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SO(4) algebraic approach to the three-body bound state problem in two dimensions

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We use the permutation symmetric hyperspherical three-body variables to cast the non-relativistic three-body Schrödinger equation in two dimensions into a set of (possibly decoupled) differential equations that define an eigenvalue problem for the hyper-radial wave function depending on an SO(4) hyper-angular matrix element. We express this hyper-angular matrix element in terms of SO(3) group Clebsch-Gordan coefficients and use the latter's properties to derive selection rules for potentials with different dynamical/permutation symmetries. Three-body potentials acting on three identical particles may have different dynamical symmetries, in order of increasing symmetry, as follows: (1) $S_3 \otimes O_L(2)$, the permutation times rotational symmetry, that holds in sums of pairwise potentials, (2) $O(2) \otimes O_L(2)$, the so-called "kinematic rotations" or "democracy symmetry" times rotational symmetry, that holds in area-dependent potentials, and (3) O(4) dynamical hyper-angular symmetry, that holds in hyper-radial three-body potentials. We show how the different residual dynamical symmetries of the non-relativistic three-body Hamiltonian lead to different degeneracies of certain states within O(4) multiplets. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4891399]

I. INTRODUCTION

The quantum three-body problem is an old one with a huge literature – the hyperspherical variables, together with the corresponding hyperspherical harmonics, form one of the best known sets of tools in the theorist's arsenal, Refs. 1–3. Classification/separation of wave functions into distinct classes under permutation symmetry is a fundamental property of (non-relativistic) quantum mechanics with non-trivial consequences in the three-body system. Permutation symmetric three-body hyperspherical harmonics in three dimensions, however, are known only in special cases such as the (small, definite values of the) total angular momentum $L = 0$, 1 ones, cf. Refs. 3 and 4. All other values of L have to be treated separately, usually by means of non-permutation symmetric hyperspherical harmonics. In that way, one loses the manifest permutation symmetry, however, as well as a certain dynamical O(2) symmetry, when the three-body potential is invariant under the so called "kinematic rotation," Ref. 2, or equivalently the "democracy," Refs. 5 and 6, transformations. This symmetry was viewed as mathematical esoterics, see Ref. 5, until recently it was shown, Refs. 7 and 19, to be a dynamical symmetry of area-dependent potentials, which class includes the so-called Y-string potential in QCD. Consequently follows the increased interest in its properties.

In two spatial dimensions, the problem of constructing permutation symmetric hyperspherical harmonics was solved in Ref. 8, however, almost as an afterthought of certain rather abstract internal geometric considerations in Refs. 9 and 10, and certain mathematical aspects of this problem were reconsidered more recently in Ref. 11. Although the need for such a theoretical tool (e.g., in anyon physics, cf. Refs. 12–17) was acute at the time of writing (mid-1990s) of Ref. 8, it never received the attention it deserves. Knowledge of three-body permutation symmetric hyperspherical harmonics in two dimensions (2D) allows one to calculate the discrete part of the energy spectrum of the three-body problem, very much as the quantum mechanical two-body problem can be solved using

SO(3) spherical harmonics in three dimensions. That line of research was not pursued in Ref. 8, nor elsewhere, to our knowledge.[41]

In the present paper, we extend the line of investigation started in Refs. 9 and 10 and continued in Refs. 8 and 11 to show how the Schrödinger equation for three-body bound states in two spatial dimensions can be reduced to an eigenvalue problem for the hyper-radial wave function, where the whole hyper-angular dependence has been reduced to an SO(4) hyperspherical harmonics matrix element that boils down to a product of two SO(3) Clebsch-Gordan coefficients. This is the basic contribution of the present paper. These results are not specific to any one particular three-body problem, i.e., they could find application in many realistic 2D three-body problems, such as the three-anyon one, Refs. 12–17, and/or other condensed matter physics problems in 2D, Ref. 18. The results of this paper have been used to study the 2D version of three-body confinement with the Δ and Y-string.[19,22]

In this way, the three-body problem in two dimensions has been effectively reduced to an SO(4) group theoretical problem (or "algebraized" in vulgate), and one eigenvalue equation for the hyper-radial wave function, a goal that was hypothesized about in three dimensions in Ref. 23 and elsewhere. In this algebraic language, one is looking for the "chain" of algebras $so(2) \oplus so_L(2) \subset so(3) \oplus so(3) \subset so(4)$ (where $so_L(2)$ is the total angular momentum part and $so(2)$ being the so-called "democracy" transformation, where the permutation group S_3 is a (discrete) subgroup of the so-called "kinematic rotations," Ref. 2, or equivalently the "democracy" transformation (continuous) group O(2), Refs. 5 and 6).

On the formal side, these SO(4) hyperspherical harmonics are directly related to the monopole harmonics of Ref. 24, as shown in Ref. 8, and to the spin-weighted spherical harmonics of Ref. 25. Our result, Eq. (31), for the hyper-angular matrix elements of SO(4) hyperspherical harmonics appears to be the first of its kind in the literature. It can be viewed as a continuation of the earlier results for the matrix elements of SO(4) hyperspherical harmonics in Refs. 26 and 27.

As an example of the utility of our results we apply our method to the three-quark confinement problem in 2D and show how we evaluated the (2D) eigen-energy splittings in the K = 2, 3 bands of the spectra of the Δ and Y-string potentials in QCD, that were presented in Refs. 19 and 22.

After defining preliminaries in Sec. II, we define our $SO(4)$ algebraic methods for solving the spectrum of the model in Sec. III. We apply our results to two classes of permutation-symmetric three-body problems in Sec. IV: (a) the three-body sum of pairwise terms, and (b) area-dependent potentials that are invariant under "democracy" O(2) transformations. Section V contains a summary and a discussion of the results.

II. PRELIMINARIES

A. Three-body variables

The $\boldsymbol{\rho}, \boldsymbol{\lambda}$ are the two Jacobi three-vectors, defined by

$$\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\mathbf{x_1} - \mathbf{x_2}), \tag{1}$$

$$\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\mathbf{x_1} + \mathbf{x_2} - 2\mathbf{x_3}). \tag{2}$$

In the relations above, we assume that all three masses are equal. In two spatial dimensions (2D), the full symmetry of the three-body kinetic energy is $O(4)$ and $SO(2)$ is its rotation symmetry. The "larger" symmetry of the non-relativistic kinetic energy is the basis of the hyper-spherical variable approach to the three-body problem.

A crucial ingredient to the solution to the three-body bound state problem are the hyper-spherical coordinates/hyper-angles.[1–3] Here, instead of two Jacobi three-vectors $\boldsymbol{\rho}, \boldsymbol{\lambda}$, defined in Eqs. (1) and (2), the hyper-spherical formalism introduces the hyper-radius R,

$$R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}, \tag{3}$$

and two hyper-angles that are defined by way of three independent scalar three-body variables, e.g., $\boldsymbol{\rho} \cdot \boldsymbol{\lambda}$, $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$. Then, one may use the hyper-space unit-vector $\hat{\mathbf{n}}$

$$\hat{\boldsymbol{n}} = (\boldsymbol{n}'_1, \boldsymbol{n}'_2, \boldsymbol{n}'_3) = \left(\frac{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}{R^2}, \frac{2\boldsymbol{\rho} \cdot \boldsymbol{\lambda}}{R^2}, \frac{2(\boldsymbol{\lambda} \times \boldsymbol{\rho})_3}{R^2} \right) \tag{4}$$

(apparently first introduced by Hopf, Ref. 29) to define a sphere with unit radius. The points on the equatorial unit circle correspond to collinear configurations ("triangles" with zero area). Two angles parametrize this sphere – they can be chosen at will.

