Exact spatiotemporal wave and soliton solutions to the generalized (3+1)-dimensional Schrödinger equation for both normal and anomalous dispersion

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We obtain exact extended traveling-wave and spatiotemporal soliton solutions to the generalized (3+1)-dimensional nonlinear Schrödinger equations for both the normal and the anomalous dispersion.

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The generalized nonlinear Schrödinger equation (GNLSE) is of paradigmatic importance to many fields of physics [1,2]. It is of tremendous importance in nonlinear optics, where it describes the full spatiotemporal (ST) optical solitons, or light bullets. Interest in NLSE is aroused by the discovery of solitary wave solutions [3]. However, stable exact soliton solutions to NLSE exist only in (1+1)-dimensions [(1+1)D]; to our knowledge there are no known exact stable solitons in higher dimensions. Here we present analytical traveling-wave and soliton solutions to the GNLSE in (3+1)D. Their stability will be addressed elsewhere.

We consider the GNLSE in (3+1)D with distributed coefficients [4–6],

$$i\partial_z u + \frac{\beta(z)}{2}(\Delta_\perp u + s\partial_t^2 u) + \chi(z)|u|^2 u = i\gamma(z)u, \quad (1)$$

which describes evolution of a slowly varying wave packet envelope $u(z,x,y,t)$ in a diffractive dispersive nonlinear Kerr medium, in the paraxial approximation. Here z is the propagation coordinate, $\Delta_\perp = \partial_x^2 + \partial_y^2$ represents the transverse Laplacean, and t is the reduced time, i.e., time in the frame of reference moving with the wave packet. The functions β, χ, and γ stand for the diffraction/dispersion, nonlinearity, and gain coefficients, respectively. All coordinates in Eq. (1) are made dimensionless by the choice of coefficients.

The parameter $s=\pm 1$ is the dispersion sign parameter. For $s=+1$ we have the anomalous dispersion, and for $s=-1$ we have the normal dispersion. Dispersions of different signs describe different physical situations [4–6], and the corresponding models (1) belong to different classes of partial differential equations. Nevertheless, we show in this Letter how to obtain classes of exact solutions for the dispersions of different signs by using the same solution method [7].

Thus far the solutions to the multidimensional NLSE have been obtained only for the anomalous dispersion [6,8]. Here we obtain solutions for the normal dispersion. For ease of comparison, both the normal and the anomalous solutions are presented in parallel. Solutions for the normal dispersion are of great interest, because the normal or positive dispersion is known to arrest or slow down the wave packet collapse associated with the multidimensional NLSE [9].

We define the complex field u of Eq. (1) in terms of its amplitude and phase,

$$u(z,x,y,t) = A(z,x,y,t)\exp[iB(z,x,y,t)]. \quad (2)$$

Substituting u into Eq. (1), we find two coupled equations for A and B,

$$\partial_z A + \frac{1}{2}\beta[2\partial_x A\partial_x B + 2\partial_y A\partial_y B + 2s\partial_t A\partial_t B + A(\Delta_\perp + s\partial_t^2)B] = \gamma A, \quad (3)$$

$$-A\partial_z B + \frac{1}{2}\beta[(\Delta_\perp + s\partial_t^2)A - A(\partial_x B)^2 - A(\partial_y B)^2 - sA(\partial_t B)^2] + \chi A^3 = 0. \quad (4)$$

To these equations we apply the balance principle [10,11] and the F-expansion technique [12,13], as developed in [7]. We seek the traveling-wave solutions to Eqs. (3) and (4), and we assume the functions to be of the form

$$A = f(z)F(\theta) + g(z)F^{-1}(\theta), \quad (5)$$

$$\theta = k(z)x + l(z)y + m(z)t + \omega(z), \quad (6)$$

$$B = a(z)(x^2+y^2+st^2) + b(z)(x+y+t) + e(z), \quad (7)$$

where f, g, k, l, m, w, a, b, e are parameter functions to be determined and F is a Jacobi elliptic function (JEF).

Substituting Eqs. (5)–(7) into Eqs. (3) and (4) and requiring that $x^q F^n$, $y^q F^n$, and $t^q F^n$, $(q=0,1,2,\ n=0,1,2,3)$ of each term be separately equal to zero, a system of algebraic or first-order ordinary differential equations is obtained that the parameters must satisfy,

$$\frac{\mathrm{d}f_j}{\mathrm{d}z} + 3a\beta f_j - \gamma f_j = 0, \tag{8}$$

$$f_j\left(\frac{\mathrm{d}k}{\mathrm{d}z} + 2ka\beta\right) = 0, \tag{9}$$

$$f_j\left(\frac{\mathrm{d}l}{\mathrm{d}z} + 2la\beta\right) = 0, \tag{10}$$

$$f_j\left(\frac{\mathrm{d}m}{\mathrm{d}z} + 2ma\beta\right) = 0, \tag{11}$$

$$-f_j\left(\frac{\mathrm{d}a}{\mathrm{d}z} + 2\beta a^2\right) = 0, \tag{12}$$

$$-f_j\left(\frac{\mathrm{d}b}{\mathrm{d}z} + 2\beta ab\right) = 0, \tag{13}$$

$$f_j\left(\frac{\mathrm{d}\omega}{\mathrm{d}z} + \beta(k+l+sm)b\right) = 0, \tag{14}$$

