.4) and the classical r-matrix (2.16). In addition, the classical r-matrix (2.16) has the unitarity property

$$r_{21}(-\lambda)=-r_{12}(\lambda), \tag{5.4}$$

and satisfies the classical Yang–Baxter equation

$$[r_{13}(\lambda),r_{23}(\mu)]+[r_{12}(\lambda-\mu),r_{13}(\lambda)+r_{23}(\mu)]=0. \tag{5.5}$$

Now we can proceed to the derivation of the relevant linear bracket relations of the Lax operator (4.16). The desired relations can be obtained by writing Eq. (4.6) in the following form, using (2.16),

$$(\mathbb{1}-\eta r_{00'}(\lambda-\mu))\,\mathcal{T}_0(\lambda)\,(\mathbb{1}-\eta r_{0'0}(\lambda+\mu))\,\mathcal{T}_{0'}(\mu)$$
$$=\mathcal{T}_{0'}(\mu)\,(\mathbb{1}-\eta r_{00'}(\lambda+\mu))\,\mathcal{T}_0(\lambda)\,(\mathbb{1}-\eta r_{0'0}(\lambda-\mu)) \tag{5.6}$$

and substituting the expansion of $\mathcal{T}(\lambda)$ (4.5) in powers of η

$$\mathcal{T}(\lambda)=K(\lambda)+\eta\,\mathcal{L}(\lambda)K(\lambda)+\frac{\eta^2}{2}\frac{d^2\mathcal{T}(\lambda)}{d\eta^2}|_{\eta=0}+\mathcal{O}(\eta^3). \tag{5.7}$$

The zero and first orders in η are identically satisfied for the matrix $K(\lambda)$ defined in (3.4). The relations we seek follow from the terms of the second order in η in (5.6). When the terms containing the second order derivatives of $\mathcal{T}$ are eliminated and Eq. (5.3) is used to eliminate the other two terms, there are ten terms remaining. Using twice the classical reflection equation (5.2) and the unitarity property (5.4) one obtains

$$\begin{aligned}
&(\mathcal{L}_0(\lambda)\mathcal{L}_{0'}(\mu)-\mathcal{L}_{0'}(\mu)\mathcal{L}_0(\lambda))\,K_0(\lambda)K_{0'}(\mu)\\
&\quad=(r_{00'}(\lambda-\mu)\mathcal{L}_0(\lambda)-\mathcal{L}_0(\lambda)r_{00'}(\lambda-\mu))\,K_0(\lambda)K_{0'}(\mu)+(\mathcal{L}_0(\lambda)K_{0'}(\mu)r_{00'}(\lambda+\mu)\\
&\qquad-K_{0'}(\mu)r_{00'}(\lambda+\mu)\mathcal{L}_0(\lambda))\,K_0(\lambda)-(r_{0'0}(\mu-\lambda)\mathcal{L}_{0'}(\mu)\\
&\qquad-\mathcal{L}_{0'}(\mu)r_{0'0}(\mu-\lambda))\,K_0(\lambda)K_{0'}(\mu)+(K_0(\lambda)r_{0'0}(\mu+\lambda)\mathcal{L}_{0'}(\mu)\\
&\qquad-\mathcal{L}_{0'}(\mu)K_0(\lambda)r_{0'0}(\mu+\lambda))\,K_{0'}(\mu).
\end{aligned} \tag{5.8}$$

Multiplying both sides of Eq. (5.8) from the right by $K_0^{-1}(\lambda)K_{0'}^{-1}(\mu)$, (5.8) can be express as

$$\begin{aligned}
[\mathcal{L}_0(\lambda),\mathcal{L}_{0'}(\mu)]=&\left[r_{00'}(\lambda-\mu)-K_{0'}(\mu)r_{00'}(\lambda+\mu)K_{0'}^{-1}(\mu),\mathcal{L}_0(\lambda)\right]\\
&-\left[r_{0'0}(\mu-\lambda)-K_0(\lambda)r_{0'0}(\mu+\lambda)K_0^{-1}(\lambda),\mathcal{L}_{0'}(\mu)\right].
\end{aligned} \tag{5.9}$$

Defining

$$r^K_{00'}(\lambda,\mu)=r_{00'}(\lambda-\mu)-K_{0'}(\mu)r_{00'}(\lambda+\mu)K_{0'}^{-1}(\mu), \tag{5.10}$$

(5.9) can be written as

$$[\mathcal{L}_0(\lambda), \mathcal{L}_{0'}(\mu)] = \left[r^K_{00'}(\lambda,\mu), \mathcal{L}_0(\lambda)\right] - \left[r^K_{0'0}(\mu,\lambda), \mathcal{L}_{0'}(\mu)\right]. \tag{5.11}$$

The commutator (5.11) is obviously anti-symmetric. It obeys the Jacobi identity because the r-matrix (5.10) satisfies the classical YB equation

$$[r^K_{32}(\lambda_3,\lambda_2), r^K_{13}(\lambda_1,\lambda_3)] + [r^K_{12}(\lambda_1,\lambda_2), r^K_{13}(\lambda_1,\lambda_3) + r^K_{23}(\lambda_2,\lambda_3)] = 0. \tag{5.12}$$

The commutator (5.11) can also be recasted as an $(\mathfrak{r},\mathfrak{s})$ Maillet algebra [48]. In the following we study the algebraic Bethe ansatz based on the linear bracket (5.11).

6. Algebraic Bethe ansatz

Our preliminary step in the implementation of the algebraic Bethe ansatz for the open Gaudin model is to bring the boundary K-matrix to the upper, or lower, triangular form. As it was pointed out in (3.4), the general form of the K-matrix (4.11) is

$$\widetilde{K}(\lambda) = \begin{pmatrix} \xi - \lambda & \widetilde{\psi}\lambda \\ \widetilde{\phi}\lambda & \xi + \lambda \end{pmatrix}. \tag{6.1}$$

It is straightforward to check that the matrix

$$U = \begin{pmatrix} -1-\nu & \widetilde{\phi} \\ \widetilde{\phi} & -1-\nu \end{pmatrix}, \tag{6.2}$$

with $\nu = \sqrt{1+\widetilde{\phi}\widetilde{\psi}}$, which does not depend on the spectral parameter λ, brings the K-matrix to the upper triangular form by the similarity transformation

$$K(\lambda) = U^{-1}\widetilde{K}(\lambda)U = \begin{pmatrix} \xi - \lambda\nu & \lambda\psi \\ 0 & \xi + \lambda\nu \end{pmatrix}, \tag{6.3}$$

where $\psi = \widetilde{\phi} + \widetilde{\psi}$. Evidently, the inverse matrix is

$$K^{-1}(\lambda) = \frac{1}{\xi^2 - \lambda^2\nu^2}\begin{pmatrix} \xi + \lambda\nu & -\lambda\psi \\ 0 & \xi - \lambda\nu \end{pmatrix}. \tag{6.4}$$

Direct substitution of the formulas above into (4.25),

$$\mathcal{L}_0(\lambda) = \begin{pmatrix} H(\lambda) & F(\lambda) \\ E(\lambda) & -H(\lambda) \end{pmatrix} = \sum_{m=1}^{N}\left(\frac{\vec{\sigma}_0\cdot\vec{S}_m}{\lambda-\alpha_m} + \frac{K_0(\lambda)\vec{\sigma}_0K_0^{-1}(\lambda)\cdot\vec{S}_m}{\lambda+\alpha_m}\right), \tag{6.5}$$

yields the following local realisation for the entries of the Lax matrix

$$E(\lambda) = \sum_{m=1}^{N}\left(\frac{S^+_m}{\lambda-\alpha_m} + \frac{(\xi+\lambda\nu)S^+_m}{(\xi-\lambda\nu)(\lambda+\alpha_m)}\right), \tag{6.6}$$

$$F(\lambda) = \sum_{m=1}^{N}\left(\frac{S^-_m}{\lambda-\alpha_m} + \frac{(\xi-\lambda\nu)^2S^-_m - \lambda^2\psi^2S^+_m - 2\lambda\psi(\xi-\lambda\nu)S^3_m}{(\xi+\lambda\nu)(\xi-\lambda\nu)(\lambda+\alpha_m)}\right), \tag{6.7}$$

$$H(\lambda) = \sum_{m=1}^{N}\left(\frac{S^3_m}{\lambda-\alpha_m} + \frac{\lambda\psi\, S^+_m + (\xi-\lambda\nu)S^3_m}{(\xi-\lambda\nu)(\lambda+\alpha_m)}\right). \tag{6.8}$$