The area of the triangle $\frac{\sqrt{3}}{2}|\boldsymbol{\rho} \times \boldsymbol{\lambda}| \simeq |(\mathbf{x_1} - \mathbf{x_2}) \times (\mathbf{x_1} - \mathbf{x_3})|$ and the hyper-radius R are related to the Smith-Iwai variables (α, ϕ),[2,9,10] as follows:

$$(\sin\alpha)^2 = \left(\boldsymbol{n}_1'^2 + \boldsymbol{n}_2'^2\right) = 1 - \left(\frac{2\boldsymbol{\rho} \times \boldsymbol{\lambda}}{R^2}\right)^2, \tag{5}$$

$$\phi = \tan^{-1}\left(\frac{n'_2}{n'_1}\right) = \tan^{-1}\left(\frac{2\boldsymbol{\rho} \cdot \boldsymbol{\lambda}}{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}\right). \tag{6}$$

The standard Delves-Simonov choice of hyper-angles is $(\chi_D = 2\chi, \theta)$,[1,3]

$$(\sin\chi_D)^2 = (\sin 2\chi)^2 = \left(\boldsymbol{n}_2'^2 + \boldsymbol{n}_3'^2\right) = 1 - \left(\frac{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}{R^2}\right)^2, \tag{7}$$

$$\theta = \tan^{-1}\left(\frac{n'_2}{n'_3}\right) = \tan^{-1}\left(\frac{|\boldsymbol{\rho} \times \boldsymbol{\lambda}|}{\boldsymbol{\rho} \cdot \boldsymbol{\lambda}}\right). \tag{8}$$

One must choose the most appropriate parametrization according to the symmetry of the potential, Ref. 7.

Only one set of three-body variables, viz., (R, α, ϕ), with the hyper-angle $\phi = \arctan\left(\frac{2\rho\cdot\lambda}{\lambda^2-\rho^2}\right)$ makes the permutation symmetry manifest, see Ref. 7. That fact makes (α, ϕ) appropriate for permutation-symmetric three-body potentials. The (other) hyper-angle α describes the "scale-invariant area" of the triangle $\cos\alpha = 2R^{-2}(\boldsymbol{\rho} \times \boldsymbol{\lambda})_3$. That makes this set also appropriate for area-dependent potentials.

B. Three-body potentials

Three-body potentials acting on three identical particles can be divided into three interesting classes according to their permutation and/or dynamical symmetries (in order of increasing symmetry): (1) $S_3 \otimes O_L(2)$, the permutation times rotational symmetry, that holds in sums of pairwise potentials, (2) $O(2) \otimes O_L(2) \subset SO(4)$, the so-called "kinematic rotations" or "democracy symmetry" times rotational symmetry, that holds in area-dependent potentials, and (3) the full SO(4) dynamical hyper-angular symmetry, that holds for hyper-radial three-body potentials which do not depend on the shape of the triangle subtended by the three particles, but only on their "mean size," the hyper-radius R.

The third class has the highest symmetry, the harmonic oscillator being one example, but it is also the least realistic one: there are simply no known hyper-radial potentials in nature. Due to its highest symmetry, its energy spectra have the highest levels of degeneracy, and can be used as the starting point for the two cases (1) and (2) with lesser symmetries. For this reason, we shall spend the least amount of space on this (third) class.

The second class corresponds to a certain dynamical O(2) symmetry, when the three-body potential is invariant under the so called "kinematic rotations," Ref. 2, or, equivalently, the "democracy" transformations, Refs. 5 and 6. This (continuous) "kinematic rotations," or "democracy" symmetry is a generalization of the (discrete) permutation symmetry of three bodies. It used to be viewed as something of mathematical esoterics, see Ref. 5, until recently Refs. 7 and 19, showed it to be the dynamical symmetry of the Y-string potential among three quarks in QCD, in particular, and of all three-body potentials that depend only on the area of the triangle subtended by the three particles, in general. Analogous "kinematic rotations," or "democracy" symmetry, for four particles in three spatial dimensions, is the non-Abelian group $SO(3)$, Ref. 20. It is not yet clear what geometrical or physical quantity is kept invariant under the corresponding democracy transformations in the four-body case, Ref. 20, let alone five- or more bodies, Ref. 21.[42]

The first class corresponds to potentials symmetric under the full S_3 permutation group. Potentials with only a two-body permutation S_2 subgroup, or a trivial (S_1) permutation symmetry will not be dealt with here.

(1) In the first class, we consider the three-body sum of pairwise distances to power $\boldsymbol{\alpha}$ (here, we use the boldface greek letter $\boldsymbol{\alpha}$ to distinguish it from the hyperangle α, introduced in Eq. (5) above),

$$V_{\boldsymbol{\alpha}} = \sigma_{\boldsymbol{\alpha}} \frac{1}{2} \sum_{i \neq j = 1}^{3} |\mathbf{x}_i - \mathbf{x}_j|^{\boldsymbol{\alpha}}. \tag{9}$$

Perhaps, the best known example of such a potential (albeit with different signs multiplying each term) is the ($\boldsymbol{\alpha} = -1$) Coulomb potential in atomic and molecular physics. More recently, potentials with different powers $\boldsymbol{\alpha} \neq 1$ have been used in few-body problems in 2D, in condensed matter physics, Ref. 18.

(2) The second class of potentials are the area-dependent ones that have the additional "democracy" dynamical O(2) symmetry, Refs. 6 and 7. Perhaps, the best known example of such a potential is the Y-string one $V_{\mathrm{Y-str.}}$, defined as

$$V_Y = \sigma_Y \min_{\mathbf{x}_0} \sum_{i=1}^{3} |\mathbf{x}_i - \mathbf{x}_0|. \tag{10}$$

The exact string potential Eq. (10) consists of the so-called Y-string term,

$$V_{\mathrm{Y}} = \sigma_{\mathrm{Y}} \sqrt{\frac{3}{2}(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2 + 2|\boldsymbol{\rho} \times \boldsymbol{\lambda}|)}, \tag{11}$$

and three other angle-dependent two-body string, or so-called V-string terms specified in Ref. 37. Manifestly, Eq. (11) depends only on the hyper-radius $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$ and on the area of the triangle $\frac{\sqrt{3}}{2}|\boldsymbol{\rho} \times \boldsymbol{\lambda}|$.

III. THE *SO*(4) ALGEBRAIC METHOD

The decomposition of the three-body spatial wave functions in terms of the $SO(4)$ "grand angular momentum" eigenfunctions is appropriate for all permutation symmetric three-body potentials, including, though not limited to the Y-string. Three-body potentials with lesser permutation symmetry can be treated in this way, as well, though with additional complications. The approximations that are used to solve the three-body Schrödinger equation depend on the potential and form a separate part of the theoretical framework.

A. *SO*(4) symmetry in the hyper-spherical approach

First, we need to define several objects that are needed in subsequent developments. The "grand angular" momentum tensor $K_{\mu\nu}$, $\mu, \nu = 1, 2, 3, 4$

$$\begin{aligned} K_{\mu\nu} &= m\left(\mathbf{X}_\mu \dot{\mathbf{X}}_\nu - \mathbf{X}_\nu \dot{\mathbf{X}}_\mu\right) \\ &= \left(\mathbf{X}_\mu \mathbf{P}_\nu - \mathbf{X}_\nu \mathbf{P}_\mu\right), \end{aligned} \tag{12}$$

where $\boldsymbol{X}_\mu = (\boldsymbol{\rho}, \boldsymbol{\lambda})$. In particular, $l_\rho \equiv K_{12}$ and $l_\lambda \equiv K_{34}$ generate $SO(2)$ rotation of vector $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$, respectively.

Next, we introduce

$$\mathbf{M} = \frac{1}{2}\left(l_\rho + l_\lambda, K_{13} - K_{24}, K_{14} + K_{23}\right), \tag{13}$$

$$\mathbf{N} = \frac{1}{2}\left(l_\rho - l_\lambda, K_{13} + K_{24}, K_{14} - K_{23}\right). \tag{14}$$

Note that $\mathbf{M}$ and $\mathbf{N}$ commute and that each of them satisfies separate $SO(3)$ commutation rules

$$\begin{aligned}\left[\mathbf{M}^i, \mathbf{M}^j\right] &= i\varepsilon^{ijk}\mathbf{M}^k,\\ \left[\mathbf{N}^i, \mathbf{N}^j\right] &= i\varepsilon^{ijk}\mathbf{N}^k,\end{aligned} \tag{15}$$

explicitly demonstrating the $so(4) = so(3) \oplus so(3)$ decomposition. In this context (of $SO(4)$ hyper-spherical harmonics), the Casimir operator eigenvalues of the two $SO(3)$ subgroups $J = M = J' = N$ must be identical, leading to the requirement $J = J' \equiv \frac{1}{2}K$ (this is easily explicitly verified by using Eqs. (12)–(14)). This constraint significantly reduces the number of hyperangular harmonics, i.e., of $SO(4)$ representations that appear in this problem.

