Analytical Light Bullet Solutions to the Generalized (3 + 1)-Dimensional Nonlinear Schrödinger Equation

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We obtain exact spatiotemporal periodic traveling wave solutions to the generalized (3 + 1)-dimensional nonlinear Schrödinger equation with distributed coefficients. We utilize these solutions to construct analytical light bullet soliton solutions of nonlinear optics.

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The nonlinear Schrödinger equation (NLSE) is one of the most useful generic mathematical models [1] that naturally arises in many fields of physics. Major interest in the NLSE was piqued by the discovery of solitary wave solutions [2,3]. Stable exact soliton solutions to the NLSE are known only in (1 + 1) dimensions [(1 + 1)D], for the simple reason that the inverse scattering method [2], responsible for the existence and stability of 1D solitons, works only in (1 + 1)D. There are no known exact stable solitons in (2 + 1)D or (3 + 1)D.

Recently, great interest has been generated when it was suggested that the (2 + 1)D generalized NLSE with varying coefficients may lead to stable 2D solitons [4]. The stabilizing mechanism has been the sign-alternating Kerr nonlinearity in a layered medium. A vigorous search for the stabilized localized solutions of the (2 + 1)D NLSE has been launched [5–7]; however, out of necessity, it has been numerical. Rare experimental solutions have been provided in Ref. [8]. We present here *analytical* periodic traveling wave and soliton solutions to the NLSE in (3 + 1)D.

Our interest is focused on the *generalized* NLSE in (3 + 1)D with distributed coefficients [9]:

$$i\partial_z u + \frac{\beta(z)}{2}(\Delta_\perp u + \partial_t^2 u) + \chi(z)|u|^2 u = i\gamma(z)u, \tag{1}$$

which describes the evolution of a slowly varying wave packet envelope $u(z, x, y, t)$ in a diffractive nonlinear Kerr medium with *anomalous* dispersion, in the paraxial approximation. Here z is the propagation coordinate, $\Delta_\perp = \partial_x^2 + \partial_y^2$ represents the transverse Laplacian, and t is the reduced time, i.e., time in the frame of reference moving with the wave packet. All coordinates are made dimensionless by the choice of coefficients. The generalized NLSE is of considerable importance, as it describes the full spatiotemporal optical solitons, or *light bullets*, in (3 + 1)D. The functions β, χ, and γ stand for the diffraction or dispersion, nonlinearity, and gain coefficients, respectively.

When the coefficients are constant, the behavior of solutions to the NLSE strongly depends on the dimensionality of the problem. In (1 + 1)D, as mentioned, one can observe stable localized wave packets. However, in (2 + 1)D, for the self-focusing nonlinearity all localized solutions either spread out with propagation (for input powers less than a critical value) or collapse at a finite distance (for powers above the critical value) [10]. This behavior is an example of *weak* collapse. In (3 + 1)D, one observes the *strong* collapse: Wave packets collapse at any power—no power threshold exists.

Utilizing an F-expansion technique [11] and a procedure for balancing terms in the expansion [12], we present in this Letter a method for finding analytical periodic traveling wave solutions to the (3 + 1)D NLSE with distributed coefficients.

We define the complex periodic wave or light bullet field u of Eq. (1) in terms of its amplitude and phase [13]:

$$u(z, x, y, t) = A(z, x, y, t)\exp[iB(z, x, y, t)]. \tag{2}$$

Substituting u into Eq. (1), we find the following coupled equations:

$$\partial_z A + \tfrac{1}{2}\beta[2\partial_x A\partial_x B + 2\partial_y A\partial_y B + 2\partial_t A\partial_t B + A(\Delta_\perp + \partial_t^2)B] = \gamma A; \tag{3}$$

$$-A\partial_z B + \tfrac{1}{2}\beta[(\Delta_\perp + \partial_t^2)A - A(\partial_x B)^2 - A(\partial_y B)^2 - A(\partial_t B)^2] + \chi A^3 = 0. \tag{4}$$

We seek traveling wave solutions to Eqs. (3) and (4) and assume the functions to be of the form:

$$A = f(z)F(\theta) + g(z)F^{-1}(\theta); \tag{5}$$

$$\theta = k(z)x + l(z)y + m(z)t + \omega(z); \tag{6}$$

$$B = a(z)(x^2 + y^2 + t^2) + b(z)(x + y + t) + e(z), \tag{7}$$

where f, g, k, l, m, ω, a, b, and e are the parameter functions to be determined and F is a Jacobi elliptic function (JEF). Substituting Eqs. (5)–(7) into Eqs. (3) and (4) and requiring that $x^q F^n$, $y^q F^n$, and $t^q F^n$ ($q = 0, 1, 2$, $n = 0, 1, 2, 3$) of each term be separately equal to zero, we obtain a system of algebraic and first-order ordinary differential equations that the parameters must satisfy [14]:

$$\frac{df_j}{dz} + 3a\beta f_j - \gamma f_j = 0; \tag{8}$$

$$f_j\left(\frac{dk}{dz} + 2ka\beta\right) = 0; \tag{9}$$

$$f_j\left(\frac{dl}{dz} + 2la\beta\right) = 0; \tag{10}$$

$$f_j\left(\frac{dm}{dz} + 2ma\beta\right) = 0; \tag{11}$$

$$f_j\left[\frac{d\omega}{dz} + \beta(k + l + m)b\right] = 0; \tag{12}$$

$$-f_j\left(\frac{da}{dz} + 2\beta a^2\right) = 0; \tag{13}$$

$$-f_j\left(\frac{db}{dz} + 2\beta ab\right) = 0; \tag{14}$$

$$-f_j\left[\frac{de}{dz} - \frac{1}{2}\beta(k^2 + l^2 + m^2)c_2 + \frac{3}{2}\beta b^2 - 3\chi f_1 f_2\right] = 0; \tag{15}$$

$$f_1[\beta(k^2 + l^2 + m^2)c_4 + \chi f_1^2] = 0; \tag{16}$$

$$f_2[\beta(k^2 + l^2 + m^2)c_0 + \chi f_2^2] = 0, \tag{17}$$

where $j = 1, 2$, $f_1 = f$, and $f_2 = g$. The constants c_0, c_2, and c_4 appearing in Eqs. (15)–(17) are related to the square of the elliptic modulus M of JEFs (see Table I). By solving Eqs. (8)–(17) self-consistently, one obtains a set of conditions on the coefficients and parameters, necessary for Eq. (1) to have exact periodic wave solutions.

We consider the most generic case, in which f and g are assumed nonzero and $\beta(z)$ and $\gamma(z)$ are arbitrary. The following set of exact solutions is found:

$$f = (\alpha)^{3/2} f_0 \exp\left(\int_0^z \gamma dz\right), \qquad g = \sqrt{\frac{c_0}{c_4}}\epsilon f; \tag{18}$$

$$k = \alpha k_0, \qquad l = \alpha l_0, \qquad m = \alpha m_0; \tag{19}$$

TABLE I. Jacobi elliptic functions.

Solution	c_0	c_2	c_4	F	$M=0$	$M=1$
1	1	$-(1+M^2)$	M^2	sn	sin	tanh
2	$1-M^2$	$2M^2-1$	$-M^2$	cn	cos	sech
3	M^2-1	$2-M^2$	-1	dn	1	sech
4	M^2	$-(1+M^2)$	1	ns	cosec	coth
5	$-M^2$	$2M^2-1$	$1-M^2$	nc	sec	cosh
6	-1	$2-M^2$	M^2-1	nd	1	cosh
7	1	$2-M^2$	$1-M^2$	sc	tan	sinh
8	$1-M^2$	$2-M^2$	1	cs	cot	cosech
9	1	$-(1+M^2)$	M^2	cd	cos	1
10	M^2	$-(1+M^2)$	1	dc	sec	1

$$\omega = \omega_0 - \alpha(k_0 + l_0 + m_0)b_0 \int_0^z \beta dz; \tag{20}$$

$$a = \alpha a_0, \qquad b = \alpha b_0; \tag{21}$$

$$e = e_0 + \frac{\alpha}{2}[(k_0^2 + l_0^2 + m_0^2)(c_2 - 6\epsilon\sqrt{c_0 c_4}) - 3b_0^2]\int_0^z \beta dz, \tag{22}$$

where $\alpha = [1 + 2a_0 \int_0^z \beta dz]^{-1}$ is the normalized *chirp* function. It is related to the wave front curvature and presents a measure of the phase chirp imposed on the wave. The subscript 0 denotes the value of the given function at $z = 0$. A parameter $\epsilon = \pm 1$ is introduced in Eqs. (18) and (22), to distinguish the two present possibilities.

One should note the universal influence of the chirp function α on the solutions. The chirp function is related only to the diffraction or dispersion coefficient; however, it affects all of the parameters. In the case when there is no chirp, $a_0 = 0$, and $\alpha = 1$, the parameters k, l, m, and b are all constant. In the presence of chirp, they all acquire the prescribed z dependence. The chirp also influences the form of the amplitude A through the dependence of f, g, and θ on α. It should also be noted that χ is not arbitrary but depends on α, β, and γ:

$$\chi = -\frac{\beta c_4}{\alpha f_0^2}(k_0^2 + l_0^2 + m_0^2)\exp\left(-2\int_0^z \gamma dz\right). \tag{23}$$

Hence, to obtain exact solutions in a *lossy* medium, the nonlinearity coefficient χ must *grow* exponentially. In our choice of independent coefficients, we could have equally well chosen χ and γ; then β would have been dependent.