The linear bracket (5.11) based on the r-matrix $r^K_{00'}(\lambda,\mu)$ (5.10), corresponding to (6.3), (6.4) and the classical r-matrix (2.16), defines the Lie algebra relevant for the open Gaudin model

$$[E(\lambda), E(\mu)] = 0, \tag{6.9}$$

$$[H(\lambda), E(\mu)] = \frac{2}{\lambda^2 - \mu^2}\left(\lambda\, E(\mu) - \frac{\xi - \lambda\nu}{\xi - \mu\nu}\mu\, E(\lambda)\right), \tag{6.10}$$

$$\begin{aligned}[E(\lambda), F(\mu)] = &\frac{2\psi\mu}{(\lambda+\mu)(\xi+\mu\nu)}\, E(\lambda) \\ &+ \frac{4}{\lambda^2-\mu^2}\left(\frac{\xi-\mu\nu}{\xi-\lambda\nu}\lambda\, H(\mu) - \frac{\xi+\lambda\nu}{\xi+\mu\nu}\mu\, H(\lambda)\right),\end{aligned} \tag{6.11}$$

$$[H(\lambda), H(\mu)] = \frac{-\psi}{\lambda+\mu}\left(\frac{\lambda}{\xi-\lambda\nu}\, E(\mu) - \frac{\mu}{\xi-\mu\nu}\, E(\lambda)\right), \tag{6.12}$$

$$\begin{aligned}[H(\lambda), F(\mu)] = &\frac{\psi}{\lambda+\mu}\left(\frac{2\lambda}{\xi-\lambda\nu}\, H(\mu) - \frac{\psi\mu^2}{\xi^2-\mu^2\nu^2}\, E(\lambda)\right) \\ &- \frac{2}{\lambda^2-\mu^2}\left(\lambda\, F(\mu) - \frac{\xi+\lambda\nu}{\xi+\mu\nu}\mu\, F(\lambda)\right),\end{aligned} \tag{6.13}$$

$$\begin{aligned}[F(\lambda), F(\mu)] = &\frac{2\psi}{\lambda+\mu}\left(\frac{\lambda}{\xi+\lambda\nu}\, F(\mu) - \frac{\mu}{\xi+\mu\nu}\, F(\lambda)\right) \\ &- \frac{2\psi^2}{\lambda+\mu}\left(\frac{\lambda^2}{\xi^2-\lambda^2\nu^2} H(\mu) - \frac{\mu^2}{\xi^2-\mu^2\nu^2} H(\lambda)\right).\end{aligned} \tag{6.14}$$

Our next step is to introduce the new generators $e(\lambda), h(\lambda)$ and $f(\lambda)$ as the following linear combinations of the original generators

$$\begin{aligned}&e(\lambda) = \frac{-\xi+\lambda\nu}{\lambda} E(\lambda), \quad h(\lambda) = \frac{1}{\lambda}\left(H(\lambda) - \frac{\psi\lambda}{2\xi}\, E(\lambda)\right), \\ &f(\lambda) = \frac{1}{\lambda}\left((\xi+\lambda\nu)F(\lambda) + \psi\lambda H(\lambda)\right).\end{aligned} \tag{6.15}$$

The key observation is that in the new basis we have

$$[e(\lambda), e(\mu)] = [h(\lambda), h(\mu)] = \left[f(\lambda), f(\mu)\right] = 0. \tag{6.16}$$

Therefore there are only three nontrivial relations

$$[h(\lambda), e(\mu)] = \frac{2}{\lambda^2-\mu^2}\left(e(\mu) - e(\lambda)\right), \tag{6.17}$$

$$\begin{aligned}\left[h(\lambda), f(\mu)\right] = &\frac{-2}{\lambda^2-\mu^2}\left(f(\mu) - f(\lambda)\right) - \frac{2\psi\nu}{(\lambda^2-\mu^2)\xi}\left(\mu^2 h(\mu) - \lambda^2 h(\lambda)\right) \\ &- \frac{\psi^2}{(\lambda^2-\mu^2)\xi^2}\left(\mu^2 e(\mu) - \lambda^2 e(\lambda)\right),\end{aligned} \tag{6.18}$$

$$\begin{aligned}\left[e(\lambda), f(\mu)\right] = &\frac{2\psi\nu}{(\lambda^2-\mu^2)\xi}\left(\mu^2 e(\mu) - \lambda^2 e(\lambda)\right) \\ &- \frac{4}{\lambda^2-\mu^2}\left((\xi^2-\mu^2\nu^2)h(\mu) - (\xi^2-\lambda^2\nu^2)h(\lambda)\right).\end{aligned} \tag{6.19}$$

In the Hilbert space $\mathcal{H}$ (2.3), in every $V_m = \mathbb{C}^{2s+1}$ there exists a vector $\omega_m \in V_m$ such that

$$S_m^3\omega_m = s_m\omega_m \quad \text{and} \quad S_m^+\omega_m = 0. \tag{6.20}$$

We define a vector Ω_+ to be

$$\Omega_+ = \omega_1 \otimes \cdots \otimes \omega_N \in \mathcal{H}. \tag{6.21}$$

From the definitions above, the formulas (6.6)–(6.8) and (6.15) it is straightforward to obtain the action of the generators $e(\lambda)$ and $h(\lambda)$ on the vector Ω_+

$$e(\lambda)\Omega_+ = 0, \quad \text{and} \quad h(\lambda)\Omega_+ = \rho(\lambda)\Omega_+, \tag{6.22}$$

with

$$\rho(\lambda) = \frac{1}{\lambda}\sum_{m=1}^{N}\left(\frac{s_m}{\lambda-\alpha_m} + \frac{s_m}{\lambda+\alpha_m}\right) = \sum_{m=1}^{N}\frac{2s_m}{\lambda^2-\alpha_m^2}. \tag{6.23}$$

The generating function of the Gaudin Hamiltonians (6.31) in terms of the entries of the Lax matrix is given by

$$\tau(\lambda) = \mathrm{tr}_0\,\mathcal{L}_0^2(\lambda) = 2H^2(\lambda) + 2F(\lambda)E(\lambda) + [E(\lambda), F(\lambda)]. \tag{6.24}$$

From (6.11) we have that the last term is

$$[E(\lambda), F(\lambda)] = 2\frac{\xi^2+\lambda^2\nu^2}{(\xi^2-\lambda^2\nu^2)\lambda}H(\lambda) - 2H'(\lambda) + \frac{\psi}{\xi+\lambda\nu}E(\lambda), \tag{6.25}$$

and therefore the final expression is

$$\tau(\lambda) = 2\left(H^2(\lambda) + \frac{\xi^2+\lambda^2\nu^2}{(\xi^2-\lambda^2\nu^2)\lambda}H(\lambda) - H'(\lambda)\right) + \left(2F(\lambda) + \frac{\psi}{\xi+\lambda\nu}\right)E(\lambda). \tag{6.26}$$

Our aim is to implement the algebraic Bethe ansatz based on the Lie algebra (6.16)–(6.19). To this end we need to obtain the expression for the generating function $\tau(\lambda)$ in terms of the generators $e(\lambda)$, $h(\lambda)$ and $f(\lambda)$. The first step is to invert the relations (6.15)

$$E(\lambda) = \frac{-\lambda}{\xi-\lambda\nu}e(\lambda), \tag{6.27}$$

$$H(\lambda) = \lambda\left(h(\lambda) - \frac{\psi\lambda}{2\xi(\xi-\lambda\nu)}e(\lambda)\right), \tag{6.28}$$

$$F(\lambda) = \frac{\lambda}{\xi+\lambda\nu}\left(f(\lambda) - \psi\lambda\, h(\lambda) + \frac{\psi^2\lambda^2}{2\xi(\xi-\lambda\nu)}e(\lambda)\right). \tag{6.29}$$

In particular, we have

$$\begin{aligned} H^2(\lambda) &= \lambda^2\left(h^2(\lambda) - \frac{\psi\lambda}{2\xi(\xi-\lambda\nu)}\left(2h(\lambda)e(\lambda) - [h(\lambda), e(\lambda)]\right) + \frac{\psi^2\lambda^2}{4\xi^2(\xi-\lambda\nu)^2}e^2(\lambda)\right) \\ &= \lambda^2\left(h^2(\lambda) - \frac{\psi\lambda}{2\xi(\xi-\lambda\nu)}\left(2h(\lambda)e(\lambda) + \frac{e'(\lambda)}{\lambda}\right) + \frac{\psi^2\lambda^2}{4\xi^2(\xi-\lambda\nu)^2}e^2(\lambda)\right). \end{aligned} \tag{6.30}$$

Substituting (6.27)–(6.30) into (6.26) we obtain the desired expression for the generating function

$$\begin{aligned} \tau(\lambda) &= 2\lambda^2\left(h^2(\lambda) + \frac{2\nu^2}{\xi^2-\lambda^2\nu^2}h(\lambda) - \frac{h'(\lambda)}{\lambda}\right) \\ &\quad - \frac{2\lambda^2}{\xi^2-\lambda^2\nu^2}\left(f(\lambda) + \frac{\psi\lambda^2\nu}{\xi}h(\lambda) + \frac{\psi^2\lambda^2}{4\xi^2}e(\lambda) - \frac{\psi\nu}{\xi}\right)e(\lambda). \end{aligned} \tag{6.31}$$

