Analytical chirped solutions to the (3 + 1)-dimensional Gross-Pitaevskii equation for various diffraction and potential functions

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Analytical solutions to the $(3+1)$-dimensional Gross-Pitaevskii equation in the presence of chirp and for different diffraction and potential functions are found. We utilize a method we formulated to solve the Riccati equation for the chirp function that arises when the F-expansion technique and the homogeneous balance principle are applied to the Gross-Pitaevskii equation. Three specific examples of physical interest are considered in some detail.

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I. INTRODUCTION

An enormous amount of research has been invested into the Gross-Pitaevskii equation (GPE) [1], a nonlinear evolution partial differential equation that arises in various fields of physics. The equation was introduced by Gross [2] and Pitaevskii [3] for unrelated problems but was later found useful in modeling various quantum systems, including, most notably, the Bose-Einstein condensates (BECs) [4]. Various, mostly numerical [5], solutions to the GPE have been found, prominently including localized wave solutions [6]. It is well-known that stable soliton solutions to the $(1+1)$-dimensional $[(1+1)D]$ equation exist and have already been found in many instances [7]. The difficulty arises when one attempts to find stable soliton solutions to the higher-dimensional GPE.

Recent research was more concerned with solving the multidimensional GP equation with distributed, i.e., time-dependent, coefficients [8,9]. The F-expansion technique and the homogeneous balance principle have proven effective when used to solve *analytically* the GPE in $(3+1)D$. In Ref. [10], a class of traveling wave solutions to the GPE for constant values of the diffraction and the quadratic potential coefficients was found. More recently, in Ref. [11] soliton solutions were found for the sinusoidal time-varying diffraction and potential functions. We extend the analysis here, to include localized solutions with chirp for a number of physically relevant choices for the diffraction and the potential function coefficients.

We consider the GP equation in $(3+1)D$ with distributed coefficients:

$$i\partial_t u + \frac{\beta(t)}{2}\Delta u + \chi(t)|u|^2 u + \alpha(t) r^2 u = i\gamma(t)u, \quad (1)$$

where Δ stands for the 3D Laplacian operator; r is the position coordinate; and α, β, χ, and γ are, respectively, the strength of the quadratic potential, the diffraction coefficient, the nonlinearity, and the linear gain/loss coefficient [10]. We solve this equation by utilizing the F expansion and the balance principle and determine specifically the localized solutions with nonzero chirp.

It should be noted that the stability of solutions to Eq. (1) is not an overriding concern when the coefficients keep changing in time, because the solutions are then *transient* in nature at all times. Therefore, they may blow up or diminish in time or tend to constant shapes. An especially interesting case is when the coefficients are periodic sign-changing functions; the methods of dispersion or diffraction or nonlinearity management then can be used to ensure stability [7].

The paper is divided into three sections. First, the method of solution is illustrated. The new solutions are presented next. Finally, the main findings are summarized in the concluding section of the paper.

II. SOLUTION METHOD

According to the F expansion and the balance principle techniques [12–16], the solution of Eq. (1) is sought in the form [10]:

$$u(x,y,z,t) = A(x,y,z,t)\exp[iB(x,y,z,t)], \quad (2)$$

where

$$A = f(t)F(\theta) + g(t)F^{-1}(\theta), \quad (3)$$

$$\theta = k(t)x + l(t)y + m(t)z + \omega(t), \quad (4)$$

$$B = a(t)(x^2+y^2+z^2) + b(t)(x+y+z) + e(t). \quad (5)$$

Here f, g, k, l, m, ω, a, b, and e are parameter functions to be determined, and F is one of the Jacobi elliptic functions (JEFs). After substituting the above expressions in Eq. (1), the following set of ordinary differential equation (ODEs) is obtained [10]:

$$\frac{df_j}{dt} + 3a\beta f_j - \gamma f_j = 0, \quad (6)$$

$$\frac{dk}{dt} + 2ka\beta = 0, \quad (7)$$

$$\frac{dl}{dt} + 2la\beta = 0, \quad (8)$$

$$\frac{dm}{dt} + 2ma\beta = 0, \quad (9)$$

$$\frac{db}{dt} + 2\beta ab = 0, \quad (10)$$

$$\frac{d\omega}{dt} + \beta(k+l+m)b = 0, \quad (11)$$

$$\frac{de}{dt} + \frac{\beta}{2}[3b^2 - (k^2+l^2+m^2)c_2] - 3\chi f_1 f_2 = 0, \quad (12)$$

