Generalization of the Gell-Mann decontraction formula for $sl(n,\mathbb{R})$ and its applications in affine gravity

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Abstract The Gell-Mann Lie algebra decontraction formula was proposed as an inverse to the Inonu-Wigner contraction formula. We considered recently this formula in the content of the special linear algebras sl(n), of an arbitrary dimension. In the case of these algebras, the Gell-Mann formula is not valid generally, and holds only for some particular algebra representations. We constructed a generalization of the formula that is valid for an arbitrary irreducible representation of the sl(n) algebra. The generalization allows us to explicitly write down, in a closed form, all matrix elements of the algebra operators for an arbitrary irreducible representation, irrespectively whether it is tensorial or spinorial, finite or infinite dimensional, with or without multiplicity, unitary or nonunitary. The matrix elements are given in the basis of the Spin(n) subgroup of the corresponding SL(n,R) covering group, thus covering the most often cases of physical interest. The generalized Gell-Mann formula is presented, and as an illustration some details of its applications in the Gauge Affine theory of gravity with spinorial and tensorial matter manifields are given.

1 Introduction

The Inönü-Wigner contraction [9] is a well known transformation of algebras (groups) with numerous applications in various fields of physics. Just to mention a few: contractions from the Poincaré algebra to the Galilean one; from the Heisenberg algebras to the Abelian ones of the same dimensions (a symmetry background of a transition processes from relativistic and quantum mechanics to classical mechanics); contractions in the Kaluza-Klein gauge theories framework; from (Anti-)deSitter to the Poincaré algebra; various cases involving the Virasoro and Kac-

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Moody algebras; relation of strong to weak coupling regimes of the corresponding theories; relation of geometrically curved to “less curved’ and/or flat spaces...

However, existence of a transformation (i.e. algebra deformation) inverse to the Inönü-Wigner contraction, so called the “Gell-Mann formula” [4, 7, 8, 1], is far less known. The aim of the formula is to express the elements of the starting algebra as explicitly given expressions containing elements of the contracted algebra. In this way, a relation between certain representations of the two algebras is also established. This, in turn, can be very useful since, by a rule, various properties of the contracted algebras are much easier to explore (e.g. construction of representations [10], decompositions of a direct product of representations [7], etc.).

Before we write down the Gell-Mann formula in the general case, some notation is in order. Let $\mathscr{A}$ be a symmetric Lie algebra $\mathscr{A} = \mathscr{M} + \mathscr{T}$ with a subalgebra $\mathscr{M}$ such that:

$$[\mathscr{M},\mathscr{M}] \subset \mathscr{M}, \quad [\mathscr{M},\mathscr{T}] \subset \mathscr{T}, \quad [\mathscr{T},\mathscr{T}] \subset \mathscr{M}. \tag{1}$$

Further, let $\mathscr{A}'$ be its Inönü-Wigner contraction algebra w.r.t its subalgebra $\mathscr{M}$, i.e. $\mathscr{A}' = \mathscr{M} + \mathscr{U}$, where

$$[\mathscr{M},\mathscr{M}] \subset \mathscr{M}, \quad [\mathscr{M},\mathscr{U}] \subset \mathscr{U}, \quad [\mathscr{U},\mathscr{U}] = \{0\}. \tag{2}$$

The Gell-Mann formula states that the elements $T \in \mathscr{T}$ can be in certain cases expressed in terms of the contracted algebra elements $M \in \mathscr{M}$ and $U \in \mathscr{U}$ by the following rather simple expression:

$$T = i\frac{\alpha}{\sqrt{U \cdot U}}[C_2(\mathscr{M}),U] + \sigma U. \tag{3}$$

Here, $C_2(\mathscr{M})$ and $U \cdot U$ denote the second order Casimir operators of the $\mathscr{M}$ and $\mathscr{A}'$ algebras respectively, while α is a normalization constant and σ is an arbitrary parameter. For a mathematically more strict definition, cf. [4].

Probably the main reason why this formula is not widely known – in spite of its potential versatility – is the lack of its general validity. Namely, there is a number of references dealing with the question when this formula is applicable [7, 8, 1, 16]. Apart form the case of (pseudo) orthogonal algebras where, loosely speaking, the Gell-Mann formula works very well [20], there are only some subclasses of representations when the formula can be applied [7, 8]. To make the things worse, the question of its applicability is not completely resolved.

Recently, we studied the $\overline{SL}(n,\mathbb{R})$ group cases, contracted w.r.t the maximal compact $Spin(n)$ subgroups. By $\overline{SL}(n,\mathbb{R})$ we denote the double cover of $SL(n,\mathbb{R})$. Note that there faithful spinorial representations are always infinite dimensional and physically correspond to fermionic matter. In these cases the Gel-Mann formula does not hold as a general operator expression and its validity depends heavily on the $sl(n,\mathbb{R})$ algebra representation space. An exhaustive list of the cases for which the Gell-Mann formula for $sl(n,\mathbb{R})$ algebras hold was obtained [16]. In particular, we have shown that the Gell-Mann formula is not valid for any spinorial representation, nor for any representation with nontrivial $Spin(n)$ multiplicity, rendering the Gell-Mann formula here useless for most of physical applications.

