Review

Reviews in Mathematical Physics
Vol. 25, No. 10 (2013) 1343006 (16 pages)

DOI: 10.1142/S0129055X1343006X

World Scientific
www.worldscientific.com

$SL(n, R)$ IN PARTICLE PHYSICS AND GRAVITY — DECONTRACTION FORMULA AND UNITARY IRREDUCIBLE REPRESENTATIONS

IGOR SALOM* and DJORDJE ŠIJAČKI†

Institute of Physics, University of Belgrade,
P.O. Box 57, 11001 Belgrade, Serbia
**isalom@ipb.ac.rs*
†*sijacki@ipb.ac.rs*

Received 11 March 2013
Accepted 2 November 2013
Published 26 November 2013

$SL(n, R)$ and $Diff(n, R)$ groups play a prominent role in various particle physics and gravity theories, notably in chromogravity (that models the IR region of QCD), gauge affine generalizations of general relativity, and pD-branes. Applications of these groups require a knowledge of their features and especially rely on the unitary irreducible representation details. Lie algebra, topology and unitary representation issues of the covering groups of the $SL(n, R)$ and $Diff(n, R)$ groups with respect to their maximal compact $SO(n)$ subgroups are considered. Topological properties determining spinorial representations of these groups are reviewed. An especial attention is paid to the fact that, contrary to other classical Lie algebras, the $SL(n, R)$, $n \geq 3$ covering groups are groups of infinite matrices, as are all their spinorial representations. A notion of Lie algebra decontraction, also known as the Gell-Mann formula, that plays a role of an inverse to the Inonu–Wigner contraction, is recalled. Contrary to orthogonal type of algebras, the decontraction formula has a limited validity. The validity domain of this formula for $sl(n, R)$ algebras contracted with respect to their $so(n)$ subalgebras is outlined. A recent generalization of the decontraction formula, that applies to all $SL(n, R)$ covering group representations, as well as an explicit closed expression of all non-compact $sl(n, R)$ operators matrix elements for all representations is presented. A construction of the unitary $sl(n, R)$ representations is discussed within a framework than combines the Harish-Chandra results and a method of fulfilling the unitarity requirements in Hilbert spaces with non-trivial scalar product kernel.

Keywords: Gell-Mann decontraction formula; Lie algebra contraction; $SL(n)$ representations.

Mathematics Subject Classification 2010: 20C33, 20C40

1. Introduction

The Poincaré spacetime and internal $SU(n)$ symmetries, both global and local, played a crucial role in describing fundamental forces in nature, physical conservation laws, and the basic matter fields. These symmetries are the core essence of the Standard Model and Einstein's General Relativity Theory, the two pillars of contemporary fundamental physics. In this work we consider the $SL(n,R)$ symmetries in the content of particle physics and gravity theory. First, we recollect several prominent examples and extract the knowledge on the relevant required $SL(n,R)$ representations. Afterwards, we pose a general framework for constructing the $SL(n,R)$ unitary irreducible representations, and outline the basic facts about recent generalization of the Gell-Mann's decontraction formula that yields all matrix elements of the $sl(n,R)$ algebra elements for all representations.

Already in 1965, Gell-Mann, Dothan and Ne'eman proposed the $SL(3,R)$ symmetry to describe the Regge trajectories of hadron recurrences in a spectrum generating algebra approach [1]. The model was subsequently generalized to the relativistic $SL(4,R)$ one, describing both parent and daughter trajectories [2]. A construction of the unitary irreducible $SL(3,R)$ representations was a first step on the way to fulfill this proposal. Moreover, the spinorial representations, faithful representations of the $\overline{SL}(3,R)$ covering group, were essential in order to describe baryonic recurrences. After some confusion among researchers at the time, denying even an existence of the covering group on the basis of a wrong interpretation of certain Cartan's statement, it was soon clear that there are specific features of the $SL(3,R)$ symmetry (subsequently, all $SL(n,R)$, $n \geq 3$, symmetries) and its representations [3]. The covering $\overline{SL}(n,R)$, $n \geq 3$, groups are necessarily defined in infinite dimensional spaces (groups of infinite matrices), thus there are no finite spinorial representations, and their representations considered with respect to maximal compact Spin(n), i.e. $\overline{SO}(n)$ subgroups have as a rule non-trivial multiplicity. An explicit construction of all $\overline{SL}(3,R)$ unitary irreducible representations confirmed these facts [4].

A potential relevance of the $SL(n,R)$, $n=3,4$ symmetries in describing confinement of quarks was noted even at the early stage of the so-called "bag-models" featuring a volume-preserving part of the action that yields confinement. These symmetries revive on the fundamental dynamic QCD level. The adoption of QCD and its incorporation in the Standard Model were the outcome of the success of asymptotic freedom (AF) in fitting the scaling results of deep inelastic electron-nucleon scattering, coupled with the fact that color-$SU(3)$ provides an explanation for some (otherwise paradoxical) key features of the Non-Relativistic Quark Model (NRQM): "wrong" spin-statistics of the baryon (**56** in $SU(6)$) ground state, zero-triality of the entire $SU(3)$ (Eightfold-Way) physical spectrum. AF provides a successful perturbative treatment for the "ultraviolet" (UV) region, e.g., high-energy electro-weak hadronic interactions, corresponding to the current-quarks aspects of NRQM. There is also a prosperous understanding of hadronic strong interactions in the "hard" and "semi-hard" regimes. Nothing of the sort has emerged in the "infrared" (IR) frequency antipode region. After several decades, we still lack a complete proof of color-confinement.

2. Chromogravity

A chromogravity approach to the IR QCD sector [5] is based on a conjecture: (a) that gluon exchange forces (with the gluons in color-neutral combinations) make up an important component of inter-hadron interactions *in the "softest" region and in confinement*; (b) that the physical role of this component is to produce *a longer-range force*, with many of the characteristics of *gravity*, starting with the basic mathematical foundation, namely, *invariance under (pseudo) diffeomorphisms*; (c) that the simplest such n-gluon exchange, that of the two-gluon system

$$G_{\mu\nu}(x) = (\kappa)^{-2} g_{ab} B^a_\mu(x) B^b_\nu(x) \tag{1}$$

fulfills the role of an effective (pseudo) metric — "chromometric", with respect to these (pseudo) diffeomorphisms — "chromo diffeomorphisms", in the same manner that the physical metric (through its Christoffel connection) "gauges" the true diffeomorphisms. Here κ has the dimensions of mass, $\mu, \nu, \ldots$ are Lorentz 4-vector indices, $a, b, \ldots$ are $SU(3)$ adjoint representation (octet) indices, g_{ab} is the Cartan metric for the $SU(3)$ octet, and B^a_μ is a gluon field.