An important initial observation in the implementation of the algebraic Bethe ansatz is that the vector Ω_+ (6.21) is an eigenvector of the generating function $\tau(\lambda)$, to show this we use the expression above, (6.22) and (6.23)

$$\tau(\lambda)\Omega_+ = \chi_0(\lambda)\Omega_+ = 2\lambda^2 \left(\rho^2(\lambda) + \frac{2\nu^2 \rho(\lambda)}{\xi^2 - \lambda^2\nu^2} - \frac{\rho'(\lambda)}{\lambda}\right)\Omega_+, \tag{6.32}$$

using (6.23) the eigenvalue $\chi_0(\lambda)$ can be expressed as

$$\chi_0(\lambda) = 8\lambda^2 \left(\sum_{m=1}^{N} \frac{s_m(s_m+1)}{(\lambda^2-\alpha_m^2)^2} + \sum_{m=1}^{N} \frac{s_m}{\lambda^2-\alpha_m^2}\left(\frac{\nu^2}{\xi^2-\lambda^2\nu^2} + \sum_{n\neq m}^{N} \frac{2s_n}{\alpha_m^2-\alpha_n^2}\right)\right). \tag{6.33}$$

An essential step in the algebraic Bethe ansatz is a definition of the corresponding Bethe vectors. In this case, they are symmetric functions of their arguments and are such that the off shell action of the generating function of the Gaudin Hamiltonians is as simple as possible. With this aim we attempt to show that the Bethe vector $\varphi_1(\mu)$ has the form

$$\varphi_1(\mu) = (f(\mu) + c_1(\mu))\,\Omega_+, \tag{6.34}$$

where $c_1(\mu)$ is given by

$$c_1(\mu) = -\frac{\psi\nu}{\xi}\left(1 - \mu^2\rho(\mu)\right). \tag{6.35}$$

Evidently, the action of the generating function of the Gaudin Hamiltonians reads

$$\tau(\lambda)\varphi_1(\mu) = \left[\tau(\lambda), f(\mu)\right]\Omega_+ + \chi_0(\lambda)\varphi_1(\mu). \tag{6.36}$$

A straightforward calculation show that the commutator in the first term of (6.36) is given by

$$\begin{aligned}\left[\tau(\lambda), f(\mu)\right]\Omega_+ = &-\frac{8\lambda^2}{\lambda^2-\mu^2}\left(\rho(\lambda) + \frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right)\varphi_1(\mu) \\ &+ \frac{8\lambda^2(\xi^2-\mu^2\nu^2)}{(\lambda^2-\mu^2)(\xi^2-\lambda^2\nu^2)}\left(\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2}\right)\varphi_1(\lambda).\end{aligned} \tag{6.37}$$

Therefore the action of the generating function $\tau(\lambda)$ on $\varphi_1(\mu)$ is given by

$$\tau(\lambda)\varphi_1(\mu) = \chi_1(\lambda,\mu)\varphi_1(\mu) + \frac{8\lambda^2(\xi^2-\mu^2\nu^2)}{(\lambda^2-\mu^2)(\xi^2-\lambda^2\nu^2)}\left(\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2}\right)\varphi_1(\lambda), \tag{6.38}$$

with

$$\chi_1(\lambda,\mu) = \chi_0(\lambda) - \frac{8\lambda^2}{\lambda^2-\mu^2}\left(\rho(\lambda) + \frac{\nu^2}{\xi^2-\lambda^2\nu^2}\right). \tag{6.39}$$

The unwanted term in (6.38) vanishes when the following Bethe equation is imposed on the parameter μ,

$$\rho(\mu) + \frac{\nu^2}{\xi^2-\mu^2\nu^2} = 0. \tag{6.40}$$

Thus we have shown that $\varphi_1(\mu)$ (6.34) is the desired Bethe vector of the generating function $\tau(\lambda)$ corresponding to the eigenvalue $\chi_1(\lambda,\mu)$.

We seek the Bethe vector $\varphi_2(\mu_1, \mu_2)$ as the following symmetric function

$$\varphi_2(\mu_1,\mu_2) = f(\mu_1)f(\mu_2)\Omega_+ + c_2^{(1)}(\mu_2;\mu_1)f(\mu_1)\Omega_+ \\ + c_2^{(1)}(\mu_1;\mu_2)f(\mu_2)\Omega_+ + c_2^{(2)}(\mu_1,\mu_2)\Omega_+, \tag{6.41}$$

where the scalar coefficients $c_2^{(1)}(\mu_1;\mu_2)$ and $c_2^{(2)}(\mu_1,\mu_2)$ are

$$c_2^{(1)}(\mu_1;\mu_2) = -\frac{\psi\nu}{\xi}\left(1-\mu_1^2\rho(\mu_1)+\frac{2\mu_1^2}{\mu_1^2-\mu_2^2}\right), \tag{6.42}$$

$$c_2^{(2)}(\mu_1,\mu_2) = -\frac{\psi^2}{\nu^2}\left(\frac{(\xi^2-3\mu_2^2\nu^2)\rho(\mu_1)-(\xi^2-3\mu_1^2\nu^2)\rho(\mu_2)}{\mu_1^2-\mu_2^2}\right. \\ \left. + (\xi^2-(\mu_1^2+\mu_2^2)\nu^2)\rho(\mu_1)\rho(\mu_2)\right). \tag{6.43}$$

One way to obtain the action of $\tau(\lambda)$ on $\varphi_2(\mu_1,\mu_2)$ is to write

$$\tau(\lambda)\varphi_2(\mu_1,\mu_2) \\ = \left[\left[\tau(\lambda), f(\mu_1)\right], f(\mu_2)\right]\Omega_+ + \left(f(\mu_2)+c_2^{(1)}(\mu_2;\mu_1)\right)\left[\tau(\lambda), f(\mu_1)\right]\Omega_+ \\ + \left(f(\mu_1)+c_2^{(1)}(\mu_1;\mu_2)\right)\left[\tau(\lambda), f(\mu_2)\right]\Omega_+ + \chi_0(\lambda)\varphi_2(\mu_1,\mu_2). \tag{6.44}$$

Then to substitute (6.37) in the second and third term above and use the relation

$$\left(f(\mu_1)+c_2^{(1)}(\mu_1;\mu_2)\right)\varphi_1(\mu_2) \\ = \varphi_2(\mu_1,\mu_2) - \frac{\psi\nu}{\xi}\frac{2\mu_2^2}{\mu_1^2-\mu_2^2}\varphi_1(\mu_1) \\ - \left(c_2^{(2)}(\mu_1,\mu_2) - c_1(\mu_1)c_1(\mu_2) + 2\frac{\psi\nu}{\xi}\frac{\mu_1^2c_1(\mu_2)-\mu_2^2c_1(\mu_1)}{\mu_1^2-\mu_2^2}\right)\Omega_+, \tag{6.45}$$

which follows from the definition (6.41). A straightforward calculation shows that the off shell action of the generating function $\tau(\lambda)$ on $\varphi_2(\mu_1,\mu_2)$ is given by

$$\tau(\lambda)\varphi_2(\mu_1,\mu_2) = \chi_2(\lambda,\mu_1,\mu_2)\varphi_2(\mu_1,\mu_2) + \sum_{i=1}^{2}\frac{8\lambda^2(\xi^2-\mu_i^2\nu^2)}{(\lambda^2-\mu_i^2)(\xi^2-\lambda^2\nu^2)} \\ \times\left(\rho(\mu_i)+\frac{\nu^2}{\xi^2-\mu_i^2\nu^2}-\frac{2}{\mu_i^2-\mu_{3-i}^2}\right)\varphi_2(\lambda,\mu_{3-i}), \tag{6.46}$$

with the eigenvalue

$$\chi_2(\lambda,\mu_1,\mu_2) = \chi_0(\lambda) - \sum_{i=1}^{2}\frac{8\lambda^2}{\lambda^2-\mu_i^2}\left(\rho(\lambda)+\frac{\nu^2}{\xi^2-\lambda^2\nu^2}-\frac{1}{\lambda^2-\mu_{3-i}^2}\right). \tag{6.47}$$

The two unwanted terms in the action above (6.46) vanish when the Bethe equations are imposed on the parameters μ_1 and μ_2,

$$\rho(\mu_i)+\frac{\nu^2}{\xi^2-\mu_i^2\nu^2}-\frac{2}{\mu_i^2-\mu_{3-i}^2}=0, \tag{6.48}$$

with $i = 1, 2$. Therefore $\varphi_2(\mu_1, \mu_2)$ is the Bethe vector of the generating function of the Gaudin Hamiltonians with the eigenvalue $\chi_2(\lambda, \mu_1, \mu_2)$.

