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Three-particle hyper-spherical harmonics and quark bound states

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Abstract. We construct the three-body permutation symmetric hyperspherical harmonics based on the subgroup chain $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset O(6)$ (and the subalgebra chain $u(1) \otimes so(3)_{rot} \subset u(3) \subset so(6)$). These hyperspherical harmonics represent a natural basis for solving non-relativistic three-body Schrödinger equation in three spatial dimensions. In particular, we apply the calculated three-particle harmonics to the three-quark bound state problem. We consider confining Δ- and Y-string three-quark effective potentials, and then calculate the spectrum of low-lying ($K \leq 4$) bound states.

1. Introduction

The quantum-mechanical three-body bound-state problem has been addressed by a huge literature, in which the hyperspherical harmonics, Refs. [1, 2, 3, 4], provide one of the most firmly established theoretical tools. Nevertheless, little is known about the general structure of the three-body bound-state spectrum, such as the ordering of states, even in the (simplest) case of three indentical particles. In comparison, the two-body bound state problem is well understood, see Refs. [5, 6, 7, 8], where theorems controlling the ordering of bound states in convex two-body potentials were proven more than 30 years ago. In this paper we make the first significant advance in the three-body problem after the 1990 paper by Taxil & Richard, Ref. [9].

The basic difficulty lay in the absence of a systematic construction of permutation-symmetric three-body wave functions. Classification of wave functions into distinct classes under permutation symmetry in the three-body system, should be a matter of course, and yet permutation symmetric three-body hyperspherical harmonics in three dimensions were known explicitly only in a few special cases, such as those with total orbital angular momentum $L = 0$, see Refs. [3, 10] before the recent progress made in Ref. [11]. In this paper we confine ourselves to the study of factorizable (in the hyper-radius and hyper-angles) three-body potentials for technical reasons: For this class of potentials our method allows closed-form ("analytical") results, at sufficiently small values of the grand angular momentum K (i.e. up to, and including the $K \leq 8$ shell). Factorizable potentials include homogenous potentials, which in turn include pair-wise sums of two-body power-law potentials, such as the linear (confining) "Δ-string", and the Coulomb ones, as well as the genuine three-body "Y-string" potential [12, 13].

In this paper, we shall: 1) show how the Schrödinger equation for three particles in a homogenous/factorizable potential can be reduced to a single differential equation and an algebraic/numerical problem for their coupling strengths; 2) use this result to explicitly confirm

Richard and Taxil's results, [9], for the ordering of K = 3 shell three-quark states, and thus resolve the controversy with [14]; 3) calculate the K = 4 shell's (purported "universal") spectral splittings in terms of four parameters (lowest hyperspherical harmonics expansion coefficients) that characterize the three-body potential. 4) show that the first manifest differences in the ordering of states in the Y- and Δ-string potentials appear in the K = 3 shell, and then reappear more emphatically in the K = 4 shell.

Our work is based on the recent advances in the construction of three-body wave functions with well-defined permutation symmetry, see Ref. [11].

2. Three-body problem in hyper-spherical coordinates

The three-body wave function $\Psi(\boldsymbol{\rho},\boldsymbol{\lambda})$ can be transcribed from the Euclidean relative position (Jacobi) vectors $\boldsymbol{\rho}=\frac{1}{\sqrt{2}}(\mathbf{x_1}-\mathbf{x_2})$, $\boldsymbol{\lambda}=\frac{1}{\sqrt{6}}(\mathbf{x_1}+\mathbf{x_2}-2\mathbf{x_3})$, into hyper-spherical coordinates as $\Psi(R,\Omega_5)$, where $R=\sqrt{\boldsymbol{\rho}^2+\boldsymbol{\lambda}^2}$ is the hyper-radius, and five angles Ω_5 that parametrize a hyper-sphere in the six-dimensional Euclidean space. Three ($\Phi_i;\ i=1,2,3$) of these five angles (Ω_5) are just the Euler angles associated with the orientation in a three-dimensional space of a spatial reference frame defined by the (plane of) three bodies; the remaining two hyper-angles describe the shape of the triangle subtended by three bodies; they are functions of three independent scalar three-body variables, e.g. $\boldsymbol{\rho}\cdot\boldsymbol{\lambda}$, $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$. As we saw above, one linear combination of the two variables $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$, is already taken by the hyper-radius R, so the shape-space is two-dimensional, and topologically equivalent to the surface of a three-dimensional sphere.