A natural basis in the space of an $SO(4)$ irreducible representation, labeled by the J value, is the tensor products basis

$$\left|\begin{matrix} J & J \\ m_1 & m_2 \end{matrix}\right\rangle = |Jm_1\rangle \otimes |Jm_2\rangle,$$

where we are free to choose which component of $\mathbf{M}$, and of $\mathbf{N}$, will be diagonalized and denoted as m_1 and m_2, respectively. We will take m_1 to be eigenvalue of M_1, thus, from Eq. (13), we read off $m_1 = \frac{1}{2}\left(l_\rho + l_\lambda\right) = \frac{L}{2}$, where L is the total angular momentum, a constant of the motion. One possibility, that is appropriate to the case of Delves-Simonov hyperangles (χ_D, θ), would be to take m_2 to be the eigenvalue of N_1, i.e., $m_2 = \frac{1}{2}\left(l_\rho - l_\lambda\right) = \frac{\Delta L}{2}$. (As the ΔL can have (only) integer values, we see that both the "half-integer" $N, M \in \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, and the "integer" $N, M \in 0, 1, 2, \ldots$, representations of $SO(4)$ must appear, Refs. 26 and 27.) We shall find it more convenient to take m_2 to be the eigenvalue of the operator $G \equiv N_2 = \frac{1}{2}(\boldsymbol{\rho} \cdot \mathbf{p}_\lambda - \boldsymbol{\lambda} \cdot \mathbf{p}_\rho)$, corresponding to choice of the Iwai-Smith hyperangles (α, ϕ).

One may need to know the explicit form of the hyper-spherical harmonics. They can be constructed either directly, as in Sec. III B 1 below, or indirectly, by way of their connection with Wu-Yang monopole harmonics, as in Sec. III B, following Ref. 8.

B. *SO*(4) hyper-spherical harmonics

The symmetries of the Y-string confinement potential/hamiltonian are: parity, rotation, and permutation/spatial exchange of particles, or its "generalization" the "democracy group" O(2). Therefore, only wave functions with the same $P = (-1)^{l_\rho + l_\lambda}$, L, and permutation symmetry M (mixed), S (symmetric), and A (antisymmetric) may mix with each other. There are two different states with mixed permutation symmetry: the M_ρ and M_λ. If P_{ij} is the ijth particle permutation/spatial exchange operator, then the permutation symmetry can be examined using the following transposition matrices:

$$P_{12} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{16}$$

$$P_{13} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \tag{17}$$

operating on the transposed "four-vector" $\boldsymbol{X}_\mu^{\mathrm{T}} = (\boldsymbol{\rho}, \boldsymbol{\lambda})^{\mathrm{T}}$ that furnish a basis for the two-dimensional (mixed) irrep. M of S_3.

Mitchell and Littlejohn, Ref. 8, have developed a general theory of $SO(4)$ hyperspherical harmonics for the planar three-body problem. They have shown, *inter alia*, that the two sets (Delves-Simonov and Iwai-Smith) of hyper-angles are related by a (hyper-)rotation through $\frac{\pi}{2}$ about the y-hyper-axis. Next, we shall briefly review that subject as we shall need it for subsequent developments.

1. Iwai-Smith variables SO(4) hyper-spherical harmonics

The general theory of symmetrized $SO(4)$ hyper-spherical harmonics in the Iwai-Smith basis has been developed in Ref. 8 on the basis of monopole harmonics, Ref. 24, or spin-weighted spherical harmonics, Ref. 25. They show an explicit formula, Eq. (5.9) in Ref. 8, for the "symmetric representation" of planar three-body wave functions $\psi^{S}_{\lambda mn}(\alpha, \beta)$ in terms of what they call the "principal axes gauge" $SO(4)$ hyperspherical harmonics, or what we call the Iwai-Smith hyper-angles (α, ϕ), where $\beta = \phi$, which formula reads

$$\begin{aligned} \psi^{S}_{\lambda mn}(\alpha, \beta, \theta) &= \frac{2}{\sqrt{2}\pi} e^{im\theta}\, Y^{\mathrm{PA}}_{m/2,\lambda/2,n/2}(\alpha, \beta) \\ &= \frac{\sqrt{1+\lambda}}{\sqrt{2}\pi} \mathcal{D}^{\lambda/2}_{n/2,-m/2}(-\beta, \alpha, 2\theta), \end{aligned} \tag{18}$$

where $Y^{\mathrm{PA}}_{m/2,\lambda/2,n/2}(\alpha, \beta)$ are related to the Wu-Yang (magnetic) monopole spherical harmonics, Ref. 24, or spin-weighted spherical harmonics, Ref. 25, in the "north regular" gauge, cf. Eq. (2.12) in Ref. 8,

$$Y^{\mathrm{NR}}_{ql\mu}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \mathcal{D}^{l}_{\mu,-q}(-\phi, \theta, \phi), \tag{19}$$

where $\mathcal{D}^{l}_{\mu,-q}(-\phi, \theta, \phi)$ are the Wigner $SO(3)$ rotation matrices defined by

$$\mathcal{D}^{l}_{m,m'}(\alpha, \beta, \gamma) = \langle lm| \exp(-i\alpha J_z) \exp(-i\beta J_y) \exp(-i\gamma J_z) |lm'\rangle. \tag{20}$$

In other words, we may identify $K = \lambda$, $L = m$, $G = n/2$, and write the $SO(4)$ hyper-spherical harmonics ($\mathcal{Y}^{K/2}_{L/2,G}$ in our notation)

$$\begin{aligned} \mathcal{Y}^{K/2}_{L/2,G}(\alpha, \phi, \Phi) &= \frac{2}{\sqrt{2}\pi} e^{iL\Phi}\, Y^{\mathrm{PA}}_{L/2,K/2,G}(\alpha, \phi) \\ &= \frac{\sqrt{1+K}}{\sqrt{2}\pi} \mathcal{D}^{K/2}_{G,-L/2}(-\phi, \alpha, 2\Phi). \end{aligned} \tag{21}$$

Note that when the total angular momentum L vanishes ($L = 0$) it leads to a particular simplification, because then the $SO(4)$ hyper-spherical harmonics reduce to ordinary $SO(3)$ spherical harmonics (modulo a multiplicative constant) in the shape-space hyper-angles (α, ϕ), due to the defining relation (cf. Eq. (1) in Sec. 4.17 of Ref. 28)

$$Y^{*}_{lm}(\beta, \alpha) = \sqrt{\frac{2l+1}{4\pi}} \mathcal{D}^{l}_{m,0}(\alpha, \beta, \gamma). \tag{22}$$

Therefore,

$$\begin{aligned} \psi^{S}_{\frac{K}{2}0G}(\alpha, \phi, \Phi) &= \mathcal{Y}^{K/2}_{0,G}(\alpha, \phi, \Phi) \\ &= \frac{2}{\sqrt{2}\pi}\, Y^{\mathrm{PA}}_{0,K/2,G}(\alpha, \phi) \\ &= \sqrt{\frac{2}{\pi}}\, Y^{G}_{K/2}(\alpha, \phi). \end{aligned} \tag{23}$$

As any three-body (spatial part of) potential must be invariant under overall (ordinary) rotations, it is a scalar, or equivalently, it contains only zero-angular momentum hyperspherical components.

Thus, we have shown that for $L = 0$ one may use an ordinary $SO(3)$ spherical harmonic expansion of the potential to recover the full $SO(4)$ hyperspherical harmonic expansion.

