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VALIDITY OF THE GELL-MANN FORMULA FOR $sl(n,\mathbb{R})$ AND $su(n)$ ALGEBRAS

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The so-called Gell-Mann formula, a prescription designed to provide an inverse to the Inönü–Wigner Lie algebra contraction, has a great versatility and potential value. This formula has no general validity as an operator expression. The question of applicability of Gell-Mann's formula to various algebras and their representations was only partially treated. The validity constraints of the Gell-Mann formula for the case of $sl(n,\mathbb{R})$ and $su(n)$ algebras are clarified, and the complete list of representations spaces for which this formula applies is given. Explicit expressions of the $sl(n,\mathbb{R})$ generators matrix elements are obtained for all these cases in a closed form by making use of the Gell-Mann formula.

Keywords: Gell-Mann decontraction formula; Lie algebra contraction; SL(n) representations; SU(n) representations.

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1. Introduction

The Gell-Mann formula [1–5] is a prescription aimed to serve as an "inverse" to the Inönü–Wigner contraction [6]. Let a symmetric Lie algebra $\mathcal{A} = \mathcal{M} + \mathcal{T}$:

$$[\mathcal{M},\mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M},\mathcal{T}] \subset \mathcal{T}, \quad [\mathcal{T},\mathcal{T}] \subset \mathcal{M}, \tag{1}$$

and its Inönü–Wigner contraction $\mathcal{A}' = \mathcal{M} + \mathcal{U}$:

$$[\mathcal{M},\mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M},\mathcal{U}] \subset \mathcal{U}, \quad [\mathcal{U},\mathcal{U}] = \{0\}, \tag{2}$$

be given. Following a definition that is mathematically less strict but closer to the original formulation, the Gell-Mann formula states that elements $T_\mu \in \mathcal{T}$ can be constructed as the following simple function of the contracted algebra operators $U_\mu \in \mathcal{U}$ and $M_\nu \in \mathcal{M}$:

$$T_\mu = i\frac{\alpha}{\sqrt{U_\nu U^\nu}}[C^2(\mathcal{M}), U_\mu] + i\sigma U_\mu. \tag{3}$$

Here, $C^2(\mathcal{M})$ and $U_\nu U^\nu$ denote the (positive definite) second-order Casimir operators of the $\mathcal{M}$ and $\mathcal{A}'$ algebras, respectively, while α is a normalization constant and σ is an arbitrary parameter. (For a mathematically more strict definition, cf. [1].) The formula was, to our knowledge, first introduced by Dothan and Ne'eman [5], and was advocated by Hermann.

This formula is of a great potential value due to its simplicity and the fact that many aspects of the representation theory are much simpler for the contracted groups/algebras (e.g. construction of representations [7], decompositions of a direct product of representations [2], etc.). However, this formula is valid, on the algebraic level, only in the case of contractions from $\mathcal{A} = so(m+1,n)$ and/or $\mathcal{A} = so(m,n+1)$ to $\mathcal{A}' = iso(m,n)$, with $\mathcal{M} = so(m,n)$ [8, 9]. Moreover, apart from this, the formula is also partially applicable in a broad class of other contractions provided one restricts to some classes of the algebra representations. The validity of Gell-Mann's formula in a weak sense, when an algebra representation requirement is imposed as well, was investigated long ago by Hermann [2, 3]. A partial set of classes of the algebra representations for which the Gell-Mann formula holds is listed [3]. No attempt to make this list exhaustive is made, deliberately concentrating "on what seems to be the simplest situation". This analysis excluded, from the very beginning, the cases of representations where the little group (in Wigner's terminology) is nontrivially represented, not claiming a complete answer even then.