$$\frac{da}{dt} + 2\beta a^2 - \alpha = 0, \quad (13)$$

where $j = 1,2$, $f_1 = f$, and $f_2 = g$. Additionally, two relations involving the functions f_1 and f_2 are found:

$$f_1\left[\beta(k^2 + l^2 + m^2)c_4 + \chi f_1^2\right] = 0, \tag{14}$$

$$f_2\left[\beta(k^2 + l^2 + m^2)c_0 + \chi f_2^2\right] = 0. \tag{15}$$

From the above ODEs one can see that the self-consistent solution of this system of equations can be found only if Eq. (13) for the parameter function $a(t)$ can be solved. The solutions of all other equations explicitly or implicitly depend on a, and the equation for a is a Riccati differential equation—which, in general, cannot be solved analytically. This testifies about the importance of the parameter function a, which is known as the *chirp* function. The equation for the chirp can only be solved analytically for specific coefficient functions. In Ref. [10] the solutions for constant α and β were found. In addition, the solutions for α and β being both sine or cosine functions were found in Ref. [11]. A more general result was obtained in Ref. [8], stating that if the following relation exists between the coefficient functions in Eq. (13):

$$\sqrt{-\frac{\beta}{\alpha}} = \sqrt{-\frac{\beta_0}{\alpha_0}} - \frac{2\sqrt{2}sA}{\sqrt{B}}\int_0^t \beta\, dt, \tag{16}$$

then the *general* solution to the Riccati equation (RE) can be found. Here A and $B > 0$ are arbitrary constants and $s = \pm 1$. Relation (16) is a strong mathematical statement, offering great many new solutions to RE. However, from the physics point of view, even more important is the fact that the coefficients α and β—the strength of the quadratic potential and the diffraction (dispersion) coefficient—are common quantities in BECs and therefore amenable to experimental manipulation. This opens the possibility of testing the limits of applicability of GPE to BECs. In this paper we utilize some specific solutions to RE satisfying relation (16) to obtain solutions to the GPE with distributed coefficients that might be useful in BE condensation.

Solving the system of ODEs (6)–(13), one discovers that all solutions can conveniently be expressed in terms of a single auxiliary function p, which is defined in terms of the chirp function a, as $p(t) = \exp(-2\int_0^t \beta a\, dt)$. The solutions are as follows:

$$f = f_0 p^{3/2} \exp\left(\int_0^t \gamma\, dt\right), \quad g = \epsilon\sqrt{\frac{c_0}{c_4}} f; \tag{17}$$

$$k = pk_0, \quad l = pl_0, \quad m = pm_0, \quad b = pb_0; \tag{18}$$

$$\omega = \omega_0 - (k_0 + l_0 + m_0)b_0 q; \tag{19}$$

$$e = e_0 + \tfrac{1}{2}\left[\left(k_0^2 + l_0^2 + m_0^2\right)(c_2 - 6\epsilon\sqrt{c_0 c_4}) - 3b_0^2\right]q, \tag{20}$$

where $q(t) = \int_0^t \beta p^2\, dt$. The solution for u then is:

$$u = f_0 p^{3/2} \exp\left(\int_0^t \gamma\, dt\right)\left[F(\theta) + \epsilon\sqrt{\frac{c_0}{c_4}}\frac{1}{F(\theta)}\right] \times \exp(i[a(x^2 + y^2 + z^2) + b(x + y + z) + e]), \tag{21}$$

where

$$\theta = (k_0 x + l_0 y + m_0 z)p + \omega_0 - (k_0 + l_0 + m_0)b_0 q.$$

We have chosen the form of the solution such that it is similar to the one in Ref. [11], but the expressions for the auxiliary functions $p(t)$ and $q(t)$ differ now. Here, $\epsilon = 0, \pm 1$. Note that we only consider solutions with $\epsilon = 0$ in order to avoid any singularities. The expression for the nonlinearity, as imposed by Eqs. (14) and (15), is:

$$\chi(t) = -c_4\beta(t)\left(k_0^2 + l_0^2 + m_0^2\right) f_0^{-2} p^{-1} \exp\left(-2\int_0^t \gamma\, dt\right). \tag{22}$$