C. The Schrödinger equation in hyper-spherical variables

An important property of the hyper-spherical formalism is that the three-body Schrödinger equation of the three-body systems with factorizable potentials, viz., $V(R, \alpha, \phi) = V(R)V(\alpha, \phi)$, turns into a set of infinitely many (mutually) coupled equations, that reduce to a common hyper-radial Schrödinger equation,

$$-\frac{1}{2m}\left[\frac{d^2}{dR^2} + \frac{3}{R}\frac{d}{dR} - \frac{K(K+2)}{R^2} + 2mE\right]\psi_c(R) + V_{\text{eff.}}(R)\sum_{c'} C_{c,c'}\psi_{c'}(R) = 0 \qquad (24)$$

albeit with different hyper-angular coupling coefficients $C_{c,c'}$. The coupling matrix $C_{c,c'}$ is defined as the proportionality coefficient in the hyper-angular matrix element, Eq. (25)

$$\begin{aligned} V_{\text{eff.}}(R)C_{[K'],[K]} &= \langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|V(R, \alpha, \phi)|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle \\ &= V(R)\langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|V(\alpha, \phi)|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle, \end{aligned} \qquad (25)$$

when the three-body potential can be factored into a hyper-radial $V_{3-\text{body}}(R)$ and hyper-angular part $V_{3-\text{body}}(\alpha, \phi)$. The latter can be expanded in $SO(3)$ (hyper-)spherical harmonics,

$$V_{3-\text{body}}(\alpha, \phi) = \sum_{J,M}^{\infty} v_{JM}^{3-\text{body}} Y_{JM}(\alpha, \phi). \qquad (26)$$

As a consequence of Eq. (23), this is related to the $L = 0$ $SO(4)$ hyper-spherical harmonics $\mathcal{Y}^J_{0M}(\alpha, \phi, \Phi)$ as follows:

$$V_{3-\text{body}}(\alpha, \phi) = \sqrt{\frac{\pi}{2}}\sum_{J,M}^{\infty} v_{JM}^{3-\text{body}} \mathcal{Y}^J_{0M}(\alpha, \phi, \Phi) \qquad (27)$$

leading to

$$V_{\text{eff.}}(R)C_{[K'],[K]} = V(R)\sqrt{\frac{\pi}{2}}\sum_{J,M\geq 0}^{\infty} v_{JM}^{3-\text{body}}\langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|\mathcal{Y}^J_{0M}(\alpha, \phi, \Phi))|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle. \qquad (28)$$

We separate out the $J = 0$ term

$$\begin{aligned} V_{\text{eff.}}(R)C_{[K'],[K]} = V(R)\Bigg(&\delta_{[K'],[K]}\frac{1}{\sqrt{4\pi}}v_{00}^{3-\text{body}} \\ &+ \sqrt{\frac{\pi}{2}}\sum_{J>0,M}^{\infty} v_{JM}^{3-\text{body}}\langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|\mathcal{Y}^J_{0M}(\alpha, \phi, \Phi))|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle\Bigg) \end{aligned} \qquad (29)$$

and absorb the factor $\frac{1}{\sqrt{4\pi}}v_{00}^{3-\text{body}}$ into the definition of $V_{\text{eff.}}(R) = \frac{1}{\sqrt{4\pi}}v_{00}^{3-\text{body}}V(R)$ to find

$$C_{[K'],[K]} = \delta_{[K'],[K]} + \pi\sqrt{2}\sum_{J>0,M}^{\infty}\frac{v_{JM}^{3-\text{body}}}{v_{00}^{3-\text{body}}}\langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|\mathcal{Y}^J_{0M}(\alpha, \phi, \Phi))|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle. \qquad (30)$$

Here, and in the following $\delta_{[K'],[K]} = \delta_{K',K}\delta_{L',L}\delta_{G'_3,G}$. Thus, the problem has been reduced to one of evaluating the SO(4) hyperspherical harmonic matrix elements $\langle \mathcal{Y}_{[K']}(\alpha, \phi, \Phi)|\mathcal{Y}^J_{0M}(\alpha, \phi, \Phi))|\mathcal{Y}_{[K]}(\alpha, \phi, \Phi)\rangle$.

In Table I we show, for reference purposes, the first six coefficients v^{Δ}_{J0} and $\sqrt{\frac{3}{2}}v^{\text{Y}}_{J0}$ in the expansion (27) of the Δ and Y-string potentials, respectively. For a derivation see the Appendix. One can see that the series is alternating (in sign) and converging, though fairly slowly, after the initial rapid drop-off from $J = 0$ to $J = 2$. This dominance of the v_{00} coefficient (the one corresponding to

TABLE I. Expansion coefficients v_{J0}^{Δ} and $\sqrt{\frac{3}{2}}v_{J0}^{\mathrm{Y}}$ of the Δ- and Y-string three-body potentials V_{Δ}, V_{Y}, respectively, in terms of the hyper-spherical O(3) harmonics $Y_{J0}(\alpha, \phi)$, for $J = 0, 2, \ldots, 10$.

J	v_{J0}^{Δ}	$\sqrt{\frac{3}{2}}v_{J0}^{\mathrm{Y}}$
0	10.0265	5.29221
2	0.320285	0.494019
4	0.232132	−0.129813
6	0.0158003	0.0599748
8	−0.00699939	−0.0345825
10	0.00369641	0.0225086

the $SO(4)$-invariant, or "hyper-spherical" component of the three-body potential), which is illustrated in Table I, is a general property of most conventional three-body potentials that demonstrates why the $SO(4)$ symmetry is generally a good starting point for most conventional three-body calculations in two dimensions.

D. The hyperspherical harmonic matrix element

The SO(4) hyperspherical harmonic matrix element in Eq. (30) can be evaluated using the definition of $SO(4)$ hyper-spherical harmonics Eq. (19) and the standard formula for the angular integral over the product of three Wigner D-functions, see, e.g., Eq. (5) in Sec. 4.11.1 of Ref. 28, as the product of the reduced (hyper-angular) $SO(4)$ matrix element $\langle J' || \mathcal{Y}_{0M}^{J} || J'' \rangle$ and the corresponding $SO(4)$ Clebsch-Gordan coefficient, which equals the product of two $SO(3)$ Clebsch-Gordan coefficients. That leads to

$$C_{[K'],[K]} = \delta_{[K'],[K]} + \sum_{J>0,M}^{\infty} \left(\frac{v_{JM}^{3-\mathrm{body}}}{v_{00}^{3-\mathrm{body}}} \right) \sqrt{\frac{(K'+1)(2J+1)}{(K+1)}}\, C_{J0,\frac{K'}{2}\frac{L'}{2}}^{\frac{K}{2}\frac{L}{2}} C_{JM,\frac{K'}{2}G'_3}^{\frac{K}{2}G}, \tag{31}$$

where $C_{Jm,J'm'}^{J''m''}$ are the $SO(3)$ Clebsch-Gordan coefficients in the notation of Ref. 28. This is our main algebraic result: it allows one to evaluate the discrete part of the (energy) spectrum of a three-body potential as a function of its shape-sphere harmonic expansion coefficients $v_{JM}^{3-\mathrm{body}}$.

The matrix ordinary differential equation (ODE) (24) can be diagonalized (in the hyper-angular sub-space of the Hilbert space) before being solved, because the potential $V(R)$ is common to all (matrix) components. That, in turn, is a consequence of the homogeneity of the potential, $V(\lambda R) \to \lambda^{\alpha} V(R)$, under dilations, $R \to \lambda R$, where α is the power of R in the potential[43] that simplifies the solution. Diagonalization of such a matrix is generally not a big problem numerically, but analytic diagonalization has its intrinsic limitations: if the matrix exceeds the 4×4 "size," then the secular equation becomes of the fifth order and thus generally not solvable in closed form. Due to the smallness of the ratio $\frac{v_{JM}^{3-\mathrm{body}}}{v_{00}^{3-\mathrm{body}}} \ll 1$ of the potential's expansion coefficients $v_{00}^{3-\mathrm{body}}$, $v_{JM}^{3-\mathrm{body}}$, the diagonalization of this usually small correction may be adequately dealt with the first-order perturbation theory. This is because the value of v_{00} is usually higher than v_{JM}, for any other $J \neq 0$ value for most conventional three-body potentials, see Table I. This is not to say that one cannot construct three-body potentials with a smaller, or even vanishing value of v_{00}, however.