The Gell-Mann formula is especially valuable as a tool in the problem of finding all unitary irreducible representations of the $sl(n,\mathbb{R})$ algebras in spaces over the $SO(n)$ and/or $\mathrm{Spin}(n)$ groups generated by their $so(n)$ subalgebras (applying the formula to contraction of $sl(n,\mathbb{R})$ with respect to subalgebra $so(n)$). Finding representations in the basis of the maximal compact subgroup $SO(n)$ of the $SL(n,\mathbb{R})$ group is mathematically superior, and it suites well various physical applications in particular in nuclear and particle physics, gravity [10], physics of p-branes [11] etc. As an example consider a gauge theory based on the Affine spacetime symmetry $SA(n,\mathbb{R}) = T_n \wedge \overline{SL}(n,\mathbb{R})$; bar denoting the covering group. The gauge covariant derivative, D_α, $\alpha = 0,1,\ldots,n-1$, as acting on an Affine matter field $\Psi(x)$, is given by,

$$D_\alpha \Psi_A(x) = \left(\partial_\alpha - i\Gamma^{ab}_\alpha(x)\,(Q_{ab})^B_A\right)\Psi_B(x), \quad Q_{ab} \in sl(n,\mathbb{R}),$$

where $\Gamma^{ab}_\alpha(x)$ are the $sl(n,\mathbb{R})$ connections, and A, B enumerate the matter field components. The matter-gravity vertices require the knowledge of the $sl(n,\mathbb{R})$ operators matrix elements $(Q_{ab})^B_A$ in the Hilbert space of the matter field components $\{\Psi_A(x)\}$. Operators Q_{ab} naturally split into antisymmetric generators of the compact $SO(n)$ subgroup $M_{ab} = Q_{[ab]}$ and the symmetric, so-called, sheer generators $T_{ab} = Q_{\{ab\}}$. While the matrix elements of the former are well-known, it is generally difficult task to find, for a given $sl(n,\mathbb{R})$ representation, the matrix elements of the latter. In particular, for a generic spinorial $\overline{SL}(n,R)$ matter field, an explicit form of the matrix elements of the $sl(n,\mathbb{R})$ generators, with respect to the Lorentz-like $\mathrm{Spin}(1,n-1)$ subgroup, for infinite-dimensional representation corresponding to the

Ψ field is required. The Gell-Mann formula, in principle, offers a powerful method to describe various representation details (including the matrix elements) in a simple closed analytic form.

Therefore, two obvious questions arise in this context: (i) What is the scope of applicability of the Gell-Mann formula in the $sl(n,\mathbb{R})$ case (i.e. what is the subset of irreducible representations that can be obtained using the formula)? and (ii) Can the formula be somehow generalized, as to account for all $sl(n,\mathbb{R})$ irreducible representations?

Recently [12], we have successfully answered the second question by obtaining a generalized formula of a form similar to that of (3):

$$T_{ab}^{\sigma_2,\ldots,\sigma_n} = i\sum_{m=2}^{n} \frac{1}{2}[C^2(so(m)_K), U_{ab}^{(mm)}] + \sigma_m U_{ab}^{(mm)}, \tag{4}$$

where $C^2(so(m)_K)$ is the second-order Casimir of the $so(m)$ left action subalgebra, $U_{ab}^{(mm)}$ are specifically chosen representations of the Abelian part of the contracted algebra and $\sigma_2,\sigma_3,\ldots,\sigma_n$ are the $sl(n,\mathbb{R})$ representation labels (for more details cf. [12], and a previous analysis [13] of the $n=5$ case). This generalized Gell-Mann formula expression for the noncompact "shear" generators T_{ab} holds for all cases of $sl(n,\mathbb{R})$ irreducible representations.

However, the above solution of the second problem in no way diminishes importance of the first one — i.e. when is the original formula applicable. Apart from mathematical curiosity, this question is of great value since, despite the simple form of the generalization, the original formula still has a number of advantages in applications. First, the summation that appears in the generalized formula certainly renders any practical calculation more complex. More importantly, the generalized Gell-Mann formula is no longer solely a Lie algebra operator expression, but an expression in terms of representation dependant operators $U_{ab}^{(lm)}$ and the so called "left action rotation generators" K_{ab} appearing through $C^2(so(m)_K) = \frac{1}{2}\sum_{a,b=1}^{m}(K_{ab})^2$. Therefore, it is still of a great value to know precisely when the original formula can be applied.