There are two traditional ways of parameterizing this sphere: 1) the standard Delves choice, [1], of hyper-angles (χ,θ), that somewhat obscures the full S_3 permutation symmetry of the problem; 2) the Iwai, Ref. [4], hyper-angles (α, ϕ): $(\sin\alpha)^2=1-\left(\frac{2\boldsymbol{\rho}\times\boldsymbol{\lambda}}{R^2}\right)^2$, $\tan\phi=\left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2-\boldsymbol{\lambda}^2}\right)$, reveal the full S_3 permutation symmetry of the problem: the angle α does not change under permutations, so that all permutation properties are encoded in the ϕ-dependence of the wave functions. We shall use the latter choice, as it leads to permutation-symmetric hyperspherical harmonics, see Ref. [11].

We expand the wave function $\Psi(R,\Omega_5)$ in terms of hyper-spherical harmonics $\mathcal{Y}^{\mathrm{K}}_{[\mathrm{m}]}(\Omega_5)$, $\Psi(R,\Omega_5)=\sum_{\mathrm{K},[\mathrm{m}]}\psi^{\mathrm{K}}_{[\mathrm{m}]}(R)\mathcal{Y}^{\mathrm{K}}_{[\mathrm{m}]}(\Omega_5)$, where K together with $[\mathrm{m}]=[Q,\nu,L,L_z=m]$ constitute the complete set of hyperspherical quantum numbers: K is the hyper-spherical angular momentum, L is the (total orbital) angular momentum, $L_z=m$ its projection on the z-axis, Q is the Abelian quantum number conjugated with the Iwai angle ϕ, and ν is the multiplicity label that distinguishes between hyperspherical harmonics with remaining four quantum numbers that are identical.

The hyper-spherical harmonics turn the Schrödinger equation into a set of (infinitely) many coupled equations,

$$\begin{aligned} - \quad & \frac{1}{2\mu}\left[\frac{d^2}{dR^2}+\frac{5}{R}\frac{d}{dR}-\frac{\mathrm{K}(\mathrm{K}+4)}{R^2}+2\mu E\right]\psi^{\mathrm{K}}_{[\mathrm{m}]}(R) \\ + \quad & V_{\mathrm{eff.}}(R)\sum_{\mathrm{K}',[\mathrm{m}']}C^{\mathrm{K}\ \mathrm{K}'}_{[\mathrm{m}][\mathrm{m}']}\psi^{\mathrm{K}'}_{[\mathrm{m}']}(R)=0 \end{aligned} \tag{1}$$

with a hyper-angular coupling coefficients matrix $C^{\mathrm{K}\ \mathrm{K}'}_{[\mathrm{m}][\mathrm{m}']}$ defined by

$$\begin{aligned} V_{\mathrm{eff.}}(R)C^{\mathrm{K}'\ \mathrm{K}}_{[\mathrm{m}'][\mathrm{m}]} &= \langle\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)|V(R,\alpha,\phi)|\mathcal{Y}^{\mathrm{K}}_{[\mathrm{m}]}(\Omega_5)\rangle \\ &= V(R)\langle\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)|V(\alpha,\phi)|\mathcal{Y}^{\mathrm{K}}_{[\mathrm{m}]}(\Omega_5)\rangle. \end{aligned} \tag{2}$$