There were some attempts to generalize the Gell-Mann formula for the “decontracted” algebra operators of the complex simple Lie algebras g with respect to decomposition $g = k + ik = k_c$ [22, 11], that resulted in a form of relatively complicated polynomial expressions. Recently we have managed to obtain a generalized form of this formula, first in the concrete case of $sl(5,\mathbb{R})$ algebra, and then also in the case of $sl(n,\mathbb{R})$ algebra, for any n.

In this paper we shall consider the obtained generalized expressions and illustrate applicability of the formula in the context of affine theory of gravity. In particular, we analyze the five dimensional affine gravity models.

2 Generalized formula

The $sl(n,\mathbb{R})$ algebra operators, i.e. the $SL(n,\mathbb{R})$, $\overline{SL}(n,\mathbb{R})$ group generators, can be split into two subsets: M_{ab}, $a,b = 1,2,...,n$ operators of the maximal compact subalgebra $so(n)$ (corresponding to the antisymmetric real $n\times n$ matrices, $M_{ab} = -M_{ba}$), and the, so called, sheer operators T_{ab}, $a,b = 1,2,...,n$ (corresponding to the symmetric traceless real $n\times n$ matrices, $T_{ab} = T_{ba}$). The $sl(n,\mathbb{R})$ commutation relations, in this basis, read:

$$[M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}), \tag{4}$$

$$[M_{ab}, T_{cd}] = i(\delta_{ac}T_{bd} + \delta_{ad}T_{cb} - \delta_{bc}T_{ad} - \delta_{bd}T_{ca}), \tag{5}$$

$$[T_{ab}, T_{cd}] = i(\delta_{ac}M_{db} + \delta_{ad}M_{cb} + \delta_{bc}M_{da} + \delta_{bd}M_{ca}). \tag{6}$$

The Inönü-Wigner contraction of $sl(n,\mathbb{R})$ with respect to its maximal compact subalgebra $so(n)$ is given by the limiting procedure:

$$U_{ab} \equiv \lim_{\varepsilon\to 0}(\varepsilon T_{ab}), \tag{7}$$

which leads to the following commutation relations:

$$[M_{ab}, M_{cd}] = i(\delta_{ac}M_{bd} + \delta_{ad}M_{cb} - \delta_{bc}M_{ad} - \delta_{bd}M_{ca}) \tag{8}$$

$$[M_{ab}, U_{cd}] = i(\delta_{ac}U_{bd} + \delta_{ad}U_{cb} - \delta_{bc}U_{ad} - \delta_{bd}U_{ca}) \tag{9}$$

$$[U_{ab}, U_{cd}] = 0. \tag{10}$$

Therefore, the Inönü-Wigner contraction of $sl(n,\mathbb{R})$ gives a semidirect sum $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ algebra, where $r_{\frac{n(n+1)}{2}-1}$ is an Abelian subalgebra (ideal) of “translations” in $\frac{n(n+1)}{2} - 1$ dimensions.

The generalized Gell-Mann formula for $sl(n,\mathbb{R})$, obtained in [18], reads:

$$T_{ab}^{\sigma_2\ldots\sigma_n} = i\sum_{c=2}^{n}\frac{1}{2}[C_2(so(c)_K),U_{ab}^{(cc)}] + \sigma_c U_{ab}^{(cc)}. \tag{11}$$

Operators T_{ab} live in the space $\mathscr{L}^2(Spin(n))$ of square integrable functions over the $Spin(n)$ manifold and it is known that this space is rich enough to contain all representatives from equivalence classes of the $\overline{SL}(n,\mathbb{R})$ group, i.e. $sl(n,\mathbb{R})$ algebra representations [3]. A natural discrete orthonormal basis in this space is given by properly normalized functions of the $Spin(n)$ representation matrix elements:

$$\left\{\left|\begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix}\right\rangle \equiv \int \sqrt{dim(\{J\})}D^{\{J\}}_{\{k\}\{m\}}(g^{-1})dg\,|g\rangle\right\}, \tag{12}$$
$$\left\langle\begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix}\middle|\begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix}\right\rangle = \delta_{\{J'\}\{J\}}\delta_{\{k'\}\{k\}}\delta_{\{m'\}\{m\}},$$

where dg is an (normalized) invariant Haar measure and $D^{\{J\}}_{\{k\}\{m\}}$ are the $Spin(n)$ irreducible representation matrix elements:

$$D^{\{J\}}_{\{k\}\{m\}}(g) \equiv \left\langle\begin{matrix}\{J\}\\ \{k\}\end{matrix}\middle| R(g) \middle|\begin{matrix}\{J\}\\ \{m\}\end{matrix}\right\rangle. \tag{13}$$

Here, $\{J\}$ stands for a set of the $Spin(n)$ irreducible representation labels, while $\{k\}$ and $\{m\}$ labels enumerate the $dim(D^{\{J\}})$ representation basis vectors.