The aim of this paper is to clarify the matters of the original Gell-Mann formula applicability for the class of $sl(n,\mathbb{R})$ algebras contracted with respect to their $so(n)$ maximal compact subalgebras. Note, that owing to a direct connection of the $sl(n,\mathbb{R})$ and $su(n)$ algebras, the conclusions readily convey to the latter case.

In the following, we stick to the notation and mathematical framework of the paper [12]. We briefly restate the minimal due set of these preliminaries in the appendix.

2. Validity of the Gell-Mann Formula

The Gell-Mann formula validity problem is due to the fact that the third commutation relation of (1) is not *a priori* satisfied as an operator relation when the algebra elements are given by expressions (3). In the $sl(n,\mathbb{R})$ case, the $\mathcal{T}$ subspace is spanned by $\frac{1}{2}n(n+1)-1$ shear generators T_μ. These operators transform as a second-order symmetric tensor with respect to Spin(n) subgroup, and, in the Cartesian basis, satisfy:

$$[T_{ab}, T_{cd}] = i(\delta_{ac}M_{db} + \delta_{ad}M_{cb} + \delta_{bc}M_{da} + \delta_{bd}M_{ca}). \tag{5}$$

Generally, we use indices from the beginning of the Latin alphabet for Cartesian basis and the Greek indices whenever we want to stress that expression is basis-independent.

To investigate circumstances in which this relation holds, we evaluate the commutator of two shear generators in the framework given in the appendix. In that framework, the Gell-Mann formula (3) reads:

$$T_\mu = i\alpha[C^2(so(n)_K), D^{\Box\Box}_{w\mu}] + i\sigma D^{\Box\Box}_{w\mu}, \tag{6}$$

where $C^2(so(n)_K) = \frac{1}{2}\sum_{a,b=1}^{n}(K_{ab})^2$. By making use of this formula, a few algebraic relations and some properties of the Wigner D-functions, after some algebra we obtain:

$$\begin{aligned}
[T_\mu, T_\nu] &= -2\alpha^2\big[K_{\{i}, [K_{j\}}, D^{\Box\Box}_{w\nu}]\big][K_j, D^{\Box\Box}_{w\mu}]K_i - (\mu \leftrightarrow \nu)\\
&= -\alpha^2\sum_J\sum_{\lambda,\lambda'}\Big(C^{\Box\Box\,\Box\Box\,J}_{\ \mu\ \ \ \nu\ \ \lambda} - C^{\Box\Box\,\Box\Box\,J}_{\ \nu\ \ \ \mu\ \ \lambda}\Big)\\
&\quad\times\Big(2\big(C^2(J) - 2C^2(\Box\Box)\big)\Big\langle\Big\langle {J \atop \lambda'}\Big|1\otimes K_i\Big|{\Box\Box \atop w}\Big\rangle\Big|{\Box\Box \atop w}\Big\rangle\\
&\quad + \Big\langle\Big\langle {J \atop \lambda'}\Big|[1\otimes K_i, C^2(K_{(I+II)})]\Big|{\Box\Box \atop w}\Big\rangle\Big|{\Box\Box \atop w}\Big\rangle\Big)D^J_{\lambda'\lambda}K_i,
\end{aligned} \tag{7}$$

where a summation over repeated Latin indices i and j that label the K generators in any real basis (such that $C^2(K) = K_iK_i$ is assumed). The $C^2(K_{(I+II)})$ operator here denotes the second-order Casimir operator acting in the tensor product of two $\Box\Box$ representations, i.e. $C^2(K_{(I+II)}) = \sum_i(K_i\otimes 1 + 1\otimes K_i)^2$.