Incorporating these solutions back into Eq. (2), we obtain the general periodic traveling wave solutions to the generalized NLSE:

$$u = (\alpha)^{3/2} f_0 \exp\left(\int_0^z \gamma dz\right)\left[F(\theta) + \sqrt{\frac{c_0}{c_4}}\epsilon F^{-1}(\theta)\right] \times \exp\{i[a(x^2 + y^2 + t^2) + b(x + y + t) + e]\}, \tag{24}$$

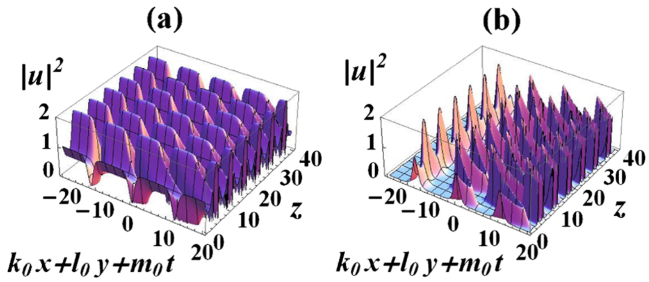

FIG. 1 (color online). Periodic traveling wave solutions with chirp, as functions of the propagation distance. (a) Intensity $|u|^2$ of solution 1 and (b) of solution 2 from Table I, presented as functions of $k_0x + l_0y + m_0t$ and z. Coefficients and parameters: $\beta(z) = \cos(z)$, $\gamma(z) = \gamma_0 = 0$, $\epsilon = 0$, $M = 0.9999$, $a_0 = 0.1$, $b_0 = 1$, $e_0 = 0$, $k_0 = l_0 = m_0 = 1$, and $\omega_0 = 0$.

where $\theta = \omega_0 + kx + ly + mt - (k + l + m)b_0 \int_0^z \beta dz$. Apart from the solutions given in Eqs. (18)–(22), one can alternatively assume that $g = 0$, in which case one obtains the exact same equations to which Eqs. (18)–(22) would reduce for $\epsilon = 0$. Thus, the parameter ϵ in Eq. (24) can take the three values: ± 1 and 0.

The form of solutions depends on what JEFs are utilized. Table I lists some of the JEFs (labeled from 1 to 10) that may appear in the solutions. Note a relation among the constants c_0, c_2, and c_4:

$$c_0 + c_4 = \pm c_2. \tag{25}$$

As long as one chooses the constants according to the relations listed in Table I and substitutes the appropriate $F(\theta)$ into Eq. (24), one obtains the exact periodic traveling wave solutions to the generalized (3 + 1)D NLSE. The parameter M varies between 0 and 1. When $M \to 0$, JEFs degenerate into trigonometric functions, and the periodic traveling wave solutions become the periodic trigonometric solutions. When $M \to 1$, JEFs degenerate into hyperbolic functions, and the periodic traveling wave solutions become the light bullet soliton solutions. As long as $0 < M < 1$, there is no problem with the periodic solutions; one can choose any of the listed functions. However, when $M = 0$ or $M = 1$, only some of the functions may be utilized, because of the developing singularities.

As an example, we present some of the periodic wave and light bullet soliton solutions, taking the diffraction (dispersion) coefficient β to be of the form $\beta = \beta_0 \cos(k_b z)$ and the gain (loss) coefficient γ to be a small constant. This choice leads to alternating regions of positive and negative values of both β and χ, which is required for an eventual stability of soliton solutions. In Fig. 1, we depict the periodic wave solutions made up from the single F functions 1 and 2 from the table, with the chirp and for $\epsilon = 0$. Figure 2 shows the periodic wave solution made from the combination of the F functions 1 and 4 for $\epsilon = 1$, without and with the chirp. As can be seen, the presence of ϵ significantly changes the nature of solutions. Figures 3 and 4 present the light bullet soliton solutions, again without and with the chirp. The effect of the particular periodic chirp function is to produce a periodic variation along the propagation direction and a monotonic asymmetric change in the transverse directions. We note that the soliton solution is similar to a single period of the periodic wave solutions for M close to 1. The period of the solution stretches to infinity as M approaches 1, while the solitons merge into each other as M decreases.

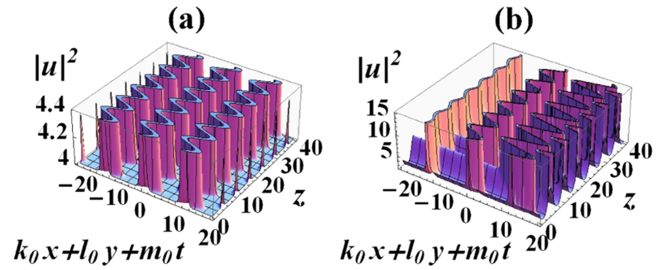

FIG. 2 (color online). Combined intensity distributions of the periodic wave solutions 1 and 4, as functions of the propagation distance, with $\epsilon = 1$. (a) Distribution without and (b) with chirp. Other parameters are the same as in Fig. 1.