In the basis (12) sets of labels $\{J\}$ and $\{m\}$ determine transformation properties of a basis vector under the $Spin(n)$ subgroup: $\{J\}$ label irreducible representation of $Spin(n)$, while numbers $\{m\}$ label particular vector within that representation. The set of parameters $\{k\}$ serve to enumerate $Spin(n)$ multiplicity of representation $\{J\}$ within the given representation of $\overline{SL}(n,\mathbb{R})$. These parameters $\{k\}$ are mathematically related to the left action of $Spin(n)$ subgroup in representation space $\mathscr{L}^2(Spin(n))$.

Operators $U_{ab}^{(cc)}$ appearing in (11) are concrete (normalized) representations (in $\mathscr{L}^2(Spin(n))$ space) of the Inönü-Wigner contractions of shear generators T_{ab}. In basis (12) these operators act in the following way:

$$\left\langle\begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix}\middle| U_{ab}^{(cd)} \middle|\begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix}\right\rangle = \sqrt{\tfrac{dim(\{J\})}{dim(\{J'\})}}C^{\{J\}\square\square\{J'\}}_{\{k\}(cd)\{k'\}}C^{\{J\}\square\square\{J'\}}_{\{m\}(ab)\{m'\}}, \tag{14}$$

where $\square\square$ denotes $Spin(n)$ representation that corresponds to second order symmetric tensors (shear generators, as well as their Inönü-Wigner contractions, transform in this way w.r.t. $Spin(n)$ subgroup) and C stands for Clebsch-Gordan coefficients of $Spin(n)$.

In (11) we also used notation $C_2(so(c)_K) \equiv \frac{1}{2}\sum_{a,b=1}^{c}(K_{ab})^2$, where K_{ab} are generators of $Spin(n)$ group left action in basis (12). These operators behave exactly as the rotation generators M_{ab}, but, instead of acting on right-hand $\{m\}$ indices, they act on the lower left-hand side indices $\{k\}$ that label multiplicity:

$$\left\langle \begin{matrix}\{J'\}\\ \{k'\}\end{matrix}\{m'\} \middle| K_{ab} \middle| \begin{matrix}\{J\}\\ \{k\}\end{matrix}\{m\} \right\rangle = \delta_{\{J'\}\{J\}}\delta_{\{m'\}\{m\}}\sqrt{C_2(\{J\})}\, C^{\{J\}\ \begin{smallmatrix}\square\\ \square\end{smallmatrix}\ \{J'\}}_{\{k\}\ (ab)\ \{k'\}}\,. \tag{15}$$

Finally, the set of $n-1$ indices $\sigma_2, \sigma_3, \ldots \sigma_n$ in (11) label the particular representation of the $\overline{SL}(n,\mathbb{R})$. The formula (11) covers all cases: infinite and finite dimensional representations, spinorial and tensorial, with and without multiplicity, unitary and non unitary.

We note that the term $c = n$ in (11) is, essentially, the original Gell-Mann formula, since $C_2(so(n)_K) = C_2(so(n)_M)$. The rest of the terms can be seen as necessary corrections securing the formula validity in the entire representation space. The additional terms vanish for some particular representations thus yielding the original formula.

An immediate mathematical benefit of the generalized formula is the expression for matrix elements of shear generators in basis (12) [18]:

$$\begin{aligned} &\left\langle \begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix} \middle| T_{ab} \middle| \begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix} \right\rangle = \tfrac{i}{2}\sqrt{\tfrac{dim(\{J\})}{dim(\{J'\})}}\, C^{\{J\}\ \square\square\ \{J'\}}_{\{m\}\ ab\ \{m'\}} \\ &\times \textstyle\sum_{c=2}^{n} \sqrt{\tfrac{c-1}{c}} \Big(C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \tilde{\sigma}_c \Big) C^{\{J\}\ (\square\square)^{n-c+1}\ \{J'\}}_{\{k\}\ \ (0)^{c-2}\ \ \{k'\}}\,. \end{aligned} \tag{16}$$

