Vladimir Dobrev *Editor*

Lie Theory and Its Applications in Physics

Varna, Bulgaria, June 2015

Springer

Permutation-Symmetric Three-Body O(6) Hyperspherical Harmonics in Three Spatial Dimensions

Igor Salom and V. Dmitrašinović

Abstract We have constructed the three-body permutation symmetric O(6) hyperspherical harmonics which can be used to solve the non-relativistic three-body Schrödinger equation in three spatial dimensions. We label the states with eigenvalues of the $U(1) \otimes SO(3)_{rot} \subset U(3) \subset O(6)$ chain of algebras and we present the corresponding $K \leq 4$ harmonics. Concrete transformation properties of the harmonics are discussed in some detail.

1 Introduction

Hyperspherical harmonics are an important tool for dealing with quantum-mechanical three-body problem, being of a particular importance in the context of bound states [1–6]. However, before our recent progress [7], a systematical construction of permutation-symmetric three-body hyperspherical harmonics was, to our knowledge, lacking (with only some particular cases being worked out – e.g. those with total orbital angular momentum $L = 0$, see Refs. [5, 8]).

In this note, we report the construction of permutation-symmetric three-body O(6) hyperspherical harmonics using the $U(1) \otimes SO(3)_{rot} \subset U(3) \subset O(6)$ chain of algebras, where $U(1)$ is the "democracy transformation", or "kinematic rotation" group for three particles, $SO(3)_{rot}$ is the 3D rotation group, and $U(3)$, $O(6)$ are the usual Lie groups. This particular chain of algebras is mathematically very natural, since the $U(1)$ group of "democracy transformations" is the only nontrivial (Lie) subgroup of full hyperspherical $SO(6)$ symmetry (the symmetry of nonrelativistic kinetic energy) that commutes with spatial rotations. Historically, this chain was also suggested in the recent review of the Russian school's work, Ref. [9], and indicated

I. Salom (✉) · V. Dmitrašinović
Institute of Physics, Belgrade University, Pregrevica 118, Zemun, P.O. Box 57,
11080 Beograd, Serbia
e-mail: isalom@ipb.ac.rs

V. Dmitrašinović
e-mail: dmitrasin@ipb.ac.rs

by the previous discovery of the dynamical $O(2)$ symmetry of the Y-string potential, Ref. [10]. The name "democracy transformations" comes from the close relation of these transformations with permutations: (cyclic) particle permutations form a discrete subgroup of this $U(1)$ group.

2 Three-Body Hyperspherical Coordinates

A natural set of coordinates for parametrization of three-body wave function $\Psi(\boldsymbol{\rho}, \boldsymbol{\lambda})$ (in the center-of-mass frame of reference) is given by the Euclidean relative position Jacobi vectors $\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x}_1 - \mathbf{x}_2)$, $\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x}_1 + \mathbf{x}_2 - 2\mathbf{x}_3)$. The overall six components of the two vectors can be seen as specifying a position in a six-dimensional configuration space $x_\mu = (\boldsymbol{\lambda}, \boldsymbol{\rho})$, which, in turn, can be parameterized by hyperspherical coordinates as $\Psi(R, \Omega_5)$. Here $R = \sqrt{\rho^2 + \lambda^2}$ is the hyper-radius, and five angles Ω_5 parametrize a hyper-sphere in the six-dimensional Euclidean space. Three $(\Phi_i; \ i = 1, 2, 3)$ of these five angles (Ω_5) are just the Euler angles associated with the orientation in a three-dimensional space of a spatial reference frame defined by the (plane of) three bodies; the remaining two hyper-angles describe the shape of the triangle subtended by three bodies; they are functions of three independent scalar three-body variables, e.g. $\boldsymbol{\rho} \cdot \boldsymbol{\lambda}$, ρ^2, and λ^2. Due to the connection $R = \sqrt{\rho^2 + \lambda^2}$, this shape-space is two-dimensional, and topologically equivalent to the surface of a three-dimensional sphere. A spherical coordinate system can be further introduced in this shape space. Among various (in principle infinitely many) ways that this can be accomplished, the one due to Iwai [6] stands out as the one that fully observes the permutation symmetry of the problem. Namely, of the two Iwai (hyper)spherical angles (α, ϕ): $(\sin \alpha)^2 = 1 - \left(\frac{2\boldsymbol{\rho} \times \boldsymbol{\lambda}}{R^2}\right)^2$, $\tan \phi = \left(\frac{2\boldsymbol{\rho} \cdot \boldsymbol{\lambda}}{\rho^2 - \lambda^2}\right)$, the angle α does not change under permutations, so that all permutation properties are encoded in the ϕ-dependence of the wave functions.

