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Vortex solitons in the (2 + 1)-dimensional nonlinear Schrödinger equation with variable diffraction and nonlinearity coefficients

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Abstract
Using Hirota's bilinear method, we determine approximate analytical localized solutions of the $(2 + 1)$-dimensional nonlinear Schrödinger equation with variable diffraction and nonlinearity coefficients. Our results indicate that a new family of vortices can be formed in the Kerr nonlinear media in the cylindrical geometry. Variable diffraction and nonlinearity coefficients allow utilization of the soliton management method. We present solitary solutions for two types of distributed coefficients: trigonometric and exponential. It is demonstrated that the soliton profiles found are structurally stable, but slowly expanding with propagation.

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(Some figures may appear in color only in the online journal)

1. Introduction

After the prediction of self-trapping [1] of optical solitons in nonlinear media [2, 3], there have been many theoretical and experimental studies of stabilization of such solutions, under different conditions for nonlinearity. It was demonstrated that a bright soliton in $(1 + 1)$ dimensions in a Kerr medium is stable for positive or self-focusing nonlinearity in the nonlinear Schrödinger equation (NLSE) [3]. However, in $(2+1)$ dimension ($(2+1)$D) or $(3+1)$D in a homogeneous bulk Kerr medium, one has *not* found stable soliton-like axisymmetric cylindrical beams or stable optical wave packets [4–6].

In recent years, dispersion management (DM) of optical solitons has attracted much interest and is expected to be a major concept in future soliton-based communication systems. It was shown theoretically and experimentally that the strong DM regime provides for stable propagation of pulses over very long distances. DM solitons are robust to the Gordon–Haus timing jitter, which makes them favorable against the standard solitons [7, 8]. Additionally, quasi-phase matching in quadratic media produces averaged equations that contain Kerr-type nonlinearities [9, 10] and thus DM solitons have potential to be realized in a wide array of systems.

Recently, a model amenable to DM was developed for the propagation of an optical beam in a nonlinear waveguide array [11]. The width of the beam and the amplitude of discrete spatial solitons, called the diffraction-managed solitons [11], evolve in time periodically. In this context, the solitons considered in this paper are close to the diffraction-managed solitons. However, they do expand slowly with propagation and attenuate in intensity, remaining structurally stable. We term them solitons, although a more precise definition would be the solitary vortex beams. A comprehensive review of nonlinear phenomena connected with the optical solitons in continuous and discrete systems can be found in [3].

Although well studied in the one-dimensional (1D) case, the 2D and 3D extensions of this problem are far less explored. The major complication is that the solutions to

NLSE in two and three dimensions are unstable against collapse. In recent years, it has been demonstrated that the nonlinearity management can prevent the collapse of solitons in 2D Kerr-type optical media [12, 13]. It has also been confirmed, by means of direct simulations, that an axisymmetric cylindrical beam in (2 + 1)D can be stabilized in a layered medium, as well as in 2D Bose–Einstein condensates [14, 15], if a variable sign-changing nonlinearity coefficient is used in different layers [12, 13]. As a result, it was observed that the beam could survive over large propagation distances if the Kerr coefficient in a layered medium is allowed to vary between successive self-focusing and self-defocusing nonlinearities, i.e. between the positive and negative values [7]. From these facts one can reasonably expect that the DM can play a balancing role also in the 2D case, and the stable 2D DM solitons can exist.

In modern optical propagation technology, DM and nonlinearity management are broadly applied. In a managed system, the parameters are allowed to vary along the propagation direction of the optical pulse. While many authors investigate the propagation of one-dimensional temporal solitons with different parameters in optical fibers, relatively few contributors study multi-dimensional spatial solitons with distributed parameters, the exception being our recent work on the analytical solution of (3 + 1)D NLSE with distributed coefficients [16, 17]. However, spatial solitary waves that propagate in an optical material may analogously experience distributed diffraction, nonlinearity and gain. Prior to our analysis, a few points should be considered.

Firstly, any real material may experience parameter perturbations, owing to the inhomogeneities in the material and the fluctuations in the environment (temperature, etc). In order to accurately describe the propagation of the light beam, we utilize the NLSE with distributed coefficients. Secondly, diffraction is related to the linear refractive index, as well as to the wavelength of the laser and the width of the input beam. That is, when the linear refractive index is inhomogeneous, or is controlled artificially, the diffraction varies. Thirdly, other methods can be used to change the diffraction of the beam. For example, the periodic structure of the material, such as photonic lattices, can affect the diffraction of an optical beam. Depending on the local dispersion and the local value of the wavevector, an optical beam may experience normal, anomalous or even vanishing diffraction, and therefore diffraction management is possible [18–20]. There are indeed many other ways to achieve the desired perturbation of coefficients [21–26], some of which are important in the application of Bose–Einstein condensates [22, 26].

The system under consideration is described by the generalized NLSE with varying coefficients. Based on the Hirota bilinear method, we find exact *but approximate* solutions in the form of 2D vortex solitons. Although the present work is of primary interest in the generation of robust optical solitons, this investigation has interesting implication in the study of Bose–Einstein condensates. The quantum mechanical *nonlinear* equation governing the evolution of the condensate, known as the Gross–Pitaevskii equation [3], is identical with the classical NLSE [27] for the evolution of optical beams in an external potential, although the interpretation of variables and parameters of these two equations is different.