In Eq. (1) we used the factorizability of the potential $V(R,\alpha,\phi)=V(R)V(\alpha,\phi)$ to reduce this set to one (common) hyper-radial Schrödinger equation. The hyper-angular part $V(\alpha,\phi)$ can

be expanded in terms of O(6) hyper-spherical harmonics with zero angular momenta $L = m = 0$ (due to the rotational invariance of the potential),

$$V(\alpha,\phi) = \sum_{\mathrm{K},Q}^{\infty} v_{\mathrm{K},Q}^{3-\mathrm{body}} \mathcal{Y}_{00}^{\mathrm{K}Q\nu}(\alpha,\phi) \tag{3}$$

where $v_{\mathrm{K},Q}^{3-\mathrm{body}} = \int \mathcal{Y}_{00}^{\mathrm{K}Q\nu *}(\Omega_5)\, V(\alpha,\phi)\, d\Omega_{(5)}$ leading to

$$\begin{aligned} V_{\mathrm{eff.}}(R) C^{\mathrm{K}'' \mathrm{K}'}_{[\mathrm{m}''][\mathrm{m}']} = \quad & V(R) \sum_{\mathrm{K},Q}^{\infty} v_{\mathrm{K},Q}^{3-\mathrm{body}} \\ \langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5)| \quad & \mathcal{Y}_{00}^{\mathrm{K}Q\nu}(\alpha,\phi) |\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle \end{aligned} \tag{4}$$

There is no summation over the multiplicity index in Eq. (3), because no multiplicity arises for harmonics with $L < 2$. Here we separate out the $\mathrm{K} = 0$ term and absorb the factor $\frac{v_{00}^{3-\mathrm{body}}}{\pi\sqrt{\pi}}$ into the definition of $V_{\mathrm{eff.}}(R) = \frac{v_{00}^{3-\mathrm{body}}}{\pi\sqrt{\pi}} V(R)$ to find

$$\begin{aligned} C^{\mathrm{K}'' \mathrm{K}'}_{[\mathrm{m}''][\mathrm{m}']} = \delta_{\mathrm{K}'',\mathrm{K}'} \delta_{[\mathrm{m}''],[\mathrm{m}']} + \pi\sqrt{\pi} \textstyle\sum_{\mathrm{K}>0,Q}^{\infty} \frac{v_{\mathrm{K},Q}^{3-\mathrm{body}}}{v_{00}^{3-\mathrm{body}}} \\ \times \langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5)| \mathcal{Y}_{00}^{\mathrm{K}Q\nu}(\alpha,\phi) |\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle. \end{aligned} \tag{5}$$

Homogenous potentials, such as the Δ and Y-string ones, which are linear in R, and the Coulomb one, have first coefficients $v_{00}^{3-\mathrm{body}}$ in the h.s. expansion that are one order of magnitude larger than the rest $v_{\mathrm{K}>0,Q}^{3-\mathrm{body}}$. This reflects the fact that, on the average, these potential energies depend more on the overall size of the system than on its shape, thus justifying the perturbative approach taken in Ref. [9], with the first term in Eq. (5) taken as the zeroth-order approximation.[1]

In such cases Eqs. (1) decouple, leading to zeroth order solutions for $\psi^{\mathrm{K}}_{0[\mathrm{m}]}(R)$ that are independent of $[m]$ and thus have equal energies within the same K shell, and different energies in different K shells. Two known exceptions are potentials with the homogeneity degree $\mathrm{k} = -1, 2$, that lead to "accidental degeneracies" and have to be treated separately.

The first-order corrections are obtained by diagonalization of the block matrices $C^{\mathrm{K}\,\mathrm{K}}_{[\mathrm{m}][\mathrm{m}']}$, $\mathrm{K} = 1, 2, ...$, while the off-diagonal couplings $C^{\mathrm{K}\,\mathrm{K}'}_{[\mathrm{m}][\mathrm{m}']}, \mathrm{K} \neq \mathrm{K}'$ appear only in the second-order corrections. Rather than calculating perturbative first-order energy shifts, a better approximation is obtained when the diagonalized block matrices are plugged back into Eq. (1), which equations then decouple into a set of (separate) individual ODEs in one variable, that differ only in the value of the effective coupling constant:

$$\left[\frac{d^2}{dR^2} + \frac{5}{R}\frac{d}{dR} - \frac{\mathrm{K}(\mathrm{K}+4)}{R^2} + 2\mu(E - V^{\mathrm{K}}_{[\mathrm{m_d}]}(R))\right] \psi^{\mathrm{K}}_{[\mathrm{m_d}]}(R) = 0, \tag{6}$$

where $V^{\mathrm{K}}_{[\mathrm{m_d}]}(R) = C^{\mathrm{K}}_{[\mathrm{m_d}]} V_{\mathrm{eff.}}(R)$, with $C^{\mathrm{K}}_{[\mathrm{m_d}]}$ being the eigenvalues of matrix $C^{\mathrm{K}\,\mathrm{K}}_{[\mathrm{m}][\mathrm{m}']}$.

