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Exact traveling wave solutions to coupled generalized nonlinear Schrödinger equations

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Abstract
Exact extended traveling wave solutions are found for the system of generalized nonlinear Schrödinger equations for co- and counterpropagating beams with self-phase and cross-phase modulation. A number of stable periodic solutions are obtained whose signal does not decrease in time in the absence of externally induced loss.

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(Some figures may appear in colour only in the online journal)

The generalized nonlinear Schrödinger equation (NLSE) is a generic model that is very important for NL optics, where it describes full spatiotemporal optical solitons or light bullets [1–4]. In recent years, there has been tremendous development in obtaining stable spatiotemporal soliton solutions for a large number of transverse dimensions [4, 5]. The traveling wave and soliton solutions to the generalized NLSE in $(3+1)$ dimensions ($(3+1)$D) with cubic nonlinearity were first developed in [6] for anomalous dispersion and then generalized in [7] to normal dispersion.

Of particular interest is to extend the method of finding these exact solutions to multicomponent systems. A system of great interest is the case of two co- or counterpropagating beams interacting with each other through Kerr nonlinearity [8]. Such a system is described by two coupled generalized NLSEs, also known as the generalized Manakov model [9]. The interaction of two counterpropagating beams may produce various forms of instability and bifurcation into more chaotic regimes. It is of interest to find these exact solutions and then check their stability by propagating them numerically and analytically [6, 10]. Here, however, we will only concern ourselves with finding novel exact solutions; the stability will be discussed in a subsequent paper. We consider only the $(1+1)$D case.

In this paper, we generalize the results of [6] and [7] to the case of two-component co- or counterpropagating beams in Kerr-like media. We consider the generalized nonlinear Schrödinger equations for two components, with cross-phase modulation (XPM) included, in $(1+1)$D:

$$\mathrm{i}\,\partial_z u_1 + \frac{\beta(z)}{2}\partial_x^2 u_1 + \chi(z)\left(|u_1|^2 + c|u_2|^2\right)u_1 = \mathrm{i}\,\gamma(z)u_1, \quad (1)$$

$$s\,\mathrm{i}\,\partial_z u_2 + \frac{\beta(z)}{2}\partial_x^2 u_2 + \chi(z)\left(|u_2|^2 + c|u_1|^2\right)u_2 = s\,\mathrm{i}\,\gamma(z)u_2, \quad (2)$$

which describes a system of two interacting light beams, u_1 and u_2, in a medium with Kerr nonlinearity. Here z is the propagation (i.e. longitudinal) coordinate and x the transverse variable. All coordinates are made dimensionless by the choice of coefficients. The functions β, χ and γ stand for the diffraction/dispersion, nonlinearity and gain coefficients, respectively. The coefficient s determines whether the two beams are copropagating (in which case $s=1$) or counterpropagating (in which case $s=-1$). The constant c determines the ratio of the coupling strengths of XPM to the self-phase modulation (SPM).

We assume a general ansatz of the following form:

$$u_1(z,x) = A_1(z,x)\exp(\mathrm{i}\,B_1(z,x)), \quad (3)$$

$$u_2(z,x) = A_2(z,x)\exp(\mathrm{i}\,B_2(z,x)). \quad (4)$$

The forms of A_1, A_2, B_1 and B_2 are assumed to be

$$A_1 = f_1(z)F(\theta) + f_2(z)G(\theta), \tag{5}$$

$$A_2 = g_1(z)G(\theta) + g_2(z)F(\theta), \tag{6}$$

$$\theta = k(z)x + \omega(z), \tag{7}$$

$$B_1 = a_1(z)x^2 + b_1(z)x + e_1(z), \tag{8}$$

$$B_2 = a_2(z)x^2 + b_2(z)x + e_2(z), \tag{9}$$

where F and G are two suitable Jacobi elliptic functions (JEFs) [11] (to be determined) satisfying the following differential equations:

$$\left(\frac{\mathrm{d}F}{\mathrm{d}\theta}\right)^2 = c_0 + c_2F^2 + c_4F^4, \tag{10}$$

$$\left(\frac{\mathrm{d}G}{\mathrm{d}\theta}\right)^2 = d_0 + d_2F^2 + d_4F^4. \tag{11}$$

The values of the parameters c_0, c_2 and c_4 for each JEF are given in table 1 of [5].

By applying the F-expansion method and using the principle of harmonic balance as described in [6], with the provision that we now treat two components as polynomials in two JEFs F and G instead of one, we obtain the following equations for f_1, f_2, g_1, g_2, ω, and k:

$$\frac{\mathrm{d}f_j}{\mathrm{d}z} + a_1\beta f_j - \gamma f_j = 0, \tag{12}$$

$$s\frac{\mathrm{d}g_j}{\mathrm{d}z} + a_2\beta g_j - s\gamma g_j = 0, \tag{13}$$

$$\frac{\mathrm{d}k}{\mathrm{d}z} + 2ka_1\beta = 0, \tag{14}$$

$$s\frac{\mathrm{d}k}{\mathrm{d}z} + 2ka_2\beta = 0, \tag{15}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}z} + \beta kb_1 = 0, \tag{16}$$

$$s\frac{\mathrm{d}\omega}{\mathrm{d}z} + \beta kb_2 = 0, \tag{17}$$

where $j = 1, 2$. From equations (14) and (15) one obtains $a_1 = sa_2 = a$ and from (16) and (17) one obtains $b_1 = sb_2 = b$.