$$f_j\left(\frac{\mathrm{d}e}{\mathrm{d}z} + \frac{\beta}{2}[(2+s)b^2 - (k^2+l^2+sm^2)c_2] - 3\chi f_1 f_2\right) = 0, \tag{15}$$

$$f_1[\beta(k^2+l^2+sm^2)c_4 + \chi f_1^2] = 0, \tag{16}$$

$$f_2[\beta(k^2+l^2+sm^2)c_0 + \chi f_2^2] = 0, \tag{17}$$

where $j=1,2$, $f_1=f$, and $f_2=g$. The constants c_0, c_2, c_4 in Eqs. (15)–(17) are related to the elliptic modulus M of JEFs [7]. By solving Eqs. (8)–(17) self-consistently, one obtains a set of conditions on the coefficients and parameters, necessary for Eq. (1) to have exact traveling-wave solutions [8]. Note that the solutions for the normal and anomalous dispersion closely parallel each other, the parameter s appearing only in certain combinations with the parameters k, l, and m in Eqs. (14)–(17). Nonetheless, they still describe distinctly different physical phenomena.

We consider the most generic case, in which f and g are assumed nonzero and $\beta(z)$ and $\gamma(z)$ are arbitrary. We also assume $k^2+l^2+sm^2\neq 0$ for $s=-1$; otherwise the only solution for nonzero χ is $f=g=0$. The following set of exact solutions is found:

$$f = (\alpha)^{3/2} f_0 \exp\left(\int_0^z \gamma \mathrm{d}z\right), \qquad g = \epsilon\sqrt{\frac{c_0}{c_4}}f, \tag{18}$$

$$k = \alpha k_0, \qquad l = \alpha l_0, \qquad m = \alpha m_0, \tag{19}$$

$$\omega = \omega_0 - \alpha(k_0 + l_0 + sm_0)b_0\int_0^z \beta \mathrm{d}z, \tag{20}$$

$$a = \alpha a_0, \qquad b = \alpha b_0, \tag{21}$$

$$e = e_0 + \frac{\alpha}{2}\cdot[(k_0^2 + l_0^2 + sm_0^2)(c_2 - 6\epsilon\sqrt{c_0 c_4}) - (2+s)b_0^2]\int_0^z \beta \mathrm{d}z; \tag{22}$$

where $\alpha=[1+2a_0\int_0^z\beta dz]^{-1}$ is the normalized chirp function. The subscript 0 denotes the value of the given function at $z=0$. A parameter $\epsilon=\pm 1$ is introduced in Eqs. (18) and (22) to distinguish the two present possibilities. It should also be noted that χ is not arbitrary but depends on α, β, and γ,

$$\chi = -\beta c_4(k_0^2 + l_0^2 + sm_0^2) f_0^{-2} \exp\left(-2\int_0^z \gamma \mathrm{d}z\right) \Big/ \alpha. \tag{23}$$

Hence, in a lossy medium the nonlinearity coefficient χ will grow exponentially.

Incorporating these solutions back into Eqs. (5)–(7), we obtain the general periodic traveling-wave solutions to the GNLSE,

$$u = (\alpha)^{3/2} f_0 \exp\left(\int_0^z \gamma \mathrm{d}z\right)\left[F(\theta) + \epsilon\sqrt{\frac{c_0}{c_4}}\frac{1}{F(\theta)}\right] \times \exp i[a(x^2+y^2+st^2) + b(x+y+t) + e], \tag{24}$$

where

$$\theta = \omega_0 + kx + ly + mt - (k+l+sm)b_0\int_0^z \beta \mathrm{d}z. \tag{25}$$

Apart from the solutions given in Eqs. (18)–(22) one can alternatively assume that $g=0$, in which case one obtains the exact same equations to which Eqs. (18)–(22) would reduce for $\epsilon=0$. Thus, the parameter ϵ in Eq. (24) can assume three values: ± 1 and 0.

The form of solutions depends on what JEFs are utilized [14]. The elliptic modulus M varies between 0 and 1. When $M\to 0$, JEFs degenerate into trigonometric functions, and the periodic traveling-wave solutions become the periodic trigonometric solutions.