The $SO(3)$ Clebsch-Gordan coefficients are subject to the usual "triangular" conditions $J' + J'' \geq J \geq |J' - J''|$. As $J = \frac{1}{2}K'$ and $J'' = \frac{1}{2}K$ we find constraints on the values of $J' = J$, where J is the "multipole order" of the interaction: $K' + K \geq 2J \geq |K' - K|$. So, for example, with the lowest non-trivial multipole order $J = 2$, we have additional constraints, discussed in Sec. IV A below. This leads to some remarkable simplifications, for example, the facts (a) that the three lowest K bands eigen-energies are entirely determined by two numbers: the expansion coefficients v_{00} and v_{20}, which has been known at least since Refs. 30 and 31 and (b) that the $K = 3$ band of states introduces just one new free parameter, the $v_{3\pm3}$, which has been known at least since Refs. 32–34.

Moreover, due to angular momentum conservation reflected in the first Clebsch-Gordan coefficient of Eq. (31), $K/2$ and $K'/2$ values must be either both integer, or both half-integer. In turn, this has consequence that all non-diagonal terms with $[K] = [K'] \pm n$, where n is an odd integer, are forbidden.

IV. APPLICATION TO THREE-QUARK CONFINEMENT

Our own interest in this matter stems from the three-quark confinement by the Δ and/or Y-strings, Refs. 7 and 37–39. Lattice QCD appears to demand one of two confining potentials: either the so-called Y-junction string three-quark potential, Eq. (10), as suggested in Refs. 35 and 36, or the sum of two-body ("Δ-string") potentials

$$V_{\Delta-\mathrm{str.}} = \sigma_{\Delta} \sum_{i>j=1}^{3} |\mathbf{x}_i - \mathbf{x}_j|. \tag{32}$$

The Y-string potential contains certain two-body terms when one of the angles in the triangle subtended by the quarks is greater than $\frac{2}{3}\pi$, cf. Subsection 2 of the Appendix. In the present paper, we shall ignore such terms, which generally make (very) small contributions to the energies of low-lying states, as shown explicitly in Ref. 37. We were led to the permutation symmetric hyperangles in the process of our study of the dynamical symmetry of the Y-string Refs. 7 and 37–39: the residual $\mathrm{O}_L(2) \otimes \mathrm{O}(2) \subset \mathrm{SO}(4)$ dynamical symmetry of the non-relativistic Y-string Hamiltonian is best visualized in terms of permutation-adapted hyper-angles.

Present methods can relate the regularities in the spectrum to the permutation, or dynamical symmetry properties of the potential. Moreover, one can use this method in systems with "only" two (rather than three) identical particles, i.e., in potentials that are only partially permutation symmetric, such as the Coulomb bound state(s) of two electrons and one positron (or vice versa).

A. The string potential's hyper-angular matrix elements

The S_3 permutation group symmetrized hyper-spherical harmonics correspond to different $\mathrm{SU}(6)_{\mathrm{FS}}$ symmetry multiplets (Young diagrams/tableaux) of the three-quark system: $S \leftrightarrow 56$, $A \leftrightarrow 20$, and $M \leftrightarrow 70$. For more about the $\mathrm{SU}(6)_{\mathrm{FS}}$ symmetry multiplets and their relation to the S_3 permutation group, see Ref. 40. Thus, we may use the "democracy" index G to classify the wave functions, i.e., the symmetrized hyper-spherical harmonics, according to their S_3 permutation group, or equivalently to their $\mathrm{SU}(6)_{\mathrm{FS}}$ symmetry properties.

1. The Y-string and other area-dependent potentials

In the case of area-dependent potentials,

$$\begin{aligned} V_{\mathrm{area-dep.}}(\alpha, \phi = 0) &= \sum_{n=0,2,\ldots}^{\infty} v_{n0}^{\mathrm{area-dep.}} Y_{n0}(\alpha, \phi = 0) \\ &= \sqrt{\frac{\pi}{2}} \sum_{J=K/2=0,2,\ldots;G=0}^{\infty} v_{J,0}^{\mathrm{area-dep.}} \mathcal{Y}_{0,0}^{J}(\alpha, \phi = 0, \Phi = 0) \end{aligned} \tag{33}$$

the expansion coefficients $v_{n0}^{\mathrm{area-dep.}}$, corresponding to $v_{JM}^{3-\mathrm{body}}$ in Eq. (27), have non-vanishing values only for the zero value of the "hyper-magnetic" quantum number $M = 0$, due to the independence of the area-dependent potentials on the azimuthal angle ϕ, see Subsection 2 of the Appendix.

Therefore, the hyper-angular matrix $C_{[K'],[K]}$ of an area-dependent three-body potential becomes

$$\begin{aligned} C_{[K'],[K]} &= \delta_{[K'],[K]} + \frac{v_{20}}{v_{00}} \sqrt{4\pi} \langle \mathcal{Y}_{[K']}(\alpha,\phi) | Y_{20}(\alpha,\phi) | \mathcal{Y}_{[K]}(\alpha,\phi) \rangle + \cdots \\ &= \delta_{[K'],[K]} + \frac{v_2}{v_0} \sqrt{4\pi} \sqrt{\frac{\pi}{2}} \langle \mathcal{Y}^{\frac{1}{2}K'}_{\frac{1}{2}L',G'}(\alpha,\phi) | \mathcal{Y}^{2}_{00}(\alpha,\phi) | \mathcal{Y}^{\frac{1}{2}K}_{\frac{1}{2}L,G}(\alpha,\phi) \rangle + \cdots . \end{aligned} \tag{34}$$

We use the potential's $SO(4)$ transformation properties to express its matrix element in terms of general $SO(4)$ Clebsch-Gordan coefficients; to that end we note that such area-dependent potentials are eigenfunctions of two $SO(4)$ generators: $\mathbf{M}^1 = \frac{1}{2}L$ and $\mathbf{N}^2 = G$ with both eigenvalues being equal to zero. Thus, a residual dynamical $SO_L(2) \otimes SO(2)$ symmetry of ordinary rotations (in the physical/geometric configuration space) and hyper-rotations (in the shape space) remains in this system. In addition to this, space parity transformation and permutation of two particles Eq. (16), which are also symmetries of this potential, extend the residual symmetry to $O_L(2) \otimes O(2)$.

Therefore, the hyper-angular matrix element of the Y-string and other area-dependent potentials is, in the lowest order, proportional to the product of the reduced matrix element $\langle J' || \mathcal{Y}^2 || J \rangle$, that is explicitly determined in Eq. (31), and the corresponding $SO(4)$ Clebsch-Gordan coefficient

$$\left\langle \begin{matrix} J' & J' \\ m_1' & m_2' \end{matrix} \middle| \mathcal{Y}^2_{00} \middle| \begin{matrix} J & J \\ m_1 & m_2 \end{matrix} \right\rangle = \langle J' \, || \mathcal{Y}^2 | \, | J \rangle \left\langle \begin{matrix} J' & J' \\ m_1' & m_2' \end{matrix} \middle| \begin{matrix} 2 & 2; & J & J \\ 0 & 0; & m_1 & m_2 \end{matrix} \right\rangle .$$

This $SO(4)$ Clebsch-Gordan coefficient is the product of two $SO(3)$ Clebsch-Gordan coefficients

$$\begin{aligned} \left\langle \begin{matrix} J' & J' \\ m_1' & m_2' \end{matrix} \middle| \begin{matrix} 2 & 2; & J & J \\ 0 & 0; & m_1 & m_2 \end{matrix} \right\rangle &= (J, m_1, 2, 0 | J', m_1')(J, m_2, 2, 0 | J', m_2') \\ &= \delta_{m_1 m_1'} \delta_{m_2 m_2'} (J, m_1, 2, 0 | J', m_1)(J, m_2, 2, 0 | J', m_2). \end{aligned} \tag{35}$$