In order to demonstrate application of this result in the context of five dimensional affine gravity models, we introduce a concrete $n = 5$ adapted notation (for all $n = 5$ notation we adhere to that of our paper [17]). As a basis for $Spin(5)$ representations we pick vectors:

$$\left\{ \left| \begin{matrix}\overline{J}_1 & \overline{J}_2\\ J_1 & J_2\\ m_1 & m_2\end{matrix} \right\rangle, \overline{J}_i = 0, \frac{1}{2}, \ldots; \overline{J}_1 \geq \overline{J}_2; m_i = -J_i, \ldots J_i \right\}. \tag{17}$$

with respect to decomposition $so(5) \supset so(4) = so(3) \oplus so(3)$. Basis of $\overline{SL}(5,\mathbb{R})$ representation space, corresponding to (12) is then given by vectors:

$$\left\{ \left| \begin{matrix}\overline{J}_1 & \overline{J}_2 & & \\ K_1 & K_2 & J_1 & J_2\\ k_1 & k_2 & m_1 & m_2\end{matrix} \right\rangle \right\}. \tag{18}$$

The reduced matrix elements of the $sl(5,\mathbb{R})$ shear (noncompact) operators, derived from an alternative form of Gell-Mann formula that we have given in the paper [17], read:

$$
\begin{aligned}
&\left\langle \begin{smallmatrix} \overline{J}'_1\,\overline{J}'_2 \\ K'_1 K'_2 \\ k'_1\, k'_2 \end{smallmatrix} \right\| T \left\| \begin{smallmatrix} \overline{J}_1\,\overline{J}_2 \\ K_1 K_2 \\ k_1\, k_2 \end{smallmatrix} \right\rangle = \sqrt{\tfrac{dim(\overline{J}_1,\overline{J}_2)}{dim(\overline{J}'_1,\overline{J}'_2)}} \\
&\times \Bigg(\Big(\sigma_1 + i\sqrt{\tfrac{4}{5}} (\overline{J}'_1(\overline{J}'_1+2) + \overline{J}'_2(\overline{J}'_2+1) - \overline{J}_1(\overline{J}_1+2) - \overline{J}_2(\overline{J}_2+1)) \Big) C^{\begin{smallmatrix} \overline{J}_1\overline{J}_2 & \overline{1}\,\overline{1} & \overline{J}'_1\overline{J}'_2 \\ K_1K_2 & 00 & K'_1K'_2 \\ k_1k_2 & 00 & k'_1k'_2 \end{smallmatrix}} \\
&+ i\big(\sigma_2 + K'_1(K'_1+1) + K'_2(K'_2+1) - K_1(K_1+1) - K_2(K_2+1)\big) C^{\begin{smallmatrix} \overline{J}_1\overline{J}_2 & \overline{1}\,\overline{1} & \overline{J}'_1\overline{J}'_2 \\ K_1K_2 & 11 & K'_1K'_2 \\ k_1k_2 & 00 & k'_1k'_2 \end{smallmatrix}} \qquad (19) \\
&- i(\delta_1 + k_1 - k_2) C^{\begin{smallmatrix} \overline{J}_1\overline{J}_2 & \overline{1}\,\overline{1} & \overline{J}'_1\overline{J}'_2 \\ K_1K_2 & 11 & K'_1K'_2 \\ k_1k_2 & 1\,{-1} & k'_1k'_2 \end{smallmatrix}} - i(\delta_1 - k_1 + k_2) C^{\begin{smallmatrix} \overline{J}_1\overline{J}_2 & \overline{1}\;\overline{1} & \overline{J}'_1\overline{J}'_2 \\ K_1K_2 & 1\;1 & K'_1K'_2 \\ k_1k_2 & {-1}\,1 & k'_1k'_2 \end{smallmatrix}} \\
&+ i(\delta_2 + k_1 + k_2) C^{\begin{smallmatrix} \overline{J}_1\overline{J}_2 & \overline{1}\,\overline{1} & \overline{J}'_1\overline{J}'_2 \\ K_1K_2 & 11 & K'_1K'_2 \\ k_1k_2 & 11 & k'_1k'_2 \end{smallmatrix}} + i(\delta_2 - k_1 - k_2) C^{\begin{smallmatrix} \overline{J}_1\overline{J}_2 & \overline{1}\;\overline{1} & \overline{J}'_1\overline{J}'_2 \\ K_1K_2 & 1\;1 & K'_1K'_2 \\ k_1k_2 & {-1}{-1} & k'_1k'_2 \end{smallmatrix}} \Bigg),
\end{aligned}
$$

where $dim(\overline{J}_1,\overline{J}_2) = (2\overline{J}_1 - 2\overline{J}_2 + 1)(2\overline{J}_1 + 2\overline{J}_2 + 3)(2\overline{J}_1 + 2)(2\overline{J}_2 + 1)/6$ is the dimension of the $so(5)$ irreducible representation characterized by $(\overline{J}_1,\overline{J}_2)$. In this notation, $\overline{SL}(5,\mathbb{R})$ irreducible representations are labelled by parameters $\sigma_1,\sigma_2,\delta_1$ and δ_2, that appear in the formula (19).