This relation, involving coefficients χ, β, and γ of the original GPE (1), places a restriction on the solutions obtainable by the present method. Hence, we can determine analytical solutions of Eq. (1) by the present method, only when the four coefficients of that equation satisfy the two relations contained in Eqs. (16) and (22). One should note that the method offers traveling wave solutions to Eq. (1) as well as localized solutions, depending on the choice of JEF and on the value of the elliptic modulus of JEF. We confine here attention only to the localized solutions, by choosing $F = \mathrm{sech}$. Moreover, note that our choice of the value of γ can have an effect on the behavior of the solution. For instance, choosing a negative value will eventually cause the soliton to die away with time, as will be seen in one of the examples.

III. RESULTS

We consider the solution to GPE for the three physically relevant cases: first, when $\beta = \frac{1}{2}(1 + e^{-\delta t})$; second, when $\beta = \cos(\Omega t)$; and, third, when $\beta = \tilde{\beta}(1 - \frac{D}{B_1 t - B_0})$. The first case presents a continuous change in the diffraction coefficient from 1 to 1/2, which might be difficult to realize in actual physical systems but provides simple analytical solutions. The second, sign-reversing, case is relevant for periodic systems with dispersion or diffraction management [7]; it is important for displaying stable but breathing solutions. The third case introduces the diffraction coefficient in the form of a Feshbach resonance function, which is important for BECs.

A. Case 1: $\beta = \frac{1}{2}(1 + e^{-\delta t})$

For this case, α is determined according to the relation (16), to be

$$\alpha = -\frac{1 + e^{-\delta t}}{2\left(1 + \frac{\sqrt{2}(1 - e^{-\delta t} + \delta t)}{\delta}\right)^2}. \tag{23}$$

After solving the relevant ODEs, we arrive at the following form for the auxiliary function $p(t)$:

$$p = -\frac{2^{5/4} e^{-\frac{\delta t}{2}}\sqrt{\frac{-2 + e^{\delta t}(2 + (\sqrt{2} + 2t)\delta)}{\delta}}}{-2\sqrt{2} + \sqrt{2}t\delta + 2t\delta a_0 + \ln\left[\frac{\delta}{-\sqrt{2} + e^{\delta t}(\sqrt{2} + \delta + \sqrt{2}t\delta)}\right](\sqrt{2} + 2a_0)}. \tag{24}$$

The solution to the RE, which yields the appropriate chirp function, has the following form:

$$a(t) = \frac{-\delta}{2 - 2e^{-\delta t} + \sqrt{2}\delta + 2t\delta} - \frac{\delta\sqrt{2}e^{\delta t}}{[-\sqrt{2} + e^{\delta t}(\sqrt{2} + \delta + \sqrt{2}t\delta)]\zeta(t)}, \quad (25)$$

where

$$\zeta(t) = \delta t + \ln\left|\frac{\delta}{-\sqrt{2} + e^{\delta t}(\sqrt{2} + \delta + \sqrt{2}t\delta)}\right| - \frac{2}{1 + \sqrt{2}a_0}.$$

Figure 1 shows how the solution of GPE looks like for this case. One may note that after an initial rapid change, the pulse settles into a slowly evolving bright solitary solution. It should be stressed that this solution, as well as other localized solutions in this paper, are not solitons in the usual sense of the word. This comes about because the coefficients in GPE are continuously changing time-dependent functions, which also cause the chirp function to continuously change. These changes in turn cause continuous reshaping of the solitary wave—relatively mild in cases 1 and 3 but more dramatic in case 2. Hence, as mentioned, the solutions are transient in nature at all times. Similar findings have been reported in Ref. [10]. Note also that, although these solutions are localized when viewed as functions of the transverse variable θ, they are *not* localized when viewed in real space. There, a direction always exists in which these solutions are *extended* [10].