The paper is organized as follows. The (2 + 1)D NLSE with varying coefficients in a Kerr-type nonlinear medium is analyzed in section 2, where the dispersion and the nonlinear coefficients vary with the propagation distance. The solitary solutions, obtained by using Hirota's bilinear method, are presented in section 2. Numerical simulations and the discussion are provided in section 3. The concluding remarks with a simple summary are given in section 4.

2. Model and discussion

The slowly varying optical field envelope in a Kerr-type nonlinear medium is governed by the scaled generalized NLSE:

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{1}{2}\beta(z)\nabla_{\perp}u+\chi(z)\,|u|^{2}\,u=0, \tag{1}$$

where $\nabla_{\perp}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$ is the transverse Laplacian operator. We presume the cylindrical geometry. Here φ is the azimuthal angle and we consider only $\varphi\geqslant 0$. Further, z is the propagation (longitudinal) coordinate normalized to the diffraction length and $\beta(z)$ and $\chi(z)$ are the distributed diffraction and the nonlinearity coefficients, respectively.

We search for the axisymmetric cylindrical-beam solutions of equation (1) in the form $u(r,\varphi,z)=\Phi(\varphi)U(r,z)$ that separates the azimuthal part. We assume that the azimuthal part of the solution is of the form $\Phi(\varphi)=\cos(s\varphi)+\mathrm{i}\mu\sin(s\varphi)$ [28], where s is a non-negative real constant—the so-called topological charge (TC). Usually, TC is assumed to be an integer, but it could also be fractional, allowing the possibility of fractional angular momentum; such a possibility has been discussed theoretically [29] and demonstrated experimentally [30]. The parameter $\mu\in[0,1]$ determines the modulation depth of the beam intensity. Note that the azimuthal part is only an *approximate* solution of equation (1), valid for weak nonlinearities or for large values of μ (close to 1); this is because the $|u|^2$ term in the nonlinearity retains the φ-dependence and spoils the assumed separation of variables. Inserting the ansatz for $u(r,\varphi,z)$ into equation (1), integrating over φ from 0 to 2π, and when s is an integer or half-integer, one readily derives an *averaged* equation for U:

$$\mathrm{i}\frac{\partial U}{\partial z}+\frac{1}{2}\beta(z)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{s^2}{r^2}\right)U$$
$$+\frac{1}{2}\chi(z)(1+\mu^2)\,|U|^2\,U=0. \tag{2}$$

We aim at obtaining analytical solutions of equation (2). It is not possible to obtain such solutions for arbitrary distributed coefficients. In fact, no *stable* solitary solutions are known even for the simplest choice of constant coefficients. However, under the conditions of management of coefficients and for their specific forms, it is possible to obtain *approximate* analytical solutions. This is most easily accomplished using the Hirota bilinear method.

To this end, we make use of the following transformation: $U=r\frac{g(r,z)}{f(r,z)}$, where $\mathrm{g}(r,z)$ is a complex function and $\mathrm{f}(r,z)$ is a real one. Substituting into equation (2), we obtain a relation connecting s with β and the bilinear forms as

$$\beta(z)(1-s^2)=0, \tag{3a}$$

$$H_1[gf] = 0, \tag{3b}$$

$$H_2[ff] = (1+\mu^2)\frac{\chi(z)}{\beta(z)}r^2 gg^*, \tag{3c}$$

where the bilinear operators are $H_1 = \mathrm{i}r\mathrm{D}_z + \frac{1}{2}r\beta(z)\mathrm{D}_r^2 + \frac{3}{2}\beta(z)\mathrm{D}_r$, $H_2 = \mathrm{D}_r^2$, and the asterisk indicates the complex conjugation. Here, D_r is Hirota's bilinear derivative operator [31–33], defined as

$$\mathrm{D}_z^n \mathrm{D}_r^m f\cdot g = \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^n \left(\frac{\partial}{\partial r} - \frac{\partial}{\partial r'}\right)^m \times f(z,r)g(z',r')|_{r'=r,z'=z}.$$

The Hirota derivative is best comprehended by an analogy to the Leibnitz derivative of a product: while in the Leibnitz rule there exists the *plus* sign in the binomial, in the Hirota derivative there is the *minus* sign.