The summation index J in (7) runs over all irreducible representations of the Spin(n) group that appear in the tensor product $\Box\Box\otimes\Box\Box$, and λ, λ' count the vectors of these representations. Since all irreducible representations terms, apart those for which the Clebsch–Gordan coefficient $C^{\Box\Box\,\Box\Box\,J}_{\ \mu\ \ \ \nu\ \ \lambda}$ is antisymmetric with respect to $\mu\leftrightarrow\nu$ vanish, we are left with only two values that J takes: one corresponding to the antisymmetric second-order tensor $\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$ and the other one corresponding to the representation that we denote as $\begin{smallmatrix}\Box\Box\\ \Box\Box\end{smallmatrix}$. The fact that in the case of $sl(n,\mathbb{R})$ algebras, there is another representation term, in addition to $\begin{smallmatrix}\Box\\ \Box\end{smallmatrix}$, in the antisymmetric product of two $\Box\Box$ representations (i.e. representations that correspond to Abelian U operators), is *in the root of the Gell-Mann formula validity problem.* Note that in the case of the $so(m+1,n)\to iso(m,n)$, i.e. $so(m,n+1)\to iso(m,n)$ contractions, where the Gell-Mann formula works on the algebraic level, the contracted U operators transform as $\Box$ and the antisymmetric product of two such representations certainly belongs to the $\yng(1,1)$ representation (i.e. to the representation that corresponds to $\mathcal{M} = so(m,n)$ subalgebra operators).

The $so(n)$ Casimir operator values satisfy $C^2(\yng(2,2)) = 2C^2(\yng(2)) = 4n$, implying that one of the two terms vanishes in (7) when $J = \yng(2,2)$, leaving us with:

$$\begin{aligned}\frac{1}{2\alpha^2}[T_\mu, T_\nu] = 4(n+2)&\sum_{\lambda,\lambda'} C^{\yng(2)\,\yng(2)\,\yng(1,1)}_{\mu\ \ \nu\ \ \lambda} \left\langle\left\langle \begin{smallmatrix}\yng(1,1)\\ \lambda'\end{smallmatrix} \right| 1\otimes K_i \left| \begin{smallmatrix}\yng(2)\\ w\end{smallmatrix}\right\rangle \right| \left.\begin{smallmatrix}\yng(2)\\ w\end{smallmatrix}\right\rangle D^{\yng(1,1)}_{\lambda'\lambda} K_i \\ &- \sum_{\lambda,\lambda'} C^{\yng(2)\,\yng(2)\,\yng(1,1)}_{\mu\ \ \nu\ \ \lambda} \left\langle\left\langle \begin{smallmatrix}\yng(1,1)\\ \lambda'\end{smallmatrix} \right| [1\otimes K_i, C^2(K_{(I+II)})] \left| \begin{smallmatrix}\yng(2)\\ w\end{smallmatrix}\right\rangle \right| \left.\begin{smallmatrix}\yng(2)\\ w\end{smallmatrix}\right\rangle D^{\yng(1,1)}_{\lambda'\lambda} K_i \\ &- \sum_{\lambda,\lambda'} C^{\yng(2)\,\yng(2)\,\yng(2,2)}_{\mu\ \ \nu\ \ \lambda} \left\langle\left\langle \begin{smallmatrix}\yng(2,2)\\ \lambda'\end{smallmatrix} \right| [1\otimes K_i, C^2(K_{(I+II)})] \left| \begin{smallmatrix}\yng(2)\\ w\end{smallmatrix}\right\rangle \right| \left.\begin{smallmatrix}\yng(2)\\ w\end{smallmatrix}\right\rangle D^{\yng(2,2)}_{\lambda'\lambda} K_i,\end{aligned} \tag{8}$$

where we used that $C^2(\yng(1,1)) = 2n-4$.

As the coefficient α can be adjusted freely, all that is needed for the Gell-Mann formula to be valid is that (8) is proportional to the appropriate linear combination of the Spin(n) generators, as determined by the Wigner–Eckart theorem, i.e.:

$$[T_\mu, T_\nu] \sim \sum_\lambda C^{\yng(2)\,\yng(2)\,\yng(1,1)}_{\mu\ \ \nu\ \ \lambda} M_\lambda = \sum_{\lambda,i} C^{\yng(2)\,\yng(2)\,\yng(1,1)}_{\mu\ \ \nu\ \ \lambda} D^{\yng(1,1)}_{i\lambda} K_i. \tag{9}$$