Nevertheless, in the construction of hyperspherical harmonics, we will, unlike the most of the previous attempts in this context, refrain from use of any explicit set of angles, and express harmonics as functions of Cartesian Jacobi coordinates.

3 O(6) Symmetry of the Hyperspherical Approach

The motivation for hyperspherical approach to the three-body problem comes from the fact that the equal-mass three-body kinetic energy T is O(6) invariant and can be written as

$$T = \frac{m}{2}\dot{R}^2 + \frac{K_{\mu\nu}^2}{2mR^2}. \tag{1}$$

Here, $K_{\mu\nu}$, $(\mu, \nu = 1, 2, \ldots, 6)$ denotes the $SO(6)$ "grand angular" momentum tensor

$$
\begin{aligned}
K_{\mu\nu} &= m \left(\mathbf{x}_\mu \dot{\mathbf{x}}_\nu - \mathbf{x}_\nu \dot{\mathbf{x}}_\mu \right) \\
&= \left(\mathbf{x}_\mu \mathbf{p}_\nu - \mathbf{x}_\nu \mathbf{p}_\mu \right).
\end{aligned}
\tag{2}
$$

$K_{\mu\nu}$ has 15 linearly independent components, that contain, among themselves three components of the "ordinary" orbital angular momentum: $\mathbf{L} = \mathbf{l}_\rho + \mathbf{l}_\lambda = m \left(\boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} + \boldsymbol{\lambda} \times \dot{\boldsymbol{\lambda}} \right)$.

It is due to this symmetry of the kinetic energy that the decomposition of the wave function and potential energy into $SO(6)$ hyperspherical harmonics becomes a natural way to tackle the three-body quantum problem.

In this particular physical context, the six dimensional hyperspherical harmonics need to have some desirable properties. Quite generally, apart from the hyperangular momentum K, which labels the O(6) irreducible representation, all hyperspherical harmonics must carry additional labels specifying the transformation properties of the harmonic with respect to (w.r.t.) certain subgroups of the orthogonal group. The symmetries of most three-body potentials, including the three-quark confinement ones, are: parity, rotations and permutations (spatial exchange of particles).

Therefore, the goal is to find three-body hyperspherical harmonics with well defined transformation properties with respect to thee symmetries. Parity is directly related to K value: $P = (-1)^K$, the rotation symmetry implies that the hyperspherical harmonics must carry usual quantum numbers L and m corresponding to $SO(3)_{rot} \supset SO(2)$ subgroups and permutation properties turn out to be related with a continuous $U(1)$ subgroup of "democracy transformations", as will be discussed below.

4 Labels od Permutation-Symmetric Three-Body Hyperspherical Harmonics

We introduce the complex coordinates:

$$
X_i^{\pm} = \lambda_i \pm i\rho_i, \quad i = 1, 2, 3.
\tag{3}
$$

Nine of 15 hermitian $SO(6)$ generators $K_{\mu\nu}$ in these new coordinates become

$$
iL_{ij} \equiv X_i^+ \frac{\partial}{\partial X_j^+} + X_i^- \frac{\partial}{\partial X_j^-} - X_j^+ \frac{\partial}{\partial X_i^+} - X_j^- \frac{\partial}{\partial X_i^-},
\tag{4}
$$

$$
2Q_{ij} \equiv X_i^+ \frac{\partial}{\partial X_j^+} - X_i^- \frac{\partial}{\partial X_j^-} + X_j^+ \frac{\partial}{\partial X_i^+} - X_j^- \frac{\partial}{\partial X_i^-}.
\tag{5}
$$

Here L_{ij} have physical interpretation of components of angular momentum vector $\mathbf{L}$. The symmetric tensor Q_{ij} decomposes as $(5) + (1)$ w.r.t. rotations, while the trace:

$$Q \equiv Q_{ii} = \sum_{i=1}^{3} X_i^+ \frac{\partial}{\partial X_i^+} - \sum_{i=1}^{3} X_i^- \frac{\partial}{\partial X_i^-} \tag{6}$$

is the only scalar under rotations, among all of the $SO(6)$ generators. Therefore, the only mathematically justified choice is to take eigenvalues of this operator for an additional label of the hyperspherical harmonics. Besides, this trace Q is the generator of the forementioned democracy transformations, a special case of which are the cyclic permutations – which in addition makes this choice particularly convenient on an route to construction of permutation-symmetric hyperspherical harmonics. The remaining five components of the symmetric tensor Q_{ij}, together with three antisymmetric tensors L_{ij} generate the $SU(3)$ Lie algebra, which together with the single scalar Q form an $U(3)$ algebra, Ref. [9].