Equation (3*a*) fixes TC to 1. Therefore, the present method supplies as solutions only the simply charged soliton vortices. We focus on obtaining solutions to equations (3*b*) and (3*c*). Following the standard procedure for finding soliton solutions, we expand the functions $g(r,z)$ and $f(r,z)$ in a power series in terms of a small parameter ε:

$$g(r,z) = \varepsilon g_1(r,z) + \varepsilon^3 g_3(r,z) + \cdots + \varepsilon^{2j+1} g_{2j+1}(r,z) + \cdots,$$
$$f(r,z) = 1 + \varepsilon^2 f_2(r,z) + \varepsilon^4 f_4(r,z) + \cdots + \varepsilon^{2j} f_{2j}(r,z) + \cdots.$$

Substituting $g(r,z)$ and $f(r,z)$ into the bilinear equations (3) and collecting terms pertaining to the same powers of ε, we obtain the following system of linear partial and ordinary differential equations:

$$\varepsilon^1 : H_1[g_1 1] = 0, \tag{4a}$$

$$\varepsilon^2 : H_2[1 f_2 + f_2 1] = (1+\mu^2)\frac{\chi(z)}{\beta(z)}r^2 g_1 g_1^*, \tag{4b}$$

$$\varepsilon^3 : H_1[g_1 f_2 + g_3 1] = 0, \tag{4c}$$

$$\varepsilon^4 : H_2[1 f_4 + f_4 1 + f_2 f_2] = (1+\mu^2)\frac{\chi(z)}{\beta(z)}r^2(g_1 g_3^* + g_3 g_1^*). \tag{4d}$$

Now, in order to obtain the first-order nonstationary soliton solution, we assume that $g(r,z)$ is truncated to $g_1(r,z)$, and $f(r,z)$ truncated to $f_2(r,z)$, i.e. $g_j(r,z) = 0$ for $j = 3, 5, \ldots$ and $f_k(r,z) = 0$ for $k = 4, 6, \ldots$. Thus, we find that

$$g_1 = \frac{a_{10}}{\left(c_1 - 2\mathrm{i}\int\beta(z)\,\mathrm{d}z\right)^2}\exp\left[\frac{r^2}{c_1 - 2\mathrm{i}\int\beta(z)\,\mathrm{d}z} + D_1\right], \tag{5a}$$

and

$$\frac{\mathrm{d}f_2^2}{\mathrm{d}r^2} = (1+\mu^2)\frac{\chi(z)}{\beta(z)}r^2 g_1 g_1^*. \tag{5b}$$

Equation (5*b*) can be solved analytically, and the solution is

$$f_2 = (1+\mu^2)\frac{\chi(z)}{\beta(z)}D(z)\left(\frac{\mathrm{e}^{A(z)r^2}}{2A(z)^2} - \frac{\sqrt{\pi}r\,\mathrm{erfi}\left(r\sqrt{A(z)}\right)}{4A(z)^{\frac{3}{2}}}\right), \tag{6}$$

where

$$D(z) = \frac{a_{10}^2\mathrm{e}^{2D_1}}{c_1^2 + 4\left(\int_0^z\beta(z)\,\mathrm{d}z\right)^2}, \tag{6a}$$

$$A(z) = \frac{2c_1}{c_1^2 + 4\left(\int_0^z\beta(z)\,\mathrm{d}z\right)^2}, \tag{6b}$$

and erfi is the imaginary error function. Here, c_1 and D_1 are the integration constants. Even though the closed-form solutions for g_1 and f_2 are found, involving only the diffraction and the nonlinearity coefficients, we should mention that such a truncation *cannot* be exact; it is immediately seen that equations (4*a*) and (4*c*), as well as equations (4*b*) and (4*d*), are *incompatible* with each other, after $g_3 = 0$ and $f_4 = 0$ is substituted. Our procedure then is to solve exactly the lower-order equations (4*a*) and (4*b*), leaving the error to reside in the higher-order equations (4*c*) and (4*d*). Thus, our procedure leads to *analytical* but *approximate* solution of equation (1). The estimation of error in our procedure and the comparison of analytical to numerical solutions are provided below.

In the end, we arrive at the exact first-order solution of equation (1) for the solitary mode:

$$u(r,\varphi,z) = [\cos(\varphi) + \mathrm{i}\mu\sin(\varphi)]r\frac{g_1}{1+f_2}. \tag{7}$$

3. Numerical analysis of the soliton solution

To address the features of soliton solutions, in the following we consider the typical two nonlinear media, with the diffraction and nonlinear coefficients chosen as: (i) $\beta(z) = \beta_0\cos(z)$, $\chi(z) = \chi_0\cos(z)$; (ii) $\beta(z) = \beta_0\exp(z)$, $\chi(z) = \chi_0$ [34]. Such trigonometrically and exponentially distributed coefficients are commonly used in the nonlinear fields of study. Without losing generality, we set the initial conditions to $a_{10} = \beta_0 = \chi_0 = 1$ and the integration constants to $c_1 = 2$, $D_1 = 4$. Our current interest is in the qualitative features of the solution given in equation (7), rather than the particular systems in detail. We plot all of our solutions using *Matlab*. The numerical modeling of equation (1), as well as the comparison with the evolving approximate exact solutions, is presented in one figure only, figure 8; in more detail, it will be presented elsewhere.

Figure 1 displays the intensity distribution of the solution u as a function of z, with the coefficients $\beta(z) = \beta_0\cos(z)$, $\chi(z) = \chi_0\cos(z)$. Here μ is set to 0, and to simplify presentation, the azimuthal angle $\varphi = 0$ is chosen. As the coefficients are trigonometric functions, the soliton management is allowed. The sign-changing nonlinearity is produced, leading to a more stable solution. The intensity distribution of the beam keeps changing cyclically with the propagation distance, in a self-similar manner. The period is uniform. The periodicity of the two functions, β and χ, has to be the same, because otherwise the ratio of the two functions can be both positive and negative in equation (6), which would produce singularities in u, since the magnitude of f_2 would diverge.