3 Gauge Affine action

The space-time symmetry of the affine models of gravity (prior to any symmetry breaking) is given by the General Affine Group $GA(n,\mathbb{R}) = T^n \wedge GL(n,\mathbb{R})$ (or, sometimes, by the Special Affine Group $SA(n,\mathbb{R}) = T^n \wedge SL(n,\mathbb{R})$). In the quantum case, the General Affine Group is replaced by its double cover counterpart $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$, which contains double cover of $\overline{GL}(n,\mathbb{R})$ as a subgroup. This subgroup here plays the role that Lorentz group has in the Poincaré symmetry case. Thus it is clear that knowledge of $\overline{GL}(n,\mathbb{R})$ representations is a must-know for any serious analysis of affine gravity models. On the other hand, the essential nontrivial representation determining part of the $\overline{GL}(n,\mathbb{R}) = R_+ \otimes \overline{SL}(n,\mathbb{R})$ group is its $\overline{SL}(n,\mathbb{R})$ subgroup (R_+ is subgroup of dilatations). We will make use of the $\overline{SL}(n,\mathbb{R})$ matrix elements expression in order to obtain coefficients for some of the gauge field–matter interaction vertices.

A standard way to introduce interactions into affine gravity models is by localization of the global affine symmetry $\overline{GA}(n,\mathbb{R}) = T^n \wedge \overline{GL}(n,\mathbb{R})$. Thus, quite generally, affine Lagrangian consists of a gravitational part (i.e. kinetic terms for gauge potentials) and Lagrangian of the matter fields: $L = L_g + L_m$. Gravitational part L_g is a function of gravitational gauge potentials and their derivatives, and also of the dilaton field φ (that ensures action invariance under local dilatations). In the case of the standard Metric Affine [6, 5], i.e. Gauge Affine Gravity [13], the gravitational potentials are tetrads $e^a{}_\mu$, metrics g_{ab} and affine connection $\Gamma^a_{b\mu}$, so that we can write: $L_g = L_g(e,\partial e, g, \partial g, \Gamma, \partial\Gamma, \varphi)$. More precisely, due to action invariance under local affine transformations, gravitational part of Lagrangian must be a function of the form $L_g = L_g(e,g,T,R,N,\varphi)$, where $T^a_{\mu\nu} = \partial_\mu e^a{}_\nu + \Gamma^a_{b\mu} e^b{}_\nu - (\mu \leftrightarrow \nu)$, $R^a{}_{b\mu\nu} = \partial_\mu \Gamma^a_{b\nu} + \Gamma^c_{b\mu}\Gamma^a_{c\nu} - (\mu \leftrightarrow \nu)$, $N_{\mu ab} = D_\mu g_{ab}$ are, respectively, torsion, curvature and nonmetricity. Assuming, as usual, that equations of motion are linear in second derivatives of gauge fields, we are confined to no higher than quadratic powers of the torsion, curvature and nonmetricity. Covariant derivative is of the form $D_\mu = \partial_\mu - i\Gamma_a{}^b{}_\mu Q_b{}^a$, where $Q_b{}^a$ denote generators of $\overline{GL}(n,\mathbb{R})$ group. The matter Lagrangian (assuming minimal coupling for all fields except the dilaton one) is a function of some number of affine fields ϕ^I and their covariant derivatives, together with metrics and tetrads (affine connection enters only through covariant derivative): $L_m = L_m(\phi^I, D\phi^I, e, g)$.

With all these general remarks, we will consider a class of affine Lagrangians, in arbitrary number of dimensions n, of the form:

$$
\begin{aligned}
& L(e_\mu{}^a, \partial_\nu e_\mu{}^a, \Gamma_{b\mu}{}^a, \partial_\nu \Gamma_{b\mu}{}^a, g_{ab}, \Psi_A, \partial_\nu \Psi_A, \Phi_A, \partial_\nu \Phi_A, \varphi, \partial_\nu \varphi) \; = \\
& e\Big[\varphi^2 R - \varphi^2 T^2 - \varphi^2 N^2 + \\
& \bar{\Psi} i g^{ab} \gamma_a e_b{}^\mu D_\mu \Psi + \tfrac{1}{2} g^{ab} e_a{}^\mu e_b{}^\nu (D_\mu \Phi)^+ (D_\nu \Phi) + \tfrac{1}{2} g^{ab} e_a{}^\mu e_b{}^\nu D_\mu \varphi D_\nu \varphi \Big]. \qquad (20)
\end{aligned}
$$

The terms in the first row represent general gravitational part of the Lagrangian, that is invariant w.r.t. affine transformations (dilatational invariance is obtained with the aid of field φ, of mass dimension $n/2-1$). Here T^2 and N^2 stand for linear combination of terms quadratic in torsion and nonmetricity, respectively, formed by irreducible components of these fields. For the scope of this paper, we need not fix these terms any further. This is a general form of gravitational kinetic terms, invariant for an arbitrary space-time dimension $n \geq 3$.