As our next step we propose the Bethe vector $\varphi_3(\mu_1, \mu_2, \mu_3)$ in the form of the following symmetric function of its arguments

$$\begin{aligned}\varphi_3(\mu_1,\mu_2,\mu_3) = {} & f(\mu_1)f(\mu_2)f(\mu_3)\Omega_+ + c_3^{(1)}(\mu_1;\mu_2,\mu_3)f(\mu_2)f(\mu_3)\Omega_+ \\ & + c_3^{(1)}(\mu_2;\mu_3,\mu_1)f(\mu_3)f(\mu_1)\Omega_+ + c_3^{(1)}(\mu_3;\mu_1,\mu_2)f(\mu_1)f(\mu_2)\Omega_+ \\ & + c_3^{(2)}(\mu_1,\mu_2;\mu_3)f(\mu_3)\Omega_+ + c_3^{(2)}(\mu_2,\mu_3;\mu_1)f(\mu_1)\Omega_+ \\ & + c_3^{(2)}(\mu_3,\mu_1;\mu_2)f(\mu_2)\Omega_+ + c_3^{(3)}(\mu_1,\mu_2,\mu_3)\Omega_+, \end{aligned} \tag{6.49}$$

where the three scalar coefficients above are given by

$$c_3^{(1)}(\mu_1;\mu_2,\mu_3) = -\frac{\psi\nu}{\xi}\left(1 - \mu_1^2\rho(\mu_1) + \frac{2\mu_1^2}{\mu_1^2-\mu_2^2} + \frac{2\mu_1^2}{\mu_1^2-\mu_3^2}\right), \tag{6.50}$$

$$\begin{aligned} & c_3^{(2)}(\mu_1,\mu_2;\mu_3) \\ & = -\frac{\psi^2}{\nu^2}\left(\frac{\xi^2-3\mu_2^2\nu^2}{\mu_1^2-\mu_2^2}\left(\rho(\mu_1)-\frac{2}{\mu_1^2-\mu_3^2}\right) - \frac{\xi^2-3\mu_1^2\nu^2}{\mu_1^2-\mu_2^2}\left(\rho(\mu_2)-\frac{2}{\mu_2^2-\mu_3^2}\right)\right) \\ & \quad - \frac{\psi^2}{\nu^2}(\xi^2-(\mu_1^2+\mu_2^2)\nu^2)\left(\rho(\mu_1)-\frac{2}{\mu_1^2-\mu_3^2}\right)\left(\rho(\mu_2)-\frac{2}{\mu_2^2-\mu_3^2}\right), \end{aligned} \tag{6.51}$$

$$\begin{aligned} & c_3^{(3)}(\mu_1,\mu_2,\mu_3) \\ & = -\frac{\psi^3}{\nu^3\xi}\left(\frac{4\xi^4+\left(\xi^2+\mu_1^2\nu^2\right)\left(4\mu_1^2-5(\mu_2^2+\mu_3^2)\right)\nu^2}{(\mu_1^2-\mu_2^2)(\mu_1^2-\mu_3^2)}\rho(\mu_1)\right. \\ & \quad + \frac{4\xi^4+\left(\xi^2+\mu_2^2\nu^2\right)\left(4\mu_2^2-5(\mu_3^2+\mu_1^2)\right)\nu^2}{(\mu_2^2-\mu_3^2)(\mu_2^2-\mu_1^2)}\rho(\mu_2) \\ & \quad \left. + \frac{4\xi^4+\left(\xi^2+\mu_3^2\nu^2\right)\left(4\mu_3^2-5(\mu_1^2+\mu_2^2)\right)\nu^2}{(\mu_3^2-\mu_1^2)(\mu_3^2-\mu_2^2)}\rho(\mu_3)\right) \\ & \quad - \frac{\psi^3}{\nu^3\xi}\left(\frac{\xi^2\nu^2\left(\mu_1^4+\mu_2^4-\mu_1^2\mu_2^2+2\mu_3^2\left(\mu_1^2+\mu_2^2\right)-5\mu_3^4\right)-\left(2\xi^4-\mu_1^2\mu_2^2\nu^4\right)\left(\mu_1^2+\mu_2^2-2\mu_3^2\right)}{(\mu_1^2-\mu_3^2)(\mu_2^2-\mu_3^2)}\rho(\mu_1)\rho(\mu_2)\right. \\ & \quad + \frac{\xi^2\nu^2\left(\mu_2^4+\mu_3^4-\mu_2^2\mu_3^2+2\mu_1^2\left(\mu_2^2+\mu_3^2\right)-5\mu_1^4\right)-\left(2\xi^4-\mu_2^2\mu_3^2\nu^4\right)\left(\mu_2^2+\mu_3^2-2\mu_1^2\right)}{(\mu_2^2-\mu_1^2)(\mu_3^2-\mu_1^2)}\rho(\mu_2)\rho(\mu_3) \\ & \quad \left. + \frac{\xi^2\nu^2\left(\mu_3^4+\mu_1^4-\mu_3^2\mu_1^2+2\mu_2^2\left(\mu_3^2+\mu_1^2\right)-5\mu_2^4\right)-\left(2\xi^4-\mu_3^2\mu_1^2\nu^4\right)\left(\mu_3^2+\mu_1^2-2\mu_2^2\right)}{(\mu_3^2-\mu_2^2)(\mu_1^2-\mu_2^2)}\rho(\mu_3)\rho(\mu_1)\right) \\ & \quad - \frac{\psi^3}{\nu^3}\xi\left(2\xi^2-\left(\mu_1^2+\mu_2^2+\mu_3^2\right)\nu^2\right)\rho(\mu_1)\rho(\mu_2)\rho(\mu_3). \end{aligned} \tag{6.52}$$

A lengthy but straightforward calculation based on appropriate generalisation of (6.44) and (6.45) shows that the action of the generating function $\tau(\lambda)$ on $\varphi_3(\mu_1, \mu_2, \mu_3)$ is given by

$$\tau(\lambda)\varphi_3(\mu_1,\mu_2,\mu_3) = \chi_3(\lambda,\mu_1,\mu_2\mu_3)\varphi_3(\mu_1,\mu_2,\mu_3) + \sum_{i=1}^{3}\frac{8\lambda^2(\xi^2-\mu_i^2\nu^2)}{(\lambda^2-\mu_i^2)(\xi^2-\lambda^2\nu^2)}$$

$$\times \left(\rho(\mu_i) + \frac{\nu^2}{\xi^2 - \mu_i^2 \nu^2} - \sum_{j \neq i}^{3} \frac{2}{\mu_i^2 - \mu_j^2} \right) \varphi_3(\lambda, \{\mu_j\}_{j \neq i}), \quad (6.53)$$

where the eigenvalue is

$$\chi_3(\lambda, \mu_1, \mu_2, \mu_3) = \chi_0(\lambda) - \sum_{i=1}^{3} \frac{8\lambda^2}{\lambda^2 - \mu_i^2} \left(\rho(\lambda) + \frac{\nu^2}{\xi^2 - \lambda^2 \nu^2} - \sum_{j \neq i}^{3} \frac{1}{\lambda^2 - \mu_j^2} \right). \quad (6.54)$$

The three unwanted terms in (6.53) vanish when the Bethe equation are imposed on the parameters μ_i,

$$\rho(\mu_i) + \frac{\nu^2}{\xi^2 - \mu_i^2 \nu^2} - \sum_{j \neq i}^{3} \frac{2}{\mu_i^2 - \mu_j^2} = 0, \quad (6.55)$$

with $i = 1, 2, 3$.

As a symmetric function of its arguments the Bethe vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$ is given explicitly in Appendix B. It is possible to check that the off shell action of the generating function $\tau(\lambda)$ on the Bethe vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$ is given by

$$\begin{aligned} &\tau(\lambda)\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4) \\ &= \chi_4(\lambda, \mu_1, \mu_2 \mu_3)\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4) + \sum_{i=1}^{4} \frac{8\lambda^2(\xi^2 - \mu_i^2 \nu^2)}{(\lambda^2 - \mu_i^2)(\xi^2 - \lambda^2 \nu^2)} \\ &\times \left(\rho(\mu_i) + \frac{\nu^2}{\xi^2 - \mu_i^2 \nu^2} - \sum_{j \neq i}^{4} \frac{2}{\mu_i^2 - \mu_j^2} \right) \varphi_4(\lambda, \{\mu_j\}_{j \neq i}), \end{aligned} \quad (6.56)$$

with the eigenvalue

$$\chi_4(\lambda, \mu_1, \mu_2, \mu_3, \mu_4) = \chi_0(\lambda) - \sum_{i=1}^{4} \frac{8\lambda^2}{\lambda^2 - \mu_i^2} \left(\rho(\lambda) + \frac{\nu^2}{\xi^2 - \lambda^2 \nu^2} - \sum_{j \neq i}^{4} \frac{1}{\lambda^2 - \mu_j^2} \right). \quad (6.57)$$

The four unwanted terms on the right hand side of (6.56) vanish when the Bethe equation are imposed on the parameters μ_i,

$$\rho(\mu_i) + \frac{\nu^2}{\xi^2 - \mu_i^2 \nu^2} - \sum_{j \neq i}^{4} \frac{2}{\mu_i^2 - \mu_j^2} = 0, \quad (6.58)$$

with $i = 1, 2, 3, 4$.