We now analyze these requirements, skipping some straightforward technical details. The third term on the right-hand side in (8), containing D functions of the representation $\yng(2,2)$, is to vanish. Since it is not possible to choose vectors w so that this term vanishes identically as an operator, the remaining possibility is to restrain the space (A.3) to some subspace $V = \{|v\rangle\} \subset \mathcal{L}^2(\mathrm{Spin}(n))$. More precisely, for this term to vanish, there must exist a subalgebra $\mathbf{L} \subset so(n)_K$, spanned by some $\{K_\alpha\}$, such that $K_\alpha \in \mathbf{L} \Rightarrow K_\alpha |v\rangle = 0$. Requiring additionally that this subspace V ought to close under an action of the shear generators, and that the first two terms of (8) ought to yield (9), we arrive at the following two necessary conditions:

(1) The algebra $\mathbf{L}$, must be a symmetric subalgebra of $so(n)$, i.e.

$$[\mathbf{L}, \mathbf{N}] \subset \mathbf{N}, \quad [\mathbf{N}, \mathbf{N}] \subset \mathbf{L}; \quad \mathbf{N} = \mathbf{L}^\perp, \tag{10}$$

(2) The vector $\left|\begin{smallmatrix}\yng(2)\\ w\end{smallmatrix}\right\rangle$ ought to be invariant under the L subgroup action (subgroup of Spin(n) corresponding to $\mathbf{L}$), i.e.

$$K_\alpha \in \mathbf{L} \Rightarrow K_\alpha \left|\begin{smallmatrix}\yng(2)\\ w\end{smallmatrix}\right\rangle = 0. \tag{11}$$

The second necessary condition is satisfied by requiring that the space V is given by Spin$(n)/L$. In Wigner's terminology, this means that L is the little group of the contracted algebra representation, and that necessarily it is to be represented trivially. Besides, the little group is to be an invariant subgroup of the Spin(n)

group. This coincides with one class of the solutions found by Hermann [3]. However, we demonstrated here that there are no other solutions in the $sl(n, \mathbb{R})$ algebra case, in particular, there are no solutions with little group represented nontrivially.

As for the first necessary condition, an inspection of the tables of symmetric spaces, yields two possibilities: $L = \mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$, where $\mathrm{Spin}(1) \equiv 1$, and, for $n = 2k$, $L = U(k)$ (U is the unitary group). However, this second possibility certainly does not imply another solution, since it turns out that there is no vector satisfying the second above property.

Thus, *the only remaining possibility* is as follows,

$$L = \mathrm{Spin}(m) \times \mathrm{Spin}(n-m), \quad m = 1, 2, \ldots, n-1, \quad \mathrm{Spin}(1) \equiv 1. \tag{12}$$

It is rather straightforward, however somewhat lengthy, to show that proportionality of (8) and (9) really holds in this case. The vector $\left|{\square\square \atop w}\right\rangle$ exists, and it is the one corresponding to traceless diagonal $n \times n$ matrix $\mathrm{diag}(\frac{1}{m}, \ldots, \frac{1}{m}, -\frac{1}{n-m}, \ldots, -\frac{1}{n-m})$.

3. Special Case: $SL(2, \mathbb{R})$

The analysis accomplished above cannot be applied directly to the $n = 2$ case, thus the $sl(2, \mathbb{R})$ case must be treated separately. The maximal compact subgroup $SO(2)$, that is, its double cover Spin(2), has only one generator M, and therefore it has only one-dimensional irreducible representations. In this case, there are two Abelian generators $U_\pm$ of the contracted group:

$$[M, U_\pm] = \pm U_\pm, \quad [U_+, U_-] = 0. \tag{13}$$

Based on these relations, it is easy to verify that the $T_\pm$ operators obtained by the Gell-Mann construction as:

$$T_\pm = i[M^2, U_\pm] + i\sigma U_\pm \tag{14}$$

automatically satisfy the $sl(2, \mathbb{R})$ commutation relation:

$$[T_+, T_-] = -2M. \tag{15}$$

Therefore, we demonstrate that the Gell-Mann formula applies to the $sl(2, \mathbb{R})$ case as well.