The corresponding (non-vanishing) $SO(3)$ Clebsch-Gordan coefficients are those with: $J' = J$, $J' = J \pm 1$, and $J' = J \pm 2$. It is clear, however, that some of these matrix elements are often not necessary. For example, the product $(J, m_1, 2, 0 | J \pm 1, m_1)(J, m_2, 2, 0 | J \pm 1, m_2)$ vanishes, due to symmetries of Clebsch-Gordan coefficients, when either $m_1 = 0$ (angular momentum of the state is zero) or $m_2 = 0$ (this also holds for higher order corrections from Eq. (33) – when either one of m_1 and m_2 is zero, the difference $|J - J'|$ must be even, i.e., $|K - K'|$ is a multiple of 4). Even when neither is the case, the $(J, m_2, 2, 0 | J + 1, m_2)$ Clebsch-Gordan coefficient connects states with values of K that differ by two units, which is important only when the $(\mathrm{K} + 2)$ band energy is degenerate with some K-band hyper-radially excited state, which happens only in the harmonic oscillator and $1/R$ hyper-Coulomb potentials.

Moreover, the Clebsch-Gordan coefficient $(J, m_2, 2, 0 | J \pm 2, m_2)$ is physically significant in situations when the absolute value of the difference of K's for the two states equals four: $|K - K'| = 4$ and the unperturbed levels are degenerate, something that only happens in the higher shells/bands of the harmonic oscillator and the $1/R$ hyper-Coulomb potential. Thus, for most practical purposes, we shall only need the $J' = J$ terms.

2. The Δ-string potential

The Δ-string potential contains all of the "ordinary" multipoles present in the area-dependent potentials, though not in the same proportion. The first distinctly "two-body potential" contribution transforms as $Y_{3\pm3}(\alpha, \phi) = \sqrt{\frac{\pi}{2}} \mathcal{Y}^3_{0\pm3}$. The corresponding coefficient for the Δ-string potential is $v^{\Delta}_{3\pm3} = 0.141232$, see Subsection 1 of the Appendix. This breaks the residual dynamical $O(2) \otimes O_L(2)$ symmetry down to $S_3 \otimes O_L(2)$. Consequently, the Clebsch-Gordan coefficients appearing in Eq. (31) are different as well, so they bring about different selection rules: the $v_{3\pm3}$ term can only contribute to $K \geq 3$ matrix elements.

B. Results

The numerical results have been discussed in Refs. 19 and 22, so here we shall only discuss the "missing steps" in their derivation.

1. K = 2 band results

We have calculated the hyper-angular matrix elements $\langle \mathcal{Y}^2_{00} \rangle_{\mathrm{ang}}$ for the SU(6) multiplets (states with the same permutation group S_3 properties) of the four lowest K(=0,1,2,3) bands: as explained

TABLE II. The values of the three-body potential hyper-angular matrix elements $\pi\sqrt{2}\langle\mathcal{Y}^2_{00}\rangle_{\rm ang}$, for various K = 2 states (for all allowed orbital waves L). The correspondence between the S_3 permutation group irreps. and SU(6)$_{FS}$ symmetry multiplets of the three-quark system: $S \leftrightarrow 56$, $A \leftrightarrow 20$, and $M \leftrightarrow 70$.

K	$[SU(6), L^P]$	$\pi\sqrt{2}\langle\mathcal{Y}^2_{00}\rangle_{\rm ang}$
2	$[70, 0^+]$	$-\frac{1}{\sqrt{5}}$
2	$[56, 2^+]$	$-\frac{1}{\sqrt{5}}$
2	$[70, 2^+]$	$\frac{1}{2\sqrt{5}}$
2	$[20, 0^+]$	$\frac{2}{\sqrt{5}}$

earlier, the $K = 0$, 1 bands are affected only by the v_{00} coefficient. The calculated energy splittings of $K = 2$ band states depend only on the Clebsch-Gordan coefficient $(J, m_2, 2, 0|J, m_2)$ with various values of $m_1 = G$ and $m_2 = L/2$ belonging to different SU(6) multiplets being listed in Table II. Our main concern is the energy splitting pattern among the states within the K = 2 hyper-spherical $SO(4)$ multiplet. The hyper-radial matrix elements of the linear hyper-radial potential are identical for all the (hyper-radial ground) states in one K band. Therefore, the 2D energy differences among various sub-states of a particular K band multiplet are integer multiples of the energy splitting "unit" Δ_K, just as they are in 3D.

Note that the two K = 2 $SU(6)$, or S_3 multiplets $[70, 0^+]$ and $[56, 2^+]$ are degenerate in 2D, as opposed to 3D, where they are split. Moreover, the 3D $[20, 1^+]$ multiplet has $L = 0$ in 2D. Otherwise, the 2D and 3D states coincide and their energy splitting patterns agree. This is but another manifestation of the Bowler-Tynemouth theorem[32,33] for this class of three-body potentials.

The 2D splitting pattern is similar, but not identical to the 3D one: the 2D multiplets $[20, 0^+]$, $[70, 2^+]$, $[20, 0^+]$, $[56, 2^+]$, are split just like the 3D multiplets $[20, 1^+]$, $[70, 2^+]$, $[20, 0^+]$, $[56, 2^+]$, but the $[70, 0^+]$ and the $[56, 2^+]$ are degenerate in 2D, whereas they are split in 3D. This indicates the differences between 2D and 3D in this problem.

2. $K = 3$ band results

The calculated energies of states with of K = 3 and various values L are listed in Table II and displayed in Fig. 2 of Ref. 22. With an area-dependent (i.e., ϕ-independent) potential in 2D, we find that the K = 3 band $SU(6)$, or S_3 multiplets have one of (only) two possible energies: the ($[70, 1^-]$, $[56, 3^-]$, $[20, 3^-]$) are degenerate, as are ($[70, 3^-]$, $[56, 1^-]$, $[20, 1^-]$) (at a different energy) (Tables III and IV). Note that the 3D $[70, 2^-]$ multiplet has no analogon in 2D. Upon introduction of a ϕ-dependent ("two-body") potential component proportional to $v^{\Delta}_{3\pm 3}$, and upon diagonalization of the $C_{[K'],[K]}$ matrix, one finds further splittings among the previously degenerate states $[70, 1^-]$,

TABLE III. The values of the three-body potential hyper-angular diagonal matrix elements $\langle Y_{20}(\alpha, \phi)\rangle_{\rm ang}$, for various K = 3 states (for all allowed orbital waves L).

K	$[SU(6), L^P]$	$\pi\sqrt{2}\langle\mathcal{Y}^2_{00}\rangle_{\rm ang}$
3	$[20, 1^-]$	$-\frac{1}{\sqrt{5}}$
3	$[56, 1^-]$	$-\frac{1}{\sqrt{5}}$
3	$[70, 1^-]$	$\frac{1}{\sqrt{5}}$
3	$[56, 3^-]$	$\frac{1}{\sqrt{5}}$
3	$[70, 3^-]$	$-\frac{1}{\sqrt{5}}$
3	$[20, 3^-]$	$\frac{1}{\sqrt{5}}$

TABLE IV. The values of the off-diagonal matrix elements of the hyper-angular part of the three-body potential $\langle \mathcal{Y}^{K}_{\frac{1}{2}L_f,G_{3f}} | \mathcal{Y}^{3}_{0\pm3} | \mathcal{Y}^{K}_{\frac{1}{2}L_i,G_{3i}} \rangle_{\text{ang}}$, for various K = 3 states (for all allowed orbital waves L).