We proceed to solve equations (12)–(17) to obtain the following solutions for f_j, g_j, k, ω, a_j and b_j:

$$f_{1,2}(z) = (\alpha)^{1/2} f_{1,2}\exp\left(\int_0^z \gamma\,\mathrm{d}z\right), \tag{18}$$

$$g_{1,2}(z) = (\alpha)^{1/2} g_{1,2}\exp\left(\int_0^z \gamma\,\mathrm{d}z\right), \tag{19}$$

$$k(z) = \alpha k_0, \tag{20}$$

$$\omega(z) = \omega_0 - \alpha k_0 b_0 \int_0^z \beta\,\mathrm{d}z, \tag{21}$$

$$a_1(z) = sa_2(z) = a(z) = \alpha a_0, \tag{22}$$

$$b_1(z) = sb_2(z) = b(z) = \alpha b_0, \tag{23}$$

where $j = 1, 2$ and $\alpha = (1 + 2a_{1,0}\int_0^z \beta\,\mathrm{d}z)^{-1} = (1 + 2a_{1,0}\int_0^z \beta\,\mathrm{d}z)^{-1}$ is the chirp function. Note that the chirp functions for a_1 and a_2 are identical. When $f_2 = g_1 = g_2 = 0$, equations (18)–(22) reduce to those in [6] for the case of a single 1D beam ($N = 1$) with $\epsilon = 0$.

One now needs to find the relationship between $f_{1,0}$, $f_{2,0}$, $g_{1,0}$ and $g_{2,0}$, as well as the formula for e. This is determined from equations which are analogous to those for χ and e in [6]. In the case of the co- and counterpropagating beams these equations will induce constraints on the forms of F and G that we can use, as well as the associated parameters c_0, c_2, c_4, d_0, d_2 and d_4.

For the equation analogous to the formula for e in [6], we obtain

$$\frac{\mathrm{d}e_1}{\mathrm{d}z} + \frac{\beta}{2}\left(b^2 - kc_2\right) = 0, \tag{24}$$

$$s\frac{\mathrm{d}e_2}{\mathrm{d}z} + \frac{\beta}{2}\left(b^2 - kc_2\right) = 0, \tag{25}$$

and similar formulae with d_2 instead of c_2. It follows that $c_2 = d_2$ and

$$e_1(z) = e_{1,0} + \frac{1}{2}(c_2k_0^2 - b_0^2)\alpha\int_0^z \beta\,\mathrm{d}z, \tag{26}$$

$$e_2(z) = e_{2,0} + s\frac{1}{2}(c_2k_0^2 - b_0^2)\alpha\int_0^z \beta\,\mathrm{d}z. \tag{27}$$