B. Case 2: $\beta = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\Omega t)^{2n} = \cos(\Omega t)$

From relation (16), we find α to be

$$\alpha = -\frac{\cos(\Omega t)}{\left(1 + 2\sqrt{2}\,\frac{\sin(\Omega t)}{\Omega}\right)^2}. \quad (26)$$

For this case, the expression for p becomes

$$p = \frac{2\sqrt{2 + \frac{4\sqrt{2}\sin(\Omega t)}{\Omega}}}{2\sqrt{2} + \ln\left[1 + \frac{2\sqrt{2}\sin(\Omega t)}{\Omega}\right](\sqrt{2} + 2a_0)}. \quad (27)$$

Next, we find $q(t)$:

$$q = \frac{e^r(2\Omega^2((2a_0 + \sqrt{2})\ln(\xi_2(t)) + 2\sqrt{2})\mathrm{Ei}(2\ln(\xi_2(t)) - r) - e^{-r}\xi_1(t))}{\Omega^2((a_0(2a_0(\sqrt{2}a_0 + 3) + 3\sqrt{2}) + 1)\ln(\xi_2(t)) + 4a_0(a_0 + \sqrt{2}) + 2)} + \xi_3(t),$$

where

$$\xi_1(t) = 8(\sqrt{2}a_0 + 1)\Omega\sin(\Omega t) - 4(2a_0 + \sqrt{2})\cos(2\Omega t) + (2a_0 + \sqrt{2})(\Omega^2 + 4),$$

$$\xi_2(t) = \frac{2\sqrt{2}\sin(\Omega t)}{\Omega} + 1, \quad \xi_3(t) = \frac{-4e^{-\frac{4}{\sqrt{2}a_0+1}}\mathrm{Ei}\left(\frac{4}{\sqrt{2}a_0+1}\right) + \sqrt{2}a_0 + 1}{\sqrt{2}(\sqrt{2}a_0 + 1)^2}, \quad r = -\frac{4}{\sqrt{2}a_0 + 1},$$

and $\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\,dt$. The chirp function then has the following form [8]:

$$a(t) = \frac{2\sqrt{2}a_0 - (\sqrt{2}a_0 + 1)\ln\left(\frac{2\sqrt{2}\sin(\Omega t)}{\Omega} + 1\right)}{\left(\frac{2\sqrt{2}\sin(\Omega t)}{\Omega} + 1\right)\left((2a_0 + \sqrt{2})\ln\left(\frac{2\sqrt{2}\sin(\Omega t)}{\Omega} + 1\right) + 2\sqrt{2}\right)}. \quad (28)$$

Note that here a proper choice of Ω had to be made to ensure that the solutions do not blow up. Figure 2 displays the behavior of the solution. It is seen that the choice of periodic β produces a *breathing* localized solution. This solution looks like a regular breather when the parameter function b of the solution is equal to 0 but wiggles back and forth, keeping an asymmetric profile when $b \neq 0$.

C. Case 3: $\beta = \tilde{\beta}\left(1 - \frac{D}{B_1 t - B_0}\right)$

Recall that this form of β is the one that usually arises from the dependence of the scattering length on the magnetic field close to the Feshbach resonance of cold BEC atoms. Note that

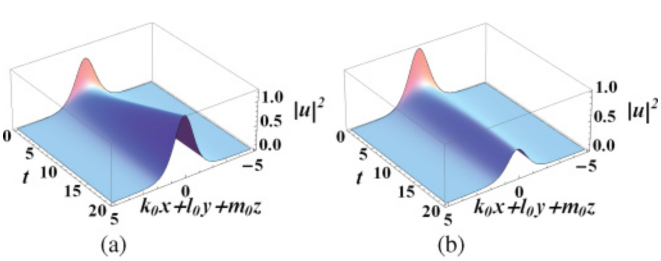

FIG. 1. (Color online) Intensity distribution $|u|^2$ for the solution of case 1. (a) No gain/loss. (b) $\gamma = -0.05$. Here $F = \mathrm{sech}$. The parameters are $a_0 = f_0 = k_0 = l_0 = m_0 = \omega_0 = 1$, $b_0 = \epsilon = 0$, and $\delta = 5$.

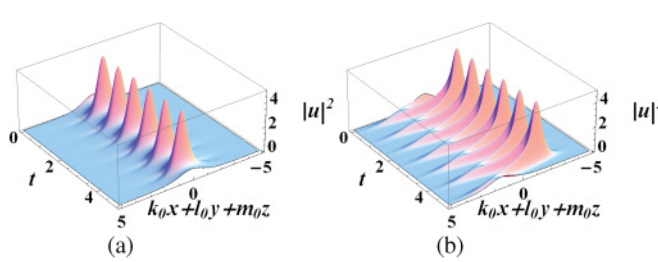

FIG. 2. (Color online) Intensity distribution in case 2, with $b_0 = 0$ in (a) and $b_0 = 5$ in (b). Here $F = \mathrm{sech}$. In both cases, $\Omega = 8$. Other parameters: $a_0 = f_0 = k_0 = l_0 = m_0 = \omega_0 = 1$, $\gamma = \epsilon = 0$.