K	$[SU(6)_f, L_f^P]$	$[SU(6)_i, L_i^P]$	$\pi\sqrt{2}\langle \mathcal{Y}^{K}_{\frac{1}{2}L_f,G_{3f}} \| \mathcal{Y}^{3}_{0\pm3} \| \mathcal{Y}^{K}_{\frac{1}{2}L_i,G_{3i}} \rangle_{\text{ang}}$
3	[20, 1$^-$]	[56, 1$^-$]	$-\frac{6}{\sqrt{35}}$
3	[56, 1$^-$]	[20, 1$^-$]	$-\frac{6}{\sqrt{35}}$
3	[70, 1$^-$]	[all, 1$^-$]	0
3	[56, 3$^-$]	[20, 3$^-$]	$\frac{2}{\sqrt{35}}$
3	[70, 3$^-$]	[all, 1$^-$]	0
3	[20, 3$^-$]	[56, 3$^-$]	$\frac{2}{\sqrt{35}}$

[56, 3$^-$], and [20, 3$^-$], as well as among the [70, 3$^-$], [56, 1$^-$], and [20, 1$^-$]

$$
\begin{aligned}
&[20, 1^-] \;\; v_{00} - \frac{1}{\sqrt{5}} v_{20} + \frac{2}{\sqrt{35}} v_{33},\\
&[56, 1^-] \;\; v_{00} - \frac{1}{\sqrt{5}} v_{20} - \frac{2}{\sqrt{35}} v_{33},\\
&[70, 1^-] \;\; v_{00} + \frac{1}{\sqrt{5}} v_{20},\\
&[70, 3^-] \;\; v_{00} - \frac{1}{\sqrt{5}} v_{20},\\
&[20, 3^-] \;\; v_{00} + \frac{1}{\sqrt{5}} v_{20} + \frac{6}{\sqrt{35}} v_{33},\\
&[56, 3^-] \;\; v_{00} + \frac{1}{\sqrt{5}} v_{20} - \frac{6}{\sqrt{35}} v_{33}.
\end{aligned} \tag{36}
$$

For the K = 3 band in 3D, the energy splittings have been calculated by Bowler and Tynemouth[32,33] for two-body anharmonic potentials perturbing the harmonic oscillator and confirmed and clarified by Richard and Taxil, Ref. 34, in the hyper-spherical formalism with linear two-body potentials (the Δ-string). One should compare the above results, Eqs. (36), with Eq. (45) in Ref. 34, in particular. Comparing the sizes of the $v^{\Delta}_{3\pm3}$-induced splittings in 3D and 2D, one finds comparable values: 1/3 in 2D vs. 2/7 in 3D.

In hindsight, Richard and Taxil's, Ref. 34, separation of $V_4(R)$ and $V_6(R)$ potentials' contributions is particularly illuminating (prescient?): the former corresponds precisely to our "area-dependent" term v_{20} and the latter to the "two-body" contribution $v_{3\pm3}$. As both the Y- and Δ strings contain the former, whereas only the Δ string contains the latter, we see that the latter to be the source of different degeneracies/splittings in the spectra of these two types of potentials. This was not noted by Richard and Taxil, Ref. 34, however, so our contribution here is the (first) demonstration of this fact in 2D. The 3D case ought to be analogous, but has not been worked out in detail, yet.

V. SUMMARY AND DISCUSSION

In summary, we have reduced the non-relativistic three-body bound state problem in a permutation symmetric potential in two dimensions to a single ordinary differential equation for the hyper-radial wave function with coefficients determined by $SO(4)$ group-theoretical arguments multiplying the (homogenous) hyper-radial potential. That one equation can be solved just like the radial Schrödinger equation in 3D. The breaking of the $SO(4)$ symmetry determines the spectrum.

In 2D, the "hyper-spherical symmetry" is $SO(4)$, and the residual dynamical symmetry of the potential is $O(2) \otimes O_L(2) \subset SO(4)$, where $O_L(2)$ is the rotational symmetry associated with the (total) angular momentum. Due to the fact that $so(4) \simeq so(3) \oplus so(3)$, one may use many of the $so(3)$ algebra results, such as the $SO(3)$ Clebsch-Gordan coefficients. Thus, we looked at the "algebra chain" $so(2) \oplus so_L(2) \subset so(3) \oplus so_L(2) \subset so(4)$.

We formulated the problem in terms of $SO(4)$-group covariant three-body variables and then brought the Schrödinger equation into a form that can be (exactly) solved. More specifically, we expanded the three-body Schrödinger equation and its eigen-functions, as well as the potential in $SO(4)$ hyperspherical harmonics. Then we showed how the energy eigenvalues (the energy spectrum) can be calculated as functions of the three-body potential's (hyper-)spherical harmonics expansion coefficients, of the $SO(4)$ reduced matrix element(s) and of the $SO(4)$ Clebsch-Gordan coefficients, that are related to the ordinary $SO(3)$ Clebsch-Gordan coefficients.

The dynamical $O(2)$ symmetry of the Y-string potential was discovered in Ref. 7, with the permutation group $S_3 \subset O(2)$ as its subgroup. The existence of an additional dynamical symmetry strongly suggested an algebraic approach, such as those used in Refs. 32 and 33, which were not based on the Y-string, however. A careful perusal of Refs. 32 and 33 showed that an $O(2)$ group had been used as an enveloping structure for the permutation group $S_3 \subset O(2)$, but was not interpreted as a (possible) dynamical symmetry. With this in mind we started an algebraic study of the Y-string-like ("collective") area-dependent potentials. We first established the basic consequences of this dynamical symmetry of the Y-string potential. The 3D case is substantially more complicated than the 2D one: for this reason we have limited ourselves to the two dimensions in this paper.

We have used these results to calculate the energy splittings of various SU(6)/S_3 multiplets in the K = 2 and K = 3 bands of the Y- and Δ string spectra, and found close correspondence with the splittings calculated by other methods in three dimensions. It is only in the K = 3 band that a difference appears between the spectra of these two confining models. That is, the first explicit consequence of the dynamical $O(2)$ symmetry of the Y-string.

Our results can be used in other three-body problems in two dimensions, such as the three-anyon problem, Refs. 13–17, and some other condensed matter physics problems, Ref. 18. There is also hope that one can extend these methods to three dimensions and thus simplify the hyper-spherical harmonics approach to the three-body problem in general.

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APPENDIX: THREE-BODY POTENTIALS IN TERMS OF HYPER-SPHERICAL VARIABLES

1. The sum of two-body α-power potentials

The Δ-string potential $V_{\Delta-\text{str.}}$, Eq. (32), is proportional to the sum of pairwise distances between the bodies. It can be viewed as a special case ($\boldsymbol{\alpha} = 1$) of the three-body sum of pairwise distances to power $\boldsymbol{\alpha}$ Eq. (9).

In terms of Iwai-Smith hyper-angles, Eq. (32) reads

$$V_{\Delta-\text{str.}}(R,\alpha,\phi) = \sigma_\Delta R\left(\sqrt{1+\sin(\alpha)\sin\left(\frac{\pi}{6}-\phi\right)} + \sqrt{1+\sin(\alpha)\sin\left(\phi+\frac{\pi}{6}\right)} + \sqrt{1-\sin(\alpha)\cos(\phi)}\right). \tag{A1}$$

In order to find the general hyper-spherical harmonic expansion of the sum of α-power two-body potentials, we note that it factors into the hyper-radial $V_{\boldsymbol{\alpha}}(R) = \sigma_{\boldsymbol{\alpha}} R^{\boldsymbol{\alpha}}$ and the hyper-angular part

$V_{\alpha}(\alpha, \phi)$

$$V_{\alpha}(R, \alpha, \phi) = V_{\alpha}(R) V_{\alpha}(\alpha, \phi) = V_{\alpha}(R) \sum_{J,M}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi), \tag{A2}$$

where

$$v_{JM}^{\alpha} = \int_0^{2\pi} d\phi \int_0^{\pi} V_{\alpha}(\alpha, \phi)\, Y_{JM}^{*}(\alpha, \phi) \sin(\alpha)\, d\alpha. \tag{A3}$$