The gluon color-$SU(3)$ gauge field transforms under an infinitesimal local $SU(3)$ variation according to

$$\delta_\epsilon B^a_\mu = \partial_\mu \epsilon^a + B^b_\mu \{\lambda_b\}^a_c \epsilon^c = \partial_\mu \epsilon^a + i f^a_{bc} B^b \epsilon^c \tag{2}$$

(we use the adjoint representation $\{\lambda_b\}^a_c = -i f^a_{bc} = i f^a_{bc}$). To deal with the non-perturbative IR region, we expand the gauge field operator around a constant global vacuum solution N^a_μ,

$$\partial_\mu N^a_\nu - \partial_\nu N^a_\mu = i f^a_{bc} N^b_\mu N^c_\nu, \tag{3}$$

$$B^a_\mu = N^a_\mu + A^a_\mu. \tag{4}$$

Such a vacuum solution might be of the instanton type, for instance, that at large distances is required to approach a constant value.

The leading part of the color-$SU(3)$ infinitesimal gauge variation of the pseudo-metric field $G_{\mu\nu}$ in the infrared region reads [5]

$$\delta_\xi G_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \partial_\mu(\xi^\sigma G_{\sigma\nu}) + \partial_\nu(\xi^\sigma G_{\mu\sigma}), \tag{5}$$

where, $\xi_\mu = \eta_{ab} \epsilon^a N^b_\mu$, and where one can reidentify δ_ξ as a variation under a formal diffeomorphism of the R^4 manifold. This $G_{\mu\nu}$ variation simulates the infinitesimal variation of a "world tensor" $G_{\mu\nu}$ under Einstein's covariance group, $x^\sigma \to x^\sigma + \xi^\sigma$. ξ^σ thus has to be defined as a contravariant vector, and $G_{\mu\nu}$ is invertible, thanks to the constant part N^a_μ. Note that as the μ, ν indices are "true" Lorentz indices, acted upon by the physical Lorentz group, the manifold has to be Riemannian (only Riemannian manifolds, with or without torsion, have tangents with orthogonal or pseudo-orthogonal symmetry). Thus

$$D_\sigma G_{\mu\nu} = 0, \tag{6}$$

the commutator of two such variations,

$$[\delta_{\xi_1}, \delta_{\xi_2}] G_{\mu\nu} = \delta_{\xi_3} G_{\mu\nu}, \tag{7}$$

where

$$\xi_{3\mu} := (\partial_\nu \xi_{1\mu})\xi_2^\nu + (\partial_\mu \xi_{1\nu})\xi_2^\nu - (\partial_\nu \xi_{2\mu})\xi_1^\nu - (\partial_\mu \xi_{2\nu})\xi_1^\nu \tag{8}$$

indeed closes on the covariance group's commutation relations.

In the general case, the QCD "gluon-made" operators which mutually connect various hadron states are characterized by color-singlet quanta. The corresponding color-singlet n-gluon field operator has the following form

$$G^{(n)}_{\mu_1\mu_2\cdots\mu_n} = d^{(n)}_{a_1 a_2 \cdots a_n} B^{a_1}_{\mu_1} B^{a_2}_{\mu_2} \cdots B^{a_n}_{\mu_n} \tag{9}$$

where

$$\begin{aligned} d^{(2)}_{a_1 a_2} &= g_{a_1 a_2}, \\ d^{(3)}_{a_1 a_2 a_3} &= d_{a_1 a_2 a_3}, \\ d^{(n)}_{a_1 a_2 \cdots a_n} &= d_{a_1 a_2 b_1} g^{b_1 c_1} d_{c_1 b_2 a_3} \cdots g^{b_{n-4} c_{n-4}} d_{c_{n-4} b_{n-3} a_{n-2}} g^{b_{n-3} c_{n-3}} d_{c_{n-3} a_{n-1} a_n}, \\ & \qquad n > 3, \end{aligned} \tag{10}$$

B^a_μ is the dressed gluon field, $g_{a_1 a_2}$ is the $SU(3)$ Cartan metric, and $d_{a_1 a_2 a_3}$ is the $SU(3)$ totally symmetric $8\times 8\times 8 \to 1$ tensor. The set of all $G^{(n)}_{\mu_1\mu_2\cdots\mu_n}$ operators, $n = 1, 2, \ldots,$ forms a basis of a vector space of colorless purely gluonic configurations. Again, in the infrared region approximation the infinitesimal color-$SU(3)$ variation can be rewritten in terms of effective pseudo-diffeomorphisms,

$$\delta_\epsilon G^{(n)}_{\mu_1\mu_2\cdots\mu_n} = \partial_{\{\mu_1} \xi^{(n-1)}_{\mu_2\mu_3\cdots\mu_n\}} \equiv \delta_\xi G^{(n)}_{\mu_1\mu_2\cdots\mu_n}, \tag{11}$$

where $\{\mu_1\mu_2\cdots\mu_n\}$ denotes symmetrization of indices, and

$$\xi^{(n-1)}_{\mu_1\mu_2\cdots\mu_{n-1}} \equiv d^{(n)}_{a_1 a_2 \cdots a_n} N^{a_1}_{\mu_1} N^{a_2}_{\mu_2} \cdots N^{a_{n-1}}_{\mu_{n-1}} \epsilon^{a_n}. \tag{12}$$

A subsequent application of two $SU(3)$-induced variations closes algebraically

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] G^{(n)}_{\mu_1\mu_2\cdots\mu_n} = \delta_{\epsilon_3} G^{(n)}_{\mu_1\mu_2\cdots\mu_n} \quad \text{i.e. } [\delta_{\xi_1}, \delta_{\xi_2}] G^{(n)}_{\mu_1\mu_2\cdots\mu_n} = \delta_{\xi_3} G^{(n)}_{\mu_1\mu_2\cdots\mu_n} \tag{13}$$

thus yielding an infinitesimal nonlinear realization of the $Diff(4, R)$ Chromodiffeomorphisms group in the space of fields $\{G^{(n)}_{\mu_1\mu_2\cdots\mu_n} \mid n = 2, 3, \ldots\}$.

2.1. *Matter particles and fields*

The simplest way to describe hadronic matter fields is by making use of nonlinear realizations of the $Diff(4, R)$ chromodiffeomorphisms group over its maximal linear subgroup, i.e. over the $GA(4, R) \supset SA(4, R)$ [8]. Here, $GA(4, R)$, $SA(4, R)$ are the semidirect product groups of the translation group T_4 and the $GL(4, R)$, $SL(4, R)$ groups, respectively.

In the following, we consider the relevant groups in an n-dimensional space time, i.e. the $Diff(n, R)$, T_n, $SL(n, R)$, $SO(n)$ groups, thus setting up a mathematical framework applicable to gravity and extended objects considerations in higher dimensions as well, and we focus on the $SL(n, R)$ group, since this group determines the non-Abelian features of the $GL(n, R)$ group as well.

The matter particles and the matter fields in quantum theory are described by the affine group, $SA(n, R) = T_n \wedge SL(n, R)$, representations in Hilbert spaces of states and fields, respectively.