The spectrum of three-body systems in homogenous potentials is now reduced to finding the eigenvalues of a single differential operator, just as in the two-body problem with a radial potential. The matrix elements in Eq. (5) can be readily evaluated using the permutation-symmetric O(6) hyper-spherical harmonics and the integrals that are spelled out in Ref. [11].

[1] (note that the h.s. matrix elements $\langle \mathcal{Y}^{\mathrm{K}''}_{[\mathrm{m}'']}(\Omega_5)| \mathcal{Y}_{00}^{\mathrm{K}Q\nu}(\alpha,\phi) |\mathcal{Y}^{\mathrm{K}'}_{[\mathrm{m}']}(\Omega_5)\rangle$ under the sum are always less than $\frac{1}{\pi\sqrt{\pi}}$).

This is our main (algebraic) result: combined with the hyperspherical harmonics recently obtained in Ref. [11], it allows one to evaluate the discrete part of the (energy) spectrum of a three-body potential as a function of its shape-sphere harmonic expansion coefficients $v^{3-\mathrm{body}}_{\mathrm{K},Q}$. Generally, these matrix elements obey selection rules: they are subject to the "triangular" conditions $\mathrm{K}' + \mathrm{K}'' \geq \mathrm{K} \geq |\mathrm{K}' - \mathrm{K}''|$ plus the condition that $\mathrm{K}' + \mathrm{K}'' + \mathrm{K} = 0, 2, 4, \ldots$, and the angular momenta satisfy the selection rules: $L' = L''$, $m' = m''$. Moreover, Q is an Abelian (i.e. additive) quantum number that satisfies the simple selection rule: $Q'' = Q' + Q$. That reduces the sum in Eq. (5) to a finite one, that depends on a finite number of coefficients $v^{3-\mathrm{body}}_{\mathrm{K},Q}$; for small values of K, this number is also small.

A matrix such as that in Eq. (5) is generally sparse in the permutation-symmetric basis, so its diagonalization is not a serious problem, and, for sufficiently small K values it can even be accomplished in closed form: for example, for $\mathrm{K} \leq 5$, all results depend only on four coefficients $(v_{00}, v_{40}, v_{6\pm6}, v_{80})$, and there is at most three-state mixing, so the eigenvalue equations are at most cubic ones, with well-known solutions. For brevity's sake we confine ourselves to $\mathrm{K} \leq 4$ states here.

3. Results

1) In the $\mathrm{K} = 2$ band/shell of the three-body energy spectrum the eigen-energies depend on two coefficients (v_{00}, v_{40}), and the splittings among various levels depend only on the (generally small, see Table 1) ratio v_{40}/v_{00}. This means that the eigen-energies form a fixed pattern ("ordering") that does not depend on the shape of the three-body potential. The actual size of the $\mathrm{K} = 2$ shell energy splitting depends on the small parameter v_{40}/v_{00}, provided that the potential is permutation symmetric. This fact was noticed almost 40 years ago, Refs. [15, 16], and it suggested that similar patterns might exist in higher-K shells.

The advantage of permutation-symmetric hyperspherical harmonics over the conventional ones is perhaps best illustrated here: the $\mathrm{K} = 2$ shell splittings in the Y- and Δ-string potentials were obtained, after some complicated calculations using conventional hyperspherical harmonics in Ref. [17], whereas here they follow from the calculation of four (simple) hyper-angular matrix elements.