We note that any S^3 permutation symmetric sum of two-body potentials (with the sole exception of the harmonic oscillator) has a specific "triple-periodic" azimuthal ϕ hyper-angular dependence with the angular period of $\frac{2}{3}\pi$. That provides additional selection rules for the magnetic quantum number M dependent terms in this expansion, besides the $J = 0, 2, \ldots$ rule for $M = 0$ terms discussed below in Subsection 2

$$\sum_{JM}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi) = \sum_{J=0,2,\ldots}^{\infty} v_{J0}^{\alpha} Y_{J0}(\alpha, \phi) + \sum_{J;M=\pm 3}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi) + \sum_{J;M=\pm 6}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi) + \cdots. \tag{A4}$$

Reality of the potential $V = \Re e(V)$ and the "azimuthal parity" under $(\phi \to -\phi)$ lead to the fact that only the "zonal harmonics" coefficients, Eq. (A5) survive, whereas the "sectorial harmonics" coefficients, Eq. (A6) vanish

$$v_{JM} = \frac{1}{2}\left(v_{JM} + v_{JM}^{*}\right), \tag{A5}$$

$$0 = \frac{1}{2}\left(v_{JM} - v_{JM}^{*}\right). \tag{A6}$$

The aforementioned reflection symmetry with respect to the "hyper-equatorial plane" $(\cos(\alpha) \to -\cos(\alpha))$, adds new selection rules for each of the new sub-series. For example,

$$\sum_{J;M=\pm 3}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi) = \sum_{J=3,5,7,\ldots;M=\pm 3}^{\infty} v_{JM}^{\alpha} Y_{JM}(\alpha, \phi). \tag{A7}$$

The first such non-vanishing coefficient for the Δ-string potential is $v_{3\pm 3}^{\Delta} = 0.141232$. Thus, we see that the number of non-vanishing coefficients in the Iwai-Smith parametrization of the shape sphere is decimated, as compared with the number of the Delves-Simonov parametrization coefficients which fact ought to improve the speed of convergence of corresponding numerical calculations.

2. Area-dependent potentials and their dynamical symmetry

The Y-string potential $V_{\mathrm{Y-str.}}$ is defined in Eq. (10). The complexity of the Y-string potential is perhaps best seen when expressed in terms of three-body Jacobi (relative) coordinates $\boldsymbol{\rho}, \boldsymbol{\lambda}$: The exact string potential Eq. (10) consists of the so-called Y-string term, Eq. (11), which is valid when

$$\begin{cases} 2\boldsymbol{\rho}^2 - \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 - 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}, \\ 2\boldsymbol{\rho}^2 + \sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda} \geq -\rho\sqrt{\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2 + 2\sqrt{3}\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}, \\ 3\boldsymbol{\lambda}^2 - \boldsymbol{\rho}^2 \geq -\frac{1}{2}\sqrt{(\boldsymbol{\rho}^2 + 3\boldsymbol{\lambda}^2)^2 - 12(\boldsymbol{\rho}\cdot\boldsymbol{\lambda})^2}, \end{cases}$$

and three other angle-dependent two-body string, or so-called V-string terms (for their explicit functional form, see Ref. 37). These additional terms are relevant only when the above angular conditions are not met – which occurs only in a small part of the configuration space – and contribute to the same "sub-leading" multipoles $v_{J\pm 3}$ (the lowest one being $v_{3\pm 3}$) as the sum of two-body terms in Eq. (A7). In this sense, the V-string terms are indistinguishable from the Δ-string contributions, except by the size of their contributions, which is smaller than the Δ-string's. Of course, they contribute to the "leading" multipoles υ_{J0}, as well. Thus, their effect can be thought of as one of slightly changing the values of the Y-string multipoles. For this reason, we shall ignore these two-body V-string terms hereafter.

The $|\boldsymbol{\rho} \times \boldsymbol{\lambda}|$ term in Eq. (11) is proportional to the area of the triangle subtended by the three quarks. Next, we show that $V_{\rm Y}$ is a function of both Delves-Simonov hyper-angles (χ, θ),

$$V_{\rm Y}(R, \chi, \theta) = \sigma_{\rm Y} R \sqrt{\frac{3}{2}\left(1 + \sin 2\chi |\sin\theta|\right)}, \tag{A8}$$

whereas it is a function of only one Smith-Iwai hyper-angle – the "polar angle" α

$$V_{\rm Y}(R, \alpha, \phi) = \sigma_{\rm Y} R \sqrt{\frac{3}{2}\left(1 + |\cos\alpha|\right)}. \tag{A9}$$

This independence of the "azimuthal" Smith-Iwai hyper-angle ϕ means that the associated component G of the hyper-angular momentum is a constant-of-the-motion of the Y-string; this result holds in all area-dependent potentials, Ref. 7.

Equation (A9) can be further re-written as a (non-polynomial) function of (the absolute value of) only one $SO(3)$ (hyper-)spherical harmonic in the shape (hyper-)space: using the following formula for $Y_{10}(\alpha, \phi)$:

$$\cos\alpha = \sqrt{\frac{4\pi}{3}} Y_{10}(\alpha, \phi), \tag{A10}$$

that leads to

$$V_{\rm Y}(R, \alpha, \phi) = \sigma_{\rm Y} R \sqrt{\frac{3}{2}\left(1 + \sqrt{\frac{4\pi}{3}} |Y_{10}(\alpha, \phi)|\right)}. \tag{A11}$$

Now, the absolute value of $|Y_{10}(\alpha, \phi)|$ can be expressed as $\sqrt{Y_{10}^{*}(\alpha, \phi) Y_{10}(\alpha, \phi)}$ and the $SO(3)$ Clebsch-Gordan expansion can be applied to $Y_{10}^{*}(\alpha, \phi) Y_{10}(\alpha, \phi)$, which contains only the (even) values of $J = 0, 2$

$$\begin{aligned} |\cos\alpha| &= \sqrt{\frac{4\pi}{3}} \sqrt{Y_{10}^{*}(\alpha, \phi) Y_{10}(\alpha, \phi)} \\ &= \sqrt{\frac{4\pi}{3}} \sqrt{Y_{00}^{2}(\alpha, \phi) + \frac{2}{\sqrt{5}} Y_{00}(\alpha, \phi) Y_{20}(\alpha, \phi)} \\ &= \sqrt{\frac{1}{3}} \sqrt{1 + \frac{2}{\sqrt{5}} \frac{Y_{20}(\alpha, \phi)}{Y_{00}(\alpha, \phi)}}. \end{aligned} \tag{A12}$$

The square root in Eq. (A12) can be expanded in a Taylor-like series, the first two terms of which coincide with the expansion in Legendre polynomials, or $SO(3)$ spherical harmonics, and in $SO(4)$ hyper-spherical harmonics. Therefore, the exact Legendre polynomial expansion of Eq. (A12) runs over even-order $J = 0, 2, 4, \ldots$, zero "hyper-magnetic" quantum number $G = M = 0$ $SO(3)$ (hyper-)spherical harmonics. This is not an accident: all three-body potentials are invariant under the reflection symmetry with respect to the "hyper-equator" $\cos(\alpha) \to -\cos(\alpha)$, which together with the independence of $V_{\rm Y}$ on the azimuthal hyper-angle ϕ leads to the fact that this series cannot depend on the "hyper-magnetic quantum number" $G = M$ and consequently to the aforementioned "selection rule": it is a sum over even values of J only

$$V_{\rm Y}(R, \alpha, \phi) = \sigma_{\rm Y} R \sqrt{\frac{3}{2}} \sum_{J=0,2,\ldots}^{\infty} \upsilon_{J0}^{\rm Y} Y_{J0}(\alpha, \phi), \tag{A13}$$

where $\upsilon_{J0}^{\rm Y}$, $J = 0, 2, \ldots$.

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[41] One recent attempt at solving the three-body Schrödinger equation in 2D can be found in Ref. 11, albeit without using the permutation symmetric hyperspherical harmonics.
[42] It has been argued by the referee of this paper that these “democracy” transformations are identical, in fluid mechanics, i.e., in the $N \to \infty$ limit, to the indistinguishability of fluid particles, which, in turn has been shown to lead to Kelvins circulation theorem.
[43] Clearly, this does not hold for non-polynomial potentials, e.g., in the Yukawa potential, so this is a particular property of the homogenous polynomial hyper-radial potentials.