4. Matrix Elements

The approach presented in this paper allows us additionally to write down explicitly the matrix elements of the $sl(n, \mathbb{R})$ generators in the cases when the Gell-Mann formula is valid. The possible cases are determined by the numbers n and m. The corresponding representation space (not irreducible in general) is the one over the coset space $\mathrm{Spin}(n)/\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$. The proportionality factor α is determined to be:

$$\alpha = \frac{1}{2}\sqrt{\frac{m(n-m)}{n}}, \tag{16}$$

and, in a matrix notation for $\square\square$ representation:

$$\left|\begin{matrix}\square\square\\ w\end{matrix}\right\rangle = \sqrt{\frac{m(n-m)}{n}}\mathrm{diag}\left(\frac{1}{m},\dots,\frac{1}{m},-\frac{1}{n-m},\dots,-\frac{1}{n-m}\right). \tag{17}$$

The Gell-Mann formula (3), (6), and the matrix representation of the contracted Abelian generators U (A.5) yield:

$$\left\langle\begin{matrix}J'\\ m'\end{matrix}\right| T_\mu \left|\begin{matrix}J\\ m\end{matrix}\right\rangle = i\sqrt{\frac{m(n-m)}{4n}}\sqrt{\frac{\dim(J)}{\dim(J')}}(C^2(J')-C^2(J)+\sigma)C^{J\,\square\square\,J'}_{0\;\,0\;\,0}\,C^{J\,\square\square\,J'}_{m\;\,\mu\;\,m'}. \tag{18}$$

The zeroes in the indices of Clebsch–Gordan coefficients here denote vectors that are invariant with respect to $\mathrm{Spin}(m)\times\mathrm{Spin}(n-m)$ transformations (in that spirit $\left|\begin{matrix}\square\square\\ w\end{matrix}\right\rangle = \left|\begin{matrix}\square\square\\ 0\end{matrix}\right\rangle$). In the formula (18), the space reduction from $\mathcal{L}^2(\mathrm{Spin}(n))$ to $\mathcal{L}^2(\mathrm{Spin}(n)/\mathrm{Spin}(m)\times\mathrm{Spin}(n-m))$ implies a reduction of the basis (A.3), i.e. $\left|\begin{matrix}J\\ 0\,m\end{matrix}\right\rangle \to \left|\begin{matrix}J\\ m\end{matrix}\right\rangle$ (only the vectors invariant with respect to left $\mathrm{Spin}(m)\times\mathrm{Spin}(n-m)$ action remain).

The expression (18), together with the action of the $\mathrm{Spin}(n)$ generators (A.4) provides an explicit form of the $SL(n,\mathbb{R})$ generators representation, that is labeled by a free parameter σ. Such representations are multiplicity-free with respect to the maximal compact $\mathrm{Spin}(n)$ subgroup, and all of them are *a priori* tensorial. One can obtain from these representations, for certain σ parameter values, the $sl(n,\mathbb{R})$ spinorial representations as well as by explicitly evaluating the Clebsch–Gordan coefficient and performing an appropriate analytic continuation in terms of the $\mathrm{Spin}(n)$ labels.

5. Conclusion

In this paper, we clarified the issue of the Gell-Mann formula validity for the $sl(n,\mathbb{R}) \to r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ algebra contraction. We have shown that the only $sl(n,\mathbb{R})$ representations obtainable in this way are given in Hilbert spaces over the symmetric spaces $\mathrm{Spin}(n)/\mathrm{Spin}(m)\times\mathrm{Spin}(n-m)$, $m=1,2,\dots,n-1$. Moreover, by making use of the Gell-Mann formula in these spaces, we have obtained a closed form expressions of all irreducible representations matrix elements of the noncompact operators generating $SL(n,\mathbb{R})/SO(n)$ cosets. The matrix elements of both compact and noncompact operators of the $sl(n,\mathbb{R})$ algebra are given by (A.4) and