Ring-shaped vortex beams are obtained when $\mu = 1$ is chosen in equation (7). A typical example of such a vortex ring is shown in figure 2, along with the axisymmetric radial

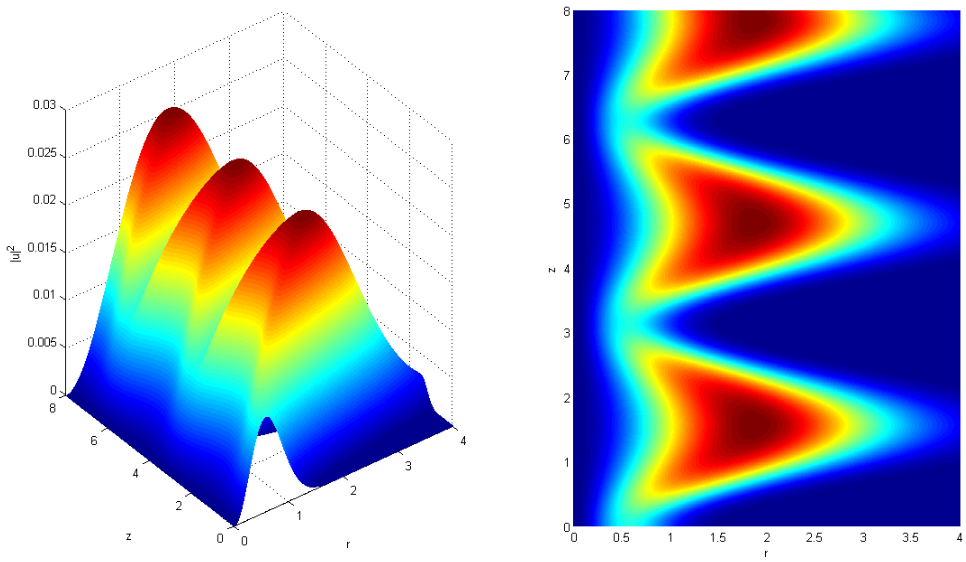

Figure 1. Evolution of the first-order solitary wave (left), and the corresponding intensity contour plot in the r–z plane (right), with $\beta(z) = \beta_0 \cos(z)$, $\chi(z) = \chi_0 \cos(z)$, $a_{10} = \beta_0 = \chi_0 = 1$, $c_1 = 2$, $D_1 = 4$.

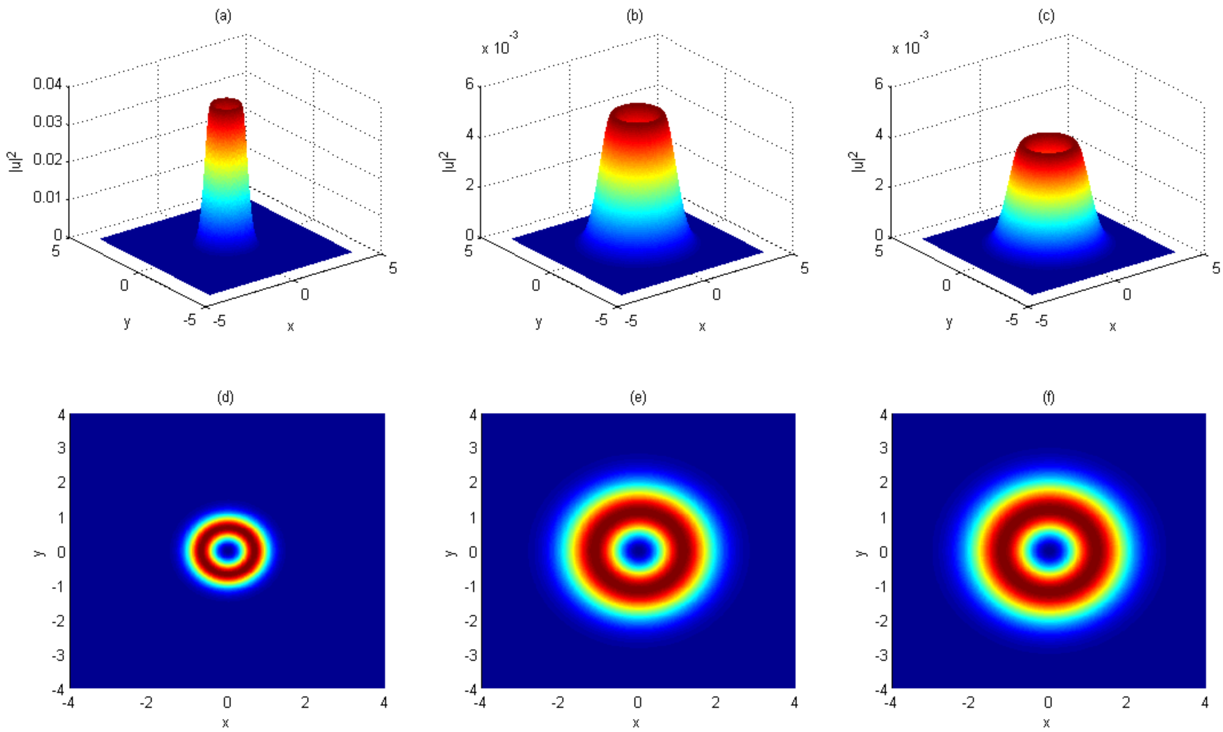

Figure 2. Intensity distributions of the slowly expanding vortex soliton in the $x - y$ plane at different propagation distances $z = 0$, 10, 20 with $\mu = 1$; the other parameters are the same as in figure 1.