The commutation relations of the $sa(n, R)$ algebra of the $SA(n, R)$ group read

$$\begin{aligned} [P_a, P_b] &= 0, \\ [Q_{ab}, P_c] &= ig_{ac}P_b, \\ [Q_{ab}, Q_{cd}] &= ig_{bc}Q_{ad} - ig_{ad}Q_{cb}, \end{aligned} \tag{14}$$

the structure constants g_{mn} being either $\delta_{ab} = (+1, +1, \dots, +1)$, $a, b, c, d = 1, 2, \dots, n$ for the $SO(n)$ subgroup or $\eta_{ab} = (+1, -1, \dots, -1)$, $a, b, c, d = 0, 1, \dots, n-1$ for the n-dimensional Lorentz subgroup $SO(1, n-1)$ of the $SL(n, R)$ group. The maximal compact $SO(n)$ subgroup of the $SL(n, R)$ group is generated by the metric preserving antisymmetric operators $J_{ab} = Q_{[ab]}$, while the remaining non-compact traceless symmetric operators $T_{ab} = Q_{(ab)}$, the shear operators, generate the (non-trivial) n-volume preserving transformations. The $SL(n, R)$ commutation relations are given as follows

$$\begin{aligned} [M_{ab}, M_{cd}] &= -i\eta_{ac}M_{bd} + i\eta_{ad}M_{bc} + i\eta_{bc}M_{ad} - i\eta_{bd}M_{ac}, \\ [M_{ab}, T_{cd}] &= -i\eta_{ac}T_{bd} - i\eta_{ad}T_{bc} + i\eta_{bc}T_{ad} + i\eta_{bd}T_{ac}, \\ [T_{ab}, T_{cd}] &= +i\eta_{ac}M_{bd} + i\eta_{ad}M_{bc} + i\eta_{bc}M_{ad} + i\eta_{bd}M_{ac}. \end{aligned} \tag{15}$$

The quantum mechanical symmetry group is given as the $U(1)$ minimal extensions of the corresponding classical symmetry group. In practice, one finds it by taking the universal covering group of the classical group (topology changes), and by solving the algebra commutation relations for possible central charges (algebra deformation). There are no non-trivial central charges of the $sa(n, R)$ and $sl(n, R)$ algebras, and the remaining important question for quantum applications is the one of the affine symmetry covering group. The translational part of the $SA(n, R)$ group is contractible to a point and thus irrelevant for the covering question. The $SL(n, R)$ subgroup is, according to the Iwasawa decomposition, given by $SL(n, R) = SO(n, R) \times A \times N$, where A is a subgroup of Abelian transformations (e.g., diagonal matrices) and N is a nilpotent subgroup (e.g., upper triangular matrices). Both A and N subgroups are contractible to point. Therefore, the covering features are determined by the topological properties of the maximal compact subgroup of the group in question. In our case, that is the $SO(n,R)$ group, i.e. more precisely its central subgroup. The universal covering group of the $SO(n)$, $D \geq 3$ group is its double covering group isomorphic to Spin(n). In other words $SO(n) \simeq \mathrm{Spin}(n)/Z_2$.

The universal covering group of a given group is a group with the same Lie algebra and with a simply-connected group manifold. A finite dimensional covering, $\overline{SL}(n,R)$, exists provided one can embed $SL(n,R)$ into a group of finite complex matrices that contain Spin(n) as subgroup. A scan of the Cartan classical algebras points to the $SL(n,C)$ groups as a natural candidate for the $SL(n,R)$ groups covering. However, there is no match of the defining dimensionalities of the $SL(n,R)$ and Spin(n) groups for $n \geq 3$,

$$\dim(SL(n,C)) = n < 2^{[\frac{n-1}{2}]} = \dim(\mathrm{Spin}(n)), \tag{16}$$

except for $n = 8$. In the $n = 8$ case, one finds that the orthogonal subgroup of the $SL(8,R)$ and $SL(8,C)$ groups is $SO(8,R)$ and not Spin(8). Thus, there are no finite dimensional covering groups of the $SL(n,R)$ groups for any $n \geq 3$. An explicit construction of all spinorial, unitary and non-unitary multiplicity-free [6], and unitary non-multiplicity-free [4], $SL(3,R)$ representations shows that they are all defined in infinite dimensional spaces.

The universal (double) covering groups of the $\overline{SL}(n,R)$ and $\overline{SA}(n,R)$, $n \geq 3$ groups are groups of infinite complex matrices. All their spinorial representations are infinite dimensional and when reduced with respect to Spin(n) subgroups contain representations of unbounded spin values.

2.2. *Representations on states*

The $\overline{SA}(n,R)$ Hilbert space representations are, owing to the semidirect product group structure, induced as in the Poincaré case from the corresponding little group (stability subgroup) representations. The correct quantum mechanical interpretation requires the little group representations to be unitary. The unitary irreducible $\overline{SA}(n,R)$ Hilbert space representations are obtained as follows: (i) determine the vectors characterized by the maximal set of labels of the Abelian translational subgroup generators, (ii) determine the corresponding little groups as subgroups of the $SL(n,R)$ groups that leave these vectors invariant, and (iii) induce the unitary irreducible $\overline{SA}(n,R)$ representations from T_n and little groups representations. In contradistinction to the Poincaré case, the little groups that describe affine particles are more complex in structure due to the fact that a orthogonal type of group is enlarged here to the linear one.

The little group of the $\overline{SA}(n,R)$ Hilbert-space particle states is of the form $T^{\sim}_{n-1} \wedge \overline{SL}(n-1,R)$, where the Abelian invariant subgroup $T^{\sim}_{n-1}$ of the little group is generated by Q_{1j}, $j = 2,3,\ldots,n$. Owing to the fact that the little group itself is given as a semidirect product, there is number of possibilities. The simplest one is when the $T^{\sim}_{n-1}$ subgroup is represented trivially, $D(T^{\sim}_{n-1}) \rightarrow 1$, i.e. $D(Q_{1j}) \rightarrow 0$, the remaining part of the little group is $\overline{SL}(n-1, R)$, and the corresponding "affine particle" is described by the unitary irreducible $\overline{SL}(n-1, R)$ representations. These representations are infinite dimensional, even in the tensorial case, due to non-compactness of the $SL(n, R)$ group.