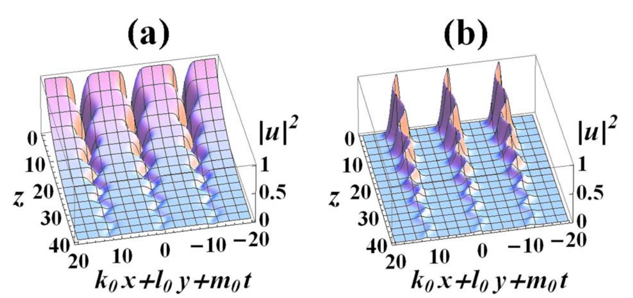

Fig. 1. (Color online) Traveling-wave solutions as functions of the propagation distance for $a_0=0$ (without chirp) and $\epsilon=0$. (a) Intensity $|u|^2$ for $F=sn$ and (b) for $F=cn$, presented as functions of $k_0x+l_0y+m_0t$ and z. Coefficients: $\beta(z)=\cos(z)$, $\gamma(z)=\gamma_0=-0.05$, $M=0.9999$, $b_0=1$, $e_0=0$, $k_0=l_0=m_0=1$, $\omega_0=0$.

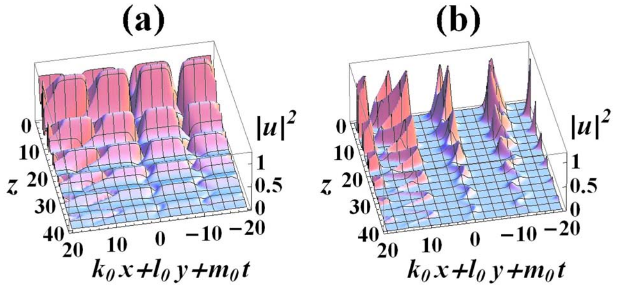

Fig. 2. (Color online) Traveling-wave solutions with the chirp as functions of the propagation distance. The setup and parameters are the same as in Fig. 1 except for $a_0=0.1$.

When $M\to 1$, JEFs degenerate into hyperbolic functions, and the traveling-wave solutions become the ST soliton solutions [7].

We should note that for $M=1$ the solutions introduced by Eqs. (5)–(7) describe spatially extended ST solitons. Even though the amplitude A as a function of the transverse variable θ is localized, it is not when viewed in the plane of transverse coordinates x and y. This is easily seen if one rotates the x and y axes about the z axis for some angle α, to arrive at a set of new coordinates x' and y'. By choosing the angle as $\tan(\alpha)=-k/l$, the variable θ will not contain y', and by choosing $\tan(\alpha)=k/l$, it will not contain x'. Thus the amplitude A will not explicitly depend on y' (or x') and the soliton will be extended along the y' axis. Hence, the solutions obtained with the present method cannot be of the light bullet type.

We display some of the traveling-wave and ST soliton solutions for the normal dispersion, taking the diffraction/dispersion coefficient β to be of the form $\beta=\beta_0\cos(k_b z)$ and the gain/loss coefficient γ to be a small negative constant. This choice leads to alternating regions of positive/negative values of both β and χ, which are required for an eventual stability of soliton solutions. In Figs. 1 and 2 we depict the periodic wave solutions made up from the single F functions *sn* and *cn*, without and with the chirp, for $\epsilon=0$. Figures 3 and 4 repeat the same sequence of plots as

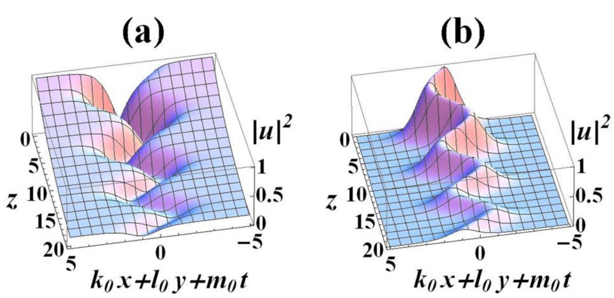

Fig. 3. (Color online) Soliton solutions without chirp. The setup and parameters are as in Fig. 1 except for $M=1$.

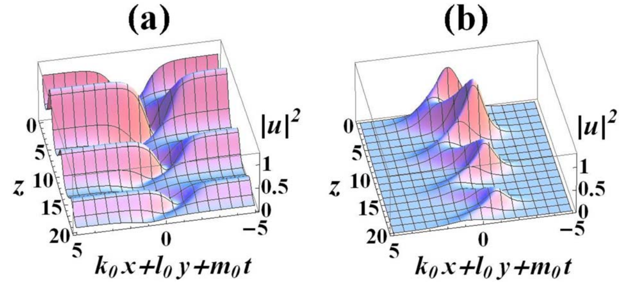

Fig. 4. (Color online) Soliton solutions with chirp. Setup is the same as in Fig. 3 except for $a_0=0.1$.

Figs. 1 and 2 but show the ST soliton solutions instead.

In conclusion, we have solved analytically the (3+1)-dimensional GNLSE with distribution diffraction, dispersion, nonlinearity, and gain for both the normal and anomalous dispersion. A number of exact traveling-wave solutions are found, and exact spatiotemporal soliton solutions are obtained.

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