intensity distribution of the solitary waves. The intensity is zero at the center, as it should be for a vortex pattern. However, in the course of evolution this nonstationary vortex ring expands and gets attenuated in the radial direction. The larger the radius of the stripes, the weaker the optical intensity of the soliton ring. Moreover, since $\mu = 1$, the distributions of the optical field and the intensity are independent of the azimuthal angle; $|u|^2$ displays a simple isotropic vortex ring shape. It is also seen that the solution is azimuthally stable—no azimuthal modulations are produced during propagation.

Figures 3 and 4 display the propagation dynamics of the first-order soliton modes, which exhibit similar patterns. The example in figure 3 is obtained for $\mu = 0$ in equation (7). It is shown that the structure of this solitary mode is formed with two dipole lobes, coming from the azimuthal modulation. The intensity remains zero at the center, as the solitary wave carries an intrinsic vorticity. For each azimuthal lobe, the maximum intensity is located at the center. The structure expands radially. The structure in figure 4 is obtained for $\mu = 0.8$; it consists of two modulated rings. As is visible

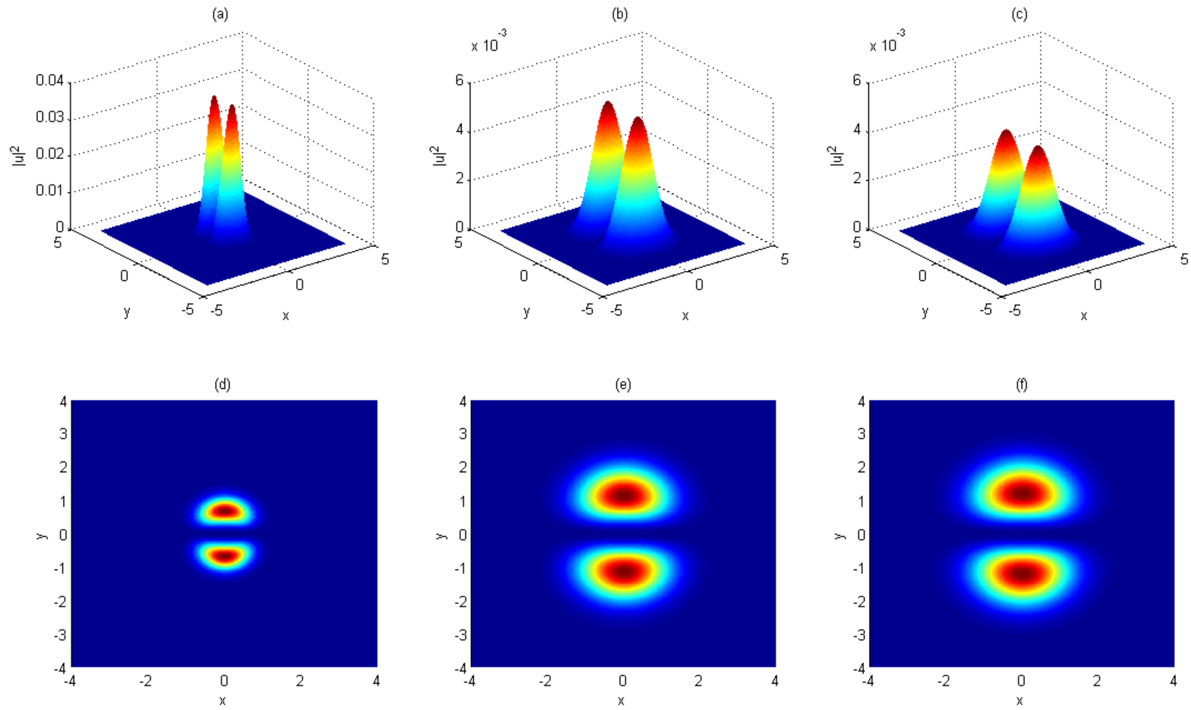

Figure 3. Intensity distributions of a slowly expanding bright dipole soliton in the $x-y$ plane at different propagation distances $z=0$, 10, 20. The parameters are the same as in figure 2, but for $\mu=0$.