2.3. *Representations on fields*

The representations of the $\overline{SA}(n, R)$ group generalize the known Poincaré group representations on fields and are given as follows,

$$(D(a, \bar{\Lambda})\Phi_i)(x) = (D(\bar{\Lambda}))_i^j \Phi_j(\Lambda^{-1}(x - a)) \quad (a, \bar{\Lambda}) \in T_n \wedge \overline{SL}(1, n-1), \tag{17}$$

where i, j enumerate a basis of the representation space of the field components. There are two physical requirements that have to be satisfied in the affine case in order to provide the due particle-field connection: (i) representations of the affine-particle little group $\overline{SL}(n-1, R)$ have to be unitary and thus (due to the little group's non-compactness) infinite dimensional, and (ii) representations of the Lorentz subgroup $\mathrm{Spin}(1, n-1)$ have to be finite dimensional and thus non-unitary as required by their Poincaré subgroup interpretation. This is achieved by making use of the so called "deunitarizing" automorphism of the $\overline{SL}(n, R)$ group [7]:

$$\mathcal{A} : \overline{SL}(n, R) \to \overline{SL}(n, R), \tag{18}$$

$$J_{ij}^{\mathcal{A}} = J_{ij}, \quad K_j^{\mathcal{A}} = iN_j, \quad N_j^{\mathcal{A}} = iK_j, \tag{19}$$

$$T_{ij}^{\mathcal{A}} = T_{ij}, \quad T_{00}^{\mathcal{A}} = T_{00}, \quad i, j = 1, 2, \ldots, D-1, \tag{20}$$

so that (J_{ij}, iK_i) generate the new compact $\mathrm{Spin}(n)^{\mathcal{A}}$ and (J_{ij}, iN_i) generate $\mathrm{Spin}(1, D-1)^{\mathcal{A}}$. Here, the $\overline{SL}(n-1, R)$, the stability subgroup of $\overline{SA}(n, R)$, is represented unitarily, while the Lorentz subgroup is represented by finite dimensional non-unitary representations. An efficient way of constructing explicitly the $\overline{SL}(n, R)$ infinite dimensional representations is based on the so called "decontraction" formula, which is an inverse of the Wigner–Inönü contraction, and will be treated below.

3. Affine Gravity and Spinorial Wave Equations

The metric affine [9], and gauge affine [10, 11] theories of gravity are generalizations of the Poincaré gauge theory where the Lorentz group $\mathrm{Spin}(1, n-1)$ is replaced by the $\overline{SL}(n, R)$ group. The customary way to develop such a theory in a particle physics framework is to start by the Dirac equation and then gauging the relevant global symmetry. In our case that means to start by a Dirac-like equation for an infinite-component spinorial affine field $\Psi(x)$,

$$(iX^a \partial_a - M)\Psi(x) = 0, \tag{21}$$

$$\Psi(x) \sim D^{(\mathrm{spin})}(\overline{SL}(n, R)). \tag{22}$$

The X^a, $a = 0, 1, \ldots, n-1$ vector operator, acting in the space of the Ψ field components, is an appropriate generalization of the Dirac γ matrices to the affine case. The $\overline{SL}(n,R)$ affine covariance requires that the following commutation relations are satisfied

$$[M_{ab}, X_c] = i\eta_{bc}X_a - i\eta_{ac}X_b, \tag{23}$$

$$[T_{ab}, X_c] = i\eta_{bc}X_a + i\eta_{ac}X_b. \tag{24}$$

The first relation ensures Lorentz covariance, and is generally a easy one to fulfill. The second relation, required by the full affine covariance, turns out to be rather difficult to accomplish.

We focus here on the $\overline{SL}(n,R)$ representations constrains required by the group algebraic consistency of this Dirac-like equation. In order to obtain all (physically relevant) unitary irreducible $\overline{SL}(n,R)$ representations, and in particular the spinorial ones fitting the Dirac-like equation construction, one works in Hilbert spaces of square integrable functions over the maximal compact subgroup, $\mathcal{L}^2(\mathrm{Spin}(n))$. The Hilbert space basis vectors in Dirac's notation are $\{|{}^{\{J\}}_{\{k\}\{m\}}\rangle\}$, where $\{J\}$ and $\{m\}$ are the representation labels of $\mathrm{Spin}(n)$ and its subgroups $\mathrm{Spin}(n-1), \mathrm{Spin}(n-2), \ldots, \mathrm{Spin}(n)$, respectively; while $\{k\}$ are labels of $\mathrm{Spin}(n-1), \mathrm{Spin}(n-2), \ldots, \mathrm{Spin}(n)$ groups acting to the left which are used to describe eventual multiplicity of the $\mathrm{Spin}(n)$ representations within a given $\overline{SL}(n,R)$ representation. We can split an $\overline{SL}(n,R)$ representation in terms of its $\mathrm{Spin}(n)$ subrepresentations, in a symbolic notation, as follows:

$$D(\overline{SL}(n,R)) \sim \sum_{\{J\},\{k\}} D^{\{J\}}(\mathrm{Spin}(n), \{k\}). \tag{25}$$

Representations of the shear operators T_{ab} are such that their matrix elements apriory have non-trivial $\{k\}$ dependence, i.e. they are proportional, as presented below, to the $C^{\{J''\}\,\square\!\square\,\{J'\}}_{\{k''\}\,\{k\}\,\{k'\}}$ $\mathrm{Spin}(n)$ Clebsch–Gordan coefficients. There are two distinct cases: (i) the $\mathrm{Spin}(n)$ multiplicity free representations when all $\{k\}$ labels are zero, and (ii) representations with non-trivial multiplicity. In the first case, the zero-value $\{k\}$ labels imply that the $\{J\}$ labels are integer, and thus all these $D(\overline{SL}(n,R))$ representations are tensorial. In the second case, when there are no constraints on the $\{k\}$ labels, one can have both tensorial and spinorial $D(\overline{SL}(n,R))$ representations.

To sum up, from the considered physical examples we conclude that applications of the $\overline{SL}(n,R)$ symmetry requires knowledge of the spinorial and tensorial unitary (infinite dimensional) representations with non-trivial $\mathrm{Spin}(n)$, $\mathrm{Spin}(1,n-1)$ subgroup multiplicity. In the following, we present an effective method of constructing all $\overline{SL}(n,R)$ representations, and set up a framework that allowes one to fulfill the unitarity and irreducibility issues as well.

4. Gell-Mann Decontraction Formula

To solve the problem of finding $\overline{SL}(n,\mathbb{R})$ representations in the basis of its (pseudo)orthogonal subgroup we will employ the so called Gell-Mann (decontraction) formula [12–16]. The aim of this formula is to provide an inverse to the well-known Inönü–Wigner contraction procedure [17]. More concretely, let a symmetric Lie algebra $\mathcal{A} = \mathcal{M} + \mathcal{T}$:

$$[\mathcal{M},\mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M},\mathcal{T}] \subset \mathcal{T}, \quad [\mathcal{T},\mathcal{T}] \subset \mathcal{M}, \tag{26}$$

and its Inönü–Wigner contraction $\mathcal{A}' = \mathcal{M} + \mathcal{U}$:

$$[\mathcal{M},\mathcal{M}] \subset M, \quad [\mathcal{M},\mathcal{U}] \subset U, \quad [\mathcal{U},\mathcal{U}] = \{0\}, \tag{27}$$

be given. Following a mathematically less rigorous definition (more strict definition can be found in [12]), the Gell-Mann formula states that, in certain cases, elements $T_\mu \in \mathcal{T}$ can be constructed as the following simple function of the contracted algebra operators $U_\mu \in \mathcal{U}$ and $M_\nu \in \mathcal{M}$:

$$T_\mu = i\frac{\alpha}{\sqrt{U_\nu U^\nu}}[C^2(\mathcal{M}), U_\mu] + i\sigma U_\mu. \tag{28}$$

Here, $C^2(\mathcal{M})$ and $U_\nu U^\nu$ denote the (positive definite) second order Casimir operators of the $\mathcal{M}$ and $\mathcal{A}'$ algebras, respectively, while α is a normalization constant and σ is an arbitrary parameter. The formula was, to our knowledge, first introduced by Dothan and Ne'eman [16], and was advocated by Hermann [13].