(18), respectively. In particular, it turns out that, due to Gell-Mann's formula validity conditions, no representations with $so(n)$ subalgebra representations multiplicity can be obtained in this way. Moreover, the matrix expressions of the noncompact operators as given by (18) do not account *a priori* for the $sl(n,\mathbb{R})$ spinorial representations. An explicit construction of spinorial representations requires an additional analytic continuation of the matrix elements explicit expressions to half-integer values of the representation labels. Due to mutual connection of the $sl(n,\mathbb{R})$ and $su(n)$ algebras, the results of this paper apply to the corresponding $su(n)$ case as well. The $SU(n)/SO(n)$ generators differ from the corresponding $sl(n,\mathbb{R})$ operators by the imaginary unit multiplicative factor, while the spinorial representations issue in the $su(n)$ case is pointless due to the fact that the $SU(n)$ is a simply connected (there exists no double cover) group.

In many physics applications (e.g. those in [18]) one is interested in the unitary irreducible representations. The unitarity question goes beyond the scope of the present work, and it relates to the Hilbert space properties, i.e. the vector space scalar product. An efficient method to study unitarity is to start with a Hilbert space $L^2(\mathrm{Spin}(n),\kappa)$ of square integrable functions with a scalar product given in terms of an arbitrary kernel κ, and to impose the unitarity constraints both on the scalar products itself and on the noncompact operators matrix elements in that scalar product (cf. [19]). The simplest series of the $sl(n,\mathbb{R})$ unitary irreducible representations, the Principal series, of the representations constructed above are obtained when $\sigma = i\sigma_I$, $\sigma_I \in \mathbb{R}\backslash\{0\}$, i.e. when σ takes an arbitrary nonzero pure imaginary value.

To conclude, we obtained recently a representation dependent generalization of the Gell-Mann formula for all $sl(n,\mathbb{R})$ algebras [12] to cover the cases of representations with nontrivial multiplicity. The $sl(n,\mathbb{R})$ noncompact operators representations obtained in that work together with the results of this work cover all $sl(n,\mathbb{R})$ representation cases.

Appendix A

In this paper, rather than following the approach of Hermann [3], we follow our approach of [12]. That is, we work in the representation space of square integrable functions $\mathcal{L}^2(\mathrm{Spin}(n))$, over the maximal compact subgroup $\mathrm{Spin}(n)$, i.e. the $SO(n)$ universal covering group, with a standard invariant Haar measure. This representation space is large enough to provide for all inequivalent irreducible representations of the contracted group, and, by a theorem of Harish-Chandra [14–17], is also rich enough to contain representatives from all equivalence classes of the $\overline{SL}(n,\mathbb{R})$ group, i.e. $sl(n,\mathbb{R})$ algebra, representations.

The generators of the contracted group are generically represented, in this space, as follows. The $so(n)$ subalgebra operators act, in a standard way:

$$M_{ab}|\phi\rangle = -i\frac{d}{dt}\exp(itM_{ab})\Big|_{t=0}|\phi\rangle,$$

where action of a $\mathrm{Spin}(n)$ element g' on an arbitrary vector $|\phi\rangle \in \mathcal{L}^2(\mathrm{Spin}(n))$ is given via action from the left on basis vectors $|g\rangle$ of this space:

$$g'|\phi\rangle = g' \int \phi(g)|g\rangle dg = \int \phi(g)|g'g\rangle dg, \quad g', g \in \mathrm{Spin}(n). \tag{A.1}$$

The contracted noncompact Abelian operators $U_\mu\,(2, 3)$, act in the same basis as multiplicative Wigner-like D-functions (the $SO(n)$ group matrix elements expressed as functions of the group parameters):

$$U_\mu \to |u| D^{\square\square}_{w\mu}(g^{-1}) \equiv |u| \left\langle \begin{matrix} \square\square \\ w \end{matrix} \right| \left(D^{\square\square}(g) \right)^{-1} \left| \begin{matrix} \square\square \\ \mu \end{matrix} \right\rangle, \tag{A.2}$$