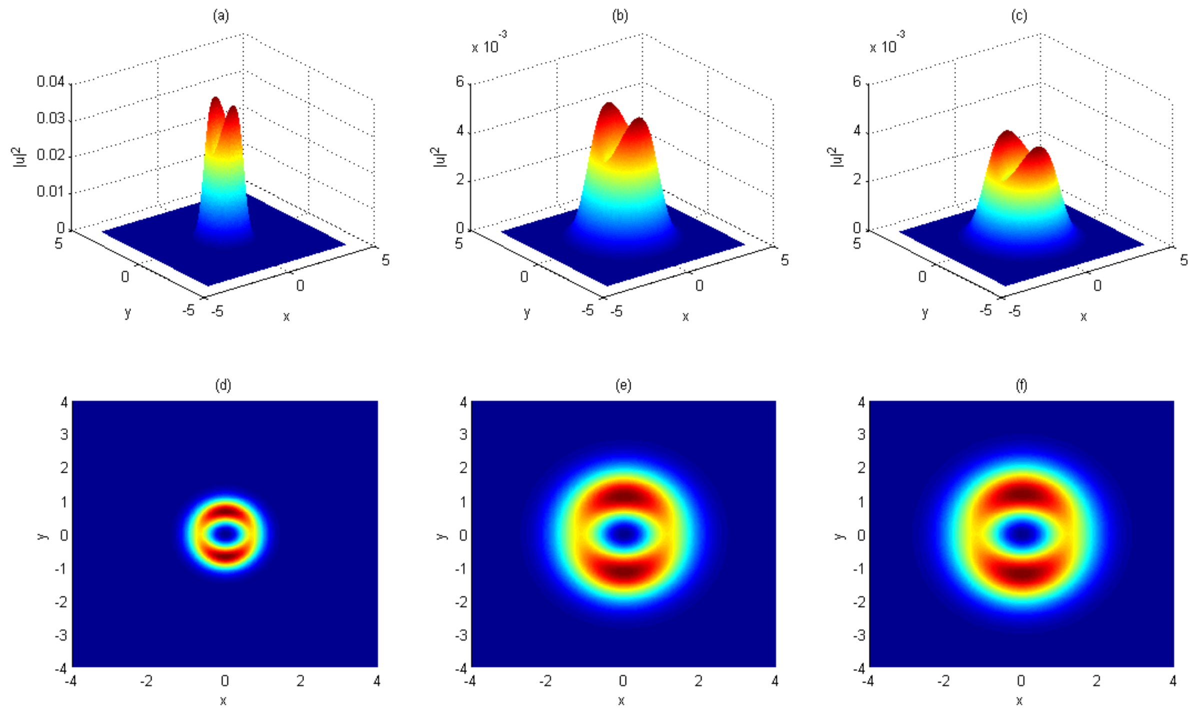

Figure 4. Intensity distributions of an expanding bright soliton in the $x-y$ plane at different propagation distances $z=0$, 10, 20. The parameters are the same as in figure 2, but for $\mu=0.8$.

in figures 3 and 4, the shape of the solitary mode changes as μ increases. With an increase in μ, the components of the dipole solitary wave change their structure from the spiky pattern to two modulated vortex rings.

Figure 5 displays the intensity distributions of the solution u obtained for the choice of coefficients: $\beta(z)=\beta_0\exp(z)$, $\chi(z)=\chi_0$. The figure presents the view along the angle $\varphi=0$, as in figure 1; μ is set to 0. As $\beta(z)$ is an exponential function, the evolution of the beam is different from that in figure 1, exhibiting only one simple bright soliton pulse. One does not see the periodic pattern. The profiles of the waves plotted in figures 1 and 5 are diffraction managed. It is seen that after the application of diffraction management the soliton remains stable but slowly expanding

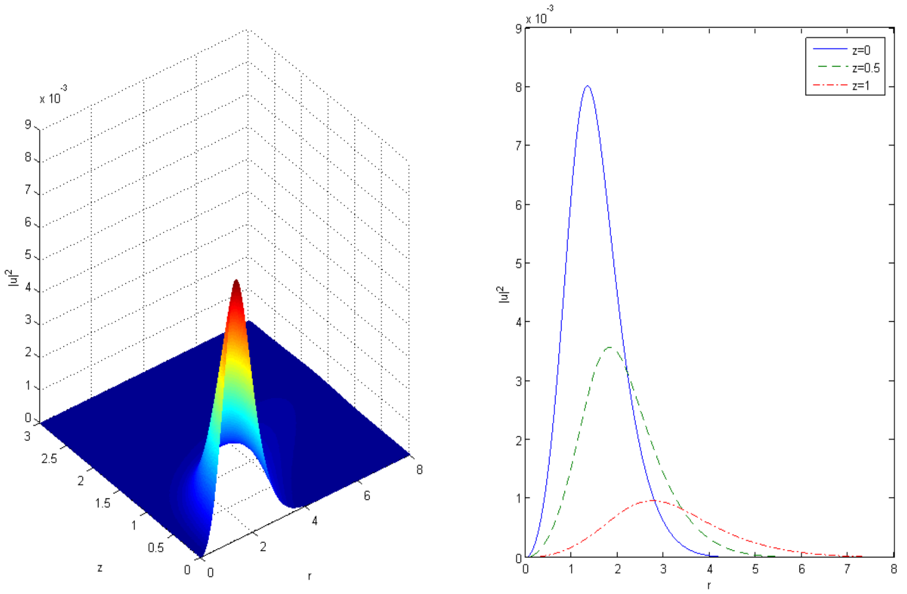

Figure 5. Evolution of the first-order solitary wave (left); examples of the cuts through the corresponding intensity at different propagation distances $z = 0, 0.5, 1$ (right) with $\beta(z) = \beta_0 \exp(z)$, $\chi(z) = \chi_0$. The other parameters are the same as in figure 1.