The importance of this formula in our case is immediate, since it is not difficult to obtain representations of the contracted algebra $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ (here $r_{\frac{n(n+1)}{2}-1}$ denotes $\frac{n(n+1)}{2} - 1$ dimensional Abelian algebra and $\uplus$ stands for semidirect sum).

To represent the contracted algebra we will work in the representation space of square integrable functions $\mathcal{L}^2(\mathrm{Spin}(n))$ over the maximal compact subgroup $\mathrm{Spin}(n)$, i.e. the $SO(n)$ universal covering group, with a standard invariant Haar measure. This representation space is large enough to provide for all inequivalent irreducible representations of the contracted group, and, by a theorem of Harish-Chandra [18–21], is also rich enough to contain representatives from all equivalence classes of the $\overline{SL}(n,\mathbb{R})$ group, i.e. $sl(n,\mathbb{R})$ algebra, representations.

The generators of the contracted group are generically represented, in this space, as follows. The $so(n)$ subalgebra operators $M_{ab}, a, b = 1, 2, \dots, n$, act in the standard way:

$$M_{ab}|\phi\rangle = -i\frac{d}{dt}\exp(itM_{ab})|_{t=0}|\phi\rangle, \tag{29}$$

where action of a $\mathrm{Spin}(n)$ element g' on an arbitrary vector $|\phi\rangle \in \mathcal{L}^2(\mathrm{Spin}(n))$ is given via action from the left on basis vectors $|g\rangle$ of this space:

$$g'|\phi\rangle = g'\int \phi(g)|g\rangle dg = \int \phi(g)|g'g\rangle dg, \quad g', g \in \mathrm{Spin}(n). \tag{30}$$

The contracted non-compact Abelian operators U_μ (27) and (28), act in the same basis as multiplicative Wigner-like D-functions (the $SO(n)$ group matrix elements expressed as functions of the group parameters):

$$U_\mu \to |u| D^{\square\square}_{w\mu}(g^{-1}) \equiv |u| \left\langle \begin{matrix}\square\square\\ w\end{matrix} \right| \left(D^{\square\square}(g)\right)^{-1} \left| \begin{matrix}\square\square\\ \mu\end{matrix} \right\rangle, \tag{31}$$

$|u|$ being a constant norm, g being an $SO(n)$ element, and $\square\square$ denoting (in a parallel to the Young tableaux) the symmetric second order tensor representation of $SO(n)$. The norm $|u|$ parametrizes representation of U, but will turn out to be irrelevant in our case, as it cancels with the denominator in (28). The $\left|\begin{matrix}\square\square\\ \mu\end{matrix}\right\rangle$ vector from representation $\square\square$ space is denoted by the index of the operator U_μ, whereas the vector $\left|\begin{matrix}\square\square\\ w\end{matrix}\right\rangle$ can be an arbitrary vector belonging to $\square\square$ (the choice of w determines, in Wigner terminology, the little group of the representation in question). Taking an inverse of g in (31) insures the correct transformation properties.

A natural discrete orthonormal basis in the $\mathcal{L}^2(\mathrm{Spin}(n))$ space is given by properly normalized Wigner D-functions:

$$\left\{ \left| \begin{matrix} J \\ k\, m \end{matrix} \right\rangle \equiv \int \sqrt{\dim(J)}\, D^J_{km}(g^{-1}) dg |g\rangle \right\}, \quad \left\langle \begin{matrix} J \\ k\, m \end{matrix} \middle| \begin{matrix} J' \\ k'\, m' \end{matrix} \right\rangle = \delta_{JJ'} \delta_{kk'} \delta_{mm'}, \tag{32}$$

where dg is an (normalized) invariant Haar measure. Here, J stands for a set of $\mathrm{Spin}(n)$ irreducible representation labels, while the k and m labels numerate the representation basis vectors.

An action of the $so(n)$ operators in this basis is well known, and it can be written in terms of the Clebsch–Gordan coefficients of the $\mathrm{Spin}(n)$ group as follows,

$$\left\langle \begin{matrix} J' \\ k'\, m' \end{matrix} \middle| M_{ab} \middle| \begin{matrix} J \\ k\, m \end{matrix} \right\rangle = \delta_{JJ'} \sqrt{C^2(J)}\; C^{J \begin{smallmatrix}\square\\ \square\end{smallmatrix} J'}_{m (ab) m'} . \tag{33}$$

The matrix elements of the U_μ operators in this basis are readily found to read:

$$\begin{aligned} \left\langle \begin{matrix} J' \\ k'\, m' \end{matrix} \middle| U^{(w)}_\mu \middle| \begin{matrix} J \\ k\, m \end{matrix} \right\rangle &= |u| \left\langle \begin{matrix} J' \\ k'\, m' \end{matrix} \middle| D^{-1\,\square\square}_{w\mu} \middle| \begin{matrix} J \\ k\, m \end{matrix} \right\rangle \\ &= |u| \sqrt{\frac{\dim(J)}{\dim(J')}}\, C^{J\, \square\square\, J'}_{k\;\; w\;\; k'}\, C^{J\, \square\square\, J'}_{m\;\; \mu\;\; m'} . \end{aligned} \tag{34}$$

A closed form of the matrix elements of the whole contracted algebra $r_{\frac{n(n+1)}{2}-1} \uplus so(n)$ representations is thus explicitly given in this space by (33) and (34). To obtain representations of $sl(n,\mathbb{R})$, apart from (33), we also need to know how to represent non-compact shear generators T_μ in this space. That is given by the Gell-Mann formula (28):

$$T^{(w,\sigma)}_\mu = i\alpha [C^2(so(n)), D^{\square\square}_{w\mu}] + i\sigma D^{\square\square}_{w\mu} . \tag{35}$$

Though it seems that our goal is accomplished, it unfortunately turns out that formula (35) does not hold in the entire space $\mathcal{L}^2(\mathrm{Spin}(n))$ and for arbitrary choice of vector w (in the sense that commutator of two so constructed shear generators will not yield the correct result).