Based on the results presented above we can conclude that the local realisation (6.6)–(6.8) of the Lie algebra (6.15)–(6.19) yields the spectrum $\chi_M(\lambda, \mu_1, \ldots, \mu_M)$ of the generating function of the Gaudin Hamiltonians

$$\chi_M(\lambda, \mu_1, \ldots, \mu_M) = \chi_0(\lambda) - \sum_{i=1}^{M} \frac{8\lambda^2}{\lambda^2 - \mu_i^2} \left(\rho(\lambda) + \frac{\nu^2}{\xi^2 - \lambda^2 \nu^2} - \sum_{j \neq i}^{M} \frac{1}{\lambda^2 - \mu_j^2} \right), \quad (6.59)$$

and the corresponding Bethe equations which should be imposed on the parameters μ_i

$$\rho(\mu_i) + \frac{\nu^2}{\xi^2 - \mu_i^2 \nu^2} - \sum_{j \neq i}^{M} \frac{2}{\mu_i^2 - \mu_j^2} = 0, \tag{6.60}$$

where $i = 1, 2, \ldots, M$. Moreover, from (4.29) and (6.59) it follows that the eigenvalues of the Gaudin Hamiltonians (4.32) and (4.33) can be obtained as the residues of $\chi_M(\lambda, \mu_1, \ldots, \mu_M)$ at poles $\lambda = \pm\alpha_m$

$$\begin{aligned}\mathcal{E}_m &= \frac{1}{4} \mathrm{Res}_{\lambda=\alpha_m} \chi_M(\lambda, \mu_1, \ldots, \mu_M) \\ &= \frac{s_m(s_m+1)}{2\alpha_m} + \alpha_m s_m \left(\frac{\nu^2}{\xi^2 - \alpha_m^2 \nu^2} + \sum_{n \neq m}^{N} \frac{2 s_n}{\alpha_m^2 - \alpha_n^2} \right) - 2\alpha_m s_m \sum_{i=1}^{M} \frac{1}{\alpha_m^2 - \mu_i^2},\end{aligned} \tag{6.61}$$

and

$$\begin{aligned}\widetilde{\mathcal{E}}_m &= -\frac{1}{4} \mathrm{Res}_{\lambda=-\alpha_m} \chi_M(\lambda, \mu_1, \ldots, \mu_M) \\ &= \frac{s_m(s_m+1)}{2\alpha_m} + \alpha_m s_m \left(\frac{\nu^2}{\xi^2 - \alpha_m^2 \nu^2} + \sum_{n \neq m}^{N} \frac{2 s_n}{\alpha_m^2 - \alpha_n^2} \right) - 2\alpha_m s_m \sum_{i=1}^{M} \frac{1}{\alpha_m^2 - \mu_i^2}.\end{aligned} \tag{6.62}$$

Evidently, the respective eigenvalues (6.61) and (6.62) of the Hamiltonians (4.32) and (4.33) coincide. When all the spin s_m are set to one half, these energies coincide with the expressions obtained in [42] (up to normalisation). The Bethe equations are also equivalent, the correspondence between our variables and the one used in [42] being given by (the left hand sides correspond to our variables, the left hand sides to the ones used in [42]):

$$\mu_j = \frac{\lambda_j}{1 - \xi^{(1)}}\ ; \quad \alpha_m = \frac{\theta_m}{1 - \xi^{(1)}}\ ; \quad \frac{\xi}{\nu} = \frac{\xi}{1 - \xi^{(1)}}. \tag{6.63}$$

However, explicit and compact form of the relevant Bethe vector $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M, requires further studies and will be reported elsewhere. As it is evident form the formulas for the Bethe vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$ given in Appendix B, the main problem lies in the definition the scalar coefficients $c_M^{(m)}(\mu_1, \ldots, \mu_m; \mu_{m+1}, \ldots, \mu_M)$, with $m = 1, 2, \ldots, M$. Some of them can be obtained straightforwardly, but, in particular, the coefficient $c_M^{(M)}(\mu_1, \mu_2, \ldots, \mu_M)$ still represents a challenge, at least in the present form of the Bethe vectors.

7. Conclusion

Following Sklyanin's proposal in the periodic case [3], here we have derived the generating function of the Gaudin Hamiltonians with boundary terms. Our derivation is based on the quasi-classical expansion of the linear combination of the transfer matrix of the XXX Heisenberg spin chain and the central element, the so-called Sklyanin determinant. The corresponding Gaudin Hamiltonians with boundary terms are obtained as the residues of the generating function. Then we have studied the appropriate algebraic structure, including the classical reflection equation.

Our approach to the algebraic Bethe ansatz is based on the relevant Lax matrix which satisfies certain linear bracket and simultaneously provides the local realisation for the corresponding Lie algebra. By defining the appropriate Bethe vectors we have obtained the strikingly simple off shell action of the generating function of the Gaudin Hamiltonians. Actually, the action of the generating function is as simple as it could possible be since it almost coincides with the one in the case when the boundary matrix is diagonal [20]. In this way we have implemented the algebraic Bethe ansatz, obtaining the spectrum of the generating function and the corresponding Bethe equations.

Although the off shell action of the generating function which we have established is very simple, it would be important to obtain more compact formula for the Bethe vector $\varphi_M(\mu_1, \mu_2, \ldots, \mu_M)$, for an arbitrary positive integer M. In particular, simpler expression for the scalar coefficients $c_M^{(m)}(\mu_1, \ldots, \mu_m; \mu_{m+1}, \ldots, \mu_M)$, with $m = 1, 2, \ldots, M$ would be of utmost importance. Such a formula would be crucial for the off shell scalar product of the Bethe vectors and these results could lead to the correlations functions of Gaudin model with boundary. Moreover, it would be of considerable interest to establish a relation between Bethe vectors and solutions of the corresponding Knizhnik–Zamolodchikov equations, along the lines it was done in the case when the boundary matrix is diagonal [20].

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Appendix A. Basic definitions

We consider the spin operators S^α with $\alpha = +, -, 3$, acting in some (spin s) representation space $\mathbb{C}^{2s+1}$ with the commutation relations

$$[S^3, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^3, \tag{A.1}$$

and Casimir operator

$$c_2 = (S^3)^2 + \frac{1}{2}(S^+S^- + S^-S^+) = (S^3)^2 + S^3 + S^-S^+ = \vec{S} \cdot \vec{S}.$$

In the particular case of spin $\frac{1}{2}$ representation, one recovers the Pauli matrices

$$S^\alpha = \frac{1}{2}\sigma^\alpha = \frac{1}{2}\begin{pmatrix} \delta_{\alpha 3} & 2\delta_{\alpha +} \\ 2\delta_{\alpha -} & -\delta_{\alpha 3} \end{pmatrix}.$$

We consider a spin chain with N sites with spin s representations, i.e. a local $\mathbb{C}^{2s+1}$ space at each site and the operators

$$S_m^\alpha = \mathbb{1} \otimes \cdots \otimes \underbrace{S^\alpha}_{m} \otimes \cdots \otimes \mathbb{1}, \tag{A.2}$$

with $\alpha = +, -, 3$ and $m = 1, 2, \ldots, N$.

Appendix B. Bethe vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$

Here we present explicit formulas of the Bethe vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$. The vector $\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4)$ is a symmetric function of its arguments and is given by

$$\begin{aligned}
&\varphi_4(\mu_1, \mu_2, \mu_3, \mu_4) \\
&\quad = f(\mu_1)f(\mu_2)f(\mu_3)f(\mu_4)\Omega_+ + c_4^{(1)}(\mu_4; \mu_1, \mu_2, \mu_3)f(\mu_1)f(\mu_2)f(\mu_3)\Omega_+ \\
&\qquad + c_4^{(1)}(\mu_3; \mu_1, \mu_2, \mu_4)f(\mu_1)f(\mu_2)f(\mu_4)\Omega_+ \\
&\qquad + c_4^{(1)}(\mu_2; \mu_1, \mu_3, \mu_4)f(\mu_1)f(\mu_3)f(\mu_4)\Omega_+ \\
&\qquad + c_4^{(1)}(\mu_1; \mu_2, \mu_3, \mu_4)f(\mu_2)f(\mu_3)f(\mu_4)\Omega_+ + c_4^{(2)}(\mu_3, \mu_4; \mu_1, \mu_2)f(\mu_1)f(\mu_2)\Omega_+ \\
&\qquad + c_4^{(2)}(\mu_2, \mu_4; \mu_1, \mu_3)f(\mu_1)f(\mu_3)\Omega_+ + c_4^{(2)}(\mu_2, \mu_3; \mu_1, \mu_4)f(\mu_1)f(\mu_4)\Omega_+ \\
&\qquad + c_4^{(2)}(\mu_1, \mu_4; \mu_2, \mu_3)f(\mu_2)f(\mu_3)\Omega_+ + c_4^{(2)}(\mu_1, \mu_3; \mu_2, \mu_4)f(\mu_2)f(\mu_4)\Omega_+ \\
&\qquad + c_4^{(2)}(\mu_1, \mu_2; \mu_3, \mu_4)f(\mu_3)f(\mu_4)\Omega_+ + c_4^{(3)}(\mu_2, \mu_3, \mu_4; \mu_1)f(\mu_1)\Omega_+ \\
&\qquad + c_4^{(3)}(\mu_1, \mu_2, \mu_4; \mu_2)f(\mu_2)\Omega_+ + c_4^{(3)}(\mu_1, \mu_2, \mu_4; \mu_3)f(\mu_3)\Omega_+ \\
&\qquad + c_4^{(3)}(\mu_1, \mu_2, \mu_3; \mu_4)f(\mu_4)\Omega_+ + c_4^{(4)}(\mu_1, \mu_2, \mu_3, \mu_4)\Omega_+,
\end{aligned} \tag{B.1}$$