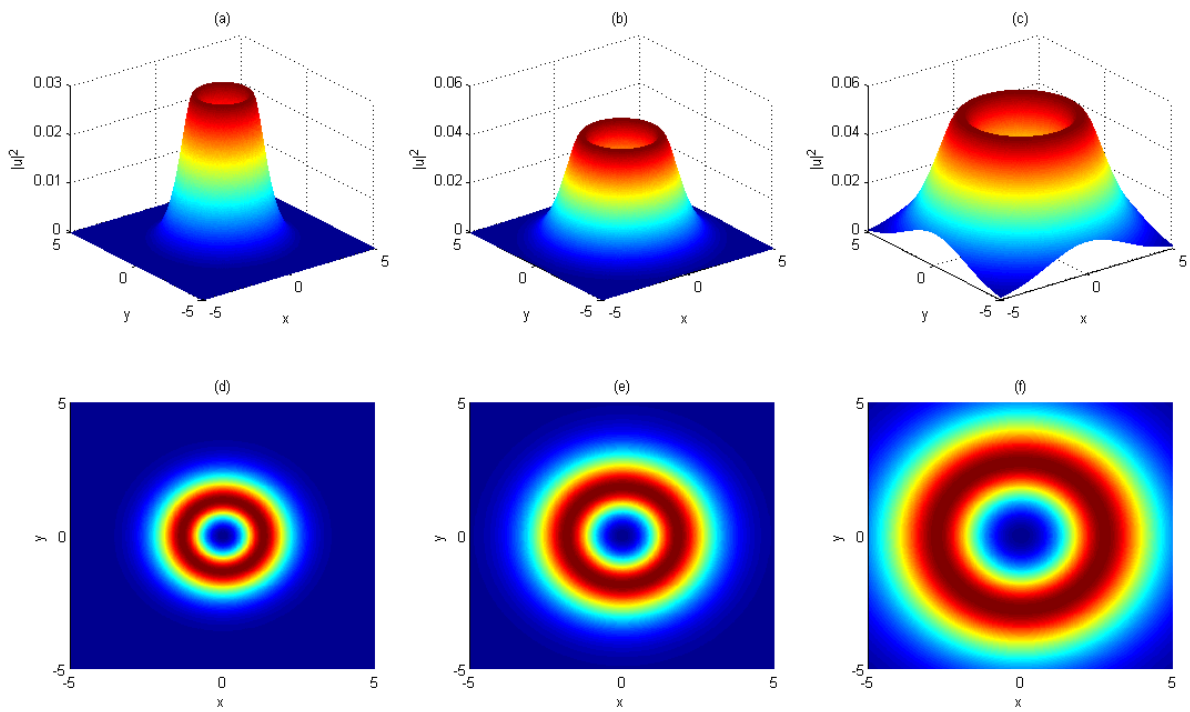

Figure 6. Intensity distributions of expanding bright solitons in the $x - y$ plane at different propagation distances $z = 0, 0.5, 1$ with $\mu = 1$. The other parameters are the same as in figure 5.

for large propagation distances. This shows clearly the effect of the oscillating and exponential diffraction coefficients on the wave stabilization. The diffraction management of the type considered can prolong the life of a soliton significantly.

For $\mu = 1$ and 0.8 in equation (7), similar to figures 2 and 4, we obtain a ring-shaped beam and a modulated vortex ring, respectively. Two typical examples of such solitary waves are shown in figures 6 and 7. As in the trigonometrically modulated system presented in figures 2 and 4, this vortex in the course of evolution also expands and gets attenuated radially. The larger the radii for the stripes, the weaker the optical intensities of soliton rings. However, different from the trigonometrically modulated system, the soliton intensity reduces more quickly with propagation.

The approximate analytical solutions have been checked by direct numerical integration of equation (1). We used a 2D split-step fast Fourier technique and consider as an initial condition the field from equation (7), of the form

$$u(0, r, \varphi) = [\cos(\varphi) + \mathrm{i}\mu \sin(\varphi)] \frac{g_{10}}{1 + f_{20}} r.$$

Here g_{10} and f_{20} are the initial values of g_1 and f_2, respectively. The comparison between the simulations and analytical predictions for this case is shown in figure 8.