In [22], we have carried out a detailed analysis of the scope of validity of Gell-Mann formula in the $sl(n,\mathbb{R})$ case. The conclusion was that the only $sl(n,\mathbb{R})$ representations obtainable in this way are given in Hilbert spaces over the symmetric spaces $\mathrm{Spin}(n)/\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$, $m = 1, 2, \ldots, n-1$. The narrowing of the space from $\mathcal{L}^2(\mathrm{Spin}(n))$ to $\mathcal{L}^2(\mathrm{Spin}(n)/\mathrm{Spin}(m)\times\mathrm{Spin}(n-m))$ in the terms of basis (32) means reduction to a subspace spanned by vectors $\left|\begin{smallmatrix} J \\ 0m \end{smallmatrix}\right\rangle$, where zero denotes the vector component invariant with respect to $\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$. Furthermore, vector w in (35) must be chosen to be the one invariant with respect to the action of the group $\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$.

With these constraints, expression (35) becomes a proper representation of shear generators. This formula then leads to explicit expression for matrix elements of shear generators in $\mathcal{L}^2(\mathrm{Spin}(n)/\mathrm{Spin}(m) \times \mathrm{Spin}(n-m))$:

$$\left\langle \begin{matrix} J' \\ m' \end{matrix} \middle| T^{(\sigma)}_{\mu} \middle| \begin{matrix} J \\ m \end{matrix} \right\rangle$$

$$= i\sqrt{\frac{m(n-m)}{4n}}\sqrt{\frac{\dim(J)}{\dim(J')}}\,(C^2(J') - C^2(J) + \sigma)\, C^{J\;\square\!\square\;J'}_{0\;\;\;0\;\;\;0}\, C^{J\;\square\!\square\;J'}_{m\;\;\;\mu\;\;\;m'}. \tag{36}$$

The zeroes in the indices of Clebsch–Gordan coefficients again denote vectors that are invariant with respect to $\mathrm{Spin}(m) \times \mathrm{Spin}(n-m)$ transformations (in that spirit $\left|\begin{smallmatrix} \square\!\square \\ w \end{smallmatrix}\right\rangle = \left|\begin{smallmatrix} \square\!\square \\ 0 \end{smallmatrix}\right\rangle$). We also used shorthand notation $\left|\begin{smallmatrix} J \\ 0m \end{smallmatrix}\right\rangle \equiv \left|\begin{smallmatrix} J \\ m \end{smallmatrix}\right\rangle$.

The expression (41), together with the action of the $\mathrm{Spin}(n)$ generators (33), provides an explicit form of the $SL(n,\mathbb{R})$ generators representation, valid for arbitrary value of parameter σ. However, such representations are multiplicity free with respect to the maximal compact $\mathrm{Spin}(n)$ subgroup, and all of them are tensorial: multiplicity is lost with fixing of the left index of basis vectors (32) and only tensor representations of $\mathrm{Spin}(n)$ possess components invariant with respect to any $\mathrm{Spin}(m) \times \mathrm{Spin}(n-m), m \geq 1$ subgroup.[a]

To obtain more general class of $sl(n,\mathbb{R})$ representations (and, in particular, those with multiplicity) the Gell-Mann formula had to be generalized.

5. Generalization of the Gell-Mann Formula

One of the key steps to obtain generalized Gel-Mann formula is introduction of, so called, left action generators K:

$$K_\mu \equiv g^{\nu\lambda} D^{\substack{\square \\ \square}}_{\mu\nu} M_\lambda, \tag{37}$$

[a] In principle, some classes of spinorial multiplicity free representations can be obtained by appropriate analytic continuation of the Clebsch–Gordan coefficient in terms of the $\mathrm{Spin}(n)$ labels.

where $g^{\nu\lambda}$ is the Cartan metric tensor of $SO(n)$. The K_μ operators have the following matrix elements in the basis (32):

$$\langle K_{ab}\rangle = \left\langle \begin{matrix} J' \\ k'm' \end{matrix} \middle| K_{ab} \middle| \begin{matrix} J \\ km \end{matrix} \right\rangle = \delta_{JJ'}\sqrt{C^2(J)}\; C^{J\,\begin{smallmatrix}\square\\ \square\end{smallmatrix}\,J'}_{k\,(ab)\,k'}\,. \tag{38}$$

In other words, they behave exactly as the rotation generators M_μ (33), with a difference that they act on the lower left-hand side indices. The operators K_μ and M_μ mutually commute, but the corresponding Casimir operators match (in particular $\sum K_\mu^2 = \sum M_\mu^2$).

In terms of these new operators we can write down the following expression:

$$T_{ab}^{\sigma_2\ldots\sigma_n} = i\sum_{c>d}^{n}\{K_{cd}, D^{\square\square}_{(cd)(ab)}\} + i\sum_{c=2}^{n}\sigma_c D^{\square\square}_{(cc)(ab)}. \tag{39}$$

In [23, 24], we have shown that this is indeed the sought for generalization of the Gell-Mann formula, as this expression satisfies $sl(n,\mathbb{R})$ commutation relations in the entire space $\mathcal{L}^2(\mathrm{Spin}(n))$. In this expression σ_c is a set of $n-1$ arbitrary parameters that essentially (up to some discrete parameters) label $sl(n,\mathbb{R})$ irreducible representations. General validity of the new formula is reflected in the fact that there are now $n-1$ free parameters, i.e. representation labels, matching the $sl(n,\mathbb{R})$ algebra rank, compared to just one parameter of the original Gell–Mann formula.

An alternative form of (39) that looks more like the original formula (28) is:

$$T_{ab}^{\sigma_2\ldots\sigma_n} = i\sum_{c=2}^{n}\frac{1}{2}[C_2(so(c)_K), U_{ab}^{(cc)}] + \sigma_c U_{ab}^{(cc)}, \tag{40}$$

where $C_2(so(c)_K)$ is the second order Casimir of the $so(c)$ left action subalgebra, i.e. $C_2(so(c)_K) = \frac{1}{2}\sum_{a,b=1}^{c}(K_{ab})^2$. It is almost as simple as the original Gell-Mann formula, with a crucial advantage of being valid in the whole representation space over $\mathcal{L}^2(\mathrm{Spin}(n))$. Thus, due to Harish-Chandra theorems, the generalized Gell-Mann formula expression for the non-compact "shear" generators T_{ab} holds for all cases of $sl(n,\mathbb{R})$ irreducible representations, irrespective of their $so(n)$ subalgebra multiplicity (multiplicity free of the original Gell-Mann formula, and non-trivial multiplicity) and whether they are tensorial or spinorial. The price paid is that the generalized Gell-Mann formula is no longer solely a Lie algebra operator expression, but an expression in terms of representation dependant operators K_{ab} and $U_{ab}^{(cd)}$.

We also note that the very term in (40) when $c=n$ is, essentially, the original Gell-Mann formula (since $C_2(so(n)_K) = C_2(so(n)_M)$), whereas the rest of the terms can be seen as necessary corrections securing the formula validity in the entire representation space. The additional terms vanish for some representations yielding the original formula.