where the four scalar coefficients are

$$c_4^{(1)}(\mu_1; \mu_2, \mu_3, \mu_4) = -\frac{\psi\nu}{\xi}\left(1 - \mu_1^2\rho(\mu_1) + \sum_{i=2}^{4} \frac{2\mu_1^2}{\mu_1^2 - \mu_i^2}\right), \tag{B.2}$$

$$\begin{aligned}
&c_4^{(2)}(\mu_1, \mu_2; \mu_3, \mu_4) \\
&\quad = -\frac{\psi^2}{\nu^2}\left(\frac{\xi^2 - 3\mu_2^2\nu^2}{\mu_1^2 - \mu_2^2}\left(\rho(\mu_1) - \sum_{i=3}^{4}\frac{2}{\mu_1^2 - \mu_i^2}\right)\right. \\
&\qquad \left. - \frac{\xi^2 - 3\mu_1^2\nu^2}{\mu_1^2 - \mu_2^2}\left(\rho(\mu_2) - \sum_{j=3}^{4}\frac{2}{\mu_2^2 - \mu_j^2}\right)\right) - \frac{\psi^2}{\nu^2}(\xi^2 - (\mu_1^2 + \mu_2^2)\nu^2) \\
&\qquad \times \left(\rho(\mu_1) - \sum_{i=3}^{4}\frac{2}{\mu_1^2 - \mu_i^2}\right)\left(\rho(\mu_2) - \sum_{j=3}^{4}\frac{2}{\mu_2^2 - \mu_j^2}\right),
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
&c_4^{(3)}(\mu_1, \mu_2, \mu_3; \mu_4) \\
&\quad = -\frac{\psi^3}{\nu^3\xi}\left(\frac{4\xi^4 + \left(\xi^2 + \mu_1^2\nu^2\right)\left(4\mu_1^2 - 5(\mu_2^2 + \mu_3^2)\right)\nu^2}{(\mu_1^2 - \mu_2^2)(\mu_1^2 - \mu_3^2)}\left(\rho(\mu_1) - \frac{2}{\mu_1^2 - \mu_4^2}\right)\right. \\
&\qquad + \frac{4\xi^4 + \left(\xi^2 + \mu_2^2\nu^2\right)\left(4\mu_2^2 - 5(\mu_3^2 + \mu_1^2)\right)\nu^2}{(\mu_2^2 - \mu_3^2)(\mu_2^2 - \mu_1^2)}\left(\rho(\mu_2) - \frac{2}{\mu_2^2 - \mu_4^2}\right) \\
&\qquad \left. + \frac{4\xi^4 + \left(\xi^2 + \mu_3^2\nu^2\right)\left(4\mu_3^2 - 5(\mu_1^2 + \mu_2^2)\right)\nu^2}{(\mu_3^2 - \mu_1^2)(\mu_3^2 - \mu_2^2)}\left(\rho(\mu_3) - \frac{2}{\mu_3^2 - \mu_4^2}\right)\right) \\
&\qquad - \frac{\psi^3}{\nu^3\xi}\left(\frac{\xi^2\nu^2\left(\mu_1^4 + \mu_2^4 - \mu_1^2\mu_2^2 + 2\mu_3^2\left(\mu_1^2 + \mu_2^2\right) - 5\mu_3^4\right) - \left(2\xi^4 - \mu_1^2\mu_2^2\nu^4\right)\left(\mu_1^2 + \mu_2^2 - 2\mu_3^2\right)}{(\mu_1^2 - \mu_3^2)(\mu_2^2 - \mu_3^2)}\right.
\end{aligned}$$

$$\times\left(\rho(\mu_1)-\frac{2}{\mu_1^2-\mu_4^2}\right)\left(\rho(\mu_2)-\frac{2}{\mu_2^2-\mu_4^2}\right)$$
$$+\frac{\xi^2\nu^2\left(\mu_2^4+\mu_3^4-\mu_2^2\mu_3^2+2\mu_1^2\left(\mu_2^2+\mu_3^2\right)-5\mu_1^4\right)-\left(2\xi^4-\mu_2^2\mu_3^2\nu^4\right)\left(\mu_2^2+\mu_3^2-2\mu_1^2\right)}{(\mu_2^2-\mu_1^2)(\mu_3^2-\mu_1^2)}$$
$$\times\left(\rho(\mu_2)-\frac{2}{\mu_2^2-\mu_4^2}\right)\left(\rho(\mu_3)-\frac{2}{\mu_3^2-\mu_4^2}\right)$$
$$+\frac{\xi^2\nu^2\left(\mu_3^4+\mu_1^4-\mu_3^2\mu_1^2+2\mu_2^2\left(\mu_3^2+\mu_1^2\right)-5\mu_2^4\right)-\left(2\xi^4-\mu_3^2\mu_1^2\nu^4\right)\left(\mu_3^2+\mu_1^2-2\mu_2^2\right)}{(\mu_3^2-\mu_2^2)(\mu_1^2-\mu_2^2)}$$
$$\times\left(\rho(\mu_3)-\frac{2}{\mu_3^2-\mu_4^2}\right)\left(\rho(\mu_1)-\frac{2}{\mu_1^2-\mu_4^2}\right)\Bigg)-\frac{\psi^3}{\nu^3}\xi\left(2\xi^2-\left(\mu_1^2+\mu_2^2+\mu_3^2\right)\nu^2\right)$$
$$\times\left(\rho(\mu_1)-\frac{2}{\mu_1^2-\mu_4^2}\right)\left(\rho(\mu_2)-\frac{2}{\mu_2^2-\mu_4^2}\right)\left(\rho(\mu_3)-\frac{2}{\mu_3^2-\mu_4^2}\right). \quad \text{(B.4)}$$