Overall, labelling of the O(6) hyperspherical harmonics with labels K, Q, L and m corresponds to the subgroup chain $U(1) \otimes SO(3)_{rot} \subset U(3) \subset SO(6)$. Yet, these four quantum numbers are in general insufficient to uniquely specify an $SO(6)$ hyperspherical harmonic and an additional quantum number must be introduced to account for the remaining multiplicity. This is the multiplicity that necessarily occurs when $SU(3)$ unitary irreducible representations are labelled w.r.t. the chain $SO(2) \subset SO(3) \subset SU(3)$ (where $SO(3)$ is "matrix embedded" into $SU(3)$), and thus is well documented in the literature. In this context the operator:

$$\mathcal{V}_{LQL} \equiv \sum_{ij} L_i Q_{ij} L_j \tag{7}$$

(where $L_i = \frac{1}{2}\varepsilon_{ijk}L_{jk}$ and Q_{ij} is given by Eq. (5)) has often been used to label the multiplicity of $SU(3)$ states. This operator commutes both with the angular momentum L_i, and with the "democracy rotation" generator Q:

$$\left[\mathcal{V}_{LQL}, L_i\right] = 0; \quad \left[\mathcal{V}_{LQL}, Q\right] = 0$$

Therefore we demand that the hyperspherical harmonics be eigenstates of this operator:

$$\mathcal{V}_{LQL}\mathcal{Y}_{L,m}^{KQ\nu} = \nu \mathcal{Y}_{L,m}^{KQ\nu};$$

Thus, ν will be the fifth label of the hyperspherical harmonics, beside the (K, Q, L, m).

5 Tables of Hyperspherical Harmonics of Given K, Q, L, M and ν

Below we explicitly list all hyperspherical harmonics for $K \leq 4$, labelled by the quantum numbers (K, Q, L, m, ν) (we will not delve here into lengthy details of the derivation of the expressions). We list only the harmonics with $m = L$ and $Q \geq 0$, as the rest can be easily obtained by acting on them with standard lowering operators and by using the permutation symmetry properties of hyperspherical harmonics: $\mathcal{Y}_{L,m}^{KQ\nu}(\lambda, \rho) = (-1)^{K-L} \mathcal{Y}_{L,m}^{K-Q-\nu}(\lambda, -\rho)$. We use the (more compact) spherical complex coordinates: $X_0^{\pm} \equiv \lambda_3 \pm i\rho_3$, $X_{(\pm)}^{\pm} \equiv \lambda_1 \pm i\rho_1 + (\pm)(\lambda_2 \pm i\rho_2)$, $|X^{\pm}|^2 = X_+^{\pm} X_-^{\pm} + (X_0^{\pm})^2$, while we are also explicitly writing out the $K \leq 3$ harmonics in terms of Jacobi coordinates.

$$\mathcal{Y}_{0,0}^{0,0,0}(X) = \frac{1}{\pi^{3/2}}$$

$$\mathcal{Y}_{1,1}^{1,1,-1}(X) = \frac{\sqrt{\tfrac{3}{2}} X_+^+}{\pi^{3/2} R} = \frac{\sqrt{\tfrac{3}{2}} (\lambda_1 + i(\lambda_2 + \rho_1 + i\rho_2))}{\pi^{3/2} \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2}}$$