The form (40) also allows us to find matrix elements of T_{ab} operators. After some calculation the following expression is obtained:

$$\begin{aligned}\left\langle \begin{matrix}\{J'\}\\ \{k'\}\{m'\}\end{matrix} \middle| T_{\{w\}} \middle| \begin{matrix}\{J\}\\ \{k\}\{m\}\end{matrix}\right\rangle &= \frac{i}{2}\sqrt{\frac{\dim(\{J\})}{\dim(\{J'\})}}\, C^{\{J\}\,\Box\Box\,\{J'\}}_{\{m\}\,\{w\}\,\{m'\}}\\ &\quad\times\sum_{c=2}^{n}\sqrt{\frac{c-1}{c}}\left(C_2(so(c)_{\{k'\}}) - C_2(so(c)_{\{k\}}) + \tilde{\sigma}_c\right)\\ &\quad\times C^{\{J\}(\Box\Box)^{n-c+1}\{J'\}}_{\{k\}\quad(0)^{c-2}\quad\{k'\}}.\end{aligned} \tag{41}$$

(For the notation used for indices of Clebsh–Gordan coefficients please cf. [24, 25].)

The relation of the labeling of (41) and the one of (39), i.e. (40), is achieved provided $\sigma_c = \tilde{\sigma}_c + \sum_{d=2}^{c-1} \tilde{\sigma}_d/d$. The Clebsch–Gordan coefficient with indices $\{m\}, \{w\}, \{m'\}$ in (41) can be evaluated in an arbitrary basis (which is stressed by denoting the appropriate index by w instead by ab). The other Clebsch–Gordan coefficient can be evaluated in any basis labeled according to the Spin(n) $\supset$ Spin($n-1$) $\supset \cdots \supset$ Spin(2) subgroup chain (e.g., Gel'fand–Tsetlin basis) and can be, nowadays, rather easily evaluated, at least numerically.

6. Unitarity

A convenient way to parametrize any non-compact semisimple Lie group is given by means of the Iwasawa decomposition according to which the group G can be written as a product $G = NAK$, where N is a nilpotent subgroup of G, and its elements are upper triangular matrices with ones on the diagonal, A is an Abelian subgroup of G, and for $SL(n, R)$ we take its elements to be of the form $a = \text{diag}(e^{\lambda}, e^{\mu}, e^{\nu}, \dots, e^{-(\lambda+\mu+\nu+\cdots)})$, and finally K is the maximal compact subgroup $SO(n)$. An element $g \in G$ can thus be written as a product $g = nak$, where $n \in N, a \in A, k \in K$. The Iwasawa decomposition is unique and the product of some element $k \in K$ and an arbitrary element $g \in G$ is in general an arbitrary element of G which can be uniquely written as $kg = na(k, g)k \cdot g$, where $n \in N, a(k, g) \in A$ and $k \cdot g \in K$. Owing to the Iwasawa decomposition every element $g \in SL(n, R)$ can be uniquely written as $g = ne^h k$. The Abelian subgroup of $SL(n, R)$ has $n-1$ generators $A_1, A_2, \dots$, and if $\lambda_1, \lambda_2, \dots$ are the corresponding group parameters, respectively, one has $h = \lambda_1 A_1 + \lambda_2 A_2 + \cdots$. Let α be a linear, in general complex, function such that $\alpha(h) = \lambda_1 \alpha(A_1) + \lambda_2 \alpha(A_2) + \cdots$, and let us denote $\alpha(A_1), \alpha(A_2), \dots$ by $\sigma_1, \sigma_2, \dots$, respectively. Existence of the mapping α is guaranteed by the 1-dimensionality of the irreducible representations of the Abelian subgroup A. The mapping α can be extended in a natural way to a mapping from the group NA into the complex numbers since N is an invariant subgroup in NA.

The set of cosets $\overline{SL}(n, R)/NA$ is in one-to-one correspondence with the group $K = SO(n)$ and can be parametrized by the elements of K. In the coset space $\overline{SL}(n, R)/NA$ one has as well a measure, which we choose to be the invariant measure

dk on K. Let $H = L^2(K)$ be the separable Hilbert space of functions on K which are square integrable with respect to the invariant measure on K, i.e. $H = \{f(k) \mid k \in K\}$, such that $\int dk f^*(k) f(k) < \infty$, and let $\int dk = 1$.

Every non-trivial unitary representation of a non-compact group is necessarily infinite dimensional and this partly accounts for the complexity which occurs when one deals with unitary representations. The class of real semisimple Lie groups is especially complex. Harish-Chandra [18–21] defines a representation $U(g)$ of $G = \overline{SL}(n, R)$ on H in the following way: $U(g)$ is a homomorphic continuous mapping from G into the set of linear transformations on H given by

$$(U(g)f)(k) = e^{(h(k,g))} f(k \cdot g), \tag{42}$$

where $g \in G,\ f \in H,\ k \in K,\ e^h \in A$ and where $(U(g)f)(k)$ denotes the value of $U(g)f$ at the point k. Harish-Chandra now defines the concept of infinitesimal equivalence of two representations in the following way: Two representations are infinitesimally equivalent if there exists a similarity transformation of one representation into the other, with a non-singular, not necessarily unitary operator. In the case of equivalence there exist a unitary operator by means of which the transformation between the two representations is carried out. If both of two infinitesimally equivalent representations are unitary, then they are equivalent. Suppose now that $U(g)$ is a representation of a group G on a Hilbert space H. Suppose further that H_1 and H_2 are the two closed invariant subspaces of H, such that $H_2 \subset H_1 \subset H$, and $H_1 \neq H_2$. Then $U(g)$ induces a representation $U'(g)$ on the quotient H_1/H_2 in a natural way. The representation $U'(g)$ is said to be deducible from the representation $U(g)$. Harish-Chandra has proved that every unirrep is infinitesimally equivalent to some irreducible representation deducible from some representation $U(g)$ of the above form. Thus it is always possible to construct a bilinear form $(\tilde{f}, \tilde{g})$ in some quotient space H_1/H_2, where $\tilde{f}, \tilde{g} \in H_1/H_2$. One can extend the domain of this bilinear form to all H_1 uniquely by defining $(,)$ to vanish on H_2. Unitarity now means that $(U(g)f, U(g)f) = (f, f),\ f \in H_1,\ g \in G$, and the additional conditions that the bilinear form is a scalar product are hermiticity and positive definiteness $(f, g) = (g, f)^*$ and $(f, f) \geq 0\ \forall f, g \in H_1$. It is convenient to extend the domain of the scalar product to the whole space H. Being interested in obtaining all unirreps of $\overline{SL}(n, R)$, we will start with the most general scalar product: $(f, g) = \int\int dk_1 dk_2 f^*(k_1) \kappa(k_1, k_2) g(k_2),\ f, g \in H$, where $\kappa(k_1, k_2)$ is a kernel, the integration is over K, and dk is an invariant measure. The problem of finding all unitary representations of $\overline{SL}(n, R)$ becomes now the problem of finding all scalar products, i.e. kernels for which the representation $U(g)$ is unitary. We start with the most general scalar product of the Hilbert space. We find, by making use of the fact that dk is an invariant measure and of the additivity properties of Spin(n) Wigner's functions following expressions for the scalar product in terms of the matrix elements of the kernel and the expansion coefficients

$$(f, g) = \sum_{\{J\}\{k\}\{k'\}(m)} f^{\{J\}*}_{\{k'\}\{m\}} g^{\{J\}}_{\{k\}\{m\}} \kappa^{\{J\}}_{\{k'\}\{k\}}. \tag{43}$$