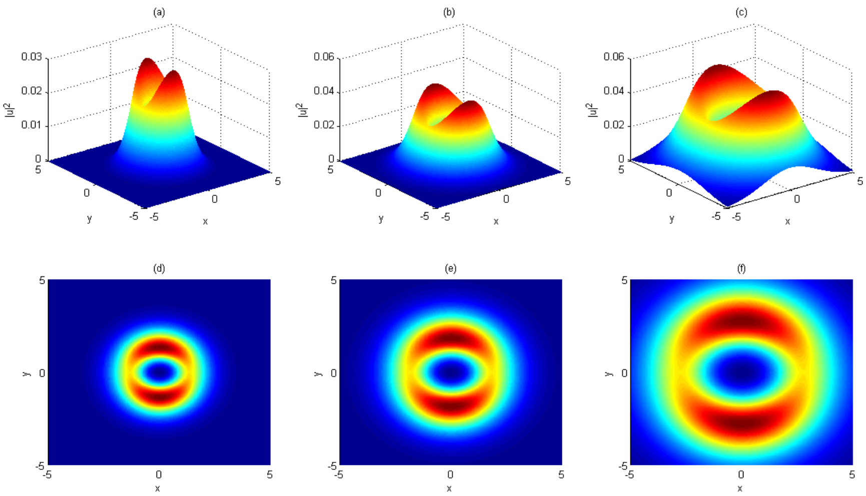

Figure 7. Intensity distributions of expanding bright solitons in the $x - y$ plane at different propagation distances $z = 0, 0.5, 1$ with $\mu = 0.8$; the other parameters are the same as in figure 5.

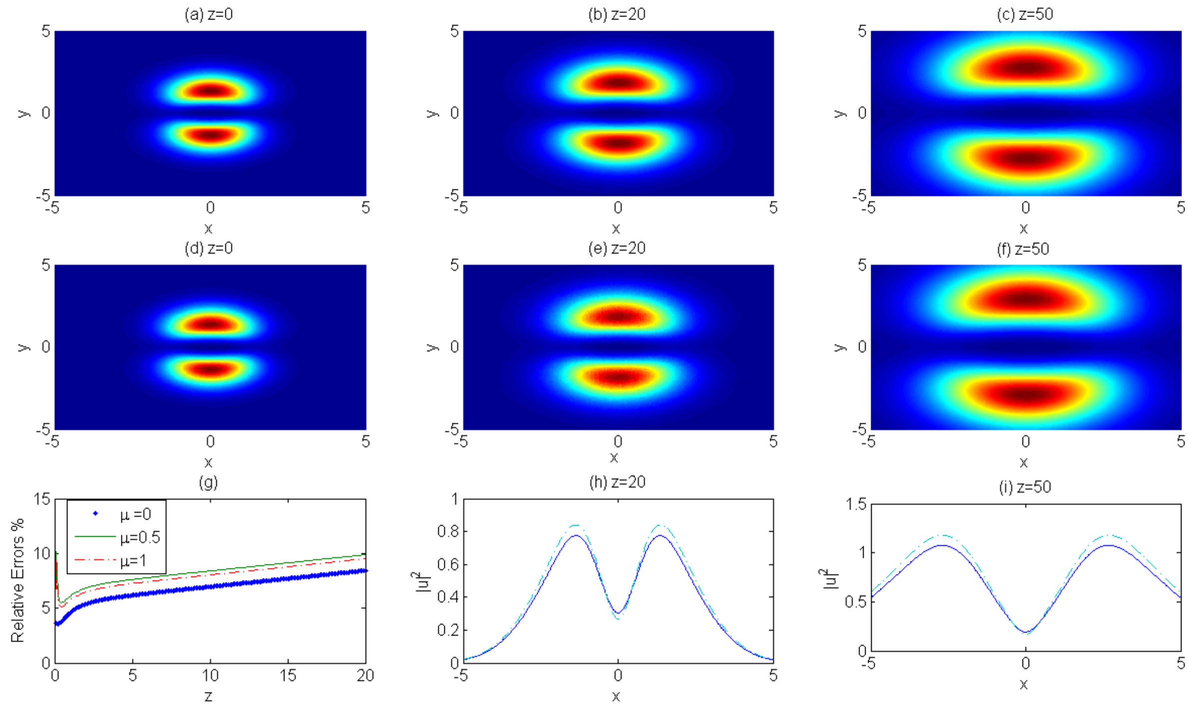

Figure 8. Comparison of approximate analytical solutions with numerical simulations. (a)–(c) Approximate analytical solutions. (d)–(f) Numerical simulations with a white noise of variance $\sigma^2 = 6$. (h, i) Simulated intensity (solid lines) together with the approximate analytical predictions (dashed lines) for $\mu = 0.3$. (g) The percentage relative error of amplitude for different $\mu = 0, 0.5, 1$. The other parameters are the same as in figure 1.

The figure also highlights the significant features of the numerical evolution of a 2D vortex solitary wave. The percentage relative error during the propagation is displayed in figure 8(g) for different values of μ. These findings confirm the fact that the diffraction management of the type considered here can prolong the life of vortex solitary waves.

4. Conclusion

In summary, using Hirota's bilinear method, approximate first-order soliton solutions to the $(2 + 1)$D NLSE with varying coefficients have been obtained and discussed in some detail. These nonlinear excitations can be classified as vortex bright spatial solitons, with structurally stable profiles slowly expanding along the propagation direction. The present results indicate that DM and nonlinearity management can be used to support quasistable soliton beams. They may provide insights into the low-energetic spatial soliton transmission with high fidelity in the Kerr devices.

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