An important feature that distinguishes our solutions from the others reported in the literature [9,14,15], apart from the dimensionality, is the appearance of the general spatiotemporal chirp function in both the phase and the amplitude. Another important feature is that the z modulation of both the diffraction or dispersion coefficient and the nonlinearity coefficient, connected as they are through Eq. (23), strongly affects the form and the behavior of solutions.

In the end, we comment on the *stability* of solutions to the generalized NLSE in (3 + 1)D. The current situation is somewhat confusing and even controversial: Some authors point out that it is possible to obtain stable solutions without modulating the dispersion [6], others disagree [7], and some others still claim that an additional trapping potential is necessary [16]. Most of them consider the stability of radially symmetric structures and do not include the modulation of diffraction. When only the dispersion is modulated, 3D light bullets seem to be unstable [7,16]. In the presence of energy dissipation, or through a feedback control, again the stabilization seems possible [17].

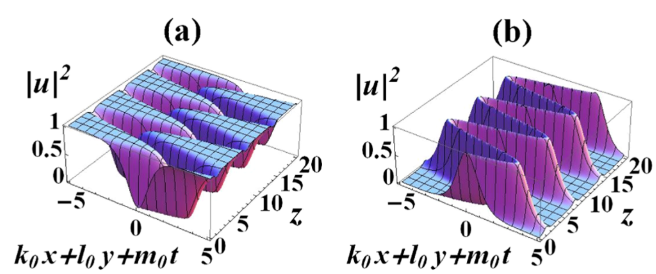

FIG. 3 (color online). Light bullet soliton solutions without chirp. The setup and parameters are as in Fig. 1, except for $M = 1$ and $a_0 = 0$.

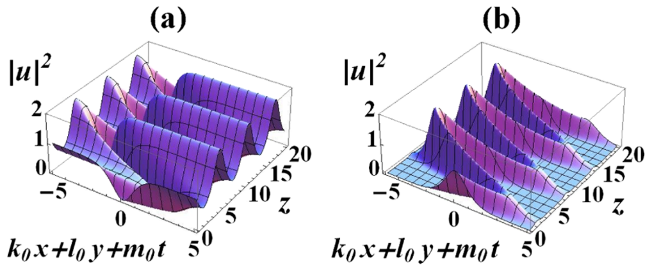

FIG. 4 (color online). Light bullet soliton solutions with chirp. Setup is the same as in Fig. 3, except for $a_0 = 0.1$.

Our situation is different: The solutions are not radially symmetric, and the modulation of both the diffraction or dispersion and the nonlinearity is effected concurrently. The issue of stability is involved and requiring a separate report (in preparation). We perform numerical solution of Eq. (1), with initial fields coming from Eq. (24), and with alternating positive and negative regions of nonlinearity. We utilize a split-step beam propagation method, adapted for the z-dependent coefficients. Our preliminary results indicate no collapse. Instead, stable propagation over tens of diffraction lengths is observed. An example of such behavior is displayed in Fig. 5, which essentially presents a numerical rerun of Fig. 3(b).

In conclusion, we have solved analytically the $(3+1)$-dimensional generalized nonlinear Schrödinger equation with distributed diffraction, dispersion, nonlinearity, and gain. A number of exact periodic wave solutions are found, and novel exact light bullet solutions are obtained. The influence of the spatiotemporal chirp function on the phase and the amplitude of solutions is displayed.

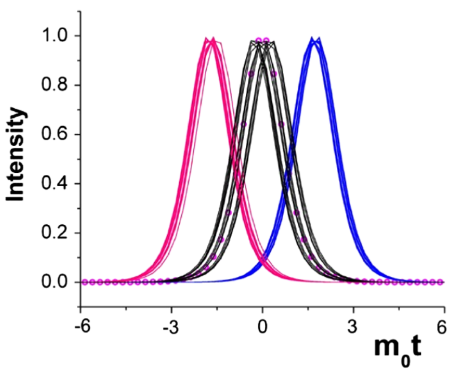

FIG. 5 (color online). Numerical simulation of the light bullet from Fig. 3(b). Initial data from Eq. (24) are propagated according to Eq. (1) for 90 diffraction lengths along the z axis. Only the dependence on t is shown. The initial profile is noted by open circles. The curves to the left present intensity profiles at the left turning point, the curves to the right the profiles at the right turning point. The curves at the center are snapshots of the profiles passing approximately through the point $t = 0$ (i.e., the frames closest to $t = 0$ are recorded). Three sets of 15 profiles are overlapped at different z points, to show that no instabilities develop.

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