$$\mathcal{Y}_{1,1}^{2,0,0}(X) = \frac{\sqrt{3}\left(X_+^- X_0^+ - X_+^+ X_0^-\right)}{\pi^{3/2} R^2} = \frac{2\sqrt{3}\left(\lambda_3(\rho_2 - i\rho_1) + i(\lambda_1 + i\lambda_2)\rho_3\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{2,2}^{2,0,0}(X) = \frac{\sqrt{3} X_+^+ X_+^-}{\pi^{3/2} R^2} = \frac{\sqrt{3}(\lambda_1 + i(\lambda_2 + \rho_1 + i\rho_2))(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{0,0}^{2,2,0}(X) = \frac{\sqrt{2}|X^+|^2}{\pi^{3/2} R^2}$$
$$= \frac{\sqrt{2}\left(2i\lambda_1\rho_1 + 2i\lambda_2\rho_2 + 2i\lambda_3\rho_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \rho_1^2 - \rho_2^2 - \rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{2,2}^{2,2,-3}(X) = \frac{\sqrt{\tfrac{3}{2}}\left(X_+^+\right)^2}{\pi^{3/2} R^2} = \frac{\sqrt{\tfrac{3}{2}}(\lambda_1 + i(\lambda_2 + \rho_1 + i\rho_2))^2}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{1,1}^{3,1,3}(X) = \frac{\sqrt{6}\left(X_+^- |X^+|^2 - \tfrac{1}{2} R^2 X_+^+\right)}{\pi^{3/2} R^3}$$
$$= \frac{\sqrt{6}\left((\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2)\left((\lambda_1 + i\rho_1)^2 + (\lambda_2 + i\rho_2)^2 + (\lambda_3 + i\rho_3)^2\right)\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)} -$$
$$\frac{\frac{\sqrt{6}}{2}(\lambda_1 + i(\lambda_2 + \rho_1 + i\rho_2))}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)}$$

$$\mathcal{Y}_{2,2}^{3,1,-5}(X) = \frac{\sqrt{5}X_+^+\left(X_+^-X_0^+ - X_+^+X_0^-\right)}{\pi^{3/2}R^3}$$

$$= \frac{2\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(\lambda_3\left(\rho_2 - i\rho_1\right) + i\left(\lambda_1 + i\lambda_2\right)\rho_3\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{3,3}^{3,1,-2}(X) = \frac{\sqrt{15}\left(X_+^+\right)^2 X_+^-}{2\pi^{3/2}R^3}$$

$$= \frac{\sqrt{15}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^2\left(\lambda_1 + i\lambda_2 - i\rho_1 + \rho_2\right)}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{1,1}^{3,3,-1}(X) = \frac{\sqrt{3}X_+^+\left|X^+\right|^2}{\pi^{3/2}R^3}$$

$$= \frac{\sqrt{3}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)\left(2i\lambda_1\rho_1 + 2i\lambda_2\rho_2 + 2i\lambda_3\rho_3 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \rho_1^2 - \rho_2^2 - \rho_3^2\right)}{\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{3,3}^{3,3,-6}(X) = \frac{\sqrt{5}\left(X_+^+\right)^3}{2\pi^{3/2}R^3} = \frac{\sqrt{5}\left(\lambda_1 + i\left(\lambda_2 + \rho_1 + i\rho_2\right)\right)^3}{2\pi^{3/2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \rho_1^2 + \rho_2^2 + \rho_3^2\right)^{3/2}}$$

$$\mathcal{Y}_{0,0}^{4,0,0}(X) = -\frac{\sqrt{3}\left(R^4 - 2\left|X^-\right|^2\left|X^+\right|^2\right)}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,0,-\sqrt{105}}(X) =$$
$$\frac{-12\sqrt{14}R^2X_+^+X_+^- + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^-\right)^2\left|X^+\right|^2 + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^+\right)^2\left|X^-\right|^2}{14\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,0,\sqrt{105}}(X) =$$
$$\frac{-12\sqrt{14}R^2X_+^+X_+^- + \sqrt{105\left(11+\sqrt{105}\right)}\left(X_+^-\right)^2\left|X^+\right|^2 + \sqrt{105\left(11-\sqrt{105}\right)}\left(X_+^+\right)^2\left|X^-\right|^2}{14\pi^{3/2}R^4}$$

$$\mathcal{Y}_{3,3}^{4,0,0}(X) = \frac{3\sqrt{5}X_+^+X_+^-\left(X_+^-X_0^+ - X_+^+X_0^-\right)}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{4,4}^{4,0,0}(X) = \frac{3\sqrt{\frac{5}{2}}\left(X_+^+\right)^2\left(X_+^-\right)^2}{2\pi^{3/2}R^4}$$

$$\mathcal{Y}_{1,1}^{4,2,2}(X) = \frac{3\left(X_+^-X_0^+ - X_+^+X_0^-\right)\left|X^+\right|^2}{\pi^{3/2}R^4}$$