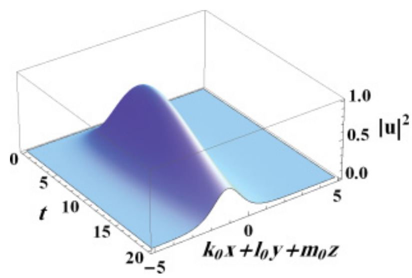

FIG. 3. (Color online) Intensity distribution in case 3 with no gain/loss ($\gamma = 0$). Here $F = \text{sech}$. Other parameters: $D = 10$, $B_0 = -1$, $a_0 = B_1 = f_0 = \omega_0 = e_0 = k_0 = l_0 = m_0 = \tilde{\beta} = 1$, and $b_0 = 0$.

here the magnetic field has a linear dependance on time. Also, note the following important relations from [8]:

$$\alpha(t) = -\tilde{\beta}\frac{1-\frac{D}{B_1t-B_0}}{\left[1-2\sqrt{2}\tilde{\beta}t+\frac{2\sqrt{2}\tilde{\beta}D}{B_1}\ln\left|\frac{B_1t-B_0}{B_0}\right|\right]^2}, \quad (29)$$

$$\phi(t) = 2\sqrt{2}\tilde{\beta}\int_0^t \frac{1-\frac{D}{B_1\tau-B_0}}{1-2\sqrt{2}\tilde{\beta}\tau+\frac{2\sqrt{2}\tilde{\beta}D}{B_1}\ln\left|\frac{B_1\tau-B_0}{B_0}\right|}\,d\tau, \quad (30)$$

$$a(t) = \frac{1}{\sqrt{2}-2\tilde{\beta}t+\frac{4\tilde{\beta}D}{B_1}\ln\left|\frac{B_1t-B_0}{B_0}\right|} + \frac{e^{\phi(t)}}{\frac{\sqrt{2}}{a_0\sqrt{2}-1}+2\tilde{\beta}\int_0^t\left(1-\frac{D}{B_1\tau-B_0}\right)e^{\phi(\tau)}\,d\tau}. \quad (31)$$

In this paper, we will not deal with the singularities that result from the resonance form of β. Therefore, we will choose parameters B_1 and B_0 such that the denominator remains finite. This can be accomplished, for example, if we choose $B_1 = 1$ and $B_0 = -1$. Note that here we do not state an explicit form of the parameter function p, as the expression for the integrals cannot be found in terms of simple elementary functions. We choose to keep the closed-form solutions in integral form and visualize these solutions instead. The behavior of the solution is shown in Fig. 3. The parameter values are properly chosen such that the solution does not blow up. The solution starts from small initial values but rapidly grows and then continuously attenuates. The fact that the solitary solution attenuates in time should not be alarming because the BE condensate lasts and the GP model describing it is valid only in a limited time interval.

IV. CONCLUSION

We have found exact localized solutions to GPE for a few examples of the diffraction and potential functions in potentially useful and realizable forms. We utilized the F-expansion technique and the homogeneous balance principle to obtain these solutions.

For the first case, it can be inferred that the intensity linearly grows with time, even when there is no gain imposed on the system. To prevent the intensity from becoming arbitrarily large, some loss should be added to the system. In the example shown this is achieved by choosing $\gamma = -0.05$.

The second case yields solutions akin to breathing solitons. Such solutions propagate stably, with a periodic change in the profile. It can be clearly observed that the addition of the parameter b_0 causes the periodic change in the soliton's direction and shape. The amplitude of the solitons does not change when there is no gain or loss added to the system.

Finally, the third case produces a localized solitary wave essentially confined to a finite interval of time. The solution quickly grows and then decays in time, the reason being the presence of a resonance in the diffraction coefficient. This is clearly observed in Fig. 3. For other values of parameters the solution might collapse. If the parameter b_0 were chosen to differ from 0, the same solitary wave would curve and change direction. However, the wave would not wiggle back and forth in this case, as in the second case, because the diffraction function is not periodic in time. We emphasize that the three aforementioned cases are possible to implement in real physical systems.

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