The Lagrangian matter terms, invariant w.r.t. the local $\overline{GA}(n,\mathbb{R})$, $n \geq 3$, transformations, are written in the second row. The field Ψ denotes a spinorial $\overline{GL}(n,\mathbb{R})$ field – components of that field transform under some appropriate spinorial $\overline{GL}(n,\mathbb{R})$ irreducible representations. All spinorial $\overline{GL}(n,\mathbb{R})$ representations are necessarily infinite dimensional [12], and thus the field Ψ will have infinite number of components. The concrete spinorial irreducible representation of field Ψ is given by a set of $n-1$ $\overline{SL}(n,\mathbb{R})$ labels $\{\sigma_c^\Psi\}$ together with the dilatation charge d_Ψ. The field Φ is a representative of a tensorial $\overline{GL}(n,\mathbb{R})$ field, transforming under a tensorial $\overline{GL}(n,\mathbb{R})$ representation (i.e. one transforming w.r.t. single-valued representation of the $SO(n)$ subgroup) labelled by parameters $\{\sigma_c^\Phi\}$ and d_Φ. Since, as it is briefly argued later, the noncompact $\overline{SL}(n-1,\mathbb{R})$ affine subgroup is to be represented unitarily, the tensorial field Φ is also to transform under an infinite-dimensional representation and to have an infinite number of components. The remaining dilaton field φ is scalar with respect to $\overline{SL}(n,\mathbb{R})$ subgroup, and thus has only one component.

Interaction of affine connection with matter fields is determined by terms containing covariant derivatives. We write these terms in a component notation, where the component labelling is done with respect to the physically important Lorenz $Spin(1,n-1)$ subgroup of $\overline{GL}(n,\mathbb{R})$. Such a labelling allows, in principle, to identify affine field components with Lorentz fields of models based on the Poincaré symmetry. Namely, the affine models of gravity necessarily imply existence of some symmetry breaking mechanism that reduces the global symmetry to the Poincaré one, reflecting the subgroup structure $T^n \wedge \overline{SO}(1,n-1) \subset T^n \wedge \overline{GL}(n,\mathbb{R})$. Therefore, we consider the field Ψ (and similarly for Φ field) as a sum of its Lorentz components:

$$\sum_{\substack{\{J\}\\ \{k\}\{m\}}} \Psi^{\{J\}}_{\{k\}\{m\}} \big|^{\{J\}}_{\{k\}\{m\}}\big\rangle.$$

The interaction term connecting fields g^{cd}, $e_d^{\ \mu}$, Γ^{ab}_μ, $\bar{\Psi}^{\{J\}}_{\{k\}\{m\}}$, $\Psi^{\{J'\}}_{\{k'\}\{m'\}}$ is now:

$$g^{cd} e_d^{\ \mu} \Gamma^{ab}_\mu \bar{\Psi}^{\{J\}}_{\{k\}\{m\}} \Psi^{\{J'\}}_{\{k'\}\{m'\}} \sum_{\substack{\{J''\}\\ \{k''\}\{m''\}}} \big\langle^{\{J\}}_{\{k\}\{m\}}\big|\gamma_c\big|^{\{J''\}}_{\{k''\}\{m''\}}\big\rangle \big\langle^{\{J''\}}_{\{k''\}\{m''\}}\big|Q_{ab}\big|^{\{J'\}}_{\{k'\}\{m'\}}\big\rangle, \quad (21)$$

while the interaction of tensorial field with connection is given by:

$$-\tfrac{i}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_\nu \partial_\mu \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \Phi^{\{J'\}}_{\{k'\}\{m'\}} \big\langle^{\{J\}}_{\{k\}\{m\}}\big|Q_{ab}\big|^{\{J'\}}_{\{k'\}\{m'\}}\big\rangle + \quad (22)$$

$$\tfrac{i}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_\nu \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \partial_\mu \Phi^{\{J'\}}_{\{k'\}\{m'\}} \big\langle^{\{J'\}}_{\{k'\}\{m'\}}\big|Q_{ab}\big|^{\{J\}}_{\{k\}\{m\}}\big\rangle^* + \quad (23)$$

$$\tfrac{1}{2} g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma^{ab}_\mu \Gamma^{a'b'}_\nu \Phi^{\dagger\{J\}}_{\{k\}\{m\}} \partial_\mu \Phi^{\{J'\}}_{\{k'\}\{m'\}} \cdot$$

$$\sum_{\substack{\{J''\}\\ \{k''\}\{m''\}}} \big\langle^{\{J\}}_{\{k\}\{m\}}\big|Q_{ab}\big|^{\{J''\}}_{\{k''\}\{m''\}}\big\rangle \big\langle^{\{J''\}}_{\{k''\}\{m''\}}\big|Q_{a'b'}\big|^{\{J'\}}_{\{k'\}\{m'\}}\big\rangle. \quad (24)$$

The scalar dilaton field interact only with the trace of affine connection:

$$\tfrac{1}{2} g^{ab} e_a^{\ \mu} e_b^{\ \nu} (\partial_\mu - i\Gamma_a{}^a{}_\mu d_\varphi)\varphi(\partial_\nu - i\Gamma_a{}^a{}_\nu d_\varphi)\varphi, \quad (25)$$

where d_φ denotes dilatation charge of φ field.