$$\mathcal{Y}_{2,2}^{4,2,2}(X) = \frac{\sqrt{\frac{3}{7}} X_+^+ \left(5 X_+^- \left| X^+ \right|^2 - 2 R^2 X_+^+ \right)}{\pi^{3/2} R^4}$$

$$\mathcal{Y}_{3,3}^{4,2,-13}(X) = \frac{3 \sqrt{\frac{5}{2}} \left(X_+^+ \right)^2 \left(X_+^- X_0^+ - X_+^+ X_0^- \right)}{2 \pi^{3/2} R^4}$$

$$\mathcal{Y}_{4,4}^{4,2,-5}(X) = \frac{\sqrt{15} \left(X_+^+ \right)^3 X_+^-}{2 \pi^{3/2} R^4}$$

$$\mathcal{Y}_{0,0}^{4,4,0}(X) = \frac{\sqrt{3} \left| X^+ \right|^4}{\pi^{3/2} R^4}$$

$$\mathcal{Y}_{2,2}^{4,4,-3}(X) = \frac{3 \sqrt{\frac{5}{14}} \left(X_+^+ \right)^2 \left| X^+ \right|^2}{\pi^{3/2} R^4}$$

$$\mathcal{Y}_{4,4}^{4,4,-10}(X) = \frac{\sqrt{15} \left(X_+^+ \right)^4}{4 \pi^{3/2} R^4}$$

6 Permutation Symmetric Hyperspherical Harmonics

There is a small step remaining from obtaining the hyperspherical harmonics labelled by quantum numbers (K, Q, L, m, ν) to achieving our goal, which is to construct hyperspherical functions with well-defined values of parity $P = (-1)^K$, rotational group quantum numbers (L, m), and permutation symmetry M (mixed), S (symmetric), and A (antisymmetric).[1] In this section we clarify how to obtain the latter as linear combinations of the former.

Properties under particle permutations of the functions $\mathcal{Y}_{J,m}^{KQ\nu}(\boldsymbol{\lambda}, \boldsymbol{\rho})$ are inferred from the transformation properties of the coordinates $X_i^{\pm}$: under the transpositions (two-body permutations) $\{\mathcal{T}_{12}, \mathcal{T}_{23}, \mathcal{T}_{31}\}$ of pairs of particles (1,2), (2,3) and (3,1), the Jacobi coordinates transform as:

---

[1]The mixed symmetry representation of the S_3 permutation group being two-dimensional, there are two different state vectors (hyperspherical harmonics) in each mixed permutation symmetry multiplet, usually denoted by M_ρ and M_λ.

$$\mathcal{T}_{12} : \ \boldsymbol{\lambda} \to \boldsymbol{\lambda}, \quad \boldsymbol{\rho} \to -\boldsymbol{\rho},$$

$$\mathcal{T}_{23} : \ \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} + \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} + \frac{\sqrt{3}}{2}\boldsymbol{\lambda}, \qquad (8)$$

$$\mathcal{T}_{31} : \ \boldsymbol{\lambda} \to -\frac{1}{2}\boldsymbol{\lambda} - \frac{\sqrt{3}}{2}\boldsymbol{\rho}, \quad \boldsymbol{\rho} \to \frac{1}{2}\boldsymbol{\rho} - \frac{\sqrt{3}}{2}\boldsymbol{\lambda}.$$

That induces the following transformations of complex coordinates $X_i^{\pm}$:

$$\mathcal{T}_{12} : \ X_i^{\pm} \to X_i^{\mp},$$

$$\mathcal{T}_{23} : \ X_i^{\pm} \to e^{\pm \frac{2i\pi}{3}} X_i^{\mp}, \qquad (9)$$

$$\mathcal{T}_{31} : \ X_i^{\pm} \to e^{\mp \frac{2i\pi}{3}} X_i^{\mp}.$$

None of the quantum numbers K, L and m change under permutations of particles, whereas the values of the "democracy label" Q and multiplicity label ν are inverted under all transpositions: $Q \to -Q, \nu \to -\nu$.