2) Historically, extensions of this kind of calculations to higher ($\mathrm{K} \geq 3$) bands, for general three-body potentials turned out more difficult than expected: Bowler et. al, Ref. [14], published a set of predictions for the $\mathrm{K} = 3, 4$ bands, which were later questioned by Richard and Taxil's [9], $\mathrm{K} = 3$ hyperspherical harmonic calculation; see also Refs. [18]. This controversy had not been resolved to the present day, to our knowledge, so we address that problem first: In the $\mathrm{K} = 3$ case the energies depend on three coefficients $(v_{00}, v_{40}, v_{6\pm6})$, and there is no mixing of multiplets, so all eigen-energies can be expressed in simple closed form that agrees with Ref. [9] and depends on two small parameters $v_{40}/v_{00}, v_{6\pm6}/v_{00}$.

Note that the third coefficient $v_{6\pm6}$ vanishes in the simplified Y-string potential without two-body terms and thus causes the first observable difference between Y- and Δ-string potentials: the splittings between $[20, 1^-]$, and $[56, 1^-]$, as well as between $[20, 3^-]$, and $[56, 3^-]$. The vanishing of $v_{6\pm6}$ implies that the Y-string potential is independent of the Iwai angle ϕ, and that consequently there is a (new) dynamical "kinematic rotations/democracy transformations" O(2) symmetry, [12, 13] associated with it.

3) In the $\mathrm{K} = 4$ band $SU(6)$, or S_3 multiplets have one of the following 12 values of the diagonalized C-matrix $C^{\mathrm{K}}_{[\mathrm{m_d}]} \times \frac{v_{00}}{\pi\sqrt{\pi}}$, from which one can evaluate the eigen-energies. We use the baryon-spectroscopic notation: $[\mathrm{dim.}, L^P]$, where dim. is the dimension of the $SU_{\mathrm{FS}}(6)$ representation and the correspondence with the representations of the permutation group S_3

is given as $70 \leftrightarrow M$, $20 \leftrightarrow A$, $56 \leftrightarrow S$.

$$
\begin{aligned}
\left[70,0^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{\sqrt{3}}{2}v_{40}+\frac{1}{2\sqrt{5}}v_{80}\right)\\
\left[56,0^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}\qquad\qquad\quad+\frac{2}{\sqrt{5}}v_{80}\right)\\
\left[70,1^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}\qquad\qquad\quad-\frac{1}{\sqrt{5}}v_{80}\right)\\
\left[70,2^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{35}\Big(7\sqrt{3}v_{40}+2\sqrt{5}v_{80}\right.\\
&\left.-3\sqrt{3v_{40}^{2}-2\sqrt{15}v_{40}v_{80}+5v_{80}^{2}+120v_{6\pm6}^{2}}\Big)\right)\\
\left[70',2^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{35}\Big(7\sqrt{3}v_{40}+2\sqrt{5}v_{80}\right.\\
&\left.+3\sqrt{3v_{40}^{2}-2\sqrt{15}v_{40}v_{80}+5v_{80}^{2}+120v_{6\pm6}^{2}}\Big)\right)\\
\left[56,2^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{12\sqrt{3}}{35}v_{40}+\frac{\sqrt{5}}{7}v_{80}\right)\\
\left[20,2^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}\qquad\qquad\quad-\frac{1}{\sqrt{5}}v_{80}\right)\\
\left[20,3^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{3\sqrt{3}}{14}v_{40}-\frac{\sqrt{5}}{14}v_{80}\right)\\
\left[70,3^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}-\frac{5\sqrt{3}}{14}v_{40}+\frac{1}{14\sqrt{5}}v_{80}\right)\\
\left[56,4^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{5\sqrt{3}}{14}v_{40}+\frac{3}{14\sqrt{5}}v_{80}\right)\\
\left[70,4^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{42\sqrt{5}}\Big(-2v_{80}\right.\\
&\left.-\sqrt{1215v_{40}^{2}-54\sqrt{15}v_{40}v_{80}+9v_{80}^{2}+1280v_{6\pm6}^{2}}\Big)\right)\\
\left[70',4^{+}\right] &: \frac{1}{\pi\sqrt{\pi}}\left(v_{00}+\frac{1}{42\sqrt{5}}\Big(-2v_{80}\right.\\
&\left.+\sqrt{1215v_{40}^{2}-54\sqrt{15}v_{40}v_{80}+9v_{80}^{2}+1280v_{6\pm6}^{2}}\Big)\right).
\end{aligned}
$$