$$c_4^{(4)}(\mu_1,\mu_2,\mu_3,\mu_4)$$
$$=-\frac{2\psi^4}{\nu^4}\left(\frac{9\xi^4+\xi^2\nu^2\left(27\mu_1^2-7(\mu_2^2+\mu_3^2+\mu_4^2)\right)+3\mu_1^2\nu^4\left(8\mu_1^2-7(\mu_2^2+\mu_3^2+\mu_4^2)\right)}{(\mu_1^2-\mu_2^2)(\mu_1^2-\mu_3^2)(\mu_1^2-\mu_4^2)}\rho(\mu_1)\right.$$
$$+\frac{9\xi^4+\xi^2\nu^2\left(27\mu_2^2-7(\mu_1^2+\mu_3^2+\mu_4^2)\right)+3\mu_2^2\nu^4\left(8\mu_2^2-7(\mu_2^2+\mu_3^2+\mu_4^2)\right)}{(\mu_2^2-\mu_1^2)(\mu_2^2-\mu_3^2)(\mu_2^2-\mu_4^2)}\rho(\mu_2)$$
$$+\frac{9\xi^4+\xi^2\nu^2\left(27\mu_3^2-7(\mu_1^2+\mu_2^2+\mu_4^2)\right)+3\mu_3^2\nu^4\left(8\mu_3^2-7(\mu_1^2+\mu_2^2+\mu_4^2)\right)}{(\mu_3^2-\mu_1^2)(\mu_3^2-\mu_2^2)(\mu_3^2-\mu_4^2)}\rho(\mu_3)$$
$$\left.+\frac{9\xi^4+\xi^2\nu^2\left(27\mu_4^2-7(\mu_1^2+\mu_2^2+\mu_3^2)\right)+3\mu_4^2\nu^4\left(8\mu_4^2-7(\mu_1^2+\mu_2^2+\mu_3^2)\right)}{(\mu_4^2-\mu_1^2)(\mu_4^2-\mu_2^2)(\mu_4^2-\mu_3^2)}\rho(\mu_4)\right)$$
$$-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(2(\mu_1^4+\mu_1^2\mu_2^2+\mu_2^4)-3(\mu_1^2\mu_3^2+\mu_1^2\mu_4^2+\mu_2^2\mu_3^2+\mu_2^2\mu_4^2-2\mu_3^2\mu_4^2)\right)$$
$$-18\xi^2\nu^2\mu_1^2\mu_2^2(\mu_3^2+\mu_4^2)+\xi^2\nu^2\Big(-2(\mu_1^6-6\mu_1^4\mu_2^2-6\mu_1^2\mu_2^4+\mu_2^6)-4(\mu_1^2+\mu_2^2)^2$$
$$\times(\mu_3^2+\mu_4^2)+7(\mu_1^2+\mu_2^2)(\mu_3^2+\mu_4^2)^2+2\mu_3^2\mu_4^2(5(\mu_1^2+\mu_2^2)-7(\mu_3^2+\mu_4^2))\Big)$$
$$+\nu^4\left(14\mu_1^2\mu_2^2(\mu_3^4+\mu_4^4)-(4\mu_1^2\mu_2^2+7\mu_3^2\mu_4^2)(\mu_1^2+\mu_2^2)(\mu_3^2+\mu_4^2)\right)$$
$$+2\nu^4\left(4\mu_3^2\mu_4^2(\mu_1^2+\mu_2^2)^2-3\mu_1^2\mu_2^2(\mu_1^2-\mu_2^2)^2-\mu_1^2\mu_2^2(3\mu_1^2\mu_2^2+5\mu_3^2\mu_4^2)\right)\Big)$$
$$\times\frac{\rho(\mu_1)\rho(\mu_2)}{(\mu_1^2-\mu_3^2)(\mu_1^2-\mu_4^2)(\mu_2^2-\mu_3^2)(\mu_2^2-\mu_4^2)}$$
$$-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(2(\mu_1^4+\mu_1^2\mu_3^2+\mu_3^4)-3(\mu_1^2\mu_2^2+\mu_1^2\mu_4^2+\mu_2^2\mu_3^2+\mu_3^2\mu_4^2-2\mu_2^2\mu_4^2)\right)$$
$$-18\xi^2\nu^2\mu_1^2\mu_3^2(\mu_2^2+\mu_4^2)+\xi^2\nu^2\Big(-2(\mu_1^6-6\mu_1^4\mu_3^2-6\mu_1^2\mu_3^4+\mu_3^6)-4(\mu_1^2+\mu_3^2)^2$$
$$\times(\mu_2^2+\mu_4^2)+7(\mu_1^2+\mu_3^2)(\mu_2^2+\mu_4^2)^2+2\mu_2^2\mu_4^2(5(\mu_1^2+\mu_3^2)-7(\mu_2^2+\mu_4^2))\Big)$$

$$
\begin{aligned}
&+\nu^4\left(14\mu_1^2\mu_3^2(\mu_2^4+\mu_4^4)-(4\mu_1^2\mu_3^2+7\mu_2^2\mu_4^2)(\mu_1^2+\mu_3^2)(\mu_2^2+\mu_4^2)\right)\\
&+2\nu^4\left(4\mu_2^2\mu_4^2(\mu_1^2+\mu_3^2)^2-3\mu_1^2\mu_3^2(\mu_1^2-\mu_3^2)^2-\mu_1^2\mu_3^2(3\mu_1^2\mu_3^2+5\mu_2^2\mu_4^2)\right)\Big)\\
&\times\frac{\rho(\mu_1)\rho(\mu_3)}{(\mu_1^2-\mu_2^2)(\mu_1^2-\mu_4^2)(\mu_3^2-\mu_2^2)(\mu_3^2-\mu_4^2)}\\
&-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(2(\mu_1^4+\mu_1^2\mu_4^2+\mu_4^4)-3(\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_4^2+\mu_3^2\mu_4^2-2\mu_2^2\mu_3^2)\right)\\
&-18\xi^2\nu^2\mu_1^2\mu_4^2(\mu_2^2+\mu_3^2)+\xi^2\nu^2\Big(-2(\mu_1^6-6\mu_1^4\mu_4^2-6\mu_1^2\mu_4^4+\mu_4^6)-4(\mu_1^2+\mu_4^2)^2\\
&\times(\mu_2^2+\mu_3^2)+7(\mu_1^2+\mu_4^2)(\mu_2^2+\mu_3^2)^2+2\mu_2^2\mu_3^2(5(\mu_1^2+\mu_4^2)-7(\mu_2^2+\mu_3^2))\Big)\\
&+\nu^4\left(14\mu_1^2\mu_4^2(\mu_2^4+\mu_3^4)-(4\mu_1^2\mu_4^2+7\mu_2^2\mu_3^2)(\mu_1^2+\mu_4^2)(\mu_2^2+\mu_3^2)\right)\\
&+2\nu^4\left(4\mu_2^2\mu_3^2(\mu_1^2+\mu_4^2)^2-3\mu_1^2\mu_4^2(\mu_1^2-\mu_4^2)^2-\mu_1^2\mu_4^2(3\mu_1^2\mu_4^2+5\mu_2^2\mu_3^2)\right)\Big)\\
&\times\frac{\rho(\mu_1)\rho(\mu_4)}{(\mu_1^2-\mu_2^2)(\mu_1^2-\mu_3^2)(\mu_4^2-\mu_2^2)(\mu_4^2-\mu_3^2)}\\
&-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(2(\mu_2^4+\mu_2^2\mu_3^2+\mu_3^4)-3(\mu_1^2\mu_2^2+\mu_2^2\mu_4^2+\mu_1^2\mu_3^2+\mu_3^2\mu_4^2-2\mu_1^2\mu_4^2)\right)\\
&-18\xi^2\nu^2\mu_2^2\mu_3^2(\mu_1^2+\mu_4^2)+\xi^2\nu^2\Big(-2(\mu_2^6-6\mu_2^4\mu_3^2-6\mu_2^2\mu_3^4+\mu_3^6)-4(\mu_2^2+\mu_3^2)^2\\
&\times(\mu_1^2+\mu_4^2)+7(\mu_2^2+\mu_3^2)(\mu_1^2+\mu_4^2)^2+2\mu_1^2\mu_4^2(5(\mu_2^2+\mu_3^2)-7(\mu_1^2+\mu_4^2))\Big)\\
&+\nu^4\left(14\mu_2^2\mu_3^2(\mu_1^4+\mu_4^4)-(4\mu_2^2\mu_3^2+7\mu_1^2\mu_4^2)(\mu_2^2+\mu_3^2)(\mu_1^2+\mu_4^2)\right)\\
&+2\nu^4\left(4\mu_1^2\mu_4^2(\mu_2^2+\mu_3^2)^2-3\mu_2^2\mu_3^2(\mu_2^2-\mu_3^2)^2-\mu_2^2\mu_3^2(3\mu_2^2\mu_3^2+5\mu_1^2\mu_4^2)\right)\Big)\\
&\times\frac{\rho(\mu_2)\rho(\mu_3)}{(\mu_2^2-\mu_1^2)(\mu_2^2-\mu_4^2)(\mu_3^2-\mu_1^2)(\mu_3^2-\mu_4^2)}\\
&-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(2(\mu_2^4+\mu_2^2\mu_4^2+\mu_4^4)-3(\mu_1^2\mu_2^2+\mu_2^2\mu_3^2+\mu_1^2\mu_4^2+\mu_3^2\mu_4^2-2\mu_1^2\mu_3^2)\right)\\
&-18\xi^2\nu^2\mu_2^2\mu_4^2(\mu_1^2+\mu_3^2)+\xi^2\nu^2\Big(-2(\mu_2^6-6\mu_2^4\mu_4^2-6\mu_2^2\mu_4^4+\mu_4^6)-4(\mu_2^2+\mu_4^2)^2\\
&\times(\mu_1^2+\mu_3^2)+7(\mu_2^2+\mu_4^2)(\mu_1^2+\mu_3^2)^2+2\mu_1^2\mu_3^2(5(\mu_2^2+\mu_4^2)-7(\mu_1^2+\mu_3^2))\Big)\\
&+\nu^4\left(14\mu_2^2\mu_4^2(\mu_1^4+\mu_3^4)-(4\mu_2^2\mu_4^2+7\mu_1^2\mu_3^2)(\mu_2^2+\mu_4^2)(\mu_1^2+\mu_3^2)\right)\\
&+2\nu^4\left(4\mu_1^2\mu_3^2(\mu_2^2+\mu_4^2)^2-3\mu_2^2\mu_4^2(\mu_2^2-\mu_4^2)^2-\mu_2^2\mu_4^2(3\mu_2^2\mu_4^2+5\mu_1^2\mu_3^2)\right)\Big)\\
&\times\frac{\rho(\mu_2)\rho(\mu_4)}{(\mu_2^2-\mu_1^2)(\mu_2^2-\mu_3^2)(\mu_4^2-\mu_1^2)(\mu_4^2-\mu_3^2)}\\
&-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(2(\mu_3^4+\mu_3^2\mu_4^2+\mu_4^4)-3(\mu_2^2\mu_3^2+\mu_1^2\mu_3^2+\mu_2^2\mu_4^2+\mu_1^2\mu_4^2-2\mu_1^2\mu_2^2)\right)\\
&-18\xi^2\nu^2\mu_3^2\mu_4^2(\mu_1^2+\mu_2^2)+\xi^2\nu^2\Big(-2(\mu_3^6-6\mu_3^4\mu_4^2-6\mu_3^2\mu_4^4+\mu_4^6)-4(\mu_3^2+\mu_4^2)^2
\end{aligned}
$$