The equations analogous to the algebraic equations for χ in [6] are

$$f_1\left(\beta c_4k^2 + \chi f_1^2 + c\chi g_2^2\right) = 0, \tag{28}$$

$$f_2\left(\beta d_4k^2 + \chi f_2^2 + c\chi g_1^2\right) = 0, \tag{29}$$

$$g_1\left(\beta d_4k^2 + \chi g_1^2 + c\chi f_2^2\right) = 0, \tag{30}$$

$$g_2\left(\beta c_4k^2 + \chi g_2^2 + c\chi f_1^2\right) = 0. \tag{31}$$

Also, an additional set of constraints emerges

$$3\chi f_1^2f_2 + 2c\chi f_1g_1g_2 + c\chi f_2g_2^2 = 0, \tag{32}$$

$$3\chi f_2^2f_1 + 2c\chi f_2g_1g_2 + c\chi f_1g_1^2 = 0, \tag{33}$$

$$3\chi g_1^2g_2 + 2c\chi g_1f_1f_2 + c\chi g_2f_2^2 = 0, \tag{34}$$

$$3\chi g_2^2g_1 + 2c\chi g_2f_1f_2 + c\chi g_1f_1^2 = 0. \tag{35}$$

From equations (28)–(31), one obtains

$$(c-1)f_1^2 = (c-1)g_2^2, \tag{36}$$

$$(c-1)f_2^2 = (c-1)g_1^2, \tag{37}$$

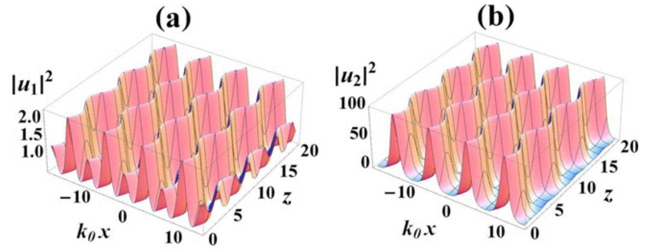

Figure 1. Traveling wave solutions for $F = \mathrm{dn}$ and $G = \mathrm{nd}$ constant as functions of time. Intensity (a) $|u_1|^2$ and (b) $|u_2|^2$ is presented as a function of k_0x and z for $\beta(z) = \beta_0 \cos \Omega z$. Coefficients: $M = 0.99$, $b_0 = 0$, $e_0 = 0$, $k_0 = l_0 = m_0 = 1$, $\omega_0 = 0$, $\Omega = 1$, $\beta_0 = 1$, $\gamma = 0$, $\epsilon = \phi = 1$ and $\delta = -1$.

from which $f_1 = \pm g_2$ and $f_2 = \pm g_1$, or $c = 1$. However, when $c = 1$ a contradiction among the equations is obtained. For $f_1 = \pm g_2$ and $f_2 = \pm g_1$, after some analysis we obtain

$$c = 3, \tag{38}$$

$$g_2 = \delta f_1, \tag{39}$$

$$f_2 = \epsilon g_1, \tag{40}$$

$$\frac{f_1}{g_1} = \phi \sqrt{\frac{c_4}{d_4}}, \tag{41}$$

where both δ, $\phi = \pm 1$ and $\epsilon = -\delta$. Also, for nontrivial solutions we must have $c_4 \neq d_4$. The combinations of F and G that satisfy $c_2 = d_2$ and $c_4 \neq d_4$ are (not including symmetries between F and G): $F = \mathrm{sn}$, $G = \mathrm{ns}$ or dc; $F = \mathrm{cd}$, $G = \mathrm{ns}$ or dc; $F = \mathrm{dn}$, $G = \mathrm{nd}$ and $F = \mathrm{sc}$, $G = \mathrm{cs}$. In all the cases, we must have $M \neq 1$ and $M \neq 0$ for our solutions to be nontrivial, where M is the elliptic modulus of the JEFs. Since $M = 1$ corresponds to the soliton solutions, such solutions are not provided by the present method.

Thus, we find that only for $c = 3$ one obtains the traveling wave solutions of the form given in equations (5)–(9) by the present method. This, of course, does not preclude the existence of solutions for other values of c. For example, the value of $c = 1$ corresponds to the simple Manakov model. It is known that the Manakov model is integrable by the inverse scattering method [9]. As to the feasibility of constructing materials for which $c = 3$, in [12] it has been found that by using periodically poled photorefractive media, one can eliminate SPM, and since this process can be graded a full range of real values of c can be achieved.

We now present the traveling wave solutions obtained. Of most interest are solutions in the case $F = \mathrm{dn}$, $G = \mathrm{nd}$ since for $M \neq 1$ functions dn and nd do not have singularities and are therefore physically preferable and experimentally realizable. We present the results in figure 1. The general form of the solutions is the same for both co- and counterpropagating beams. We see that the solutions remain periodic and do not decay in the absence of loss.

In figure 2, the effects of chirp on the solutions are displayed. The effects are similar to those described in [6] for a one-component system, namely a modulation of the amplitude and a deformation of the wave along the k_0x-axis.

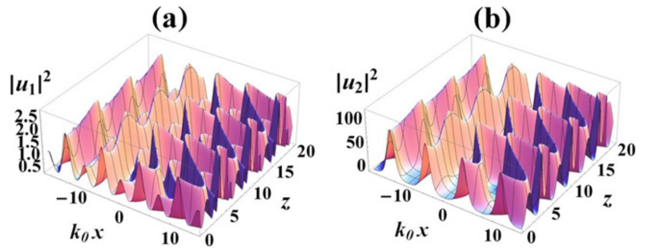

Figure 2. Traveling wave solutions as functions of time. The parameters are the same as in figure 1 except for $a_0 = 0.1$.

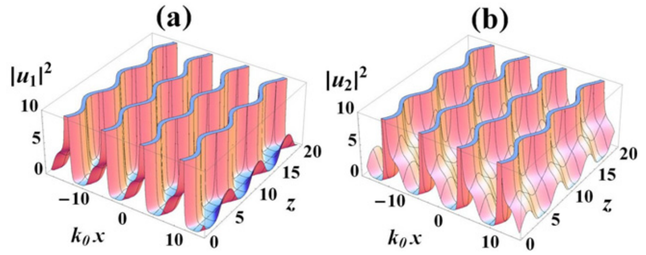

Figure 3. Traveling wave solutions as functions of time. The parameters are the same as in figure 1 except for $F = \mathrm{sn}$ and $G = \mathrm{dc}$.

Finally, in figure 3 a combination of two different types of functions is presented, giving us novel solutions to the coupled NLSE, albeit with a singularity.

In conclusion, we have obtained exact traveling-wave solutions to the two-component (1 + 1)D NLSE with Kerr nonlinearity and an XPM three times stronger than the SPM.

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