The hermiticity of the scalar product yields

$$\kappa^{\{J\}*}_{\{k'\}\{k\}} = \kappa^{\{J\}}_{\{k\}\{k'\}}. \tag{44}$$

Therefore κ is a hermitian matrix and can be diagonalized. Thus without any loss of generality we write κ in the form $\kappa(\{J\};\{k\})$. The positive definiteness of the scalar product yields

$$\kappa(\{J\};\{k\}) \geq 0. \tag{45}$$

Finally we find that the hermiticity condition of an arbitrary group generator Q, i.e. the unitarity of the representation, $(f, Qg) = (g, Qf)^*$ reads

$$\kappa(\{J'\};\{k'\})\langle^{\{J'\}}_{\{k'\}\{m'\}}|\, Q\, |^{\{J\}}_{\{k\}\{m\}}\rangle = \kappa(\{J\};\{k\})\langle^{\{J\}}_{\{k\}\{m\}}|\, Q\, |^{\{J'\}}_{\{k'\}\{m'\}}\rangle^*. \tag{46}$$

We now substitute in this equations the explicit expressions for the non-compact generators as given by making use of the generalized Gell-Mann formula, and allow the representation labels values to be arbitrary complex numbers, e.g., $\sigma_i = \sigma_{iR} + i\sigma_{iI}$, $i = 1, 2, \ldots, n$, and what is left is to solve above equations and determine all possible solutions for the representation labels σ_i and the corresponding kernels of the scalar products, thus determining all $\overline{SL}(n, R)$ unitary representations. The irreducibility of the representations is most effectively achieved by using the little group technique.

Let us present explicitly the simplest case when the scalar product kernel is given by the Dirac δ function. The kernel matrix elements are now trivial, i.e. $\kappa(\{J\};\{k\}) = 1$, for all $\{J\},\ \{k\})$, and the unitarity equations yield $\sigma_i = i\sigma_{iI}$, where σ_{iI} is an arbitrary real number for all $i = 2, 3, \ldots, n$. The corresponding $\overline{SL}(n, R)$ unitary representations constitute the principal series of representations, for which, due to the generalized Gell-Mann formula, we obtained all matrix elements of the non-compact $\overline{SL}(n, R)$ generators.

Acknowledgments

This work was supported in part by MPNTR, Projects OI-171031 and OI-171004.

References

[1] Y. Dothan, M. Gell-Mann and Y. Ne'eman, Series of hadron energy levels as representations of non-compact groups, *Phys. Lett.* **17** (1965) 148–151.

[2] Dj. Šijački, $\overline{SL}(3, R)$ unitary irreducible representations in an algebraic approach to hadronic physics, Ph.D. thesis, Duke University (1974).

[3] Y. Ne'eman, Spinor-type fields with linear, affine and transformations, *Ann. Inst. Henri Poincaré A* **28** (1978) 369–378.

[4] Dj. Šijački, The unitary irreducible representations of $\overline{SL}(3, R)$, *J. Math. Phys.* **16** (1975) 298–311.

[5] Dj. Šijački and Y. Ne'eman, QCD as an effective strong gravity, *Phys. Lett. B* **247** (1990) 571–575.

[6] Dj. Šijački, All $\overline{SL}(3, R)$ ladder representations, *J. Math. Phys.* **31** (1990) 1872–1876.

[7] Y. Ne'eman and Dj. Šijački, GL(4,R) group-topology, covariance and curved-space spinors, *Int. J. Mod. Phys. A* **2** (1987) 1655–1668.
[8] Dj. Šijački, Spinors for spinning p-branes, *Class. Quant. Grav.* **25** (2008) 065009.
[9] F. W. Hehl, G. D. Kerlick and P. von der Heyde, On a new metric affine theory of gravitation, *Phys. Lett. B* **63** (1976) 446–448.
[10] Y. Ne'eman and Dj. Šijački, Unified affine gauge theory of gravity and strong interactions with finite and infinite $\overline{GL}(4, R)$ spinor fields, *Ann. Phys.* $(N.Y.)$ **120** (1979) 292–315; Errata, *ibid.* **125** (1980) 227.
[11] Dj. Šijački, Affine particles and fields, *Int. J. Geom. Methods Mod. Phys.* **2** (2005) 159–188.
[12] M. Hazewinkel (ed.), *Encyclopaedia of Mathematics, Supplement I* (Springer, 1997), p. 269.
[13] R. Hermann, *Lie Groups for Physicists* (W. A. Benjamin Inc, 1965).
[14] R. Hermann, The Gell-Mann formula for representations of semisimple groups, *Comm. Math. Phys.* **2** (1966) 155–164.
[15] G. Berendt, Contraction of Lie algebras: On the reversal problem, *Acta Phys. Austriaca* **25** (1967) 207–211.
[16] Y. Dothan and Y. Ne'eman, Band spectra generated by non-compact algebra, in *Symmetry Groups in Nuclear and Particle Physics* (Benjamin Inc., New York, 1966), pp. 287–310; CALT-68-41 preprint.
[17] E. Inönü and E. P. Wigner, On the contraction of groups and their representations, *Proc. Nat. Acad. Sci.* **39** (1953) 510–524.
[18] Harish-Chandra, Representations of semisimple Lie groups on a Banach space, *Proc. Nat. Acad. Sci.* **37** (1951) 170–173.
[19] Harish-Chandra, Representations of semisimple Lie groups on a Banach space, II, *Proc. Nat. Acad. Sci.* **37** (1951) 362–365.
[20] Harish-Chandra, Representations of semisimple Lie groups on a Banach space, III, *Proc. Nat. Acad. Sci.* **37** (1951) 366–369.
[21] Harish-Chandra, Representations of semisimple Lie groups on a Banach space, IV, *Proc. Nat. Acad. Sci.* **37** (1951) 691–694.
[22] I. Salom and Dj. Šijački, Validity of the Gell-Mann formula for sl(n, R) and su(n) algebras, *Int. J. Geom. Met. Mod. Phys.* **10** (2013) 1350017, 10 pp.
[23] I. Salom and Dj. Šijački, Generalization of the Gell-Mann formula for sl$(5, R)$ and su(5) algebras, *Int. J. Geom. Meth. Mod. Phys.* **7** (2010) 455–470.
[24] I. Salom and Dj. Šijački, Generalization of the Gell-Mann decontraction formula for sl(n, R) and su(R) algebras, *Int. J. Geom. Met. Mod. Phys.* **8** (2011) 395–410.
[25] I. Salom, Decontraction formula for sl(n, R) algebras and applications in theory of gravity, Ph.D. thesis, University of Belgrade (2011) (in Serbian).