Apart from the changes in labels, transpositions of two particles generally also result in the appearance of an additional phase factor multiplying the hyperspherical harmonic. For values of K, Q, L and m with no multiplicity, we readily derive (Ref. [7]) the following transformation properties of h.s. harmonics under (two-particle) particle transpositions:

$$\mathcal{T}_{12} : \ \mathcal{Y}_{L,m}^{KQ\nu} \to (-1)^{K-J} \mathcal{Y}_{L,m}^{K,-Q,-\nu},$$

$$\mathcal{T}_{23} : \ \mathcal{Y}_{L,m}^{KQ\nu} \to (-1)^{K-L} e^{\frac{2Qi\pi}{3}} \mathcal{Y}_{L,m}^{K,-Q,-\nu}, \qquad (10)$$

$$\mathcal{T}_{31} : \ \mathcal{Y}_{L,m}^{KQ\nu} \to (-1)^{K-L} e^{-\frac{2Qi\pi}{3}} \mathcal{Y}_{L,m}^{K,-Q,-\nu}.$$

There are three distinct irreducible representations of the S_3 permutation group - two one-dimensional (the symmetric S and the antisymmetric A ones) and a two-dimensional (the mixed M one). In order to determine to which representation of the permutation group any particular h.s. harmonic $\mathcal{Y}_{L,m}^{KQ\nu}$ belongs, one has to consider various cases, with and without multiplicity, see Ref. [7]; here we simply state the results of the analysis conducted therein. The following linear combinations of the h.s. harmonics,

$$\mathcal{Y}_{L,m,\pm}^{K|Q|\nu} \equiv \frac{1}{\sqrt{2}} \left(\mathcal{Y}_{L,m}^{K|Q|\nu} \pm (-1)^{K-L} \mathcal{Y}_{L,m}^{K,-|Q|,-\nu} \right). \qquad (11)$$

are no longer eigenfunctions of Q operator but are (pure sign) eigenfunctions of the transposition $\mathcal{T}_{12}$ instead:

$$\mathcal{T}_{12} : \mathcal{Y}_{L,m,\pm}^{K|Q|\nu} \to \pm \mathcal{Y}_{L,m,\pm}^{K|Q|\nu}.$$

They are the appropriate h.s. harmonics with well-defined permutation properties:

1. for $Q \not\equiv 0 \pmod 3$, the harmonics $\mathcal{Y}_{L,m,\pm}^{K|Q|\nu}$ belong to the mixed representation M, where the $\pm$ sign determines the M_ρ, M_λ component,
2. for $Q \equiv 0 \pmod 3$, the harmonic $\mathcal{Y}_{L,m,+}^{K|Q|\nu}$ belongs to the symmetric representation S and $\mathcal{Y}_{L,m,-}^{K|Q|\nu}$ belongs to the antisymmetric representation A.

The above rules define the representation of S_3 for any given h.s. harmonic.

7 Summary

In this paper we have reported on our recent construction of permutation symmetric three-body $SO(6)$ hyperspherical harmonics. In the Sect. 5 we have displayed explicit forms the harmonic functions labelled by quantum numbers K, Q, L, m and ν, postponing explanation of their derivation to [7]. In Sect. 6 we demonstrated that simple linear combinations $\mathcal{Y}_{L,m,\pm}^{K|Q|\nu}$ of these functions have well defined permutation properties. To our knowledge, this is the first time that such hyperspherical harmonics are constructed in full generality.

References

1. T.H. Gronwall, Phys. Rev. **51**, 655 (1937).
2. J.H. Bartlett, Phys. Rev. **51**, 661 (1937).
3. L.M. Delves, Nucl. Phys. **9**, 391 (1958); ibid. **20**, 275 (1960).
4. F.T. Smith, J. Chem. Phys. **31**, 1352 (1959); F.T. Smith, Phys. Rev. **120**, 1058 (1960); F.T. Smith, J. Math. Phys. **3**, 735 (1962); R.C. Whitten and F.T. Smith, J. Math. Phys. **9**, 1103 (1968).
5. Yu.A. Simonov, Sov. J. Nucl. Phys. **3**, 461 (1966) [Yad. Fiz. **3**, 630 (1966)].
6. T. Iwai, J. Math. Phys. **28**, 964, 1315 (1987).
7. I. Salom and V. Dmitrašinović, "Permutation-symmetric three-particle hyper-spherical harmonics based on the $S_3 \otimes SO(3)_{rot} \subset U(1) \otimes SO(3)_{rot} \subset U(3) \subset O(6)$ chain of algebras", in preparation (2015).
8. N. Barnea and V.B. Mandelzweig, Phys. Rev. A **41**, 5209 (1990).
9. V.A. Nikonov and J. Nyiri, International Journal of Modern Physics **A**, Vol. 29, No. 20, 1430039 (2014).
10. V. Dmitrašinović, T. Sato and M. Šuvakov , Phys. Rev. D **80**, 054501 (2009).