$|u|$ being a constant norm, g being an $SO(n)$ element, and $\square\square$ denoting (in a parallel to the Young tableaux) the symmetric second-order tensor representation of $SO(n)$. The norm $|u|$ parametrizes representation of U, but will turn out to be irrelevant in our case, as it cancels with the denominator in (3). The $\left| \begin{matrix} \square\square \\ \mu \end{matrix} \right\rangle$ vector from representation $\square\square$ space is denoted by the index of the operator U_μ, whereas the vector $\left| \begin{matrix} \square\square \\ w \end{matrix} \right\rangle$ can be an arbitrary vector belonging to $\square\square$ (the choice of w determines, in Wigner terminology, the little group of the representation in question). Taking an inverse of g in (A.2) insures the correct transformation properties.

A natural discrete orthonormal basis in the $\mathcal{L}^2(\mathrm{Spin}(n))$ space is given by properly normalized Wigner D-functions:

$$\left\{ \left| \begin{matrix} J \\ km \end{matrix} \right\rangle \equiv \int \sqrt{\dim(J)} D^J_{km}(g^{-1}) dg\, |g\rangle \right\}, \quad \left\langle \begin{matrix} J \\ km \end{matrix} \middle| \begin{matrix} J' \\ k'm' \end{matrix} \right\rangle = \delta_{JJ'} \delta_{kk'} \delta_{mm'}, \tag{A.3}$$

where dg is an (normalized) invariant Haar measure. Here, J stands for a set of $\mathrm{Spin}(n)$ irreducible representation labels, while the k and m labels numerate the representation basis vectors.

An action of the $so(n)$ operators in this basis is well-known, and it can be written in terms of the Clebsch–Gordan coefficients of the $\mathrm{Spin}(n)$ group as follows,

$$\left\langle \begin{matrix} J' \\ k'm' \end{matrix} \right| M_{ab} \left| \begin{matrix} J \\ km \end{matrix} \right\rangle = \delta_{JJ'} \sqrt{C^2(J)}\, C^{J\, \square\, J'}_{m\,(ab)\,m'}. \tag{A.4}$$

The matrix elements of the U_μ operators in this basis are readily found to read:

$$\begin{aligned} &\left\langle \begin{matrix} J' \\ k'm' \end{matrix} \right| U^{(w)}_\mu \left| \begin{matrix} J \\ km \end{matrix} \right\rangle \\ &\quad = |u| \left\langle \begin{matrix} J' \\ k'm' \end{matrix} \right| D^{-1\,\square\square}_{w\mu} \left| \begin{matrix} J \\ km \end{matrix} \right\rangle = |u| \sqrt{\frac{\dim(J)}{\dim(J')}} C^{J\,\square\square\,J'}_{k\;\,w\;\,k'} C^{J\,\square\square\,J'}_{m\;\,\mu\;\,m'}. \end{aligned} \tag{A.5}$$

A closed form of the matrix elements of the whole contracted algebra $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ (a semidirect sum of a $\frac{n(n+1)}{2} - 1$-dimensional Abelian algebra and $so(n)$) representations is thus explicitly given in this space by (A.4) and (A.5).

Moreover, we introduce the so-called, left action generators K as:

$$K_\mu \equiv g^{\nu\lambda} D^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{\mu\nu} M_\lambda, \tag{A.6}$$

where $g^{\nu\lambda}$ is the Cartan metric tensor of $SO(n)$. The K_μ operators behave exactly as the rotation generators M_μ, it is only that they act on the lower left-hand side indices of the basis (A.3):

$$\langle K_{ab}\rangle = \left\langle \begin{matrix} J' \\ k'\,m' \end{matrix} \right| K_{ab} \left| \begin{matrix} J \\ k\,m \end{matrix} \right\rangle = \delta_{JJ'} \sqrt{C^2(J)}\; C^{\;J\;\begin{smallmatrix}\square\\\square\end{smallmatrix}\;J'}_{\;k\,(ab)\,k'}\,. \tag{A.7}$$

The operators K_μ and M_μ mutually commute. However, the corresponding Casimir operators match and, in particular, we will use $\sum K_\mu^2 = \sum M_\mu^2$ in the expression for the Gell-Mann formula (3).

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