$$\times(\mu_1^2+\mu_2^2)+7(\mu_3^2+\mu_4^2)(\mu_1^2+\mu_2^2)^2+2\mu_1^2\mu_2^2(5(\mu_3^2+\mu_4^2)-7(\mu_1^2+\mu_2^2))\Big)$$
$$+\nu^4\left(14\mu_3^2\mu_4^2(\mu_1^4+\mu_2^4)-(4\mu_3^2\mu_4^2+7\mu_1^2\mu_2^2)(\mu_3^2+\mu_4^2)(\mu_1^2+\mu_2^2)\right)$$
$$+2\nu^4\left(4\mu_1^2\mu_2^2(\mu_3^2+\mu_4^2)^2-3\mu_3^2\mu_4^2(\mu_3^2-\mu_4^2)^2-\mu_3^2\mu_4^2(3\mu_3^2\mu_4^2+5\mu_1^2\mu_2^2)\right)\Big)$$
$$\times\frac{\rho(\mu_3)\rho(\mu_4)}{(\mu_3^2-\mu_1^2)(\mu_3^2-\mu_2^2)(\mu_4^2-\mu_1^2)(\mu_4^2-\mu_2^2)}$$
$$-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(3\mu_4^4-2\mu_4^2(\mu_1^2+\mu_2^2+\mu_3^2)+\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2\right)$$
$$-\xi^2\nu^2\Big(7\mu_4^6-4\mu_4^4(\mu_1^2+\mu_2^2+\mu_3^2)+\mu_4^2\times(3(\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2)-2(\mu_1^4$$
$$+\mu_2^4+\mu_3^4))+\mu_1^4\mu_2^2+\mu_1^4\mu_3^2+\mu_1^2\mu_2^4+\mu_1^2\mu_3^4+\mu_2^4\mu_3^2+\mu_2^2\mu_3^4-4\mu_1^2\mu_2^2\mu_3^2\Big)$$
$$-\nu^4\Big(2\mu_4^4(\mu_1^2\mu_2^2+\mu_1^2\mu_3^2+\mu_2^2\mu_3^2)-\mu_4^2(\mu_1^4\mu_2^2+\mu_1^2\mu_2^4+\mu_1^4\mu_3^2+\mu_1^2\mu_3^4$$
$$+\mu_2^4\mu_3^2+\mu_2^2\mu_3^4+6\mu_1^2\mu_2^2\mu_3^2)\Big)-\nu^4\left(2\mu_1^2\mu_2^2\mu_3^2(\mu_1^2+\mu_2^2+\mu_3^2)\right)\Big)$$
$$\times\frac{\rho(\mu_1)\rho(\mu_2)\rho(\mu_3)}{(\mu_1^2-\mu_4^2)(\mu_2^2-\mu_4^2)(\mu_3^2-\mu_4^2)}$$
$$-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(3\mu_3^4-2\mu_3^2(\mu_1^2+\mu_2^2+\mu_4^2)+\mu_1^2\mu_2^2+\mu_1^2\mu_4^2+\mu_2^2\mu_4^2\right)$$
$$-\xi^2\nu^2\Big(7\mu_3^6-4\mu_3^4(\mu_1^2+\mu_2^2+\mu_4^2)+\mu_3^2(3(\mu_1^2\mu_2^2+\mu_1^2\mu_4^2+\mu_2^2\mu_4^2)-2(\mu_1^4$$
$$+\mu_2^4+\mu_4^4))+\mu_1^4\mu_2^2+\mu_1^4\mu_4^2+\mu_1^2\mu_2^4+\mu_1^2\mu_4^4+\mu_2^4\mu_4^2+\mu_2^2\mu_4^4-4\mu_1^2\mu_2^2\mu_4^2\Big)$$
$$-\nu^4\Big(2\mu_3^4(\mu_1^2\mu_2^2+\mu_1^2\mu_4^2+\mu_2^2\mu_4^2)-\mu_3^2(\mu_1^4\mu_2^2+\mu_1^2\mu_2^4+\mu_1^4\mu_4^2+\mu_1^2\mu_4^4$$
$$+\mu_2^4\mu_4^2+\mu_2^2\mu_4^4+6\mu_1^2\mu_2^2\mu_4^2)\Big)-\nu^4\left(2\mu_1^2\mu_2^2\mu_4^2(\mu_1^2+\mu_2^2+\mu_4^2)\right)\Big)$$
$$\times\frac{\rho(\mu_1)\rho(\mu_2)\rho(\mu_4)}{(\mu_1^2-\mu_3^2)(\mu_2^2-\mu_3^2)(\mu_4^2-\mu_3^2)}$$
$$-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(3\mu_2^4-2\mu_2^2(\mu_1^2+\mu_3^2+\mu_4^2)+\mu_1^2\mu_3^2+\mu_1^3\mu_4^2+\mu_3^2\mu_4^2\right)$$
$$-\xi^2\nu^2\Big(7\mu_2^6-4\mu_2^4(\mu_1^2+\mu_3^2+\mu_4^2)+\mu_2^2(3(\mu_1^2\mu_3^2+\mu_1^2\mu_4^2+\mu_3^2\mu_4^2)-2(\mu_1^4$$
$$+\mu_3^4+\mu_4^4))+\mu_1^4\mu_3^2+\mu_1^4\mu_4^2+\mu_1^2\mu_3^4+\mu_1^2\mu_4^4+\mu_3^4\mu_4^2+\mu_3^2\mu_4^4-4\mu_1^2\mu_3^2\mu_4^2\Big)$$
$$-\nu^4\Big(2\mu_2^4(\mu_1^2\mu_3^2+\mu_1^2\mu_4^2+\mu_3^2\mu_4^2)-\mu_2^2(\mu_1^4\mu_3^2+\mu_1^2\mu_3^4+\mu_1^4\mu_4^2+\mu_1^2\mu_4^4$$
$$+\mu_3^4\mu_4^2+\mu_3^2\mu_4^4+6\mu_1^2\mu_3^2\mu_4^2)\Big)-\nu^4\left(2\mu_1^2\mu_3^2\mu_4^2(\mu_1^2+\mu_3^2+\mu_4^2)\right)\Big)$$
$$\times\frac{\rho(\mu_1)\rho(\mu_3)\rho(\mu_4)}{(\mu_1^2-\mu_2^2)(\mu_3^2-\mu_2^2)(\mu_4^2-\mu_2^2)}$$
$$-\frac{\psi^4}{\nu^4}\Big(3\xi^4\left(3\mu_1^4-2\mu_1^2(\mu_2^2+\mu_3^2+\mu_4^2)+\mu_2^2\mu_3^2+\mu_2^3\mu_4^2+\mu_3^2\mu_4^2\right)$$

$$
\begin{aligned}
&-\xi^2\nu^2\Big(7\mu_1^6-4\mu_1^4(\mu_2^2+\mu_3^2+\mu_4^2)+\mu_1^2(3(\mu_2^2\mu_3^2+\mu_2^2\mu_4^2+\mu_3^2\mu_4^2)-2(\mu_2^4\\
&+\mu_3^4+\mu_4^4))+\mu_2^4\mu_3^2+\mu_2^4\mu_4^2+\mu_2^2\mu_3^4+\mu_2^2\mu_4^4+\mu_3^4\mu_4^2+\mu_3^2\mu_4^4-4\mu_2^2\mu_3^2\mu_4^2\Big)\\
&-\nu^4\Big(2\mu_1^4(\mu_2^2\mu_3^2+\mu_2^2\mu_4^2+\mu_3^2\mu_4^2)-\mu_1^2(\mu_2^4\mu_3^2+\mu_2^2\mu_3^4+\mu_2^4\mu_4^2+\mu_2^2\mu_4^4\\
&+\mu_3^4\mu_4^2+\mu_3^2\mu_4^4+6\mu_2^2\mu_3^2\mu_4^2)\Big)-\nu^4\Big(2\mu_2^2\mu_3^2\mu_4^2(\mu_2^2+\mu_3^2+\mu_4^2)\Big)\Big)\\
&\times\frac{\rho(\mu_2)\rho(\mu_3)\rho(\mu_4)}{(\mu_2^2-\mu_1^2)(\mu_3^2-\mu_1^2)(\mu_4^2-\mu_1^2)}\\
&-\frac{\psi^4}{\nu^4}\xi^2\Big(3\xi^2-\Big(\mu_1^2+\mu_2^2+\mu_3^2+\mu_4^2\Big)\nu^2\Big)\rho(\mu_1)\rho(\mu_2)\rho(\mu_3)\rho(\mu_4)
\end{aligned}
\tag{B.5}
$$

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