In the above interaction terms we note an appearance of matrix elements of $\overline{GL}(n,\mathbb{R})$ generators, written in a basis of the Lorenz subgroup $Spin(1,n-1)$. The dilatation generator (that is, the trace Q^a_a) acts merely as multiplication by dilatation charge, so it is really the $\overline{SL}(n,\mathbb{R})$ matrix elements that should be calculated. (An infinite dimensional generalization of Dirac's gamma matrices also appear in the term (21); more on these matrices can be found in papers of Šijački [21].) However, before presenting examples of the matrix elements evaluations, and thus calculations of the vertex coefficients, it is due to note that the correct physical interpretation of the $\overline{SL}(n,\mathbb{R})$ representations requires these representations to be unitary w.r.t. its $\overline{SL}(n-1,\mathbb{R})$ subgroup and to be nonunitary w.r.t. its lorentz-like $Spin(1,n-1)$ subgroup. It turns out that these requirements can be properly satisfied by making use of the so called deunitarizing automorphism [12].

4 Gauge Affine symmetry vertex coefficients evaluation

Now we return to evaluation of vertex coefficients for interaction between various Lorentz components of the $\overline{GL}(n,\mathbb{R})$ fields. The nontrivial part is to find matrix elements of $\overline{SL}(n,\mathbb{R})$ shear generators in expressions (21)-(24), and, to do that in $n=5$ case we will use expression (19). However, this formula is given in the basis of the compact $Spin(n)$ subgroup, and not in the basis of the physically important Lorentz group $Spin(1,n-1)$. On the other hand, it turns out that taking into account deunitarizing automorphism exactly amounts to keeping reduced matrix element from (16) and replacing the remaining Clebsch-Gordan coefficient of the $Spin(n)$ group by the corresponding coefficient of the Lorenz group $Spin(1,n-1)$ [15].

As the first example, let the field Φ correspond to an unitary multiplicity free $\overline{SL}(5,\mathbb{R})$ representation, defined by labels $\sigma_2 = -4, \delta_1 = \delta_2 = 0$, with σ_1 arbitrary real. The representation space is spanned by vectors (18) satisfying $\overline{J}_1 = \overline{J}_2 = \overline{J} \in \mathbb{N}_0 + \frac{1}{2}; K_1 = K_2 = 0; J_1 = J_2 = J \leq \overline{J}$. This is a simplest class of multiplicity free representations that is unitary assuming usual scalar product. If we denote $\Phi^a, a = 1\ldots 5$ the five Φ components with $\overline{J}_1 = \overline{J}_2 = \frac{1}{2}$ (in this sense Φ^a corresponds to a Lorenz 5-vector) then the interaction vertex (22) connecting fields $\Phi^{a\dagger}$, $\partial_\mu \Phi^d$ and affine shear connection Γ_ν^{bc} is:

$$\frac{i}{2} g^{ef} e_e^{\ \mu} e_f^{\ \nu} \Phi^{a\dagger} \Gamma_\nu^{bc} \partial_\mu \Phi^d \frac{\sqrt{5}}{14}\, \sigma_1 (\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{n}\eta_{ad}\eta_{bc}). \tag{26}$$

To obtain this result we used an easily derivable formula for Clebsch-Gordan coefficient connecting Lorentz vector and symmetric second order Lorenz tensor representations:

$$C^{L\square\square\square\square}_{a\ (bc)\ d} = \sqrt{\frac{n}{2(n+2)(n-1)}} (\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{n}\eta_{ad}\eta_{bc}), \tag{27}$$

where we labelled $Spin(1,n-1)$ irreducible representations by Young diagrams, as in [18]. More importantly, we also used value of the reduced matrix element:

$$\left\langle \begin{smallmatrix} \overline{\frac{1}{2}\,\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \right\| Q \left\| \begin{smallmatrix} \overline{\frac{1}{2}\,\frac{1}{2}} \\ 0\,0 \\ 0\,0 \end{smallmatrix} \right\rangle = \sqrt{\frac{2}{7}}\sigma_1, \tag{28}$$

that we obtained by using formula (19) (based on this formula, a Mathematica program was generated that directly calculates $sl(5,\mathbb{R})$ matrix elements [15], taking into account $Spin(5)$ Clebsch-Gordan coefficients found in [19]).