Table 1 shows that the ordering of K = 4 states is not universally valid even for these two convex confining potentials. This, of course, is a consequence of different ratios v_{40}/v_{00}, $v_{6\pm6}/v_{00}$ and v_{80}/v_{00}. That goes to show that one cannot expect strongly restrictive ordering theorems

Table 1. Expansion coefficients v_{KQ} of the Y- and Δ-string potentials in terms of O(6) hyper-spherical harmonics $\mathcal{Y}_{0,0}^{K,0,0}$, for $K = 0$, 4, 8, respectively, and of the hyper-spherical harmonics $\mathcal{Y}_{0,0}^{6,\pm6,0}$.

(K,Q)	(0,0)	(4,0)	(6,±6)	(8,0)
v_{KQ}(Y)	8.18	-0.44	0	-0.09
$v_{KQ}(\Delta)$	16.04	-0.44	-0.14	-0.06

to hold for three-body systems in general, the way they hold in the two-body problem, Ref. [8]. Nevertheless, even the present results can be useful, as they indicate that certain groups of multiplets are jointly lifted, or depressed in the spectrum, subject to the value of the ratio v_{40}/v_{00}, with ordering within each group being subject to the finer structure of the potential, i.e., to higher coefficients ratios $v_{6\pm6}/v_{00}$ and v_{80}/v_{00}.

Of course, similar conclusions hold also for K = 3 spectrum splitting, but are less pronounced, as that shell depends only on two numbers: the ratios v_{40}/v_{00} and $v_{6\pm6}/v_{00}$. As the difference between Δ and Y-string potentials is most pronounced in the value of $v_{6\pm6}$, that is the case where the distinction between these two potentials is most clearly seen.

4. Summary and Conclusions

We have reduced the non-relativistic (quantum) three-identical-body problem to a single ordinary differential equation for the hyper-radial wave function with coefficients multiplying the linear hyper-radial potential determined by O(6) group-theoretical arguments. That equation can be solved in the same way as the radial Schrödinger equation in 3D. The breaking of the O(6) symmetry by the three-body potential determines the ordering of states in the spectrum.

In three dimensions (3D) the hyper-spherical symmetry group is O(6), and the residual dynamical symmetry of the potential is $S_3 \otimes SO(3)_{rot} \subset O(2) \otimes SO(3)_{rot} \subset O(6)$, where $SO(3)_{rot}$ is the rotational symmetry associated with the (total orbital) angular momentum L. We showed how the energy eigenvalues can be calculated in terms of the three-body potential's (hyper-)spherical harmonics expansion coefficients v_{KQ}.

The ordering of bound states has its most immediate application in the physics of three confined quarks, where the question was raised originally, Refs. [9, 14, 15, 16, 17]. We have used these results to calculate the energy splittings of various SU(6)/S_3 multiplets in the K $\leq$ 4 shells of the Y- and Δ-string potential spectra. The dynamical $O(2)$ dynamical symmetry of the Y-string potential was discovered in Ref. [12], with the permutation group $S_3 \subset O(2)$ as its subgroup. The existence of an additional dynamical symmetry strongly suggested an algebraic approach to this problem, such as that used in two-dimensional space, in Ref. [13]. We have shown that the first clear difference between the spectra of these two models of confinement appears in the K $\geq$ 3 shell. That is also the first explicit consequence of the dynamical O(2) symmetry of the "Y-string" potential. We stress the analytical nature of our results, in contrast to the numerical results of Refs. [18].

The next step would be to apply the method to linear combinations of homogenous potentials, which can only be done numerically, however. Several "realistic" two-body potentials, such as the Lennard-Jones inter-atomic one, as well as the "Coulomb + linear" quark-quark one, are simple linear combinations of (only) two homogenous potentials.

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