It is no more difficult to obtain coefficients of the vertices of the form (24). Lagrangian term (24) connecting Lorenz 5-vector Φ components Φ_5, $\Phi_5^\dagger$ and affine connection component $\Gamma_{(55)\mu}$ is:

$$\frac{1}{15} \left(\sigma_1^2 - 25\right) g^{cd} e_c^{\ \mu} e_d^{\ \nu} \Gamma_\mu^{55} \Gamma_\nu^{55} \Phi_5^\dagger \partial_\mu \Phi_5. \tag{29}$$

Next we will consider an example where Φ field corresponds to a representation with multiplicity. Let us, again, consider 5-vector component $\overline{J}_1 = \overline{J}_2 = \frac{1}{2}$ of Φ, only this time without any restriction to the values of $\sigma_1, \sigma_2, \delta_1, \delta_2$. In general, this will correspond to a representation with non trivial multiplicity. Quantum numbers $\{k\} = (K_1, K_2, k_1, k_2)$, that label multiplicity, now can take values : $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$ and $(0,0,0,0)$. Therefore, this *a priori* corresponds to 5 observable 5-vector fields, differentiated by the $\{k\}$ values, and these five vector fields mutually interact by gravitational interaction. Part of the Lagrangian term (22), responsible for this interaction, has the form:

$$\frac{i}{2} g^{ef} e_e^{\ \mu} e_f^{\ \nu} \Phi^{a\dagger}_{\{k'\}} \Gamma_\nu^{bc} \partial_\mu \Phi^{d}_{\{k\}} \left\langle \begin{smallmatrix} \overline{\frac{1}{2}} & \overline{\frac{1}{2}} \\ K'_1 & K'_2 \\ k'_1 & k'_2 \end{smallmatrix} \right\| Q \left\| \begin{smallmatrix} \overline{\frac{1}{2}} & \overline{\frac{1}{2}} \\ K_1 & K_2 \\ k_1 & k_2 \end{smallmatrix} \right\rangle \frac{\sqrt{5}}{\sqrt{56}} (\eta_{ab}\eta_{dc} + \eta_{ac}\eta_{db} - \frac{2}{5}\eta_{ad}\eta_{bc}). \tag{30}$$

The reduced matrix element is obtained from the generalized Gell-Mann formula:

$$\begin{aligned}
&\left\langle \begin{smallmatrix} \overline{\frac{1}{2}} & \overline{\frac{1}{2}} \\ \frac{1}{2} & \frac{1}{2} \\ k'_1 & k'_2 \end{smallmatrix} \right\| Q \left\| \begin{smallmatrix} \overline{\frac{1}{2}} & \overline{\frac{1}{2}} \\ \frac{1}{2} & \frac{1}{2} \\ k_1 & k_2 \end{smallmatrix} \right\rangle = \\
&\frac{1}{4\sqrt{14}} \Bigg(-2\sigma_1 C_3{}^{\frac{1}{2}\,0\,\frac{1}{2}}_{k_1\,0\,k'_1} C_3{}^{\frac{1}{2}\,0\,\frac{1}{2}}_{k_2\,0\,k'_2} + 15\sigma_2 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,0\,k'_1} C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,0\,k'_2} - \\
&-15 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,-1\,k'_1} \Big((k_1 + k_2 - \delta_2) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,-1\,k'_2} + (-k_1 + k_2 + \delta_1) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,1\,k'_2} \Big) \\
&-15 C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_1\,1\,k'_1} \Big((k_1 - k_2 + \delta_1) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,-1\,k'_2} - (k_1 + k_2 + \delta_2) C_3{}^{\frac{1}{2}\,1\,\frac{1}{2}}_{k_2\,1\,k'_2} \Big) \Bigg), \\
&\left\langle \begin{smallmatrix} \overline{\frac{1}{2}\ \frac{1}{2}} \\ 0\ 0 \\ 0\ 0 \end{smallmatrix} \right\| Q \left\| \begin{smallmatrix} \overline{\frac{1}{2}} & \overline{\frac{1}{2}} \\ K_1 & K_2 \\ k_1 & k_2 \end{smallmatrix} \right\rangle = 0, \qquad \left\langle \begin{smallmatrix} \overline{\frac{1}{2}\ \frac{1}{2}} \\ 0\ 0 \\ 0\ 0 \end{smallmatrix} \right\| Q \left\| \begin{smallmatrix} \overline{\frac{1}{2}\ \frac{1}{2}} \\ 0\ 0 \\ 0\ 0 \end{smallmatrix} \right\rangle = \sqrt{\tfrac{2}{7}}\, \sigma_1,
\end{aligned} \tag{31}$$

where C_3 denotes an usual $Spin(3)$ Clebsch-Gordan coefficient.

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